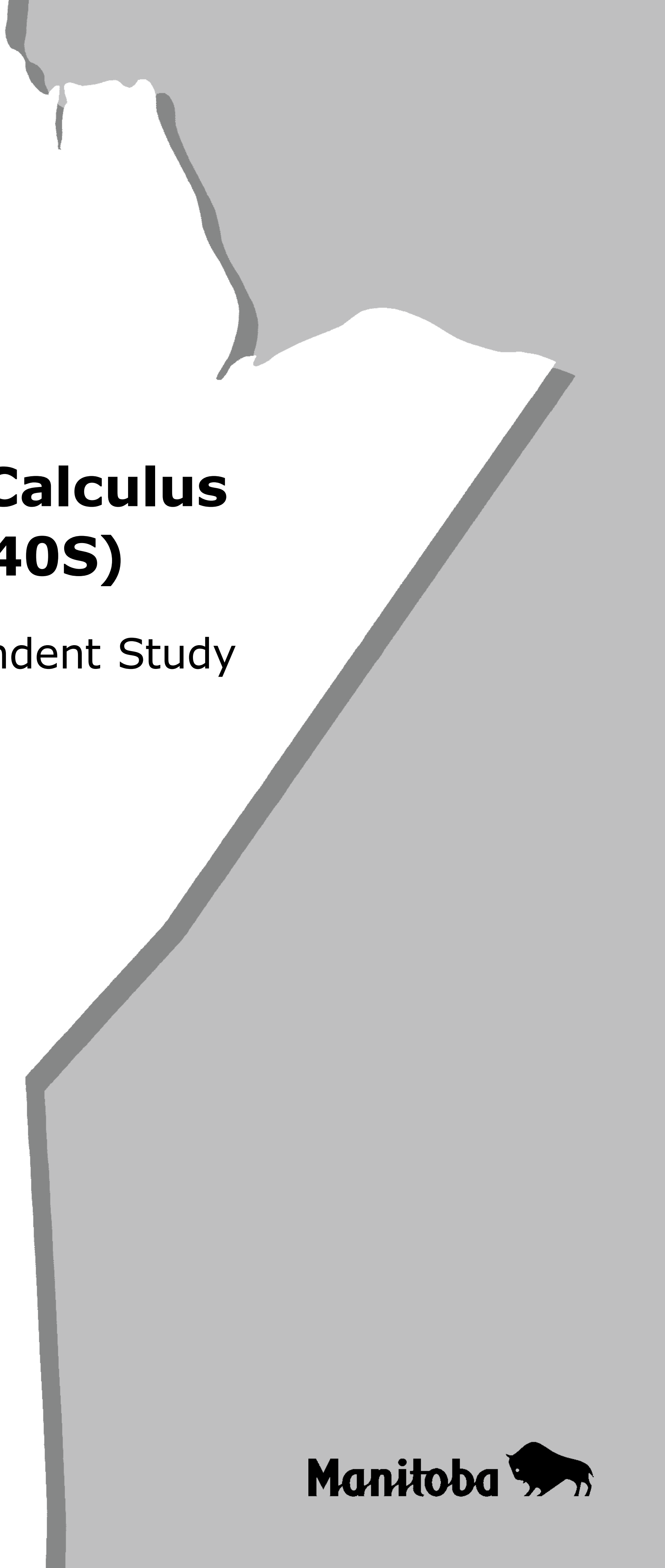
Grade 12 Pre-Calculus Mathematics (40S)

A Course for Independent Study



Manitoba

GRADE 12 PRE-CALCULUS (40S)

A Course for Independent Study

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GRADE 12 PRE-CALCULUS (40S)

Introduction

INTRODUCTION

Overview

Welcome to Grade 12 Pre-Calculus Mathematics! This course is a continuation of the concepts you have studied in previous years, as well as an introduction to new topics. It builds upon the pre-calculus topics you were introduced to in *Grade 11 Pre-Calculus Mathematics*. You will put to use many of the skills that you have already learned to solve problems and learn new skills along the way. This course helps you develop the skills, ideas, and confidence you will need to continue studying math in the future.

As a student enrolled in an independent study course, you have taken on a dual role—that of a student and a teacher. As a student, you are responsible for mastering the lessons and completing the learning activities and assignments. As a teacher, you are responsible for checking your work carefully, noting areas in which you need to improve and motivating yourself to succeed.

What Will You Learn in This Course?

In this course, problem solving, communication, reasoning, and mental math are some of the themes you will discover in each module. You will engage in a variety of activities that promote the connections between symbolic math ideas and the world around you.

There are three main areas that you will be exploring: Number, Patterns and Relations, and Shape and Space.

How Is This Course Organized?

The Grade 12 Pre-Calculus Mathematics course consists of the following eight modules:

- Module 1: Permutations, Combinations, and the Binomial Theorem
- Module 2: Function Transformations
- Module 3: Reflections
- Module 4: Polynomials
- Module 5: Trigonometric Functions and the Unit Circle
- Module 6: Trigonometric Equations and Identities
- Module 7: Exponents and Logarithms
- Module 8: Radical and Rational Functions

Each module in this course consists of several lessons, which contain the following components:

- **Lesson Focus:** The Lesson Focus at the beginning of each lesson identifies one or more specific learning outcomes (SLOs) that are addressed in the lesson. The SLOs identify the knowledge and skills you should have achieved by the end of the lesson.
- **Introduction:** Each lesson begins by outlining what you will be learning in that lesson.
- **Lesson:** The main body of the lesson consists of the content and processes that you need to learn. It contains information, explanations, diagrams, and completed examples.
- **Learning Activities:** Each lesson has a learning activity that focuses on the lesson content. Your responses to the questions in the learning activities will help you to practise or review what you have just learned. Once you have completed a learning activity, check your responses with those provided in the Learning Activity Answer Key found at the end of the applicable module. Do not send your learning activities to the Distance Learning Unit for assessment.
- **Assignments:** Assignments are found throughout each module within this course. At the end of each module, you will mail or electronically submit all your completed assignments from that module to the Distance Learning Unit for assessment. All assignments combined will be worth a total of 55 percent of your final mark in this course.
- **Summary:** Each lesson ends with a brief review of what you just learned.

There are two online resources for this course.

- Glossary
- Graph Paper

The online resources are found in the learning management system (LMS). If you do not have access to the Internet, contact the Distance Learning Unit at 1-800-465-9915 to get a copy of these online resources.

What Resources Will You Need for This Course?

You do not need a textbook for this course. All of the content is provided directly within the course.

Required Resources

Required resources for this course are a scientific calculator and graph paper. Graph paper is available as one of the online resources.

You will also require access to an email account if you plan to

- communicate with your tutor/marker by email
- use the learning management system (LMS) to submit your completed assignments

Optional Resources

It would be helpful if you had access to the following resources:

- **A computer with spreadsheet and graphing capabilities:** Access to a computer with spreadsheet software and graphing capabilities may be helpful to you for exploration and checking your understanding. However, none of these are required or allowed when writing either your midterm or final examination.
- **A computer with Internet access:** Use of the Internet may be suggested as a resource in some places, but if you do not have access to an online computer you can still complete the related learning activities and assignments without it.
- **A photocopier:** With access to a photocopier/scanner, you could make a copy of your assignments before submitting them so that if your tutor/marker wants to discuss an assignment with you over the phone, each of you will have a copy. It would also allow you to continue studying or to complete further lessons while your original work is with the tutor/marker. Photocopying or scanning your assignments will also ensure that you keep a copy in case the originals are lost.

Who Can Help You with This Course?

Taking an independent study course is different from taking a course in a classroom. Instead of relying on the teacher to tell you to complete a learning activity or an assignment, you must tell yourself to be responsible for your learning and for meeting deadlines. There are, however, two people who can help you be successful in this course: your tutor/marker and your learning partner.

Your Tutor/Marker

Tutor/markers are experienced educators who tutor Independent Study Option (ISO) students and mark assignments and examinations. When you are having difficulty with something in this course, contact your tutor/marker, who is there to help you. Your tutor/marker's name and contact information were sent to you with this course. You can also obtain this information in the learning management system (LMS).

Your Learning Partner

A learning partner is someone **you choose** who will help you learn. It may be someone who knows something about mathematics, but it doesn't have to be. A learning partner could be someone else who is taking this course, a teacher, a parent or guardian, a sibling, a friend, or anybody else who can help you. Most importantly, a learning partner should be someone with whom you feel comfortable and who will support you as you work through this course.

Your learning partner can help you keep on schedule with your coursework, read the course with you, check your work, look at and respond to your learning activities, or help you make sense of assignments. You may even study for your examination(s) with your learning partner. If you and your learning partner are taking the same course, however, your assignment work should not be identical.

One of the best ways that your learning partner can help you is by reviewing your midterm and final practice examinations with you. These are found in the learning management system (LMS), along with their answer keys. Your learning partner can administer your practice examination, check your answers with you, and then help you learn the things that you missed.

How Will You Know How Well You Are Learning?

You will know how well you are learning in this course by how well you complete the learning activities, assignments, and examinations.

Learning Activities

The learning activities in this course will help you to review and practise what you have learned in the lessons. You will not submit the completed learning activities to the Distance Learning Unit. Instead, you will complete the learning activities and compare your responses to those provided in the Learning Activity Answer Key found at the end of each module.

Each learning activity has two parts—Part A has BrainPower questions and Part B has questions related to the content in the lesson.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for you before trying the other questions. Each question should be completed quickly and without using a calculator, and most should be completed without using pencil and paper to write out the steps. Some of the questions will relate directly to content of the course. Some of the questions will review content from previous courses—content that you need to be able to answer efficiently.

Being able to do these questions in a few minutes will be helpful to you as you continue with your studies in mathematics. If you are finding it is taking you longer to do the questions, you can try one of the following:

- work with your learning partner to find more efficient strategies for completing the questions
- ask your tutor/marker for help with the questions
- search online for websites that help you practice the computations so you can become more efficient at completing the questions

None of the assignment questions or examination questions will require you to do the calculations quickly or without a calculator. However, it is for your benefit to complete the questions as they will help you in the course. Also, being able to successfully complete the BrainPower exercises will help build your confidence in mathematics. BrainPower questions are like a warm-up you would do before competing in a sporting event.

Part B: Course Content Questions

One of the easiest and fastest ways to find out how much you have learned is to complete Part B of the learning activities. These have been designed to let you assess yourself by comparing your answers with the answer keys at the end of each module. There is at least one learning activity in each lesson. You will need a notebook or loose-leaf pages to write your answers.

Make sure you complete the learning activities. Doing so will not only help you to practise what you have learned, but will also prepare you to complete your assignments and the examinations successfully. Many of the questions on the examinations will be similar to the questions in the learning activities. Remember that you **will not submit learning activities to the Distance Learning Unit**.

Assignments

Lesson assignments are located throughout the modules, and include questions similar to the questions in the learning activities of previous lessons. The assignments have space provided for you to write your answers on the question sheets. **You need to show all your steps as you work out your solutions, and make sure your answers are clear (include units, where appropriate).**

Once you have completed all the assignments in a module, you will submit them to the Distance Learning Unit for assessment. The assignments are worth a total of 55 percent of your final course mark. You must complete each assignment in order to receive a final mark in this course. **You will mail or electronically submit these assignments to the Distance Learning Unit along with the appropriate cover page once you complete each module.** Check page 15 for Marking Guidelines that will be used for assignments and examinations.

The tutor/marker will mark your assignments and return them to you. Remember to keep all marked assignments until you have finished the course so that you can use them to study for your examinations.

Resource Sheet

When you write your midterm and final examinations, you will be allowed to take an Examination Resource Sheet with you into the examination. This sheet will be one letter-sized page, 8½ ″ by 11″, with both sides in your handwriting or typewritten. It is to be submitted with your examination. The Examination Resource Sheet is not worth any marks.

Creating your own resource sheet is an excellent way to review. It also provides you with a convenient reference and quick summary of the important facts of each module. Each student is asked to complete a resource sheet for each module to help with studying and reviewing.

The lesson summaries are written for you to use as a guide, as are the module summaries at the end of each module. Refer to these when you create your own resource sheet. Also, refer to the online glossary found in the learning management system (LMS) to check the information on your resource sheet.

After completing each module's resource sheet, you should summarize the sheets from all of the modules to prepare your Examination Resource Sheet. When preparing your Midterm Examination Resource Sheet, remember that your midterm examination is based on Modules 1 to 4. When preparing your Final Examination Resource Sheet, remember that your final examination is based on the entire course, Modules 1 to 8.

Midterm and Final Examinations

This course contains a midterm examination and a final examination.

- The **midterm examination** is based on Modules 1 to 4, and is worth 20 percent of your final course mark. You will write the midterm examination when you have completed Module 4.
- The **final examination** is cumulative, so it is based on Modules 1 to 8 and is worth 25 percent of your final course mark. The examination content will consist of 20% from Modules 1 to 4 and 80% from Modules 5 to 8. You will write the final examination when you have completed Module 8.

The two examinations are worth a total of 45 percent of your final course mark. You will write both examinations under supervision.

In order to do well on the examinations, you should review all of the work that you have completed from Modules 1 to 4 for your midterm examination and Modules 1 to 8 for your final examination, including all learning activities and assignments. You can use your Examination Resource Sheet to bring any formulas and other important information into the examination with you.

You will be required to bring the following supplies when you write both examinations: pens/pencils (2 or 3 of each), blank paper, a ruler, a scientific calculator, and your Examination Resource Sheet. Both examinations are 3 hours in duration.

Practice Examinations and Answer Keys

To help you succeed in your examinations, you will have an opportunity to complete a Midterm Practice Examination and a Final Practice Examination. These examinations, along with the answer keys, are found in the learning management system (LMS). If you do not have access to the Internet, contact the Distance Learning Unit at 1-800-465-9915 to obtain a copy of the practice examinations.

These practice examinations are similar to the actual examinations you will be writing. The answer keys enable you to check your answers. This will give you the confidence you need to do well on your examinations.

Requesting Your Examinations

You are responsible for making arrangements to have the examinations sent to your proctor from the Distance Learning Unit. Please make arrangements before you finish Module 4 to write the midterm examination. Likewise, you should begin arranging for your final examination before you finish Module 8.

To write your examinations, you need to make the following arrangements:

- **If you are attending school,** your examination will be sent to your school as soon as all the applicable assignments have been submitted. You should make arrangements with your school's ISO school facilitator to determine a date, time, and location to write the examination.

- **If you are not attending school,** check the Examination Request Form for options available to you. Examination Request Forms can be found on the Distance Learning Unit's website, or look for information in the learning management system (LMS). Two weeks before you are ready to write the examination, fill in the Examination Request Form and mail, fax, or email it to

  Distance Learning Unit
  500–555 Main Street
  PO Box 2020
  Winkler MB R6W 4B8
  Fax: 204-325-1719
  Toll-Free Telephone: 1-800-465-9915
  Email: distance.learning@gov.mb.ca

How Much Time Will You Need to Complete This Course?

Learning through independent study has several advantages over learning in the classroom. You are in charge of how you learn and you can choose how quickly you will complete the course. You can read as many lessons as you wish in a single session. You do not have to wait for your teacher or classmates.

From the date of your registration, you have a maximum of **12 months** to complete the course, but the pace at which you proceed is up to you. Read the following suggestions on how to pace yourself.

Chart A: Semester 1

If you want to start this course in September and complete it in January, you can follow the timeline suggested below.

Module	Completion Date
Module 1	Middle of September
Module 2	End of September
Module 3	Middle of October
Module 4	End of October
Midterm Examination	Beginning of November
Module 5	Middle of November
Module 6	End of November
Module 7	Middle of December
Module 8	Middle of January
Final Examination	End of January

Chart B: Semester 2

If you want to start the course in February and complete it in May, you can follow the timeline suggested below.

Module	Completion Date
Module 1	Middle of February
Module 2	End of February
Module 3	Beginning of March
Module 4	Middle of March
Midterm Examination	End of March
Module 5	Beginning of April
Module 6	Middle of April
Module 7	End of April
Module 8	Beginning of May
Final Examination	Middle of May

Chart C: Full School Year (Not Semestered)

If you want to start the course in September and complete it in May, you can follow the timeline suggested below.

Module	Completion Date
Module 1	End of September
Module 2	End of October
Module 3	End of November
Module 4	End of December
Midterm Examination	Middle of January
Module 5	Middle of February
Module 6	Middle of March
Module 7	Beginning of April
Module 8	Beginning of May
Final Examination	Middle of May

Timelines

Do not wait until the last minute to complete your work, since your tutor/marker may not be available to mark it immediately. It may take a few weeks for your tutor/marker to assess your work and return it to you or your school.

If you need this course to graduate this school year, all coursework must be received by the Distance Learning Unit on or before the first Friday in May, and all examinations must be received by the Distance Learning Unit on or before the last Friday in May. Any coursework or examinations received after these deadlines may not be processed in time for a June graduation. Assignments or examinations submitted after these recommended deadlines will be processed and marked as they are received.

When and How Will You Submit Completed Assignments?

When to Submit Assignments

While working on this course, you will submit completed assignments to the Distance Learning Unit eight times. The following chart shows you exactly what assignments you will be submitting at the end of each module.

Submission of Assignments

Submission	Assignments You Will Submit
1	**Module 1: Permutations, Combinations, and the Binomial Theorem** Module 1 Cover Sheet Assignment 1.1: The Fundamental Counting Principle and Permutations Assignment 1.2: Permutations and Combinations Assignment 1.3: The Binomial Theorem
2	**Module 2: Function Transformations** Module 2 Cover Sheet Assignment 2.1: Transformations of Functions Assignment 2.2: Combinations of Transformations Assignment 2.3: Operations on and Compositions of Functions
3	**Module 3: Reflections** Module 3 Cover Sheet Assignment 3.1: Reflections on the x-axis and in the y-axis Assignment 3.2: Inverse Functions and Relations
4	**Module 4: Polynomials** Module 4 Cover Sheet Assignment 4.1: Polynomial Functions Assignment 4.2: Factoring and Graphing Polynomials
5	**Module 5: Trigonometric Functions and the Unit Circle** Module 5 Cover Sheet Assignment 5.1: Degrees, Radians, and the Unit Circle Assignment 5.2: The Six Trigonometric Ratios Assignment 5.3: Trigonometric Functions and Their Graphs
6	**Module 6: Trigonometric Equations and Identities** Module 6 Cover Sheet Assignment 6.1: Solving Trigonometric Equations Assignment 6.2: Using Elementary Identities Assignment 6.3: Sum and Difference and Double Angle Identities
7	**Module 7: Exponents and Logarithms** Module 7 Cover Sheet Assignment 7.1: Exponential Functions and Logarithms Assignment 7.2: Dealing with Logarithms Assignment 7.3: Solving Exponential and Logarithmic Equations
8	**Module 8: Radical and Rational Functions** Module 8 Cover Sheet Assignment 8.1: Radical Functions Assignment 8.2: Solving Radical Equations Assignment 8.3: Rational Functions

How to Submit Assignments

In this course, you have the choice of submitting your assignments either by mail or electronically.

- **Mail:** Each time you **mail** something, you must include the print version of the applicable Cover Sheet (found at the end of this Introduction). Complete the information at the top of each Cover Sheet before submitting it along with your assignments.
- **Electronic submission:** You do not need to include a cover sheet when submitting assignments electronically.

Submitting Your Assignments by Mail

If you choose to mail your completed assignments, please photocopy/scan all the materials first so that you will have a copy of your work in case your package goes missing. You will need to place the applicable module Cover Sheet and assignment(s) in an envelope, and address it to

Distance Learning Unit
500–555 Main Street
PO Box 2020
Winkler MB R6W 4B8

Your tutor/marker will mark your work and return it to you by mail.

Submitting Your Assignments Electronically

Assignment submission options vary by course. Sometimes assignments can be submitted electronically and sometimes they must be submitted by mail. Specific instructions on how to submit assignments were sent to you with this course. In addition, this information is available in the learning management system (LMS).

If you are submitting assignments electronically, make sure you have saved copies of them before you send them. That way, you can refer to your assignments when you discuss them with your tutor/marker. Also, if the original hand-in assignments are lost, you are able to resubmit them.

Your tutor/marker will mark your work and return it to you electronically.

> The Distance Learning Unit does not provide technical support for hardware-related issues. If troubleshooting is required, consult a professional computer technician.

Marking Guidelines

All assignments will be marked following the guidelines below, adapted from the provincial marking guide. It is recommended you keep this sheet available and use it to review your assignments before submitting them.

- **Concept errors** (related to the learning outcome specific to the question) will result in a 1 mark deduction.
- The following will result in a 0.5 mark deduction:
  - arithmetic error
  - procedural error
  - terminology error in explanation
  - lack of clarity in explanation
- **Communication errors** (not conceptually related to the learning outcomes of the question) will result in a 0.5 mark deduction. Each communication error will only be deducted once per assignment/examination.

E1 — answer given as a complex fraction
— final answer not stated
— impossible solution(s) not rejected in final answer and/or in steps leading to final answer

E2 — changing an equation to an expression or vice versa
— equating the two sides when proving an identity

E3 — variables omitted in an equation or identity
— variables introduced without being defined

E4 — "$\sin x^2$" written instead of "$\sin^2 x$"
— missing brackets but still implied

E5 — units of measure omitted in final answer
— incorrect units of measure
— answer stated in degrees instead of radians or vice versa

E6 — rounding error
— rounding too early

E7 — notation error
— transcription error

E8 — answer outside the given domain
— bracket error made when stating domain or range
— domain or range written in incorrect order

E9 — endpoints or arrowheads omitted or incorrect
— scale values on axes not indicated
— coordinate points labelled incorrectly

E10 — asymptotes drawn as solid lines
— asymptotes missing but still implied
— graph crosses or curls away from asymptotes

What Are the Guide Graphics For?

Guide graphics are used throughout this course to identify and guide you in specific tasks. Each graphic has a specific purpose, as described below.

Lesson Introduction: The introduction sets the stage for the lesson. It may draw upon prior knowledge or briefly describe the organization of the lesson. It also lists the learning outcomes for the lesson. Learning outcomes describe what you will learn.

Learning Partner: Ask your learning partner to help you with this task.

Learning Activity: Complete a learning activity. This will help you to review or practise what you have learned and to prepare for an assignment or an examination. You will not submit learning activities to the Distance Learning Unit. Instead, you will compare your responses to those provided in the Learning Activity Answer Key found at the end of every module.

Assignment: Complete an assignment. You will submit your completed assignments to the Distance Learning Unit for assessment at the end of a given module.

Mail or Electronic Submission: Mail or electronically submit your completed assignments to the Distance Learning Unit for assessment at this time.

Phone Your Tutor/Marker: Telephone your tutor/marker.

Resource Sheet: Indicates material that may be valuable to include on your resource sheet.

Examination: Write your midterm or final examination at this time.

Note: Take note of and remember this important information or reminder.

Getting Started

Take some time right now to skim through the course material, locate your cover sheets, and familiarize yourself with how the course is organized. Get ready to learn!

Remember: If you have questions or need help at any point during this course, contact your tutor/marker or ask your learning partner for help.

Good luck with the course!

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Module 1: Permutations, Combinations, and the Binomial Theorem Cover Sheet

Please complete this sheet and place it on top of your assignments to assist in proper recording of your work. Submit the package to:

Drop-off/Courier Address

Distance Learning Unit
555 Main Street
Winkler MB R6W 1C4

Mailing Address

Distance Learning Unit
500–555 Main Street
PO Box 2020
Winkler MB R6W 4B8

Contact Information

Legal Name: ____________________ Preferred Name: ____________________

Phone: ____________________ Email: ____________________

Mailing Address: ____

City/Town: ______________________________ Postal Code: __________

Attending School: ☐ No ☐ Yes

School Name: ____

Has your contact information changed since you registered for this course? ☐ No ☐ Yes

Note: Please keep a copy of your assignments so that you can refer to them when you discuss them with your tutor/marker.

For Student Use	For Office Use Only	
Module 1 Assignments	**Attempt 1**	**Attempt 2**
Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	____________ Date Received	____________ Date Received
☐ Assignment 1.1: The Fundamental Counting Principle and Permutations	________ /45	________ /45
☐ Assignment 1.2: Permutations and Combinations	________ /49	________ /49
☐ Assignment 1.3: The Binomial Theorem	________ /24	________ /24
	Total: ____ **/118**	**Total:** ____ **/118**

For Tutor/Marker Use
Remarks:

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Module 2: Function Transformations
Cover Sheet

Please complete this sheet and place it on top of your assignments to assist in proper recording of your work. Submit the package to:

Drop-off/Courier Address
Distance Learning Unit
555 Main Street
Winkler MB R6W 1C4

Mailing Address
Distance Learning Unit
500–555 Main Street
PO Box 2020
Winkler MB R6W 4B8

Contact Information

Legal Name: ____________________ Preferred Name: ____________________

Phone: ____________________ Email: ____________________

Mailing Address: ____

City/Town: ______________________________ Postal Code: __________

Attending School: ☐ No ☐ Yes

School Name: ____

Has your contact information changed since you registered for this course? ☐ No ☐ Yes

Note: Please keep a copy of your assignments so that you can refer to them when you discuss them with your tutor/marker.

For Student Use	**For Office Use Only**	
Module 2 Assignments	**Attempt 1**	**Attempt 2**
Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	__________ Date Received	__________ Date Received
☐ Assignment 2.1: Transformations of Functions	______ /28	______ /28
☐ Assignment 2.2: Combinations of Transformations	______ /20	______ /20
☐ Assignment 2.3: Operations on and Compositions of Functions	______ /54	______ /54
	Total: ____ **/102**	**Total:** ____ **/102**

For Tutor/Marker Use
Remarks:

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Module 3: Reflections
Cover Sheet

Please complete this sheet and place it on top of your assignments to assist in proper recording of your work. Submit the package to:

Drop-off/Courier Address
Distance Learning Unit
555 Main Street
Winkler MB R6W 1C4

Mailing Address
Distance Learning Unit
500–555 Main Street
PO Box 2020
Winkler MB R6W 4B8

Contact Information

Legal Name: ____________________ Preferred Name: ____________________

Phone: ____________________ Email: ____________________

Mailing Address: ____

City/Town: ______________________________ Postal Code: __________

Attending School: ☐ No ☐ Yes

School Name: ____

Has your contact information changed since you registered for this course? ☐ No ☐ Yes

Note: Please keep a copy of your assignments so that you can refer to them when you discuss them with your tutor/marker.

For Student Use	For Office Use Only	
Module 3 Assignments	**Attempt 1**	**Attempt 2**
Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	____________ Date Received	____________ Date Received
☐ Assignment 3.1: Reflections in the *x*-axis and in the *y*-axis	______ /34	______ /34
☐ Assignment 3.2: Inverse Functions and Relations	______ /31	______ /31
	Total: ____ **/65**	**Total:** ____ **/65**

For Tutor/Marker Use
Remarks:

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Module 4: Polynomials
Cover Sheet

Please complete this sheet and place it on top of your assignments to assist in proper recording of your work. Submit the package to:

Drop-off/Courier Address

Distance Learning Unit
555 Main Street
Winkler MB R6W 1C4

Mailing Address

Distance Learning Unit
500–555 Main Street
PO Box 2020
Winkler MB R6W 4B8

Contact Information

Legal Name: ____________________ Preferred Name: ____________________

Phone: ____________________ Email: ____________________

Mailing Address: ____

City/Town: ____________________ Postal Code: __________

Attending School: ☐ No ☐ Yes

School Name: ____

Has your contact information changed since you registered for this course? ☐ No ☐ Yes

Note: Please keep a copy of your assignments so that you can refer to them when you discuss them with your tutor/marker.

For Student Use	For Office Use Only	
Module 4 Assignments	**Attempt 1**	**Attempt 2**
Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	________ Date Received	________ Date Received
☐ Assignment 4.1: Polynomial Functions	______ /40	______ /40
☐ Assignment 4.2: Factoring and Graphing Polynomials	______ /42	______ /42
	Total: ____ /82	**Total: ____ /82**

For Tutor/Marker Use
Remarks:

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Module 5: Trigonometric Functions and the Unit Circle
Cover Sheet

Please complete this sheet and place it on top of your assignments to assist in proper recording of your work. Submit the package to:

Drop-off/Courier Address	**Mailing Address**
Distance Learning Unit 555 Main Street Winkler MB R6W 1C4	Distance Learning Unit 500–555 Main Street PO Box 2020 Winkler MB R6W 4B8

Contact Information

Legal Name: ______________________ Preferred Name: ______________________

Phone: ______________________ Email: ______________________

Mailing Address: ____

City/Town: ______________________ Postal Code: ____________

Attending School: ☐ No ☐ Yes

School Name: ____

Has your contact information changed since you registered for this course? ☐ No ☐ Yes

Note: Please keep a copy of your assignments so that you can refer to them when you discuss them with your tutor/marker.

For Student Use	**For Office Use Only**	
Module 5 Assignments	**Attempt 1**	**Attempt 2**
Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	____________ Date Received	____________ Date Received
☐ Assignment 5.1: Degrees, Radians, and the Unit Circle	______ /31	______ /31
☐ Assignment 5.2: The Six Trigonometric Ratios	______ /32	______ /32
☐ Assignment 5.3: Trigonometric Functions and Their Graphs	______ /26	______ /26
	Total: ____ /89	**Total: ____ /89**

For Tutor/Marker Use
Remarks:

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Module 6: Trigonometric Equations and Identities
Cover Sheet

Please complete this sheet and place it on top of your assignments to assist in proper recording of your work. Submit the package to:

Drop-off/Courier Address

Distance Learning Unit
555 Main Street
Winkler MB R6W 1C4

Mailing Address

Distance Learning Unit
500–555 Main Street
PO Box 2020
Winkler MB R6W 4B8

Contact Information

Legal Name: ____________________ Preferred Name: ____________________

Phone: ____________________ Email: ____________________

Mailing Address: ____

City/Town: ____________________ Postal Code: __________

Attending School: ☐ No ☐ Yes

School Name: ____

Has your contact information changed since you registered for this course? ☐ No ☐ Yes

Note: Please keep a copy of your assignments so that you can refer to them when you discuss them with your tutor/marker.

For Student Use	For Office Use Only	
Module 6 Assignments Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	**Attempt 1** __________ Date Received	**Attempt 2** __________ Date Received
☐ Assignment 6.1: Solving Trigonometric Equations	______ /21	______ /21
☐ Assignment 6.2: Using Elementary Identities	______ /44	______ /44
☐ Assignment 6.3: Sum and Difference and Double Angle Identities	______ /39	______ /39
	Total: ____ **/104**	**Total:** ____ **/104**
For Tutor/Marker Use		
Remarks:		

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Module 7: Exponents and Logarithms
Cover Sheet

Please complete this sheet and place it on top of your assignments to assist in proper recording of your work. Submit the package to:

Drop-off/Courier Address	**Mailing Address**
Distance Learning Unit 555 Main Street Winkler MB R6W 1C4	Distance Learning Unit 500–555 Main Street PO Box 2020 Winkler MB R6W 4B8

Contact Information

Legal Name: ____________________________ Preferred Name: __________________________

Phone: __________________________________ Email: ___________________________________

Mailing Address: ____

City/Town: ____ Postal Code: _______________

Attending School: ☐ No ☐ Yes

School Name: ____

Has your contact information changed since you registered for this course? ☐ No ☐ Yes

Note: Please keep a copy of your assignments so that you can refer to them when you discuss them with your tutor/marker.

For Student Use	**For Office Use Only**	
Module 7 Assignments	**Attempt 1**	**Attempt 2**
Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	______________ Date Received	______________ Date Received
☐ Assignment 7.1: Exponential Functions and Logarithms	________ /48	________ /48
☐ Assignment 7.2: Dealing with Logarithms	________ /46	________ /46
☐ Assignment 7.3: Solving Exponential and Logarithmic Equations	________ /37	________ /37
	Total: ____ **/131**	**Total:** ____ **/131**

For Tutor/Marker Use
Remarks:

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Module 8: Radical and Rational Functions
Cover Sheet

Please complete this sheet and place it on top of your assignments to assist in proper recording of your work. Submit the package to:

Drop-off/Courier Address	**Mailing Address**
Distance Learning Unit 555 Main Street Winkler MB R6W 1C4	Distance Learning Unit 500–555 Main Street PO Box 2020 Winkler MB R6W 4B8

Contact Information

Legal Name: ____________________________ Preferred Name: __________________________

Phone: _________________________________ Email: ___________________________________

Mailing Address: ____

City/Town: ____ Postal Code: ______________

Attending School: ☐ No ☐ Yes

School Name: ____

Has your contact information changed since you registered for this course? ☐ No ☐ Yes

Note: Please keep a copy of your assignments so that you can refer to them when you discuss them with your tutor/marker.

For Student Use	**For Office Use Only**	
Module 8 Assignments	**Attempt 1**	**Attempt 2**
Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	__________ Date Received	__________ Date Received
☐ Assignment 8.1: Radical Functions	________ /34	________ /34
☐ Assignment 8.2: Solving Radical Equations	________ /13	________ /13
☐ Assignment 8.3: Rational Functions	________ /34	________ /34
	Total: ____ **/81**	**Total:** ____ **/81**

For Tutor/Marker Use
Remarks:

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GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Module 1
Permutations, Combinations, and the Binomial Theorem

MODULE 1:
PERMUTATIONS, COMBINATIONS, AND THE BINOMIAL THEOREM

Introduction

Welcome to your first module of Grade 12 Pre-Calculus Mathematics! In previous mathematics courses, you may have been asked problem-solving questions such as, "How many ways can you assign first, second, and third place in a race if three people are racing?" In this module, you will learn how to answer questions like these quickly and efficiently.

You will learn a method of "fast counting," called permutations and combinations, that allows you to count large amounts of objects or arrangements accurately and quickly. Anyone can count the number of objects in a set one at a time, but what happens if there are over a million objects to count? You need a method that will produce the answer without it being necessary to count the objects one at a time. Learning these "fast" counting processes is the objective of this module.

This counting method also allows you to expand binomials quickly and efficiently. You will learn how to do this according to a theorem called the Binomial Theorem.

Assignments in Module 1

When you have completed the assignments for Module 1, submit your completed assignments to the Distance Learning Unit either by mail or electronically through the learning management system (LMS). The staff will forward your work to your tutor/marker.

Lesson	Assignment Number	Assignment Title
2	Assignment 1.1	The Fundamental Counting Principle and Permutations
4	Assignment 1.2	Permutations and Combinations
5	Assignment 1.3	The Binomial Theorem

Resource Sheet

When you write your midterm examination, you are encouraged to take a Midterm Examination Resource Sheet with you into the examination. This sheet will be one letter-sized page, 8½ ″ by 11″, with both sides in your handwriting or typewritten. You will submit it with your examination, but you do not receive any marks for it.

Many students have found that preparing a resource sheet is an excellent way to review. It provides you with a summary of the important facts of each module. You should complete a resource sheet for each module to help with your studying and reviewing. Lesson summaries and module summaries are included for you to use as a guide.

You may use the list of instructions provided below to help you with preparing your resource sheet for the material in Module 1. On this sheet, you should record math terms and definitions, formulas, sample questions, or a list of places where you often make mistakes. You should also identify special areas that require extra attention or review by writing the page numbers.

After you have completed each module's resource sheet, you may summarize the sheets from Modules 1, 2, 3, and 4 to prepare your Midterm Examination Resource Sheet. The midterm examination for this course is based on Modules 1 to 4.

Resource Sheet for Module 1

As you go through the lessons of this module, you may want to consider the following suggestions regarding the creation of a resource sheet.

1. List all the important math terms, and define them if necessary.
2. List all the formulas and perhaps a sample problem that shows how the formula is used.
3. If necessary, write the solutions to some problems, showing in detail how you did the calculations.
4. Copy any questions that represent the key points of the lesson, and perhaps include the solutions as well.
5. Identify the problems you found most difficult, and copy the page numbers onto the resource sheet so that you can review them before writing the examination. You may also copy the problems and the solutions onto your resource sheet, and later write them onto your Midterm Examination Resource Sheet.
6. Write any comments, ideas, shortcuts, or other reminders that may be helpful during an examination.

LESSON 1: THE FUNDAMENTAL COUNTING PRINCIPLE

Lesson Focus

In this lesson, you will

☐ learn how to use the Fundamental Counting Principle to solve problems

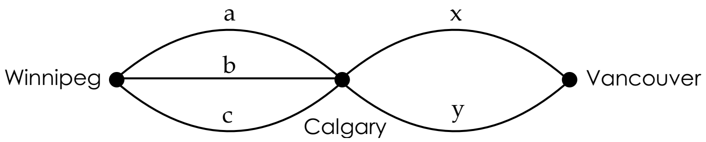
Lesson Introduction

Let us suppose that you have three pairs of jeans and two sweaters in your closet. How many different possible outfits, consisting of one pair of jeans and one sweater, could you wear? Would you have enough outfits so that you could wear a different outfit on each day of the week? The Fundamental Counting Principle, which you will be learning about in this lesson, allows you to determine the answer to these types of questions. This principle allows you to quickly count a large number of possible results for a given situation.

Using the Fundamental Counting Principle

Consider the following situation:

Melissa is planning to drive from Winnipeg to Vancouver via Calgary. If there are three roads from Winnipeg to Calgary and two roads from Calgary to Vancouver, how many different routes are there from Winnipeg to Vancouver passing through Calgary?



You can count the routes one at a time and end up with the following possibilities:

ax bx cx

ay by cy

Therefore, there are six different routes.

Or, you can use the Fundamental Counting Principle, which produces the same answer of six, by answering the following questions, step by step.

Step 1. How many decisions must Melissa make?

Two decisions: which road to take from Winnipeg to Calgary and which road to take from Calgary to Vancouver.

You can represent these two decisions by two blanks.

$$\underset{\mathrm{W} \longrightarrow \mathrm{C}}{\underline{\qquad\qquad}} \qquad \underset{\mathrm{C} \longrightarrow \mathrm{V}}{\underline{\qquad\qquad}}$$

Step 2. In how many ways can Melissa make each of these decisions?

The first decision has three choices of roads. Fill in the first blank with a "3."

$$\underset{\mathrm{W} \longrightarrow \mathrm{C}}{\underline{\qquad 3 \qquad}} \qquad \underset{\mathrm{C} \longrightarrow \mathrm{V}}{\underline{\qquad\qquad}}$$

After making the first decision, she has two choices of roads for her second decision. Fill in the second blank with a "2."

$$\underset{\mathrm{W} \longrightarrow \mathrm{C}}{\underline{\qquad 3 \qquad}} \qquad \underset{\mathrm{C} \longrightarrow \mathrm{V}}{\underline{\qquad 2 \qquad}}$$

Step 3. Do you see the answer 6? For each of the three choices for the first road, there are two choices for the second road. Hence, there are 3(2) = 6 choices for both decisions to be made.

Note: In the tabular listing below, for each of the *three* columns there are two rows.

ax	bx	cx
ay	by	cy

The third step illustrates the **Fundamental Counting Principle**, which states:

> If a decision can be made in *m* different ways and a second decision can be made in *n* different ways, then the two decisions can be made in this order, in *mn* different ways.

This principle can be extended to include *any number* of successive decisions. Include this principle on your resource sheet.

Example 1

Melissa is planning to drive from Winnipeg to Vancouver via Calgary. There are three roads from Winnipeg to Calgary and two roads from Calgary to Vancouver. How many different "round-trip" routes are there from Winnipeg to Vancouver, passing through Calgary and returning to Winnipeg via Calgary, if no road is used more than once?

Solution

There are *four* decisions to be made. There are *two* selections on the way to Vancouver and *two* selections on the return trip; therefore, four blanks are needed.

$$\underset{W \longrightarrow C}{\underline{\qquad\qquad}} \qquad \underset{C \longrightarrow V}{\underline{\qquad\qquad}} \qquad \underset{V \longrightarrow C}{\underline{\qquad\qquad}} \qquad \underset{C \longrightarrow W}{\underline{\qquad\qquad}}$$

a) There are *three* choices of roads from Winnipeg to Calgary.

b) There are *two* choices of roads from Calgary to Vancouver.

c) There is only *one* choice of road from Vancouver to Calgary since the other road was used to get from Calgary to Vancouver, and you are not to use a road more than once.

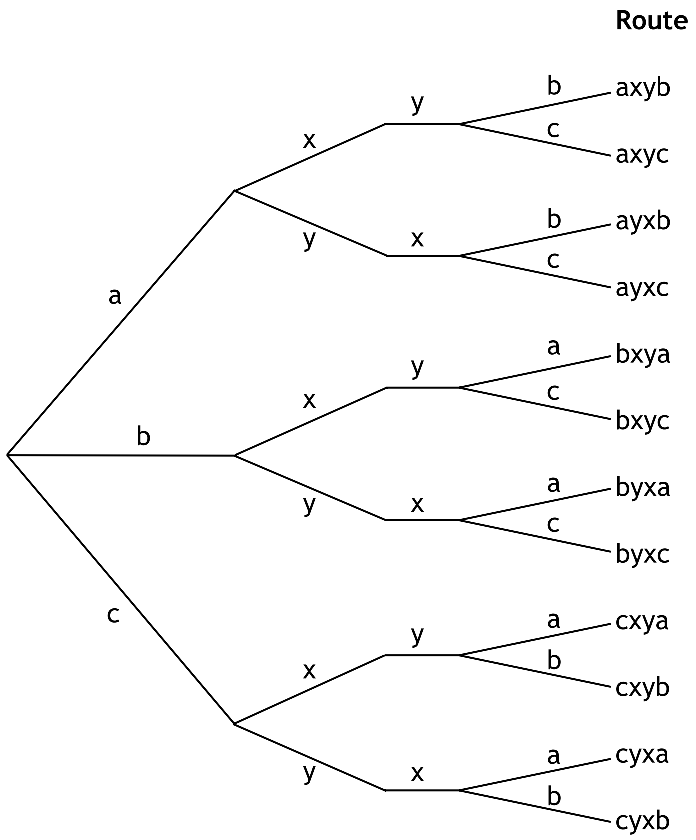
d) Similarly, there are *two* choices of roads from Calgary to Winnipeg because one of the three roads has been used on the trip from Winnipeg to Calgary.

Using the Fundamental Counting Principle, you get:

$$\underset{W \longrightarrow C}{\underline{\qquad 3 \qquad}} \qquad \underset{C \longrightarrow V}{\underline{\qquad 2 \qquad}} \qquad \underset{V \longrightarrow C}{\underline{\qquad 1 \qquad}} \qquad \underset{C \longrightarrow W}{\underline{\qquad 2 \qquad}} = 12$$

In other words, if you multiply all your choices together, 3(2)(1)(2), you get an answer of 12.

To reinforce your confidence in this principle, the 12 possible routes are listed below in a **tree diagram**. The letters *a*, *b*, and *c* represent the roads between Winnipeg and Calgary, and *x* and *y* represent the roads between Calgary and Vancouver.



Generally, you are asked for the *number of choices* without having to list them. The reason for this is that the counts are usually quite large, and it would be quite a chore to list thousands or even millions of possible choices.

Example 2

You bought five digits at your local hardware store to create a house number. The digits are 1, 2, 7, 8, and 9. How many different three-digit house numbers can you form using these digits?

Solution

How many decisions are there to be made? Three: one for each digit.

$$\underline{\qquad\qquad} \quad \underline{\qquad\qquad} \quad \underline{\qquad\qquad}$$

There are five choices for the first digit. After the first digit is selected, there are four choices for the second digit. After the first two digits are selected, there are three choices for the last digit. Therefore, there are:

$$\underline{\qquad 5 \qquad} \quad \underline{\qquad 4 \qquad} \quad \underline{\qquad 3 \qquad}$$

= 60 possible house numbers

Example 3

Using the digits 1, 2, 3, 4, and 5, how many three-digit house numbers

a) are even?

b) are odd?

c) are greater than 300?

d) are greater than 300 and even?

Solution

a) If there is a **restriction on the problem**, it is usually best to begin filling in the blanks starting at the restriction. For example, the restriction on this problem is that the number must be even. For the number to be even, the last digit must be even. Therefore, the last digit must be 2 or 4 (i.e., you have two choices).

$$\underline{} \qquad \underline{} \qquad \underset{\text{even}}{\underline{\quad 2 \quad}}$$

There are four remaining digits to be placed into the first blank and then three remaining digits for the second blank. Therefore,

$$\underline{\quad 4 \quad} \qquad \underline{\quad 3 \quad} \qquad \underline{\quad 2 \quad}$$

= 24 possible even numbers

b) For the house number to be odd, the last digit must be odd. Therefore, the last digit must be 1, 3, or 5 (i.e., you have three choices). Again, there are four remaining digits to be placed into the first blank and then three remaining digits for the second blank. The solution is:

$$\underline{\quad 4 \quad} \qquad \underline{\quad 3 \quad} \qquad \underset{\text{odd}}{\underline{\quad 3 \quad}}$$

= 36 possible odd numbers

Note: There is an alternate solution for part (b) using Example 2 and part (a) above. You could have found the number of odd numbers by subtracting the number of even numbers from the total number of numbers. Remember, if a number is not even, it must be odd!

In other words, the number of odd house numbers is 60 − 24 = 36. You arrive at the same answer using either calculation.

You can determine how many objects do not have a certain property by subtracting the number that do have this property from the total. Problems that can be solved using this technique are called **complementary problems**. It may be helpful for you to include this strategy on your resource sheet.

c) To be greater than 300, the first digit must be 3, 4, or 5 (i.e., you have three choices for this first digit, followed by four and three choices for the last two digits).

3	4	3
≥ 3		

= 36 possible numbers greater than 300

d) This is a tricky question because there is a restriction on the *first and last digit*! Which one do you fill in first? Suppose you decide to fill in the last digit first.

		2
		even

How many choices do you have for the first digit? It depends on whether the last digit is the digit 2. If so, then you have three choices for the first digit—namely, 3, 4, and 5. However, if the last digit is the digit 4, you have only two choices for the first digit—namely, 3 or 5, since the digit 4 was already used.

If you began by filling the first digit, you would encounter the same problem in trying to fill the last digit. To resolve this type of problem, you have to count the number of choices by dividing the situation into two separate cases. Then, you need to add the cases together to determine the total number of cases. This is another common process called **solving problems using cases**. It may be helpful to add this strategy to your resource sheet.

Case 1: If the last digit is the digit 2:

		1	← one choice of the digit 2
		even	

You now have three choices—namely 3, 4, or 5—for the first digit.

3		1
≥ 3		even

Now, you can complete the solution of this case. Since two digits are already used, there are three remaining possible digits for the middle space.

3	3	1	= 9 possibilities
≥ 3	remaining	even	

Case 2: If the last digit is the digit 4:

		1	← one choice of the digit 4
		even	

You now have two choices—namely, 3 or 5—for the first digit.

2		1
≥ 3 (not 4)		even

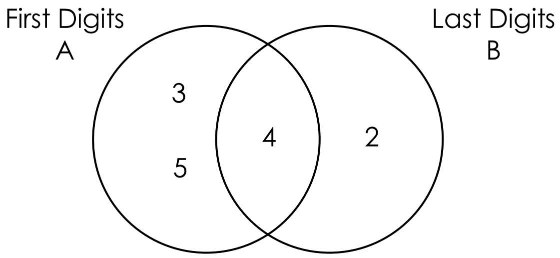
The complete solution of this case is:

2	3	1	= 6 possibilities
≥ 3 (not 4)	remaining	even	

The total number of house numbers that are greater than 300 and even is the sum of the possibilities in the two cases:

$9 + 6 = 15$ possible numbers

The following Venn diagram illustrates why part (d) had to be counted by cases. Circle A contains all possible first digits to satisfy the problem and circle B contains all possible last digits to satisfy the problem. The "4" cannot be used for both the first digit and the last digit at the same time. Since these two circles contain the common element 4, it follows that the count should be done in two separate cases. If you tried to count all the possibilities in one case, you would either miss some of the possibilities or count some possibilities more than once.



The Fundamental Counting Principle states that you need to multiply the number of possibilities for different events that happen at the same time. However, when you have cases that do not happen at the same time, you need to add the number of possibilities of each case.

Example 4

Using the digits 0, 1, 3, 4, 5, and 8, how many 4-digit numbers are possible if repetition of digits is not allowed?

Solution

Use the Fundamental Counting Principle.

Four positions to fill and six numbers to choose from. However, zero cannot be placed in the first position because this would make a 3-digit number.

$$\underset{\substack{\uparrow \\ \text{not zero}}}{\underline{\quad 5 \quad}} \times \underset{\substack{\uparrow \\ \text{can be any of the} \\ \text{5 remaining digits} \\ \text{after filling the} \\ \text{first position}}}{\underline{\quad 5 \quad}} \times \underline{\quad 4 \quad} \times \underline{\quad 3 \quad} = 300$$

Learning Activity 1.1

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. What is the reciprocal of $\dfrac{\frac{2}{5}}{10}$?
2. Find the next four numbers in the pattern: 7, 1, 9, 3, 11, 5, 13, . . .
3. Is $x = -1$ a solution to the inequality $x^2 + 4x + 3 < 0$?
4. Rationalize the denominator: $\dfrac{4}{2\sqrt{3}}$

continued

Learning Activity 1.1 (continued)

5. Evaluate: $f(x) = \dfrac{(x+3)^2}{2x-1}$, if $x = -2$.
6. In which direction does the parabola $y + x^2 - 2x = 0$ open?
7. List all factors of 70.
8. Evaluate: $\left|3\frac{2}{5} - 6\frac{4}{5}\right|$

Part B: The Fundamental Counting Principle

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Refer to Example 3(d).
   a) List the 15 possible house numbers.
   b) How many different house numbers use the digits 2, 3, and 4?
2. Using the digits 1, 2, 3, 4, and 5 without repetition, how many three-digit house numbers
   a) have the digit 4 as the middle digit?
   b) do not have the digit 4 as the middle digit?
   c) begin and end with an odd digit?
   d) are between 200 and 500?
   e) are less than 300 and odd?
3. There are 72 girls in a school with 108 students. How many of the students are boys?
4. Winnipeg Stadium has five gates. In how many ways can you enter the stadium and leave the stadium
   a) by a different gate?
   b) by any gate?
5. In how many ways can a school president, a vice-president, and a secretary be selected from the 23 students comprising the student council?

continued

Learning Activity 1.1 (continued)

6. Nayeli, a recent university graduate, is buying clothes for her new job. If she buys 2 new pairs of pants and 4 new blouses, how many different "new" outfits is it possible for her to wear if each consists of 1 new blouse and 1 new pair of pants?
7. In how many ways can five different books be arranged on a bookshelf?
8. In how many ways can four women and three men be seated on a bench
   a) if the men and women must alternate seats?
   b) if the men and women can sit wherever?
9. A university student must take a language course in slot 1, a mathematics course in slot 2, and a science course in slot 3. If there are four different language courses, eight different mathematics courses, and three different science courses, how many different timetables are possible?

Now that you have completed your first lesson and your first Learning Activity in Grade 12 Pre-Calculus, it will be helpful for you to contact your tutor/marker. Learning Activity 1.2 will ask you to do so. As mentioned in the Introduction to this course, your tutor/marker is available if you need help learning any of the material. Also, your tutor/marker will be marking all of your assignments and examinations. Your tutor/marker's contact information is on the cover sheet that came with this course. If you are unable to find your cover sheet, phone the Independent Study Option at 1–800–465–9915 and someone will provide you with the information you need.

Learning Activity 1.2

Learning Activity 1.2 is the only one that doesn't include a BrainPower section, although it still has two parts.

This learning activity involves you having a conversation with your tutor/marker. Having this conversation with your tutor/marker has two important purposes. First, it introduces you to a very valuable resource—your tutor/marker. He or she is available for you to answer questions, explain concepts, and guide you through this course. You can discuss your math learning and progress. Feel free to contact your tutor/marker by phone or email at anytime during this course.

continued

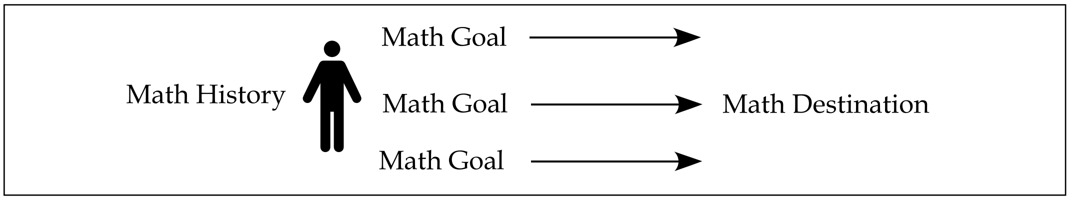
Learning Activity 1.2 (continued)

The second important purpose of this assignment is to get you thinking about your math goals. You may have a future career in mind and this course is getting you one step closer to it by completing a prerequisite for a future required course. There may be specific skills or topics you are interested in learning about and they are covered in this course.

If you are unsure of your math goals or why they are important, consider this:

- goals give you a sense of direction and purpose in taking this course
- goals help motivate you to learn and do your best, even when it's tough
- when you accomplish your goals, there is a great sense of achievement and success

Good goals need to be realistic, specific, and they should reflect what is important to you. They should give you direction and take you further down the path from where you have been to where you want to go.



Goals can be long term or short term, but they are the pathway that takes **you** from where you were/are, closer to where you want to go.

Part A: Contacting Your Tutor/Marker

Your first task is to contact your tutor/marker by phone.

Fill in the following blanks using information provided with your course:

My tutor/marker's name is ____

I can phone my tutor/marker at 1-866-______________________________

My tutor/marker's email address is _________________________________

Be ready to discuss the following topics and the reasons for your answers with your tutor/marker and learning partner. If you like, make some notes before you call in order to help you feel prepared. Feel free to add any other questions or comments that you may have.

continued

Learning Activity 1.2 (continued)

1. I am taking this course by distance education because:

____

____

____

____

2. What I like about math and can do mathematically is (include favourite topic, skill, where you use math, etc.):

____

____

____

____

3. What I dislike about math or have difficulty doing is:

____

____

____

____

4. Previous math experiences that influence the way I feel about math are:

____

____

____

____

continued

Learning Activity 1.2 (continued)

5. The next math course I would like to take is:

____

____

____

____

6. What I am hoping this course will help me accomplish and learn for the future:

____

____

____

____

7. What I am doing and how I organize things to help me succeed in this course is:

____

____

____

____

During your phone conversation, jot down a sentence or two about what you and your tutor/marker talk about, in the spaces above. For example, if you are taking this course because it doesn't fit into your schedule at school or because you travel with your basketball team, missing a large number of classes, state that in the space below question 1.

continued

Learning Activity 1.2 (continued)

Part B: Your Math Pathway

Use the answers to the questions from the conversation with your tutor/marker as a starting point to fill in the following diagram. In the Math History box, jot down point-form notes about your prior experience and knowledge about math (questions 2, 3, and 4). In the Math Destination box, jot down what completing this course will help you accomplish in the future (questions 5 and 6).

In the Pathway box, write down what you will need to do to move along the pathway from your History to your Destination.

Math History	Pathway	Math Destination

For example, if you want to start your own business in the future, what skills will you need that will make you feel confident in your ability to do this? Will you be able to balance your finances and complete your taxes? Alternatively, your goal may be to find the right approach for you in acquiring the math skills that will prepare you for a post-secondary program of your choice. Your study plan may involve setting up a schedule to ensure you complete your assignments on time. You may need to find your calculator manual and learn how to use your scientific or graphing calculator in ways that will maximize its benefits towards helping you reach your destination. You may decide to set up regular appointments with your learning partner, research a topic on the Internet, or read a textbook about a certain math concept or skill. Your pathway is unique to you.

continued

Learning Activity 1.2 (continued)

As you move through this course and work on achieving your goals, self-assessment is important for you to determine whether you are getting closer to your destination. It helps you determine whether the steps along your pathway are taking you in the right direction. You will need to periodically ask yourself: Am I doing my assignments? Are my note-taking skills improving? How often have I contacted my tutor/marker or worked with my learning partner? Have I found useful homework websites if necessary? Is my schedule working? What do I need to change or adjust so I can get to my destination?

Repeatedly going through this cycle of looking at where you have been, where you want to go, and where you currently are is recommended. At any time, you may want to revise your goals or set new ones as you evaluate your own progress and learning.

- Look back/history—reflect on what you know, how far you have come.
- Look around/pathway—assess if you are achieving your goals, determine if new learning or understanding has occurred, and check your progress.
- Look forward/destination—determine what you want to know, set goals.

Each time you go through these steps, you will become better at mathematics!

It is important that you keep this diagram handy as you will revisit it at other points in this course.

Lesson Summary

In this lesson, you learned about the Fundamental Counting Principle. The Fundamental Counting Principle can help you determine how many ways there are to arrange a group of objects or how many ways an event can happen. In the next lesson, you will develop a formula to help you count the number of arrangements of a group of objects.

Notes

LESSON 2: PERMUTATIONS

Lesson Focus

In this lesson, you will

☐ learn how to use factorial notation

☐ learn how to use a formula to solve unrestricted permutation problems

Lesson Introduction

Consider the following situations:

- You want mushrooms, pepperoni, and green peppers on your pizza
- You are trying to open your locker with the combination 4–31–16

In which of these situations does order matter?

In the first situation, the order in which the ingredients go on your pizza does not matter. A mushroom, pepperoni, and green pepper pizza is the same as a green pepper, pepperoni, and mushroom pizza. However, in the second situation, the order you enter the numbers into your lock matters. If you know your locker combination contains the numbers 4, 16, and 31, but do not know in what order they should be entered, it may take you a few tries to open your locker.

Permutations deal with situations in which *order matters*. Thus, a locker combination should actually be called a locker permutation.

Permutations

Permutations and combinations, which you will be learning about later in this module, can be simplified using a notation called "factorial." In order to understand how the formula for permutations is developed, it is important that you are comfortable dealing with factorials and factorial notation first.

Factorial Notation

A **factorial** looks like a "!" and has an important mathematical meaning. This type of notation is a kind of shorthand, as you will see in the following example.

Example 1

Using the Fundamental Counting Principle, determine how many ways there are to arrange three different books on a shelf.

Solution

There are three decisions to be made for each of the three positions on the shelf, represented by three blanks.

$\underline{\qquad\qquad}\quad\underline{\qquad\qquad}\quad\underline{\qquad\qquad}$

There are three ways to make the first decision, two ways to make the second decision, and only one way to make the last decision.

$\underline{\qquad 3 \qquad}\quad\underline{\qquad 2 \qquad}\quad\underline{\qquad 1 \qquad}$ = 6 ways to arrange the books

To arrive at the answer of 6, you multiply $3 \times 2 \times 1$. If you were trying to arrange four books on a shelf, you would have to multiply $4 \times 3 \times 2 \times 1$.

What would happen if you wanted to know how many ways there were to arrange 20 books on a shelf? You would have to write out $20 \times 19 \times 18 \ldots$ all the way down to 1. Writing this out takes up a lot of room and can be very time consuming. For this reason, mathematicians use factorials.

$3 \times 2 \times 1$ is equivalent to $3!$

Note: The symbol "!" means factorial. 3! is read as "three factorial."

Observe the pattern of expanded factorials below.

$0! = 1$

$1! = 1$

$2! = 2 \times 1$

$3! = 3 \times 2 \times 1$

$4! = 4 \times 3 \times 2 \times 1$

$5! = 5 \times 4 \times 3 \times 2 \times 1$

$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$

For a natural number n,

$n! = n(n - 1)(n - 2) \ldots 3 \cdot 2 \cdot 1 \longleftarrow$ The product of the natural numbers from 1 to n.

$0! = 1$

Note: $0! = 1$ seems unusual, but it is a necessary definition to maintain consistency. Very soon you will see the rationale for this definition.

Note: n can only be one of the numbers 0, 1, 2, 3, . . . (or the counting numbers, including zero).

Include the above information on your resource sheet. It may also be helpful for you to include some examples of how to simplify factorials.

Example 2

Answer the following questions regarding factorials.

a) What is the value of $0!$?

b) Evaluate $4!$.

c) Find the product of $(2!)(3!)$.

d) Evaluate $\frac{4!}{3!}$.

e) Evaluate $\frac{7!}{4!}$.

f) Evaluate $\frac{5!}{3!2!}$.

Solutions

a) What is the value of $0!$?

$0! = 1$ (this is a defined value)

Although it may seem logical that $0! = 0$, it cannot have that value. This will be evident later in the lesson. It is sometimes necessary in mathematics to define values to fit a pattern—this is one of those times.

b) Evaluate $4!$.

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

c) Find the product of $(2!)(3!)$.

$(2!)(3!) = (2 \cdot 1)(3 \cdot 2 \cdot 1) = 12$

A common error here is to multiply the numbers and say that $(2!)(3!) = 6!$, but $6!$ is $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$, which is not even close to 12.

d) Evaluate $\frac{4!}{3!}$.

$$\frac{4!}{3!} = \frac{4 \cdot 3!}{3!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 4$$

Expanding the 4! and cancelling the 3!s is a useful method for simplifying factorials.

e) Evaluate $\frac{7!}{4!}$.

$$\frac{7!}{4!} = \frac{7 \cdot 6 \cdot 5 \cdot 4!}{4!} = 7 \cdot 6 \cdot 5 = 210$$

f) Evaluate $\frac{5!}{3!2!}$.

$$\frac{5!}{2!3!} = \frac{5 \cdot 4 \cdot 3!}{2!3!} = \frac{5 \cdot 4}{2!} = \frac{5 \cdot 4}{2 \cdot 1} = \frac{5 \cdot 4}{2} = \frac{20}{2} = 10$$

The following Learning Activity includes questions designed to enhance your understanding of basic factorial notation and how factorials are reduced. Make sure you complete this Learning Activity before you continue on with the rest of the lesson, as it will help you understand the formula for calculating permutations.

Learning Activity 1.3

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Solve for x: $x + 17 = 48$
2. Simplify: $\frac{3}{4} + \frac{1}{6} + \frac{7}{12}$
3. If $f(x) = \frac{x+6}{2}$, calculate $f(-102)$.
4. Factor: $3x^2 + 5x + 2$
5. Estimate the taxes, 13%, on a \$113 item.
6. Simplify: $\dfrac{\frac{16x^2y^3}{4xy}}{\frac{2x^3y}{y^4}}$
7. Evaluate: $\left|-\frac{4}{7} - \frac{9}{49}\right|$
8. Evaluate: $\frac{9}{2} \div \frac{3}{8}$

continued

Learning Activity 1.3 (continued)

Part B: Factorials

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Simplify each of the following factorials.

   a) $\dfrac{4!}{2!}$

   b) $\dfrac{6!}{2!4!}$

   c) $\dfrac{6 \cdot 4!}{3!} + 0!$

   d) $\dfrac{9!}{2!7!} + 1!$

   e) $\dfrac{7 \cdot 6!}{7!}$

2. Simplify without the use of a calculator.

   a) $\dfrac{8!}{7!}$

   b) $\dfrac{10!}{9!}$

   c) $\dfrac{k!}{(k-1)!}$

   d) $4!5$

   e) $8!9$

   f) $n!(n+1)$

   g) $3! + 4!$

   h) $5! + 6!$

   i) $n! + (n+1)!$

3. Solve for n in the following factorials.

   a) $\dfrac{(n+1)!}{n!} = 7$

   b) $\dfrac{(7!1!)}{2!5!} = \dfrac{(n-6)!}{(n-7)!}$

Permutations

In Example 1, you were asked to calculate how many different ways you can order books on a shelf. This was a **permutation**, or an ordered arrangement. *You can order n different items in n! ways*. Using this definition, the three books were ordered in 3! ways or 3 × 2 × 1 = 6 ways.

Factorial notation is used so that you do not have to write out a long list of numbers. If you were asked how many ways you can arrange 25 items in a row, the answer would be 25!. It is much easier to write 25! than 25 × 24 × 23 × . . ., etc., or approximately 1.551×10^{25}.

Factorial notation is useful to simplify expressions and do permutation and combination calculations. You may even have a factorial (!) button on your calculator.

What happens if you have n items, but you want to arrange only k of them? Of course, k must be less than n. In other words, what happens if you have a large number of items and you want to know how many ways you can arrange a smaller subset of the items? Consider the following example.

Example 3

In how many ways can a row of

a) three desks be filled from amongst 10 students?

b) four desks be filled from amongst eight students?

c) five desks be filled from amongst 20 students?

d) 40 desks be filled from amongst 60 students?

Solutions

The first two questions can be done easily by thinking about the Fundamental Counting Principle. How many students are available to sit in each desk?

a) $\underline{\qquad 10 \qquad} \cdot \underline{\qquad 9 \qquad} \cdot \underline{\qquad 8 \qquad} = 720$

Similarly, this could have been written as:

$$\frac{10!}{7!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 10 \cdot 9 \cdot 8 = 720$$

b) $\underline{\qquad 8 \qquad} \cdot \underline{\qquad 7 \qquad} \cdot \underline{\qquad 6 \qquad} \cdot \underline{\qquad 5 \qquad} = 1680$

Similarly, this could have been written as:

$$\frac{8!}{4!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = 8 \cdot 7 \cdot 6 \cdot 5 = 1680$$

The third question is still reasonable to compute.

c) $\underline{\quad 20 \quad} \cdot \underline{\quad 19 \quad} \cdot \underline{\quad 18 \quad} \cdot \underline{\quad 17 \quad} \cdot \underline{\quad 16 \quad} = 1\,860\,480$

Similarly, this could have been written as:

$$\frac{20!}{15!} = \frac{(20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15!)}{15!} = 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 = 1\,860\,480$$

d) You can probably see how to answer this last question, but the number of blanks (40 of them) is very unwieldy. Furthermore, the answer will be very large.

$\underline{\quad 60 \quad} \cdot \underline{\quad 59 \quad} \cdot \underline{\quad 58 \quad} \ldots \underline{\quad 22 \quad} \cdot \underline{\quad 21 \quad} = ?$

However, you can write the answer to this question in factorial notation by multiplying the answer by a special form of 1. The form chosen is a number that will produce a factorial from the product:

$\underline{\quad 60 \quad} \cdot \underline{\quad 59 \quad} \cdot \underline{\quad 58 \quad} \ldots \underline{\quad 22 \quad} \cdot \underline{\quad 21 \quad}$

The part that is missing is the product from 1 to 20. Hence,

$\underline{\quad 60 \quad} \cdot \underline{\quad 59 \quad} \cdot \underline{\quad 58 \quad} \ldots \underline{\quad 22 \quad} \cdot \underline{\quad 21 \quad} =$

$$60 \cdot 59 \cdot 58 \ldots 22 \cdot 21 \cdot \frac{20 \cdot 19 \cdot 18 \ldots 2 \cdot 1}{20 \cdot 19 \cdot 18 \ldots 2 \cdot 1} = \frac{60!}{20!}.$$

The advantage of using the factorial notation version is that this answer can be found on your calculator using the factorial key. If your calculator does not have this key on its face, it is sometimes found in the MATH menu or the PRB menu.

On a calculator, you get:

$$\frac{60!}{20!} = \frac{8.32099 \times 10^{81}}{2.43290 \times 10^{18}} = 3.42019 \times 10^{63} \text{ approximately}$$

Example 4

In how many ways can

a) 10 prizes be given away from amongst 30 contestants?

b) 20 prizes be given away from amongst 50 contestants?

c) 15 prizes be given away from amongst 40 contestants?

d) r prizes be given away from amongst n contestants?

Solutions

You should multiply by special forms of 1 so that factorial notation can be used. The answers will be left in factorial notation for now so that a formula will become apparent.

a) $$\underbrace{30 \cdot 29 \ldots 22 \cdot 21}_{\text{10 numbers}} = 30 \cdot 29 \ldots 22 \cdot 21 \cdot \frac{20 \cdot 19 \ldots 2 \cdot 1}{20 \cdot 19 \ldots 2 \cdot 1} = \frac{30!}{20!} = 1.09 \times 10^{14}$$

b) $$\underbrace{50 \cdot 49 \ldots 32 \cdot 31}_{\text{20 numbers}} = 50 \cdot 49 \ldots 32 \cdot 31 \cdot \frac{30 \cdot 29 \ldots 2 \cdot 1}{30 \cdot 29 \ldots 2 \cdot 1} = \frac{50!}{30!} = 1.15 \times 10^{32}$$

c) $$\underbrace{40 \cdot 39 \ldots 27 \cdot 26}_{\text{15 numbers}} = 40 \cdot 39 \ldots 27 \cdot 26 \cdot \frac{25 \cdot 24 \ldots 2 \cdot 1}{25 \cdot 24 \ldots 2 \cdot 1} = \frac{40!}{25!} = 5.26 \times 10^{22}$$

Look at the three examples above. Can you see the pattern? You may have noticed that the numerator is the factorial of the number of contestants (or objects), and the denominator is the factorial of the difference between the number of prizes and the number of contestants (available positions). Therefore,

d) $$n \cdot (n-1) \ldots (n-r+2) \cdot (n-r+1)$$

$$= n \cdot (n-1) \ldots (n-r+2) \cdot (n-r+1) \frac{(n-r) \cdot (n-r-1) \ldots 2 \cdot 1}{(n-r) \cdot (n-r-1) \ldots 2 \cdot 1}$$

$$= \frac{n!}{(n-r)!}$$

The arrangement of a set of objects in which the *order of the objects is important* is called a **permutation**. The number of ways n objects can be placed into r available positions, written as ${}_nP_r$, is given by the formula $\frac{n!}{(n-r)!}$.

Note: r = number of blanks or number of decisions to be made

n = how many objects you have altogether

Include the above information on your resource sheet.

It is often easier to calculate the number of permutations by writing out blanks for each decision and then multiplying using the Fundamental Counting Principle. However, when the number of blanks is large, you are better off using the formula.

Example 5

Find the value of

a) $_6P_2$
b) $_6P_3$
c) $_{10}P_3$
d) $_{16}P_4$
e) $_{23}P_{14}$
f) $_2P_8$

Solution

When the number of decisions to be made is small, you can use blanks or factorial notation. When the number of available positions is large, factorial notation is more appropriate.

a) $_6P_2$ means you have 6 objects altogether and 2 decisions to be made. Using the Fundamental Counting Principle, the solution is:

$6 \cdot 5 = 30$

Using the formula $\dfrac{n!}{(n-r)!}$, you know that $n = 6$ and $r = 2$.

$$\frac{n!}{(n-r)!} = \frac{6!}{(6-2)!} = \frac{6!}{4!} = \frac{6 \cdot 5 \cdot 4!}{4!} = 6 \cdot 5 = 30$$

b) $_6P_3$ means you have 6 objects and 3 decisions, so $6 \cdot 5 \cdot 4 = 120$.

Using the formula, you know that $n = 6$ and $r = 3$.

$$\frac{n!}{(n-r)!} = \frac{6!}{(6-3)!} = \frac{6!}{3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3!} = 6 \cdot 5 \cdot 4 = 120$$

c) $_{10}P_3$ means 10 objects and 3 decisions, so $10 \cdot 9 \cdot 8 = 720$.

Using the formula, $n = 10$ and $r = 3$.

$$\frac{n!}{(n-r)!} = \frac{10!}{7!} = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!} = 10 \cdot 9 \cdot 8 = 720$$

d) ${}_{16}P_4$ means 16 objects and 4 decisions, so $16 \cdot 15 \cdot 14 \cdot 13 = 43\,680$.

Using the formula, $n = 16$ and $r = 4$.

$$\frac{n!}{(n-r)!} = \frac{16!}{12!} = \frac{16 \cdot 15 \cdot 14 \cdot 13 \cdot 12!}{12!} = 16 \cdot 15 \cdot 14 \cdot 13 = 43\,680$$

e) ${}_{23}P_{14}$ means 23 objects and 14 decisions. You could write this out using the Fundamental Counting Principle, however, this is one situation where the formula can help save you time.

Using the formula, $n = 23$ and $r = 14$.

$$\frac{n!}{(n-r)!} = \frac{23!}{9!} = 7.12 \times 10^{16}$$

Note: It is also possible to calculate this answer by using the ${}_nP_r$ button on your calculator. See if your calculator has this function and try it out.

f) ${}_2P_8$ has no meaning.

$n = 2$ and $r = 8$

You may be tempted to put these values into the formula right away. If you do this, however, you will be trying to find the value of a negative factorial. Factorials of negative numbers have not been defined. The reason for this is that you only have 2 objects but you are trying to make 8 decisions about these 2 objects. It is only possible to make a maximum of 2 decisions about 2 objects. Therefore, the answer to this question is undefined.

You may find it beneficial to include a description on your resource sheet of why n must be greater than or equal to r in a permutation question.

Example 6

Jake and Mariah invited four other people to sit on their bench to watch the basketball game. In how many ways can these six people be seated on this bench if

a) there are no restrictions?

b) Jake is seated at the left end and Mariah is seated at the right end?

c) Jake and Mariah must be together?

Solutions

a) This is an unrestricted permutation of six people taken six at a time.

Six people and six positions yields:

$$\underline{\;6\;}\cdot\underline{\;5\;}\cdot\underline{\;4\;}\cdot\underline{\;3\;}\cdot\underline{\;2\;}\cdot\underline{\;1\;} = 720$$

Or, using the formula:

$${}_nP_r = \frac{n!}{(n-r)!}, \text{ you get } {}_6P_6 = \frac{6!}{0!} = 6! = 720$$

(Now you can see why it is consistent that 0! is defined to be 1.)

b) If you are given a permutation question with restrictions, it is easiest to calculate the answer using blanks instead of using the formula.

$$\underset{\substack{\uparrow\\ \text{Jake}}}{\underline{\;1\;}}\cdot\underbrace{\underline{\;4\;}\cdot\underline{\;3\;}\cdot\underline{\;2\;}\cdot\underline{\;1\;}}_{\text{arrange the remaining 4 people}}\cdot\underset{\substack{\uparrow\\ \text{Mariah}}}{\underline{\;1\;}} = 24$$

c) This is an example of a **group permutation**. In this example, it is easiest to treat Jake and Mariah as one person or one group that move around together.

You can consider this problem in two parts. First, you need to determine how many arrangements there are of the 5 groups. Remember, Jake and Mariah are now counted as one group with each of their four friends being counted as additional groups.

$$\overset{5}{\underline{\quad}}\cdot\overset{4}{\underline{\quad}}\cdot\overset{3}{\overbrace{\underline{\quad}\;\underline{\quad}}}\cdot\overset{2}{\underline{\quad}}\cdot\overset{1}{\underline{\quad}}$$

There are 5! = 120 possible arrangements of these five groups.

Now you can consider how many different arrangements there are of Jake and Mariah inside their particular group. Two people can be arranged in two ways and thus there are:

$$\overset{5}{\underline{\quad}}\cdot\overset{4}{\underline{\quad}}\cdot\overset{3}{\overbrace{\underline{\;2\;}\cdot\underline{\;1\;}}}\cdot\overset{2}{\underline{\quad}}\cdot\overset{1}{\underline{\quad}}$$

2! = 2 possible arrangements of Jake and Mariah.

To determine your final answer, you need to multiply these two values together. Why multiply? The reason is that for each of the 120 possible arrangements of five groups, it is possible for Jake and Mariah to be arranged in 2 different ways. Therefore, you need to multiply 120 by 2 in order to determine the total number of arrangements in this situation. This is an application of the Fundamental Counting Principle.

The final answer is: $120 \cdot 2 = 240$ possible arrangements.

Example 7

Solve for n.

a) $_nP_2 = 20$

b) $\dfrac{(n+3)!}{n!} = 24$

Solutions

a) Since $_nP_2 = 20$, it follows that $\dfrac{n!}{(n-2)!} = 20$.

$$\frac{n(n-1)(n-2)!}{(n-2)!} = 20$$

Notice that you start with the higher number, $n!$, and reduce it down to the lower number, $(n-2)!$.

Therefore, $n(n-1) = 20$.

Or,

$$n^2 - n - 20 = 0$$

$$(n+4)(n-5) = 0$$

$$n = 5 \quad \cancel{n = -4}$$

($n = -4$ is impossible since you cannot have -4 objects.)

Therefore, $n = 5$.

b) $\dfrac{(n+3)(n+2)(n+1)n!}{n!} = 24$

Again, keep writing the factors of the higher factorial until you reach the lower factorial.

$(n+3)(n+2)(n+1) = 24$

$(n+3)(n+2)(n+1) = 4(3)(2)$

By inspection, three consecutive integers with a product of 24 are 4, 3, and 2.

Hence, $n = 1$.

You know n is a number from the set {0, 1, 2, 3, . . .}, so in this case, thinking through the answer is the best method for solving it.

Learning Activity 1.4

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Your restaurant bill came to \$35.73. If you wish to leave a 15% tip, approximately how much should you leave?
2. Factor: $2x^2 - 5x - 12$
3. Solve for x: $\dfrac{1}{9^x} = 81$
4. At what speed must you travel to go 832 metres in 8 seconds?
5. Solve for x: $(x - 4)^2 = 1$
6. Expand: $(x + y)^2$
7. Write as a mixed number: $\dfrac{435}{51}$
8. What is 40% of 1560?

continued

Learning Activity 1.4 (continued)

Part B: Permutations

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Simplify as much as possible.
   a) $\dfrac{(k+3)!}{(k+2)!}$
   b) $\dfrac{7!(r+2)!}{6!(r-1)!}$

2. Solve for n.
   a) $\dfrac{(n+2)!}{(n+1)!} = 20$
   b) $(n-1)! = 72(n-3)!$
   c) $\dfrac{(n+1)!}{(n-1)!} - 30 = 0$

3. Show that:
   a) $\dfrac{n!}{(n-r-1)!(r+1)!} = \dfrac{n!}{(n-r)!r!} \cdot \dfrac{(n-r)}{r+1}$
   b) $\dfrac{n!r}{(n-r)!r!} = \dfrac{(n-1)!n}{(n-r)!(r-1)!}$

4. Find the value of r if $\dfrac{18!}{(18-r)!r!} = \dfrac{18!}{(16-r)!(r+2)!}$.

5. Find the value of each without using your calculator.
   a) ${}_5P_2$
   b) ${}_5P_3$
   c) ${}_5P_5$
   d) ${}_{100}P_2$
   e) ${}_6P_1$

continued

Learning Activity 1.4 (continued)

6. Leave your answers in factorial form.
   a) In how many ways can five seats on a bench be assigned from amongst 12 people?
   b) In how many different ways can eight vacant seats be occupied on a bus by four people, if each person occupies only one seat?
   c) In how many ways can a president, a treasurer, and a secretary be selected from amongst 10 candidates if no candidate can hold more than one position?

7. How many "words" can be made using all letters of the word MABLE? **Note:** When you are asked to form words in counting problems, unless stated otherwise, it means any permutation of the given letters. Nonsense words are acceptable (for example, ELBAM is considered a "word" in this instance).

8. How many words can be formed by using five letters of the word CROMBIE?

9. Solve for n if ${}_nP_3 = 7({}_6P_2)$.

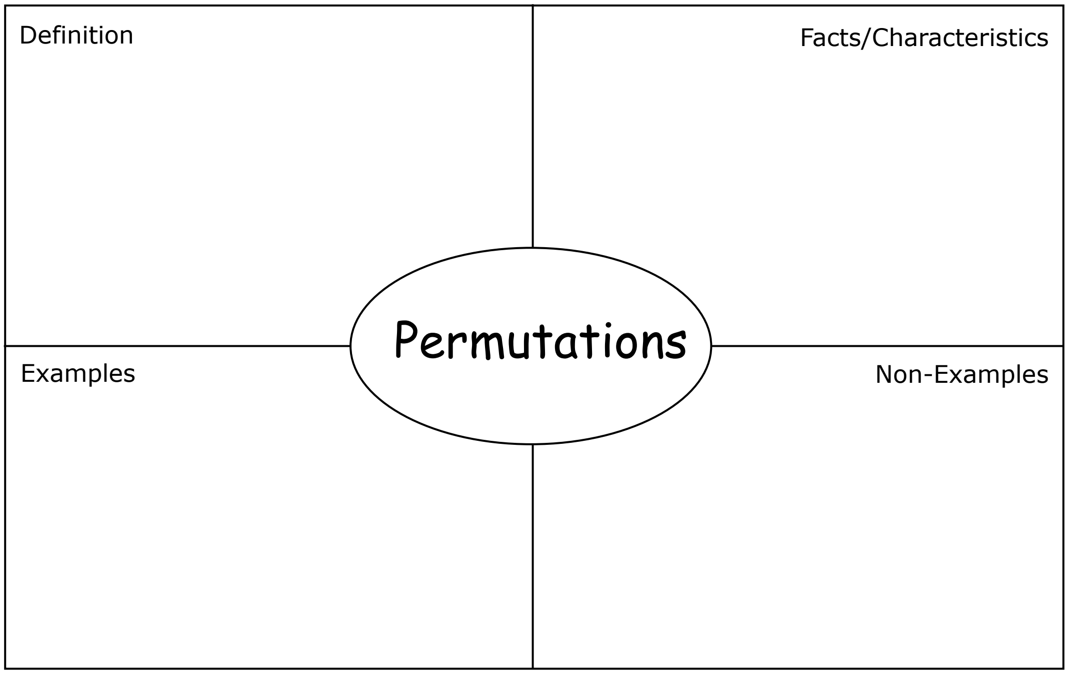
10. Explain the meaning of ${}_8P_3$. Why does ${}_3P_8$ not make sense?

11. Solve for n if ${}_nP_2 = 72$.

continued

Learning Activity 1.4 (continued)

12. Fill in the following Permutation Graphic Organizer according to the headings in each section. **Hint:** This graphic organizer is something you can add to your resource sheet, study from, or even create for different concepts, to help increase your understanding.



13. Find the different ways 7 friends (3 boys and 4 girls) can sit together at the movie theatre if
    a) there are no restrictions
    b) Jenelle must be the first person in the row
    c) Jenelle can't be the first person in the row
    d) Jenelle must be the first person in the row and Blake must be the last person in the row
    e) Jenelle and Blake must sit together
    f) a girl must sit in the first seat and the last seat

Lesson Summary

In this lesson, you learned how to count the number of arrangements of objects by using a formula involving factorials. This formula counts the number of permutations of objects or the number of arrangements of objects when order matters in the arrangement. In this lesson, you did not consider situations where some of the objects are the same; you only considered arrangements where each object was different. In the next lesson, you will learn how to calculate the number of ordered arrangements of objects when some of the objects are identical. Complete the following assignment before moving on the the next lesson.

Assignment 1.1

The Fundamental Counting Principle and Permutations

Total: 45 marks

You need to show all your steps as you work out your solutions, and make sure your answers are clear (include units, where appropriate). Check the Introduction for the Marking Guidelines that will be used for all assignments and examinations.

1. Use the Fundamental Counting Principle to answer the following question. Using the digits 1, 2, 3, 5, 6, 8, 0, with no repetition allowed, how many four-digit numbers can be constructed if
   a) there are no other restrictions other than no repeated digits? Is the order of the digits important. Explain. *(3 marks)*

   b) the number is even? *(3 marks)*

continued

Assignment 1.1: The Fundamental Counting Principle and Permutations (continued)

c) each digit of the number is even? *(2 marks)*

continued

Assignment 1.1: The Fundamental Counting Principle and Permutations (continued)

2. Use the Fundamental Counting Principle to answer the following question. In how many ways can four women and four men be seated in a row on a large bench if
   a) there are no restrictions? *(1 mark)*

   b) the men and women must alternate seats? *(2 marks)*

   c) one man and one woman are dating and want to sit together? *(2 marks)*

   d) one man and one woman used to date and now they cannot sit beside each other? *(1 mark)*

continued

Assignment 1.1: The Fundamental Counting Principle and Permutations (continued)

3. Simplify without using a calculator. All work must be shown for marks.

   a) $\frac{7!}{6!}$ *(1 mark)*

   b) $\frac{12!}{10!}$ *(1 mark)*

   c) $\frac{10!}{6!4!}$ *(1 mark)*

   d) $6!7$ *(1 mark)*

   e) $3!2!$ *(1 mark)*

   f) $\frac{5!8!}{4!7!}$ *(1 mark)*

   g) $\frac{(n+7)!}{(n+6)!}$ *(1 mark)*

   h) $\frac{n!}{(n-2)!}$ *(2 marks)*

continued

Assignment 1.1: The Fundamental Counting Principle and Permutations (continued)

4. Solve for n.

   a) $\dfrac{n!}{(n-1)!} = 10$ *(1 mark)*

   b) $\dfrac{n!}{(n-2)!} = 42$ *(3 marks)*

5. Rewrite each in factorial notation and then find the value without using your calculator. *(4 × 2 marks each = 8 marks)*

   a) ${}_{4}P_{2}$

   b) ${}_{7}P_{3}$

   c) ${}_{6}P_{6}$

   d) ${}_{101}P_{2}$

continued

Assignment 1.1: The Fundamental Counting Principle and Permutations (continued)

6. Explain your thinking.
   a) Manitoba Public Insurance requires that learners must pass a written test before receiving their Learner's License. The test consists of twenty multiple choice questions, of which sixteen must be answered correctly. Explain why this is not an application of permutations. *(1 mark)*

   b) Describe, without calculating, how to find the number of ways three different jobs can be filled from amongst eight people. *(1 mark)*

7. Find the number of different arrangements of the word FRIGHTEN if *(6 × 1 mark each = 6 marks)*
   a) there are no restrictions

   b) the first letter must be G

   c) the first letter is not a G

continued

d) the first letter must be G and the last letter must be H

e) the arrangements must end in TH

f) the arrangements begin and end in a vowel

8. The positions of captain and assistant captain have to be assigned for a school hockey team. If there are 15 players on the team, in how many ways can the captain and assistant captain be assigned? Explain your method. *(2 marks)*

Notes

LESSON 3: PERMUTATIONS OF LIKE OBJECTS

Lesson Focus

In this lesson, you will

- ☐ learn how to count the number of permutations of items in which some of the items in the set are the same
- ☐ learn how to solve a variety of permutation problems

Lesson Introduction

In Lesson 2, you calculated arrangements of objects. This was done when each object in the set was different, but this is not always the case. Sometimes you may encounter sets of objects in which some of the objects are the same. For example, in the word THAT, there is one H, one A, but two Ts. Therefore, it is not possible to calculate the number of arrangements of these letters in the same way as you did before. If you were to calculate the arrangement of the letters in THAT the way you did earlier, you would have to somehow distinguish between the two Ts. When writing words, however, it is the same word even after switching the two Ts. In this lesson, you will learn a way of calculating the number of arrangements of objects when some of the objects in the set are identical.

Permutations

Look at the following lists of all the arrangements of the letters in ZOO and the letters in APE.

ZOO	**APE**	
ZOO	APE	AEP
OZO	PEA	PAE
OOZ	EAP	EPA

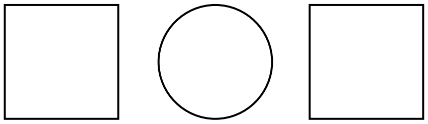
Why are there a different number of arrangements even though both words have three letters?

The reason is because there are fewer permutations of objects when some of the letters are the same. To see why, suppose you could distinguish between the Os in ZOO. Let's call them O_1 and O_2. If the Os are different, then there are 3! permutations of the letters in the word ZO_1O_2. The permutations are:

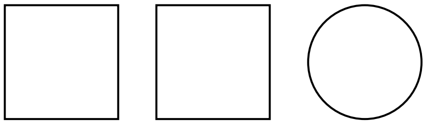
ZO_1O_2 O_1O_2Z O_1ZO_2
ZO_2O_1 O_2O_1Z O_2ZO_1

If you ignore the subscript of each letter O, you will notice that there are only three different words, not six different words. The answer 3 can be found by dividing the six permutations by the number 2, which represents the number of ways of arranging the two Os.

Another way to see this is to consider shapes. How many ways can the shapes shown below be arranged in a row?



There are three items to arrange, but two of them are identical so you will get the following arrangements.



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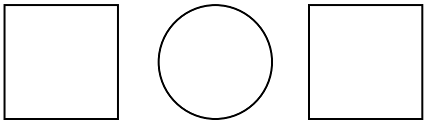
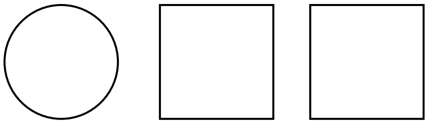


or



Since the squares are identical, there are only three different arrangements possible.

If none of these objects were identical, then there would have been 3! = 6 ways to arrange the 3 shapes. Because of the identical squares, there are only 3 different permutations or arrangements, since there are 2! ways of arranging the two squares counted separately. Notice that there are fewer permutations when you are arranging identical objects because you divide the total number of arrangements by the number of ways of arranging the objects that are identical.

Example 1

How many different “words” can be formed using the letters of the following words?

a) SEA

b) SEE

c) SEAM

d) SEEM

Solutions

a) $3! = 6$

b) Two of the letters in the word SEE are the same. There are 2! ways to arrange these two letters. Therefore, you need to divide the total number of permutations of the three letters by 2!.

$$\frac{3!}{2!} = \frac{6}{2} = 3$$

c) $4! = 24$

d) Using the reasoning above, you would expect $24 \div 2 = 12$ permutations, since the two Es can be rearranged in two ways that do not change the word.

The permutations can be arranged in an organized list like this:

	EESM	EEMS	← “words” beginning with two Es
	ESEM	ESME	← “words” beginning with ES
	EMES	EMSE	← “words” beginning with EM
SEEM	SEME	SMEE	← “words” beginning with S
MEES	MESE	MSEE	← “words” beginning with M

Example 2

How many different “words” can be formed using the letters of the following words?

a) STRESS

b) SUCCESS

Solutions

a) If the Ss were distinguishable, the six letters could be ordered in $6! = 720$ ways.

The three Ss can be ordered in $3! = 6$ ways that are actually all the same.

Therefore, the letters of the word STRESS can be ordered in $\frac{6!}{3!} = \frac{720}{6} = 120$ ways.

b) If the letters were all different, there would be 7! ways to arrange them.

However, there are three Ss and 3! ways to arrange these Ss.

There are also two Cs and 2! ways to arrange these Cs.

Therefore, the letters of the word SUCCESS can be ordered in $\frac{7!}{3!2!} = \frac{5040}{12} = 420$ ways.

In general, a set of n objects taken together, of which there are a alike of one kind, b alike of another object, and c alike of the last kind has $\frac{n!}{a!b!c!}$ permutations.

This formula could be expanded if there were four or more different kinds of repeating objects.

Include the above definition and formula on your resource sheet. You may also find it helpful to include an example of this type of situation on your resource sheet.

Example 3

How many permutations are there of the letters of each of the following words?

a) AARDVARK

b) MISSISSIPPI

Solutions

a) There are three As and two Rs in AARDVARK. Using the above formula, the number of permutations of this word are $\frac{8!}{3!2!} = \frac{40\ 320}{12} = 3360$.

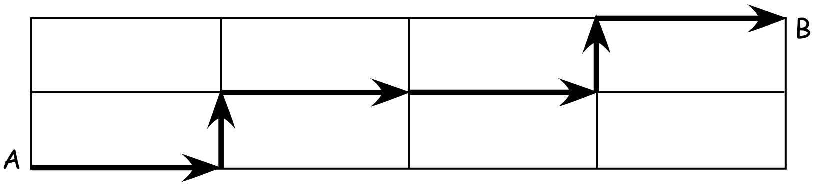
b) There are four Ss, four Is, and two Ps in MISSISSIPPI. Therefore, there are $\frac{11!}{4!4!2!} = \frac{39\ 916\ 800}{1152} = 34\ 650$ permutations of the letters.

Example 4

In how many different ways can Albert walk to Bart if the trip takes exactly six blocks?

Solution

A trip requires Albert to walk four blocks to the right and two blocks up. Each arrangement of the blocks right (R) and up (U) is a different way. Therefore, Albert must decide how to order four Rs and two Us. For example, the order RURRUR would produce the trip outlined below.



The number of ways of arranging four Rs and two Us is 6! divided by the number of repeats.

$$\frac{6!}{4!2!} = 15$$

Make sure you complete the following Learning Activity, as it will allow you to practice computing permutations with repeated objects.

Learning Activity 1.5

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Simplify: $3\sqrt[4]{81}$
2. Evaluate: $|\,3.64 - 4.78\,|$
3. Solve: $2x^2 + 3x - 9 = 0$
4. What is 40% of $\frac{7}{8}$?
5. Write as a mixed fraction: $\frac{18}{5}$
6. List all the factors of 100.
7. Simplify: $\sqrt{108}$
8. If the balance in your chequing account is \$520, what is your new balance after your paycheque of \$256 is deposited and you buy gas for \$49?

Part B: Permutations Involving Like Objects

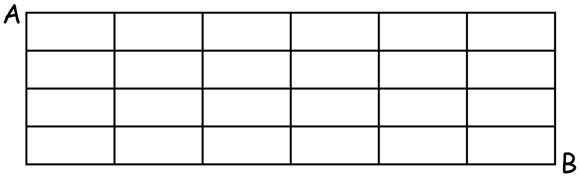
Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. How many permutations are there of the letters of the following words?
    a) SCHOOLS
    b) BOOKKEEPERS
    c) CANADIAN

continued

Learning Activity 1.5 (continued)

2. In how many distinct ways can three red flags, two blue flags, two green flags, and four yellow flags be arranged in a row?
3. If $a^3b^2c^4$ is written with exponents of one, how many arrangements of the nine letters are possible (e.g., abaccabcc would be one of these arrangements)?
4. How many different five-digit numbers can be formed using the digits 1, 2, 3, 4, and 5, if
   a) the odd digits occupy the odd places?
   b) the odd digits occupy the odd places in ascending order?
5. All the letters of the word BARRIER are arranged. Find the number of arrangements
   a) beginning with the letter R
   b) beginning with two Rs
   c) beginning with three Rs
   d) beginning with exactly one R
   e) beginning with exactly two Rs
6. How many arrangements can be made using all the letters of BABBLING BABY?
7. In how many different ways can person A walk to person B if the trip takes exactly 10 blocks?



8. A coin is flipped eight times. In how many ways could the result be five heads and three tails (e.g., HHTHTHTH would be one of the ways)?
9. In how many ways can a committee of three people be selected from five candidates identified as A, B, C, D, and E? (Careful! The selections ABC and ACB are the same committee.)

Lesson Summary

In this lesson, you learned more about permutations. You learned how to calculate permutations when some of the items in the set are identical. To do this, you needed to incorporate factorials and divide by the number of times a certain arrangement occurred. In the next lesson, you will look at a different type of counting technique called combinations.

LESSON 4: COMBINATIONS

Lesson Focus

In this lesson, you will

☐ learn how to solve problems involving combinations

☐ learn how to distinguish between permutations and combinations

☐ learn how to solve problems involving permutations and combinations

Lesson Introduction

Consider the following situation: You and your friend are going out for pizza to The Pizza Palace. The Pizza Palace has 10 different toppings for pizza available. How many ways can you choose two different toppings?

Does the order in which the toppings are placed on the pizza matter? In other words, is a pepperoni and mushroom pizza (where the pepperoni was put on first) different than a mushroom and pepperoni pizza (where the mushrooms were put on first)?

No, both of these pizzas are the same. Therefore, this is *not* an example of a permutation because permutations deal with situations where order matters.

In this lesson, you will be learning how to count the number of arrangements of objects in situations where *order does not matter*. These situations, including the pizza problem above, are called combinations.

Combinations

When you calculate permutations, you calculate the number of ways of selecting and then ordering a number of objects. However, when you are dealing with combinations, you are interested only in selecting objects; you are not interested in the order in which the objects are selected.

Combinations Versus Permutations

Question 9 from Learning Activity 1.5 involved combinations: “In how many ways can a committee of three people be selected from five candidates, A, B, C, D, and E?”

To solve this question, consider the following.

The number of permutations if these 5 people are taken 3 at a time where order matters is ${}_5P_3 = 5 \cdot 4 \cdot 3 = 60$.

However, we are selecting a committee and the order of the 3 people doesn’t matter (ABC, ACB, BAC, BCA, CAB, and CBA all represent the same committee). There are 3! or 6 ways of arranging the 3 committee members that should not be counted as separate arrangements.

This situation is the same as when you had 3 repeating letters and you had to divide the permutations by 3!, since the order of the identical letters did not matter.

Now with 3! arrangements all representing the same committee, we need to divide the permutations by 3!

number of
permutations
of 3 objects

$$\frac{{}_5P_3}{3!} = \frac{60}{3} = 10$$

divide by the
number of ways of
arranging 3 objects

Each selection in which the *order of the selection is not important* is called a **combination**.

Be sure you can distinguish between the two different ways of counting outlined below.

- The number of ways of arranging objects in a particular order is a **permutation**
- The number of ways of arranging objects in a group in no particular order is a **combination**

For example, as was pointed out at the beginning of Lesson 2, the sequence of numbers for the lock on your locker should actually be called a permutation, not a combination, since the order of the numbers to open your lock matters.

A general formula to find the number of combinations of n different objects taken r at a time, written as ${}_nC_r$ or $\binom{n}{r}$, is

$${}_nC_r = \frac{{}_nP_r}{r!} = \frac{\dfrac{n!}{(n-r)!}}{r!} \text{ or } {}_nC_r = \frac{n!}{(n-r)!r!}$$

You should add this formula to your resource sheet: ${}_nC_r = \dfrac{n!}{(n-r)!r!}$

Example 1

How many sets of two marbles can be selected from a bag with five marbles of different colours?

Solution

The order of the marbles in a set does not matter. Therefore, this is a combination question and there are ${}_5C_2 = \dfrac{5!}{3!2!} = 10$ sets of marbles.

You can see this pattern by analyzing the table below.

	blue	**yellow**	**orange**	**red**	**pink**
blue	BB	BY	BO	BR	BP
yellow	YB	YY	YO	YR	YP
orange	OB	OY	OO	OR	OP
red	RB	RY	RO	RR	RP
pink	PB	PY	PO	PR	PP

Note: Right away, you can cross off the combinations consisting of two of the same colours of marbles, as you know those colour combinations cannot exist (there is only one marble of each colour in the bag; the two marbles cannot be the same colour).

	blue	**yellow**	**orange**	**red**	**pink**
blue	BB	BY	BO	BR	BP
yellow	YB	YY	YO	YR	YP
orange	OB	OY	OO	OR	OP
red	RB	RY	RO	RR	RP
pink	PB	PY	PO	PR	PP

Now you can see that there are ten arrangements of marbles above the diagonal crossed-out line and ten arrangements of marbles below the diagonal crossed-out line.

If you compare the arrangements that are diagonal from each other (for example, YB and BY), you will notice that they are the same set of marbles; the marbles are just listed in a different order. Therefore, you can cross out one of these sets of marbles because they represent the same set.

If you do this for every colour combination, you will divide the twenty arrangements in half to arrive at an answer of ten.

	blue	**yellow**	**orange**	**red**	**pink**
blue	BB	**BY**	**BO**	**BR**	**BP**
yellow	YB	YY	**YO**	**YR**	**YP**
orange	OB	OY	OO	**OR**	**OP**
red	RB	RY	RO	RR	**RP**
pink	PB	PY	PO	PR	PP

From the chart, you can see that there are ten different combinations of marbles.

Example 2

Evaluate.

a) ${}_9C_3$

b) ${}_{100}C_2$

c) ${}_3C_3$

d) $\begin{pmatrix} 8 \\ 2 \end{pmatrix}$

e) $\begin{pmatrix} 50 \\ 3 \end{pmatrix}$

f) ${}_7C_3$

g) ${}_7C_4$

Solutions

a) ${}_9C_3$ means that you are choosing 3 objects from a set of 9 objects. It is the number of permutations of selecting 3 objects from a set of 9 objects divided by 3!. This is equivalent to:

$$\frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 84$$

Or, using the formula:

$$\frac{9!}{(9-3)!3!} = \frac{9!}{6!3!} = 84$$

b) ${}_{100}C_2$ means $\frac{100 \cdot 99}{2 \cdot 1} = 4950$

Or, using the formula:

$$\frac{100!}{(100-2)!2!} = \frac{100!}{98!2!} = \frac{(100)(99)}{2} = 4950$$

Note: Part (b) in the above example was calculated by reducing the factorial fraction $\frac{100!}{98!}$ before attempting to evaluate the answer. The reason for reducing first is that your calculator may produce an error message when you try to evaluate a factorial such as 100!. It is too large. This is an example where your understanding and facility with factorial notation comes in handy.

Another way you can handle this problem is by using a built-in function ${}_nC_r$. Most scientific calculators have this function and it calculates the final answer all at once. If your calculator has this function, you can try using it to check your answers.

c) ${}_3C_3$ means $\frac{3 \cdot 2 \cdot 1}{3!} = 1$

Or, using the formula:

$$\frac{3!}{(3-3)!3!} = \frac{3!}{0!3!} = 1$$

d) Recall: $\binom{8}{2}$ is the same as ${}_8C_2$. Therefore, $\binom{8}{2}$ means $\frac{(8 \cdot 7)}{2 \cdot 1} = 28$.

Or, using the formula:

$${}_8C_2 = \frac{8!}{6!2!} = 28$$

e) $\binom{50}{3}$ is the same as ${}_{50}C_3$. Therefore, $\binom{50}{3}$ means $\dfrac{50 \cdot 49 \cdot 48}{3 \cdot 2 \cdot 1} = 19\,600$.

Or, using the formula:

$$\frac{50!}{47!3!} = \frac{(50)(49)(48)}{(3)(2)} = 19600$$

f) ${}_7C_3$ means $\dfrac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$ or

$$\frac{7!}{(7-3)!3!} = \frac{7!}{4!3!} = 35$$

g) ${}_7C_4$ means $\dfrac{7 \cdot 6 \cdot 5 \cdot 4}{4 \cdot 3 \cdot 2 \cdot 1} = 35$ or

$$\frac{7!}{(7-4)!4!} = \frac{7!}{3!4!} = 35$$

Notice for (f) and (g) that ${}_7C_3$ is the same as ${}_7C_4$. Do you think that is a coincidence? You will look at this topic in more detail later.

Example 3

There are four books on a shelf. In how many ways can

a) the books be arranged on the shelf?

b) Jeremy and Alexis select two books each?

c) the books be separated into two groups with each group containing two books each?

Solutions

a) This is a permutation. Therefore, order matters.

$4 \cdot 3 \cdot 2 \cdot 1 = 24$ possible ways

b) Once Jeremy selects his two books, Alexis takes the remaining two books.

Jeremy can select two books in ${}_4C_2 = \dfrac{4!}{2!2!} = 6$ ways. This is the same answer as if Alexis selects the books first and Jeremy second. Consider the following chart.

If the books were labelled A, B, C, and D, then the possible distributions are:

Jeremy	AB	AC	AD	BC	BD	CD
Alexis	CD	BD	BC	AD	AC	AB

c) In this part, the books must be subdivided and are not assigned to any individual. Therefore, the number of ways is

$${}_4C_2 \div 2! = \frac{4!}{2!2!} \div 2 = 6 \div 2 = 3.$$

Consider the above chart. If the objective is only to subdivide the books into two equal groups, then the first three columns are repeated in the last three columns. Therefore, there are only three possible subdivisions:

AB	AC	AD
CD	BD	BC

Note: Whenever a group of objects is to be subdivided into equal unassigned parcels, you must divide by the number of ways of ordering these parcels. You are dividing out the repeats.

Example 4

A committee consists of five men and four women. A subcommittee of two men and three women is to be selected from the original committee. In how many ways can this subcommittee be selected?

Solution

You must select two men from a set of five and three women from a set of four. You can do this in ${}_5C_2 \cdot {}_4C_3 = \frac{5!}{3!2!} \cdot \frac{4!}{1!3!} = (10)(4) = 40$ ways.

Note: With these questions, when you see the word AND, this means both things happen. So, by the Fundamental Counting Principle, you need to MULTIPLY. When you see the word OR, this means you have different cases that do not happen at the same time; therefore, you need to ADD. You may want to add this comment to your resource sheet.

Example 5

Ten points are placed on a circle. How many lines can you draw by joining any two of these points?

Solution

You must choose two points out of 10 points to draw a line. The order of choosing the two points is arbitrary. Therefore, you can draw

${}_{10}C_2 = \dfrac{10!}{8!2!} = 45$ lines.

Example 6

A committee of four is to be selected from amongst four girls and five boys. In how many ways can the selection be made if

a) there must be at least one girl on the committee?

b) there must be more girls than boys on the committee?

Solutions

a) "At least one" means there cannot be zero girls. This can be solved as a complementary problem:

(all possible) – (no girls on committee)

$$\underbrace{{}_9C_4}_{\text{total}} - \underbrace{{}_4C_0}_{\text{girls}} \cdot \underbrace{{}_5C_4}_{\text{boys}} = 126 - (1)(5) = 121$$

b) Case 1, four girls and no boys

${}_4C_4 \cdot {}_5C_0 = 1$

Case 2, three girls and one boy

${}_4C_3 \cdot {}_5C_1 = 4(5) = 20$

There are 1 + 20 = 21 committees with a majority of girls.

Example 7

A combination lock uses three different numbers from 1 to 39 for any combination. How many combinations are possible for this lock?

Solution

The order of the selected numbers is important. Therefore, this is a permutation.

${}_{39}P_3$ means $39 \cdot 38 \cdot 37 = 54\ 834$ or $\dfrac{39!}{36!} = 54\ 834$

Example 8

Solve for the unknown algebraically.

$${}_{n-1}P_3 = 2\left({}_{n-1}C_2\right)$$

Solution

Use ${}_nP_r = \dfrac{n!}{(n-r)!}$ and ${}_nC_r = \dfrac{n!}{(n-r)!r!}$.

$$\begin{aligned} {}_{n-1}P_3 &= 2\left({}_{n-1}C_2\right) \\ \frac{(n-1)!}{(n-1-3)!} &= 2\left(\frac{(n-1)!}{(n-1-2)!2!}\right) \end{aligned}$$

Simplify.

$$\frac{(n-1)!}{(n-4)!} = \frac{2(n-1)!}{(n-3)!2!}$$

Collect factorials to same side of equal sign using algebra.

$$\frac{(n-1)!(n-3)!}{(n-4)!(n-1)!} = \frac{2}{2!}$$

Simplify.

$$\begin{aligned} \frac{\cancel{(n-1)!}\,(n-3)!}{(n-4)!\,\cancel{(n-1)!}} &= \frac{\cancel{2}}{\cancel{2}\cdot 1} \\ \frac{(n-3)!}{(n-4)!} &= 1 \end{aligned}$$

Expand and simplify to solve for n.

$$\begin{aligned} \frac{(n-3)\,\cancel{(n-4)!}}{\cancel{(n-4)!}} &= 1 \\ n-3 &= 1 \\ n &= 1+3 \\ n &= 4 \end{aligned}$$

Make sure you complete the following Learning Activity. This Learning Activity will allow you to practice calculating combinations before you are asked to complete the following assignment, which will be handed in for marks.

Learning Activity 1.6

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Factor: $2x^2 - x - 15$
2. Simplify: $\sqrt[3]{16x^2y^6z^3}$
3. Simplify: $256^{\frac{1}{4}}$
4. Simplify: $\frac{1}{8} + \frac{7}{9}$
5. Multiply: $3\frac{2}{3} \cdot 6\frac{2}{7}$
6. What is 15% of 236?
7. Convert 14.7% into a decimal.
8. Reduce to lowest terms: $\frac{24}{360}$

continued

Learning Activity 1.6 (continued)

Part B: Combinations

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Determine whether each of the following situations represents a permutation or a combination.
   a) A restaurant offers a deal of 2 meals for $20 from a list of 15 meals. In how many ways can a customer choose 2 meals off the deal menu?
   b) One hundred people are running in a marathon. How many different ways can first, second, and third place be awarded?
   c) How many different 4-letter words can be formed from the letters in FISH if no repetition of letters is allowed?
   d) There are five males and eight females on a co-ed basketball team. How many five-person shifts can be made consisting of two male members and three female members?
2. Evaluate each pair and explain your results.
   a) ${}_{10}C_3$ and ${}_{10}C_7$
   b) ${}_{5}C_3$ and ${}_{5}C_2$
   c) ${}_{10}C_4$ and ${}_{10}C_6$
3. Based on your discovery in Question 2, if ${}_{12}C_7 = x$, then what other combination must also produce an answer of x?
4. Use factorial notation to prove that ${}_{n}C_r = {}_{n}C_{n-r}$. Explain this statement with respect to making selections. Why should this statement be true?
5. Solve for n: ${}_{n}C_4 = {}_{n}C_3$
6. In how many ways can four policemen be selected for special duty from a group of 12 policemen?
7. Natalie's wardrobe includes five pairs of slacks, eight blouses, and five pairs of shoes. She wants to select three pairs of slacks, four blouses, and three pairs of shoes for her camping trip. What is the number of selections Natalie can make?

continued

Learning Activity 1.6 (continued)

8. In a class of 30 students, each student shakes hands with each of the other students once. How many handshakes are there?

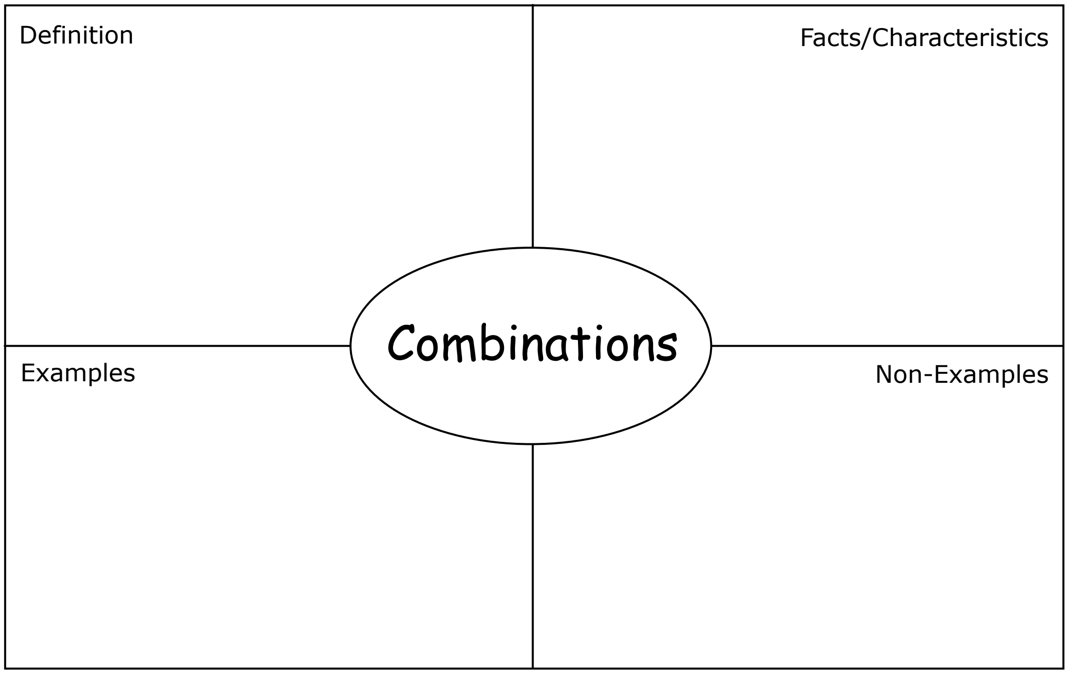
9. Consider a standard deck of 52 cards. There are four suits (clubs, diamonds, hearts, and spades) and 13 cards in each suit.
   a) How many different five-card hands can be dealt?
   b) How many of these hands contain exactly four clubs?

10. A tennis club has 10 boys and eight girls as members. From amongst these members, how many different matches are possible with
    a) a boy against a girl?
    b) two boys against two girls?

11. There are five friends sitting around a table having dinner when suddenly the doorbell rings. In how many ways can you choose nobody to answer the door?

12. We know ${}_3P_5$ does not make sense. Does ${}_3C_5$ make sense? Explain.

13. Fill in the following Combination Graphic Organizer according to the headings in each section. **Hint:** This graphic organizer is something you can add to your resource sheet, study from, or even create for different concepts to help increase your understanding.



Lesson Summary

In this lesson, you explored the difference between permutations and combinations. Permutations are selections in which order matters. Combinations are selections in which order does not matter. Before answering any questions regarding the selection of objects, you should ask yourself the question, “Does order matter?” This will help you to determine whether the question involves permutations or combinations.

Notes

Assignment 1.2

Permutations and Combinations

Total: 49 marks

You need to show all your steps as you work out your solutions, and make sure your answers are clear (include units, where appropriate). Check the Introduction for the Marking Guidelines that will be used for all assignments and examinations.

1. Determine whether each of the following situations represents a permutation or a combination. Explain. *(4 × 1 mark each = 4 marks)*
   a) 10 singers are competing at a local singing competition. How many different ways can the three top prizes be awarded?

   b) Three of your classmates are being selected to compete in a math competition. How many different ways can three people be chosen?

   c) How many different ways can 25 students be assigned to desks in a classroom?

   d) A coach must choose five players to go to the all-star tournament. How many different groups of players can be chosen?

continued

Assignment 1.2: Permutations and Combinations (continued)

2. Evaluate using factorial notation. Show all work. *(4 × 2 marks each = 8 marks)*

   a) $\binom{10}{3}$

   b) $\binom{5}{4}$

   c) ${}_8C_4$

   b) ${}_9C_6$

3. a) Evaluate ${}_6P_4$. *(1 mark)*

   b) Evaluate ${}_6C_4$. *(1 mark)*

   c) Which one of the above answers is larger? Explain using the definition of permutations and combinations. *(1 mark)*

continued

4. How many permutations are there of the letters of the following words? *(3 × 2 marks each = 6 marks)*
   a) HARDWARE

   b) RIFFRAFF

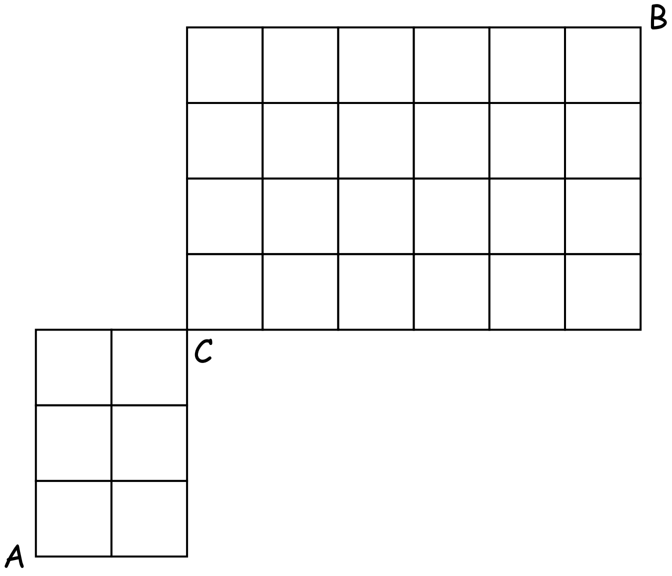
   c) MATHEMATICAL

5. In how many ways can the letters of the word MAXIMUM be arranged if the three Ms must be together? Explain your process. *(2 marks)*

continued

Assignment 1.2: Permutations and Combinations (continued)

6. In how many different ways can person A walk to person B, passing through street intersection C, if the trip takes exactly 15 blocks? *(3 marks)*



7. Lotto 6-49 is a lottery in which a person selects six different numbers from 1 to 49. The order that the numbers are selected does not matter. How many ways can a selection be made? *(1 mark)*

8. On a Grade 12 Mathematics examination, students must answer exactly five of the first six questions and three out of the last five questions. In how many ways can this be done? *(2 marks)*

continued

9. Using the digits 2, 2, 2, 3, 3, 4, 5, how many
   a) seven-digit numbers can be formed? *(2 marks)*

   b) seven-digit numbers can be formed if the number is greater than 3,400,000? *(5 marks)*

   c) seven-digit numbers can be formed if the number is even? *(3 marks)*

continued

Assignment 1.2: Permutations and Combinations (continued)

10. How many rectangles of various sizes are formed when seven vertical lines are intersected by four horizontal lines? *(2 marks)*

11. How many ways can a committee of five be chosen from 10 girls and eight boys so that the girls have a majority on the committee? *(4 marks)*

12. Solve for n in the equation using the combination formula: ${}_{n+2}C_4 = 6({}_nC_2)$ *(4 marks)*

LESSON 5: THE BINOMIAL THEOREM

Lesson Focus

In this lesson, you will

- ☐ learn how to expand a power of a binomial when the exponent is a positive integer, using the Binomial Theorem
- ☐ learn how to find and simplify any term of a binomial expansion

Lesson Introduction

In previous mathematics courses, you have learned how to expand binomials in the form $(x + y)^n$. When $n = 2$, this is simply $(x + y)(x + y)$, which you can expand by multiplying each term in the first factor by each term in the second factor. This is an application of the distributive property. Some of you may have used the acronym FOIL. What happens, however, when n is greater than two?

Expanding binomials when n is greater than two is more difficult and time consuming. One way of expanding these binomials that saves time involves the **Binomial Theorem**. The Binomial Theorem is used to expand binomials with the use of combinations. In this lesson, you will see how these concepts are related.

Another way of expanding these binomials involves Pascal's Triangle, named after the famous mathematician Blaise Pascal. In this triangle, the following patterns emerge:

- The first and last entry for any row is always 1
- The coefficients along any row are the same when read left to right or right to left
- Beginning with the second row of the triangle, the sum of any two adjacent entries forms the entry "between them" in the next row

The first four rows of Pascal's Triangle are:

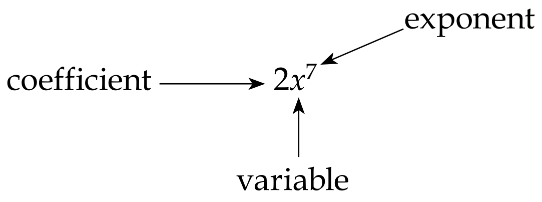
1
1 1
1 2 1
1 3 3 1

Can you determine the next three rows?

_____ _____ _____ _____ _____
_____ _____ _____ _____ _____ _____
_____ _____ _____ _____ _____ _____ _____

Introducing the Binomial Theorem

In order to understand the rest of this lesson, it is crucial that you remember the differences among a coefficient, a variable, and an exponent. Consider the following diagram.



A **coefficient** is the numeric component of an algebraic term. In the term $3x^2$, 3 is the coefficient.

A **variable** is a letter or symbol that represents an unknown value. In the term $2x^2y^3$, both x and y are variables.

An **exponent** is the number of times a number or variable is multiplied with itself in a power; it is usually written as a superscript after the number. For example, 3 is the exponent in 4^3.

You might want to add these definitions to your resource sheet.

The Binomial Theorem is useful when you are expanding binomials. In order to see the patterns in the expansion of binomials, it is helpful for you to first expand each of the binomials in the following example.

Example 1

Expand each of the following.

a) $(x + y)^2$

b) $(x + y)^3$

c) $(x + y)^4$

d) $(x + y)^5$

e) $(x + y)^6$

Solutions

a) $$\begin{aligned}(x + y)^2 &= (x + y)(x + y)\\ &= x^2 + xy + xy + y^2\\ &= x^2 + 2xy + y^2\end{aligned}$$

b) $$\begin{aligned}(x + y)^3 &= (x + y)(x + y)(x + y)\\ &= (x + y)(x + y)^2\\ &= (x + y)(x^2 + 2xy + y^2)\\ &= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3\\ &= x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

c) $$\begin{aligned}(x + y)^4 &= (x + y)(x + y)(x + y)(x + y)\\ &= (x + y)(x + y)^3\\ &= (x + y)(x^3 + 3x^2y + 3xy^2 + y^3)\\ &= x^4 + 3x^3y + 3x^2y^2 + xy^3 + x^3y + 3x^2y^2 + 3xy^3 + y^4\\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\end{aligned}$$

d) $$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

e) $$(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

Example 2

Using the expansion of the binomials in Example 1, answer the following questions.

a) What is the pattern of the sum of the exponents of each term?

b) What is the pattern of the exponents of the first term, x, in the expansion of the binomial?

c) What is the pattern of the exponents of the second term, y, in the expansion of the binomial?

d) What is the relation between the number of terms in the expansion and the exponent of the binomial?

e) How is the pattern of coefficients related to Pascal's Triangle?

Solutions

a) The sum of the exponents in each term of the expansion is always equal to the exponent of the binomial.

b) The exponent of the first term begins with the same value as the exponent of the binomial and decreases by one in each successive term and ends with an exponent of zero.

c) The exponent of the second term appears in the second term of the expansion and increases by one until it matches the exponent of the binomial.

d) The total number of terms in the expansion is one more than the power of the binomial.

e) The coefficients of the terms in the expansion of $(x + y)^2$ correspond to the third row in Pascal's triangle. The coefficients of the terms in the expansion of $(x + y)^3$ correspond to the fourth row in Pascal's triangle. This pattern continues for the other expansions as well.

Consider the expansion of $(x + y)^6$.

$$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

The goal in this topic is to be able to write out the terms of the binomial expansion without having to do all the multiplication and combining of like terms.

Compare the coefficients to Pascal's triangle in the following expansion:

1 6 15 20 15 6 1

The pattern is the same as the last row you filled in on page 74 when you completed the last three rows of the Pascal's Triangle exercise. What are the patterns you notice for the variables?

As you know, x^0 and y^0 both equal one. Also, multiplying by one does not change the value of any expression. Therefore, to further illustrate the pattern of the exponents, write the expansion of $(x + y)^6$ as follows:

$$(x + y)^6 = x^6y^0 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + x^0y^6$$

Look at the power of x, the first term in the binomial.

binomial 6 5 4 3 2 1 0

↓ ↓ ↓ ↓ ↓ ↓ ↓

$$(x + y)^6 = x^6y^0 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + x^0y^6$$

As you can see, the exponent of the first term of the binomial, x, decreases by 1 in every term. This pattern continues throughout the entire expansion of the binomial from 6 to 0.

Now look at the power of y, the second term in the binomial.

second term in
the binomial 0 1 2 3 4 5 6

↓ ↓ ↓ ↓ ↓ ↓ ↓

$$(x + y)^6 = x^6y^0 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + x^0y^6$$

As you can see, the exponent of the second term of the binomial, y, increases by 1 in every term. This pattern continues throughout the entire expansion of the binomial from 0 to 6.

Example 3

Use the patterns you summarized in Example 2 to write the terms of the expansion of $(x + y)^8$. Ignore the coefficients for the time being.

Solution

The expansion will have 8 + 1 = 9 terms. You should begin with the pattern of the first term of binomial, x.

The second term, y, begins in the first position of the expansion to the power of zero, and increases by one for each successive term.

$$x^8y^0 + x^7y^1 + x^6y^2 + x^5y^3 + x^4y^4 + x^3y^5 + x^2y^6 + x^1y^7 + x^0y^8$$

The only things missing from a complete expansion of the binomial are the coefficients of the terms. If you look more closely at the expansions in Example 1, you will probably be able to write the first two and the last two coefficients.

$$x^8y^0 + 8x^7y^1 + x^6y^2 + x^5y^3 + x^4y^4 + x^3y^5 + x^2y^6 + 8x^1y^7 + x^0y^8$$

But what about the other coefficients?

You can add the missing coefficients by considering the row in Pascal's triangle that begins with the values 1 and 8. This is the ninth row in Pascal's triangle and looks like:

1 8 28 56 70 56 28 8 1

Add a couple of rows to the Pascal's Triangle you created earlier (page 74) to confirm that you get the same values.

Therefore, the expansion of $(x + y)^8$ is:

$$x^8 + 8x^7y + 28x^6y^2 + 56x^5y^3 + 70x^4y^4 + 56x^3y^5 + 28x^2y^6 + 8x^1y^7 + y^8$$

The coefficients in the binomial expansion match one of the rows of Pascal's triangle!

Another surprising pattern in Pascal's triangle is that numbers in each of its rows correspond to a pattern of combinations.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Each term in the last row shown can be calculated using:

$${}_4C_0 \quad {}_4C_1 \quad {}_4C_2 \quad {}_4C_3 \quad {}_4C_4$$

So, the coefficients in the expansion of $(x + y)^8$ can be found using combinations.

The entire set of coefficients is:

${}_8C_0,\ {}_8C_1,\ {}_8C_2,\ {}_8C_3,\ {}_8C_4,\ {}_8C_5,\ {}_8C_6,\ {}_8C_7,\ {}_8C_8$ which can also be written as:

$\binom{8}{0}, \binom{8}{1}, \binom{8}{2}, \binom{8}{3}, \binom{8}{4}, \binom{8}{5}, \binom{8}{6}, \binom{8}{7}, \binom{8}{8}$ with values:

1, 8, 28, 56, 70, 56, 28, 8, 1

Consider the expansion of $(x + y)^5$ and $(x + y)^6$. Check to see whether the combination pattern agrees with the coefficients in these expansions.

The coefficients should be

$$\binom{5}{0}, \binom{5}{1}, \binom{5}{2}, \binom{5}{3}, \binom{5}{4}, \binom{5}{5}, \text{ and}$$

$$\binom{6}{0}, \binom{6}{1}, \binom{6}{2}, \binom{6}{3}, \binom{6}{4}, \binom{6}{5}, \binom{6}{6}, \text{ respectively.}$$

If you compute these values, you will find the coefficients of the binomials are:

1, 5, 10, 10, 5, 1 and

1, 6, 15, 20, 15, 6, 1.

If you refer back to Example 1, you will see these are the correct coefficients that you previously had to calculate by multiplying five or six binomials together. Can you see how these patterns can save time in the expansion of binomials?

This pattern in the exponents and coefficients of an expansion of a binomial is called the **Binomial Theorem** and is written as:

$$(x + y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

Overall, the following are the most important facts to remember regarding the Binomial Theorem:

- the leading and final coefficients are 1
- there is one more term in each expansion than the exponent of the binomial
- the exponent of the first term in the binomial decreases by 1 in each term of the expansion
- the exponent of the second term in the binomial increases by 1 in each term of the expansion
- the coefficient of any term, t_{k+1}, for k going from 0 to n, can be found using the combination ${}_nC_k$ (for example, the second term in the expansion of $(x + y)^5$ has a coefficient equal to ${}_5C_1$, since for t_2, t_{k+1} has k equal to 1)

- the expression for any term in the expansion of $(a + b)^n$ can be found using the formula:

  $$t_{k+1} = {}_nC_k \cdot a^{n-k} \cdot b^k$$

  This formula is used frequently, since you usually are asked to find one term rather than expand the whole binomial.

- the sum of the exponents of any term of the expansion of $(x + y)^n$ is n.
  - For example, the 3rd term in the expansion of $(x + y)^5$ is:

    ${}_5C_2(x)^3(y)^2$ and the sum of the exponents is 5 and n equals 5

- The exponent of the second factor in the expansion is the same as the number of objects being selected in the combination. For example:

  $${}_5C_2(x)^3(y)^2$$

  These are always the same.

- The exponent of the second factor in the expansion is one less than the term position.
  - Consider the 6th term in $(x + y)^8$.

    $t_{k+1} = t_6$, so $k = 5$

    $$t_6 = t_{5+1} = {}_8C_5\,(x)^{8-5}\,(y)^5 = 56x^3y^5$$

    As you can see, the exponent of the second factor in the expansion, or the exponent of y, being 5, is one less than the term position, which is 6.

It would be useful for you if you copied the above information on your resource sheet.

Example 4

Use the Binomial Theorem to expand

a) $(a - b)^5$

b) $(2x + b)^3$

c) $\left(x^2 + \dfrac{2}{x}\right)^4$

Solutions

a) Treat $(a-b)^5$ as $(a+(-b))^5$.

$$\binom{5}{0}a^5+\binom{5}{1}a^4(-b)+\binom{5}{2}a^3(-b)^2+\binom{5}{3}a^2(-b)^3+\binom{5}{4}a(-b)^4+\binom{5}{5}(-b)^5$$

$$=1a^5+5a^4(-b)+10a^3b^2+10a^2\left(-b^3\right)+5ab^4+1(-b^5)$$

$$=a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$$

Notice how the signs alternate.

b) Treat $(2x+b)^3$ as $([2x]+b)^3$.

$$\binom{3}{0}(2x)^3+\binom{3}{1}(2x)^2b+\binom{3}{2}(2x)b^2+\binom{3}{3}b^3$$

$$=8x^3+3\left(4x^2\right)b+3(2x)b^2+b^3$$

$$=8x^3+12x^2b+6xb^2+b^3$$

Notice how the symmetry of the coefficients disappears because the first term of the original binomial is $2x$, not x.

c) $\binom{4}{0}\left(x^2\right)^4+\binom{4}{1}\left(x^2\right)^3\left(\frac{2}{x}\right)+\binom{4}{2}\left(x^2\right)^2\left(\frac{2}{x}\right)^2+\binom{4}{3}\left(x^2\right)\left(\frac{2}{x}\right)^3+\binom{4}{4}\left(\frac{2}{x}\right)^4$

$$=x^8+4x^6\left(\frac{2}{x}\right)+6x^4\left(\frac{4}{x^2}\right)+4x^2\left(\frac{8}{x^3}\right)+\frac{16}{x^4}$$

$$=x^8+8x^5+24x^2+\frac{32}{x}+\frac{16}{x^4}$$

Both the coefficient pattern and the exponent pattern have changed. The coefficients are influenced by the $\frac{2}{x}$ term in the binomial. The exponent on the x decreases by 3 each time.

Besides expanding binomials, it will be beneficial if you can write specific terms of the expansion. Usually, you are not asked to write out all the terms in the expansion of a binomial.

Example 5

In the expansion of $\left(x^3 - \frac{2}{x^2}\right)^{10}$, write and simplify

a) the fifth term
b) the term containing x^{15}
c) the constant term

Solutions

a) Use the formula $t_{k+1} = {}_nC_k \cdot a^{n-k} \cdot b^k$.

In this question:

n is the exponent of the expansion = 10

Since you are finding term five, $k + 1$ must equal 5, so $k = 4$.

Also, first part in the binomial = x^3; second part in the binomial $= -\frac{2}{x^2}$.

$$\begin{aligned} t_5 &= \binom{10}{4}\left(x^3\right)^{10-4}\left(-\frac{2}{x^2}\right)^4 \\ &= (210)x^{3(6)}\left(\frac{16}{x^{2(4)}}\right) \\ &= \frac{3360\left(x^{18}\right)}{x^8} \\ &= 3360\left(x^{18-8}\right) \\ &= 3360x^{10} \end{aligned}$$

The second part in the binomial can also be written as $(-2x^{-2})$.

$$\begin{aligned} t_5 &= \binom{10}{4}\left(x^3\right)^{10-4}\left(-2x^{-2}\right)^4 \\ &= 210 \cdot x^{18}(-2)^4\left(x^{-2}\right)^4 \\ &= 210 \cdot x^{18} \cdot 16 \cdot x^{-8} \\ &= 3360x^{10} \end{aligned}$$

Either notation is correct.

b) **Method 1: Using the formula**

Use the formula $t_{k+1} = {}_nC_k \cdot a^{n-k} \cdot b^k$.

From the given information, you know that $n = 10$ and you know that the term you are looking for simplifies to a term with x^{15}. What you need to find is k.

$$t_{k+1} = {}_nC_k \cdot a^{n-k} \cdot b^k$$

$$x^{15} = \left(x^3\right)^{10-k} \cdot \left(x^{-2}\right)^k$$ No coefficients are required to solve for k.

$$x^{15} = x^{30-3k} \cdot x^{-2k}$$ Using Power Laws.

$$x^{15} = x^{30-5k}$$ Using Power Laws.

$$15 = 30 - 5k$$ Since the bases are equal, compare exponents only.

$$-15 = -5k$$

$$3 = k$$

If $k = 3$, you need to find term 4.

$$t_4 = {}_{10}C_3\left(x^3\right)^7 \cdot \left(-2x^{-2}\right)^3 = 120x^{21} \cdot (-8)x^{-6} = -960x^{15}$$

Method 2: Using patterns

Use $\left(x^3 - \dfrac{2}{x^2}\right)^{10}$ or $\left(x^3 - 2x^{-2}\right)^{10}$.

You know this expansion has 11 terms.

You also know: the t_1 exponent $= \left(x^3\right)^{10} = x^{30}$ 30

the t_{11} exponent $= \left(x^{-2}\right)^{10} = x^{-20}$ -20

$30 \downarrow -20$ Down 50 in 10 steps.

Therefore, this expansion starts at x^{30} and the exponent goes down by 5 for each successive term.

$x^{30}, x^{25}, x^{20}, x^{15}$ So term 4 contains x^{15}.

Term 4: $k = 3$, $n = 10$

$$t_4 = {}_{10}C_3\left(x^3\right)^{10-3} \cdot \left(-2x^{-2}\right)^3 = 120x^{21} \cdot \left(-8x^{-6}\right) = -960x^{15}$$

Either method will provide the correct solution for this type of question. However, Method 1 can be applied to a wider variety of questions.

c) This question is similar to part (b), except the constant term implies that the exponent of x is zero. Using the formula method:

$$x^{30-5k} = x^0$$
$$30 - 5k = 0$$
$$k = 6$$

Since $k = 6$, you need to find term 7:

$t_{k+1} = t_7 = {}_{10}C_6(x^3)^4(-2x^{-2})^6 = 13\,440$

Or by Method 2, using the pattern:

30, 25, 20, 15, 10, 5, 0

Term 7 has an exponent of 0.

Since $n = 10$, $k = 6$

$t_7 = {}_{10}C_6(x^3)^4(-2x^{-2})^6 = 13\,440$

It is important that you can find these terms without writing the entire expansion! If you need more help with this part of the lesson, contact your tutor/marker or ask your learning partner for help. Since this concept can be difficult to understand, it may be necessary for you spend extra time figuring out this concept before you continue to the rest of the lesson.

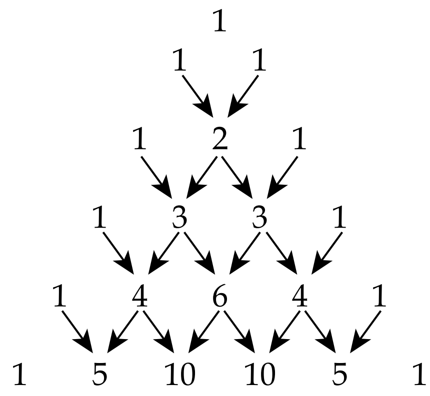
Binomial Expansion Using Pascal's Triangle

As you have discovered, the numerical coefficients in a binomial expansion of $(x + y)^n$ can be written in the form of a triangular array. This array is called Pascal's Triangle. To develop this triangle, you need to arrange the numerical coefficients of the binomial expansions of $(x + y)^n$, for increasing values of n, in a triangular array.

n	$(x + y)^n$	Coefficients	
0	$(x + y)^0$	1	← Row 0
1	$(x + y)^1$	1 1	← Row 1
2	$(x + y)^2$	1 2 1	← Row 2
3	$(x + y)^3$	1 3 3 1	← Row 3
4	$(x + y)^4$	1 4 6 4 1	← Row 4
5	$(x + y)^5$	1 5 10 10 5 1	← Row 5
6	$(x + y)^6$	1 6 15 20 15 6 1	← Row 6
7	$(x + y)^7$	1 7 21 35 35 21 7 1	← Row 7

You have already discovered some of the patterns in Pascal's Triangle.

Look at the diagram below:



This diagram shows one of the patterns in Pascal's Triangle: the two numbers diagonally above a value in Pascal's Triangle add together to produce that value.

Note: The first row consisting only of the value "1" in Pascal's Triangle, is called row 0. Row 1 is actually the second row in the formation of the triangle. The naming of the rows is consistent with the exponent of the binomial being expanded.

Include the above information on your resource sheet.

Example 6

Use Pascal's Triangle to expand $(a + b)^5$.

Solution

The numerical coefficients are: 1, 5, 10, 10, 5, and 1.

Including the exponent patters of a and b you get:

$$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Pascal’s Triangle and Combinations

Pascal’s Triangle can also be written using combinations. This is because each coefficient in the expansion of a binomial can also be written as a combination, as shown previously.

$$\binom{0}{0}$$

$$\binom{1}{0} \qquad \binom{1}{1}$$

$$\binom{2}{0} \qquad \binom{2}{1} \qquad \binom{2}{2}$$

$$\binom{3}{0} \qquad \binom{3}{1} \qquad \binom{3}{2} \qquad \binom{3}{3}$$

$$\binom{4}{0} \qquad \binom{4}{1} \qquad \binom{4}{2} \qquad \binom{4}{3} \qquad \binom{4}{4}$$

Make sure you complete the following Learning Activity. This Learning Activity is the last one in this module and it will prepare you for the upcoming assignment.

Learning Activity 1.7

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Simplify: $\frac{3}{7} + \frac{7}{8}$
2. What is 12% of 164?
3. Convert 62.36% into a fraction.
4. Simplify: $\frac{9!}{8!}$
5. Simplify: $\frac{42x^3y^4}{18x^4y}$
6. Write an equivalent fraction to $\frac{7}{19}$.
7. What kind of function is $y = x - 3x^2 + 3$?
8. How many terms will there be in the expansion of $\left(x - \frac{5}{2}\right)^7$?

continued

Part B: The Binomial Theorem

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Expand and simplify, using the Binomial Theorem.
   a) $(a+b)^{10}$
   b) $(x-y)^{9}$
   c) $(2x^2-x)^{5}$
   d) $\left(3a^3+\frac{3}{a}\right)^{4}$
2. Write and simplify the first three terms, using the Binomial Theorem.
   a) $\left(x-\frac{1}{x}\right)^{12}$
   b) $\left(3ax+\frac{x^2}{3}\right)^{6}$
   c) $\left(\frac{x}{2}-\frac{2}{x}\right)^{7}$
3. Find and simplify the seventh term of $(a+b)^8$.
4. Find and simplify.
   a) The sixth term of $(2y+x)^{11}$
   b) The term containing x^2 in $\left(\frac{x^5}{2}-\frac{2}{x^3}\right)^{10}$
   c) The term containing x^{14} in $(2x+x^2)^{11}$
   d) The $(r+1)$ term of $\left(3a-\frac{1}{6a^2}\right)^{9}$

continued

Learning Activity 1.7 (continued)

5. Write the first four terms of $\left(2x^3 - \frac{1}{4x}\right)^{11}$. Find the eighth term.

6. Find the middle term of $\left(3x - \frac{1}{3x}\right)^{8}$.

7. Why should the numerical coefficients of a binomial expansion be combinations? Use the 'meaning' of $(x + y)^5$ to explain why the coefficient of the term containing x^3y^2 is ${}_5C_2$.

Lesson Summary

In this lesson, you learned how combinations can be used to find the coefficients in the expansion of binomials. This is called the Binomial Theorem. You also learned how the Binomial Theorem relates to Pascal's Triangle. Using the Binomial Theorem or Pascal's Triangle can help you expand binomials quickly and efficiently. The Binomial Theorem is used extensively in the study of calculus.

Notes

Assignment 1.3

The Binomial Theorem

Total: 24 marks

You need to show all your steps as you work out your solutions, and make sure your answers are clear (include units, where appropriate). Check the Introduction for the Marking Guidelines that will be used for all assignments and examinations.

1. Expand and simplify, using the Binomial Theorem. (*2 × 4 marks each = 8 marks)*

   a) $(2x + 1)^7$

   b) $\left(2 + \frac{1}{x}\right)^3$

continued

Assignment 1.3: The Binomial Theorem (continued)

2. Write and simplify the first three terms of $\left(x - \frac{1}{x}\right)^{10}$. *(3 marks)*

3. Find the fifth term of $\left(\frac{y}{4} - \frac{2}{y}\right)^{7}$. *(2 marks)*

continued

Assignment 1.3: The Binomial Theorem (continued)

4. Find and simplify.

   a) The middle term of $\left(2x - \frac{1}{2x}\right)^{12}$. *(3 marks)*

   b) The term containing x^{20} in $\left(2x - x^4\right)^{14}$. *(3 marks)*

   c) The constant term in $\left(2x^4 - \frac{1}{2x^2}\right)^{12}$. *(3 marks)*

   d) The fourth term in $\left(\frac{x^3}{3} - 2\right)^9$. *(2 marks)*

Notes

MODULE 1 SUMMARY

Congratulations, you have finished the first module in the course! In this module, you learned counting techniques that allowed you to count the number of unordered selections, called combinations, and the number of ordered selections, called permutations. It can be quite complex to count the number of arrangements of objects in certain situations, but now you know how to count these arrangements quickly and easily!

You also learned how to use the Binomial Theorem, including Pascal's Triangle, to expand binomials. This method of expanding binomials is widely used, as it decreases the amount of multiplying, adding, and simplifying you have to complete, which also decreases the number of mistakes you may make.

In the next module, you will be learning about function transformations.

Submitting Your Assignments

It is now time for you to submit Assignments 1.1 to 1.3 to the Distance Learning Unit so that you can receive some feedback on how you are doing in this course. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 1 assignments and organize your material in the following order:

- ☐ Module 1 Cover Sheet (found at the end of the course Introduction)
- ☐ Assignment 1.1: The Fundamental Counting Principle and Permutations
- ☐ Assignment 1.2: Permutations and Combinations
- ☐ Assignment 1.3: The Binomial Theorem

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Notes

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Module 1
Permutations, Combinations, and the Binomial Theorem

Learning Activity Answer Keys

MODULE 1: PERMUTATIONS, COMBINATIONS, AND THE BINOMIAL THEOREM

Learning Activity 1.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. What is the reciprocal of $\dfrac{\frac{2}{5}}{10}$?
2. Find the next four numbers in the pattern: 7, 1, 9, 3, 11, 5, 13, . . .
3. Is $x = -1$ a solution to the inequality $x^2 + 4x + 3 < 0$?
4. Rationalize the denominator: $\dfrac{4}{2\sqrt{3}}$
5. Evaluate: $f(x) = \dfrac{(x+3)^2}{2x-1}$, if $x = -2$.
6. In which direction does the parabola $y + x^2 - 2x = 0$ open?
7. List all factors of 70.
8. Evaluate: $\left|3\frac{2}{5} - 6\frac{4}{5}\right|$

Answers:

1. $25 \left(\left(\dfrac{10}{\frac{2}{5}}\right) \text{ or } 10 \times \dfrac{5}{2} = \dfrac{50}{2}\right)$
2. 7, 15, 9, 17 (subtract 6, then add 8)
3. No ($(-1)^2 + 4(-1) + 3 = 0$, which is not less than 0)
4. $\dfrac{2\sqrt{3}}{3} \left(\dfrac{4}{2\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} \rightarrow \dfrac{4\sqrt{3}}{2(3)} \rightarrow \dfrac{4\sqrt{3}}{6}\right)$
5. $f(-2) = \dfrac{1}{-5}$

6. Down ($y = -x^2 + 2x$)
7. 1, 2, 5, 7, 10, 14, 35, 70
8. $3\frac{2}{5}$ $\left(\text{subtract } 6\frac{4}{5} - 3\frac{2}{5} \rightarrow 6 - 3 = 3 \text{ and } \frac{4}{5} - \frac{2}{5} = \frac{2}{5}\right)$

Part B: The Fundamental Counting Principle

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Refer to Example 3(d).
   a) List the 15 possible house numbers.

      Answer:

      Ending in the digit 2: 312, 342, 352, 412, 432, 452, 512, 532, 542

      Ending in the digit 4: 314, 324, 354, 514, 524, 534

   b) How many different house numbers use the digits 2, 3, and 4?

      Answer:

      3; namely, 342, 324, and 432. A different order creates a different house number.

2. Using the digits 1, 2, 3, 4, and 5 without repetition, how many three-digit house numbers
   a) have the digit 4 as the middle digit?

      Answer:

      only "4" is allowed in the middle

      $\underline{\ \ 4\ \ } \cdot \underline{\ \ 1\ \ } \cdot \underline{\ \ 3\ \ } = 12$

   b) do not have the digit 4 as the middle digit?

      Answer:

      4 choices for middle number

      $\underline{\ \ 4\ \ } \cdot \underline{\ \ 4\ \ } \cdot \underline{\ \ 3\ \ } = 48$

      Or, use the complementary problem strategy:

      (all possible) − (numbers with 4 in middle)

      = 60 − 12 = 48

c) begin and end with an odd digit?

Answer:

3 odd choices for first digit and 2 odd remaining for last digit

$\underline{\;3\;} \cdot \underline{\;3\;} \cdot \underline{\;2\;} = 18$

d) are between 200 and 500?

Answer:

first digit must be a “2,” “3,” or “4”

$\underline{\;3\;} \cdot \underline{\;4\;} \cdot \underline{\;3\;} = 36$

e) are less than 300 and odd?

Answer:

Case 1: first digit is 1, so last digit must be “3” or “5”:

$\underline{\;1\;} \cdot \underline{\;3\;} \cdot \underline{\;2\;} = 6$

Case 2: first digit is 2, so last digit must be “1,” “3,” or “5”:

$\underline{\;1\;} \cdot \underline{\;3\;} \cdot \underline{\;3\;} = 9$

Total number is: 6 + 9 = 15

3. There are 72 girls in a school with 108 students. How many of the students are boys?

Answer:

108 − 72 = 36 boys. Complementary problem.

4. Winnipeg Stadium has five gates. In how many ways can you enter the stadium and leave the stadium

a) by a different gate?

Answer:

$\underline{\;5\;} \cdot \underline{\;4\;} = 20$

b) by any gate?

Answer:

$\underline{\;5\;} \cdot \underline{\;5\;} = 25$

5. In how many ways can a school president, a vice-president, and a secretary be selected from the 23 students comprising the student council?

   Answer:

   $\underline{\ 23\ } \cdot \underline{\ 22\ } \cdot \underline{\ 21\ } = 10\ 626$

6. Nayeli, a recent university graduate, is buying clothes for her new job. If she buys 2 new pairs of pants and 4 new blouses, how many different "new" outfits is it possible for her to wear if each consists of 1 new blouse and 1 new pair of pants?

   Answer:

   $\underline{\ 4\ } \cdot \underline{\ 2\ } = 8$ new outfits

7. In how many ways can five different books be arranged on a bookshelf?

   Answer:

   $\underline{\ 5\ } \cdot \underline{\ 4\ } \cdot \underline{\ 3\ } \cdot \underline{\ 2\ } \cdot \underline{\ 1\ } = 120$

8. In how many ways can four women and three men be seated on a bench

   a) If the men and women must alternate seats?

   Answer:

   Label the blanks with "w" or "m," and then determine the number of men and women available for each space.

   $$\underset{w}{\underline{\ 4\ }} \cdot \underset{m}{\underline{\ 3\ }} \cdot \underset{w}{\underline{\ 3\ }} \cdot \underset{m}{\underline{\ 2\ }} \cdot \underset{w}{\underline{\ 2\ }} \cdot \underset{m}{\underline{\ 1\ }} \cdot \underset{w}{\underline{\ 1\ }} = 144$$

   b) If the men and women can sit wherever?

   Answer:

   $\underline{\ 7\ } \cdot \underline{\ 6\ } \cdot \underline{\ 5\ } \cdot \underline{\ 4\ } \cdot \underline{\ 3\ } \cdot \underline{\ 2\ } \cdot \underline{\ 1\ } = 5040$

   There are 5040 possible seating arrangements.

9. A university student must take a language course in slot 1, a mathematics course in slot 2, and a science course in slot 3. If there are four different language courses, eight different mathematics courses, and three different science courses, how many different timetables are possible?

   Answer:

   There are 4 ways to pick the language course. Following that, there are eight ways to pick a mathematics course and three ways to pick a science course.

   $\underline{\quad 4 \quad} \cdot \underline{\quad 8 \quad} \cdot \underline{\quad 3 \quad}$ = 96 possible course selections

Learning Activity 1.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Solve for x: $x + 17 = 48$
2. Simplify: $\frac{3}{4} + \frac{1}{6} + \frac{7}{12}$
3. If $f(x) = \frac{x+6}{2}$, calculate $f(-102)$.
4. Factor: $3x^2 + 5x + 2$
5. Estimate the taxes, 13%, on a $113 item.
6. Simplify: $\dfrac{\frac{16x^2y^3}{4xy}}{\frac{2x^3y}{y^4}}$
7. Evaluate: $\left|-\frac{4}{7} - \frac{9}{49}\right|$
8. Evaluate: $\frac{9}{2} \div \frac{3}{8}$

Answers:

1. $x = 31$
2. $\frac{18}{12} = \frac{3}{2} = 1\frac{1}{2}$ $\left(\frac{3}{4} \times \frac{3}{3} + \frac{1}{6} \times \frac{2}{2} + \frac{7}{12} \rightarrow \frac{9}{12} + \frac{2}{12} + \frac{7}{12}\right)$
3. $f(-102) = -48$ $\left(\frac{-102+6}{2} \rightarrow -\frac{96}{2}\right)$
4. $(3x + 2)(x + 1)$
5. $14 (10% of 113 is 11.3 and 1% is $1.13; add 11.30 + 1.13 + 1.13 + 1.13)
6. $\frac{2y^5}{x^2}$ $\left(\frac{16x^2y^3}{4xy} \cdot \frac{y^4}{2x^3y} \rightarrow 4xy^2 \cdot \frac{y^3}{2x^3}\right)$
7. $\frac{37}{49}$

8. $12 \left(\frac{9}{2} \times \frac{8}{3} \rightarrow 3 \times 4\right)$

Part B: Factorials

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Simplify each of the following factorials.

a) $\frac{4!}{2!}$

b) $\frac{6!}{2!4!}$

c) $\frac{6 \cdot 4!}{3!} + 0!$

d) $\frac{9!}{2!7!} + 1!$

e) $\frac{7 \cdot 6!}{7!}$

Answers:

a) $\frac{4!}{2!} = \frac{4 \cdot 3 \cdot 2!}{2!} = 4 \cdot 3 = 12$

b) $\frac{6!}{2!4!} = \frac{6 \cdot 5 \cdot 4!}{2!4!} = \frac{6 \cdot 5}{2!} = \frac{30}{2} = 15$

c) $\frac{6 \cdot 4!}{3!} + 0! = \frac{6 \cdot 4 \cdot 3!}{3!} + 1 = (6 \cdot 4) + 1 = 24 + 1 = 25$

d) $\frac{9!}{2!7!} + 1! = \frac{9 \cdot 8 \cdot 7!}{2!7!} + 1 = \frac{9 \cdot 8}{2!} + 1 = \frac{72}{2} + 1 = 36 + 1 = 37$

e) $\frac{7 \cdot 6!}{7!} = \frac{7!}{7!} = 1$

2. Simplify without the use of a calculator.

a) $\dfrac{8!}{7!}$

b) $\dfrac{10!}{9!}$

c) $\dfrac{k!}{(k-1)!}$

d) $4!5$

e) $8!9$

f) $n!(n+1)$

g) $3!+4!$

h) $5!+6!$

i) $n!+(n+1)!$

Answers:

a) $\dfrac{8\cdot 7!}{7!}=8$

b) $\dfrac{10\cdot 9!}{9!}=10$

c) $\dfrac{k(k-1)!}{(k-1)!}=k$

d) $4!5=5\cdot 4!=5!$ or 120

e) $8!9=9\cdot 8!=9!$

(since this is without a calculator, 9! is a good place to stop)

f) $n!(n+1)=(n+1)\cdot n!=(n+1)!$

g) $3!+3!4$

Think of this as a binomial. Each term has 3! in common, so factor.

$$\begin{aligned} 3!+3!4 &= 3!(1+4) \\ &= 3!(5) \\ &= 3\times 2\times 1\times 5 \\ &= 6\times 5 \\ &= 30 \end{aligned}$$

or

$3!+4!=6+24=30$

h) $5! + 5!6 = 5!(1 + 6) = 5!7 = 840$

or

$5! + 6! = 120 + 720 = 840$

i) $n! + n!(n + 1) = n!(1 + n + 1) = n!(n + 2)$

3. Solve for n in the following factorials.

a) $\dfrac{(n+1)!}{n!} = 7$

Answer:

$$\frac{(n+1)(n)!}{n!} = 7$$
$$n + 1 = 7$$
$$n = 6$$

b) $\dfrac{(7!1!)}{2!5!} = \dfrac{(n-6)!}{(n-7)!}$

Answer:

$$\frac{7 \cdot 6 \cdot \cancel{5!}1}{2 \cdot 1 \cdot \cancel{5!}} = \frac{(n-6)\,\cancel{(n-7)!}}{\cancel{(n-7)!}}$$
$$\frac{42}{2} = n - 6$$
$$21 = n - 6$$
$$27 = n$$

Learning Activity 1.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Your restaurant bill came to \$35.73. If you wish to leave a 15% tip, approximately how much should you leave?
2. Factor: $2x^2 - 5x - 12$
3. Solve for x: $\frac{1}{9^x} = 81$
4. At what speed must you travel to go 832 metres in 8 seconds?
5. Solve for x: $(x - 4)^2 = 1$
6. Expand: $(x + y)^2$
7. Write as a mixed number: $\frac{435}{51}$
8. What is 40% of 1560?

Answers:

1. ≈ \$5.25 (10% of \$35.00 is \$3.50 and 5% of \$35.00 is \$1.75)
2. $(2x + 3)(x - 4)$
3. $x = -2 \left(81 = 9^2 = \left(\frac{1}{9}\right)^{-2} = \frac{1}{9^{-2}}\right)$
4. 104 m/s (832 ÷ 8 = 104)
5. $x = 5$ or $x = 3$ $\left(x - 4 = \pm\sqrt{1} \rightarrow x - 4 = 1 \text{ or } x - 4 = -1\right)$
6. $x^2 + 2xy + y^2$
7. $8\frac{27}{51}$ $(8 \times 51 = 408 \rightarrow 435 - 408 = 27)$
8. 624 (10% of 1560 is 156 → 4 × 156)

Part B: Permutations

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Simplify as much as possible.

   a) $\dfrac{(k+3)!}{(k+2)!}$

   Answer:

   $$\frac{(k+3)(k+2)!}{(k+2)!} = k+3$$

   b) $\dfrac{7!(r+2)!}{6!(r-1)!}$

   Answer:

   $$\begin{aligned}\frac{(7\cdot 6!)(r+2)(r+1)(r)(r-1)!}{6!(r-1)!} &= 7(r+2)(r+1)(r)\\ &= 7\left(r^2+3r+2\right)(r)\\ &= 7r^3+21r^2+14r\end{aligned}$$

2. Solve for n.

   a) $\dfrac{(n+2)!}{(n+1)!} = 20$

   Answer:

   $$\begin{aligned}\frac{(n+2)(n+1)!}{(n+1)!} &= 20\\ (n+2) &= 20\\ n &= 18\end{aligned}$$

b) $(n-1)! = 72(n-3)!$

Answer:

Begin by writing the consecutive numbers of the larger factorial.

$$(n-1)(n-2)(n-3)! = 72(n-3)!$$

$$(n-1)(n-2) = 72$$

⟵ In this step, dividing both sides by $(n-3)!$ is acceptable because you know $(n-3)! \neq 0$.

$$n^2 - 3n + 2 = 72$$

$$n^2 - 3n - 70 = 0$$

$$(n-10)(n+7) = 0$$

$n = 10$ or $\cancel{n = -7}$ (reject)

$\therefore n = 10$

c) $\dfrac{(n+1)!}{(n-1)!} - 30 = 0$

Answer:

$$\frac{(n+1)(n)(n-1)!}{(n-1)!} - 30 = 0$$

$$(n+1)n - 30 = 0$$

$$n^2 + n - 30 = 0$$

$$(n+6)(n-5) = 0$$

$\cancel{n = -6}$ (reject) or $n = 5$

$\therefore n = 5$

3. Show that:

a) $$\frac{n!}{(n-r-1)!(r+1)!} = \frac{n!}{(n-r)!r!} \cdot \frac{(n-r)}{r+1}$$

Answer:

RHS

$$= \frac{n!(n-r)}{(n-r)(n-r-1)!r!(r+1)}$$

$$= \frac{n!}{(n-r-1)!r!(r+1)}$$

$$= \frac{n!}{(n-r-1)!(r+1)r!}$$

$$= \frac{n!}{(n-r-1)!(r+1)!}$$

= LHS

b) $$\frac{n!r}{(n-r)!r!} = \frac{(n-1)!n}{(n-r)!(r-1)!}$$

Answer:

LHS

$$\frac{n!r}{(n-r)!r!}$$

$$= \frac{n(n-1)!\cancel{r}}{(n-r)!\cancel{r}(r-1)!}$$

$$= \frac{n(n-1)!}{(n-r)!(r-1)!}$$

= RHS

4. Find the value of r if $\dfrac{18!}{(18-r)!r!} = \dfrac{18!}{(16-r)!(r+2)!}$.

Answer:

Simplifying both sides:

$$\frac{18!}{(18-r)!r!} = \frac{18!}{(16-r)!(r+2)!}$$

$$\frac{1}{(18-r)(17-r)(16-r)!r!} = \frac{1}{(16-r)!(r+2)(r+1)r!}$$

Divide 18! from both sides and expand the denominator.

$$\frac{1}{(18-r)(17-r)} = \frac{1}{(r+2)(r+1)}$$

Multiply to cancel common terms from each side.

$$r^2 + 3r + 2 = 306 - 35r + r^2$$

Simplify.

$$38r = 304$$

$$r = 8$$

5. Find the value of each without using your calculator.

a) ${}_5P_2$
b) ${}_5P_3$
c) ${}_5P_5$
d) ${}_{100}P_2$
e) ${}_6P_1$

Answer:

a) ${}_5P_2$ means $5 \times 4 = 20$
b) ${}_5P_3$ means $5 \times 4 \times 3 = 60$
c) ${}_5P_5$ means $5 \times 4 \times 3 \times 2 \times 1 = 120$
d) ${}_{100}P_2$ means $100 \times 99 = 9900$
e) ${}_6P_1$ means 6

Note: The formula could also be used.

a) ${}_5P_2 = \dfrac{5!}{3!} = 20$

b) ${}_5P_3 = \dfrac{5!}{2!} = 60$

c) ${}_5P_5 = \dfrac{5!}{0!} = 120$

d)
$$\begin{aligned} {}_{100}P_2 &= \frac{100!}{98!} \\ &= \frac{100(99)(98!)}{98!} \\ &= 100(99) \\ &= 9900 \end{aligned}$$

e) ${}_6P_1 = \dfrac{6!}{5!} = 6$

6. Leave your answers in factorial form.
   a) In how many ways can five seats on a bench be assigned from amongst 12 people?

      Answer:

      $${}_{12}P_5 = \frac{12!}{7!}$$

   b) In how many different ways can eight vacant seats be occupied on a bus by four people, if each person occupies only one seat?

      Answer:

      $${}_{8}P_4 = \frac{8!}{4!}$$

   c) In how many ways can a president, a treasurer, and a secretary be selected from amongst 10 candidates if no candidate can hold more than one position?

      Answer:

      $${}_{10}P_3 = \frac{10!}{7!}$$

7. How many "words" can be made using all letters of the word MABLE? **Note:** When you are asked to form words in counting problems, unless stated otherwise, it means any permutation of the given letters. Nonsense words are acceptable (for example, ELBAM is considered a "word" in this instance).

   Answer:

   $${}_{5}P_5 = \frac{5!}{0!} = 5! = 120$$

   or

   $$\underline{\ 5\ } \cdot \underline{\ 4\ } \cdot \underline{\ 3\ } \cdot \underline{\ 2\ } \cdot \underline{\ 1\ } = 120$$

8. How many words can be formed by using five letters of the word CROMBIE?

   Answer:

   $${}_{7}P_5 = \frac{7!}{2!} = 2520$$

   or

   $$\underline{\ 7\ } \cdot \underline{\ 6\ } \cdot \underline{\ 5\ } \cdot \underline{\ 4\ } \cdot \underline{\ 3\ } = 2520$$

9. Solve for n if ${}_nP_3 = 7({}_6P_2)$.

Answer:

$${}_nP_3 = 7({}_6P_2)$$

$$\frac{n!}{(n-3)!} = 7 \cdot \frac{6!}{4!}$$

$$\frac{n(n-1)(n-2)(n-3)!}{(n-3)!} = \frac{7 \cdot 6 \cdot 5 \cdot 4!}{4!}$$

$$n(n-1)(n-2) = 7(6)(5)$$

Since the left side represents three consecutive numbers, by comparison, $n = 7$.

10. Explain the meaning of ${}_8P_3$. Why does ${}_3P_8$ not make sense?

Answer:

${}_8P_3$ means you are arranging eight objects using three at a time where the order of the objects matters. The symbol ${}_3P_8$ does not make sense because you cannot permute (or arrange) three objects eight at a time since you only have three objects.

11. Solve for n if ${}_nP_2 = 72$.

Answer:

$$\frac{n!}{(n-2)!} = 72$$

$$n(n-1) = 9(8)$$

$$n^2 - 1n = 72$$

$$n^2 - 1n - 72 = 0$$

$$(n-9)(n+8) = 0$$

$$n = 9 \text{ or } \cancel{n = -8} \text{ (reject)}$$

$$n = 9$$

12. Fill in the following Permutation Graphic Organizer according to the headings in each section. **Hint:** This graphic organizer is something you can add to your resource sheet, study from, or even create for different concepts, to help increase your understanding.

    Answer:

    Your answer may not match the following exactly, but should include the same information.

Permutations

Definition	Facts/Characteristics
The arrangement of a set of objects in which the *order of the objects is important* is called a *permutation.*	The number of ways n objects can be placed into r available positions, written as ${}_nP_r$ is given by the formula $\frac{n!}{(n-r)!}$.
Examples	**Non-Examples**
How many ways can 10 people be awarded first, second, and third place in a race? ${}_{10}P_3 = \frac{10!}{(10-3)!} = \frac{10!}{7!} = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!} = 720$ or just write $10 \cdot 9 \cdot 8 = 720$	How many sets of four objects can be chosen from six objects? **Note:** In this example, any group of the same four objects (pulled in any order) will be the same set. Therefore, this is *not* a permutation since the order of the four objects doesn't matter.

13. Find the different ways 7 friends (3 boys and 4 girls) can sit together at the movie theatre if

    a) there are no restrictions

    Answer:

$$ {}_7P_7 = \frac{7!}{0!} = 5040 \text{ or} $$

$$ \underline{\ 7\ } \cdot \underline{\ 6\ } \cdot \underline{\ 5\ } \cdot \underline{\ 4\ } \cdot \underline{\ 3\ } \cdot \underline{\ 2\ } \cdot \underline{\ 1\ } = 5040 $$

    b) Jenelle must be the first person in the row

    Answer:

$$ \underbrace{\underline{\ 1\ }}_{\text{Jenelle}} \cdot \underline{\ 6\ } \cdot \underline{\ 5\ } \cdot \underline{\ 4\ } \cdot \underline{\ 3\ } \cdot \underline{\ 2\ } \cdot \underline{\ 1\ } = 720 $$

c) Jenelle can’t be the first person in the row

Answer:

(All possible) − (the number of times Jenelle is the first person in the row)

= 5040 − 720

= 4320

d) Jenelle must be the first person in the row and Blake must be the last person in the row

Answer:

$$\underbrace{\underline{\ 1\ }}_{\text{Jenelle}} \cdot \underline{\ 5\ } \cdot \underline{\ 4\ } \cdot \underline{\ 3\ } \cdot \underline{\ 2\ } \cdot \underline{\ 1\ } \cdot \underbrace{\underline{\ 1\ }}_{\text{Blake}} = 5! = 120$$

e) Jenelle and Blake must sit together

Answer:

Treat Jenelle and Blake as one object that can be arranged in 2! ways.

$$\underline{\ 6\ } \cdot \underline{\ 5\ } \cdot \overbrace{\underset{2}{\underline{\qquad}} \cdot \underset{1}{\underline{\qquad}}}^{4} \cdot \underline{\ 3\ } \cdot \underline{\ 2\ } \cdot \underline{\ 1\ } = 6! \times 2! = 1440$$

f) a girl must sit in the first seat and the last seat

Answer:

$$\underbrace{\underline{\ 4\ }}_{\text{Girl}} \cdot \underline{\ 5\ } \cdot \underline{\ 4\ } \cdot \underline{\ 3\ } \cdot \underline{\ 2\ } \cdot \underline{\ 1\ } \cdot \underbrace{\underline{\ 3\ }}_{\text{Girl}} = 5!(4)(3)$$

$$= 120(12) = 1440$$

Learning Activity 1.5

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Simplify: $3\sqrt[4]{81}$
2. Evaluate: $|\,3.64 - 4.78\,|$
3. Solve: $2x^2 + 3x - 9 = 0$
4. What is 40% of $\frac{7}{8}$?
5. Write as a mixed fraction: $\frac{18}{5}$
6. List all the factors of 100.
7. Simplify: $\sqrt{108}$
8. If the balance in your chequing account is $520, what is your new balance after your paycheque of $256 is deposited and you buy gas for $49?

Answers:

1. $9\ \left(3 \times \sqrt{9} = 3 \times 3\right)$
2. 1.14 (4.78 − 3.64)
3. $x = \frac{3}{2}$ and $x = -3\ \left((2x - 3)(x + 3) = 0\right)$
4. $\frac{7}{20}\ \left(\frac{\cancel{40}}{100} \times \frac{7}{\cancel{8}} = \frac{5}{100} \times 7\right)$
5. $3\frac{3}{5}$
6. 1, 2, 4, 5, 10, 20, 25, 50, 100
7. $6\sqrt{3}\ \left(\sqrt{108} = \sqrt{2 \times 2 \times 3 \times 3 \times 3} = \sqrt{2 \times 2 \times 3 \times 3} \times \sqrt{3}\right)$
8. $727 (520 + 256 = 776; now, subtract 50 and add 1)

Part B: Permutations Involving Like Objects

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. How many permutations are there of the letters of the following words?
   a) SCHOOLS

      Answer:

      7 letters with 2 Ss and 2 Os

      $$\frac{7!}{2!2!} = 1260$$

   b) BOOKKEEPERS

      Answer:

      11 letters with 2 Os, 2 Ks, and 3 Es

      $$\frac{11!}{2!2!3!} = 1\ 663\ 200$$

   c) CANADIAN

      Answer:

      8 letters with 3 As and 2 Ns

      $$\frac{8!}{3!2!} = 3360$$

2. In how many distinct ways can three red flags, two blue flags, two green flags, and four yellow flags be arranged in a row?

   Answer:

   $$\frac{11!}{3!2!2!4!} = 69\ 300$$

3. If $a^3b^2c^4$ is written with exponents of one, how many arrangements of the nine letters are possible (e.g., abaccabcc would be one of these arrangements)?

   Answer:

   $$\frac{9!}{3!2!4!} = 1260$$

4. How many different five-digit numbers can be formed using the digits 1, 2, 3, 4, and 5, if

   a) the odd digits occupy the odd places?

      Answer:

      $$\underbrace{3}_{\text{odd}} \cdot \underbrace{2}_{\text{even}} \cdot \underbrace{2}_{\text{odd}} \cdot \underbrace{1}_{\text{even}} \cdot \underbrace{1}_{\text{odd}} = 12$$

   b) the odd digits occupy the odd places in ascending order?

      Answer:

      Since the odd numbers cannot change order, it follows that you must divide by the number of ways of ordering the odd numbers—by 3! = 6. Therefore, 12 ÷ 6 = 2.

      Or, $\underbrace{1}_{\text{odd}} \cdot \underbrace{2}_{\text{even}} \cdot \underbrace{1}_{\text{odd}} \cdot \underbrace{1}_{\text{even}} \cdot \underbrace{1}_{\text{odd}} = 2$

5. All the letters of the word BARRIER are arranged. Find the number of arrangements

   a) beginning with the letter R

      Answer:

      $$\underbrace{1}_{\text{R}} \cdot \underbrace{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}_{\text{there are 2 Rs}} \div (2!) = 360$$

   b) beginning with two Rs

      Answer:

      $$\underbrace{1}_{\text{R}} \cdot \underbrace{1}_{\text{R}} \cdot \underbrace{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}_{\text{there is only 1 R}} = 120$$

   c) beginning with three Rs

      Answer:

      $$\underbrace{1}_{\text{R}} \cdot \underbrace{1}_{\text{R}} \cdot \underbrace{1}_{\text{R}} \cdot \underbrace{1 \cdot 3 \cdot 2 \cdot 1}_{\text{there are no Rs}} = 6$$

   d) beginning with exactly one R

      Answer:

      Exactly one R means that the first letter is an R and the second is not an R.

      $$\underbrace{1}_{\text{R}} \cdot \underbrace{4}_{\text{not R}} \cdot \underbrace{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}_{\text{there are 2 Rs}} \div (2!) = 240$$

e) beginning with exactly two Rs

Answer:

Exactly two Rs means that the first two letters are Rs and the third is not an R.

$$\underbrace{1}_{\text{R}} \cdot \underbrace{1}_{\text{R}} \cdot \underbrace{4}_{\text{not R}} \cdot \underbrace{4 \cdot 3 \cdot 2 \cdot 1}_{\text{there is 1 R}} = 96$$

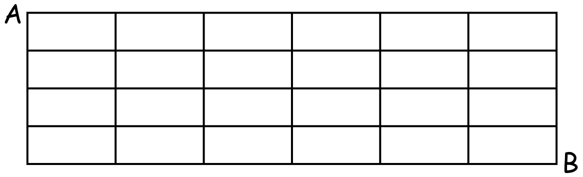
6. How many arrangements can be made using all the letters of BABBLING BABY?

Answer:

12 letters with 5 Bs and 2 As

$$\frac{12!}{5!2!} = 1\,995\,840$$

7. In how many different ways can person A walk to person B if the trip takes exactly 10 blocks?



Answer:

The answer is the number of ways of ordering six Rights and four Downs (for example, RRRRRRDDDD). Therefore, $\frac{10!}{6!4!} = 210$.

8. A coin is flipped eight times. In how many ways could the result be five heads and three tails (e.g., HHTHTHTH would be one of the ways)?

Answer:

$$\frac{8!}{5!3!} = 56$$

9. In how many ways can a committee of three people be selected from five candidates identified as A, B, C, D, and E? (Careful! The selections ABC and ACB are the same committee.)

   Answer:

   Since the order of the three selected objects is immaterial, you should count it as if the order is important and then cancel out all the arrangements of three objects (i.e., divide by 3! = 6).

   Therefore, the answer is $\frac{{}_5P_3}{3!} = \frac{60}{6} = 10$ ways.

   (This problem is actually called a combination and will be explained more fully in the next lesson.)

Learning Activity 1.6

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Factor: $2x^2 - x - 15$
2. Simplify: $\sqrt[3]{16x^2y^6z^3}$
3. Simplify: $256^{\frac{1}{4}}$
4. Simplify: $\frac{1}{8} + \frac{7}{9}$
5. Multiply: $3\frac{2}{3} \cdot 6\frac{2}{7}$
6. What is 15% of 236?
7. Convert 14.7% into a decimal.
8. Reduce to lowest terms: $\frac{24}{360}$

Answers:

1. $(2x + 5)(x - 3)$
2. $2y^2z\sqrt[3]{2x^2}\ \left(\sqrt[3]{8y^6z^3} \cdot \sqrt[3]{2x^2}\right)$
3. $4\ \left(\sqrt[4]{256} = \sqrt[4]{4 \times 4 \times 4 \times 4} = 4\right)$
4. $\frac{65}{72}\ \left(\frac{9}{72} + \frac{56}{72}\right)$
5. $\frac{484}{21}\ \left(\frac{11}{3} \cdot \frac{44}{7}\right)$
6. 35.4 (10% of 236 is 23.6, so 5% is 11.8; now add 23.6 + 11.8)
7. 0.147
8. $\frac{1}{15}\ \left(\frac{24 \div 4}{360 \div 4} \rightarrow \frac{6 \div 3}{90 \div 3} \rightarrow \frac{2 \div 2}{30 \div 2} \rightarrow \frac{1}{15}\right)$

Part B: Combinations

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Determine whether each of the following situations represents a permutation or a combination.
   a) A restaurant offers a deal of 2 meals for $20 from a list of 15 meals. In how many ways can a customer choose 2 meals off the deal menu?
   b) One hundred people are running in a marathon. How many different ways can first, second, and third place be awarded?
   c) How many different 4-letter words can be formed from the letters in FISH if no repetition of letters is allowed?
   d) There are five males and eight females on a co-ed basketball team. How many five-person shifts can be made consisting of two male members and three female members?

   Answers:

   a) This is a combination because order doesn't matter.
   b) This is a permutation because order matters.
   c) This is a permutation because order matters.
   d This is a combination because order doesn't matter.

2. Evaluate each pair and explain your results.
   a) ${}_{10}C_3$ and ${}_{10}C_7$
   b) ${}_{5}C_3$ and ${}_{5}C_2$
   c) ${}_{10}C_4$ and ${}_{10}C_6$

   Answers:

   a) Both equal 120
   b) Both equal 10
   c) Both equal 210

   Each pair must be equal because if you select three persons out of 10 **to take** some place, you are simultaneously selecting seven persons out of 10 **to leave** behind. Every selection has this dual action of selecting some objects and choosing not to select the remaining objects.

   This is confirmed by the calculations since the numbers are identical in the two cases.

For example:

$$_{10}C_3 = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)(3 \cdot 2 \cdot 1)}$$

$$_{10}C_7 = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1)(7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)}$$

3. Based on your discovery in Question 2, if $_{12}C_7 = x$, then what other combination must also produce an answer of x?

   Answer:

   $_{12}C_5$ will also equal x.

   You might want to make note of this relationship on your resource sheet. It can help to simplify some calculations. For example, $_{200}C_{198}$ is equal to $_{200}C_2$, which is just $\frac{200 \times 199}{2 \times 1}$ or 19 900.

4. Use factorial notation to prove that $_nC_r = {}_nC_{n-r}$. Explain this statement with respect to making selections. Why should this statement be true?

   Answer:

   The number of ways of choosing r out of n objects is the same as choosing $n - r$ out of n objects. The factorial proof is:

$$\begin{aligned} _nC_{n-r} &= \frac{n!}{(n-(n-r))!(n-r)!} \\ &= \frac{n!}{(n-n+r)!(n-r)!} \\ &= \frac{n!}{r!(n-r)!} \\ &= {}_nC_r \end{aligned}$$

   As explained in Question 2, every time you select r objects out of n objects for some task, you are selecting $n - r$ objects that will be left behind. Therefore, the count should be the same.

5. Solve for n: ${}_nC_4 = {}_nC_3$

   Answer:

   $${}_nC_4 = {}_nC_3 \rightarrow \frac{n!}{4!(n-4)!} = \frac{n!}{3!(n-3)!} \rightarrow \frac{n!(n-3)!}{n!(n-4)!} = \frac{4!}{3!} \rightarrow \frac{(n-3)(n-4)!}{(n-4)!}$$

   $$= \frac{4(3!)}{3!} \rightarrow n-3 = 4 \rightarrow n = 7$$

   By comparing the denominators in $\frac{n!}{4!(n-4)!} = \frac{n!}{3!(n-3)!}$, you might have guessed that $n = 7$ here. The interesting concept this example shows is that when two combinations are equated, the sum of the two r values must always equal the n value.

6. In how many ways can four policemen be selected for special duty from a group of 12 policemen?

   Answer:

   ${}_{12}C_4 = 495$

7. Natalie’s wardrobe includes five pairs of slacks, eight blouses, and five pairs of shoes. She wants to select three pairs of slacks, four blouses, and three pairs of shoes for her camping trip. What is the number of selections Natalie can make?

   Answer:

   ${}_5C_3 \cdot {}_8C_4 \cdot {}_5C_3 = 10(70)10 = 7000$

8. In a class of 30 students, each student shakes hands with each of the other students once. How many handshakes are there?

   Answer:

   Two students must be selected for every handshake. The order of selection is unimportant. Therefore, there are ${}_{30}C_2 = 435$ handshakes.

9. Consider a standard deck of 52 cards. There are four suits (clubs, diamonds, hearts, and spades) and 13 cards in each suit.

   a) How many different five-card hands can be dealt?

      Answer:

      You must select five cards in any order.

      ${}_{52}C_5$ = 2 598 960 different hands.

   b) How many of these hands contain exactly four clubs?

      Answer:

      There are four suits and there are 13 cards in each suit. Therefore, there are ${}_{13}C_4 \cdot {}_{39}C_1 = 27\ 885$ hands.

10. A tennis club has 10 boys and eight girls as members. From amongst these members, how many different matches are possible with

    a) a boy against a girl?

       Answer:

       Choose one of each: ${}_{10}C_1 \cdot {}_{8}C_1 = 10(8) = 80$

    b) two boys against two girls?

       Answer:

       Choose two of each: ${}_{10}C_2 \cdot {}_{8}C_2 = 45(28) = 1260$

11. There are five friends sitting around a table having dinner when suddenly the doorbell rings. In how many ways can you choose nobody to answer the door?

    Answer:

    This is a question that requires some "outside the box" thinking.

    There are 5 people that could answer the door and you are choosing none of them. Logically, the answer is ${}_{5}C_0 = \frac{5!}{(0!)(5!)} = 1,$ according to the ${}_{n}C_r$ formula.

    If no one answers the door, you might think that can happen in zero ways. On the other hand, the only way for nobody to answer the door is if everybody remains seated, and this can happen in only one way.

12. We know ${}_3P_5$ does not make sense. Does ${}_3C_5$ make sense? Explain.

Answer:

${}_3P_5$ does not make sense because you can't arrange 3 elements 5 at a time. Similarly, ${}_3C_5$ does not exist because you can't choose a group of 5 from 3 available elements. Another thing to note here is that $n \geq r$ for permutations and combinations. If this were not the case, then the formulae would yield negative factorials.

13. Fill in the following Combination Graphic Organizer according to the headings in each section. **Hint:** This graphic organizer is something you can add to your resource sheet, study from, or even create for different concepts to help increase your understanding.

Answer:

Your answer may not match the following exactly, but should include the same information.

Combinations

Definition	Facts/Characteristics
The arrangement of ways of selecting objects when the *order of the objects is not important* is called a *combination*	A general formula to find the number of combinations of n different objects taken r at a time, written ${}_nC_r$ or $\binom{n}{r}$, is ${}_nC_r = \frac{n!}{(n-r)!r!}$.
Examples	Non-Examples
How many ways can 5 people be chosen from 25 in a random draw? ${}_{25}C_5 = \frac{25!}{(25-5)!5!} = \frac{25!}{20!5!} = \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20!}{20!5!} =$ $\frac{(25 \cdot 24 \cdot 23 \cdot 22 \cdot 21)}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 53\,130$	How many different ways can five people sit in a row on a bench? **Note:** In this example, order matters as different seating arrangements exist when the five people are moved around. Therefore, this is not a combination.

Learning Activity 1.7

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Simplify: $\frac{3}{7}+\frac{7}{8}$
2. What is 12% of 164?
3. Convert 62.36% into a fraction.
4. Simplify: $\frac{9!}{8!}$
5. Simplify: $\frac{42x^3y^4}{18x^4y}$
6. Write an equivalent fraction to $\frac{7}{19}$.
7. What kind of function is $y = x - 3x^2 + 3$?
8. How many terms will there be in the expansion of $\left(x-\frac{5}{2}\right)^7$?

Answers:

1. $\frac{73}{56}\ \left(\frac{24}{56}+\frac{49}{56}\right)$
2. 19.68 (10% of 164 is 16.40 and 1% is 1.64; add 16.40 + 1.64 + 1.64)
3. $\frac{6236}{10\,000}\ \left(\frac{62.36}{100}=\frac{6236}{10\,000}\right)$
4. $9\ \left(\frac{9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}\right)$
5. $\frac{7y^3}{3x}$
6. Answers will vary. One possible answer is $\frac{14}{38}$.
7. This is a quadratic function.
8. 8 (one more than the power of the binomial)

Part B: The Binomial Theorem

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Expand and simplify, using the Binomial Theorem.

   a) $(a+b)^{10}$

   Answer:

$$a^{10} + \binom{10}{1}a^9b + \binom{10}{2}a^8b^2 + \binom{10}{3}a^7b^3 + \binom{10}{4}a^6b^4 + \binom{10}{5}a^5b^5 +$$

$$\binom{10}{6}a^4b^6 + \binom{10}{7}a^3b^7 + \binom{10}{8}a^2b^8 + \binom{10}{9}ab^9 + b^{10}$$

$$= a^{10} + 10a^9b + 45a^8b^2 + 120a^7b^3 + 210a^6b^4 + 252a^5b^5 +$$

$$210a^4b^6 + 120a^3b^7 + 45a^2b^8 + 10ab^9 + b^{10}$$

   b) $(x-y)^9$

   Answer:

$$x^9 + \binom{9}{1}x^8(-y)^1 + \binom{9}{2}x^7(-y)^2 + \binom{9}{3}x^6(-y)^3 + \binom{9}{4}x^5(-y)^4 +$$

$$\binom{9}{5}x^4(-y)^5 + \binom{9}{6}x^3(-y)^6 + \binom{9}{7}x^2(-y)^7 + \binom{9}{8}x(-y)^8 + (-y)^9$$

$$= x^9 - 9x^8y + 36x^7y^2 - 84x^6y^3 + 126x^5y^4 - 126x^4y^5 +$$

$$84x^3y^6 - 36x^2y^7 + 9xy^8 - y^9$$

c) $\left(2x^2 - x\right)^5$

Answer:

$$\left(2x^2\right)^5 + \binom{5}{1}\left(2x^2\right)^4(-x) + \binom{5}{2}\left(2x^2\right)^3(-x)^2 + \binom{5}{3}\left(2x^2\right)^2(-x)^3 +$$

$$\binom{5}{4}\left(2x^2\right)(-x)^4 + (-x)^5$$

$$= 32x^{10} + 5\left(16x^8\right)(-x) + 10\left(8x^6\right)\left(x^2\right) + 10\left(4x^4\right)\left(-x^3\right) +$$

$$5\left(2x^2\right)\left(x^4\right) + \left(-x^5\right)$$

$$= 32x^{10} - 80x^9 + 80x^8 - 40x^7 + 10x^6 - x^5$$

d) $\left(3a^3 + \dfrac{3}{a}\right)^4$

Answer:

$$\left(3a^3\right)^4 + \binom{4}{1}\left(3a^3\right)^3\left(\frac{3}{a}\right) + \binom{4}{2}\left(3a^3\right)^2\left(\frac{3}{a}\right)^2 + \binom{4}{3}\left(3a^3\right)\left(\frac{3}{a}\right)^3 + \left(\frac{3}{a}\right)^4$$

$$= 81a^{12} + 4\left(27a^9\right)\left(\frac{3}{a}\right) + 6\left(9a^6\right)\left(\frac{9}{a^2}\right) + 4\left(3a^3\right)\left(\frac{27}{a^3}\right) + \left(\frac{81}{a^4}\right)$$

$$= 81a^{12} + 324a^8 + 486a^4 + 324 + \frac{81}{a^4}$$

2. Write and simplify the first three terms, using the Binomial Theorem.

a) $\left(x - \dfrac{1}{x}\right)^{12}$

Answer:

$$x^{12} + \binom{12}{1}x^{11}\left(-\frac{1}{x}\right) + \binom{12}{2}x^{10}\left(-\frac{1}{x}\right)^2 + \ldots$$

$$= x^{12} + 12x^{11}\left(-\frac{1}{x}\right) + 66x^{10}\left(\frac{1}{x^2}\right) - \ldots$$

$$= x^{12} - 12x^{10} + 66x^8 - \ldots$$

b) $\left(3ax + \frac{x^2}{3}\right)^6$

Answer:

$$(3ax)^6 + \binom{6}{1}(3ax)^5\left(\frac{x^2}{3}\right) + \binom{6}{2}(3ax)^4\left(\frac{x^2}{3}\right)^2 + \ldots$$

$$= 729a^6x^6 + 6\left(243a^5x^5\right)\left(\frac{x^2}{3}\right) + 15\left(81a^4x^4\right)\left(\frac{x^4}{9}\right) + \ldots$$

$$= 729a^6x^6 + 486a^5x^7 + 135a^4x^8 + \ldots$$

c) $\left(\frac{x}{2} - \frac{2}{x}\right)^7$

Answer:

$$\left(\frac{x}{2}\right)^7 + \binom{7}{1}\left(\frac{x}{2}\right)^6\left(-\frac{2}{x}\right) + \binom{7}{2}\left(\frac{x}{2}\right)^5\left(-\frac{2}{x}\right)^2 + \ldots$$

$$= \frac{x^2}{128} + 7\left(\frac{x^6}{64}\right)\left(-\frac{2}{x}\right) + 21\left(\frac{x^5}{32}\right)\left(\frac{4}{x^2}\right) + \ldots$$

$$= \frac{x^7}{128} - \frac{7x^5}{32} + \frac{21x^3}{8} - \ldots$$

3. Find and simplify the seventh term of $(a + b)^8$.

   Answer:

   The exponent of b in the seventh term will be one less than seven, or 6. Therefore, the exponent of a must be 2, as the sum of the exponents must be 8. Hence, the seventh term is:

$$t_{k+1} = \binom{n}{k}(x)^{n-k}(y)^k$$

   Where,

   $n = 8$

   $k + 1 = 7$

   $k = 6$

$$t_7 = \binom{8}{6}a^2b^6 = 28a^2b^6$$

4. Find and simplify.

   a) The sixth term of $(2y + x)^{11}$

   Answer:

   Using the formula $t_{k+1} = {}_nC_k \cdot a^{n-k} \cdot b^k$,

   $$\binom{11}{5}(2y)^{11-5}x^5 = 462\left(64y^6\right)x^5 = 29\ 568y^6x^5 = 29\ 586x^5y^6$$

   b) The term containing x^2 in $\left(\frac{x^5}{2} - \frac{2}{x^3}\right)^{10}$

   Answer:

   The pattern of the exponent is:

   $$\left(x^5\right)^{10}, \left(x^5\right)^9\left(\frac{1}{x^3}\right), \left(x^5\right)^8\left(\frac{1}{x^3}\right)^2, \ldots \text{ or } x^{50}, x^{42}, x^{34}, \ldots$$

   The exponent is decreasing by 8. Therefore, to reach x^2 from x^{50}, you need $\frac{48}{8} = 6$ decreases. The requested term is the seventh term.

   Now use the t_{k+1} formula.

   $$\binom{10}{6}\left(\frac{x^5}{2}\right)^4\left(-\frac{2}{x^3}\right)^6 = 840x^2$$

   c) The term containing x^{14} in $(2x + x^2)^{11}$

   Answer:

   The pattern of the exponent of x is: x^{11}, $x^{10}x^2$, $x^9(x^2)^2$, . . ., or x^{11}, x^{12}, x^{13}, ... The requested term is the fourth term.

   Now use the t_{k+1} formula.

   $$\binom{11}{3}(2x)^8\left(x^2\right)^3 = 42\ 240x^{14}$$

   d) The $(r + 1)$ term of $\left(3a - \frac{1}{6a^2}\right)^9$

   Answer:

   Use the t_{k+1} formula where $k = r$.

   $$\binom{9}{r}(3a)^{9-r}\left(-\frac{1}{6a^2}\right)^r$$

5. Write the first four terms of $\left(2x^3 - \dfrac{1}{4x}\right)^{11}$. Find the eighth term.

Answer:

$$(2x^3)^{11} + \binom{11}{1}(2x^3)^{10}\left(-\frac{1}{4x}\right) + \binom{11}{2}(2x^3)^9\left(-\frac{1}{4x}\right)^2 +$$

$$\binom{11}{3}(2x^3)^8\left(-\frac{1}{4x}\right)^3 + \ldots$$

$$= 2048x^{33} + 11\left(1024x^{30}\right)\left(-\frac{1}{4x}\right) + 55\left(512x^{27}\right)\left(\frac{1}{16x^2}\right) +$$

$$165\left(256x^{24}\right)\left(-\frac{1}{64x^3}\right) + \ldots$$

$$= 2048x^{33} - 2816x^{29} + 1760x^{25} - 660x^{21} + \ldots$$

The eighth term is $\binom{11}{7}(2x^3)^{11-7}\left(-\frac{1}{4x}\right)^7 = 330(2^4x^{12})\left(-\frac{1}{2^{14}x^7}\right)$

$$= -\frac{330}{1024}x^5 = -\frac{165}{512}x^5.$$

6. Find the middle term of $\left(3x - \dfrac{1}{3x}\right)^8$.

Answer:

There are nine terms because the power is 8. The middle term will be the fifth term (with four terms above it and four terms below it).

$$\binom{8}{4}(3x)^4\left(-\frac{1}{3x}\right)^4 = 70\left(81x^4\right)\left(\frac{1}{81x^4}\right) = 70$$

7. Why should the numerical coefficients of a binomial expansion be combinations? Use the 'meaning' of $(x + y)^5$ to explain why the coefficient of the term containing x^3y^2 is ${}_5C_2$.

   Answer:

   $(x + y)^5 = (x + y)(x + y)(x + y)(x + y)(x + y)$

   To produce the term containing x^3y^2, the multiplication selects three of the five factors to contribute the xs and the other two to contribute the ys. The order of the xs and ys does not matter because multiplication is commutative. There are ${}_5C_2$ ways in which you can select the two factors that contribute the ys. (If you concentrate on the xs, there are ${}_5C_3$ ways of selecting the three factors that will contribute to the xs.) Whichever variable you choose to concentrate on, the answer is the same since ${}_5C_2 = {}_5C_3 = 10$. The numerical coefficient, 10, counts the number of x^3y^2 terms in the expansion.

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Module 2
Function Transformations

Module 2: Function Transformations

Introduction

A major portion of this course is the study of functions. You studied some functions in detail in Grade 11 Pre-Calculus Mathematics—specifically, the linear, quadratic, absolute value, and reciprocal functions. You also studied the transformations of quadratic functions. In this module, you will expand on this knowledge and study the transformations of the graphs of functions in general. The most common transformations are translations, reflections, and stretches. This module concentrates on the translations and stretches of functions. The study of reflections is addressed in Module 3.

You will also learn about operations on functions and compositions of functions. Operations on functions include adding, subtracting, multiplying, or dividing two or more functions. Compositions of functions occur when the value of one function depends on the value of a second function. This is similar to the food chain, and will be explained more in the last lesson of this module.

Note: Definitions and explanations are often given in the learning activities, as well as in the text of the lessons. For this reason, consider the learning activities as part of the lesson, not as review exercises. The answer keys for the learning activities should also be considered as part of the lesson.

Assignments in Module 2

When you have completed the assignments for Module 2, submit your completed assignments to the Distance Learning Unit either by mail or electronically through the learning management system (LMS). The staff will forward your work to your tutor/marker.

Lesson	Assignment Number	Assignment Title
2	Assignment 2.1	Transformations of Functions
3	Assignment 2.2	Combinations of Transformations
5	Assignment 2.3	Operations on and Compositions of Functions

Resource Sheet

When you write your midterm examination, you are encouraged to take a Midterm Examination Resource Sheet with you into the examination. This sheet will be one letter-sized page, 8½ ″ by 11″, with both sides in your handwriting or typewritten. You will submit it with your examination, but you do not receive any marks for it.

Many students have found that preparing a resource sheet is an excellent way to review. It provides you with a summary of the important facts of each module. You should complete a resource sheet for each module to help with your studying and reviewing. Lesson summaries and module summaries are included for you to use as a guide.

You may use the list of instructions provided below to help you with preparing your resource sheet for the material in Module 2. On this sheet, you should record math terms and definitions, formulas, sample questions, or a list of places where you often make mistakes. You should also identify special areas that require extra attention or review by writing the page numbers.

After you have completed each module's resource sheet, you may summarize the sheets from Modules 1, 2, 3, and 4 to prepare your Midterm Examination Resource Sheet. The midterm examination for this course is based on Modules 1 to 4.

Resource Sheet for Module 2

As you go through the lessons of this module, you may want to consider the following suggestions regarding the creation of a resource sheet.

1. List all the important math terms, and define them if necessary.
2. List all the formulas and perhaps a sample problem that shows how the formula is used.
3. If necessary, write the solutions to some problems, showing in detail how you did the calculations.
4. Copy any questions that represent the key points of the lesson, and perhaps include the solutions as well.
5. Identify the problems you found most difficult, and copy the page numbers onto the resource sheet so that you can review them before writing the examination. You may also copy the problems and the solutions onto your resource sheet, and later write them onto your Midterm Examination Resource Sheet.
6. Write any comments, ideas, shortcuts, or other reminders that may be helpful during an examination.

Lesson 1: Horizontal and Vertical Translations

Lesson Focus

In this lesson, you will

- ☐ learn how a translation affects the graph and properties of a function
- ☐ learn how to sketch the translation of a function
- ☐ learn how to state the translation that produced a new sketch from the given sketch of a function

Lesson Introduction

Functions make up a large part of this course. In order to become comfortable dealing with functions, you need to be familiar with the four main graphs used throughout this course. You also need to be aware of what effects various parts of a function have on the graph of a function. This is similar to what you did in Grade 11 Pre-Calculus Mathematics when you graphed quadratic functions using transformations. In this lesson, you will learn how to graph multiple types of functions by using function transformations.

Functions

When you don't know the shape of a function, you can use graphing technology to create the graph or you can use a table of values to create your own paper-and-pencil sketch.

You probably remember the shapes of $y = x$ and $y = x^2$ from previous math courses, but what do $y = x^3$ and $y = \sqrt{x}$ look like?

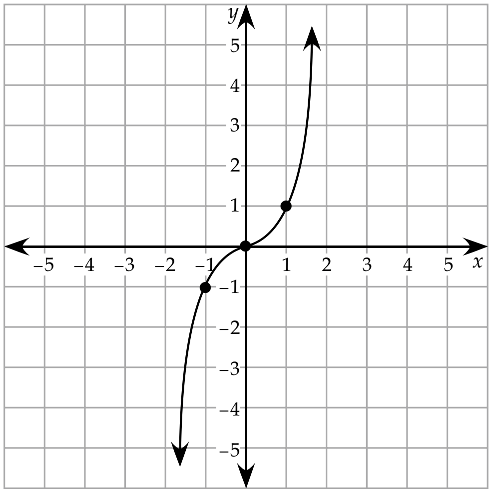
Create a table of values by choosing a variety of x-values and calculating the corresponding y-values.

Sketch $y = x^3$.

x	$y = x^3$	ordered pairs
-2	$(-2)^3 = -8$	$(-2, -8)$
-1	$(-1)^3 = -1$	$(-1, -1)$
0	$0^3 = 0$	$(0, 0)$
1	$1^3 = 1$	$(1, 1)$
2	$2^3 = 8$	$(2, 8)$

In words, for any value of x, the associated y-value is the cube of x.

Graph of $y = x^3$



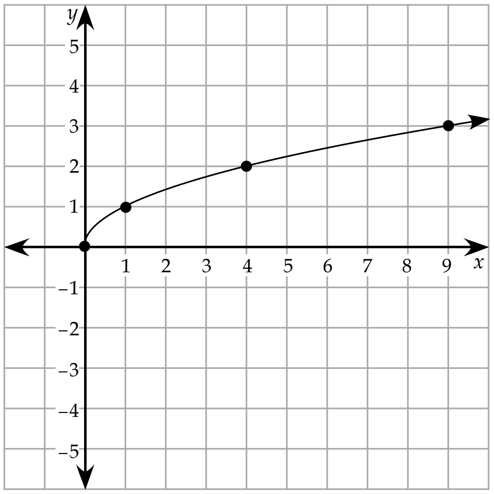
Sketch $y = \sqrt{x}$.

x	$y = \sqrt{x}$	ordered pairs
-4	$\sqrt{-4} = ?$	-4 is not in the domain
-1	$\sqrt{-1} = ?$	-1 is not in the domain
0	$\sqrt{0} = 0$	$(0, 0)$
1	$\sqrt{1} = 1$	$(1, 1)$
4	$\sqrt{4} = 2$	$(4, 2)$
9	$\sqrt{9} = 3$	$(9, 3)$

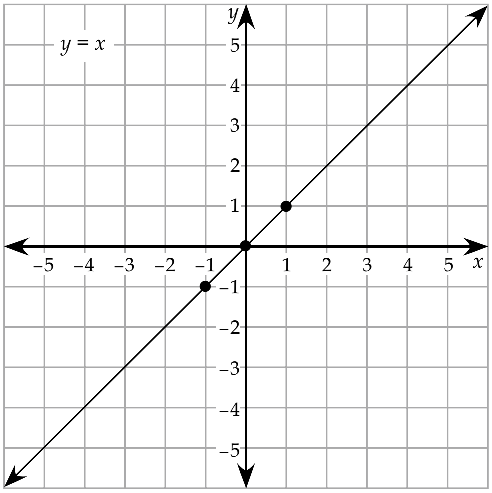
In words, for any value of x, the associated y-value is the square root of x. Since you can't take the square root of a negative number in the real number system, then there are no points of this function with negative x-values.

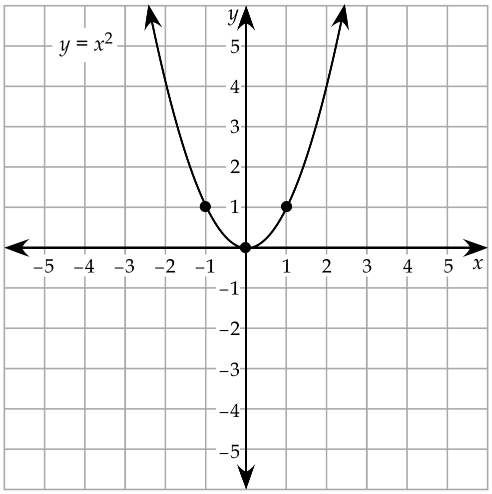
Note: To make the points easy to plot, the x-values chosen for the table are perfect squares (0, 1, 4, 9). You could have chosen an x-value of 2, then the y-value would be $y = \sqrt{2} \doteq 1.41$, and would have an ordered pair of (2, 1.41).

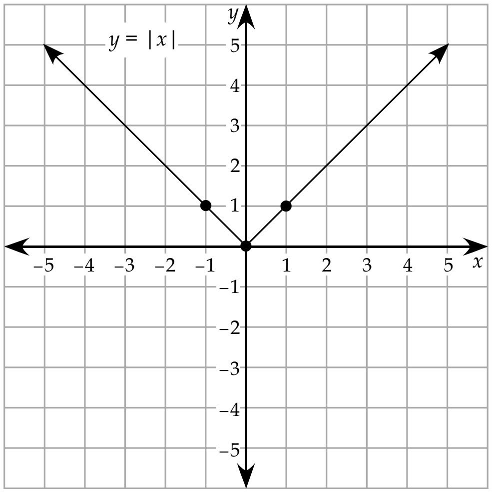
Graph of $y = \sqrt{x}$.

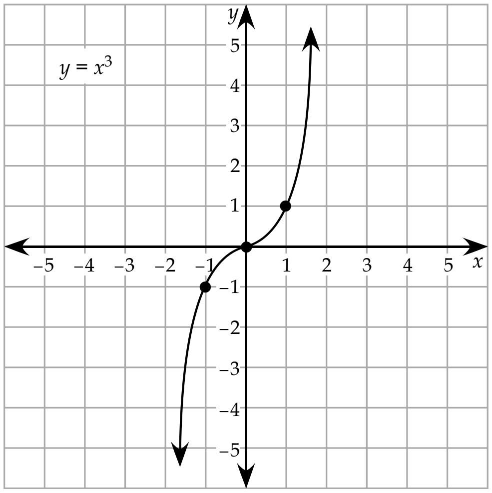


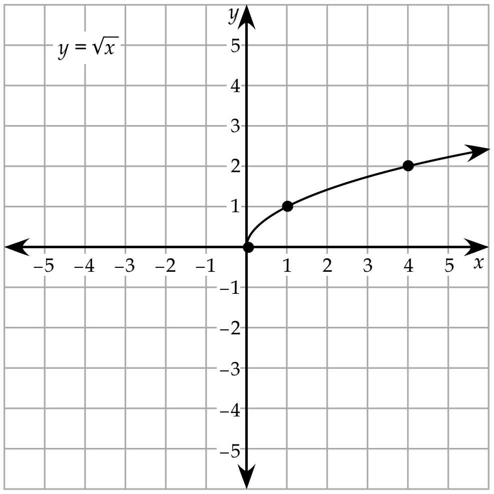
There are five main graphs used throughout this lesson (and the rest of this course) that you need to be familiar with. These include the graphs of $y = x$, $y = x^2$, $y = |x|$, $y = x^3$, and $y = \sqrt{x}$.







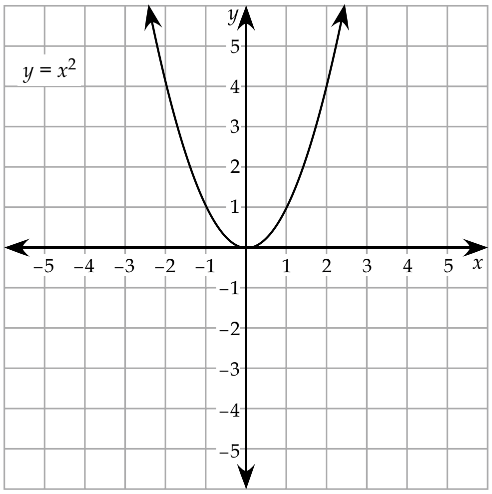


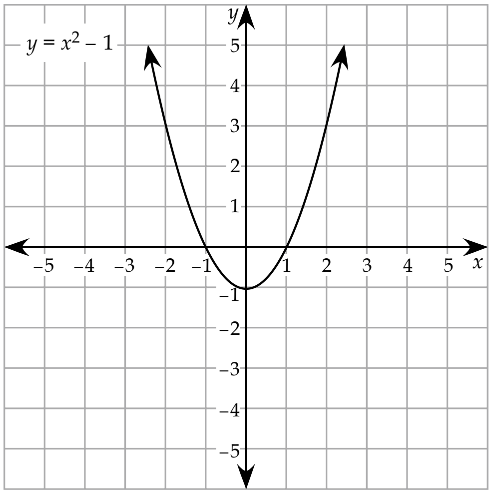


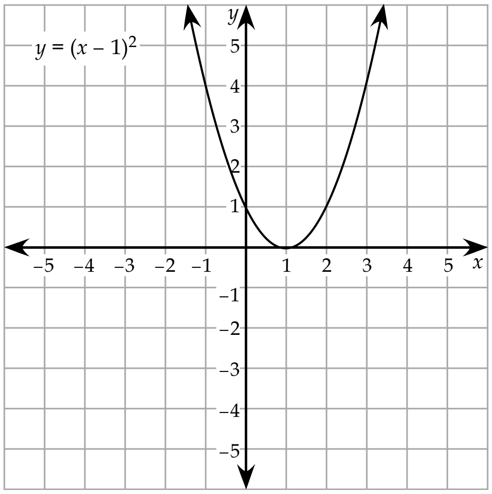
You need to familiarize yourself with the five basic graphs and be able to sketch them without going to the trouble of writing down the table of values. Instead, remember the location of the key points and the shape of each graph. Include the graphs on your resource sheet.

Translations

You have encountered translations when you sketched the following three functions in Grade 11 Pre-Calculus Mathematics.







In $y = x^2 - 1$, each y-value is 1 less than the corresponding y-value in $y = x^2$. Therefore, the graph of $y = x^2 - 1$ is 1 unit lower than the graph of $y = x^2$. It is said that the graph of $y = x^2$ has been **translated** 1 unit down. The function $y = x^2 - 1$ is a **vertical translation** of the function, $y = x^2$.

Similarly, $y = (x - 1)^2$ is a horizontal translation of $y = x^2$. The graph of $y = x^2$ has been shifted one unit to the right.

A translation is a transformation of a geometric figure in which every point is moved the same distance in the same direction.

Include the above information on your resource sheet.

Note: It is important when graphing that the coordinates of key points are labelled if they are fractions or decimals.

Note: You may find it helpful to sketch all three graphs on one set of axes, using three different colours to better visualize the translation.

Example 1

Use the sketch of $y = x^2$ to sketch

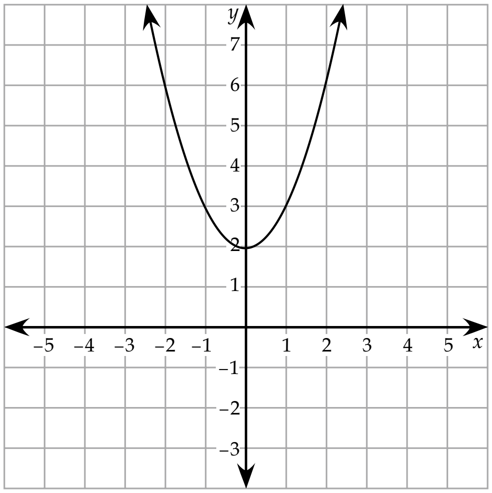
a) $y = x^2 + 2$

b) $y = (x + 2)^2$

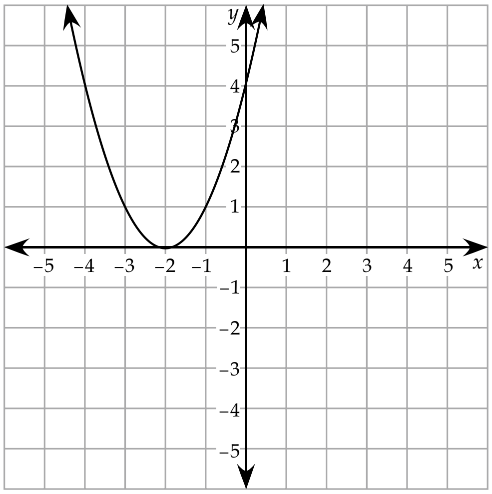
Note: A graphing calculator is not required for any part of this course. However, if you have a graphing calculator, it can be useful to check your answers.

Solutions

a)



b)



As evident from the graphs, the sketch of $y = x^2 + 2$ is a vertical translation of two units up, and the graph of $y = (x + 2)^2$ is a horizontal translation of two units to the left of the graph of $y = x^2$.

Notice that with the vertical translation, $y = x^2 + 2$, the "2" is outside the squared function, and affects the y-values by an amount of 2. With the horizontal translation, $y = (x + 2)^2$, the "2" is inside the squared function next to the x and affects the x-values.

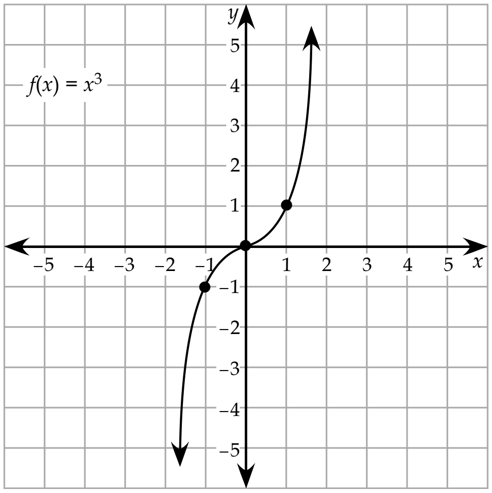
Vertical translations are often easier and more natural to understand. One way of remembering in which direction to shift a horizontal translation is to think of the value of x that makes the squared binomial zero and then move in that direction. In the above example, the sign inside the brackets was positive as $y = (x + 2)^2$ and the graph shifted to the left. The value of x that makes the squared binomial zero is −2, and thus the graph is shifted 2 units in the negative direction, or 2 units to the left.

Consider the following chart:

Quadratic Equation	**Value of x which Makes $y = 0$**	**Effects on Graph**
$y = x^2$	0	basic graph
$y = (x + 2)^2$	–2	2 units left
$y = (x - 2)^2$	2	2 units right

Example 2

Use the given sketch of $f(x) = x^3$ to sketch the following translations of the cubic function. To translate this function, try plotting the x-intercept first. Every point of the function is shifted but you can start by plotting one key point. Remember that the x-intercepts are also called the zeros of the function.



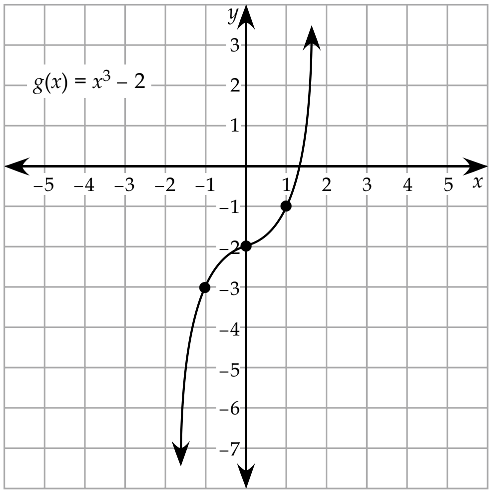
a) $g(x) = x^3 - 2$

b) $h(x) = (x - 1)^3$

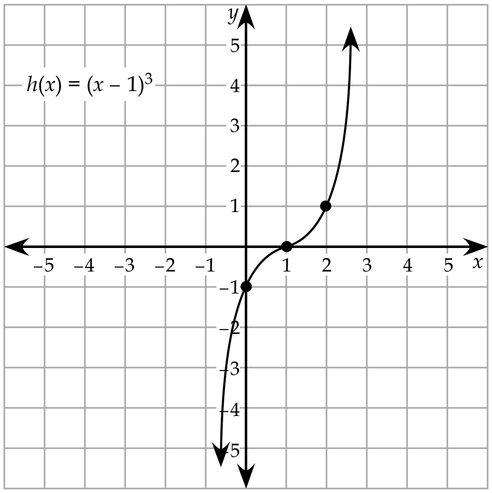
c) $k(x) = (x + 3)^3$

Solutions

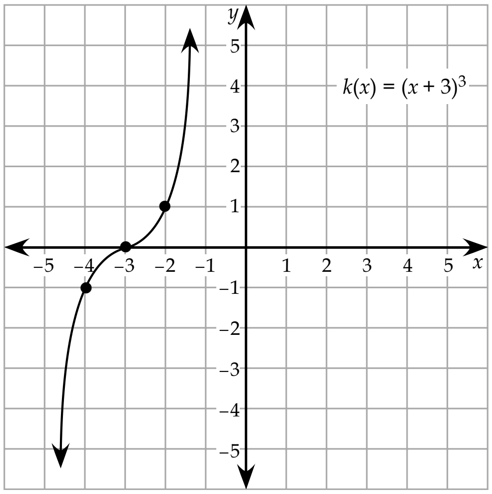
a)



b)



c)



In (a), the graph of $f(x) = x^3$ is shifted 2 units down because the "−2" is outside the brackets. This is a vertical translation.

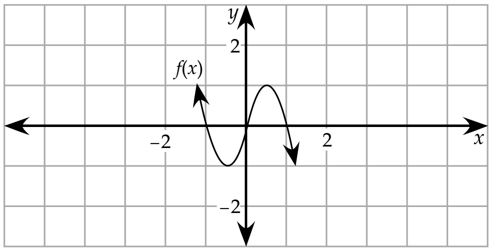
In (b), the graph of $f(x) = x^3$ is shifted 1 unit to the right because the "−1" is inside the brackets. Therefore, you need to switch the sign and then move in that direction. This results in a horizontal shift in the positive direction of 1 unit.

In (c), the graph of $f(x) = x^3$ is shifted 3 units to the left because the "+3" is inside the brackets. Therefore, you need to switch the sign and then move in that direction. This results in a horizontal shift in the negative direction of 3 units.

Example 3

In addition to the five basic functions, you also need to do translations of other functions when given the graph of the function.

Use the given sketch of $f(x)$ to sketch the following functions. Again, try translating the x- and y-intercepts as a starting point.



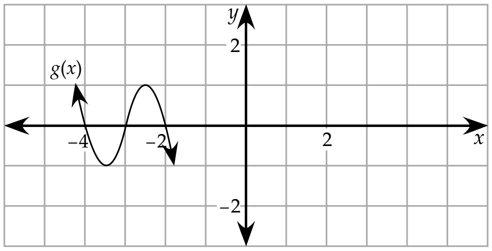
a) $g(x) = f(x + 3)$

b) $m(x) = f(x) + 2$

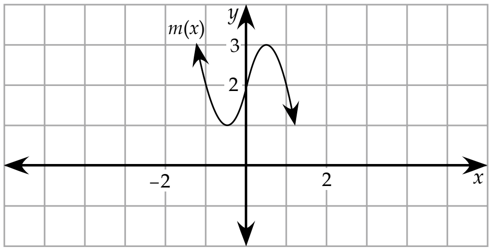
c) $n(x) = f(x - 5)$

Solutions

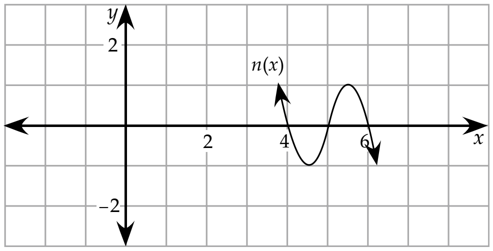
a)



b)



c)



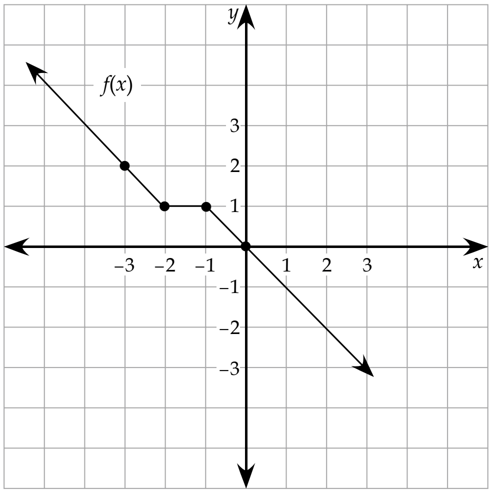
In (a), $\boldsymbol{g(x) = f(x + 3)}$. As the "+3" is inside the brackets next to the x, this results in a horizontal shift of $f(x)$ 3 units to the left.

In (b), $\boldsymbol{m(x) = f(x) + 2}$. As the "+2" is outside the brackets and so affects the y-values, this results in a vertical shift of $f(x)$ 2 units upwards.

In (c), $\boldsymbol{n(x) = f(x - 5)}$. As the "−5" is inside the brackets next to the x, this results in a horizontal shift of $f(x)$ 5 units to the right.

Example 4

Use the given sketch of $f(x)$ to sketch the following functions.



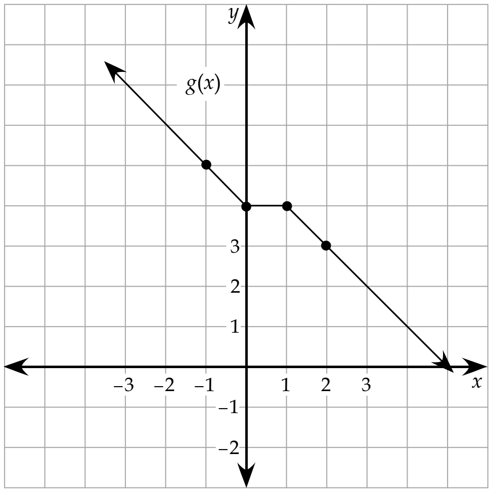
a) $g(x) = f(x - 2) + 3$

b) $h(x) = f(x + 1) - 4$

Solutions

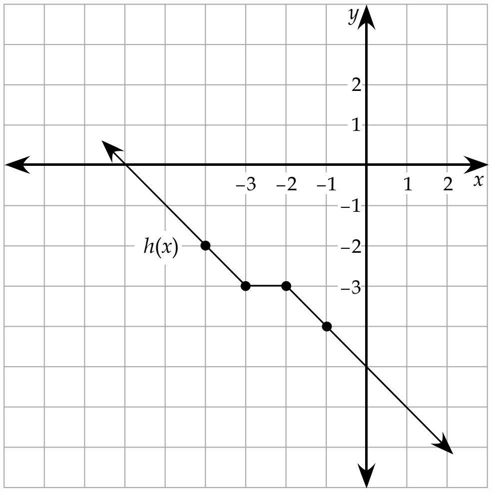
a) $g(x) = f(x - 2) + 3$

In this question, you have to graph both a vertical and a horizontal translation. It does not matter which translation you graph first because one translation affects the x-values and the other affects the y-values. Your final graph should contain a vertical shift of 3 units up and a horizontal shift of 2 units to the right. It is easiest to shift a few key points and then draw the resulting function curve. When you are doing translations, the slope of the curve does not change.



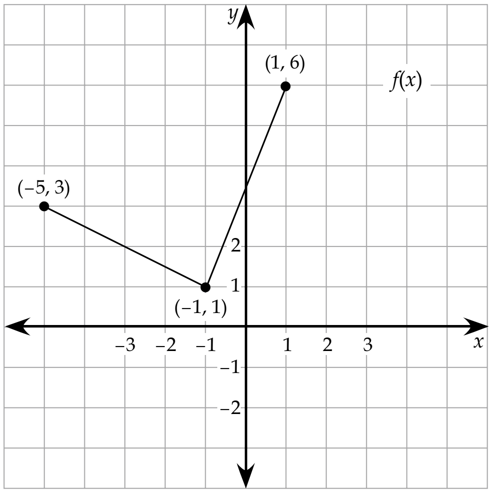
b) $h(x) = f(x + 1) - 4$

Again, it does not matter which translation you graph first. The final graph should contain a vertical shift of 4 units down and a horizontal shift of 1 unit to the left.

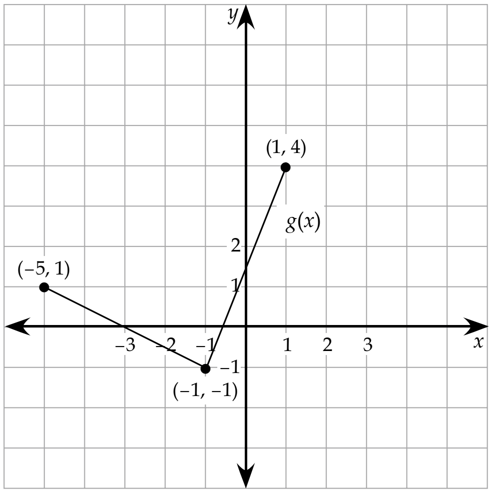


Example 5

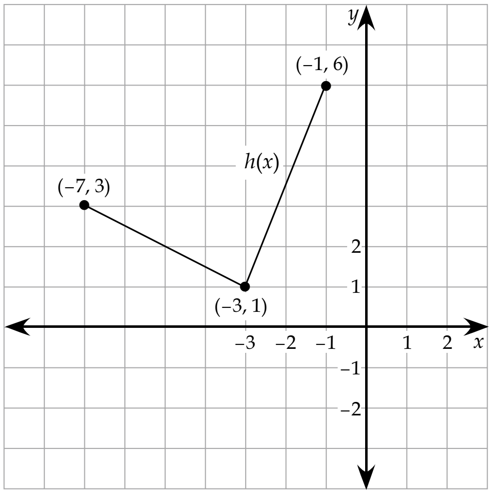
Consider the graph of $f(x)$ below. For each of the following graphs, write the equation of the new function.



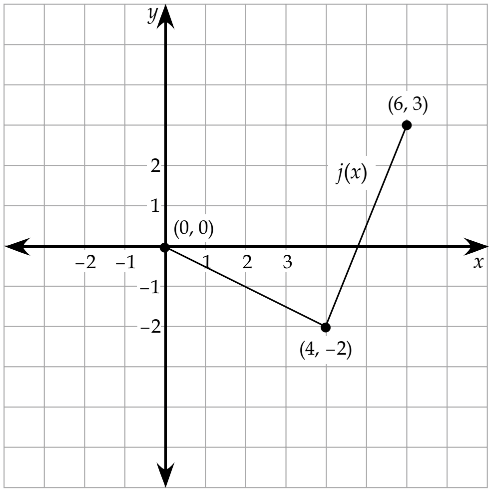
a)



b)



c)



Solutions

a) This graph is moved 2 units down. Therefore, the new function is $g(x) = f(x) - 2$.

b) This graph is moved 2 units to the left. Therefore, the new function is $h(x) = f(x + 2)$. **Note:** The sign is positive because the direction of horizontal shifts always correspond to the *opposite* sign inside the brackets.

c) This graph is moved 5 units to the right and 3 units down. Therefore, the new function is $j(x) = f(x - 5) - 3$.

Given a function, $f(x)$, the effect of a translation on $f(x)$ is summarized in the chart below.

Translation	Effect on Graph	Examples
$f(x) + k$	Vertical translation of k units (affects y as implied)	$f(x) + 3$: shift up 3 (add 3 to y-coordinate) $f(x) - 5$: shift down 5 (subtract 5 from y-coordinate)
$f(x - h)$	Horizontal translation of h units (affects x in opposite way)	$f(x + 2)$: shift left 2 (subtract 2 from x-coordinate) $f(x - 7)$: shift right 7 (add 7 to x-coordinate)

Make sure you have the above information on your resource sheet.

Learning Activity 2.1

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Simplify: $\frac{8!}{6!}$
2. In how many ways can three different flower arrangements be arranged on a table?
3. What is the reciprocal of $\frac{xy}{3}$?
4. Rationalize the denominator: $\frac{3}{1 - \sqrt{5}}$
5. The Winnipeg Blue Bombers have sold out their stadium to 29 503 fans. Halfway through the game, it starts raining. If 15 316 fans leave, how many fans remain to see the Winnipeg Blue Bombers win the game?

continued

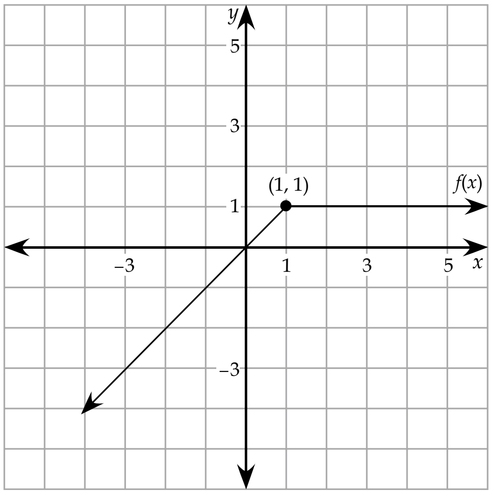
Learning Activity 2.1 (continued)

6. If $f(x) = 3x^3 - 2x^2 + 1$, evaluate $f(x)$ at $x = -3$.
7. Factor: $4x^2 - 12xy + 9y^2$
8. Which fraction is larger: $\frac{9}{29}$ or $\frac{9}{27}$?

Part B: Horizontal and Vertical Translations

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Given the sketch of $f(x)$ drawn below, sketch each of the following functions.
   a) $y = f(x - 5)$
   b) $y = f(x) - 5$
   c) $y = f(x) + 5$
   d) $y = f(x + 5)$
   e) $y = f(x - 5) - 5$
   f) $y = f(x + 5) + 5$



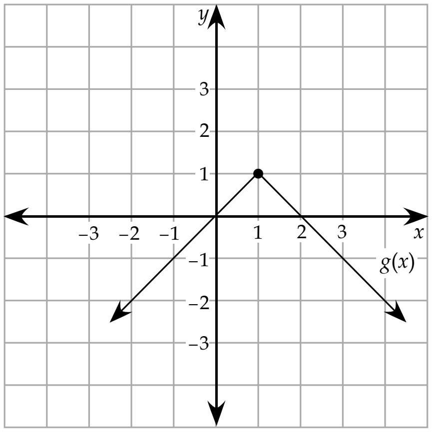
2. Let $f(x) = x^2 + 2$. Sketch each of the following functions.
   a) $f(x)$
   b) $y = f(x) - 6$
   c) $y = f(x + 1)$
   d) $y = f(x - 2) - 3$
3. For each of the functions in question 2 state the properties of the function: the domain, the range, and the values of the intercepts. (Recall: To find the y-intercept, let $x = 0$ and solve for y. To find the x-intercept, let $y = 0$ and solve for x.)

   Recall: The domain is the set of all the possible x-values of the function. The range is the set of all the possible y-values of the function.

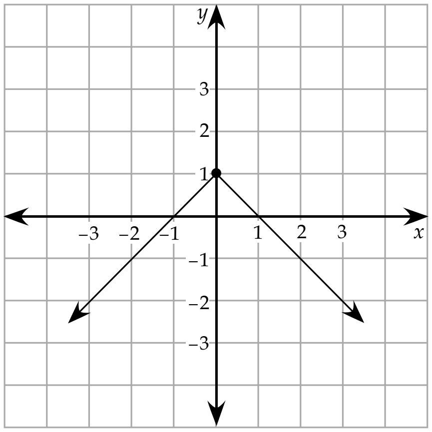
continued

Learning Activity 2.1 (continued)

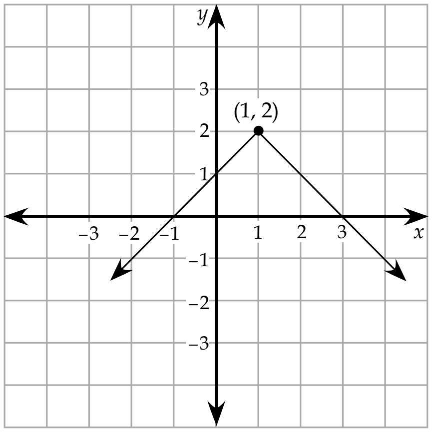
4. Each of graphs (a), (b), and (c) represents a translation of the given function, $g(x)$, shown below. Write an expression for each new function in terms of $g(x)$.



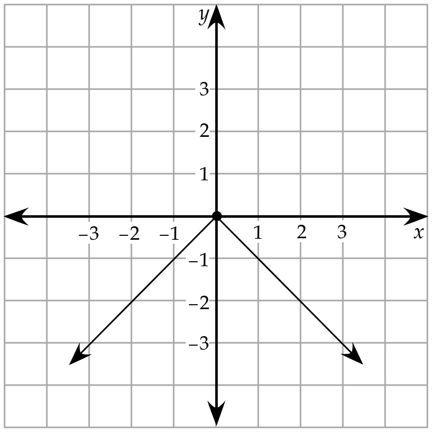
a)



b)



c)

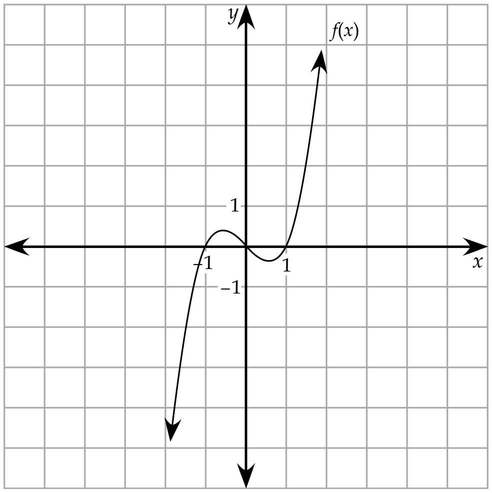


continued

Learning Activity 2.1 (continued)

5. If $f(x) = x^3 + 3x^2 - x + 6$, write an equation for the following. Do not simplify. For example, the equation of $g(x)$, which has the same graph as $f(x)$ moved two units to the left, would be $g(x) = f(x + 2)$. This could be written as $g(x) = f(x + 2) = (x + 2)^3 + 3(x + 2)^2 - (x + 2) + 6$.
   a) $h(x)$, which has the same graph as $f(x)$ moved three units down
   b) $m(x)$, which has the same graph as $f(x)$ moved two units to the right and one unit up

6. Below is the graph of $f(x) = x^3 - x$. Sketch the graph of $g(x) = (x + 2)^3 - (x + 2)$.



7. Is the translation of a function still a function?

Lesson Summary

In this lesson, you learned how to graph vertical and horizontal translations of functions. You learned that functions in the form $f(x) + k$ result in a vertical shift of the original function $f(x)$. You also learned that functions in the form $f(x + h)$ result in a horizontal shift of the original function $f(x)$. Make sure you are comfortable with these types of transformations before you continue on to the next lesson.

In the next lesson, you will learn how to graph stretches and compressions of functions.

LESSON 2: HORIZONTAL AND VERTICAL STRETCHES AND COMPRESSIONS

Lesson Focus

In this lesson, you will

☐ learn how to describe how a stretch or a compression affects the graph and properties of a function

☐ learn how to sketch functions that have been stretched or compressed

☐ learn how to state the transformation that produced a new sketch from a given sketch of a function

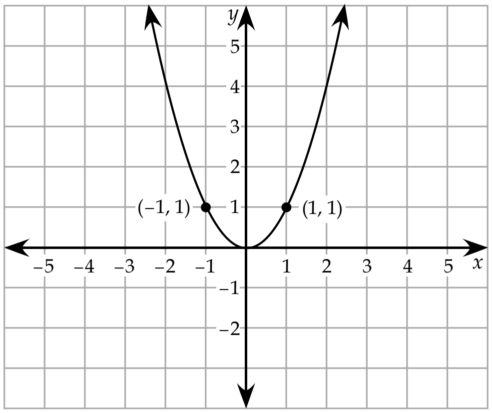
Lesson Introduction

In Lesson 1, you learned how to graph horizontal and vertical translations of functions. To do this, you built on your knowledge of translations of parabolas from Grade 11 Pre-Calculus Mathematics. In this lesson, you will build on your knowledge of stretches and compressions of parabolas to learn about stretches and compressions of many different types of functions.

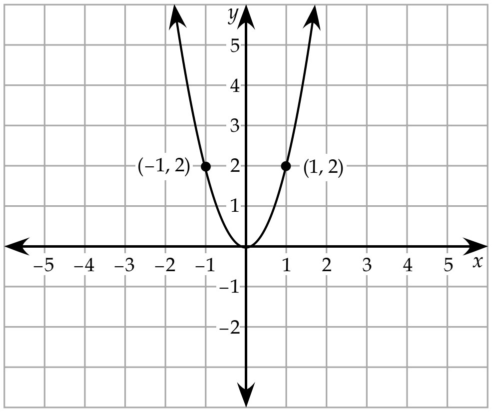
Stretches and Compressions

You encountered stretches and compressions when you sketched the following three functions in Grade 11 Pre-Calculus Mathematics.

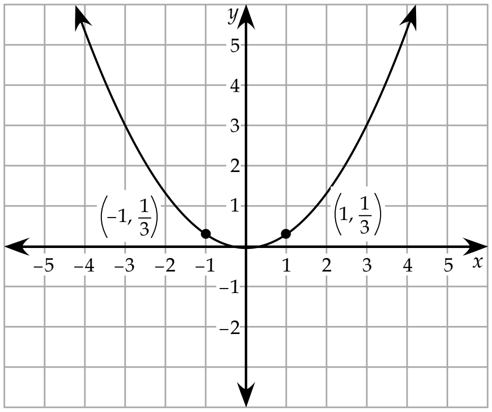
$y = x^2$ (standard)



$y = 2x^2$ (stretch)



$y = \frac{1}{3}x^2$ (compression)



As you can see from the diagrams above, unlike translations, both a **stretch** and a **compression** change the shape of a graph.

Note: You may find it helpful to sketch all three graphs on one set of axes, using three different colours to better visualize the sketch and compression.

Include this information on your resource sheet.

Vertical Stretches and Compressions

In $y = 2x^2$, each y-value is twice the corresponding y-value in $y = x^2$. Therefore, the graph of $y = x^2$ has been stretched vertically by a factor of 2 to produce the graph of $y = 2x^2$.

Similarly, $y = \frac{1}{3}x^2$ is a vertical stretch of $y = x^2$. The graph of $y = x^2$ has been stretched by a factor of $\frac{1}{3}$ to produce the graph of $y = \frac{1}{3}x^2$. A stretch like this can also be called a compression by a factor of 3.

This can be described algebraically as $y = af(x)$ or $(x, y) \rightarrow (x, ay)$. Notice that only the y-values are affected when the multiplier, a, is in front of the function, $f(x)$.

All transformations can be written algebraically (also known as mapping notation) in order to determine the new points on the graph.

In this case, $y = \frac{1}{3}x^2$ can be transformed using the key points on the original graph, the mapping notation, and a table of values.

$y = x^2$

(x, y)

x	y
-2	4
-1	1
0	0
1	1
2	4

$y = \frac{1}{3}x^2$

$\left(x, \frac{1}{3}y\right)$

x	y
-2	$\frac{4}{3}$
-1	$\frac{1}{3}$
0	0
1	$\frac{1}{3}$
2	$\frac{4}{3}$

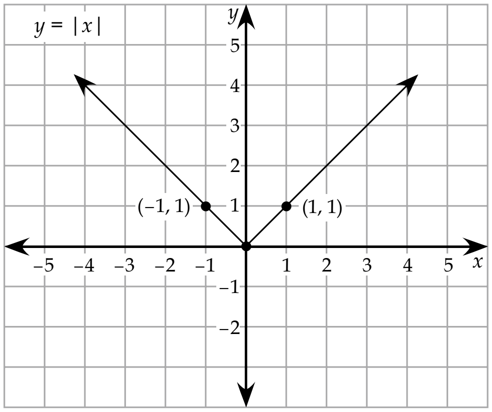
The new points can be used to draw the transformed graph above.

Example 1

Sketch $y = |x|$ and use its graph to sketch the following. State the transformation(s) and write algebraically.

a) $y = 3|x|$

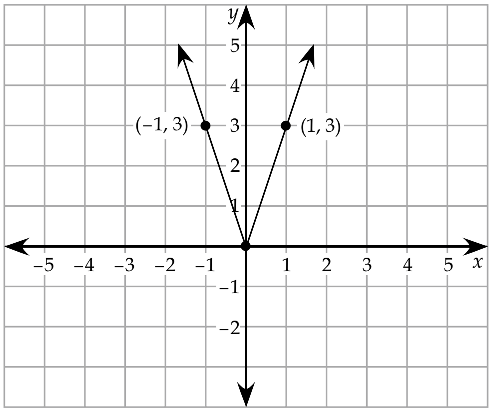
b) $y = \frac{1}{2}|x - 3|$



Solutions

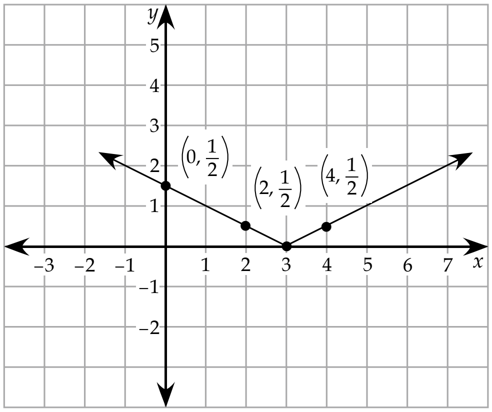
a) This function is stretched vertically by a factor of 3.

Algebraically: $(x, y) \rightarrow (x, 3y)$



b) This function is stretched vertically by a factor of $\frac{1}{2}$ (or compressed by a factor of 2) and then shifted 3 units to the right. The order doesn't matter in this example because the horizontal translation affects the x-values and the vertical stretch affects the y-values.

Algebraically: $\left(x, y\right) \rightarrow \left(x + 3, \frac{1}{2}y\right)$



Note: Similar to the order of operations you are familiar with where multiplication happens before addition, stretches, and/or compressions must be performed before translations. This matters only when both transformations are vertical or both are horizontal. You should add this comment to your resource sheet.

Example 2

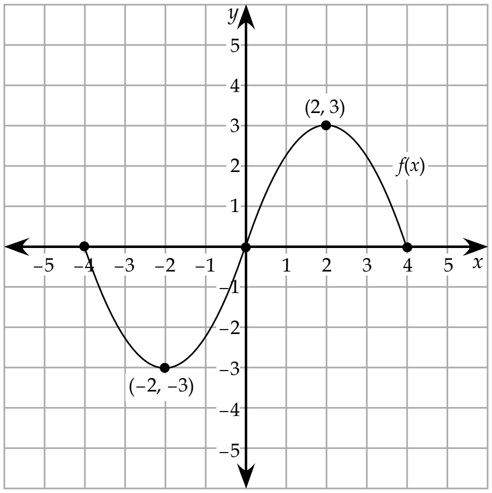
Use the graph of *f*(*x*), drawn below, to sketch the following. State the transformation(s) in words and algebraically.

a) $y = 2f(x)$

b) $y = \frac{1}{4}f(x)$

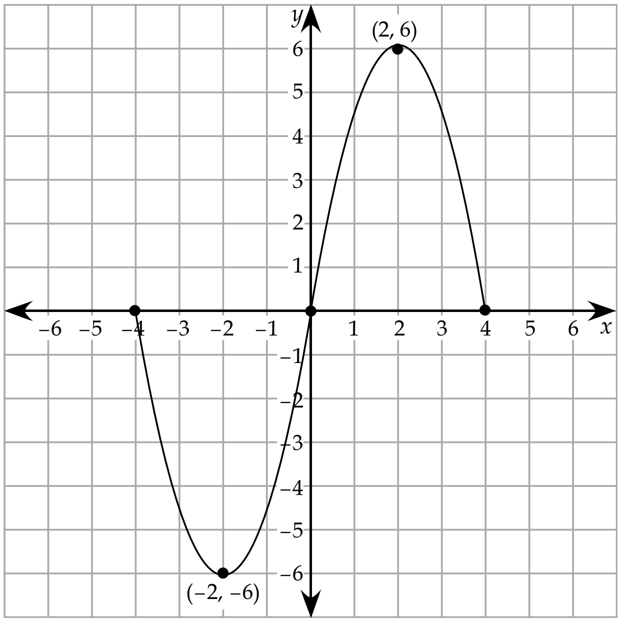
c) $y = 2f(x+2)$

d) $y = \frac{2}{3}f(x) + 2$



Solutions

a) stretched vertically by a factor of 2



Note: Using key points on the original graph, you can calculate algebraically the new points to plot.

f(x)	→	2*f*(*x*)
(*x*, *y*)	→	(*x*, 2*y*)
(−4, 0)	→	(−4, 0)
(−2, −3)		(−2, −6)
(0, 0)		(0, 0)
(2, 3)		(2, 6)
(4, 0)		(4, 0)

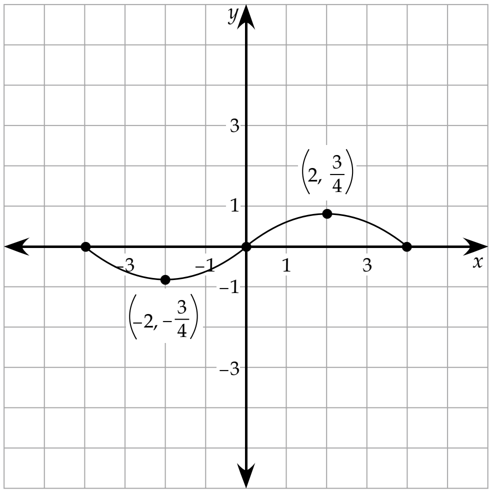
Algebraically: $(x, y) \rightarrow (x, 2y)$

According to our mapping notation, in the new graph all the *x*-values will remain the same and all the *y*-values will be multiplied by 2.

b) vertically stretched by a factor of $\frac{1}{4}$ or vertically compressed by a factor of 4

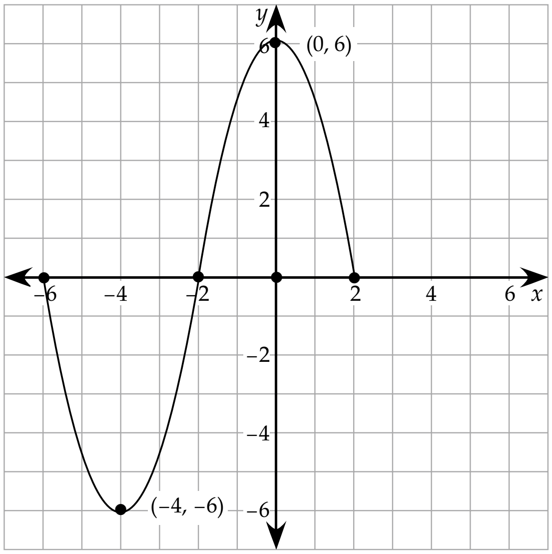
Algebraically: $(x, y) \rightarrow \left(y, \frac{1}{4}x\right)$ or

$$(x, y) \rightarrow \left(y, \frac{x}{4}\right)$$



c) stretched vertically by a factor of 2 and shifted 2 units to the left.

Algebraically: $(x, y) \rightarrow (x - 2, 2y)$

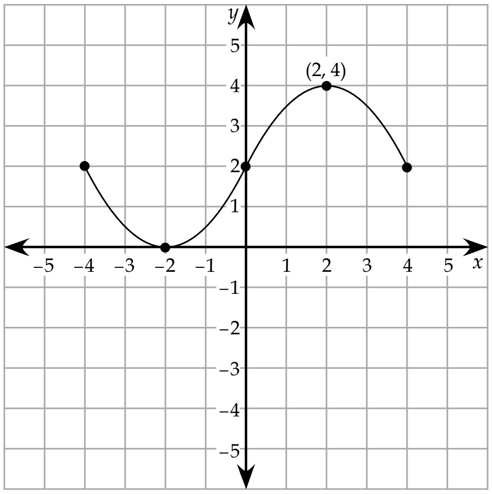


d) vertically stretched by a factor of $\frac{2}{3}$, shifted up 2 units

The order for this example is important because both transformations are vertical and both affect the y-values:

- stretch first (multiply)
- shift second (add)

Algebraically: $(x, y) \rightarrow \left(x, \frac{2}{3}y + 2\right)$

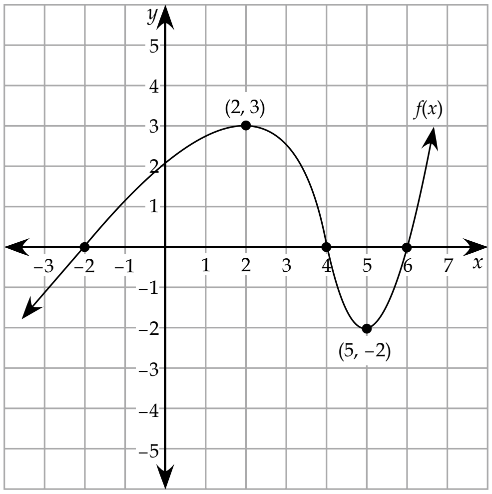


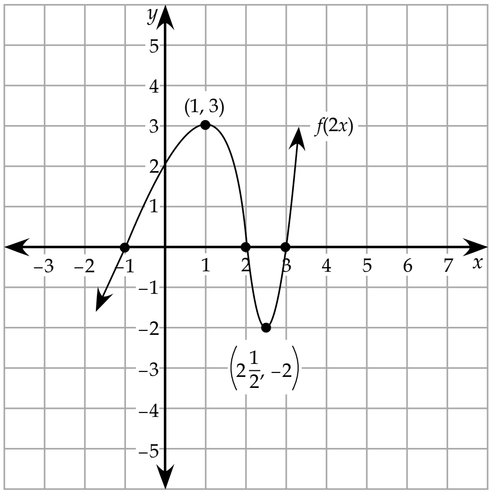
Horizontal Stretches and Compressions

Graphs of functions can also be transformed by horizontal stretches and compressions. It is helpful to analyze this transformation of a graph by examining the x-intercepts.

Example 3

Examine the x-intercepts of $f(x)$ and $f(2x)$, both shown below.





Solution

The graph of $f(2x)$ is a *horizontal compression* by a factor of 2 of the graph of $f(x)$.

The x-intercepts of $f(2x)$ are *half* the value of the x-intercepts of $f(x)$.

Why should this happen? One way of seeing this connection is to examine the x-intercepts. Since the x-intercepts of $f(x)$ are −2, 4, and 6, it follows that $y = f(x)$ will be zero when x takes on the values −2, 4, or 6.

Therefore, $y = f(2x)$ will be equal to zero when $2x$ takes on the values −2, 4, or 6. For this to happen, x must be equal to −1, 2, or 3. Therefore the graph of $f(2x)$ is a horizontal compression of the graph of $f(x)$ by a factor of 2. You divide the original x-values by 2.

Note: The x-values of $f(2x)$ are half of those of $f(x)$, but the corresponding y-values are unchanged. Also, note that you can recognize it is a horizontal transformation since the factor of 2 is with the x and is inside the brackets.

By the same reasoning, the graph of $f\left(\frac{1}{2}x\right)$ is a *horizontal compression* of the graph of $f(x)$ by a factor of $\frac{1}{2}$. Again, it is a horizontal transformation since the factor is inside the brackets and the x-values are affected. You divide the original x-values by $\frac{1}{2}$ (this is the same as multiplying the x-values by 2).

In general, this transformation can be described algebraically as follows.

If $y = f(x) \rightarrow y = f(bx)$

then, $(x, y) \rightarrow \left(\frac{1}{b}x, y\right)$

The following chart will help clarify the transformations you have learned so far.

Summary of Examples of Compressions and Stretches					
Transformation	**Placement of Coefficient**	**Scale Factor**	**Affects**	**Effect**	**Important Notes**
$2f(x)$	outside	multiply the y's by 2	y-values	vertical stretch	
$\frac{1}{2}f(x)$	outside	multiply the y's by $\frac{1}{2}$	y-values	vertical stretch	This is sometimes called a vertical compression by a factor of 2.
$f(2x)$	inside	divide the x's by 2	x-values	horizontal compression	
$f\left(\frac{1}{2}x\right)$	inside	divide the x's by $\frac{1}{2}$	x-values	horizontal compression	This is sometimes called a horizontal stretch by a factor of 2. (Recall: Dividing by $\frac{1}{2}$ is the same as multiplying by 2.)

Include the above chart on your resource sheet. It may also be beneficial if you include an example of each situation on your resource sheet as well.

Make sure you understand these transformations, either by asking your learning partner for help or by talking to your tutor/marker before you continue on to the rest of this lesson.

Example 4

Use the graph of $f(x)$, shown below, to sketch the following. State the transformation(s) in words and algebraically.

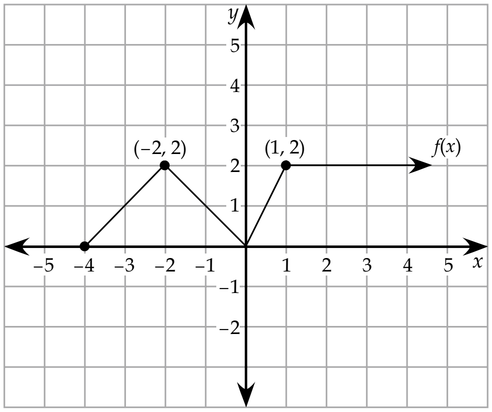
a) $y = f(2x)$

b) $y = f\left(\frac{1}{2}x\right)$

c) $y = 2f(2x)$

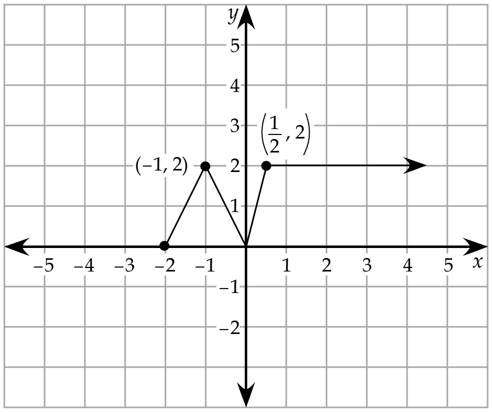
d) $y = f(3x) - 1$

e) $y = 3f(x) + 3$



Solutions

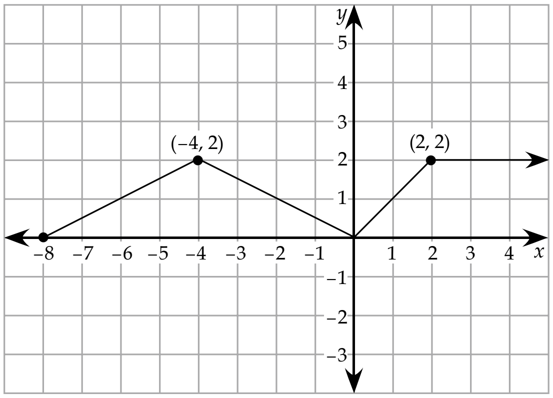
a) horizontal compression by a factor of 2



Algebraically: $(x, y) \rightarrow \left(\frac{x}{2}, y\right)$

To compress by a factor of 2 means you will need to divide x-values.

b) horizontal compression by a factor of $\frac{1}{2}$

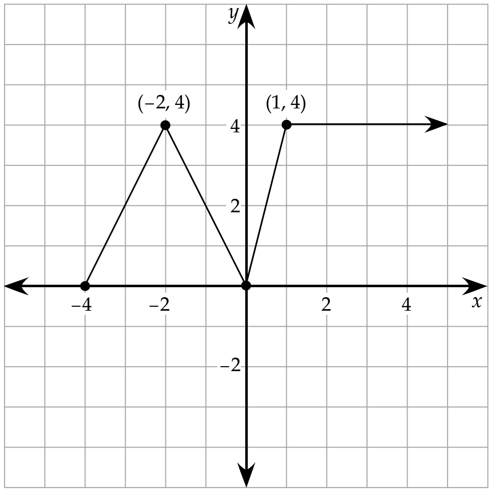


Algebraically: $(x, y) \rightarrow (2x, y)$

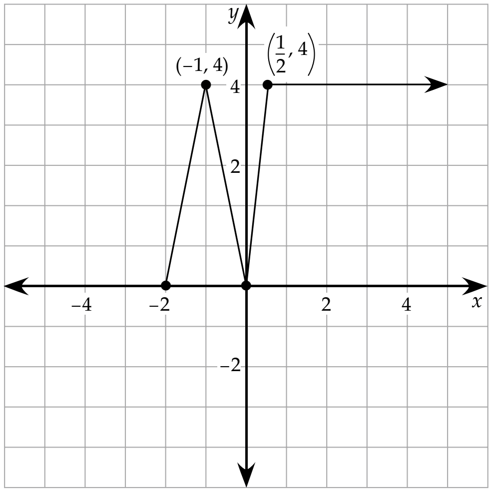
Compression by a factor of $\frac{1}{2}$ is the same as stretching by a factor of 2, which means you will need to multiply x-values.

c) Does the order in which you perform the stretches and compressions matter? To figure this out, try this question in two different ways. First, vertically stretch the graph by a factor of 2 and then horizontally compress the graph by a factor of 2.

Vertical stretch by a factor of 2

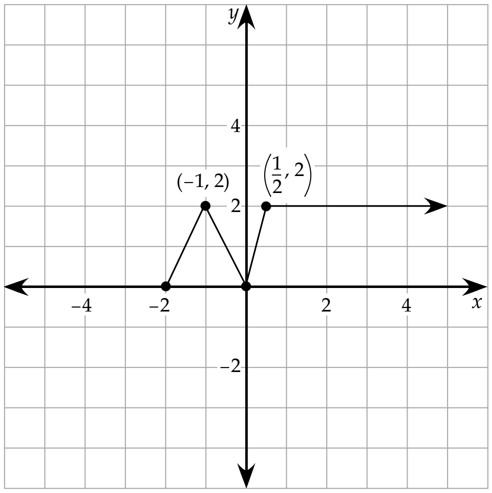


Horizontal compression by a factor of 2

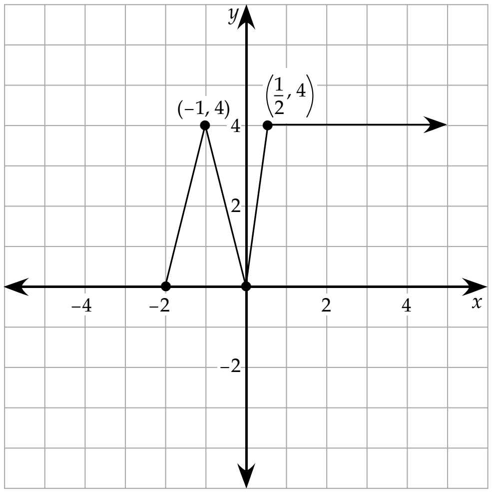


Second, horizontally compress the graph by a factor of 2 and then vertically stretch the graph by a factor of 2.

Horizontal compression by a factor of 2



Vertical stretch by a factor of 2

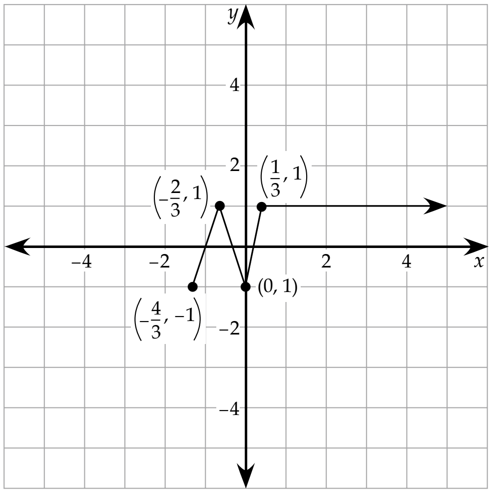


What do you notice?

Both of the final graphs are the same. In general, it does not matter which order you perform transformations if one of the transformations is horizontal and the other is vertical. Add this information to your resource sheet.

d) horizontal compression by a factor of 3 and a translation down 1 unit

Since one transformation is vertical and the other is horizontal, the order doesn't matter here.

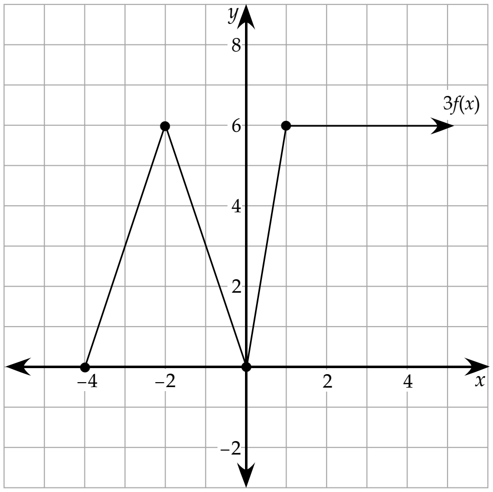


Algebraically:

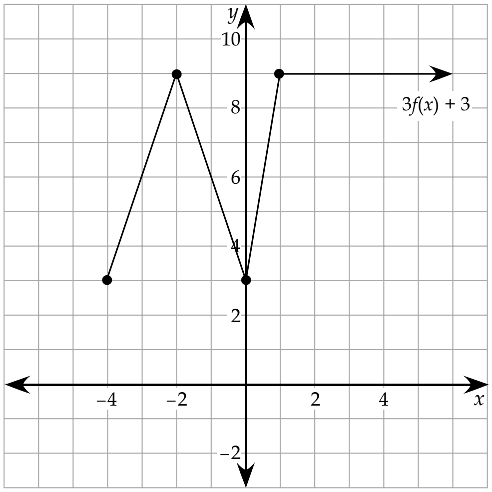
$$(x, y) \rightarrow \left(\frac{x}{3}, y - 1\right)$$

e) Does the order in which you perform the stretch and translation matter? To figure this out, try this question in two different ways. First, vertically stretch the graph by a factor of 3 and then shift the graph 3 units up.

Vertical stretch by a factor of 3

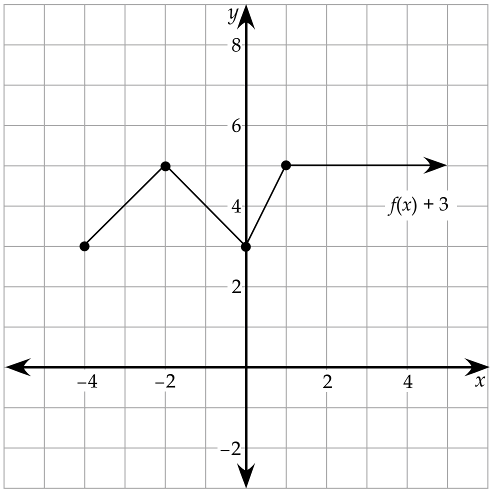


Vertical shift 3 units up



Second, vertically shift the graph 3 units up and then vertically stretch the graph by a factor of 3.

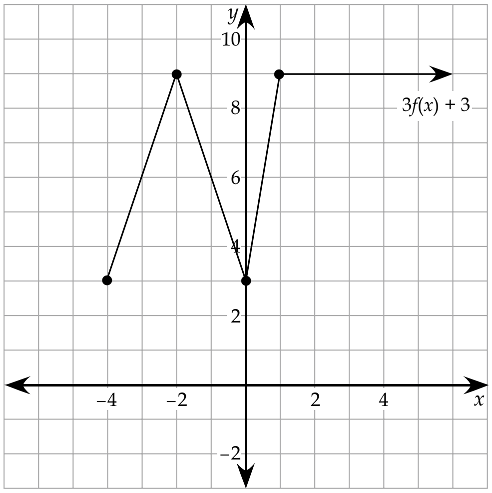
Vertical shift 3 units up



If you were to stretch this graph by a factor of 3, you would essentially by multiplying the function $f(x) + 3$ by 3. This would be the same as graphing $g(x) = 3f(x) + 9$ when you were asked to graph $g(x) = 3f(x) + 3$.

Therefore, when you have two transformations that are both vertical or both horizontal, the order in which you perform the transformations matters. Stretches and compressions need to be performed before any translations, according to the order of operations.

The correct graph of $3f(x) + 3$ is:

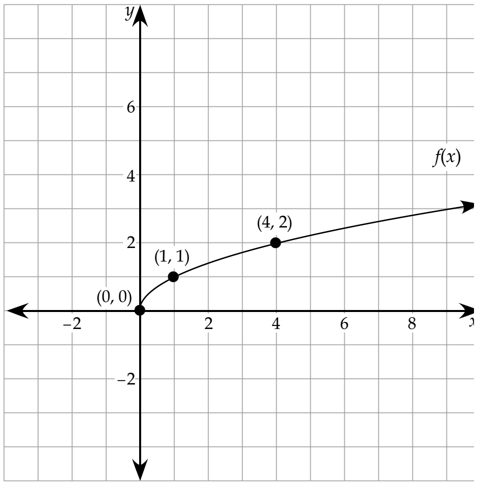


Algebraically: $(x, y) \rightarrow (x, 3y + 3)$

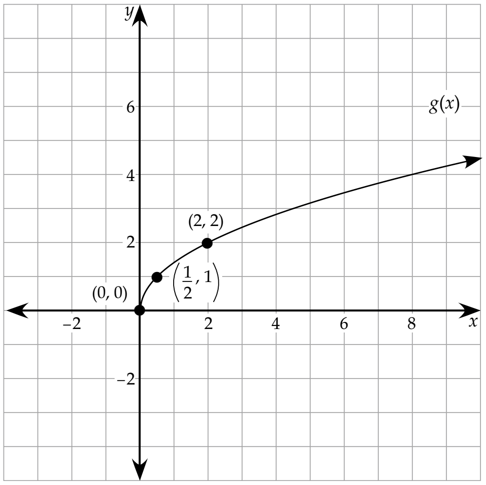
The vertical stretch is graphed before the vertical shift.

Example 5

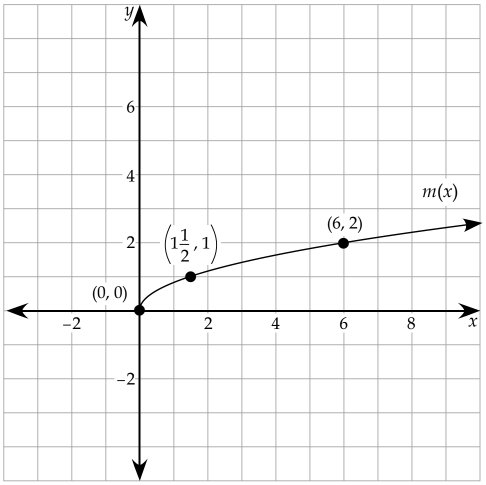
Consider the graph of $f(x)$ shown here. For each of the following graphs, consider the transformation and write the equation of the new function in terms of $f(x)$.



a)



b)



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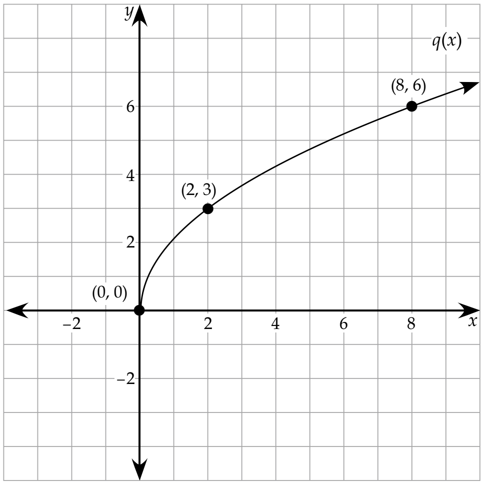


Solutions

a) The x-values of $g(x)$ are half the values of the x-values in $f(x)$. This is a horizontal stretch by $\frac{1}{2}$. This results in a horizontal compression by a factor of 2.

$$g(x) = f(2x)$$

b) The x-values of $m(x)$ are $1\frac{1}{2}$ times as large as the x-values in $f(x)$. This is a horizontal stretch by $\frac{3}{2}$. This results in a horizontal compression by a factor of $\frac{2}{3}$.

$$m(x) = f\left(\frac{2}{3}x\right)$$

c) This function has been stretched vertically as well as horizontally. The x-values of $q(x)$ are twice as large as the x-values in $f(x)$. This is a horizontal stretch by a factor of 2 (or a compression by $\left.\frac{1}{2}\right)$.

Also, the y-values of $q(x)$ are three times as large as the y-values in $f(x)$. This is a vertical stretch by a factor of 3.

$$q(x) = 3f\left(\frac{1}{2}x\right)$$

The following learning activity contains many questions where you can practice graphing stretches and compressions. Make sure you complete this learning activity and that you understand the questions and their answers before you continue on to Assignment 2.1.

Learning Activity 2.2

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. If $2x + 14 = -6$, what is the value of x?
2. Simplify: $\sqrt[3]{\dfrac{x^6}{27y^4}}$
3. Simplify: $\dfrac{1}{2} + \dfrac{6}{7}$
4. Evaluate: $|\,8.43 - 9.25\,|$
5. Solve for x: $(x + 1)^2 = 16$
6. Factor: $2x^2 - 18x - 72$
7. Convert $\dfrac{451}{10\ 000}$ into a decimal.
8. Your restaurant bill came to \$74.23. If you wish to leave a 20% tip, how much should you leave?

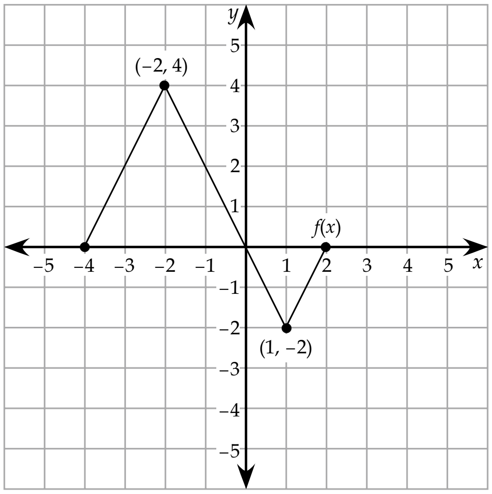
continued

Part B: Stretches and Compressions

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Compare and contrast the graph of $y = x^3$ with the graph of each of the following. Name one property that is the same and one that is different for the two graphs.
   a) $y = 5x^3$
   b) $y = \frac{1}{3}x^3$

2. Use the graph of $f(x)$, shown below, to sketch the following.
   a) $y = 2f(x)$
   b) $y = f(2x)$
   c) $y = \frac{2}{3}f(x)$
   d) $y = f\left(\frac{1}{2}x\right)$
   e) $y = 2f(3x)$



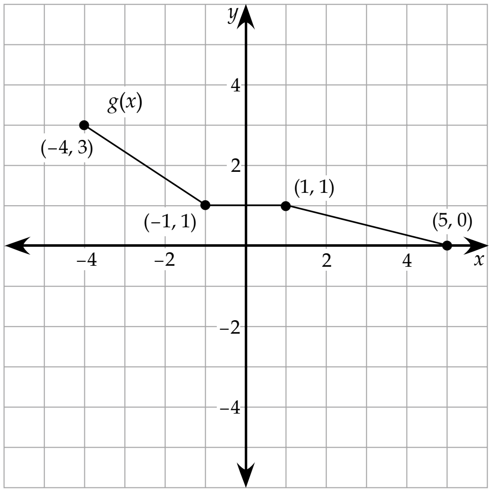
3. The x-intercepts of the function $f(x)$ are 6, 2, and −8. What are the x-intercepts of each of the following?
   a) $y = f(2x)$
   b) $y = f\left(\frac{1}{3}x\right)$
   c) $y = f(2x - 4)$

continued

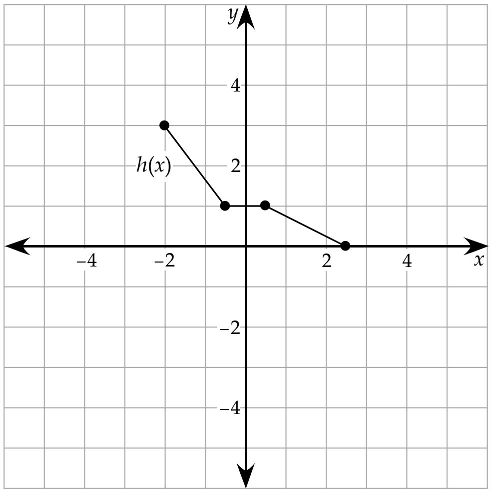
Learning Activity 2.2 (continued)

4. How should the function equation for $f(x)$ be modified if you want to perform the following transformation?
   a) translate the graph two units to the right
   b) compress the graph horizontally by a factor of 5
   c) stretch the graph vertically by a factor of two and translate it three units down
   d) stretch the graph horizontally by a factor of 3 and compress the graph vertically by a factor of 5

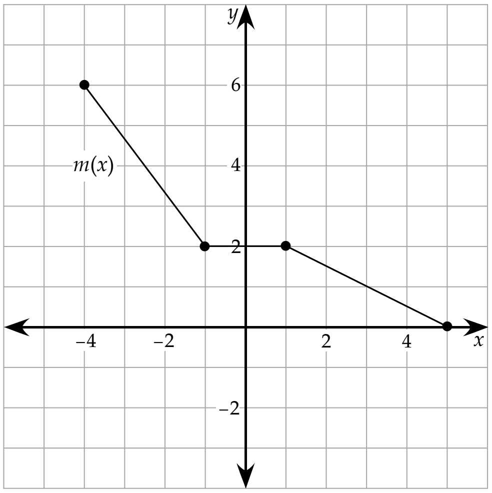
5. Each of graphs (a), (b), and (c) represents a stretch and/or a compression of the given function, $g(x)$, shown below. Write an expression for each new function in terms of $g(x)$.



a)



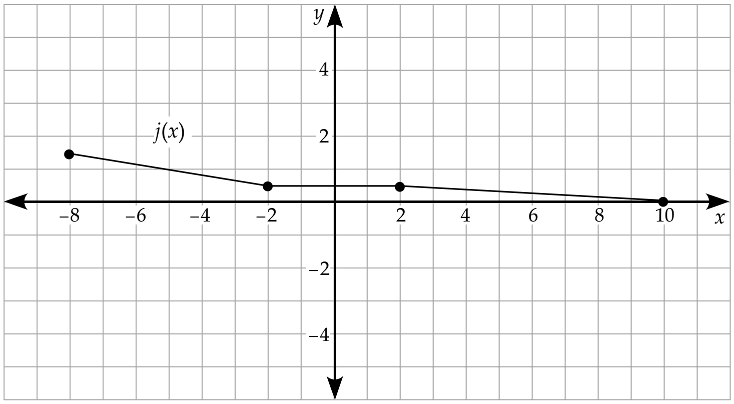
b)



continued

Learning Activity 2.2 (continued)

c)



Lesson Summary

In this lesson, you learned how to complete vertical and horizontal stretches and compressions of functions. You learned that vertical stretches (or compressions) occur when the multiplier is outside the function. You also learned that horizontal compressions (or stretches) occur when the multiplier is inside the function. In the next lesson, you will learn how to combine all the transformations you have learned so far.

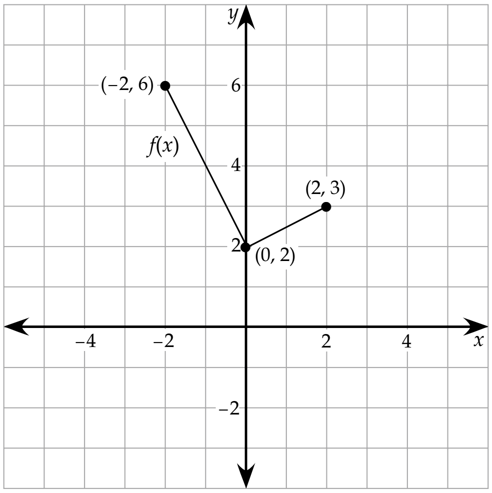
Assignment 2.1

Transformations of Functions

Total: 28 marks

You need to show all your steps as you work out your solutions, and make sure your answers are clear (include units, where appropriate). Check the Introduction for the Marking Guidelines that will be used for all assignments and examinations. **Note:** Be sure to label your scale and key points in your transformations.

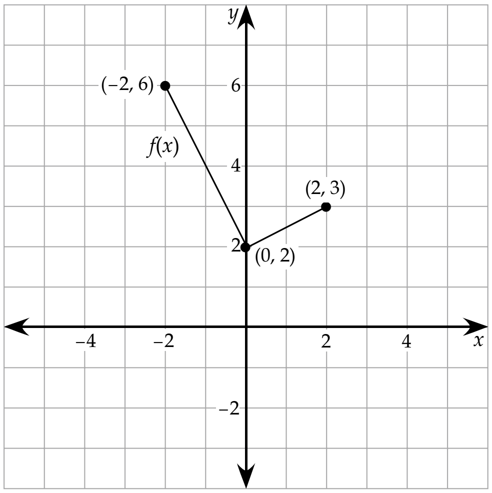
1. Given the sketch of $f(x)$ shown below, sketch each of the following functions. State the domain and range of each function.



a) $y = f(x) - 2$ *(2 marks)*

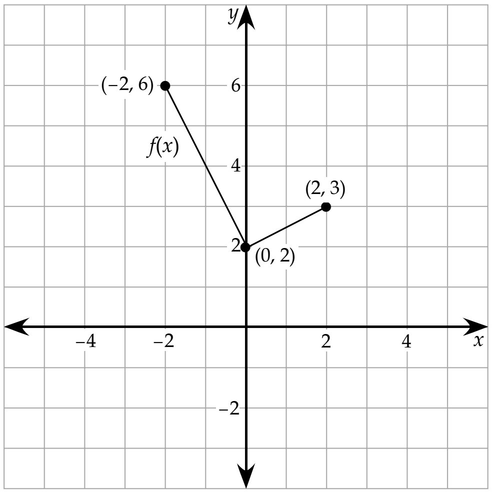
continued

Assignment 2.1: Transformations of Functions (continued)

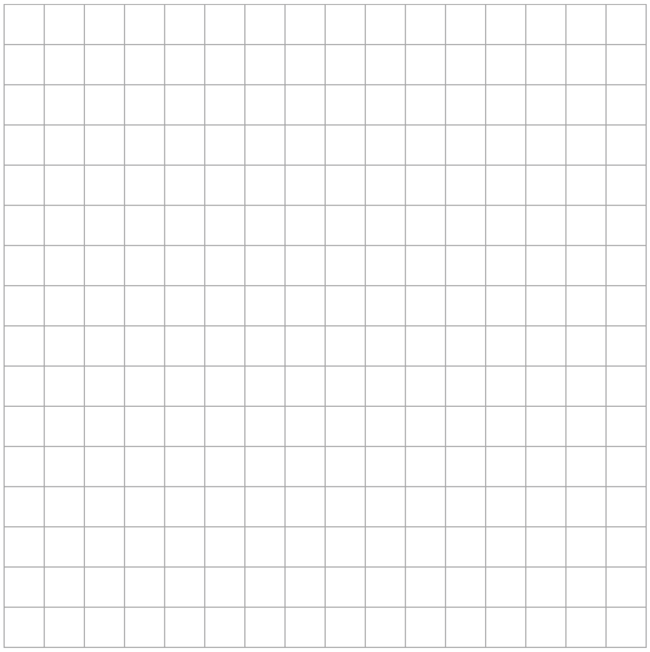


b) $y = f(x - 3)$ *(2 marks)*

continued



c) $y = f(x + 2) + 1$ *(3 marks)*



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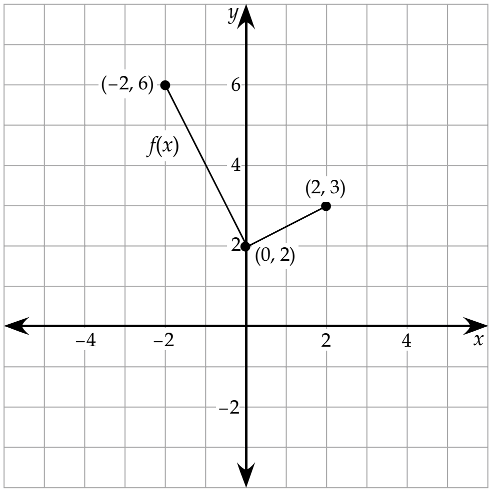
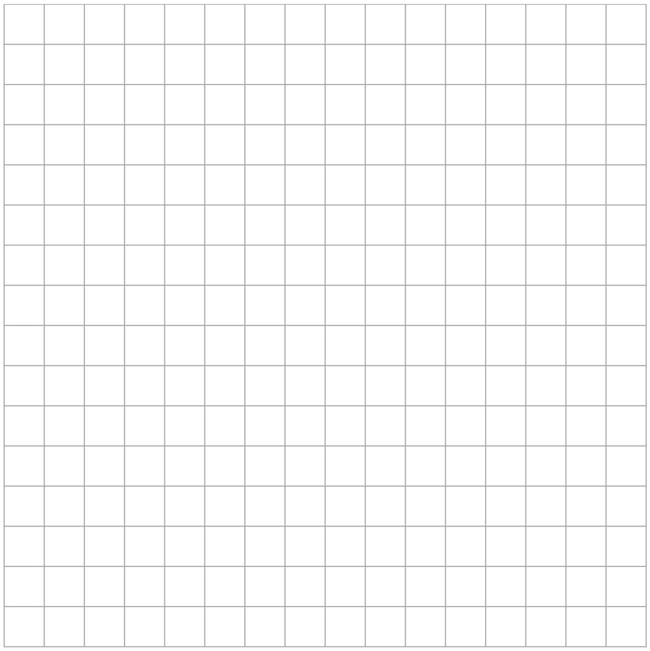


d) $y = f\left(\frac{1}{3}x\right)$ *(2 marks)*



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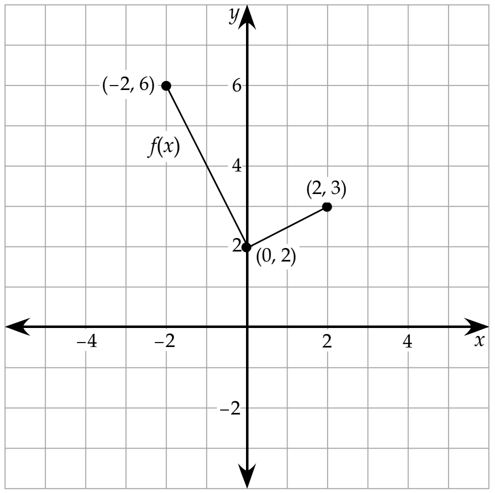
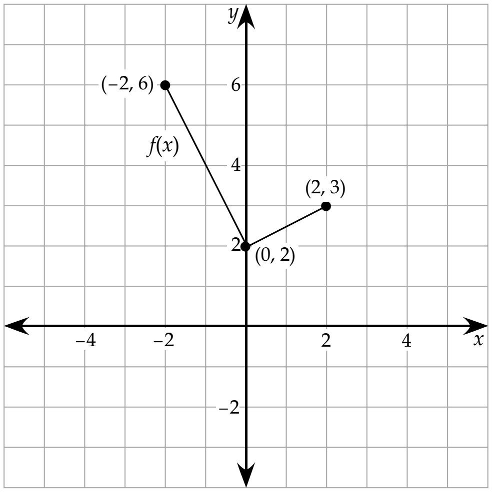
Assignment 2.1: Transformations of Functions (continued)

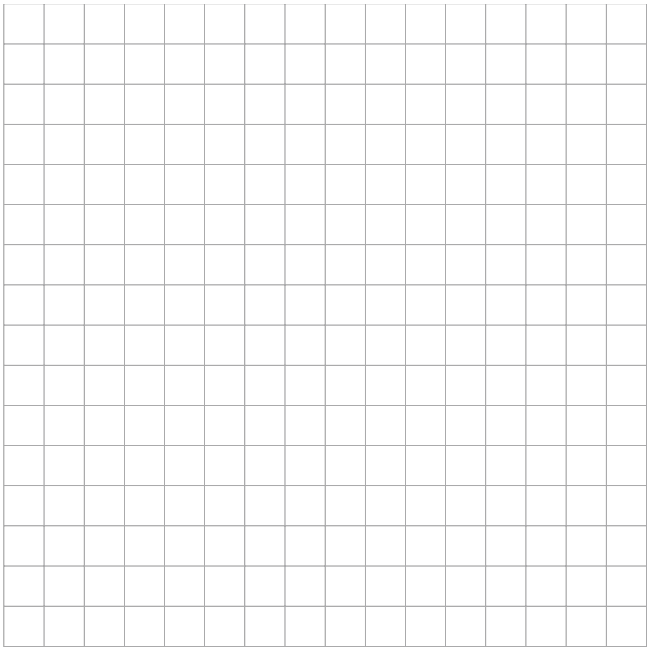


e) $y = \frac{1}{2} f(x)$ *(2 marks)*

continued



f) $y = \frac{1}{3} f(2x)$ *(3 marks)*



continued

Assignment 2.1: Transformations of Functions (continued)

2. Let $f(x) = x^3$. Sketch each of the following functions.

   a) $f(x)$ *(1 mark)*

   b) $y = f(x - 2) - 5$ *(2 marks)*

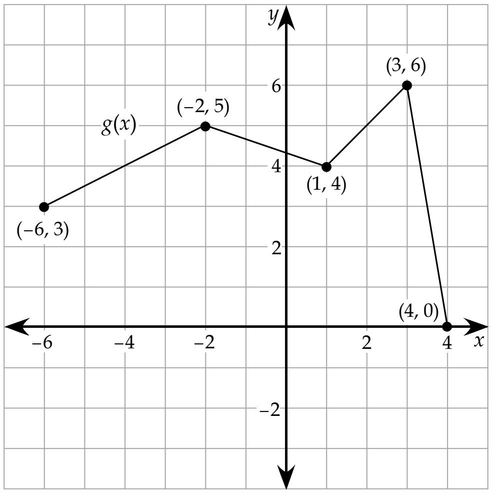
continued

Assignment 2.1: Transformations of Functions (continued)

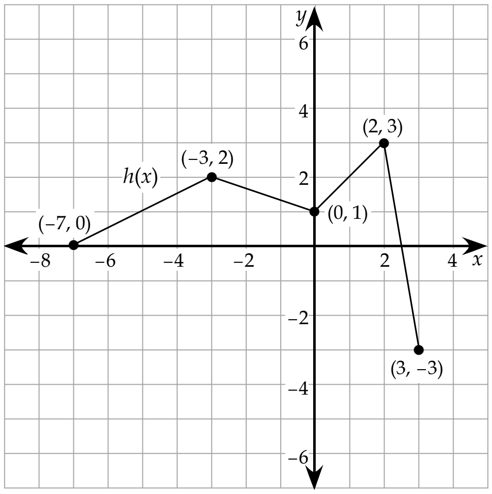
c) $y = 3f\left(\frac{1}{2}x\right)$ *(2 marks)*

continued

3. Each of graphs below represents a translation, stretch, and/or compression of the given function, $g(x)$, shown below. Write an equation for each new function in terms of $g(x)$ using correct notation.

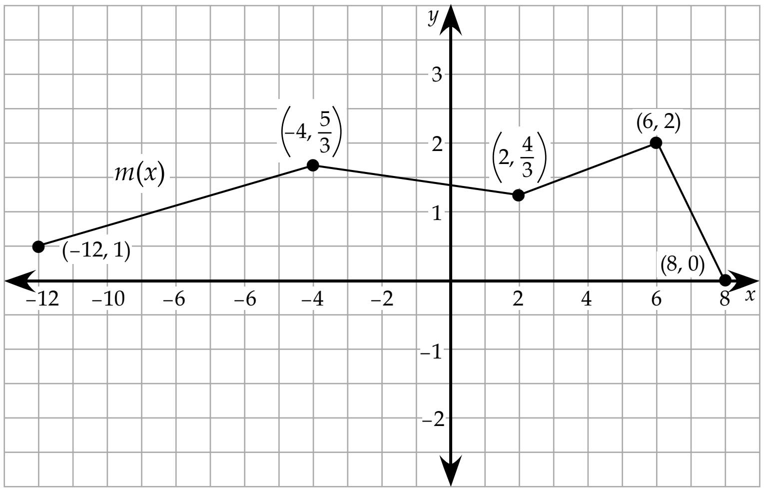


a) *(1 mark)*



continued

b) *(2 marks)*



4. How should the function equation for $f(x)$ be modified if you want to perform each of the following transformations? *(3 × 2 marks each = 6 marks)*

a) translate the graph two units to the right and three units up

b) compress the graph vertically by a factor of 5 and horizontally by a factor of 3

c) stretch the graph vertically by a factor of four and translate it three units to the left

LESSON 3: COMBINATIONS OF TRANSFORMATIONS

Lesson Focus

In this lesson, you will

- ☐ learn how to write a function as a combination of transformations of simpler functions
- ☐ learn how to draw a function by building on the simpler functions that make up the function
- ☐ learn how to state properties of a function

Lesson Introduction

Many complex functions are created by the combination of simpler functions. Therefore, if you know how to graph the simpler functions, with a few transformations, you can arrive at the graph of the more complex function. Graphing functions in this way is generally easier and does not require you to graph in other more time-consuming ways, such as creating a table of values.

Combining Transformations

Consider the equation below. It can be applied to any function, f.

$$y = af(b(x - h)) + k$$

An easy way of understanding how the variables transform the basic graph of $y = f(x)$ is displayed in the graphic below.

"b" and "h" are inside the function next to the x. Therefore, they are horizontal transformations that affect the x-values (but in the opposite way implied by the operations).

$$y = af\left(b\left(x - h\right)\right) + k$$

"a" and "k" are outside the function. Therefore, they are vertical transformations that affect the y-values (and in the same way implied by the operations).

The main ideas you need to remember are:

- These transformations affect all functions in exactly the same way (quadratic, cubic, absolute value, trigonometric, etc.)
- The coefficient or constant outside the function (next to the $f(x)$)
  - Affects the y-values
  - The operation on y-values is the same as the operation implied by "a" and "k"
- The coefficient or constant inside the function (next to the x)
  - Affects the x-values
  - The operation on x-values is the opposite operation implied by "b" and "h"

It would be a good idea to add the illustration and/or the main ideas to your resource sheet.

These main ideas state that values outside the function affect the y-values in the way you would expect. For example, $f(x) - 2$ is a translation *down* two units; **subtract** 2 from the y-values.

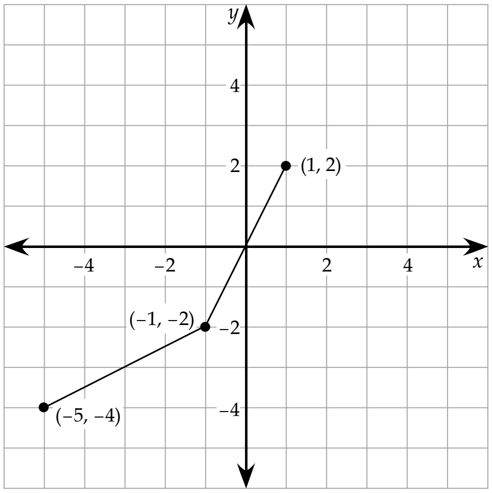
Also, the values inside the function affect the x-values in the opposite way you would expect. For example, $f\left(\frac{1}{2}x\right)$ is a horizontal compression; **divide** the x-values by $\frac{1}{2}$ (or multiply by 2) rather than multiply by $\frac{1}{2}$.

Order of Transformations

The idea of a function as a combination of simpler functions is a useful tool for graphing. However, how do you know which transformation to perform first? In Lesson 2, you learned that the vertical transformations do not influence the horizontal transformations, so they can be done in any order. Does it matter in which order translations and stretches/compressions are performed if they are both horizontal or both vertical transformations? Consider the following example.

Example 1

Use the graph of $f(x)$, shown below, to sketch $y = f(2x + 6)$ by performing the transformations in two different ways.



Solution

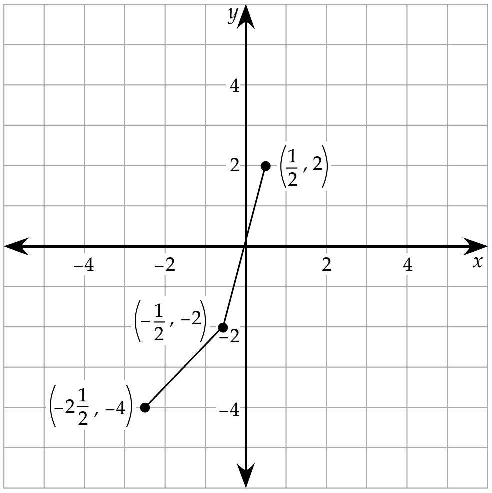
Before you graph, you need to rewrite $y = f(2x + 6)$ in the form $y = af(b(x - h)) + k$, as it makes it easier for you to identify the transformations.

$$f(2x + 6) = f(2(x + 3))$$

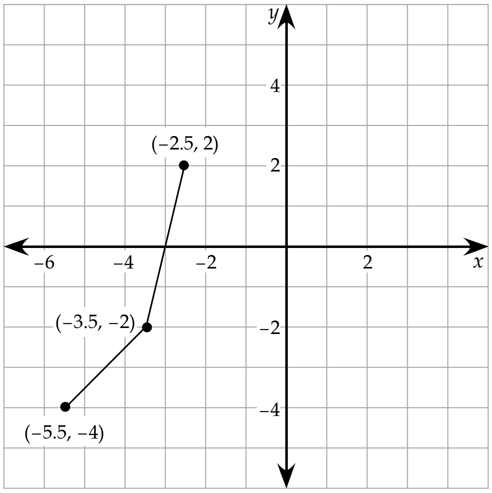
In this function, $b = 2$ and $h = -3$. Therefore, this graph is horizontally compressed by a factor of 2 and shifted 3 units to the left.

Graphing the Compression First:

Horizontally compressed by a factor of 2

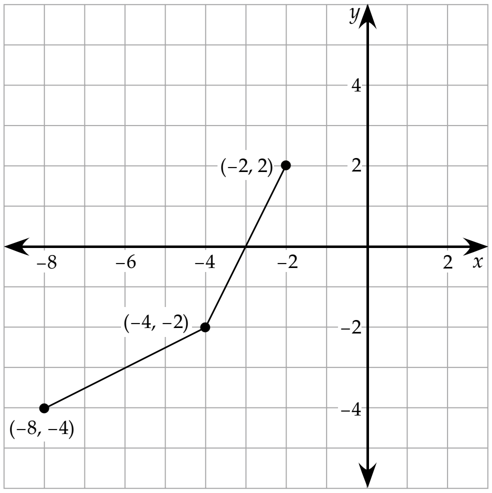


Horizontal shift 3 units to the left

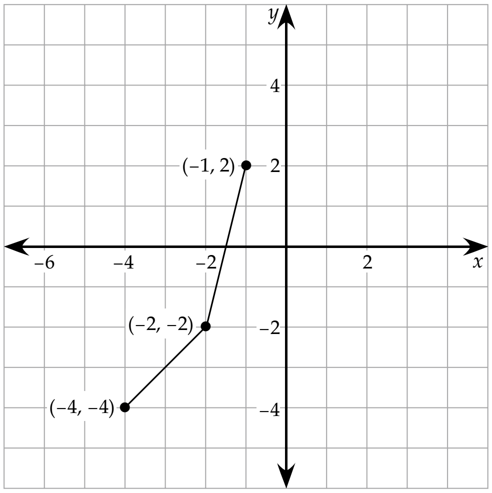


Graphing the Translation First:

Horizontal shift 3 units to the left



Horizontally compressed by a factor of 2



As you can see, the two final graphs are different. Therefore, the order in which you perform compressions/stretches and translations does matter. The correct way to perform transformations is to perform any compressions or stretches first, and then perform any translations. This is similar to the order of operations where multiplying and dividing come before adding and subtracting. You can account for all transformations and the importance of order if you do the work algebraically, using mapping notation.

Write down key points from the original graph.

$f(x)$ (x, y)
$(-5, -4)$
$(-1, -2)$
$(0, 0)$
$(1, 2)$

Determine the mapping notation of $f(2(x + 3))$.

$$(x, y) \rightarrow \left(\frac{x}{2} - 3, y\right)$$

Substitute original points into mapping notation. Note that if you simplify using order of operations, you will have found the new point using the correct order of transformation.

$$f\left(2(x+3)\right) \qquad \left(\frac{x}{2}-3,\ y\right)$$

$$(-5,\ -4) \rightarrow \left(\frac{-5}{2}-3,\ -4\right) \rightarrow (-5.5,\ -4)$$

$$(-1,\ -2) \rightarrow \left(\frac{-1}{2}-3,\ -2\right) \rightarrow (-3.5,\ -2)$$

$$(0,\ 0) \rightarrow \left(\frac{0}{2}-3,\ 0\right) \rightarrow (-3,\ 0)$$

$$(1,\ 2) \rightarrow \left(\frac{1}{2}-3,\ 0\right) \rightarrow (-2.5,\ 2)$$

You will see this more clearly in the next example.

If you haven't already, it would be a good idea for you to include the order in which you should perform transformations on your resource sheet.

Combining Transformations

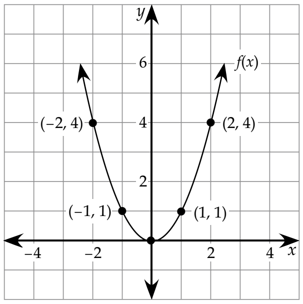
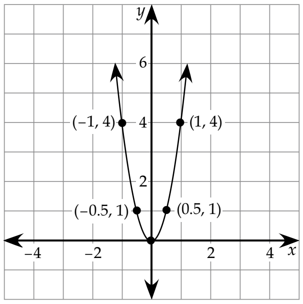
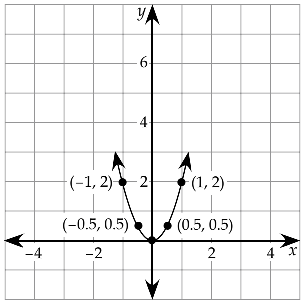
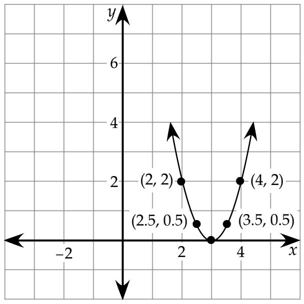
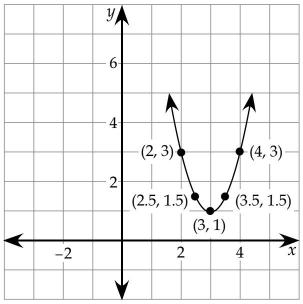
The function $f(x) = x^2$ can be transformed into the function $g(x) = \frac{1}{2}\left(2(x-3)\right)^2 + 1$ by graphing the function in a series of steps. You need to examine the operations on x given by the function equation of $g(x)$.

These steps in the table are shown in the order:

- horizontal compression
- vertical compression
- horizontal translation
- vertical translation

You can use the function notation to express the operations on the basic graph $y = x^2$, or the transformation notation to express the operations on the graph step-by-step.

Combining Transformations Chart

Step	Function	Algebraic Transformation	Description	Graph
1	$f(x) = x^2$	$(x, y) \to (x, x^2)$	The basic quadratic function, $y = x^2$.	
2	$h(x) = f(2x)$ $h(x) = (2x)^2$	$(x, y) \to \left(\frac{x}{2}, y\right)$	The x-values are halved. This results in a horizontal compression by a factor of 2.	
3	$k(x) = \frac{1}{2}(h(x))$ $k(x) = \frac{1}{2}(2x)^2$	$(x, y) \to \left(x, \frac{1}{2}y\right)$	The y-values are halved. This results in a vertical compression by a factor of 2.	
4	$p(x) = k(x - 3)$ $p(x) = \frac{1}{2}(2(x-3))^2$	$(x, y) \to (x + 3, y)$	The x-values are increased by 3. This results in a horizontal translation 3 units to the right.	
5	$g(x) = p(x) + 1$ $g(x) = \frac{1}{2}(2(x-3))^2 + 1$	$(x, y) \to (x, y + 1)$	The y-values are increased by 1. The result is the parabola shifted up 1 unit. The result is the required graph.	

The equation of this transformed function is $g(x) = \frac{1}{2}(2(x-3))^2 + 1$. If you graph this function on your graphing calculator, you will see that the transformations were completed correctly as you will get the same graph as in Step 5.

Example 2

State the transformations required on the basic quadratic $g(x) = x^2$ to obtain each function.

a) $s(x) = (x+2)^2$

b) $s(x) = \frac{1}{2}(3x)^2 + 2$

c) $s(x) = 2\left(\frac{1}{2}(x+1)\right)^2 - 5$

Solutions

a)

Similar Function	Algebraic Transformation	Explanation
$s(x) = g(x+2)$ $s(x) = (x+2)^2$	$(x, y) \to (x-2, y)$	The x-values are decreased by 2, the y-values remain unchanged. The result is the parabola shifted 2 units left.

b)

Similar Function	Algebraic Transformation	Explanation
$h(x) = g(3x)$ $h(x) = (3x)^2$	$(x, y) \to \left(\frac{1}{3}x, y\right)$	The x-values are divided by 3. The y-values remain the same. The result is a horizontal compression.
$j(x) = \frac{1}{2}(h(x))$ $j(x) = \frac{1}{2}(3x)^2$	$(x, y) \to \left(x, \frac{1}{2}y\right)$	The y-values are multiplied by $\frac{1}{2}$. The x-values remain the same. The result is a vertical compression.
$s(x) = j(x) + 2$ $s(x) = \frac{1}{2}(3x)^2 + 2$	$(x, y) \to (x, y+2)$	The y-values are increased by 2, the x-values remain unchanged. The result is the parabola shifted 2 units up.

c)

Similar Function	Algebraic Transformation	Explanation
$h(x) = g\left(\frac{1}{2}x\right)$ $h(x) = \left(\frac{1}{2}x\right)$	$(x, y) \to \left(x \div \frac{1}{2}, y\right)$	The x-values are multiplied by 2. The y-values remain the same. The result is a horizontal stretch.
$j(x) = 2(h(x))$ $j(x) = 2\left(\frac{1}{2}x\right)^2$	$(x, y) \to (x, 2y)$	The y-values are multiplied by 2. The x-values remain the same. The result is a vertical stretch.
$k(x) = j(x+1)$ $k(x) = 2\left(\frac{1}{2}(x+1)\right)^2$	$(x, y) \to (x-1, y)$	The x-values are decreased by 1, the y-values remain unchanged. The result is the parabola shifted 1 unit to the left.
$s(x) = k(x) - 5$ $s(x) = 2\left(\frac{1}{2}(x+1)\right)^2 - 5$	$(x, y) \to (x, y-5)$	The y-values are decreased by 5, the x-values remain unchanged. The result is the parabola shifted 5 units down.

Example 3

Sketch the function and state its domain and range. State the values of any intercepts that exist.

a) $y - 2 = 3(2(x+4))^2$

b) $y = 2\left|\frac{1}{3}(x-1)\right| - 5$

Solutions

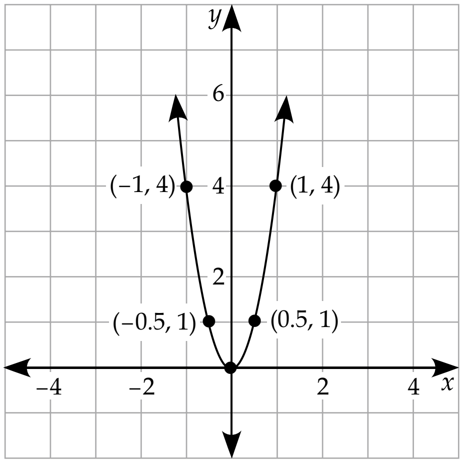
a) Remember, the order of the transformations matters. If both transformations are horizontal or vertical, stretches or compressions need to come before translations.

First, you need to rearrange the formula.

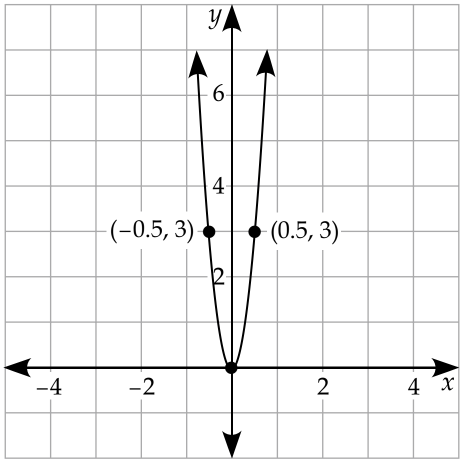
$$y = 3(2(x+4))^2 + 2$$

This quadratic graph is compressed horizontally by a factor of 2, stretched vertically by a factor of 3, moved 4 units to the left and moved 2 units up. Consider the following sequence of graphs.

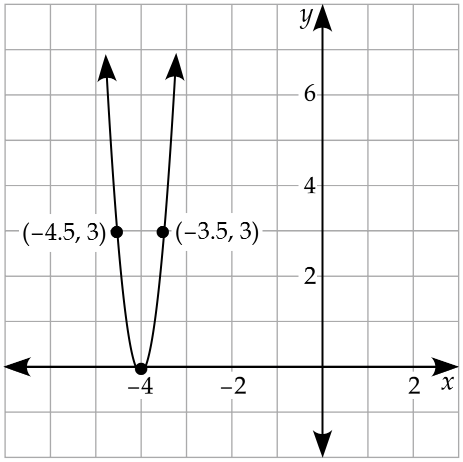
Graph of $(2x)^2$



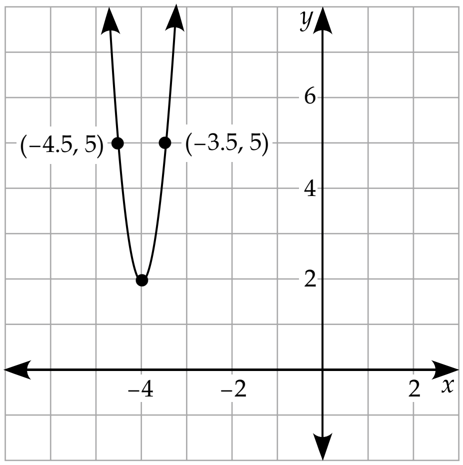
Graph of $3(2x)^2$



Graph of $3(2(x + 4))^2$



Graph of $3(2(x + 4))^2 + 2$



Domain	**Range**	***y*-intercept**	***x*-intercept**
$\{x \mid x \in \Re\}$	$\{y \mid y \geq 2\}$	194	none

Solving for the *y*-intercept:

$y = 3(2(0 + 4))^2 + 2$

$y = 3(8)^2 + 2$

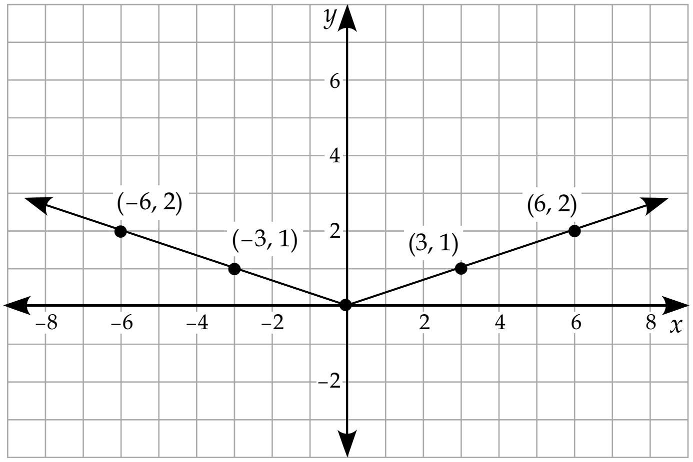
$y = 3(64) + 2$

$y = 194$

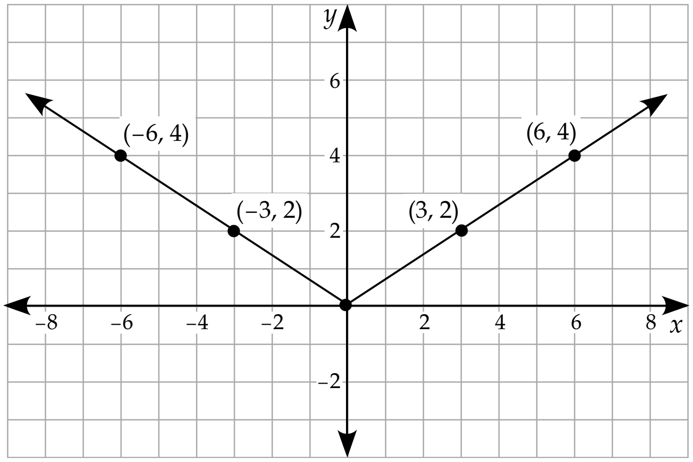
b) To complete this question, you need to know how to graph $y = | x |$. You should have this graph on your resource sheet already; if not, refer back to Lesson 1.

This graph is compressed horizontally by a factor of $\frac{1}{3}$, stretched vertically by a factor of 2, moved 1 unit to the right, and moved 5 units down. Consider the following sequence of graphs.

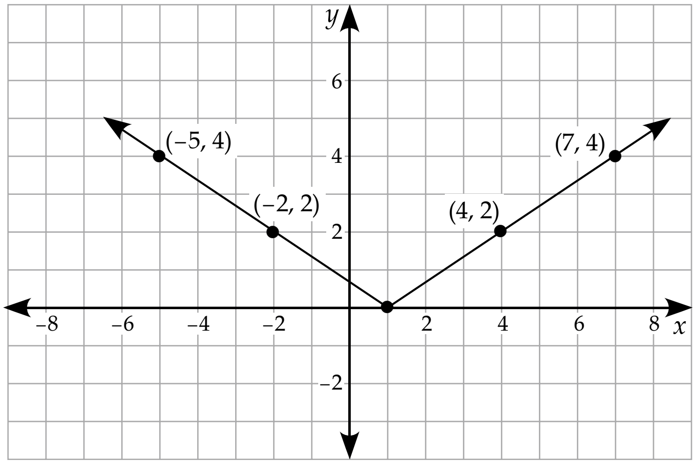
Graph of $y = \left|\frac{1}{3}x\right|$



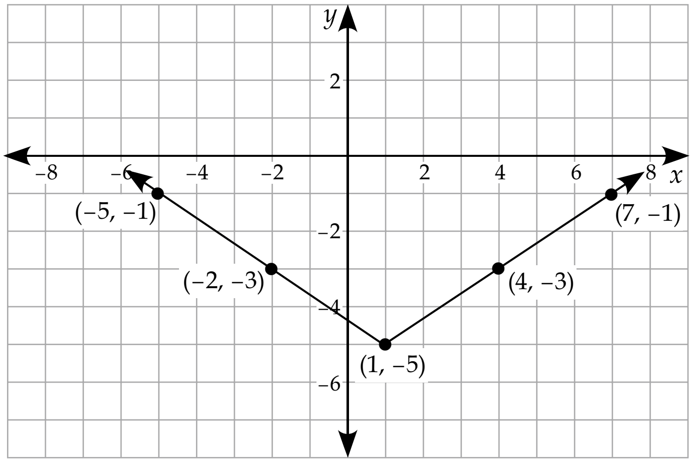
Graph of $y = 2\left|\frac{1}{3}x\right|$



Graph of $y = 2\left|\frac{1}{3}(x-1)\right|$



Graph of $y = 2\left|\frac{1}{3}(x-1)\right| - 5$



Domain	**Range**	***y*-intercept**	***x*-intercept**
$\{x \mid x \in \Re\}$	$\{y \mid y \geq -5\}$	$-\frac{13}{3}$	8.5 and −6.5

Solving for the *y*-intercept:

$$y = 2\left|\frac{1}{3}(0-1)\right| - 5$$

$$y = 2\left|-\frac{1}{3}\right| - 5$$

$$y = \frac{2}{3} - 5$$

$$y = -\frac{13}{3}$$

Solving for the *x*-intercepts:

Recall, the expression inside the absolute value brackets can be either positive or negative and the result will be the same.

$$0 = 2\left|\frac{1}{3}(x-1)\right| - 5$$

$$\frac{5}{2} = \frac{1}{3}(x-1)$$

$$\frac{5}{2}(3) = x - 1$$

$$\frac{15}{2} + 1 = x$$

$$x = \frac{17}{2} = 8.5$$

and

$$0 = 2\left|\frac{1}{3}(x-1)\right| - 5$$

$$\frac{5}{2} = -\frac{1}{3}(x-1)$$

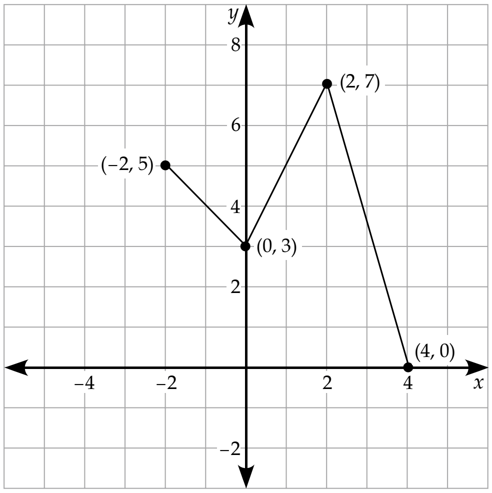
$$\frac{5}{2}(3) = -(x-1)$$

$$\frac{15}{2} = -x + 1$$

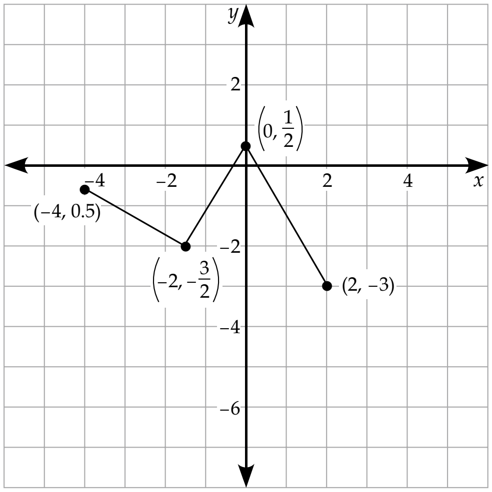
$$x = -\frac{13}{2} = -6.5$$

Example 4

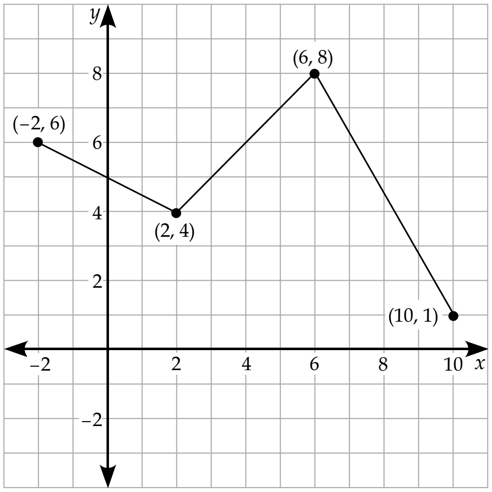
Each of the graphs below represents a transformation of $f(x)$. Write an expression for each new function in terms of $f(x)$.



a)



b)



Solutions

a) Locate key points on the graph of $f(x)$ and their corresponding points on the graph of $g(x)$.

$f(x)$	$g(x)$
(−2, 5)	(−4, −0.5)
(0, 3)	$\left(-2, -\frac{3}{2}\right)$
(2, 7)	$\left(0, \frac{1}{2}\right)$
(4, 0)	(2, −3)

As you can see from the table, all of the x-values of $g(x)$ are 2 less than the x-values of $f(x)$. Therefore, this graph has been shifted two units to the left. No horizontal compression or stretch has occurred.

Now, there is no obvious pattern between the y-coordinates. However, if you look at the x-intercept, (4, 0), you will notice that this point moves 3 units down. Therefore, as a vertical stretch or compression does not affect the x-intercept, there must have been a vertical translation 3 units down.

Note: Vertical stretches or compressions do not affect the x-intercepts because vertical stretches and compressions affect the y-coordinates. As the y-coordinate of the x-intercept will always be 0, this point will not change if you multiply 0 by any value.

It may be helpful for you to now create a table of values of what the points of $g(x)$ would have looked like before the translation of 3 units down.

$f(x)$	$g(x)$
$(-2, 5)$	$\left(-4, \frac{5}{2}\right)$
$(0, 3)$	$\left(-2, \frac{3}{2}\right)$
$(2, 7)$	$\left(0, \frac{7}{2}\right)$
$(4, 0)$	$(2, 0)$

From the chart, you can see that each of the y-values were halved. Therefore, this graph has been vertically stretched by a factor of $\frac{1}{2}$.

Putting all the transformations together, you get:

$$g(x) = \frac{1}{2}(x + 2) - 3$$

b) Locate key points on the graph of $f(x)$ and their corresponding points on the graph of $h(x)$.

$f(x)$	$g(x)$
(−2, 5)	(−2, 6)
(0, 3)	(2, 4)
(2, 7)	(6, 8)
(4, 0)	(10, 1)

It is easiest in this situation to first compare the y-values. Each of the y-values of $h(x)$ is 1 greater than the corresponding y-value of $f(x)$. This results in a vertical shift of 1 unit up. No vertical stretch or compression has occurred.

There is no obvious pattern between the x-coordinates. Therefore, it is easiest to look at the y-intercept (0, 3) and compare it to its corresponding point on $h(x)$.

A horizontal stretch or compression will not affect the y-intercept. The only transformation, in regards to the x-values, that will transform the y-intercept is a translation. The x-value in this coordinate in $h(x)$ is 2 greater than the x-value in the corresponding coordinate on $f(x)$. This results in a horizontal translation 2 units to the right.

To figure out what horizontal stretch or compression may have been applied, it may be helpful for you to now create a table of values of what the points of $h(x)$ would have looked like before the translation of 2 units to the right.

$f(x)$	$g(x)$
(−2, 5)	(−4, 6)
(0, 3)	(0, 4)
(2, 7)	(4, 8)
(4, 0)	(8, 1)

From the above table, you will notice that each of the x-values in $h(x)$ are double the x-values in the corresponding coordinates of $f(x)$. Therefore, there has been a horizontal stretch by a factor of 2 or a horizontal compression by a factor of $\frac{1}{2}$.

Combining all of the transformations together, the expression for $h(x)$ in terms of $f(x)$ is:

$$h(x) = f\left(\frac{1}{2}(x-2)\right) + 1$$

Summary of Transformations			
Name	**Transformation**	**Effect on (x, y)**	**Notes**
vertical translation	$f(x) \rightarrow f(x) + k$	$(x, y + k)$	shift up k units if $k > 0$; shift down k units if $k < 0$
horizontal translation	$f(x) \rightarrow f(x - h)$	$(x + h, y)$	shift h units right if $h > 0$; shift h units left if $h < 0$
vertical stretch	$f(x) \rightarrow af(x)$	(x, ay)	vertical stretch if $a > 1$; vertical compression if $0 < a < 1$
horizontal compression	$f(x) \rightarrow f(bx)$	$\left(\frac{1}{b}x, y\right)$	horizontal stretch if $0 < b < 1$; horizontal compression if $b > 1$

Make sure you have all of the information in the above chart on your resource sheet.

Learning Activity 2.3

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. What is the surface area of a cylinder with a diameter of 6 cm and a height of 4 cm?
2. Is $x = 2$ a solution to the inequality $-x^2 + 10x - 8 \geq 0$?
3. Simplify: $\dfrac{49x^7}{7x^3}$
4. Estimate the taxes, 13%, on a $226 item.
5. If $f(x) = \dfrac{x^2}{x - 1}$, evaluate $f(-2)$.

continued

Learning Activity 2.3 (continued)

6. Convert 0.024 into a percent.

7. Express $11 + \frac{4}{7}$ as an improper fraction.

8. Solve for x: $\frac{1}{x+2} = \frac{2}{x-1}$

Part B: Combining Transformations

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

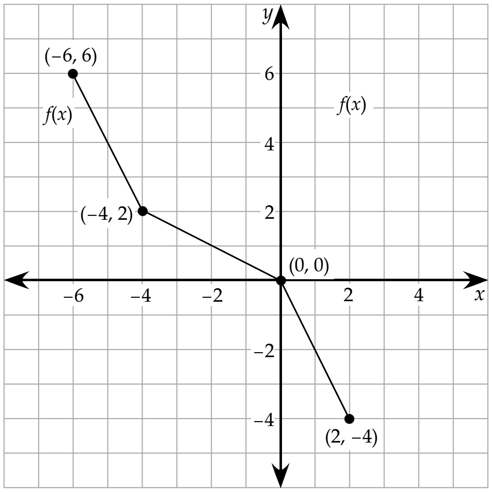
1. Assume a graph of $f(x)$ is given. Describe how the graph of $F(x)$ is obtained (using the correct order of transformations when needed).
   a) $F(x) = \frac{1}{4}(4x)$
   b) $F(x) = 2f(x) + 1$
   c) $F(x) = 4f(x+1)$
   d) $F(x) = 3f(x-2)$
   e) $F(x) = f(2x) - 5$
   f) $F(x) = f\left(\frac{1}{3}(x-1)\right) + 4$
   g) $F(x) = \frac{1}{2}\left(f(x-2)\right) - 3$
   h) $F(x) = 2\left(f\left(\frac{1}{3}(x-5)\right)\right) + 7$

2. Given the function $y = f(x)$, write the equation of the form $y = af(b(x - h)) + k$ that would result from each combination of transformations.
   a) Vertical shift 5 units up, horizontal shift 2 units to the right, horizontal compression by a factor of 3
   b) Horizontal stretch by a factor of 4, vertical compression by a factor of 2, vertical shift 4 units down
   c) Horizontal compression by a factor of 6, vertical stretch by a factor of 3, vertical shift 8 units up, horizontal shift 7 units to the left

continued

3. The point (18, −6) is on the graph of $y = f(x)$. Determine its corresponding point after each of the following transformations of $f(x)$.
   a) $y = 2f(x - 1) + 3$
   b) $y = \frac{1}{3}f(2(x - 1)) + 4$
   c) $y = \frac{1}{2}f\left(\frac{1}{3}(x + 2)\right) - 6$

4. Use the graph of $f(x)$ below to graph each of the following transformations.



   a) $g(x) = f\left(\frac{1}{2}(x - 3)\right) + 2$
   b) $h(x) = \frac{1}{2}f(4(x + 1)) - 3$

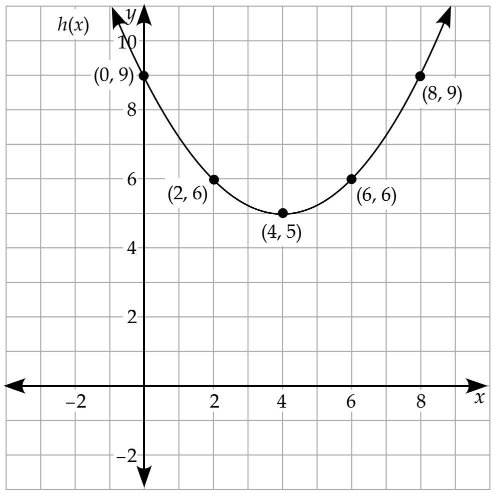
5. Sketch each function and state the domain, range, y-intercepts, and zeros of the function.
   a) $f(x) = 2(x - 2)^3 - 1$
   b) $f(x) = \left(\frac{1}{3}(x + 2)\right)^2 + 3$
   c) $f(x) = \left|\frac{1}{2}(x - 1)\right| - 4$
   d) $g(x) = 3(x + 2)^3 + 1$

continued

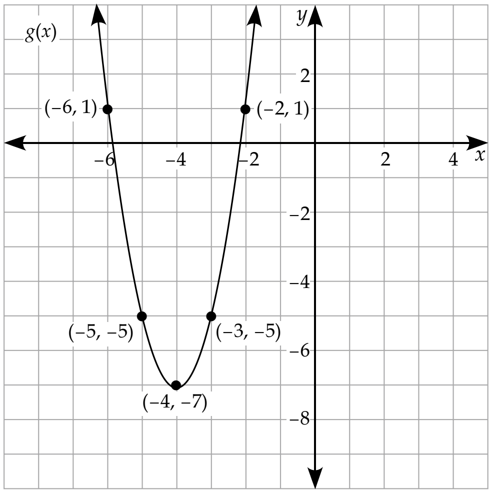
Learning Activity 2.3 (continued)

6. Each of the graphs below represents a transformation of $f(x) = x^2$. Write an expression for each new function in terms of $f(x)$.

a)



b)



Lesson Summary

In this lesson, you learned how to graph multiple transformations at one time. By analyzing the transformations, you were able to graph variations on any function including the cubic function, the quadratic function, as well as the absolute value function. In Module 4, you will learn more about graphing transformations of the square root function.

In the next lesson, you will learn about operations on functions.

Notes

Assignment 2.2

Combinations of Transformations

Total: 20 marks

You need to show all your steps as you work out your solutions, and make sure your answers are clear (include units, where appropriate). Check the Introduction for the Marking Guidelines that will be used for all assignments and examinations.

1. Assume a graph of $f(x)$ is given. Describe how the graph of $F(x)$ is obtained, using the correct order of transformations. *(2 × 2 marks each = 4 marks)*

   a) $F(x) = 3\left(f\left(\frac{1}{3}(x-1)\right)\right) + 7$

   b) $F(x) = 7\left(f\left(3(x+2)\right)\right) - 5$

2. Given the function $y = f(x)$, write the equation of the form $y = af(b(x - h)) + k$ that would result from each combination of transformations. *(2 × 2 marks each = 4 marks)*

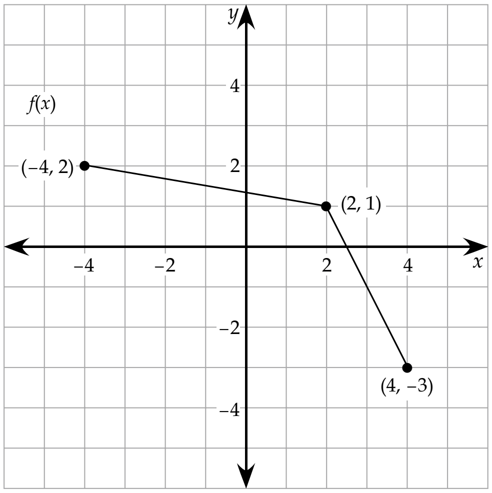
   a) Horizontal translation 3 units to the left, vertical translation 4 units down, horizontal compression by a factor of 2, and a vertical stretch by a factor of $\frac{1}{4}$

   b) Vertical compression by a factor of 6, horizontal stretch by a factor of 2, vertical translation 2 units up, and a horizontal translation 7 units to the right

continued

Assignment 2.2: Combinations of Transformations (continued)

3. Given the graph of $f(x)$ below, show each transformation algebraically and graphically.



a) $g(x) = f\big(2(x+1)\big) - 5$ *(3 marks)*

Graphically **Algebraically**

continued

Assignment 2.2: Combinations of Transformations (continued)

b) $h(x) = 2f\left(\frac{1}{2}(x-3)\right) + 4$ *(3 marks)*

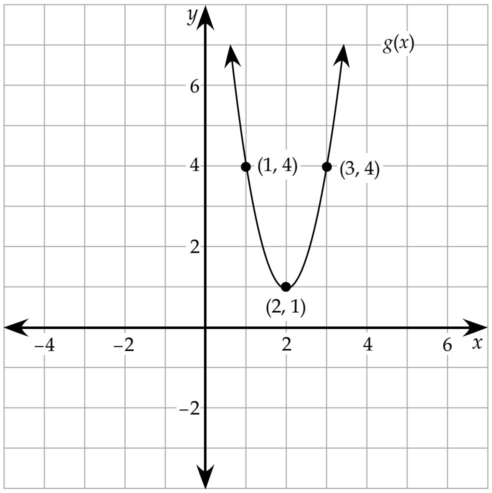
Graphically **Algebraically**

4. Sketch the function $f(x) = 2\left(\frac{1}{2}(x+3)\right)^2 - 5$ and state the domain and range of the function. *(4 marks)*

continued

Assignment 2.2: Combinations of Transformations (continued)

5. The following graph represents a transformation of $f(x) = x^2$. Write an equation for the new function in terms of $f(x)$ using function notation. *(2 marks)*



LESSON 4: OPERATIONS ON FUNCTIONS

Lesson Focus

In this lesson, you will

- ☐ learn how to write the equation of a function that is the sum, difference, product, or quotient of two functions
- ☐ learn how to sketch the graph of a function that is the sum, difference, product, or quotient of two functions
- ☐ learn how to determine the domain and range of a function that is the sum, difference, product, or quotient of two functions

Lesson Introduction

Relationships between two variables can be complex. Sometimes one function is formed from two smaller functions. Two smaller functions can be added, subtracted, multiplied, or divided to create a new function. These operations on functions can be used to model real-world situations, such as wave motion, revenue, and the world's population.

Functions

A function is a relationship between two quantities where the value of the independent variable determines the value of the dependent variable. The **domain** is the set of possible values of the independent variable. The set of all possible values of the dependent variable is called the **range**.

Evaluating a Function

A function can be thought of as a calculating machine where you input a value (domain) and the machine turns it into an output value (range), according to the operation defined in the function.

This process is called **evaluating a function**.

Example 1

If $f(x) = x^2 - 3x$, find $f(-1)$, $f(a)$, $f(x + 2)$.

Solution

$f(x) = x^2 - 3x$

$f(-1) = (-1)^2 - 3(-1) = 1 + 3 = 4$

$f(a) = a^2 - 3a$

$f(x + 2) = (x + 2)^2 - 3(x + 2) = x^2 + 4x + 4 - 3x - 6 = x^2 + x - 2$

Just as two real numbers can be combined by the operations of addition, subtraction, multiplication, and division to form other real numbers, two functions can be combined to create new functions.

Operations on Functions

For example, if $f(x) = 2x - 3$ and $g(x) = -x + 2$, you can form new functions by finding the sum, difference, product, and quotient of f and g in the following manner:

$f(x) + g(x) = (2x - 3) + (-x + 2) = 2x - x - x + 2 = x - 1$ Sum

$f(x) - g(x) = (2x - 3) - (-x + 2) = 2x - x + x - 2 = 3x - 5$ Difference

$f(x) \cdot g(x) = (2x - 3)(-x + 2) = -2x^2 + 4x + 3x - 6 = -2x^2 + 7x - 6$ Product

$\dfrac{f(x)}{g(x)} = \dfrac{2x - 3}{-x + 2}$ Quotient

The domain of an arithmetic combination of f and g consists of all real numbers that are common to the domain of f and g. There may be other restrictions. In the case of the quotient of $\dfrac{f(x)}{g(x)}$, , there is a restriction that $g(x) \neq 0$. This is because it is not possible to divide by zero. As such, the domain of $\dfrac{f(x)}{g(x)}$ is restricted to x-values that do not make $g(x)$ equal to zero.

Example 2

Consider the functions $f(x) = x^2 - 4x + 3$ and $g(x) = x - 8$

a) Determine an expression for $(f - g)(x)$.

b) Sketch $f(x)$, $g(x)$, and $(f - g)(x)$ on the same coordinate plane.

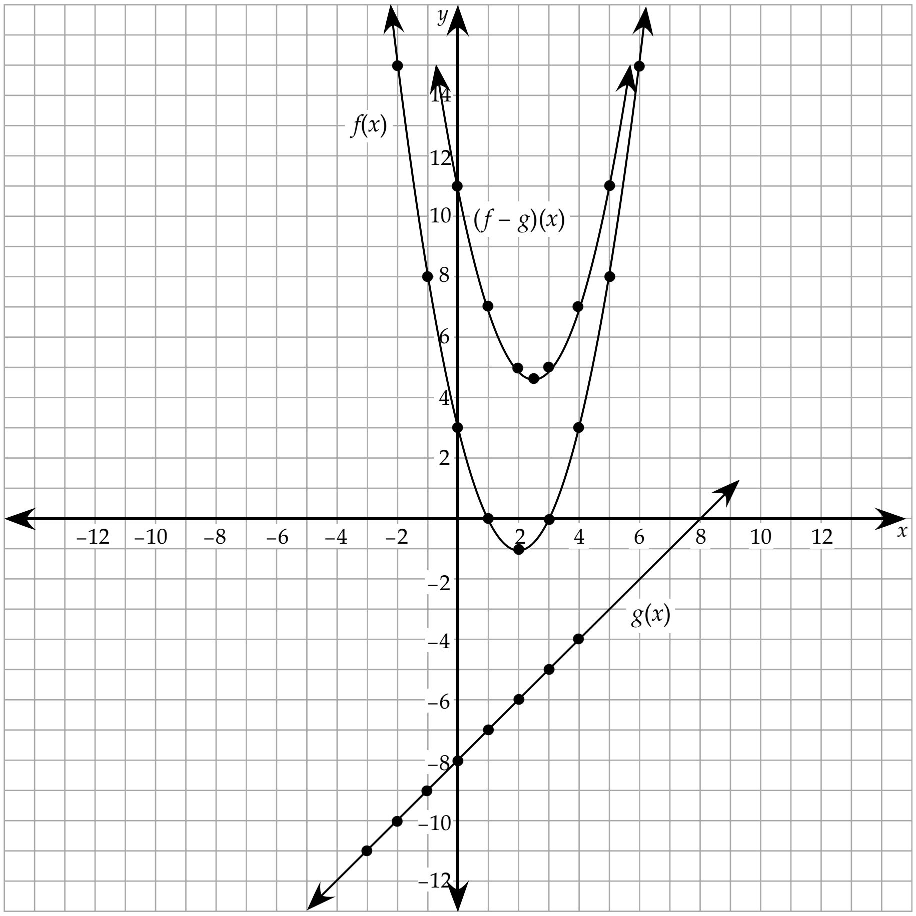
c) State the domain and range of $(f - g)(x)$.

Solutions

a) $(f - g)(x) = (x^2 - 4x + 3) - (x - 8) = x^2 - 5x + 11$

b)

x	$f(x) = x^2 - 4x + 3$	$g(x) = x - 8$	$(f - g)(x) = x^2 - 5x + 11$
−3	24	−11	35
−2	15	−10	25
−1	8	−9	17
0	3	−8	11
1	0	−7	7
2	−1	−6	5
3	0	−5	5
4	3	−4	7



c) The domain of $f(x)$ is $\{x\,|\,x \in \Re\}$.

The domain of $g(x)$ is $\{x\,|\,x \in \Re\}$.

The domain of $(f - g)(x)$ is the combined domain of $f(x)$ and $g(x)$, which is $\{x\,|\,x \in \Re\}$.

The range of $(f - g)(x)$ is $\left\{y \mid y \geq \dfrac{19}{4}\right\}$.

Note: To find this point, it is easiest to find the x-coordinate of the vertex of $(f - g)(x) = x^2 - 5x + 11$ using the formula $x = -\dfrac{b}{2a}$. Do you remember this formula from Grade 11 Pre-Calculus Mathematics? You can then find the y-coordinate of the vertex by substituting this value back into the original function.

$$x = -\frac{b}{2a} = -\frac{-5}{2(1)} = \frac{5}{2}$$

$$y = x^2 - 5x + 11 = \left(\frac{5}{2}\right)^2 - 5\left(\frac{5}{2}\right) + 11 = \frac{25}{4} - \frac{25}{2} + \frac{44}{4} = \frac{25}{4} - \frac{50}{4} + \frac{44}{4} = \frac{19}{4}$$

Example 3

Consider the functions $f(x) = 2x - 1$ and $g(x) = \sqrt{2x - 6}$.

a) Determine $(f + g)(x)$.

b) Sketch $f(x)$, $g(x)$, and $(f + g)(x)$ on the same coordinate plane.

c) State the domain and range of $(f + g)(x)$.

Solutions

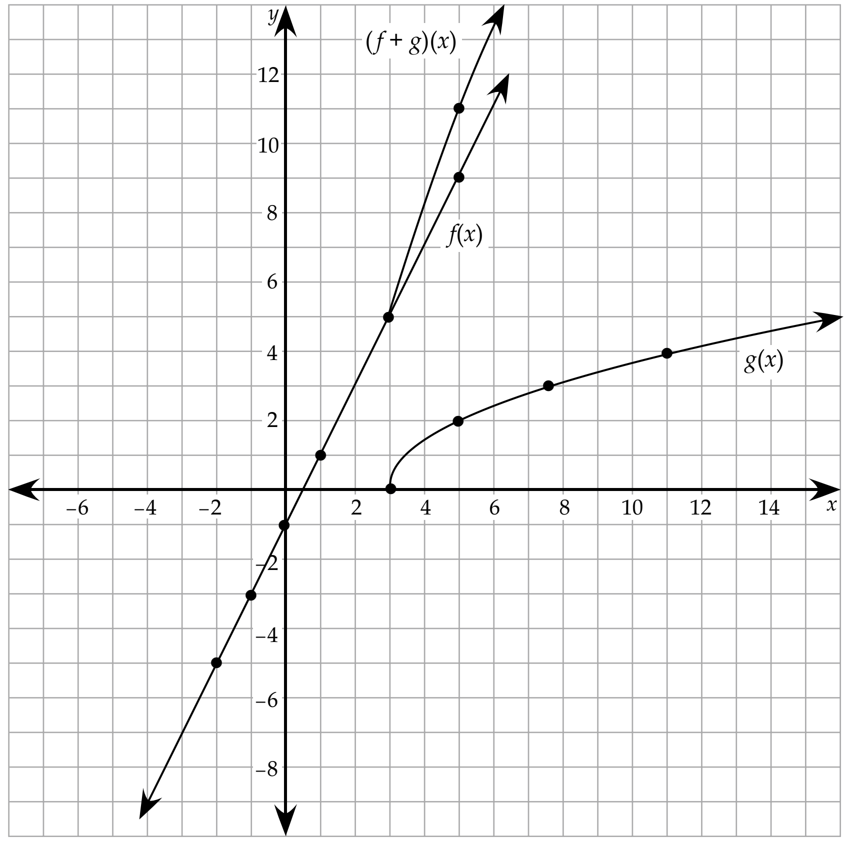
a) $(f + g)(x) = f(x) + g(x) = (2x - 1) + \left(\sqrt{2x - 6}\right) = 2x - 1 + \sqrt{2x - 6}$

b) To sketch each of these functions, you may find it easiest to construct a table of values.

x	$f(x) = 2x - 1$	$g(x) = \sqrt{2x - 6}$	$(f + g)(x) = 2x - 1 + \sqrt{2x - 6}$
−2	−5	undefined	undefined
−1	−3	undefined	undefined
0	−1	undefined	undefined
1	1	undefined	undefined
3	5	0	5
5	9	2	11
$\frac{15}{2}$	14	3	17
11	21	4	25

What connection do you notice between the y-values of $f(x)$, $g(x)$, and $(f + g)(x)$?

As you might expect, the y-values of $(f + g)(x)$ are the sum of the y-values of $f(x)$ and $g(x)$.



c) The domain of $f(x)$ is $\{x \mid x \in \Re\}$.

The domain of $g(x)$ is $\{x \mid x \geq 3\}$.

The domain of $(f + g)(x)$ is the set of x-values that are common to both the domain of $f(x)$ and $g(x)$, which is $\{x \mid x \geq 3\}$.

The range of $f(x)$ is $\{y \mid y \in \Re\}$.

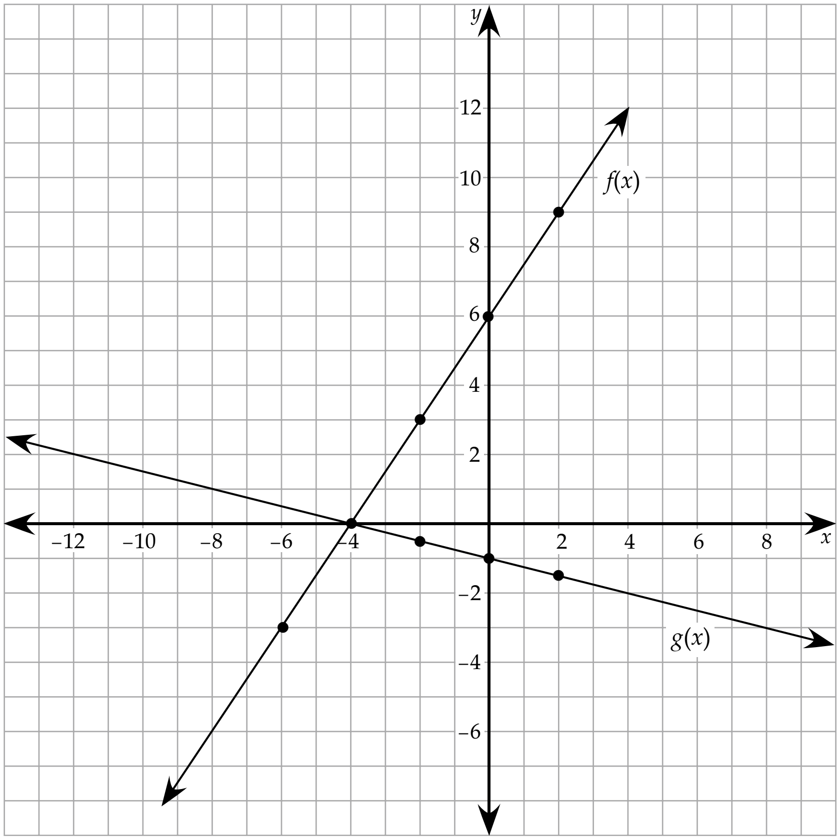
The range of $g(x)$ is $\{y \mid y \geq 0\}$.

The range of $(f + g)(x)$ is $\{y \mid y \geq 5\}$.

As you can see, the range of $(f + g)(x)$ is not a combination of the ranges of $f(x)$ and $g(x)$. It is easiest to determine the range of $(f + g)(x)$ by examining the graph.

Example 4

Consider the graphs of $f(x)$ and $g(x)$ below. Use these graphs to sketch $(f + g)(x)$.



Solution

There are two methods you can use to sketch $(f + g)(x)$.

Method 1: Table of Values

You can create a table of values for $f(x)$ and $g(x)$ and determine $(f + g)(x)$ by adding the y-values of $f(x)$ and $g(x)$ corresponding to a specific x-value.

x	$f(x)$	$g(x)$	$(f + g)(x)$
−4	0	0	0
−2	3	−0.5	2.5
0	6	−1	5
2	9	−1.5	7.5

As you can see, the points (−4, 0), (−2, 2.5), (0, 5), and (2, 7.5) are on the graph of $(f + g)(x)$, as shown below.

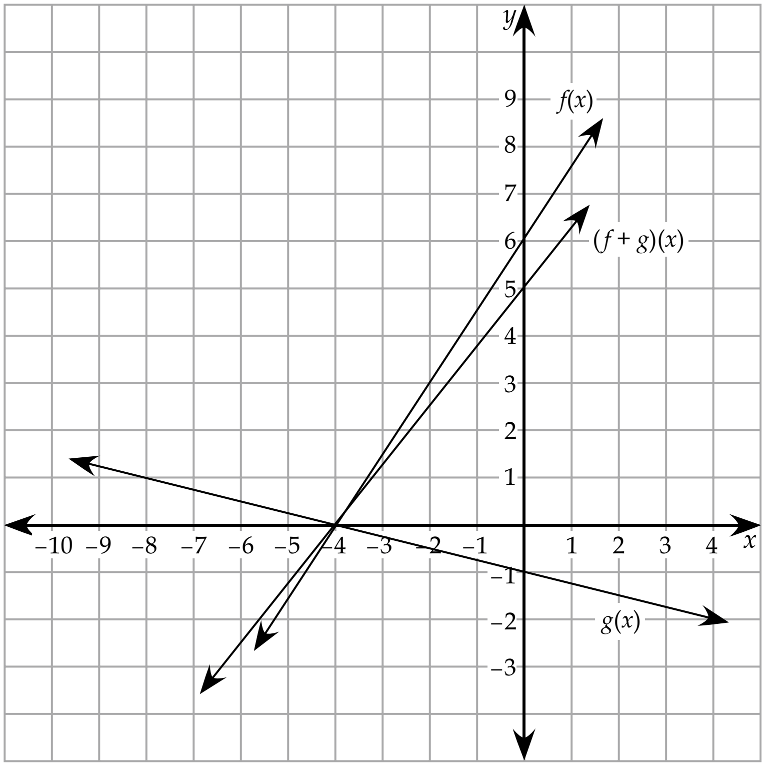
Method 2: Determine Equations

You can determine the equations of $f(x)$ and $g(x)$ and then find $(f + g)(x)$ by adding these equations.

$f(x)$ has a slope of $\frac{3}{2}$ and a y-intercept of 6; therefore, $f(x) = \frac{3}{2}x + 6$.

$g(x)$ has a slope of $-\frac{1}{4}$ and a y-intercept of −1; therefore, $g(x) = -\frac{1}{4}x - 1$.

$$(f+g)(x) = \left(\frac{3}{2}x + 6\right) + \left(-\frac{1}{4}x - 1\right) = \frac{6}{4}x - \frac{1}{4}x + 5 = \frac{5}{4}x + 5$$



It is only easiest to determine the equations of the functions, and then determine the equation after the operation on the function, if the original functions are both lines. If you are given the graph of a parabola or a graph that is not a straight line, it is usually easiest to create a table of values.

Example 5

Consider the functions $f(x) = x + 7$ and $g(x) = 2x - 3$.

a) Determine $(f \cdot g)(x)$.

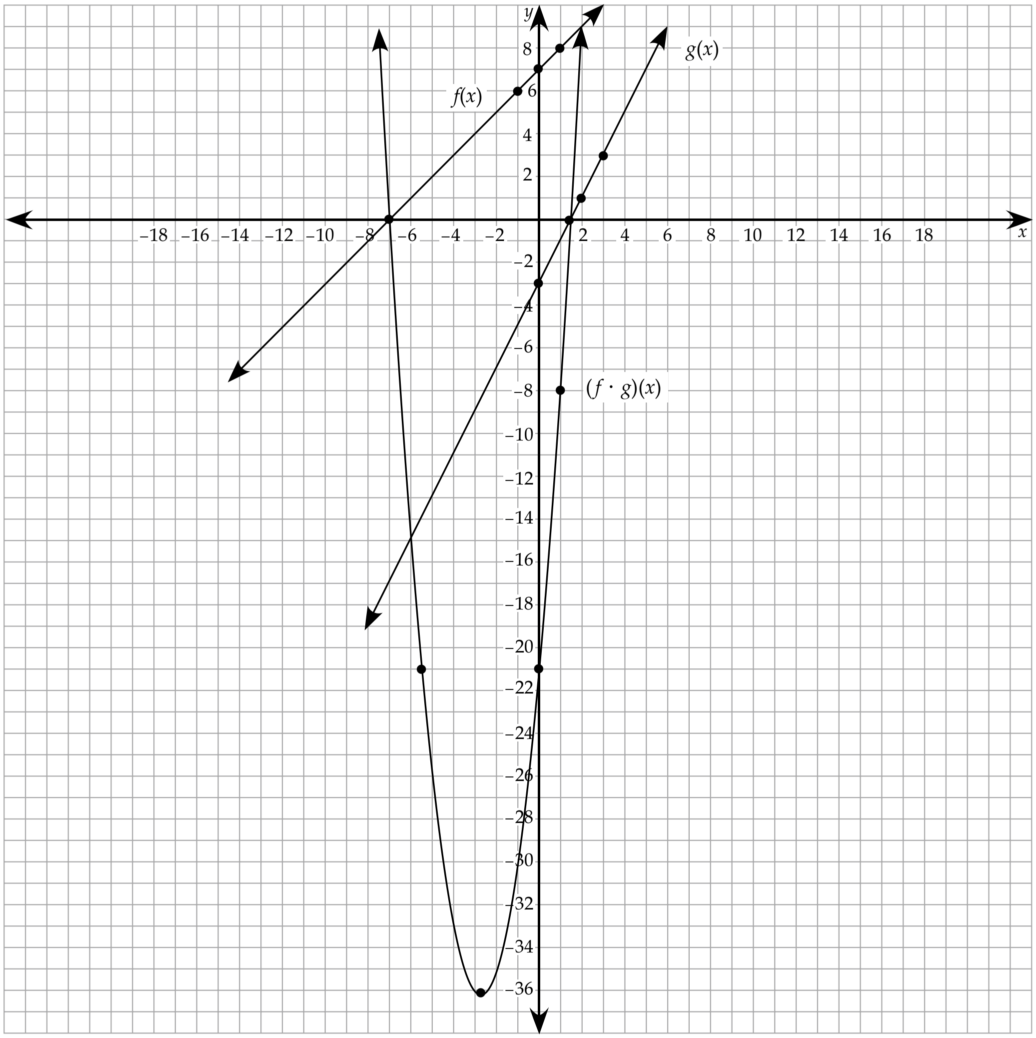
b) Sketch $f(x)$, $g(x)$, and $(f \cdot g)(x)$ on the same coordinate plane.

c) State the domain and range of $(f \cdot g)(x)$.

Solutions

a) $(f \cdot g)(x) = (x + 7)(2x - 3) = 2x^2 + 11x - 21$

b) Graph of $f(x)$, $g(x)$, and $(f \cdot g)(x)$ on the same coordinate plane



Notice the product of two linear functions is a quadratic function. Also, the x-intercepts of the parabola are the same as the x-intercepts of the linear curves. This is true because the y-value is zero at the x-intercepts of the lines and multiplying zero by any number gives zero for the product.

c) Domain: $\{x \mid x \in \Re\}$

Range: $\{y \mid y \geq -36.125\}$ since the vertex is $\left(\frac{-11}{4}, \frac{-289}{8}\right)$

Example 6

Consider the functions $f(x) = 2x - 4$ and $g(x) = 4$.

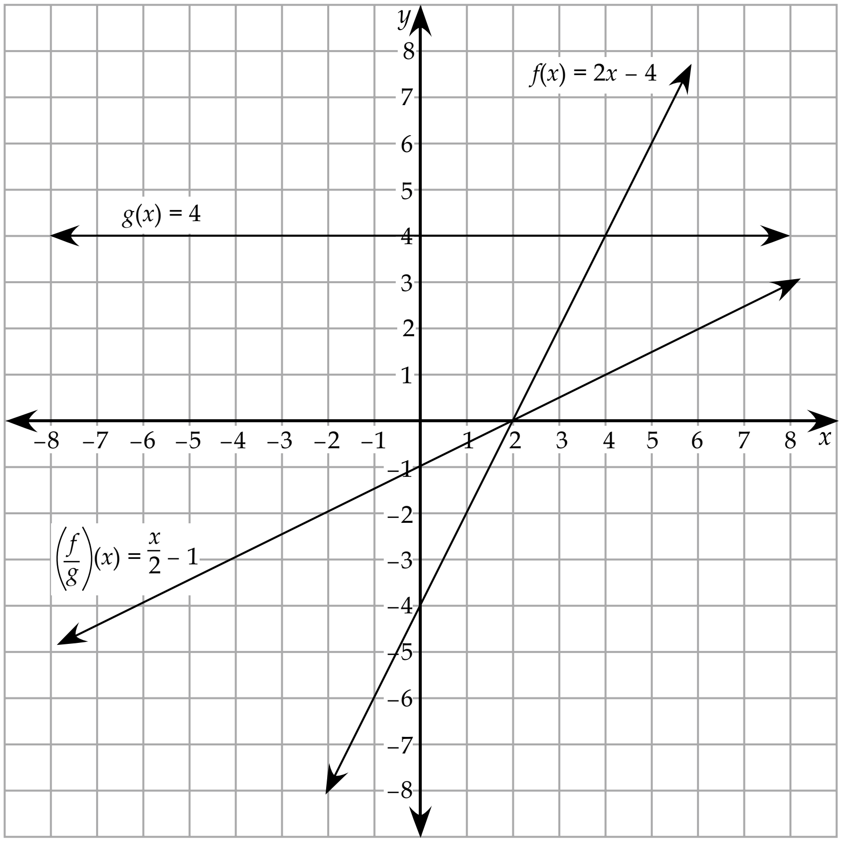
a) Determine $\left(\frac{f}{g}\right)(x)$.

b) Sketch $f(x)$, $g(x)$, and $\left(\frac{f}{g}\right)(x)$ on the same coordinate plane.

Solutions

a) $\left(\frac{f}{g}\right)(x) = \frac{2x-4}{4} = \frac{2x}{4} - \frac{4}{4} = \frac{x}{2} - 1$

b) Graph:



It is often important in the analysis of functions to be able to break down a particular function into a combination of simpler functions.

Notice that the domain of $\left(\frac{f}{g}\right)x$ is not restricted by $g(x)$ since $g(x)$ is never equal to zero. So it is always okay to divide by $g(x)$.

Example 7

If $h(x) = f(x) \cdot g(x)$ and $g(x) = x - 3$, determine $f(x)$.

a) $h(x) = x^2 - 9$

b) $h(x) = x^2 - x - 6$

c) $h(x) = x^{\frac{3}{2}} - 3\sqrt{x}$

Solutions

a) To determine $f(x)$, first factor $x^2 - 9$.

$x^2 - 9 = (x - 3)(x + 3)$

As $h(x) = f(x) \cdot g(x) = (x - 3)(x + 3)$, $f(x)$ must equal $x + 3$.

b) To determine $f(x)$, you need to factor $x^2 - x - 6$.

$x^2 - x - 6 = (x - 3)(x + 2)$

$f(x) = x + 2$

c) This question is a little bit more difficult. To answer this question, you need to factor $(x - 3)$ out of $h(x)$ to determine what is left over.

Recall: $\sqrt{x} = x^{\frac{1}{2}}$ and $x^{\frac{3}{2}} = x^{\frac{2}{2}}\left(x^{\frac{1}{2}}\right) = x\left(x^{\frac{1}{2}}\right) = x\sqrt{x}$

$$x^{\frac{3}{2}} - 3\sqrt{x} = x\sqrt{x} - 3\sqrt{x} = \left(\sqrt{x}\right)(x - 3)$$

Therefore, $f(x) = \sqrt{x}$.

Learning Activity 2.4

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Simplify: $\left(\sqrt{3x^2}\right)^4$

2. If you received a mark of 18 out of 25 on a test, what percentage does this represent?

3. What is the length of the remaining leg of a right-angled triangle if the hypotenuse measures 20 m and one leg measures 16 m?

4. What are the x-intercepts of the quadratic function $y = x^2 + 6x - 7$?

5. In how many ways can three different flower arrangements be arranged on a table?

6. If $x = -2$, calculate $(2x - 3)(2x + 4)$.

7. Write as an entire radical: $4x\sqrt{3xy}$

8. Simplify: $2\sqrt{12} - 5\sqrt{27}$

continued

Learning Activity 2.4 (continued)

Part B: Operations on Functions

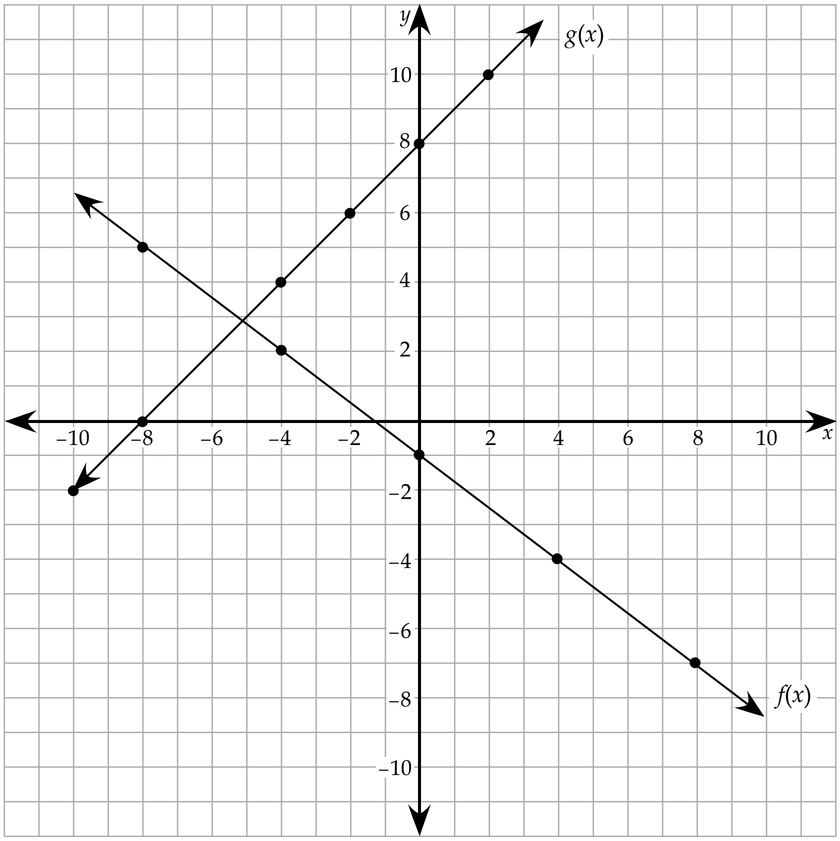
Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Using the functions $f(x) = |x - 1|$, $g(x) = x + 2$, and $h(x) = x^2 + 4x + 4$, determine a simplified function equation for the following.
   a) $(f + g)(x)$
   b) $(f - g)(x)$
   c) $(g + h)(x)$
   d) $\left(\frac{f}{h}\right)(x)$
   e) $(h \cdot g)(x)$
   f) $(h - g)(x)$
   g) $\left(\frac{h}{g}\right)(x)$
   h) $(f \cdot g)(x)$
2. Use the functions in (1) to evaluate the following.
   a) $\left(\frac{f}{h}\right)(4)$
   b) $(h \cdot g)(-3)$
   c) $(f - g)(7)$
   d) $(g + h)(-5)$
3. Let $f(x) = x - 1$ and $g(x) = x^2 + 9$. Graph each of the following, stating the domain and range.
   a) $(g - f)(x)$
   b) $(f + g)(x)$
   c) $(f - g)(x)$

continued

Learning Activity 2.4 (continued)

4. Consider the graphs of $f(x)$ and $g(x)$ shown below. Use these graphs to sketch $(f - g)(x)$ and $(f \cdot g)(x)$.



5. Let $f(x) = x + 5$ and $g(x) = x^2 + 2x - 15$. Write the function equation and state the domain.

   a) $\left(\frac{f}{g}\right)(x)$

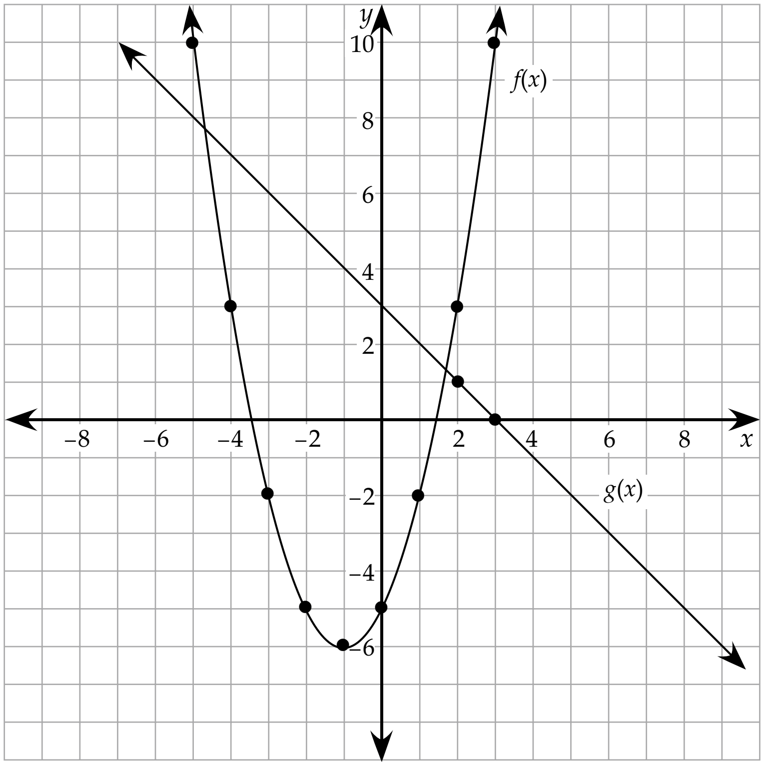
   b) $(f \cdot g)(x)$

   c) $\left(\frac{g}{f}\right)(x)$

continued

Learning Activity 2.4 (continued)

6. Use the graphs of $f(x)$ and $g(x)$ to evaluate the following.



   a) $(f+g)(2)$
   b) $(f-g)(-4)$
   c) $(f \cdot g)(0)$
   d) $\left(\dfrac{g}{f}\right)(3)$

7. If $h(x) = \dfrac{f(x)}{g(x)}$ and $g(x) = x^2 + 2x + 1$, determine $f(x)$.

   a) $h(x) = \dfrac{x+2}{x^2+2x+1}$
   b) $h(x) = 1$
   c) $h(x) = \dfrac{1}{x+1}$
   d) $h(x) = x + 8$

continued

Learning Activity 2.4 (continued)

8. A hot dog vendor sets up his shop on a downtown Winnipeg street corner. His daily costs are \$140.00 plus \$1.35 per every hot dog sold. This vendor charges \$3.75 for each hot dog and he sells up to 350 hot dogs a day.
    a) Write equations to represent the total cost, C, and the total proceeds, P, as functions of h, the number of hot dogs sold.
    b) Graph these functions on the same coordinate grid.
    c) The point at which the hot dog vendor makes just enough money to cover his expenses is $C(h) = P(h)$. Find the number of hot dogs required to break even.
    d) Develop an algebraic function for the hot dog vendor's profit.
    e) Graph the profit function.
    f) What is the maximum amount of money this hot dog vendor can earn in one day?

Lesson Summary

In this lesson, you learned how to perform operations on functions, such as adding, subtracting, multiplying, and dividing. You also analyzed the graphs of these new functions to determine relationships between the coordinates, domain, and range of the original functions and the functions created by the use of operations.

In the next lesson, you will be learning about compositions of functions. Composite functions occur when the output of one function becomes the input of another function.

LESSON 5: COMPOSITIONS OF FUNCTIONS

Lesson Focus

In this lesson, you will

- ☐ learn how to determine the equation of a composite function
- ☐ learn how to determine the value of a composite function when evaluated at a point
- ☐ learn how to sketch the graph of a composite function
- ☐ learn about the relationship between absolute value functions and composite functions
- ☐ learn how to sketch the graph of $y = |f(x)|$ when you are given the graph of $y = f(x)$
- ☐ learn about the relationship between reciprocal functions and composite functions
- ☐ learn how to sketch the graph of $y = \frac{1}{f(x)}$ when you are given the graph of $y = f(x)$

Lesson Introduction

In the last lesson, you learned how to combine functions using operations, including adding, subtracting, multiplying, and dividing. Another way of combining two functions is to form the composition of one with the other. This can be thought of has having the output of one function become the input of another.

Composition of functions is a similar concept to the food chain. For example, the change in the amount of vegetation available to consume changes the amount of herbivores able to survive. These herbivores, such as deer, depend on this vegetation to live. If the amount of vegetation decreases, fewer deer will be able to survive. This change in the number of herbivores produces a change in the number of carnivores able to survive. If there are less deer available to hunt, carnivores, such as wolves or coyotes, will have less food available to them.

Composite Functions

As stated earlier, one way of combining two functions is to form the composition of one with the other. The composition of the functions f and g is $(f \circ g)(x) = f(g(x))$. This means that you are substituting the function $g(x)$ into the function $f(x)$.

In the food-chain scenario described in the introduction, x would represent the amount of vegetation, $g(x)$ would represent the number of herbivores, and $f(x)$ would represent the number of carnivores. Overall, $f(g(x))$ would represent the number of carnivores able to survive based on the herbivore population, which in turn is based on the amount of available vegetation.

When you are composing functions, order is very important as $f(g(x))$ and $g(f(x))$ are not usually the same. For any particular value, $f(g(x))$ means you substitute an x-value into $g(x)$, evaluate $g(x)$ at this particular value, and the result is in turn substituted into $f(x)$. On the other hand, $g(f(x))$ means that you substitute a particular x-value into $f(x)$, evaluate, and then substitute the resulting value into $g(x)$.

Example 1

If $f(x) = x - 1$ and $g(x) = 2x$, find $(f \circ g)(x)$ for $x = -2, -1, 0, 1$, and 2.

Solution

x (input into g)	$g(x) = 2x$	$g(x)$ (input for f)	$f(x) = x - 1$	$f(g(x))$
-2	$2(-2) = -4$	-4	$-4 - 1 = -5$	-5
-1	$2(-1) = -2$	-2	$-2 - 1 = -3$	-3
0	$2(0) = 0$	0	$0 - 1 = -1$	-1
1	$2(1) = 2$	2	$2 - 1 = 1$	1
2	$2(2) = 4$	4	$4 - 1 = 3$	3

Notice that the range (output) of g becomes the domain (input) of f.

Algebraically:

$(f \circ g)(x) = f(g(x))$

$= f(2x)$ Substitute formula for $g(x)$.

$= (2x) - 1$ Apply formula for $f(x)$.

$= 2x - 1$ Simplify.

For the above example, find $(g \circ f)(x)$.

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) & \\ &= g(x - 1) & \text{Substitute formula for } f(x). \\ &= 2(x - 1) & \text{Apply formula for } g(x). \\ &= 2x - 2 & \text{Simplify.} \end{aligned}$$

As you can see, $f(g(x))$ and $g(f(x))$ are not always equal.

Make sure you include the definition for the composition of a function on your resource sheet.

Example 2

Consider $f(x) = x^2 + 1$ and $g(x) = \sqrt{x + 3}$.

a) Determine a function equation for $f(g(x))$ and $g(f(x))$.

b) State the domain of $f(g(x))$ and $g(f(x))$.

Solutions

a) To determine $f(g(x))$, substitute $g(x)$ into $f(x)$.

$$\begin{aligned} f(g(x)) &= f\left(\sqrt{x+3}\right) \\ &= \left(\sqrt{x+3}\right)^2 + 1 \\ &= x + 3 + 1 \\ &= x + 4 \end{aligned}$$

To determine $g(f(x))$, substitute $f(x)$ into $g(x)$.

$$\begin{aligned} g(f(x)) &= g\left(x^2 + 1\right) \\ &= \sqrt{\left(x^2 + 1\right) + 3} \\ &= \sqrt{x^2 + 4} \end{aligned}$$

b) Determine the domain of $f(g(x))$:

In order to determine the domain of $f(g(x))$, you need to consider the domain of $g(x)$ and $f(g(x))$. If $g(x)$ is not defined at a point, then it is not possible for $f(g(x))$ to be defined at that point.

The domain of $g(x) = \{x \mid x \geq -3, x \in \Re\}$ since a Real Number cannot be the square root of a negative value.

There are no restrictions on the domain of $f(g(x)) = x + 4$.

However, the domain of the composition, $f(g(x))$, is limited by the domain of the “inside” function, $g(x)$.

Therefore, the domain of $f(g(x))$ is $\{x \mid x \geq -3, x \in \Re\}$.

Determine the domain of $g(f(x))$:

The domain of $g(f(x))$ depends on the domain of $f(x)$ and the domain of $g\left(f(x)\right) = \sqrt{x^2 + 2}$. There are no restrictions on the domain of $f(x)$ and there are also no restrictions on the domain of $g(f(x))$, as $x^2 + 2$ is never less than zero.

Therefore, the domain of $g(f(x))$ is $\{x \mid x \in \Re\}$.

Example 3

Consider $f(x) = x^2 + 1$ and $g(x) = 2x - 4$. Determine the equation of each composite function and graph each composite function.

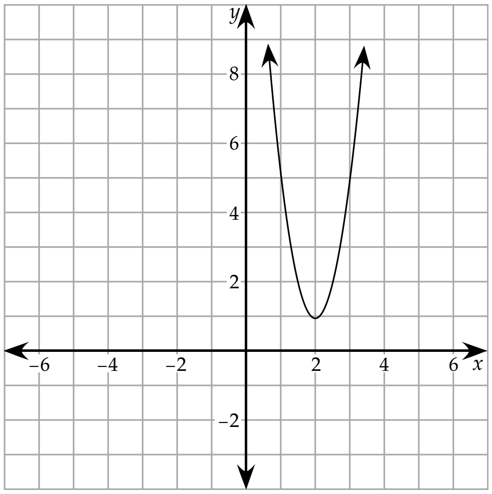
a) $f(g(x))$

b) $g(g(x))$

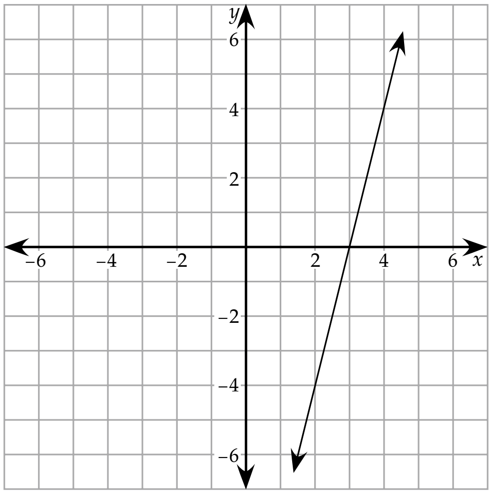
Solutions

a) $f(g(x)) = f(2x - 4) = (2x - 4)^2 + 1 = (2(x - 2))^2 + 1$

This is the standard quadratic equation being horizontally compressed by a factor of 2, shifted to the right 2 units, and shifted up 1 unit.



b) $g(g(x)) = g(2x - 4) = 2(2x - 4) - 4 = 4x - 8 - 4 = 4x - 12$



Transformations and Composition of Functions

The transformations that you have worked on in this module can be written in terms of compositions of functions. Consider the following functions:

$f(x) = x^2$

$h(x) = x - 3$

The composition $y = f(h(x))$ can be written as $y = (x - 3)^2$, which represents function $f(x)$ shifted right 3 units.

The composition $y = h(f(x))$ can be written as $y = (x^2) - 3$, which represents function $f(x)$ shifted down 3 units.

Example 4

Use the functions defined below to write the new equation after each given composition and then describe the corresponding transformation on $f(x)$.

$f(x) = \sqrt{x}$

$g(x) = x + 2$

$h(x) = 3x$

a) $y = f(h(x))$

b) $y = h(f(x))$

c) $y = g(f(x))$

d) $y = f(g(x))$

Solutions

a) $y = f\left(h(x)\right)$

Answer:

$y = \sqrt{3x}$

Transformation on $f(x)$ is a horizontal compression by a factor of 3.

b) $y = h\left(f(x)\right)$

Answer:

$y = 3\left(\sqrt{x}\right)$

Transformation on $f(x)$ is a vertical stretch by a factor of 3.

c) $y = g\left(f(x)\right)$

Answer:

$y = \sqrt{x} + 2$

Transformation on $f(x)$ is a vertical shift up 2 units.

d) $y = f\left(g(x)\right)$

Answer:

$y = \sqrt{x + 2}$

Transformation on $f(x)$ is a horizontal shift left 2 units.

Absolute Value Functions

You can think of an absolute value function as a composition of functions. For example, consider the function $h(x) = | x + 3 |$. If you let $g(x) = x + 3$ and $f(x) = | x |$, then $f(g(x)) = f(x + 3) = | x + 3 |$.

Recall: The absolute value of an expression is denoted by vertical bars surrounding an expression and is a positive number. The absolute value of an expression can be defined in two pieces as:

$$|x| = \begin{cases} x, \ x \geq 0 \\ -x, \ x < 0 \end{cases}$$

Example 5

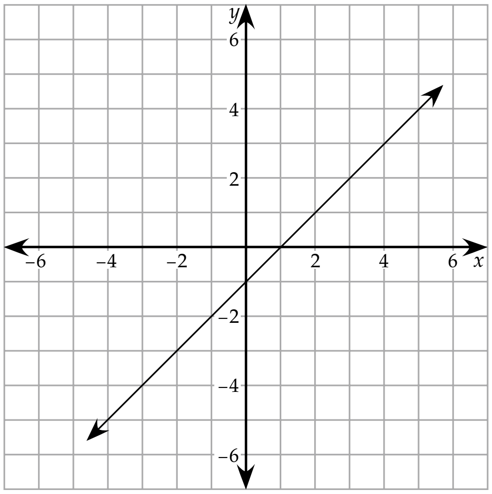
Let $f(x) = x - 1$ and $g(x) = |x|$. Determine and graph $g(f(x))$.

Solution

$g(f(x)) = g(x - 1) = |x - 1|$

Graphing $|x - 1|$:

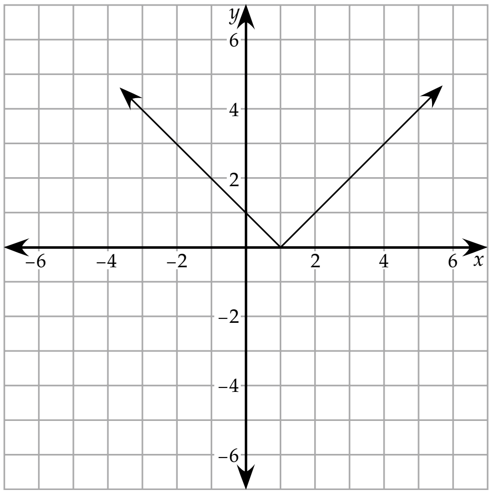
Step 1: Graph $x - 1$



Step 2: Determine the x-intercepts

The x-intercept of this function is at (1, 0). The x-intercept becomes the pivot point. Everything below the x-axis, starting at the pivot point, is now reflected to above the x-axis since the absolute value function makes all negative function values positive. The y-values remain or become positive, but the x-values remain unchanged.

Step 3: Graph $|x - 1|$:



You could also graph this function by noticing that $g(f(x))$ is a shift of one unit to the right of the standard absolute value function, $y = |x|$.

Example 6

Let $f(x) = x^2 - 4$ and $g(x) = | x |$. Determine and graph $g(f(x))$.

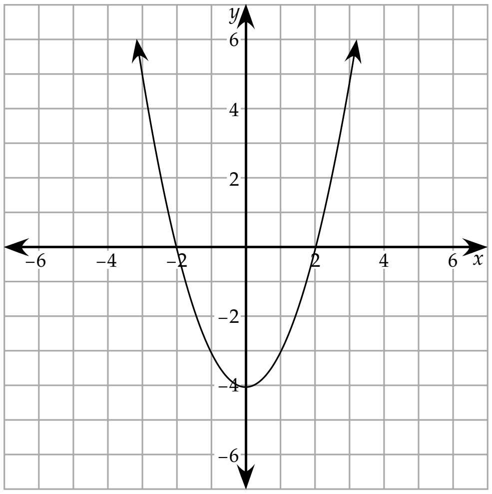
Solution

$g(f(x)) = g(x^2 - 4) = | x^2 - 4 |$

Graphing $y = | x^2 - 4 |$:

Step 1: Graph $f(x) = x^2 - 4$

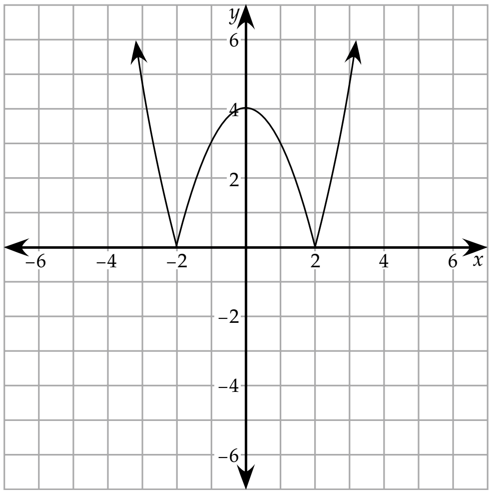
This is the standard parabola shifted down 4 units.



Step 2: Determine the x-intercepts

The x-intercepts of this function are at (−2, 0) and (2, 0). These x-intercepts become the pivot points. Every function value below the x-axis, starting at the pivot points, is now reflected up over the x-axis since the absolute value function makes all negative function values positive.

Step 3: Graph $g(x) = | f(x) |$



Notice how the curve now has sharp corners or "cusps" at the pivot points.

Reciprocal Functions

You have been exposed to reciprocal functions in Grade 11 Pre-Calculus Mathematics. You can also think of reciprocal functions as compositions. For example, consider the function $h(x) = \dfrac{1}{x-5}$. If you let $g(x) = x - 5$ and $f(x) = \dfrac{1}{x}$, then $f\big(g(x)\big) = \dfrac{1}{x-5}$.

Example 7

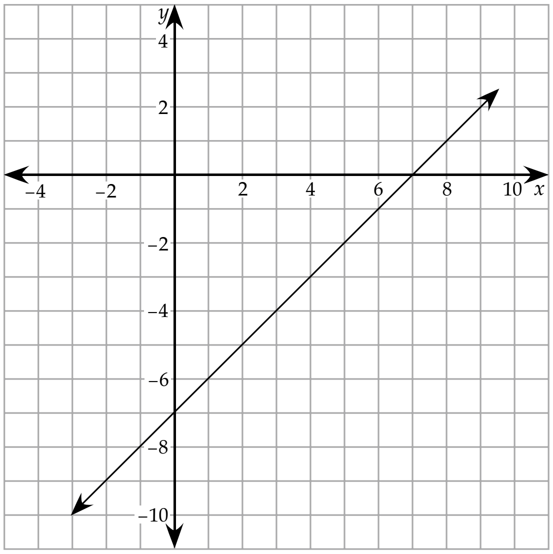
Let $g(x) = x - 7$ and $f(x) = \dfrac{1}{x}$. Determine and graph $f(g(x))$.

Solution

$$f\big(g(x)\big) = f(x-7) = \frac{1}{x-7}$$

To graph $f(g(x))$:

Step 1: Sketch the curve $f(x) = x - 7$.



Step 2: Find the reciprocal of each of the y-values.

a) Since any x-intercept has a corresponding y-value of zero and the reciprocal of zero is infinitely large and undefined, there will be no function value at the x-intercepts. This is indicated by a vertical asymptote at each x-intercept.

   This concept allows you to find the vertical asymptotes. If the x-intercept is a, the equation of the vertical asymptote is $x = a$.

   Here, the equation of the vertical asymptote is $x = 7$.

b) Now consider the points where the y-value is 1. Since the reciprocal of 1 is 1, these points will be the same on both curves. Recall that these points are called *invariant points*. The value for x, when $y = 1$ can be found by reading the graph or by solving:

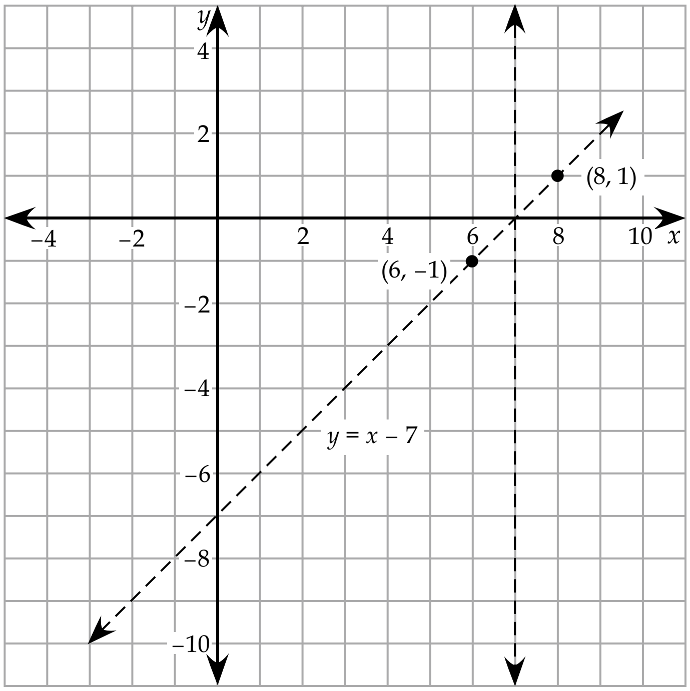
$$1 = x - 7$$
$$x = 8$$

A point on both the graph of $y = x - 7$ and $y = \dfrac{1}{x-7}$.

c) Consider the invariant point where y is -1. Again, its reciprocal will also be -1. The corresponding x-value is

$$-1 = x - 7$$
$$x = 6$$

A point on both the graph of $y = x - 7$ and $y = \dfrac{1}{x-7}$.

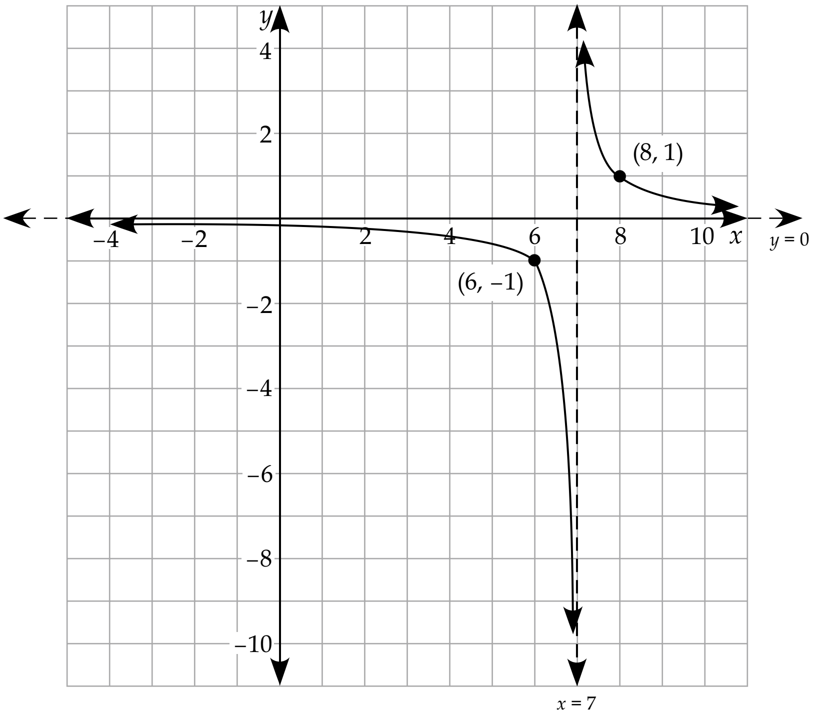


d) Very large x-values, found to the extreme left and right of the graph, have very small reciprocal values close to zero. This concept allows you to find the horizontal asymptote whose equation is $y = 0$.

e) x-values found close to and at the right of the vertical asymptote become very large positive values when their reciprocals are found. For example, when x is 7.01, the value of $y = \dfrac{1}{x-7}$ is $y = \dfrac{1}{0.01} = 100$.

Similarly, x-values found close to and on the left of the vertical asymptote become very large negative values. For example, when $x = 6.99$, the value of $y = \dfrac{1}{x-7}$ is $y = \dfrac{1}{-0.01} = -100$.

Step 3: Complete the sketch by joining these points with smooth curves.



Example 8

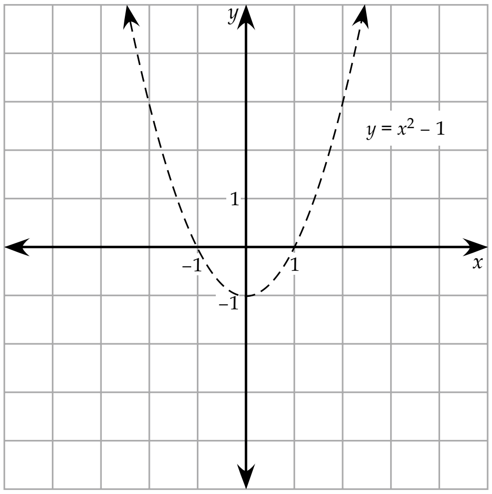
Let $g(x) = x^2 - 1$ and $f(x) = \dfrac{1}{x}$. Determine and graph $f(g(x))$.

Solution

$$f(g(x)) = f(x^2 - 1) = \frac{1}{x^2 - 1}$$

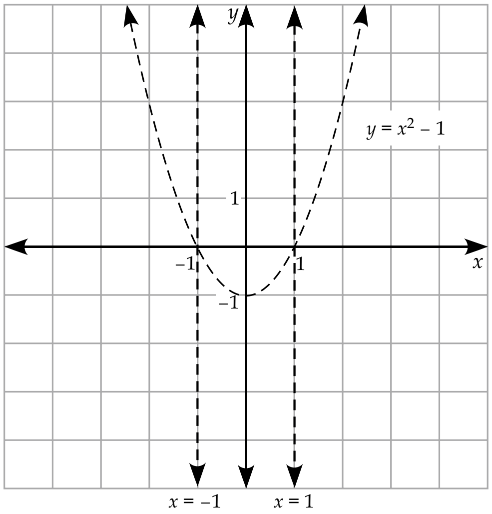
To graph $f(g(x))$:

Step 1: Sketch the curve $y = x^2 - 1$.



Step 2: Find the reciprocal of each of the y-values.

a) Since any x-intercept has a corresponding y-value of zero, there will be no defined reciprocals at the x-intercepts. This is indicated by drawing vertical asymptotes at $x = 1$ and $x = -1$.



b) Now consider the points where the y-value is 1. Since the reciprocal of 1 is 1, these points will be the same on both curves. You can simply locate these points using the graph where $y = 1$. If you choose, the value for x when $y = 1$ can be found by solving:

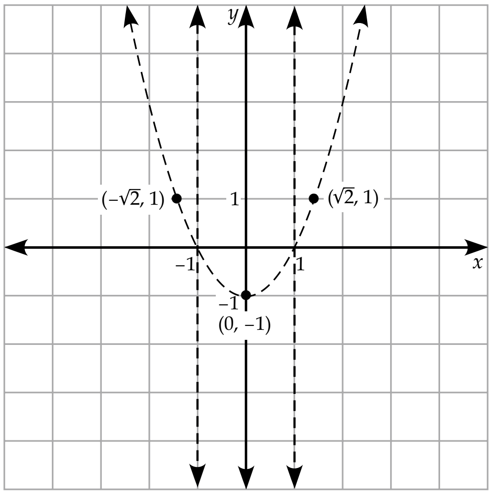
$$\begin{aligned} y &= x^2 - 1 \\ 1 &= x^2 - 1 \\ 2 &= x^2 \\ \pm\sqrt{2} &= x \end{aligned}$$

Thus, the points $\left(-\sqrt{2}, 1\right)$ and $\left(\sqrt{2}, 1\right)$ are on $y = x^2 - 1$ and on $y = \dfrac{1}{x^2 - 1}$.

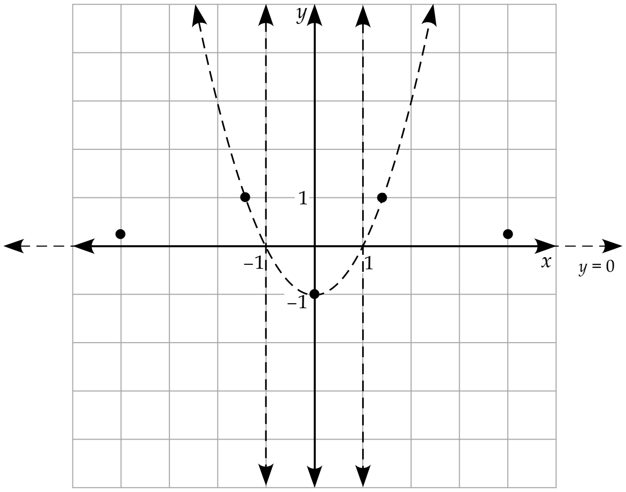
c) Consider the point where y is -1. Again, its reciprocal will be at the same point where $y = -1$. You can simply locate this point using the graph where $y = -1$. If you choose, you can find the x-value by solving:

$$\begin{aligned} y &= x^2 - 1 \\ -1 &= x^2 - 1 \\ 0 &= x^2 \\ 0 &= x \end{aligned}$$

Thus, the point $(0, -1)$ is on both the original curve and its reciprocal. The invariant points $\left(\pm\sqrt{2}, 1\right)$ and $(0, -1)$ are shown on the diagram below.

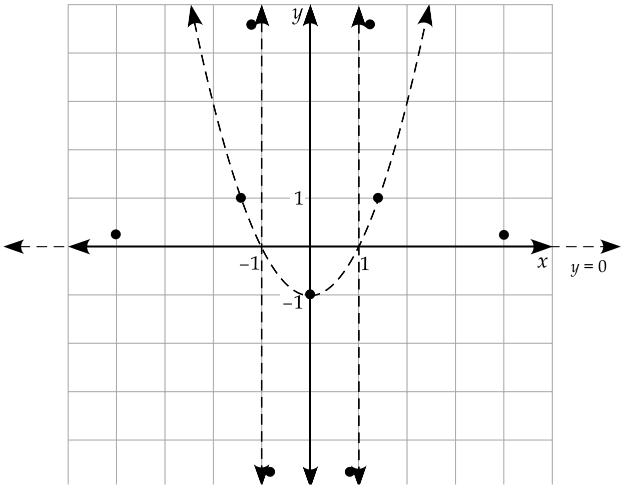


d) Very large x-values, found to the extreme left and right of the graph, have very small reciprocal function values. This concept allows you to find the horizontal asymptote whose equation is $y = 0$.

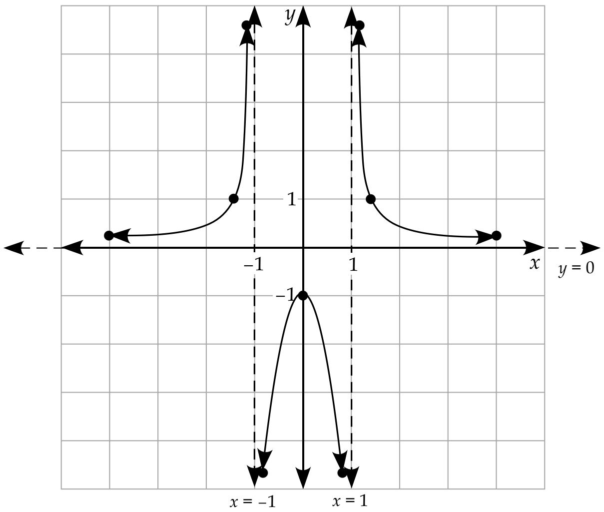


e) x-values just a little bit larger than 1 or a little bit smaller than −1 will have large reciprocal function values. For example, when $x = 1.01$ or $x = -1.01$, the value of $y = \dfrac{1}{x^2 - 1}$ is $y \doteq 50$.

Similarly, x-values just a little bit smaller than 1 or a little bit larger than −1 will have large negative reciprocal function values. For example, when $x = 0.99$ or $x = -0.99$, the value of $y = \dfrac{1}{x^2 - 1}$ is $y \doteq -50$.



Step 3: Complete the sketch by joining these points with smooth curves.



Learning Activity 2.5

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Calculate: 6!
2. Evaluate: ${}_4P_3$
3. Factor: $3xy^6 - 6x^2y$
4. List all the factors of 36.
5. What is $\frac{1}{3}$ of 324?
6. Simplify: $-|-5(4) + 2 - 8^2|$
7. Simplify: $5\sqrt{8x^3} - 3x\sqrt{50x}$
8. Simplify: $8^{\frac{1}{3}}$

continued

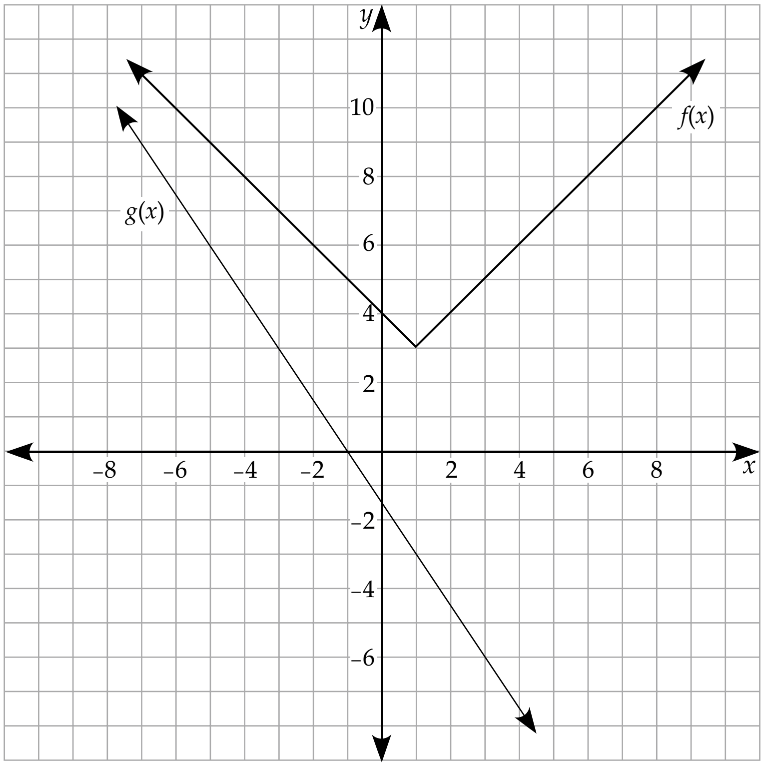
Learning Activity 2.5 (continued)

Part B: Compositions of Functions

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Given that $f(x) = 3x + 4$ and $g(x) = x^2 - 1$, find the following.
   - a) $g(f(2))$
   - b) $f(g(2))$
   - c) $g(f(-1))$
   - d) $f(g(-1))$
   - e) $g(f(a))$
   - f) $f(g(a))$
   - g) $f(f(a))$
   - h) $g(g(a))$

2. Use the graphs of $f(x)$ and $g(x)$ shown below to evaluate the following.



   - a) $f(g(3))$
   - b) $g(f(-1)$
   - c) $g(f(1))$
   - d) $f(g(-5))$

continued

3. Given $h(x) = \sqrt{x-2}$ and $g(x) = 2x,$ determine each of the following.

   a) $\dfrac{g(h(6))}{h(g(9))}$

   b) $\dfrac{g(h(5))}{h(g(5))}$

   c) $h(g(x))$

   d) $g(h(x))$

   e) The domain and range of $h(g(x))$.

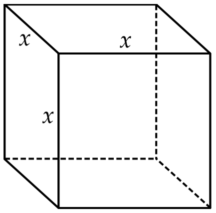
   f) The domain and range of $g(h(x))$.

4. If $f(x) = \dfrac{1}{x-2}$ and $k(x) = x+1,$ write:

   a) An equation defining the composition of f with k. Specify the domain.

   b) An equation defining the composition of k with f. Specify the domain.

5. The volume of a cube with edges of length x is given by the function $f(x) = x^3$. Find $f(5)$. Explain what $f(5)$ represents.



6. Sketch $h(x) = f(g(x))$ when $f(x) = |x|$ for each $g(x)$. State the domain and range of $h(x)$.

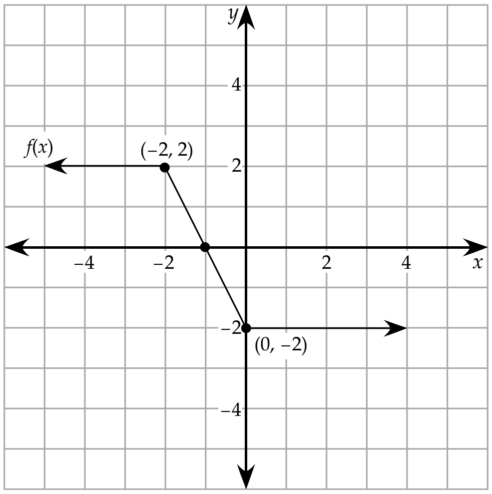
   a) $g(x) = 2x - 1$

   b) $g(x) = x^2 - 4$

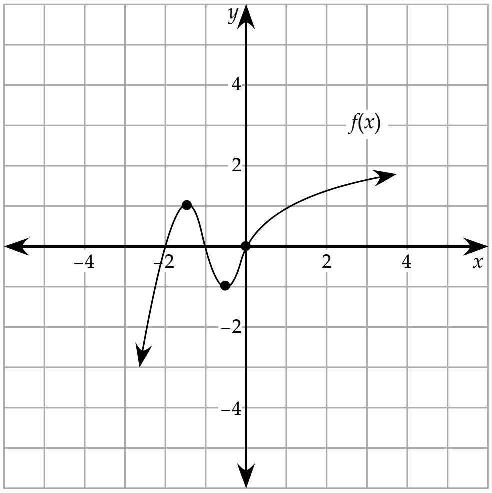
continued

7. Use the graphs of the functions shown below to sketch $y = | f(x) |$.

a)



b)



8. Sketch $h(x) = f\left(g(x)\right)$ when $f(x) = \dfrac{1}{x}$ for each $g(x)$. Specify the domain, range, and the equations of the asymptotes.

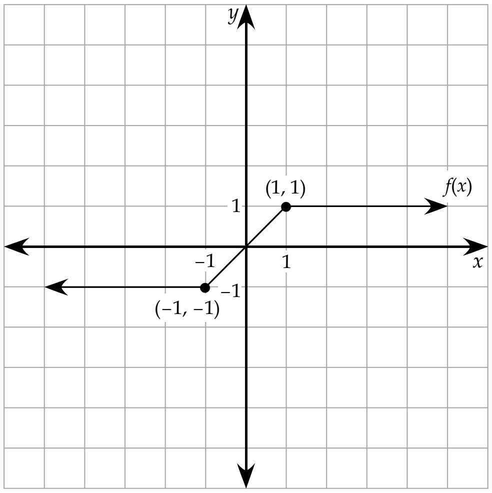
a) $g(x) = x - 2$

b) $g(x) = x^2 - 9$

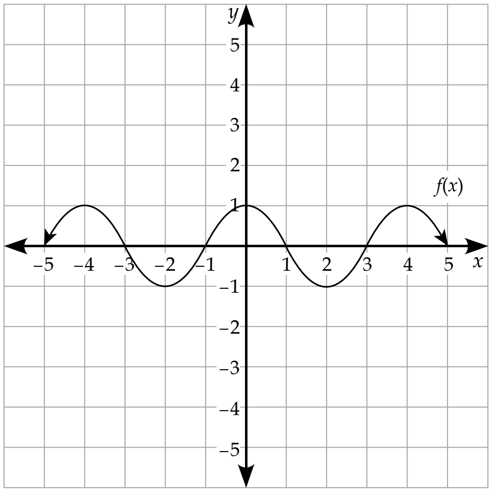
continued

9. Use the graph of $f(x)$ to sketch $\dfrac{1}{f(x)}$.

a)



b)



10. Given $g(x) = x - 3$ and $f(x) = x^3$, describe the graph of $y = f(g(x))$ in terms of the transformation of the graph of $y = f(x)$.

11. The point with coordinates (7, 11) is on the graph of the function $f(x)$. What are the coordinates of a point that must lie on the graph of $\dfrac{1}{f(x)}$?

12. Write the following functions as compositions of two functions.

a) $h(x) = (x + 4)^3 - 1$

b) $h(x) = \dfrac{1}{x + 2} + 5$

Lesson Summary

In this lesson, you learned about the composition of functions. You learned how to determine the equation of a composite function as well as how to evaluate a composite function at a point. Once you were able to find the equation of a composite function, you were able to sketch the graph of a composite function using graphing techniques you learned previously. You also learned how reciprocal functions and absolute value functions are related to composite functions. Using techniques such as transformations and finding asymptotes, you learned how to transform the graph of a composition of functions into its corresponding reciprocal graph or its absolute value graph.

This is the last lesson in Module 2. Make sure you complete the above learning activity before you complete Assignment 2.3. The learning activities are designed to provide practice questions for the assignments in this course. If you do well on the learning activities, you are likely to do well on the assignments.

Assignment 2.3

Operations on and Compositions of Functions

Total: 54 marks

You need to show all your steps as you work out your solutions, and make sure your answers are clear (include units, where appropriate). Check the Introduction for the Marking Guidelines that will be used for all assignments and examinations.

1. Using the functions $f(x) = x^2 + 3x - 18$, $g(x) = -x + 7$, and $h(x) = -x + 3$, determine a simplified function equation for the following and evaluate as indicated.

   a) $(f + g)(x)$ *(2 marks)*

   b) $(f + g)(3)$ *(1 mark)*

   c) $\left(\frac{f}{h}\right)(x)$ *(2 marks)*

continued

Assignment 2.3: Operations on and Compositions of Functions (continued)

d) $\left(\frac{f}{h}\right)(2)$ *(1 mark)*

e) $(h \cdot g)(x)$ *(2 marks)*

f) $(h \cdot g)(0)$ *(1 mark)*

continued

Assignment 2.3: Operations on and Compositions of Functions (continued)

g) $(h - g)(x)$ *(2 marks)*

h) $(h - g)(-1)$ *(1 mark)*

continued

Assignment 2.3: Operations on and Compositions of Functions (continued)

2. Let $f(x) = 2x^2 + 5x - 6$ and $g(x) = x^2 - 2x + 4$.
   a) Find $(f + g)(x)$. *(2 marks)*

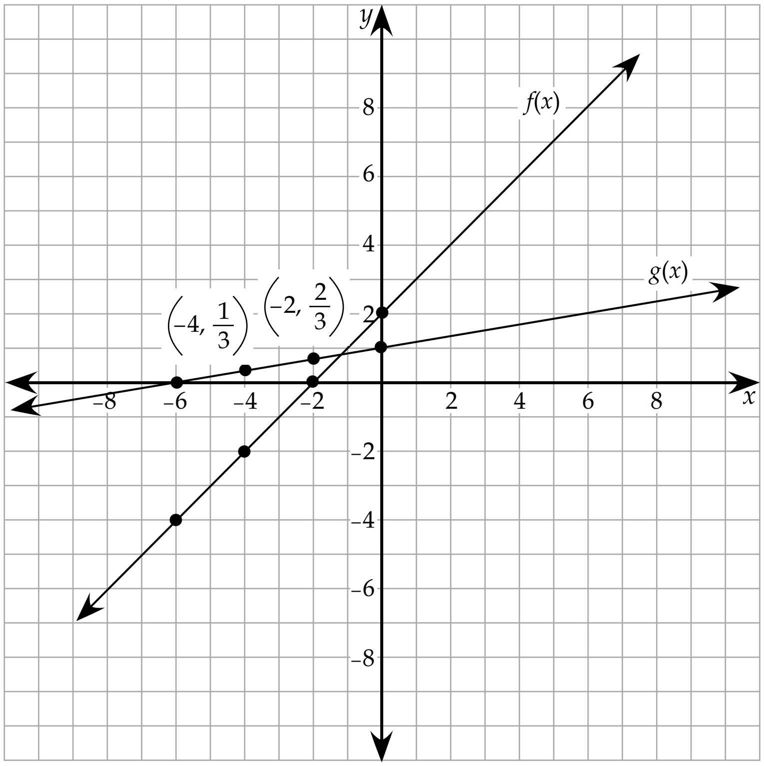
   b) Graph $(f + g)(x)$. Show your work. *(2 marks)*

   c) State the domain and range. *(1 mark)*

continued

Assignment 2.3: Operations on and Compositions of Functions (continued)

3. Consider the graphs of $f(x)$ and $g(x)$ shown below.



a) Create a table of values for $(f \cdot g)(x)$. *(2 marks)*

continued

Assignment 2.3: Operations on and Compositions of Functions (continued)

b) Sketch the graph of $(f \cdot g)(x)$. *(1 mark)*

continued

Assignment 2.3: Operations on and Compositions of Functions (continued)

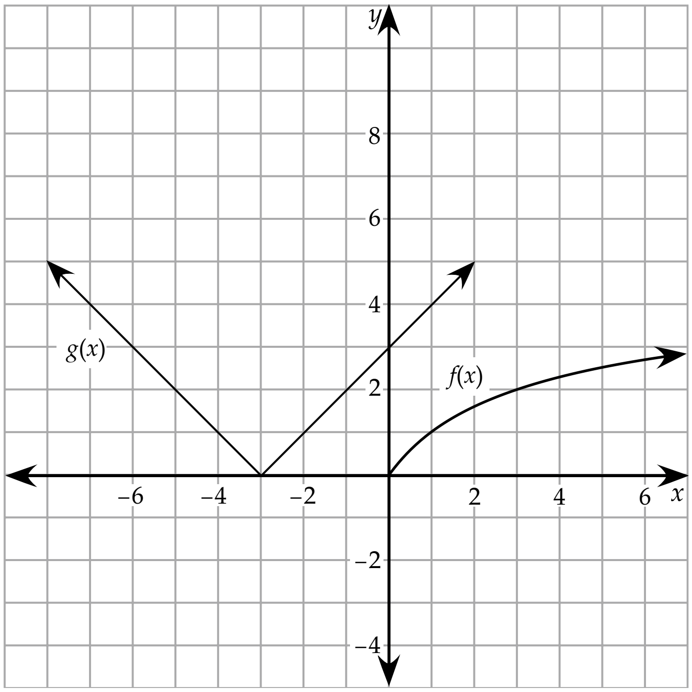
4. Let $f(x) = (2x + 4)(x + 1)$ and $g(x) = 2$. Determine the simplified equation $\left(\frac{f}{g}\right)(x)$.

   Graph $\left(\frac{f}{g}\right)(x)$, stating the domain and range. *(4 marks)*

continued

Assignment 2.3: Operations on and Compositions of Functions (continued)

5. Use the graphs of $f(x)$ and $g(x)$ shown below to evaluate the following. *(4 × 1 mark each = 4 marks)*



a) $(g + f)(-3)$

b) $(f - g)(4)$

continued

c) $(f \cdot g)(0)$

d) $\left(\frac{f}{g}\right)(1)$

6. Given that $t(x) = 2x - 6$ and $s(x) = \frac{1}{2}x + 3$, find the following.

a) $s(t(x))$ *(2 marks)*

b) $t(s(x))$ *(2 marks)*

c) $s(s(x))$ *(2 marks)*

continued

Assignment 2.3: Operations on and Compositions of Functions (continued)

d) $t(t(x))$ *(2 marks)*

e) $s(t(7))$ *(1 mark)*

f) $s(s(-3))$ *(1 mark)*

g) $t(s(8))$ *(1 mark)*

h) $t\left(t\left(-\frac{3}{4}\right)\right)$ *(1 mark)*

i) Does $s(t(x)) = t(s(x))$ for all values of x? Justify your answer using composition of functions. *(1 mark)*

continued

Assignment 2.3: Operations on and Compositions of Functions (continued)

7. If $f(x) = \sqrt{x+3}$ and $g(x) = x^2 - 1$,

   a) determine the simplified equation of $h(x) = g(f(x))$. *(2 marks)*

   b) determine the domain and range of $h(x) = g(f(x))$. *(2 marks)*

   c) sketch the graph of $h(x) = g(f(x))$. *(1 mark)*

continued

Assignment 2.3: Operations on and Compositions of Functions (continued)

8. If $f(x) = \dfrac{1}{x+2}$ and $g(x) = |x|$, determine the equation and sketch $g(f(x))$. *(4 marks)*

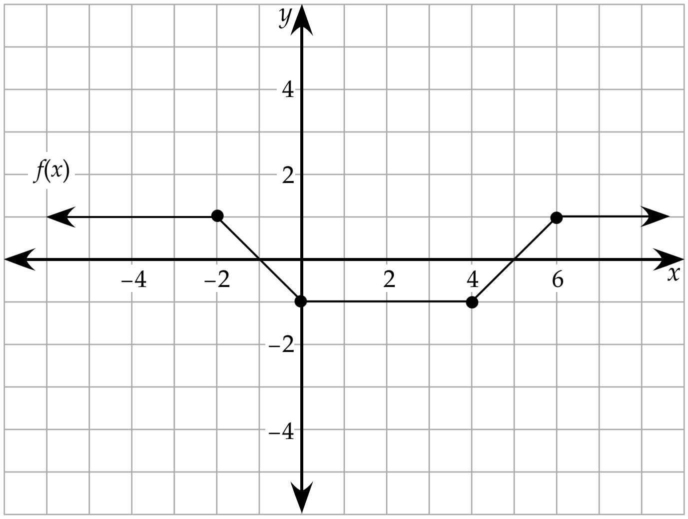
continued

Assignment 2.3: Operations on and Compositions of Functions (continued)

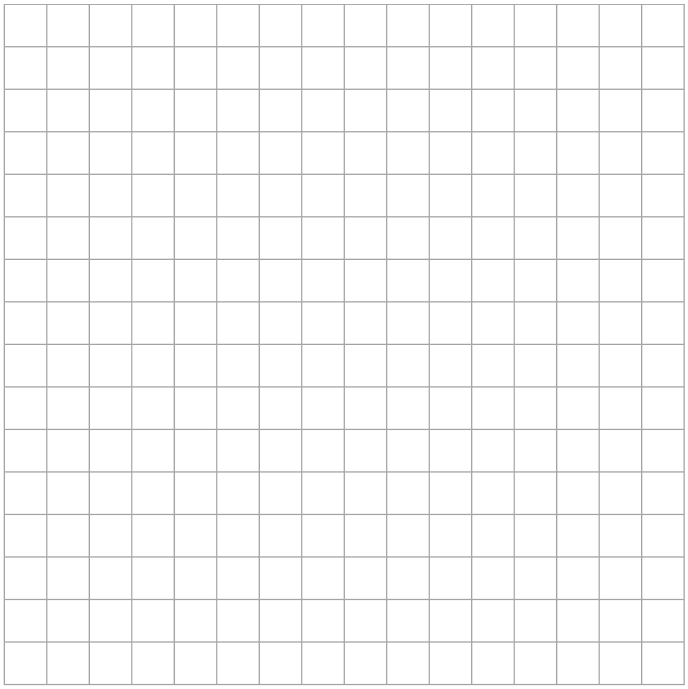
9. Use the graph of the function $f(x)$ shown below to sketch:

   a) $\left| f(x) \right|$ *(1 mark)*

   b) $\dfrac{1}{f(x)}$ *(3 marks)*



a) Graph of | $f(x)$ |



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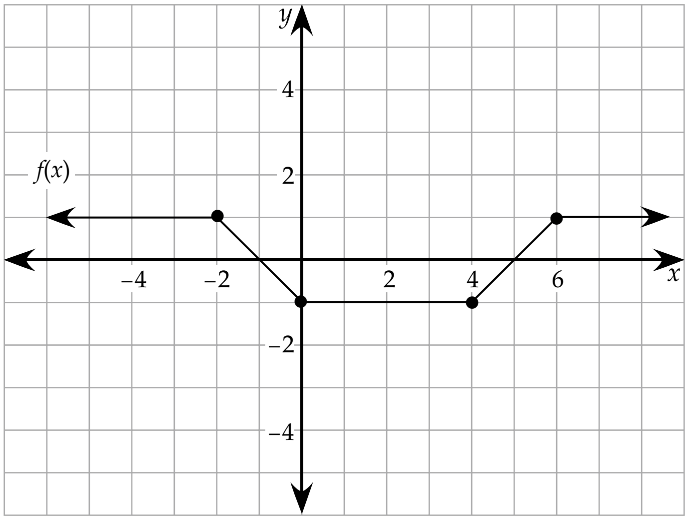
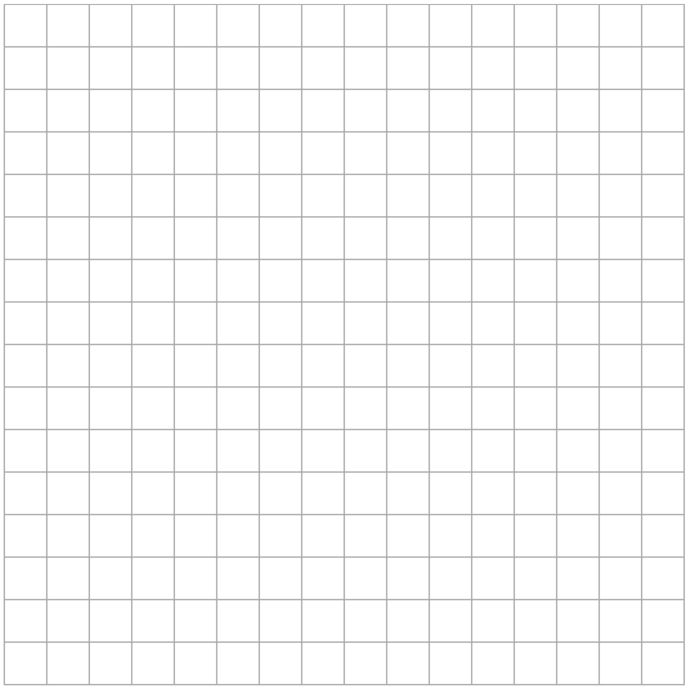
Assignment 2.3: Operations on and Compositions of Functions (continued)



b) Graph of $\frac{1}{f(x)}$



MODULE 2 SUMMARY

Throughout this module, you expanded on your knowledge of functions. You learned how to transform functions with the use of translations, stretches, and compressions, similar to how you transformed quadratic functions in Grade 11 Pre-Calculus Mathematics. You also learned how to graph the reciprocal or the absolute value of any function when you were given the graph of that function. These strategies will come in handy throughout the rest of the course as a great deal of graphing is required. These strategies will also prepare you for learning about reflections in the next module.

You also learned about operations on functions and compositions of functions. Operations and compositions are different ways of combining functions. You may not notice it, but you will often encounter operations on functions and compositions of functions when you are dealing with more complex functions.

Submitting Your Assignments

It is now time for you to submit Assignments 2.1 to 2.3 to the Distance Learning Unit so that you can receive some feedback on how you are doing in this course. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 2 assignments and organize your material in the following order:

- ☐ Module 2 Cover Sheet (found at the end of the course Introduction)
- ☐ Assignment 2.1: Transformations of Functions
- ☐ Assignment 2.2: Combinations of Transformations
- ☐ Assignment 2.3: Operations on and Compositions of Functions

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Notes

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Module 2
Function Transformations

Learning Activity Answer Keys

MODULE 2: FUNCTION TRANSFORMATIONS

Learning Activity 2.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Simplify: $\frac{8!}{6!}$
2. In how many ways can three different flower arrangements be arranged on a table?
3. What is the reciprocal of $\frac{xy}{3}$?
4. Rationalize the denominator: $\frac{3}{1-\sqrt{5}}$
5. The Winnipeg Blue Bombers have sold out their stadium to 29 503 fans. Halfway through the game, it starts raining. If 15 316 fans leave, how many fans remain to see the Winnipeg Blue Bombers win the game?
6. If $f(x) = 3x^3 - 2x^2 + 1$, evaluate $f(x)$ at $x = -3$.
7. Factor: $4x^2 - 12xy + 9y^2$
8. Which fraction is larger: $\frac{9}{29}$ or $\frac{9}{27}$?

Answers:

1. 56 $\left(\frac{8 \times 7 \times 6!}{6!} = 8 \times 7\right)$
2. 6 (3! = 6)
3. $\frac{3}{xy}$
4. $\frac{\left(3 + 3\sqrt{5}\right)}{-4}$ $\left(\frac{3}{1-\sqrt{5}} \cdot \frac{1+\sqrt{5}}{1+\sqrt{5}} \rightarrow \frac{3+3\sqrt{5}}{1-5}\right)$
5. 14 187 (first, 29 000 − 15 000 = 14 000; 500 − 316 = 184; 14 000 + 184 + 3)
6. $f(-3) = -98$ $(3(-3)^3 - 2(-3)^2 + 1 \rightarrow -81 - 18 + 1)$

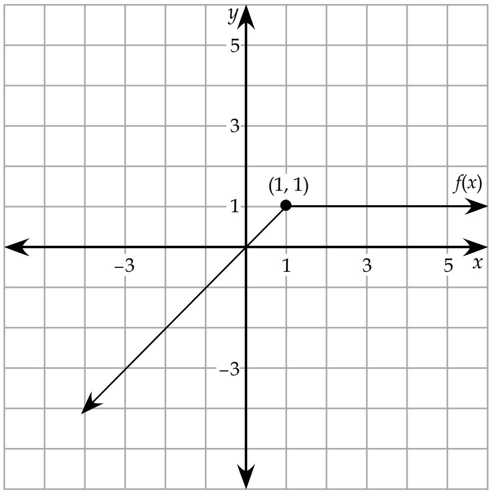
7. $(2x - 3y)^2$ $((2x - 3y)(2x - 3y))$

8. $\dfrac{9}{27}$

Part B: Horizontal and Vertical Translations

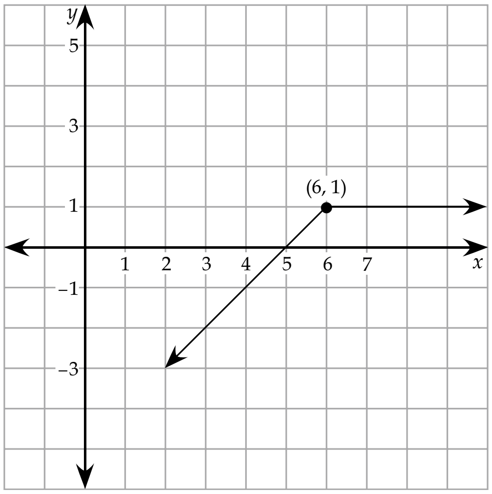
Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Given the sketch of $f(x)$ drawn below, sketch each of the following functions.
   a) $y = f(x - 5)$
   b) $y = f(x) - 5$
   c) $y = f(x) + 5$
   d) $y = f(x + 5)$
   e) $y = f(x - 5) - 5$
   f) $y = f(x + 5) + 5$

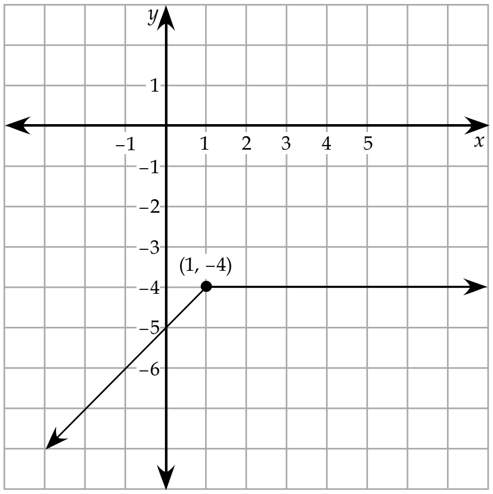


Answers:

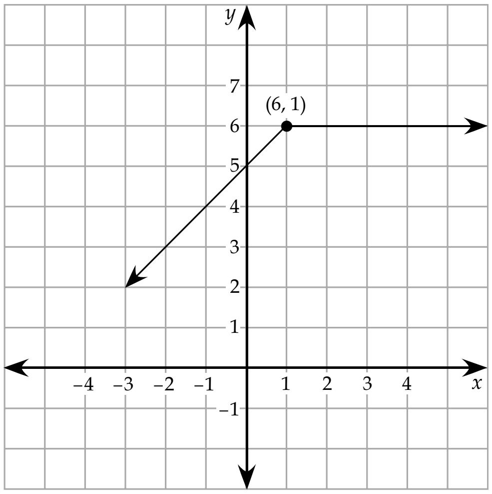
a) $y = f(x - 5)$



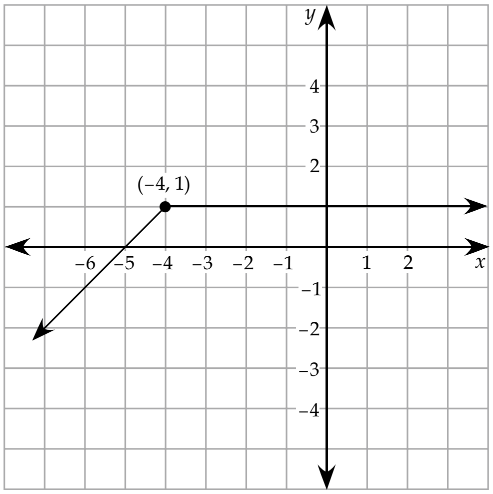
b) $y = f(x) - 5$



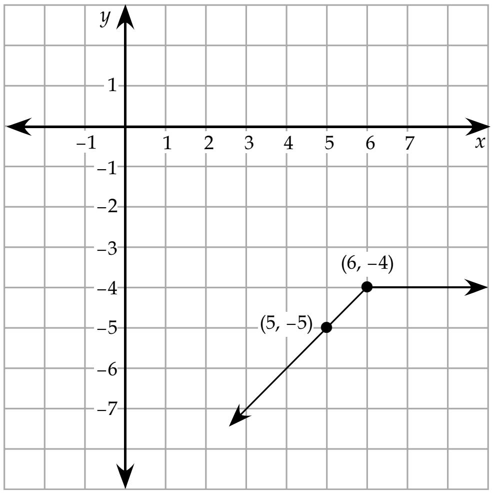
c) $y = f(x) + 5$



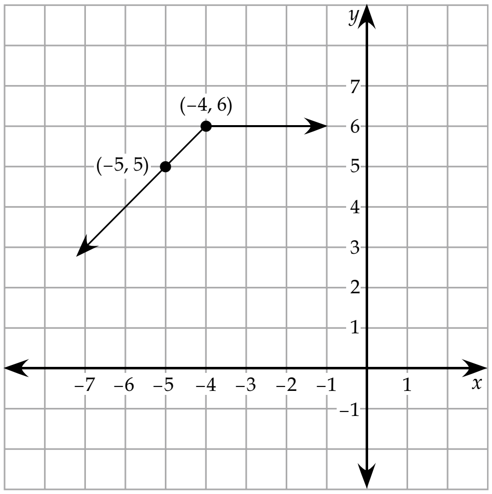
d) $y = f(x + 5)$



e) $y = f(x - 5) - 5$



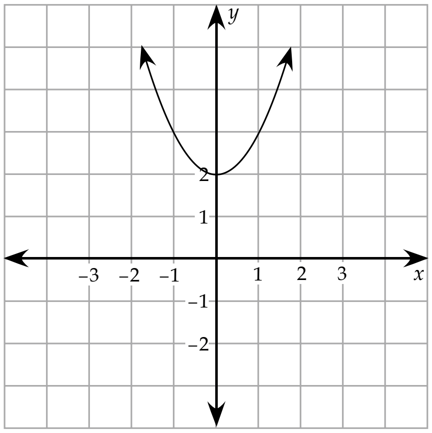
f) $y = f(x + 5) + 5$



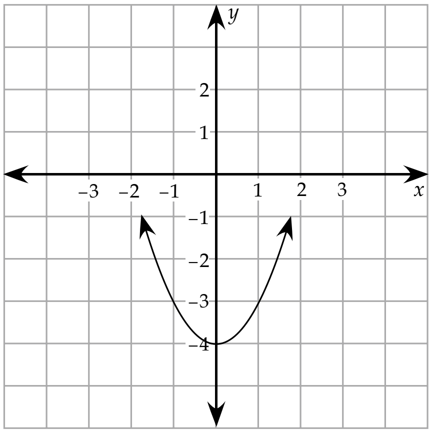
2. Let $f(x) = x^2 + 2$. Sketch each of the following functions.
   a) $f(x)$
   b) $y = f(x) - 6$
   c) $y = f(x + 1)$
   d) $y = f(x - 2) - 3$

Answers:

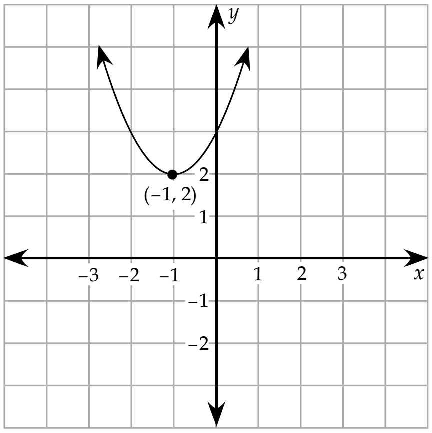
a) $f(x)$



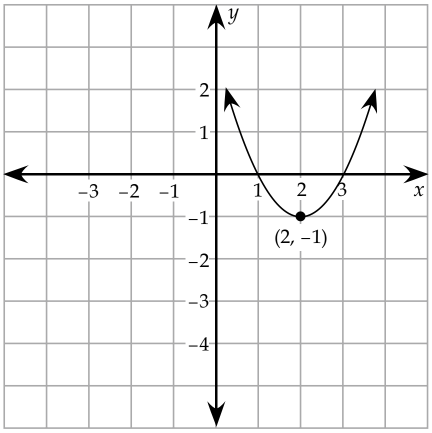
b) $y = f(x) - 6$



c) $y = f(x + 1)$



d) $y = f(x - 2) - 3$



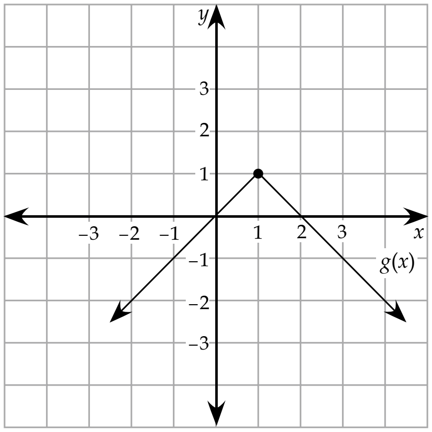
3. For each of the functions in question 2 state the properties of the function: the domain, the range, and the values of the intercepts. (Recall: To find the y-intercept, let $x = 0$ and solve for y. To find the x-intercept, let $y = 0$ and solve for x.)

   Recall: The domain is the set of all the possible x-values of the function. The range is the set of all the possible y-values of the function.

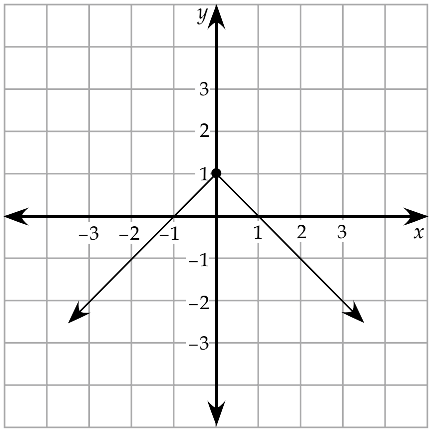
   Answers:

Question	**Function**	**Domain**	**Range**	**y-intercept**	**x-intercepts**
(a)	$f(x)$	$\Re$	$[2, \infty)$	2	$x^2 + 2 = 0$ no solution or $\varnothing$
(b)	$f(x) - 6$	$\Re$	$[-4, \infty)$	-4	$(x^2 + 2) - 6 = 0$ $x^2 - 4 = 0$ $x = \pm 2$
(c)	$f(x + 1)$	$\Re$	$[2, \infty)$	3	$(x + 1)^2 + 2 = 0$ $(x + 1)^2 = -2$ $\therefore$ no solution
(d)	$f(x - 2) - 3$	$\Re$	$[-1, \infty)$	3	$(x - 2)^2 + 2 - 3 = 0$ $(x - 2)^2 - 1 = 0$ $(x - 2)^2 = 1$ $x - 2 = \pm 1$ $x = 3$ or 1

4. Each of graphs (a), (b), and (c) represents a translation of the given function, $g(x)$, shown below. Write an expression for each new function in terms of $g(x)$.

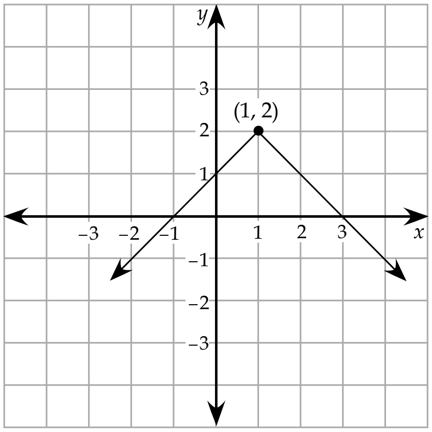


a)



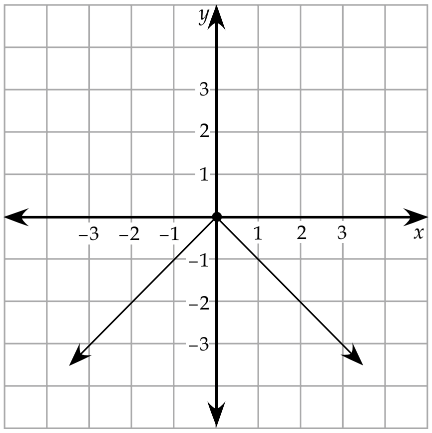
Answer: $y = g(x + 1)$

b)



Answer: $y = g(x) + 1$

c)



Answer: $y = g(x + 1) - 1$

5. If $f(x) = x^3 + 3x^2 - x + 6$, write an equation for the following. Do not simplify. For example, the equation of $g(x)$, which has the same graph as $f(x)$ moved two units to the left, would be $g(x) = f(x + 2)$. This could be written as $g(x) = f(x + 2) = (x + 2)^3 + 3(x + 2)^2 - (x + 2) + 6$.

   a) $h(x)$, which has the same graph as $f(x)$ moved three units down

   b) $m(x)$, which has the same graph as $f(x)$ moved two units to the right and one unit up

   Answers:

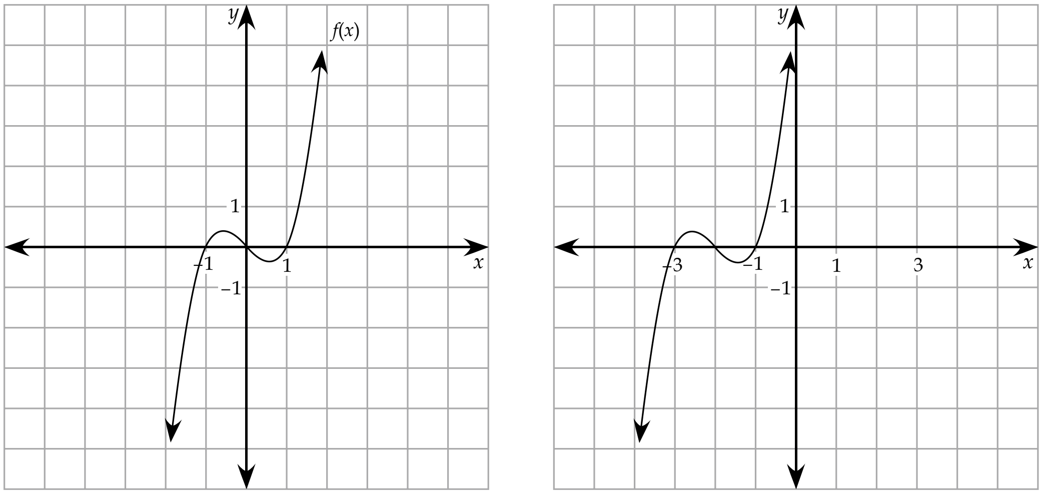
   a) $h(x) = f(x) - 3 = \left(x^3 + 3x^2 - x + 6\right) - 3$

   b) $m(x) = f(x - 2) + 1 = \left[(x - 2)^3 + 3(x - 2)^2 - (x - 2) + 6\right] + 1$

6. Below is the graph of $f(x) = x^3 - x$. Sketch the graph of $g(x) = (x + 2)^3 - (x + 2)$.

Answer:

Shift *f(x)* 2 units left.



7. Is the translation of a function still a function?

   Answer:

   The translation of a function is always a function. The new function is moved only horizontally or vertically and its slope doesn't change, so the new graph will still pass the vertical line test.

Learning Activity 2.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. If $2x + 14 = -6$, what is the value of x?
2. Simplify: $\sqrt[3]{\dfrac{x^6}{27y^4}}$
3. Simplify: $\dfrac{1}{2} + \dfrac{6}{7}$
4. Evaluate: $|\, 8.43 - 9.25 \,|$
5. Solve for x: $(x + 1)^2 = 16$
6. Factor: $2x^2 - 18x - 72$
7. Convert $\dfrac{451}{10\ 000}$ into a decimal.
8. Your restaurant bill came to $74.23. If you wish to leave a 20% tip, how much should you leave?

Answers:

1. $x = -10$ $(2x = -20)$
2. $\dfrac{x^2}{3y}\sqrt[3]{\dfrac{1}{y}}$ $\left(\sqrt[3]{\dfrac{x^6}{27y^3}} \cdot \sqrt[3]{\dfrac{1}{y}}\right)$
3. $\dfrac{19}{14}$ $\left(\dfrac{7}{14} + \dfrac{12}{14}\right)$
4. 0.82 (8.43 is 0.57 units from 9; 0.57 + 0.25)
5. $x = 3, -5$ $\left(x + 1 = \pm\sqrt{16};\ x = -1 + 4 \text{ or } x = -1 - 4\right)$
6. $2(x - 12)(x + 3)$ $[2(x^2 - 9x - 36)]$
7. 0.0451 (shift decimal of 451.0 four places left to divide by 10 000)
8. $14.84 (10% of 74.23 is 7.42, so 20% is 14.84)

Part B: Stretches and Compressions

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Compare and contrast the graph of $y = x^3$ with the graph of each of the following. Name one property that is the same and one that is different for the two graphs.

   a) $y = 5x^3$

   b) $y = \frac{1}{3}x^3$

   Answers:

   a) Same shape as $y = x^3$ except much steeper due to the vertical stretch by a factor of 5.

   b) Same shape as $y = x^3$ except much flatter due to the vertical stretch by a factor of $\frac{1}{3}$.

2. Use the graph of $f(x)$, shown below, to sketch the following.

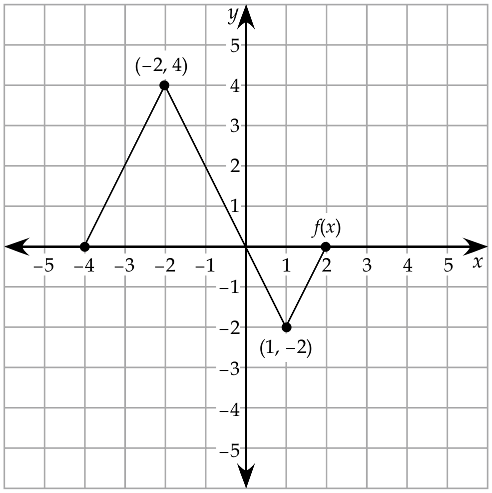
   a) $y = 2f(x)$

   b) $y = f(2x)$

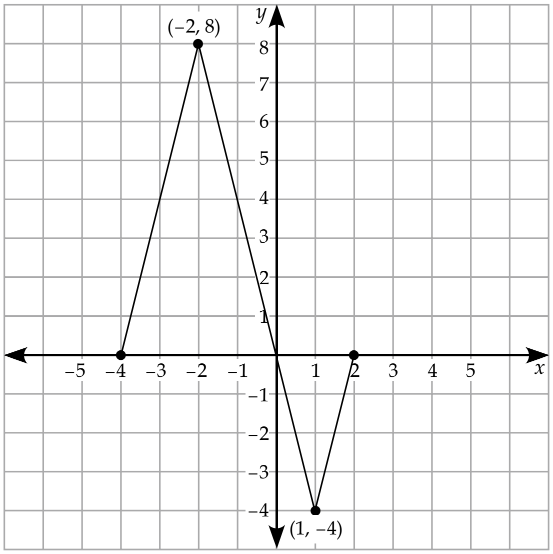
   c) $y = \frac{2}{3}f(x)$

   d) $y = f\left(\frac{1}{2}x\right)$

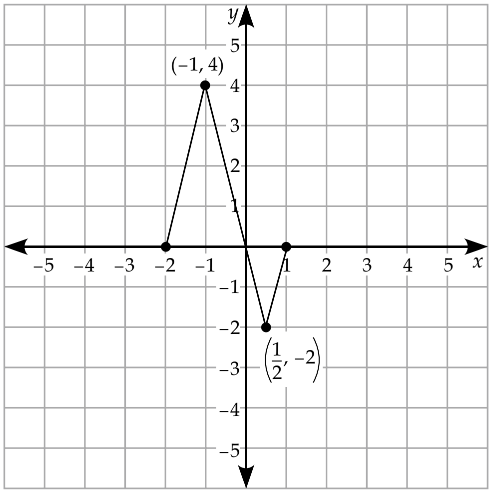
   e) $y = 2f(3x)$



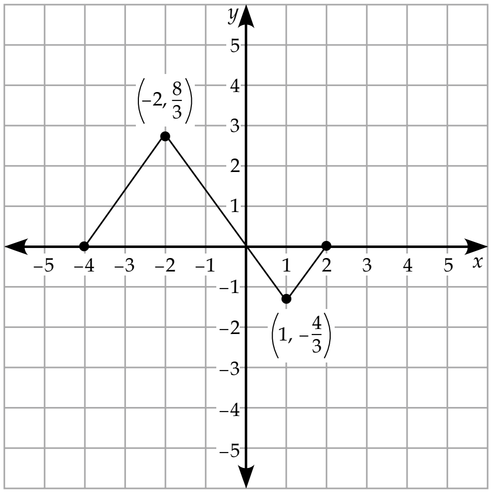
a) $y = 2f(x)$



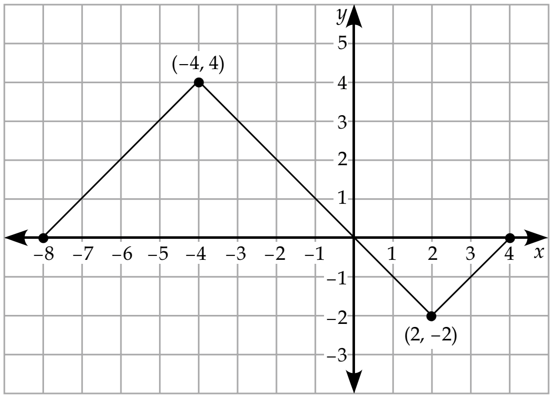
b) $y = f(2x)$



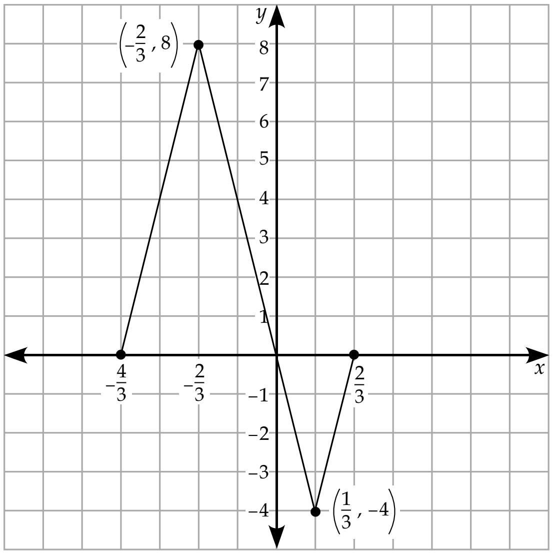
c) $y = \frac{2}{3}f(x)$



d) $y = f\left(\frac{1}{2}x\right)$



e) $y = 2f(3x)$



3. The x-intercepts of the function $f(x)$ are 6, 2, and −8. What are the x-intercepts of each of the following?

   a) $y = f(2x)$

   Answer:

   Horizontal compression by a factor of 2.

   3, 1, −4 (divide each x-intercept by 2)

   b) $y = f\left(\frac{1}{3}x\right)$

   Answer:

   Horizontal compression by a factor of $\frac{1}{3}$.

   18, 6, −24 (multiply each x-intercept by 3)

   c) $y = f(2x - 4)$

   Answer:

   Factor first to determine transformations for $y = f(2(x - 2))$.

   Horizontal compression by a factor of 2 and then shift right 2.

   5, 3, −2 (divide each x-intercept by 2 and then add 2)

4. How should the function equation for $f(x)$ be modified if you want to perform the following transformation?

   a) translate the graph two units to the right

   b) compress the graph horizontally by a factor of 5

   c) stretch the graph vertically by a factor of two and translate it three units down

   d) stretch the graph horizontally by a factor of 3 and compress the graph vertically by a factor of 5

   Answers:

   a) $y = f(x - 2)$

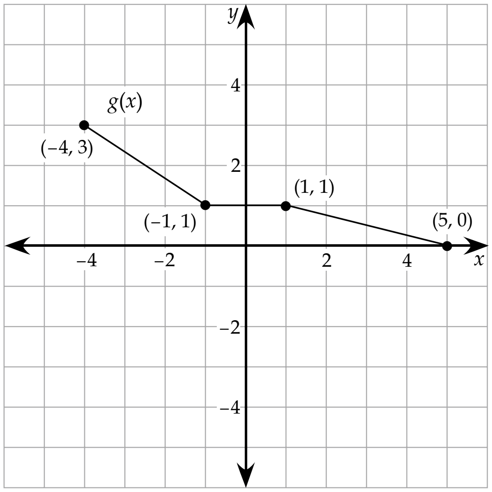
   b) $y = f(5x)$

   c) $y = 2f(x) - 3$

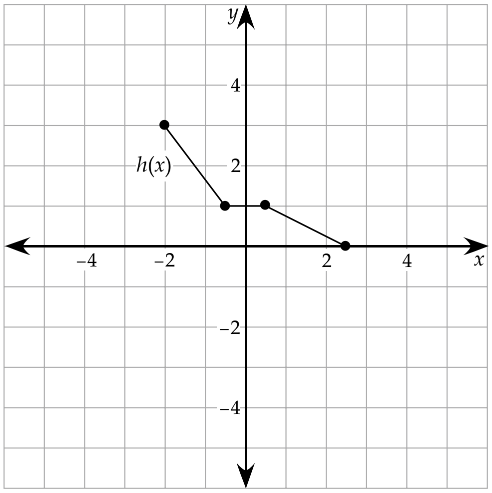
   d) It is easiest to think of horizontal compressions and vertical stretches, so convert to that form first. Horizontal compression by $\frac{1}{3}$ and vertical stretch of $\frac{1}{5}$.

   $$y = \frac{1}{5} f\left(\frac{1}{3}x\right)$$

5. Each of graphs (a), (b), and (c) represents a stretch and/or a compression of the given function, $g(x)$, shown below. Write an expression for each new function in terms of $g(x)$.



a)

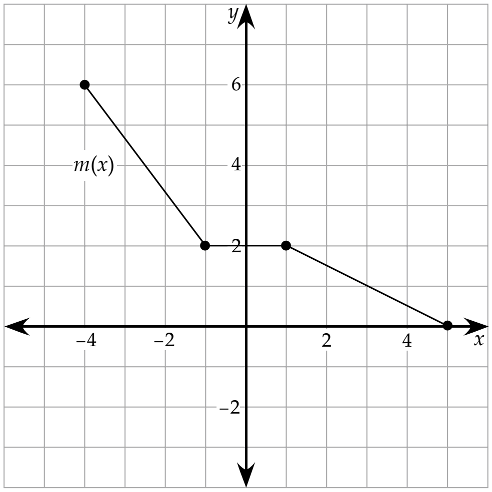


Answer:

All of the x-values of $h(x)$ are half of the x-values of $g(x)$. Therefore, this graph has undergone a horizontal compression by a factor of 2.

$h(x) = g(2x)$

b)

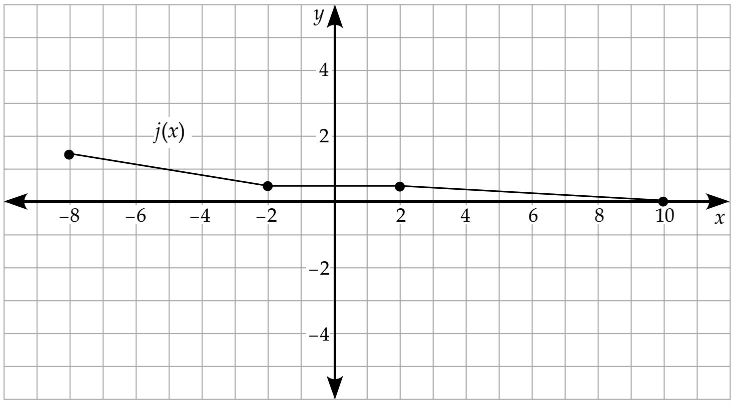


Answer:

All of the y-values of $m(x)$ are double the y-values of $g(x)$. Therefore, this graph has undergone a vertical stretch by a factor of 2.

$m(x) = 2g(x)$

c)



Answer:

All of the x-values of $j(x)$ are double the x-values of $g(x)$. Also, all of the y-values of $j(x)$ are half of the y-values of $g(x)$. This results in a horizontal stretch of a factor of 2 and a vertical stretch by a factor of $\frac{1}{2}$. Think of a horizontal compression of $\frac{1}{2}$ rather than a stretch of 2.

$$j(x) = \frac{1}{2}g\left(\frac{1}{2}x\right)$$

Learning Activity 2.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. What is the surface area of a cylinder with a diameter of 6 cm and a height of 4 cm?
2. Is $x = 2$ a solution to the inequality $-x^2 + 10x - 8 \geq 0$?
3. Simplify: $\dfrac{49x^7}{7x^3}$
4. Estimate the taxes, 13%, on a $226 item.
5. If $f(x) = \dfrac{x^2}{x-1}$, evaluate $f(-2)$.
6. Convert 0.024 into a percent.
7. Express $11 + \dfrac{4}{7}$ as an improper fraction.
8. Solve for x: $\dfrac{1}{x+2} = \dfrac{2}{x-1}$

Answers:

1. 42π cm^2 (each circle end, $\pi(3)^2$; curved sides, $2\pi(3)(4)$; SA = $9\pi + 9\pi + 24\pi$)
2. Yes ($-(2)^2 + 10(2) - 8 = -4 + 20 - 8 = 8$)
3. $7x^4$
4. $27 (10% of 226 is 22.6; 1% of 226 is 2.26; add 23 + 2 + 2)
5. $f(-2) = -\dfrac{4}{3}\left(\dfrac{(-2)^2}{-2-1} = \dfrac{4}{-3}\right)$
6. 2.4%
7. $\dfrac{81}{7}\left(\dfrac{77}{7} + \dfrac{4}{7}\right)$
8. $x = -5\left(\dfrac{x+2}{1} = \dfrac{x-1}{2} \rightarrow 2x + 4 = x - 1 \rightarrow x = -5\right)$

Part B: Combining Transformations

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Assume a graph of $f(x)$ is given. Describe how the graph of $F(x)$ is obtained (using the correct order of transformations when needed).

   a) $F(x) = \frac{1}{4}(4x)$

   Answer:

   $F(x)$ is a vertical stretch by a factor of $\frac{1}{4}$, as well as a horizontal compression by a factor of 4 of the graph of $f(x)$.

   b) $F(x) = 2f(x) + 1$

   Answer:

   $F(x)$ is a vertical stretch by a factor of 2 of the graph of $f(x)$, which is then translated 1 unit up.

   c) $F(x) = 4f(x + 1)$

   Answer:

   $F(x)$ is a vertical stretch by a factor of 4 of the graph of $f(x)$, and it is translated 1 unit to the left.

   d) $F(x) = 3f(x - 2)$

   Answer:

   $F(x)$ is a vertical stretch by a factor of 3 of the graph of $f(x)$, and it is translated 2 units to the right.

   e) $F(x) = f(2x) - 5$

   Answer:

   $F(x)$ is a horizontal compression by a factor of 2 of the graph of $f(x)$, and it is translated 5 units down.

   f) $F(x) = f\left(\frac{1}{3}(x - 1)\right) + 4$

   Answer:

   $F(x)$ is a horizontal compression by a factor of $\frac{1}{3}$ of the graph of $f(x)$, which is then translated 1 unit to the right and 4 units up.

g) $F(x) = \frac{1}{2}(f(x-2)) - 3$

Answer:

$F(x)$ is a vertical compression by a factor of 2 of the graph of $f(x)$, which is then translated 2 units to the right and 3 units down.

h) $F\left(x\right) = 2\left(f\left(\frac{1}{3}(x-5)\right)\right) + 7$

Answer:

$F(x)$ is a vertical stretch by a factor of 2 and a horizontal compression by a factor of $\frac{1}{3}$ of the graph of $f(x)$, which is then translated 5 units to the right and 7 units up.

2. Given the function $y = f(x)$, write the equation of the form $y = af(b(x - h)) + k$ that would result from each combination of transformations.

a) Vertical shift 5 units up, horizontal shift 2 units to the right, horizontal compression by a factor of 3

Answer:

$y = f(3(x - 2)) + 5$

b) Horizontal stretch by a factor of 4, vertical compression by a factor of 2, vertical shift 4 units down

Answer:

$$y = \frac{1}{2}f\left(\frac{1}{4}(x)\right) - 4$$

c) Horizontal compression by a factor of 6, vertical stretch by a factor of 3, vertical shift 8 units up, horizontal shift 7 units to the left

Answer:

$y = 3f(6(x + 7)) + 8$

3. The point (18, −6) is on the graph of $y = f(x)$. Determine its corresponding point after each of the following transformations of $f(x)$.

   a) $y = 2f(x - 1) + 3$

   Answer:

   This function is undergoing a horizontal compression by a factor of 3, a horizontal translation 2 units to the right, and a vertical translation 5 units up.

   The stretch occurs before the translation and therefore the y-coordinate is multiplied by 2 first.

   $(18, -6) \rightarrow (18, -12)$

   The translations then occur. The x-coordinate is increased by 1 and the y-coordinate is increased by 3.

   $(18, -12) \rightarrow (19, -9)$

   b) $y = \frac{1}{3}f\left(2(x - 1)\right) + 4$

   Answer:

   This function is undergoing a vertical compression by a factor of 3, a horizontal compression by a factor of 2, a horizontal shift 1 unit to the right and vertical shift 4 units up.

   The compressions occur before any translations. Therefore, as there is a horizontal compression by a factor of 2, divide the x-coordinate by 2. As there is a vertical compression by a factor of 3, multiply the y-coordinate by $\frac{1}{3}$.

   $(18, -6) \rightarrow (9, -2)$

   The translations then occur. The x-coordinate is increased by 1 and the y-coordinate is increased by 4.

   $(9, -2) \rightarrow (10, 2)$

c) $y = \frac{1}{2} f\left(\frac{1}{3}(x + 2)\right) - 6$

Answer:

This function is undergoing a vertical stretch by a factor of $\frac{1}{2}$, a horizontal compression by a factor of $\frac{1}{3}$, a horizontal translation 2 units to the left, and a vertical shift 6 units down.

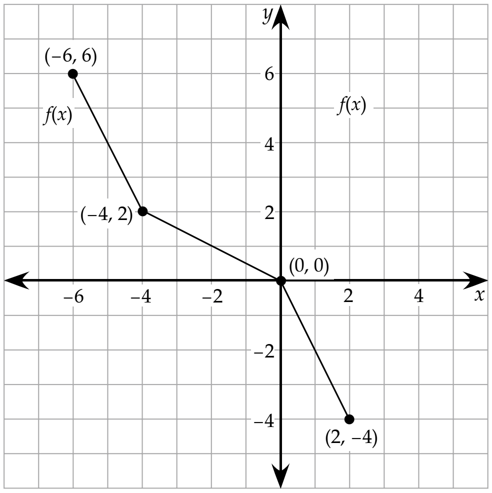
The compression and stretch occur before the translations. Therefore, as there is a horizontal stretch by a factor of 3, multiply the x-coordinate by 3. As there is a vertical stretch by a factor of $\frac{1}{2}$, multiply the y-coordinate by $\frac{1}{2}$.

$(18, -6) \rightarrow (54, -3)$

The translations then occur. The x-coordinate is decreased by 2 and the y-coordinate is decreased by 6.

$(54, -3) \rightarrow (52, -9)$

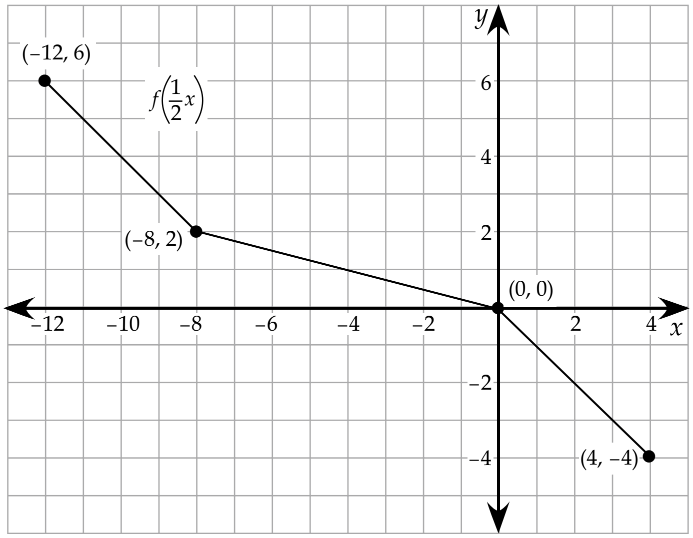
4. Use the graph of $f(x)$ below to graph each of the following transformations.



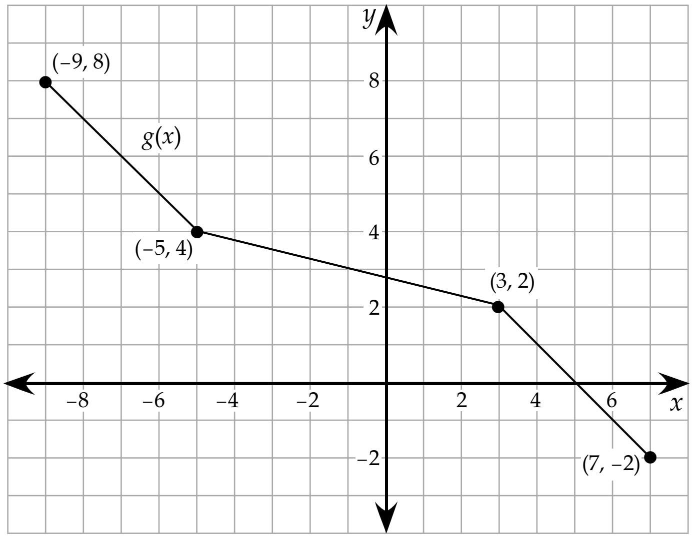
a) $g(x) = f\left(\frac{1}{2}(x - 3)\right) + 2$

Answer:

First, each point on the graph of $f(x)$ undergoes a horizontal compression by a factor of $\frac{1}{2}$ (each x-coordinate is multiplied by 2).



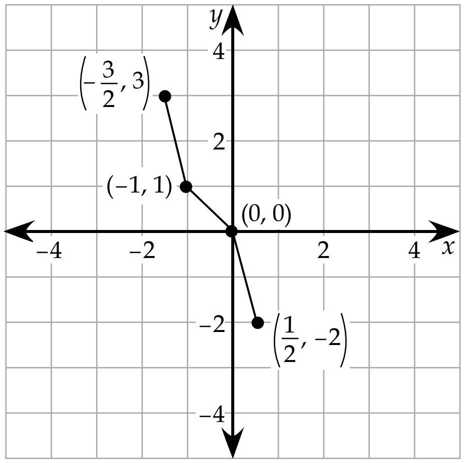
Then, each point is shifted 3 units to the right and 2 units up to achieve the graph of $g(x)$.



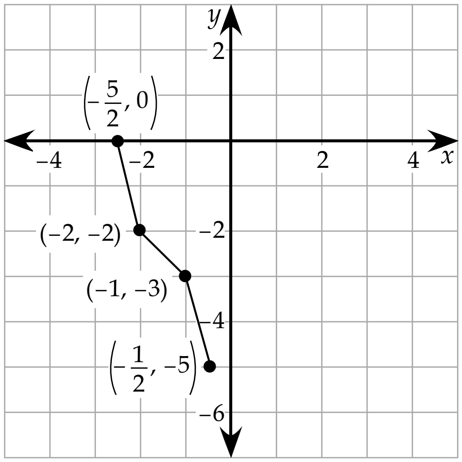
b) $h(x) = \frac{1}{2} f\left(4(x+1)\right) - 3$

Answer:

First, each point on the graph of *f*(*x*) undergoes a vertical compression by a factor of 2 (each *y*-coordinate is multiplied by $\left.\frac{1}{2}\right)$ and a horizontal compression by a factor of 4 (each *x*-coordinate is multiplied by $\left.\frac{1}{4}\right)$.



Then, each point is shifted 1 unit to the left and 3 units down to achieve the graph of *h*(*x*).

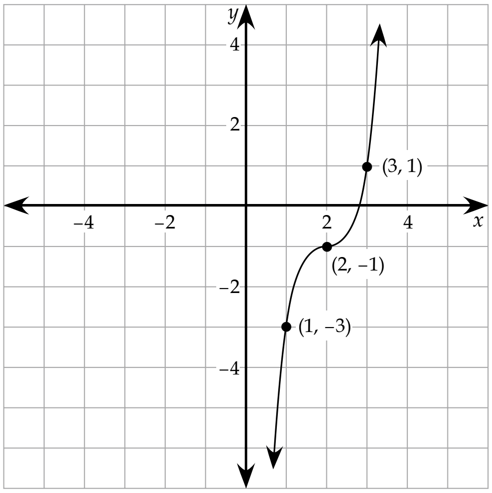


5. Sketch each function and state the domain, range, y-intercepts, and zeros of the function.

   a) $f(x) = 2(x-2)^3 - 1$

   Answer:

   First, you need to know how to graph the basic cubic curve, $y = x^3$. The vertical stretch by a factor of 2 is performed first on the graph of $y = x^3$, followed by a horizontal translation 2 units to the right and a vertical translation 1 unit down.



   Domain: $\{x \mid x \in \Re\}$

   Range: $\{y \mid y \in \Re\}$

   y-intercept: −17

   x-intercept: 2.794

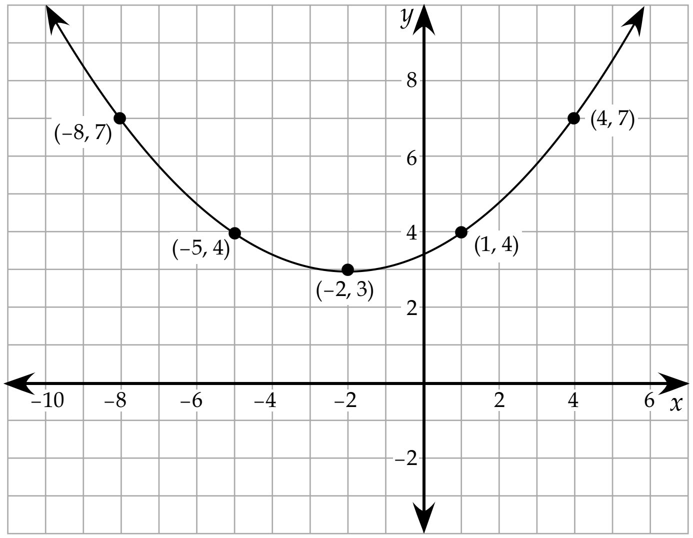
   Solving for the intercepts:

y-intercept	x-intercept
$f(0) = 2(0-2)^3 - 1$	$0 = 2(x-2)^3 - 1$
$f(0) = 2(-8) - 1 = -17$	$\frac{1}{2} = (x-2)^3$
	$\sqrt[3]{\frac{1}{2}} = x - 2$
	$\frac{1}{\sqrt[3]{2}} + 2 = x$
	$x \cong 2.794$

b) $f(x) = \left(\frac{1}{3}(x+2)\right)^2 + 3$

Answer:

The horizontal stretch by a factor of 3 is performed first on the graph of $y = x^2$, followed by a horizontal translation 2 units to the left and a vertical translation 3 units up.



Domain: $\{x \mid x \in \Re\}$

Range: $\{y \mid y \geq 3\}$

y-intercept: $\frac{31}{9}$

x-intercepts: none

Solving for the intercepts:

y-intercept

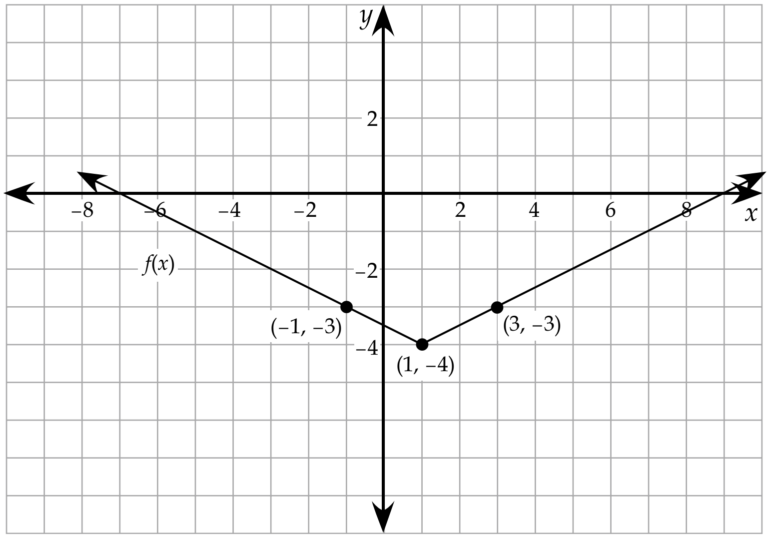
$$f(0) = \left(\frac{1}{3}(0+2)\right)^2 + 3$$

$$f(0) = \left(\frac{2}{3}\right)^2 + 3$$

$$f(0) = \frac{4}{9} + 3 = \frac{31}{9}$$

c) $f(x) = \left|\frac{1}{2}(x-1)\right| - 4$

Answer:



First, you need to know the shape of the basic absolute value curve, $y = |x|$.

To see the transformation in a table, consider the following three key points of $y = |x|$.

	Horizontal Stretch by a Factor of 2	Horizontal Shift 1 Unit Right	Vertical Shift 4 Units Down
(x, y)	$(2x, y)$	$(2x + 1, y)$	$(2x + 1, y - 4)$
$(-1, 1)$	$(-2, 1)$	$(-1, 1)$	$(-1, -3)$
$(0, 0)$	$(0, 0)$	$(1, 0)$	$(1, -4)$
$(1, 1)$	$(2, 1)$	$(3, 1)$	$(3, -3)$

Domain: $\{x \mid x \in \Re\}$

Range: $\{y \mid y \geq -4\}$

y-intercepts: $\left(0, -\frac{7}{2}\right)$

x-intercepts: $(-7, 0)$ and $(9, 0)$

Solving for the intercepts:

(Recall, as the absolute value of a negative number is positive, you need to account for both positive and negative values inside the absolute value brackets.)

y-intercept:

$$f(0) = \left|\frac{1}{2}(0-1)\right| - 4$$

$$f(0) = \left|-\frac{1}{2}\right| - 4$$

$$f(0) = -\frac{7}{2}$$

x-intercept

$$0 = \left|\frac{1}{2}(x-1)\right| - 4$$

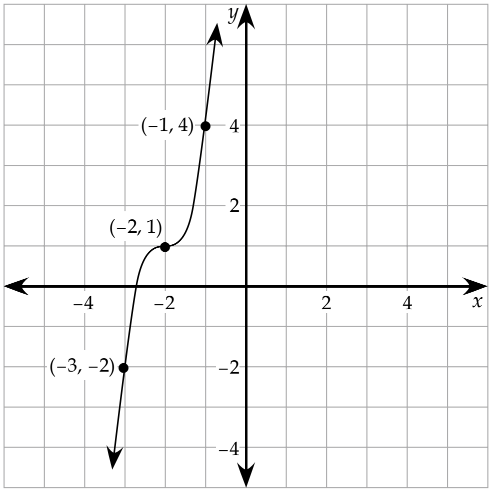
$$4 = \left|\frac{1}{2}(x-1)\right|$$

$$8 = -(x-1) \text{ or } 8 = x-1$$

$$x = -7 \text{ and } x = 9$$

d) $g(x) = 3(x+2)^3 + 1$

Answer:



To see the transformation in a table, consider the following three key points of $y = x^3$.

	Horizontal Shift 2 Units Left	**Vertical Stretch By a Factor of 3**	**Vertical Shift 1 Unit Up**
(x, y)	$(x - 2, y)$	$(x - 2, 3y)$	$(x - 2, 3y + 1)$
(−1, −1)	(−3, −1)	(−3, −3)	(−3, −2)
(0, 0)	(−2, 0)	(−2, 0)	(−2, 1)
(1, 1)	(−1, 1)	(−1, 3)	(−1, 4)

Domain: $\{x \mid x \in \Re\}$

Range: $\{y \mid y \in \Re\}$

y-intercepts: (0, 25)

x-intercepts: (−2.693, 0)

Solving for the intercepts:

y-intercept

$$g(0) = 3(0+2)^3 + 1$$

$$g(0) = 3(8) + 1 = 25$$

x-intercept

$$0 = 3(x+2)^3 + 1$$

$$-\frac{1}{3} = (x+2)^3$$

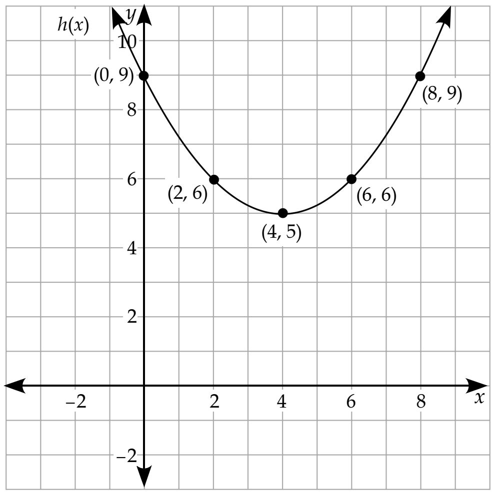
$$\sqrt[3]{-\frac{1}{3}} = x + 2$$

$$-\frac{1}{\sqrt[3]{3}} - 2 = x$$

$$x \cong -2.693$$

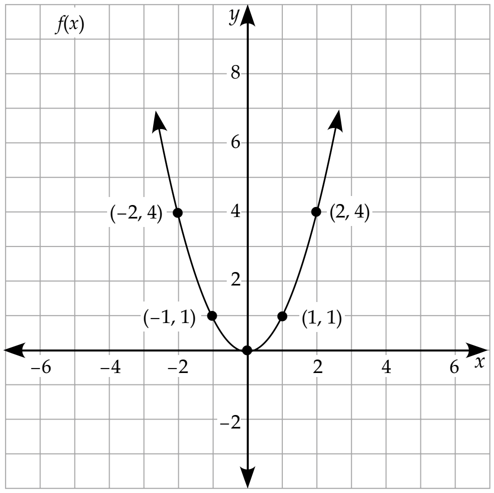
6. Each of the graphs below represents a transformation of $f(x) = x^2$. Write an expression for each new function in terms of $f(x)$.

a)



Answer:

First, sketch the basic curve, $f(x) = x^2$.



Locate key points on the graph of $f(x)$ and their corresponding points on the graph of $h(x)$.

$f(x)$	$h(x)$
(−2, 4)	(0, 9)
(−1, 1)	(2, 6)
(0, 0)	(4, 5)
(1, 1)	(6, 6)
(2, 4)	(8, 9)

As you can see from the table, all of the y-values of $h(x)$ are 5 more than the y-values of $f(x)$. Therefore, this graph has been shifted 5 units up.

Now, there is no apparent pattern between the x-coordinates. However, if you look at the y-intercept, (0, 0), you will notice that this point moves 4 units to the right. Therefore, as a horizontal stretch or compression does not affect the y-intercept, there must have been a horizontal translation 4 units to the right.

It may be helpful for you now to create a table of values of what the points of $h(x)$ would have looked like before the translation.

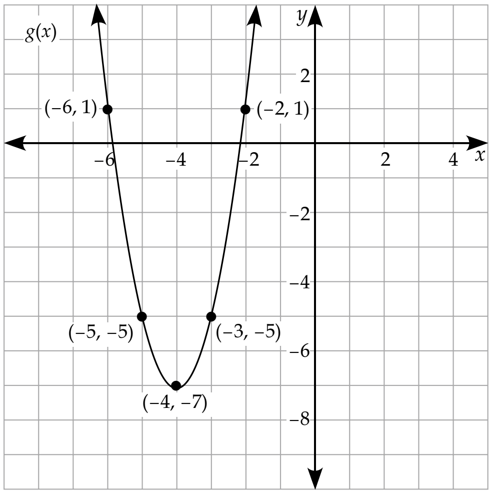
$f(x)$	$h(x)$
(−2, 4)	(−4, 9)
(−1, 1)	(−2, 6)
(0, 0)	(0, 5)
(1, 1)	(2, 6)
(2, 4)	(4, 9)

From the chart, you can see that each of the x-values were doubled. Therefore, this graph has been horizontally stretched by a factor of 2.

Putting all the transformations together, you get:

$$h(x) = f\left(\frac{1}{2}(x-4)\right) + 5 \text{ or } h(x) = \left(\frac{1}{2}(x-4)\right)^2 + 5$$

b)



Answer:

Locate key points on the graph of $f(x)$ and their corresponding points on the graph of $g(x)$.

$f(x)$	$g(x)$
(−2, 4)	(−6, 1)
(−1, 1)	(−5, −5)
(0, 0)	(−4, −7)
(1, 1)	(−3, −5)
(2, 4)	(−2, 1)

As you can see from the table, all of the x-values of $g(x)$ are 4 less than the x-values of $f(x)$. Therefore, this graph has been moved 4 units to the left.

Now, there is no apparent pattern between the y-coordinates. However, if you look at the x-intercept, (0, 0), you will notice that this point moves 7 units down. Therefore, as a vertical stretch or compression does not affect the x-intercept, there must have been a vertical translation 7 units down.

It may be helpful for you now to create a table of values of what the points of $g(x)$ would have looked like before the vertical translation.

$f(x)$	$g(x)$
$(-2, 4)$	$(-6, 8)$
$(-1, 1)$	$(-5, 2)$
$(0, 0)$	$(-4, 0)$
$(1, 1)$	$(-3, 2)$
$(2, 4)$	$(-2, 8)$

From the chart, you can see that each of the y-values were doubled. Therefore, this graph has been vertically stretched by a factor of 2.

Putting all the transformations together, you get:

$g(x) = 2f(x + 4) - 7$ or $g(x) = 2(x + 4)^2 - 7$

Learning Activity 2.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Simplify: $\left(\sqrt{3x^2}\right)^4$
2. If you received a mark of 18 out of 25 on a test, what percentage does this represent?
3. What is the length of the remaining leg of a right-angled triangle if the hypotenuse measures 20 m and one leg measures 16 m?
4. What are the x-intercepts of the quadratic function $y = x^2 + 6x - 7$?
5. In how many ways can three different flower arrangements be arranged on a table?
6. If $x = -2$, calculate $(2x - 3)(2x + 4)$.
7. Write as an entire radical: $4x\sqrt{3xy}$
8. Simplify: $2\sqrt{12} - 5\sqrt{27}$

Answers:

1. $9x^4$ $\left(\left(\left(3x^2\right)^{\frac{1}{2}}\right)^4 = \left(3x^2\right)^2 = 9x^4\right)$
2. 72% $(18 \times 4 = 72)$
3. 12 m (3 – 4 – 5 right triangle is similar to 12 – 16 – 20)
4. $x = -7$ and $x = 1$ $(y = (x + 7)(x - 1))$
5. 6 (3!)
6. 0 $(2(-2) - 3)(2(-2) + 4) = (-7)(0) = 0)$
7. $\sqrt{48x^3y}$ $\left(\sqrt{16x^2} \cdot \sqrt{3xy}\right)$
8. $-11\sqrt{3}$ $\left(2\sqrt{4}\sqrt{3} - 5\sqrt{9}\sqrt{3} \rightarrow 2(2)\sqrt{3} - 5(3)\sqrt{3} \rightarrow 4\sqrt{3} - 15\sqrt{3}\right)$

Part B: Operations on Functions

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Using the functions $f(x) = |x - 1|$, $g(x) = x + 2$, and $h(x) = x^2 + 4x + 4$, determine a simplified function equation for the following.

   a) $(f + g)(x)$

   Answer:

   $(f + g)(x) = |x - 1| + x + 2$

   b) $(f - g)(x)$

   Answer:

   $(f - g)(x) = |x - 1| - (x + 2) = |x - 1| - x - 2$

   c) $(g + h)(x)$

   Answer:

   $(g + h)(x) = x + 2 + x^2 + 4x + 4 = x^2 + 5x + 6$

   d) $\left(\dfrac{f}{h}\right)(x)$

   Answer:

   $$\left(\frac{f}{h}\right)(x) = \frac{|x - 1|}{x^2 + 4x + 4}$$

   e) $(h \cdot g)(x)$

   Answer:

   $(h \cdot g)(x) = (x^2 + 4x + 4)(x + 2) = x^3 + 2x^2 + 4x^2 + 8x + 4x + 8$
   $= x^3 + 6x^2 + 12x + 8$

   f) $(h - g)(x)$

   Answer:

   $(h - g)(x) = x^2 + 4x + 4 - (x + 2) = x^2 + 4x + 4 - x - 2 = x^2 + 3x + 2$

g) $\left(\frac{h}{g}\right)(x)$

Answer:

$$\left(\frac{h}{g}\right)(x) = \frac{\left(x^2 + 4x + 4\right)}{x + 2} = \frac{(x + 2)(x + 2)}{x + 2} = x + 2,\ x \neq -2$$

h) $(f \cdot g)(x)$

Answer:

$(f \cdot g)(x) = |\ x - 1\ |(x + 2)$

2. Use the functions in (1) to evaluate the following.

a) $\left(\frac{f}{h}\right)(4)$

Answer:

$$\left(\frac{f}{h}\right)(x) = \frac{|x - 1|}{x^2 + 4x + 4}$$

$$\left(\frac{f}{h}\right)(4) = \frac{|4 - 1|}{4^2 + 4(4) + 4} = \frac{3}{16 + 16 + 4} = \frac{3}{36} = \frac{1}{12}$$

b) $(h \cdot g)(-3)$

Answer:

$$(h \cdot g)(x) = x^3 + 6x^2 + 12x + 8$$

$$(h \cdot g)(-3) = (-3)^3 + 6(-3)^2 + 12(-3) + 8 = -27 + 54 - 36 + 8 = -1$$

c) $(f - g)(7)$

Answer:

$$(f - g)(x) = |x - 1| - x - 2$$

$$(f - g)(7) = |7 - 1| - 7 - 2 = 6 - 7 - 2 = -3$$

d) $(g + h)(-5)$

Answer:

$$(g + h)(-5)$$

$$(g + h)(x) = x^2 + 5x + 6$$

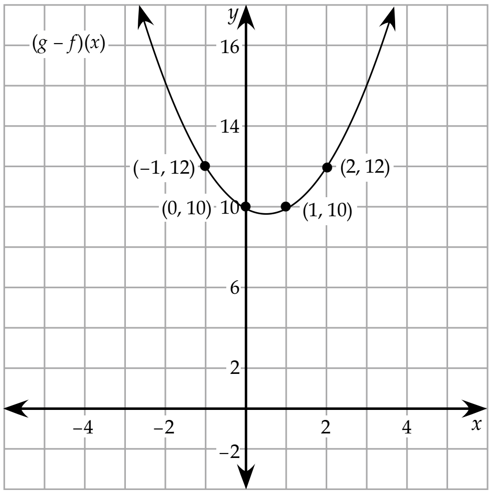
$$(g + h)(-5) = (-5)^2 + 5(-5) + 6 = 25 - 25 + 6 = 6$$

3. Let $f(x) = x - 1$ and $g(x) = x^2 + 9$. Graph each of the following, stating the domain and range.

   a) $(g - f)(x)$

   Answer:

   $(g - f)(x) = x^2 + 9 - (x - 1) = x^2 + 9 - x + 1 = x^2 - x + 10$



Domain: $\{x \mid x \in \Re\}$

Range: $\left\{y \mid y \geq \dfrac{39}{4}\right\}$

Remember, to find this value, you can find the vertex of the parabola.

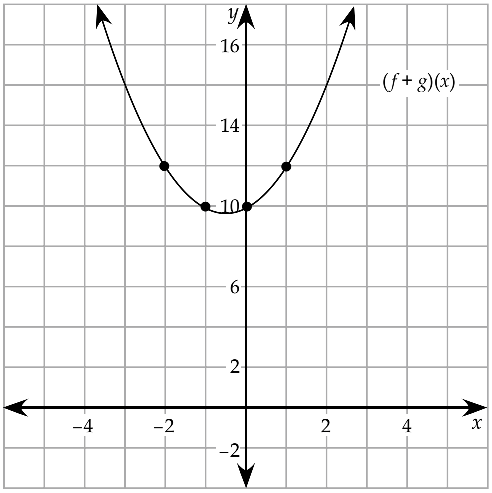
$$x = -\frac{b}{2a} = -\frac{-1}{2(1)} = \frac{1}{2}$$

$$y = \left(\frac{1}{2}\right)^2 - \frac{1}{2} + 10 = \frac{1}{4} - \frac{1}{2} + 10 = \frac{1}{4} - \frac{2}{4} + \frac{40}{4} = \frac{39}{4}$$

b) $(f + g)(x)$

Answer:

$(f + g)(x) = x - 1 + x^2 + 9 = x^2 + x + 8$



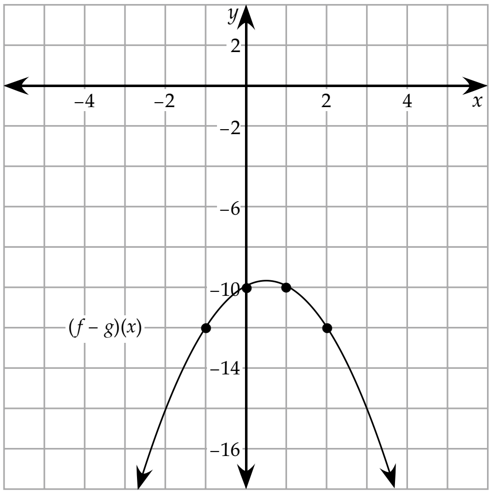
Domain: $\{x \mid x \in \Re\}$

Range: $\{y \mid y \geq 7.75\}$

c) $(f - g)(x)$

Answer:

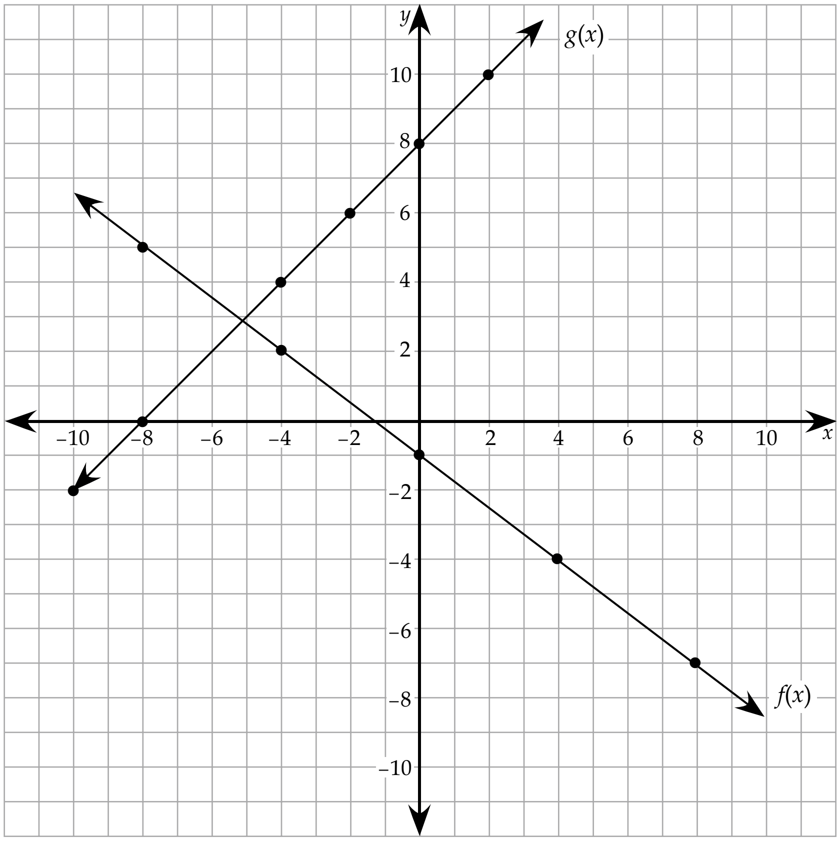
$(f - g)(x) = x - 1 - (x^2 + 9) = x - 1 - x^2 - 9 = -x^2 + x - 10$



Domain: $\{x \mid x \in \Re\}$

Range: $\left\{y \mid y \leq -\dfrac{39}{4}\right\}$

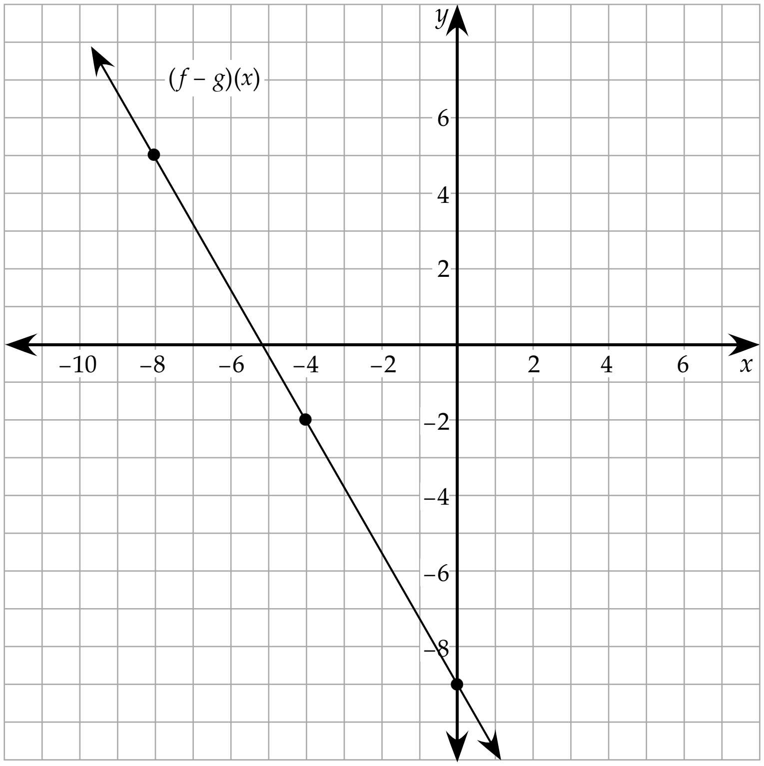
4. Consider the graphs of $f(x)$ and $g(x)$ shown below. Use these graphs to sketch $(f - g)(x)$ and $(f \cdot g)(x)$.



Answer:

Create a table of values. The sum or difference of two linear functions will still be linear, so you don't need many points.

x	f	g	$(f - g)(x)$	Coordinates
−8	5	0	5	(−8, 5)
−4	2	4	−2	(−4, −2)
0	−1	8	−9	(0, −9)



Create a table of values. The product of two linear functions will be a quadratic. Be sure to use the *x*-intercepts of the lines to find the *x*-intercepts of the parabola.

Also, use the *x*-value at the midpoint of the *x*-intercepts since the vertex of the parabola will be there.

The *x*-intercepts are at -8 and $-1\frac{1}{3}$. The midpoint between -8 and $-\frac{4}{3}$ is the average.

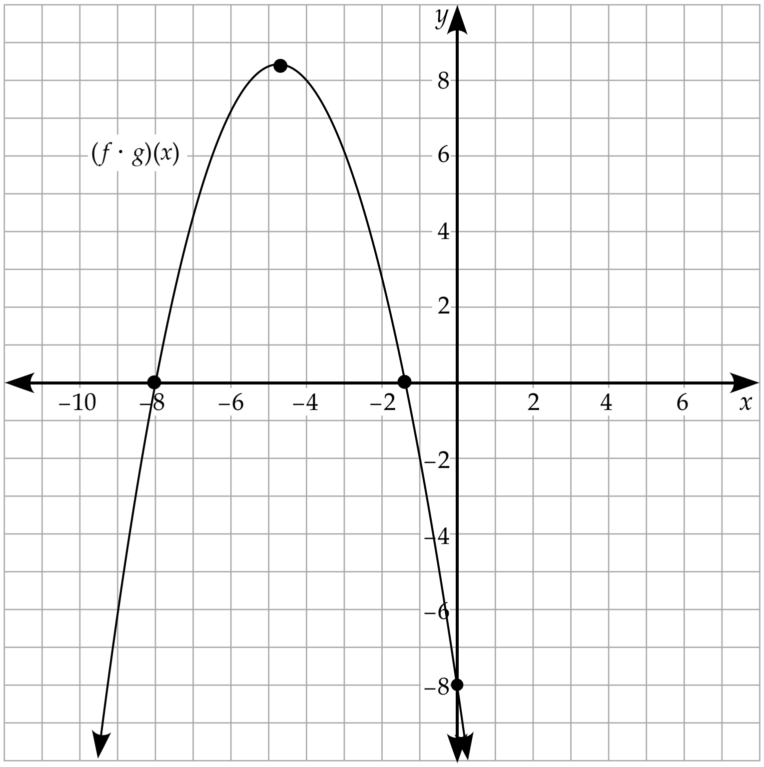
$$\left((-8)+\left(-\frac{4}{3}\right)\right)\div 2$$

$$=-\frac{28}{8}\times\frac{1}{2}$$

$$=-\frac{14}{3}\text{ or }-4\frac{2}{3}$$

x	f	g	$(f \cdot g)(x)$
-8	5	0	0
$-4\frac{2}{3}$	2.5	3.3	8.3
$-1\frac{1}{3}$	0	6.7	0

Note: The values in the table are read from the graph, so they may not be exact.



5. Let $f(x) = x + 5$ and $g(x) = x^2 + 2x - 15$. Write the function equation and state the domain.

a) $\left(\dfrac{f}{g}\right)(x)$

Answer:

$$\left(\frac{f}{g}\right)(x) = \frac{x+5}{x^2+2x-15} = \frac{(x+5)}{(x+5)(x-3)} = \frac{1}{x-3},\ x \neq 3,\ -5$$

To determine the domain, find the value of x that makes the denominator zero, since those values are not permitted.

Domain: $\{x \mid x \neq 3, -5, x \in \Re\}$

b) $(f \cdot g)(x)$

Answer:

$$\begin{aligned}(f \cdot g)(x) &= (x + 5)(x^2 + 2x - 15)\\ &= x^3 + 2x^2 - 15x + 5x^2 + 10x - 75\\ &= x^3 + 7x^2 - 5x - 75\end{aligned}$$

The domain of $f(x)$ is all real numbers and the domain of $g(x)$ is all real numbers, so the domain of the product is all real numbers.

Domain: $\{x \mid x \in \Re\}$

c) $\left(\dfrac{g}{f}\right)(x)$

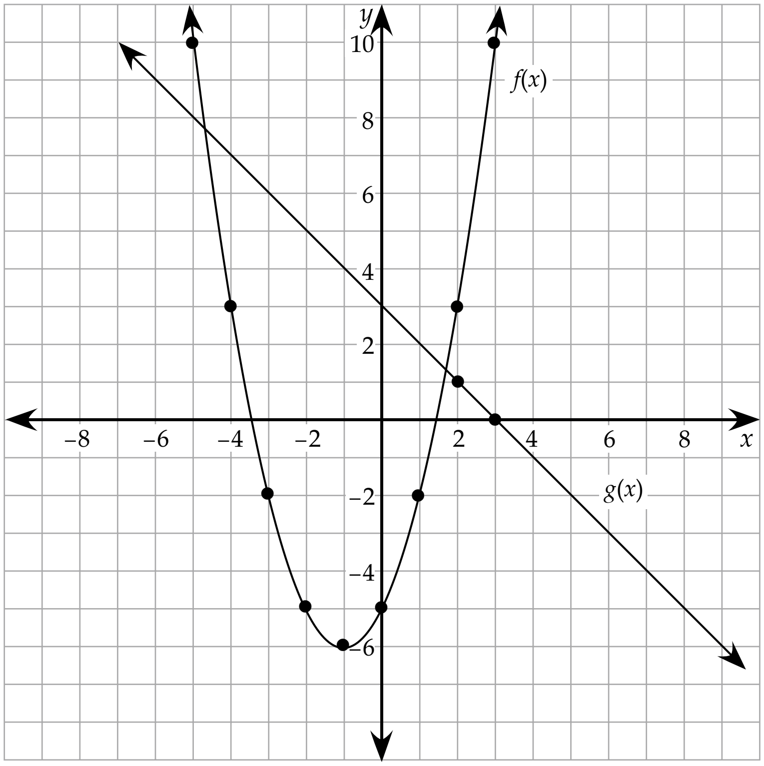
Answer:

$$\left(\frac{g}{f}\right)(x) = \frac{\left(x^2+2x-15\right)}{x+5} = \frac{(x+5)(x-3)}{x+5} = x - 3,\ x \neq -5$$

To determine the domain, find the values of x that make the denominator zero.

Domain: $\{x \mid x \neq -5, x \in \Re\}$

6. Use the graphs of $f(x)$ and $g(x)$ to evaluate the following.



a) $(f+g)(2)$

Answer:

$(f+g)(2) = f(2) + g(2) = 3 + 1 = 4$

b) $(f-g)(-4)$

Answer:

$(f-g)(-4) = f(-4) - g(-4) = 3 - 7 = -4$

c) $(f \cdot g)(0)$

Answer:

$(f \cdot g)(0) = (f(0))(g(0)) = (-5)(3) = -15$

d) $\left(\dfrac{g}{f}\right)(3)$

Answer:

$$\left(\frac{g}{f}\right)(3) = \frac{g(3)}{f(3)} = \frac{0}{10} = 0$$

7. If $h(x) = \dfrac{f(x)}{g(x)}$ and $g(x) = x^2 + 2x + 1,$ determine $f(x)$.

a) $h(x) = \dfrac{x+2}{x^2 + 2x + 1}$

Answer:

$$h(x) = \frac{f(x)}{g(x)} = \frac{x+2}{x^2 + 2x + 1}$$

$$\therefore\ f(x) = x + 2$$

b) $h(x) = 1$

Answer:

$$h(x) = \frac{f(x)}{g(x)} = 1 = \frac{f(x)}{x^2 + 2x + 1}$$

$$\therefore\ f(x) = x^2 + 2x + 1$$

c) $h(x) = \dfrac{1}{x+1}$

Answer:

$$h(x) = \frac{f(x)}{g(x)} = \frac{1}{x+1} = \frac{f(x)}{x^2 + 2x + 1} = \frac{f(x)}{(x+1)(x+1)}$$

$$\frac{1}{x+1} = \frac{f(x)}{(x+1)(x+1)}$$

$$\therefore\ f(x) = x + 1$$

d) $h(x) = x + 8$

Answer:

$$h(x) = \frac{f(x)}{g(x)} = x + 8 = \frac{f(x)}{x^2 + 2x + 1}$$

$$\therefore\ f(x) = \left(x^2 + 2x + 1\right)(x + 8)$$

8. A hot dog vendor sets up his shop on a downtown Winnipeg street corner. His daily costs are \$140.00 plus \$1.35 per every hot dog sold. This vendor charges \$3.75 for each hot dog and he sells up to 350 hot dogs a day.

   a) Write equations to represent the total cost, C, and the total proceeds, P, as functions of h, the number of hot dogs sold.

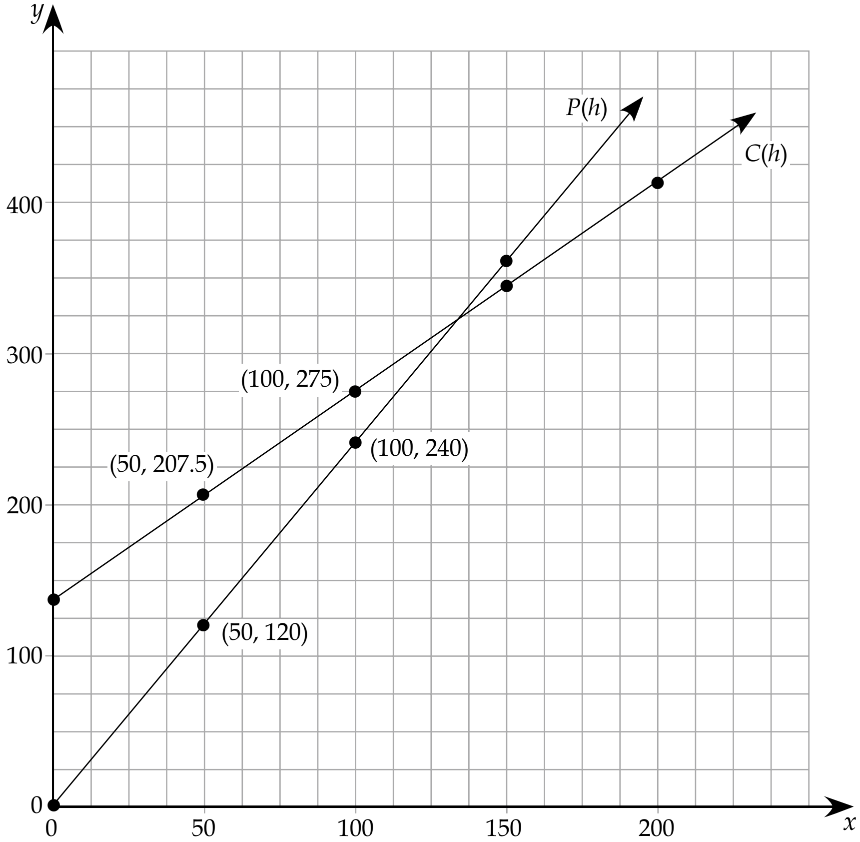
   Answer:

   $C = 140 + 1.35h$

   $P = (3.75 - 1.35)h = 2.4h$

   b) Graph these functions on the same coordinate grid.

   Answer:



As you cannot sell a negative number of hot dogs or have a business that has a negative cost, you do not need to include the negative x- and y-axes.

c) The point at which the hot dog vendor makes just enough money to cover his expenses is $C(h) = P(h)$. Find the number of hot dogs required to break even.

Answer:

$$C(h) = P(h)$$

$$140 + 1.35h = 2.4h$$

$$1.05h = 140$$

$$h = \frac{140}{1.05} = 133.33 \text{ or } 134 \text{ hot dogs}$$

d) Develop an algebraic function for the hot dog vendor's profit.

Answer:

Profit = Proceeds − Cost

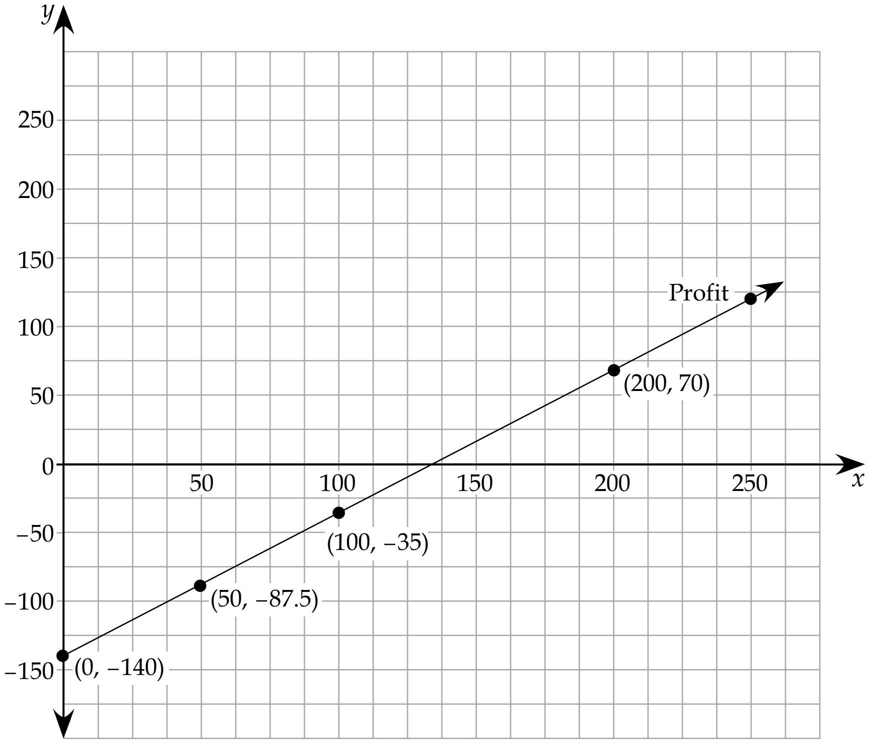
Profit = $2.4h - (140 + 1.35h)$

Profit = $2.4h - 140 - 1.35h$

Profit = $1.05h - 140$

e) Graph the profit function.

Answer:



As it is possible for the vendor to make a negative amount of money, you need to include the negative y-axis. However, it is still not possible to sell a negative number of hot dogs and thus you do not need to include the negative x-axis.

f) What is the maximum amount of money this hot dog vendor can earn in one day?

Answer:

This vendor sells a maximum of 350 hot dogs in one day. Therefore, you need to evaluate the profit formula as if the vendor sold 350 hot dogs.

Profit = 1.05(350) − 140 = \$227.50

The maximum amount of money this hot dog vendor can earn in one day is \$227.50.

Learning Activity 2.5

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Calculate: 6!
2. Evaluate: ${}_4P_3$
3. Factor: $3xy^6 - 6x^2y$
4. List all the factors of 36.
5. What is $\frac{1}{3}$ of 324?
6. Simplify: $-|\,-5(4) + 2 - 8^2\,|$
7. Simplify: $5\sqrt{8x^3} - 3x\sqrt{50x}$
8. Simplify: $8^{\frac{1}{3}}$

Answers:

1. 720 ($6 \times 5 \times 4 \times 3 \times 3 \times 1 = (6 \times 4 \times 3) \cdot (5 \times 2) = 72 \times 10$)
2. 24 ($4 \times 3 \times 2$)
3. $3xy(y^5 - 2x)$
4. 1, 2, 3, 4, 6, 9, 12, 18, 36
5. 108 ($324 \div 3$ is $300 \div 3 + 24 \div 3$ is $100 + 8$
6. -82 ($-|\,-20 + 2 - 64\,| = -|\,-82\,| = -(82)$)
7. $5x\sqrt{2x}$ $\left(5\sqrt{4x^2} \cdot \sqrt{2x} - 3x\sqrt{25}\sqrt{2x} = 10x\sqrt{2x} - 15x\sqrt{2x}\right)$
8. $2 \left(8^{\frac{1}{3}} = \sqrt[3]{8}\right)$

Part B: Compositions of Functions

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and v. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Given that $f(x) = 3x + 4$ and $g(x) = x^2 - 1$, find the following.

   a) $g(f(2))$

   Answer:

   $f(2) = 3 \cdot 2 + 4 = 10$

   $$\begin{aligned} g(f(2)) &= g(10) \\ &= 10^2 - 1 \\ &= 99 \end{aligned}$$

   b) $f(g(2))$

   Answer:

   $g(2) = 2^2 - 1 = 3$

   $$\begin{aligned} f(g(2)) &= f(3) \\ &= 3 \cdot 3 + 4 \\ &= 13 \end{aligned}$$

   c) $g(f(-1))$

   Answer:

   $f(-1) = 3(-1) + 4 = 1$

   $$\begin{aligned} g(f(-1)) &= g(1) \\ &= 1^2 - 1 \\ &= 0 \end{aligned}$$

   d) $f(g(-1))$

   Answer:

   $g(-1) = (-1)^2 - 1 = 0$

   $$\begin{aligned} f(g(-1)) &= f(0) \\ &= 3 \cdot 0 + 4 \\ &= 4 \end{aligned}$$

e) $g(f(a))$

Answer:

$f(a) = 3(a) + 4 = 3a + 4$

$$\begin{aligned} g(f(a)) &= g(3a + 4) \\ &= (3a + 4)^2 - 1 \\ &= 9a^2 + 24a + 16 - 1 \\ &= 9a^2 + 24a + 15 \end{aligned}$$

f) $f(g(a))$

Answer:

$g(a) = a^2 - 1$

$$\begin{aligned} f(g(a)) &= f(a^2 - 1) \\ &= 3(a^2 - 1) + 4 \\ &= 3a^2 - 3 + 4 \\ &= 3a^2 + 1 \end{aligned}$$

g) $f(f(a))$

Answer:

$f(a) = 3 \cdot a + 4 = 3a + 4$

$$\begin{aligned} f(f(a)) &= f(3a + 4) \\ &= 3(3a + 4) + 4 \\ &= 9a + 12 + 4 \\ &= 9a + 16 \end{aligned}$$

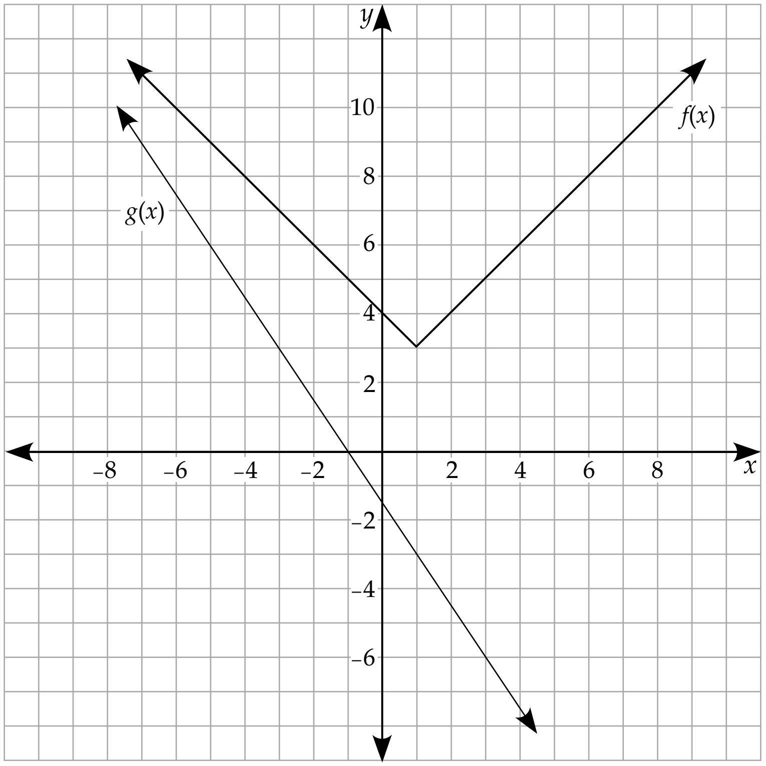
h) $g(g(a))$

Answer:

$g(a) = a^2 - 1$

$$\begin{aligned} g(g(a)) &= g(a^2 - 1) \\ &= (a^2 - 1)^2 - 1 \\ &= a^4 - 2a^2 + 1 - 1 \\ &= a^4 - 2a^2 \end{aligned}$$

2. Use the graphs of $f(x)$ and $g(x)$ shown below to evaluate the following.



a) $f(g(3))$

Answer:

$g(3) = -6$

$f(g(3)) = f(-6)$

$f(-6) = 10$

$\therefore f(g(3)) = 10$

b) $g(f(-1)$

Answer:

$f(-1) = 5$

$g(f(-1)) = g(5)$

$g(5) = -9$

$\therefore g(f(-1)) = -9$

Note: To find $g(5)$, you need to extend the graph. When you extend the graph, you will find $g(5) = -9$.

c) $g(f(1))$

Answer:

$f(1) = 3$

$g(f(1)) = g(3)$

$g(3) = -6$

$\therefore g(f(1)) = -6$

d) $f(g(-5))$

Answer:

$g(-5) = 6$

$f(g(-5)) = f(6)$

$f(6) = 8$

$\therefore f(g(-5)) = 8$

3. Given $h(x) = \sqrt{x-2}$ and $g(x) = 2x$, determine each of the following.

a) $\dfrac{g(h(6))}{h(g(9))}$

Answer:

$$\frac{g(h(6))}{h(g(9))} = \frac{g(2)}{h(18)} \qquad h(6) = \sqrt{6-2} = \sqrt{4} = 2$$

$$g(9) = 2 \cdot 9 = 18$$

$$= \frac{2 \cdot 2}{\sqrt{18-2}}$$

$$= \frac{4}{\sqrt{16}}$$

$$= \frac{4}{4}$$

$$= 1$$

b) $\dfrac{g(h(5))}{h(g(5))}$

Answer:

$$\frac{g(h(5))}{h(g(5))} = \frac{g(\sqrt{3})}{h(10)} \qquad h(5) = \sqrt{5-2} = \sqrt{3}$$

$$g(5) = 2 \cdot 5 = 10$$

$$= \frac{2 \cdot \sqrt{3}}{\sqrt{10-2}}$$

$$= \frac{2\sqrt{3}}{\sqrt{8}} \qquad \text{Simplify } \sqrt{8} = 2\sqrt{2}.$$

$$= \frac{2\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \qquad \text{Rationalize denominator.}$$

$$= \frac{\sqrt{6}}{2}$$

c) $h(g(x))$

Answer:

$$h(g(x)) = h(2x)$$

$$= \sqrt{2x-2}$$

d) $g(h(x))$

Answer:

$$g(h(x)) = g\left(\sqrt{x-2}\right)$$

$$= 2\sqrt{x-2}$$

e) The domain and range of $h(g(x))$.

Answer:

Domain: $\{x \mid x \geq 1\}$ or $[1, \infty)$

Range: $\{y \mid y \geq 0\}$ or $[0, \infty)$

f) The domain and range of $g(h(x))$.

Answer:

Domain: $\{x \mid x \geq 2\}$ or $[2, \infty)$

Range: $\{y \mid y \geq 0\}$ or $[0, \infty)$

4. If $f(x) = \dfrac{1}{x-2}$ and $k(x) = x + 1$, write:

   a) An equation defining the composition of *f* with *k*. Specify the domain.

   Answer:

$$f(k(x)) = f(x+1)$$

$$f(k(x)) = \frac{1}{(x+1)-2}$$

$$= \frac{1}{x-1}$$

Domain: $\{x \mid x \neq 1, x \in \Re\}$

   b) An equation defining the composition of *k* with *f*. Specify the domain.

   Answer:

$$k(f(x)) = k\left(\frac{1}{x-2}\right)$$

$$k(f(x)) = \left(\frac{1}{x-2}\right) + 1$$

$$= \frac{1}{x-2} + \frac{1(x-2)}{x-2}$$

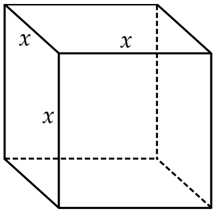
$$= \frac{1+x-2}{x-2}$$

$$= \frac{x-1}{x-2}$$

The original equation is sufficient. However, to simplify it, you can combine these terms by adding them as fractions using the common denominator of $x - 2$.

Domain: $\{x \mid x \neq 2, x \in \Re\}$

5. The volume of a cube with edges of length x is given by the function $f(x) = x^3$. Find $f(5)$. Explain what $f(5)$ represents.



Answer:

$f(x) = x^3$

$f(5) = 5^3$ or 125 units^3

$f(5)$ represents the volume of the cube when each edge is 5 units.

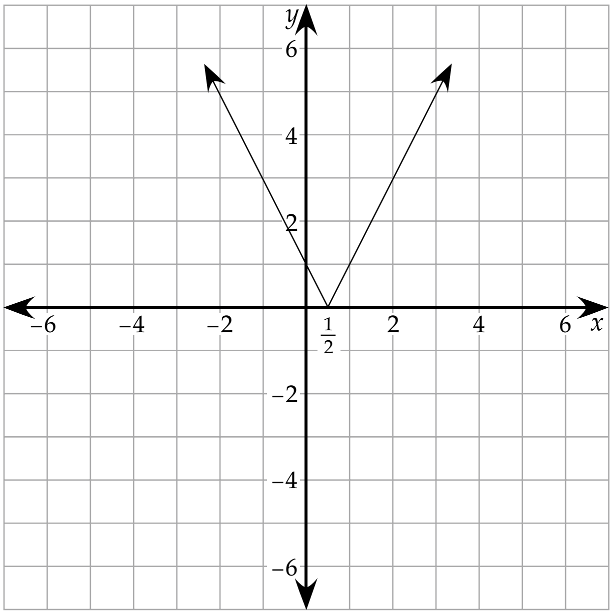
6. Sketch $h(x) = f(g(x))$ when $f(x) = | x |$ for each $g(x)$. State the domain and range of $h(x)$.

a) $g(x) = 2x - 1$

Answer:

Draw the line $y = 2x - 1$ with a y-intercept of -1 and a slope of 2. Then reflect all negative function values up to become positive function values.

$h(x) = f(g(x)) = f(2x - 1) = | 2x - 1 |$



Domain: $\{x \mid x \in \Re\}$

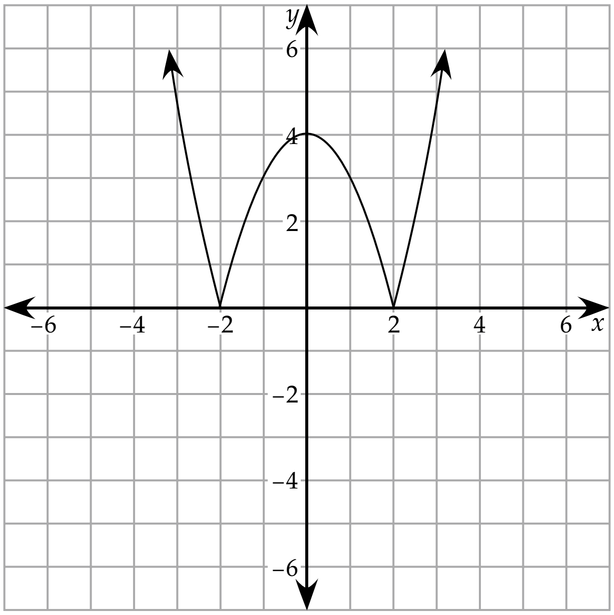
Range: $\{y \mid y \geq 0, y \in \Re\}$

b) $g(x) = x^2 - 4$

Answer:

Sketch the parabola $y = x^2 - 4$ and then reflect all negative function values up to become positive function values.

$h(x) = f(g(x)) = f(x^2 - 4) = | x^2 - 4 |$

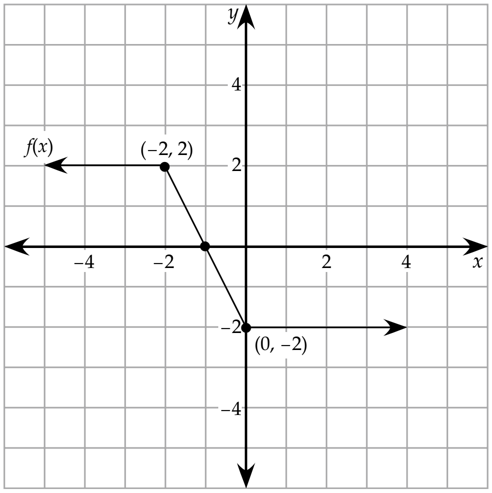


Domain: $\{x \mid x \in \Re\}$

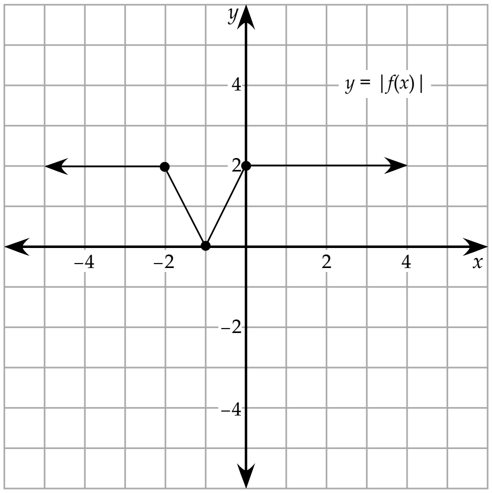
Range: $\{y \mid y \geq 0, y \in \Re\}$

7. Use the graphs of the functions shown below to sketch $y = | f(x) |$.

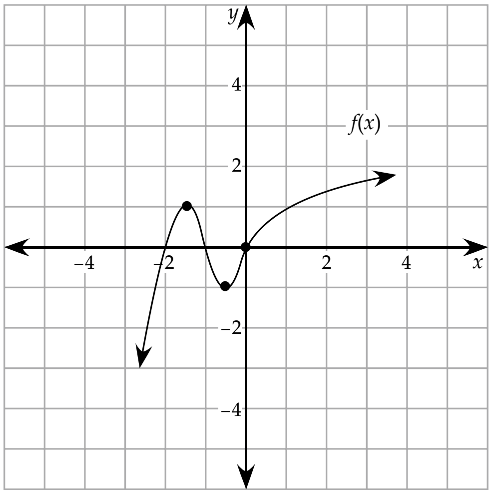
a)



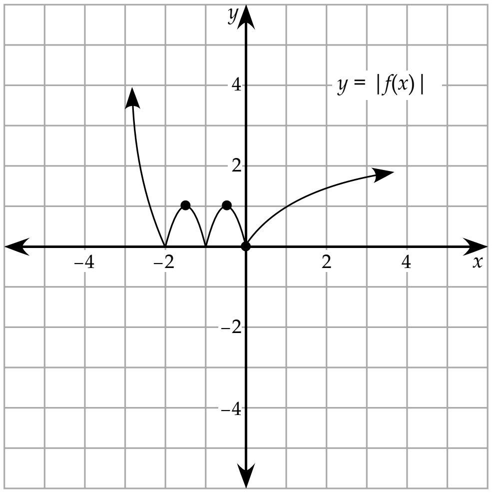
Answer:



b)



Answer:



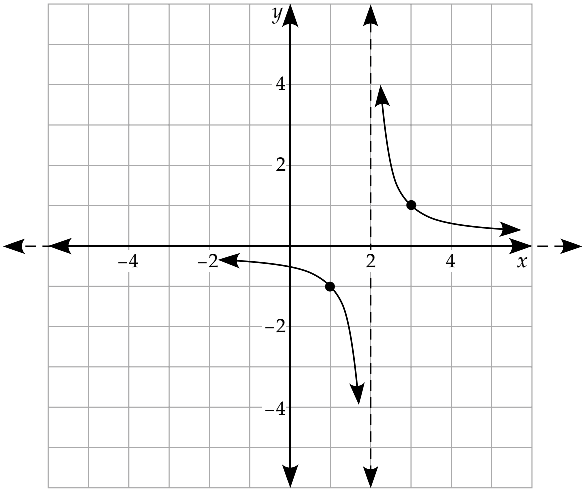
8. Sketch $h(x) = f(g(x))$ when $f(x) = \frac{1}{x}$ for each $g(x)$. Specify the domain, range, and the equations of the asymptotes.

a) $g(x) = x - 2$

Answer:

First, draw the line $y = x - 2$, and then locate the invariant points and the positions of the vertical and horizontal asymptotes.

$$h(x) = f(g(x)) = f(x-2) = \frac{1}{x-2}$$



Domain: $\{x \mid x \neq 2, x \in \Re\}$

Range: $\{y \mid y \neq 0, x \in \Re\}$

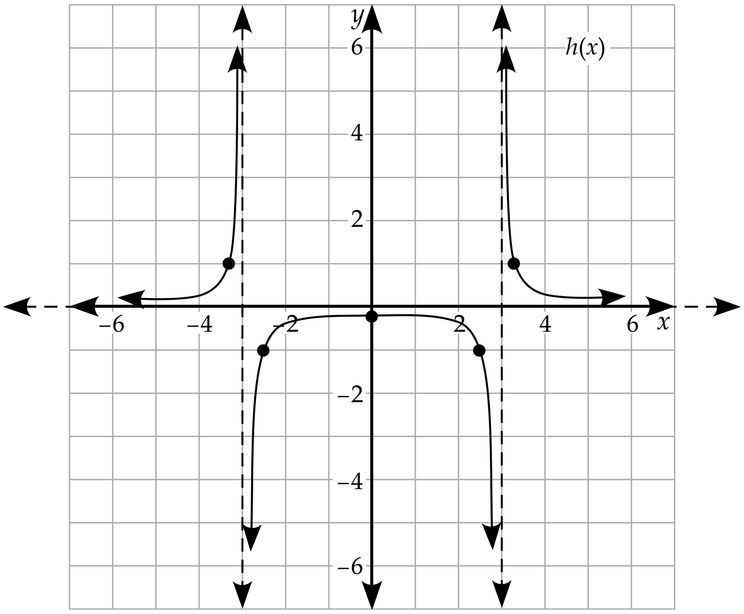
Asymptotes: $y = 0$, $x = 2$

b) $g(x) = x^2 - 9$

Answer:

First, draw the parabola $y = x^2 - 9$, and then locate the invariant points and the positions of the vertical and horizontal asymptotes. You also need to plot the reciprocal key point knowing the vertex is at (0, −3). The reciprocal function key point will be at $\left(0, -\frac{1}{3}\right)$.

$$h(x) = f\left(g(x)\right) = f\left(x^2 - 9\right) = \frac{1}{x^2 - 9}$$

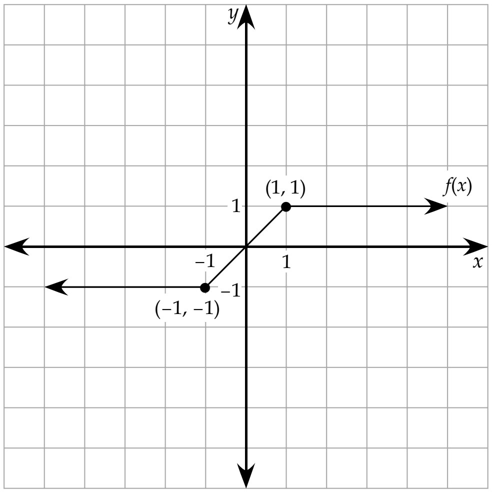


Domain: $\{x \mid x \neq 3, -3, x \in \Re\}$

Range: $\{y \mid y \neq 0, x \in \Re\}$

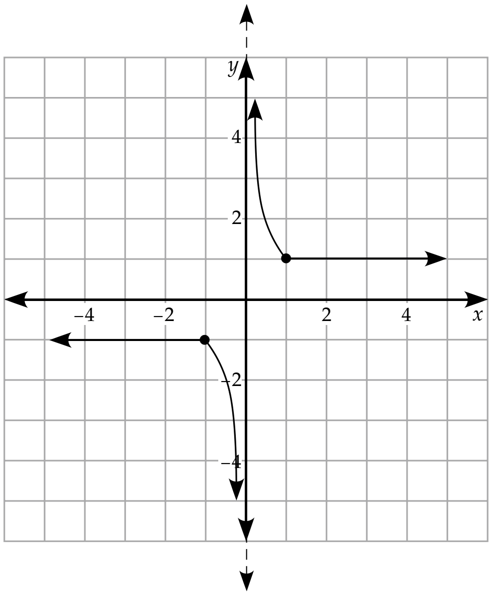
Asymptotes: $y = 0$, $x = 3$, $x = -3$

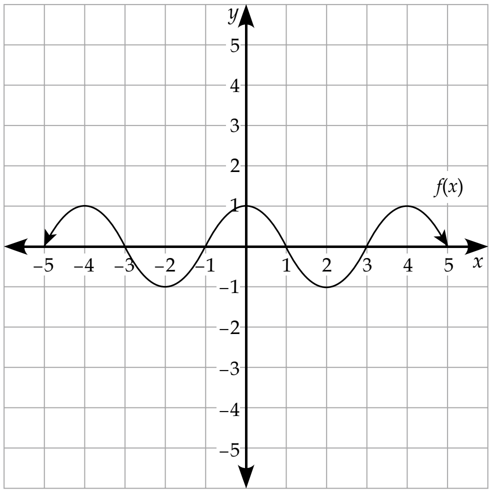
9. Use the graph of $f(x)$ to sketch $\frac{1}{f(x)}$.

a) 

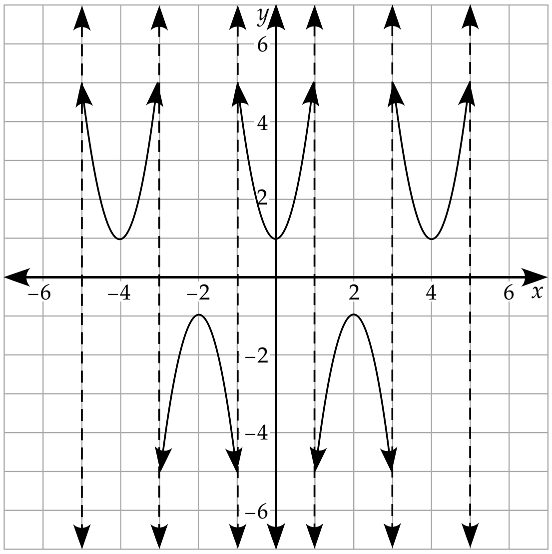
Answer:

The reciprocal function will be 1 everywhere that $f(x) = 1$.



b) 

Answer:



10. Given $g(x) = x - 3$ and $f(x) = x^3$, describe the graph of $y = f(g(x))$ in terms of the transformation of the graph of $y = f(x)$.

Answer:

The equation of the composition is $y = (x - 3)^3$. The graph of this function is $y = x^3$ shifted right 3 units.

11. The point with coordinates (7, 11) is on the graph of the function $f(x)$. What are the coordinates of a point that must lie on the graph of $\frac{1}{f(x)}$?

Answer:

$\left(7, \frac{1}{11}\right)$

12. Write the following functions as compositions of two functions.

a) $h(x) = (x + 4)^3 - 1$

Answer:

Let $h(x) = f(g(x))$. Thus, $g(x)$ is inside of $f(x)$.

If $f(x) = x^3 - 1$, then $g(x) = x + 4$.

b) $h(x) = \dfrac{1}{x + 2} + 5$

Answer:

Let $h(x) = f(g(x))$. Thus, $g(x)$ is inside of $f(x)$.

If $f(x) = \dfrac{1}{x} + 5$, then $g(x) = x + 2$.

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Module 3
Reflections

MODULE 3:
REFLECTIONS

Introduction

You first started to look at reflections in Grade 5. Throughout the years that followed, you went from reflecting images using a glass mirror to reflecting points in all four quadrants. In this module, you will expand this knowledge even further to learn about the reflections of functions on a coordinate plane.

In this module, you will learn more about different types of functions, all of which involve reflections. There are three main types of reflections including reflections through the x-axis, reflections through the y-axis, and reflections though the line, $y = x$. In addition to what you learned about in Module 2, reflections are a type of transformation. After you learn about all the different types of reflections, you will be able to graph any variation of a quadratic function, a cubic function, a square root function, and an absolute value function.

As you will learn, reflections through the line, $y = x$, are related to inverse functions. The concept of “inverse” has been part of your learning with regard to “actions” in the past. When two actions seem to undo each other, they are inverse actions. For example, the action of opening a door undoes the action of closing the door, or the action of sitting down is “undone” by the action of standing up. In arithmetic, the operation of subtraction is undone by the operation of addition, and the operation of multiplication is undone by the operation of division.

Assignments in Module 3

When you have completed the assignments for Module 3, submit your completed assignments to the Distance Learning Unit either by mail or electronically through the learning management system (LMS). The staff will forward your work to your tutor/marker.

Lesson	Assignment Number	Assignment Title
2	Assignment 3.1	Reflections in the x-axis and in the y-axis
4	Assignment 3.2	Inverse Functions and Relations

Resource Sheet

When you write your midterm examination, you are encouraged to take a Midterm Examination Resource Sheet with you into the examination. This sheet will be one letter-sized page, 8½ ″ by 11″, with both sides in your handwriting or typewritten. You will submit it with your examination, but you do not receive any marks for it.

Many students have found that preparing a resource sheet is an excellent way to review. It provides you with a summary of the important facts of each module. You should complete a resource sheet for each module to help with your studying and reviewing. Lesson summaries and module summaries are included for you to use as a guide.

You may use the list of instructions provided below to help you with preparing your resource sheet for the material in Module 3. On this sheet, you should record math terms and definitions, formulas, sample questions, or a list of places where you often make mistakes. You should also identify special areas that require extra attention or review by writing the page numbers.

After you have completed each module's resource sheet, you may summarize the sheets from Modules 1, 2, 3, and 4 to prepare your Midterm Examination Resource Sheet. The midterm examination for this course is based on Modules 1 to 4.

Resource Sheet for Module 3

As you go through the lessons of this module, you may want to consider the following suggestions regarding the creation of a resource sheet.

1. List all the important math terms, and define them if necessary.
2. List all the formulas and perhaps a sample problem that shows how the formula is used.
3. If necessary, write the solutions to some problems, showing in detail how you did the calculations.
4. Copy any questions that represent the key points of the lesson, and perhaps include the solutions as well.
5. Identify the problems you found most difficult, and copy the page numbers onto the resource sheet so that you can review them before writing the examination. You may also copy the problems and the solutions onto your resource sheet, and later write them onto your Midterm Examination Resource Sheet.
6. Write any comments, ideas, shortcuts, or other reminders that may be helpful during an examination.

LESSON 1: REFLECTIONS THROUGH THE *x*-AXIS

Lesson Focus

In this lesson, you will

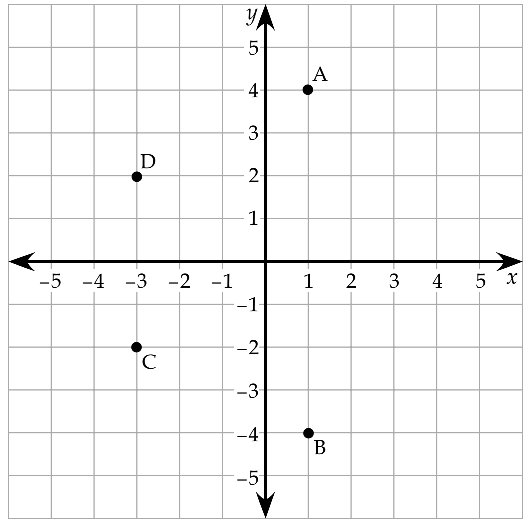
- ☐ learn how the coordinates of an ordered pair are affected when the ordered pair is reflected through the *x*-axis
- ☐ learn how to sketch the graph of a function that is reflected through the *x*-axis
- ☐ learn how to sketch the graph of $y = -f(x)$
- ☐ learn how to write the equation of a function that has been reflected vertically through the *x*-axis

Lesson Introduction

Have you ever seen a photo taken, such as the one on the right, in which a building, mountain, or landscape scene is reflected in the water? This type of reflection is similar to a reflection through the *x*-axis. You can imagine the *x*-axis to be the line between the building and the water. Every part of the building and its corresponding image in the water is equidistant from the *x*-axis.

Reflections through the *x*-axis

Consider the following graph.



How do you get from point A to point B?

You reflect the point A down through the x-axis to arrive at point B.

How do you get from point C to point D?

You reflect the point C up through the x-axis to arrive at point D.

In both cases, the first point is flipped over the *x*-axis, which is the line $y = 0$, to obtain the second point.

Also, points A and B, and points C and D, are **equidistant** from the *x*-axis. This means that points A and B are the same distance away from the *x*-axis, 4 units, and points C and D are also the same distance away from the *x*-axis, 2 units.

Consider the following chart.

Point	Coordinates	
A	(1, 4)	
B	(1, −4)	Reflection in the *x*-axis
C	(−3, −2)	
D	(−3, 2)	Reflection in the *x*-axis

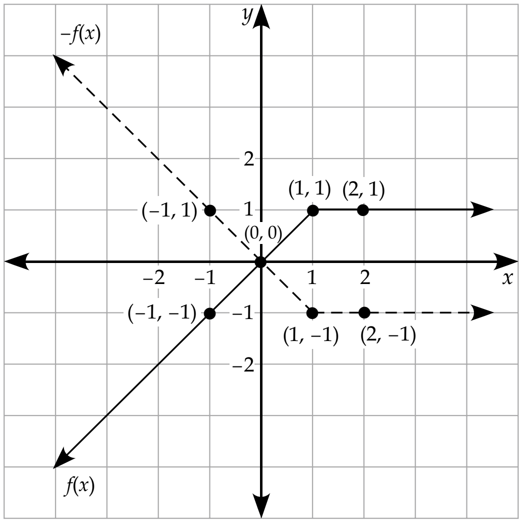
What do you notice about the coordinates of the points that are reflections in the *x*-axis?

The y-coordinates of these points have opposite signs. In other words, a reflection through the x-axis produces a change in the sign of the y-coordinate. The x-coordinate stays the same.

This can be expressed as $(x, y) \rightarrow (x, -y)$, or $f(x) \rightarrow -f(x)$.

Similar to what you learned in Module 2, coefficients of the function that are outside the function itself, such as the negative sign, affect the y-values. That is, $y = -f(x)$ represents a transformation where all of the y-values are multiplied by negative 1. This type of transformation is called a **vertical reflection**.

Consider the following graph and its corresponding table of values.



$y = f(x)$	$y = -f(x)$
(−1, −1)	(−1, 1)
(0, 0)	(0, 0)
(1, 1)	(1, −1)
(2, 1)	(2, −1)

What do you notice about the y-coordinates?

Each of the y-coordinates on $y = -f(x)$ is the opposite sign of the y-coordinate on $y = f(x)$.

Note: A reflection in the x-axis is a vertical reflection and affects the y-values. The x-values remain the same.

Why does the x-intercept stay the same?

The x-intercept stays the same because the y-coordinate of the x-intercept is zero. Multiplying negative one by zero is still zero.

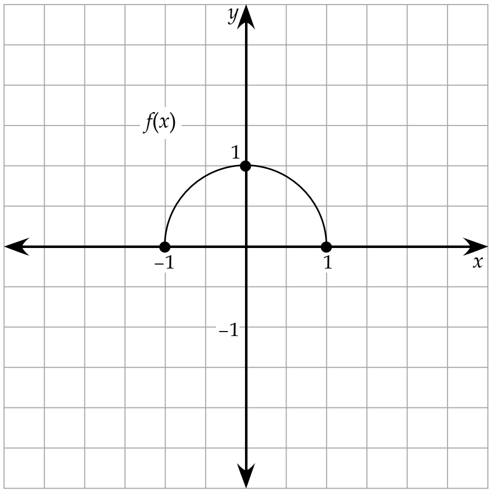
In general, any *x*-intercepts on the graph of $y = f(x)$ will stay the same (they are invariant points) on the graph of $y = -f(x)$. The *x*-intercepts don't change as the result of a vertical reflection.

Include the above information on your resource sheet. Make sure you understand the connection between a vertical reflection, which is a reflection through the *x*-axis, or the line $y = 0$, the transformation of the functions $y = f(x) \rightarrow y = -f(x)$, and the transformation of the coordinates $(x, y) \rightarrow (x, -y)$.

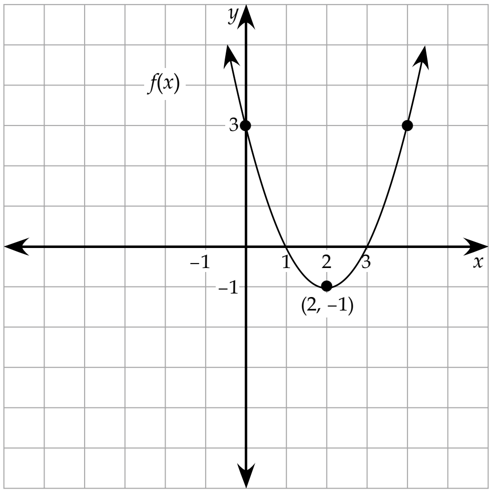
Example 1

Using the sketch of $f(x)$, sketch $y = -f(x)$.

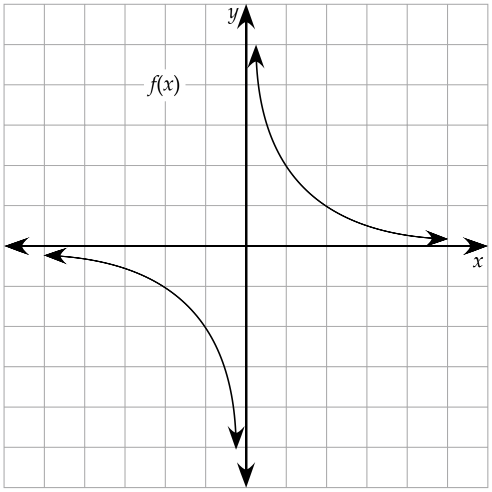
a)



b)



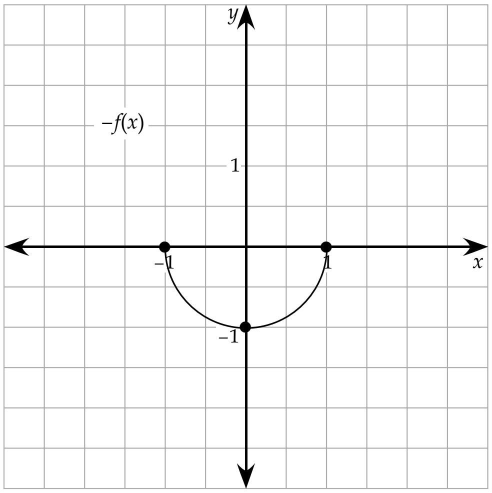
c)



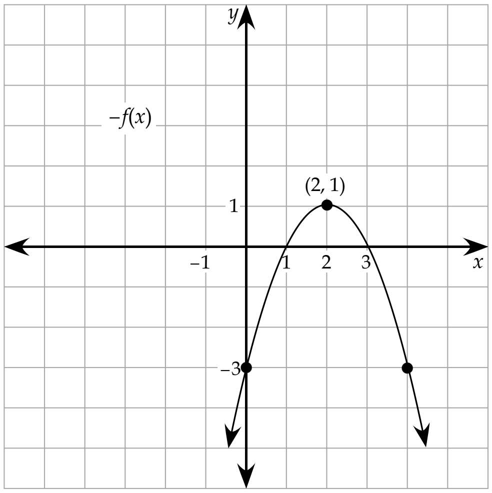
Solutions

a) The x-intercepts of $f(x)$, $(-1, 0)$, and $(1, 0)$,stay the same. The point $(0, 1)$ on the graph of $f(x)$ changes to the point $(0, -1)$ on the graph of $y = -f(x)$.

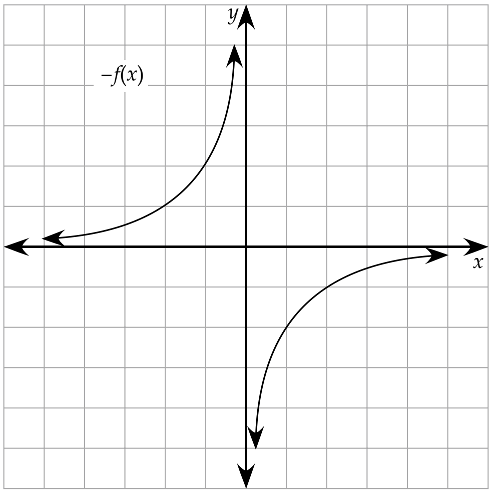
This can be done algebraically as $(x, -y)$.



b) The x-intercepts of $f(x)$, $(1, 0)$, and $(3, 0)$, stay the same. The point $(2, -1)$ on the graph of $f(x)$ changes to the point $(2, 1)$ on the graph of $y = -f(x)$. The y-intercept at 3 reflects to become a y-intercept at -3.



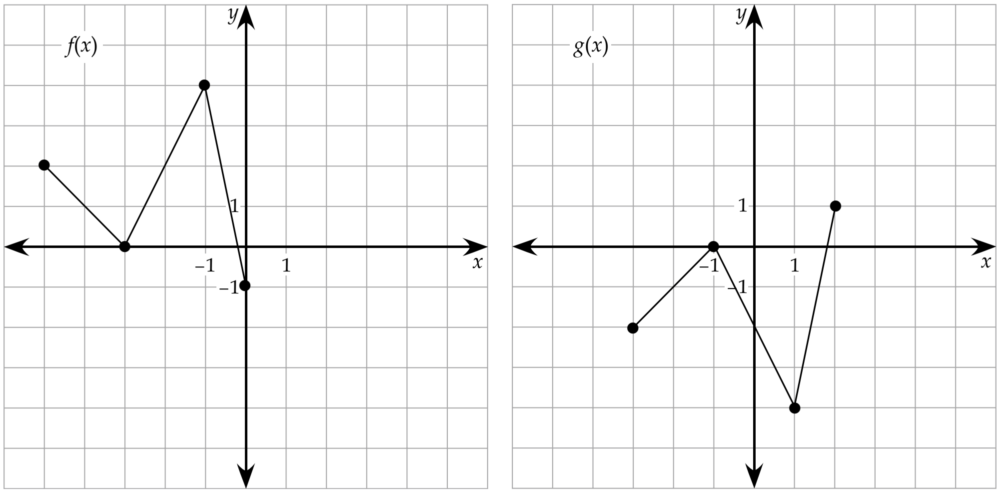
c) There are no x-intercepts on this graph. The easiest way to graph $y = -f(x)$ is to choose a couple of key points and then simply to flip $f(x)$ over the x-axis.



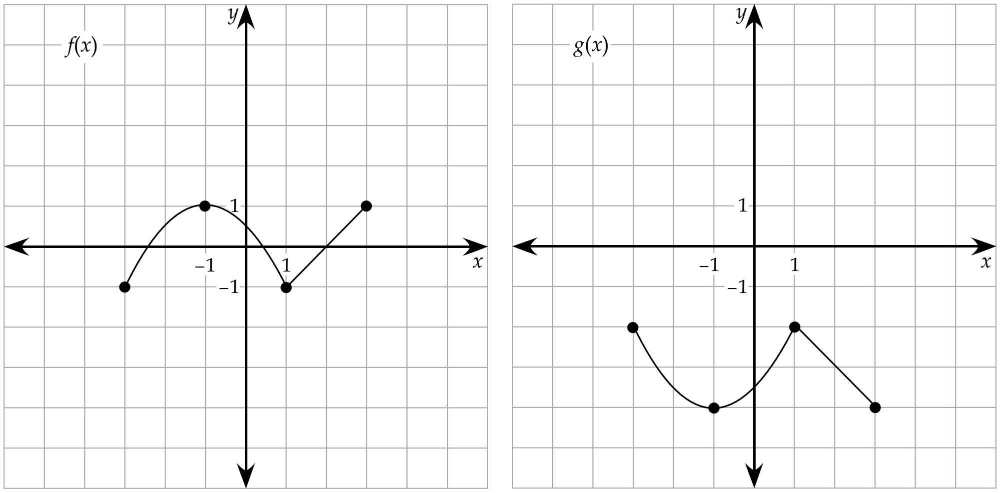
Example 2

For each of the graphs of $f(x)$ below, write the equation of the transformed function, $g(x)$, in terms of $f(x)$.

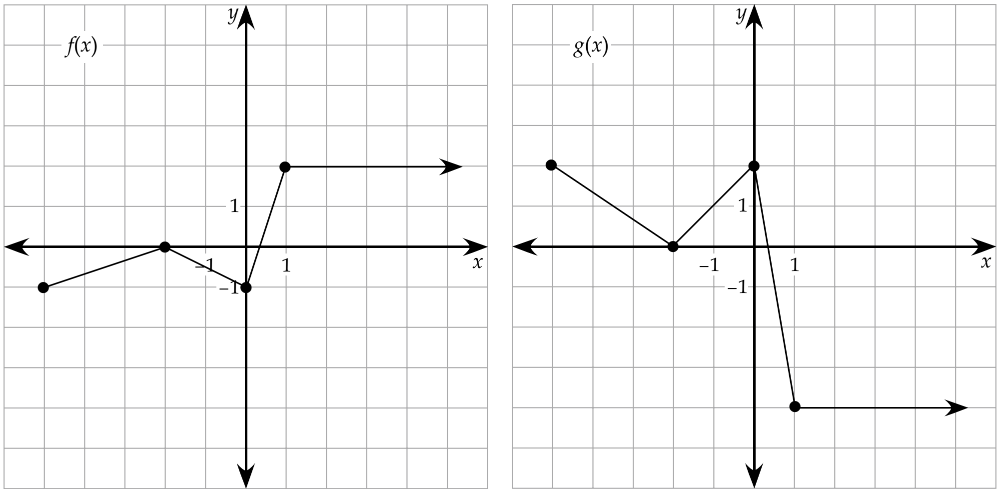
a)



b)



c)



Solutions

a) $g(x)$ is a vertical reflection of $f(x)$ in the x-axis. However, the graph was also shifted 2 units to the right.

Note: When they are in the same direction, stretches, compressions, and reflections are generally performed before horizontal and vertical shifts.

Thus, $g(x) = -f(x - 2)$.

b) $g(x)$ is a vertical reflection of $f(x)$.

However, the graph was also shifted vertically.

After the reflection, this graph was shifted 3 units down.

Thus, $g(x) = -f(x) - 3$. Alternatively, this could be written $g(x) = -(f(x) + 3)$.

There are two ways you can represent this vertical shift: either before or after the reflection. The easiest way to represent a vertical shift is to assume it occurred after the reflection in the x-axis. Otherwise, brackets are required.

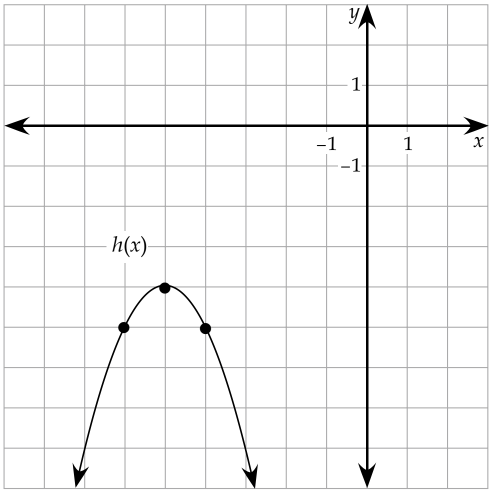
c) $g(x)$ is a reflection of $f(x)$ in the x-axis. The graph was also vertically stretched by a factor of 2.

Thus, $g(x)= -2f(x)$.

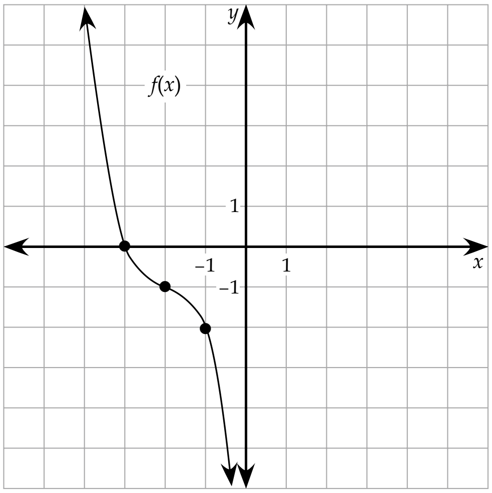
Example 3

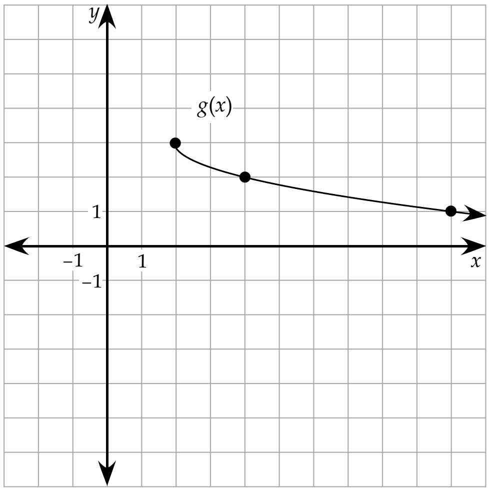
For each of the graphs below, write the equation of the transformed function in terms of the basic function of the same type.

a)



b)



c) 

Solutions

a) This graph is a transformation of the basic quadratic function $y = x^2$.

However, this graph has undergone the following transformations:

- A vertical reflection through the line $y = 0$ (the x-axis)
- A vertical shift 4 units down
- A horizontal shift 5 units to the left

Therefore, the resulting equation is $h(x) = -(x + 5)^2 - 4$.

b) This graph is a transformation of the basic cubic function $y = x^3$.

This graph has undergone the following transformations:

- A vertical reflection through the line $y = 0$ (the x-axis)
- A vertical shift 1 unit down
- A horizontal shift 2 units to the left

Therefore, the resulting equation is $f(x) = -(x + 2)^3 - 1$.

c) This graph is a transformation of the square root function $y = \sqrt{x}$. A square root function can be transformed just like any other function. Using $y = a\sqrt{b(x - h)} + k$ as a guide, you can determine the various transformations the square root function has undergone and then fill in the corresponding values for the variables. If you do not remember how the different variables affect a function, refer back to Module 2, Lesson 3. You will learn more about the square root function in Module 8.

This graph has undergone the following transformations:

- A vertical reflection through the line $y = 0$ (the x-axis)
- A horizontal stretch by a factor of 2
- A horizontal shift 2 units to the right
- A vertical shift 3 units up

Therefore, the resulting equation is $g(x) = -\sqrt{\frac{1}{2}(x-2)} + 3$.

The vertical reflection through the x-axis can be summarized in the following table. If you haven't already added this information to your resource sheet, now would be a good time to do so.

Transformation	**Vertical Reflection**	**Reflection through Line**	**Effect on (x, y)**
$-f(x)$	Reflect through x-axis	Reflect through $y = 0$	$(x, -y)$

Learning Activity 3.1

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

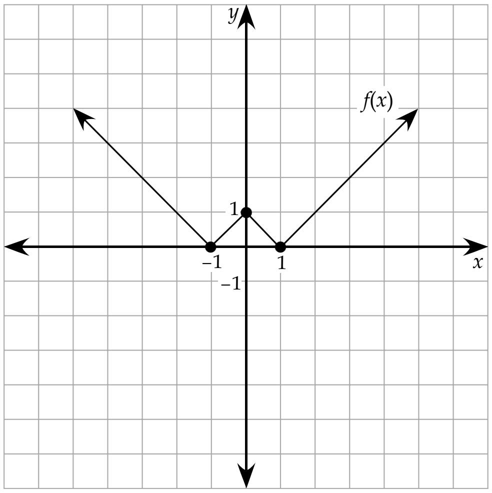
1. What is the first term in the expansion of $(2x^2 - 3y)^6$?
2. If the graph of $y = -\frac{1}{2}x^3 - 2$ is translated 3 units down, what is an equation of the translated cubic?
3. What is the reciprocal of $x - 3$?
4. Simplify: $((7x^3y^2)^4)^0$
5. A 171-page book contains 9 chapters. Assuming each chapter has the same number of pages, how many pages are there in each chapter?
6. Estimate the taxes, 13%, on a \$1050 item.
7. If $f(x) = -x^3 - 2x$, find $f(-2)$.
8. Factor: $81y^4 - 100z^4$.

continued

Part B: Reflections in the *x*-axis

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Using the sketch of $f(x)$, sketch the following.
   a) $y = -f(x)$
   b) $y = -f(x) + 2$
   c) $y = -f(x - 3) + 1$



2. If the x-intercepts of a function, $f(x)$, are 3, -5, and 0, find the x-intercepts after each transformation (if possible).
   a) $y = -f(x)$?
   b) $y = -f(x) + 1$?
   c) $y = -f(x - 2)$?

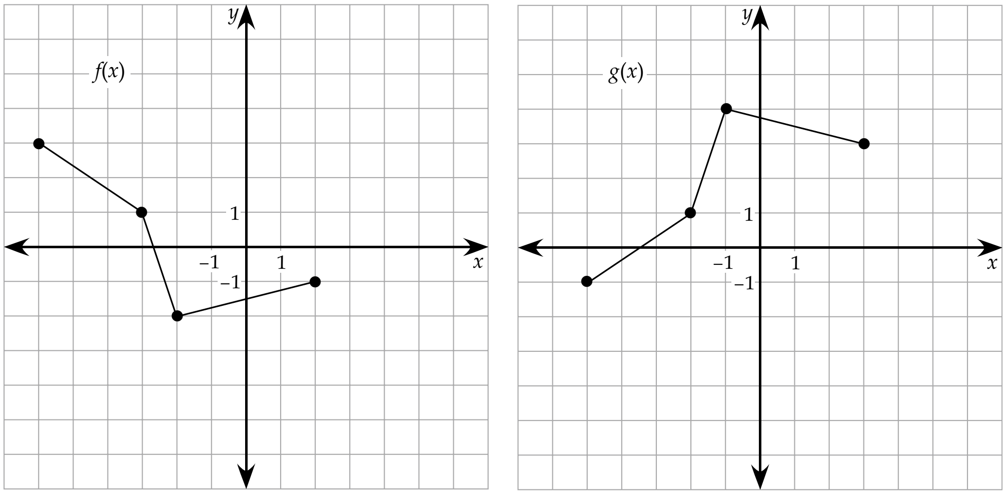
3. If the y-intercepts of a relation, $f(x)$, are 3, -5, and 0, find the y-intercepts after each transformation (if possible).
   a) $y = -f(x)$?
   b) $y = -f(x) + 1$?
   c) $y = -f(x - 2)$?

continued

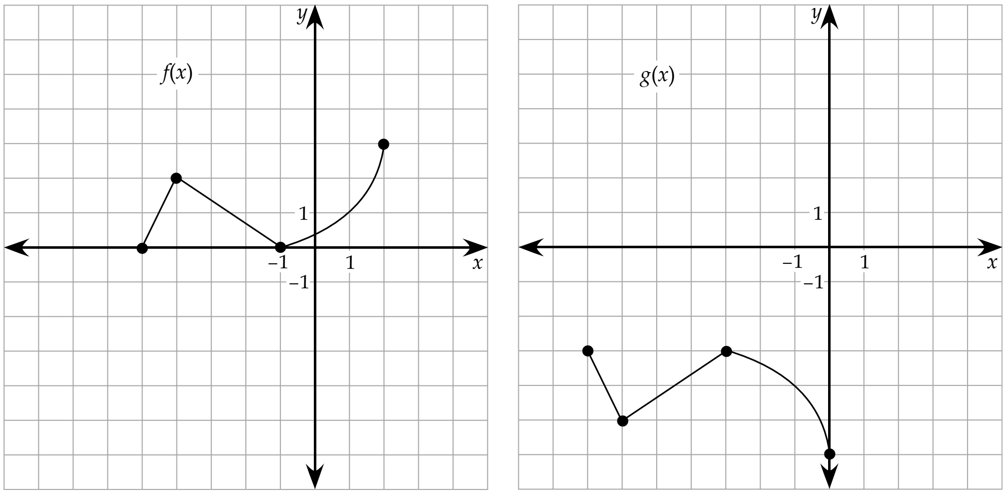
Learning Activity 3.1 (continued)

4. For each of the graphs of $f(x)$ below, write the equation of the transformed function, $g(x)$, in terms of $f(x)$.

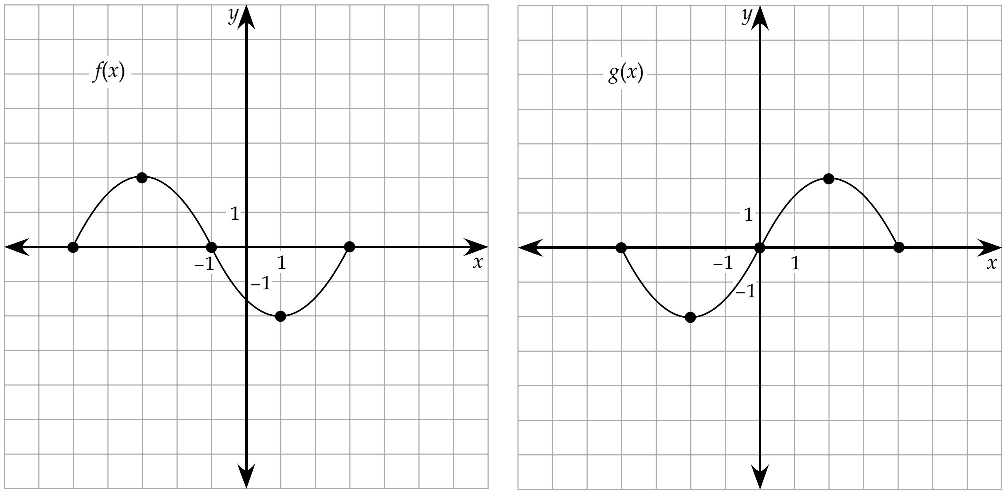
a)



b)



c)



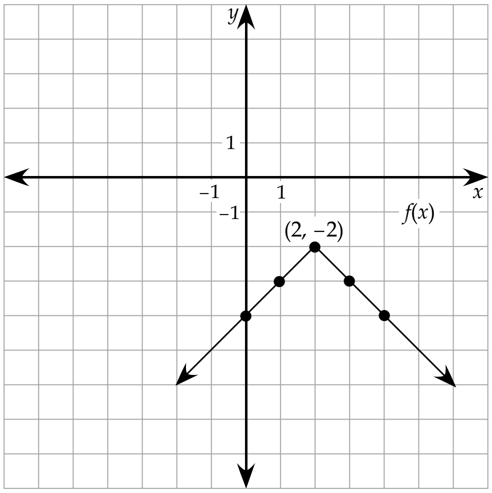
continued

Learning Activity 3.1 (continued)

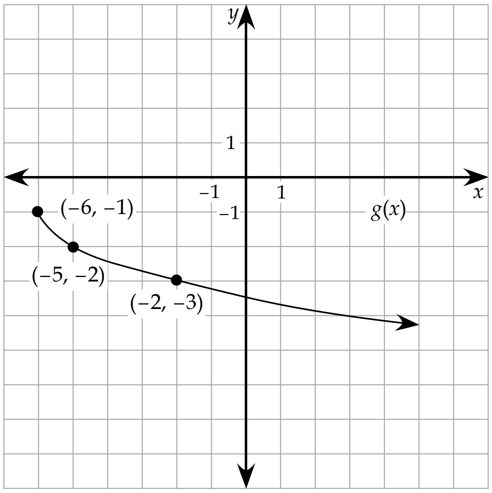
5. Sketch the graph of $y = -|\,2(x + 1)\,|$. State the domain and range.

6. For each of the graphs below, write the equation of the transformed function in terms of the basic function of the same type.

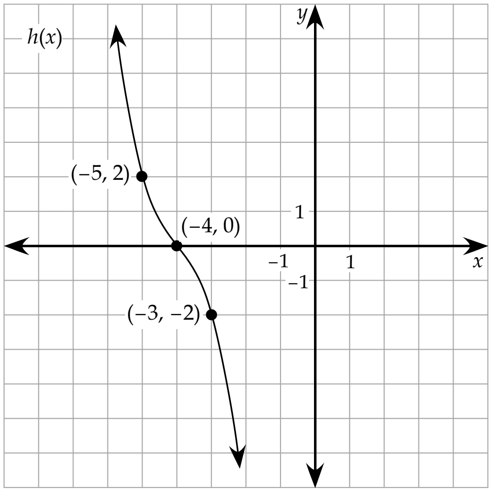
a)



b)



c)



7. Maya and Rayne are debating how to graph the function, $y = -2\sqrt{x}$. Maya believes that the vertical reflection through the x-axis should be graphed first, followed by the vertical stretch. Rayne believes that the vertical stretch should be graphed first, followed by the reflection through the x-axis. Graph the above function using both of their methods. Which method is correct?

Lesson Summary

In this lesson, you learned how to graph vertical reflections of functions in the x-axis. This is the same as graphing functions reflected through the line $y = 0$. You also learned how the y-coordinates of points of a function are multiplied by negative one when they are reflected through the x-axis. Putting all of this information together, you were able to graph any function that was in the form $y = -f(x)$ and you were also able to state the equation of the graph of a function that was reflected through the line $y = 0$.

Notes

LESSON 2: REFLECTIONS THROUGH THE Y-AXIS

Lesson Focus

In this lesson, you will

- ☐ learn how the coordinates of an ordered pair are affected when the ordered pair is reflected through the y-axis
- ☐ learn how to sketch the graph of a function that is reflected through the y-axis
- ☐ learn how to sketch the graph of $y = f(-x)$
- ☐ learn how to write the equation of a function that has been reflected horizontally through the y-axis

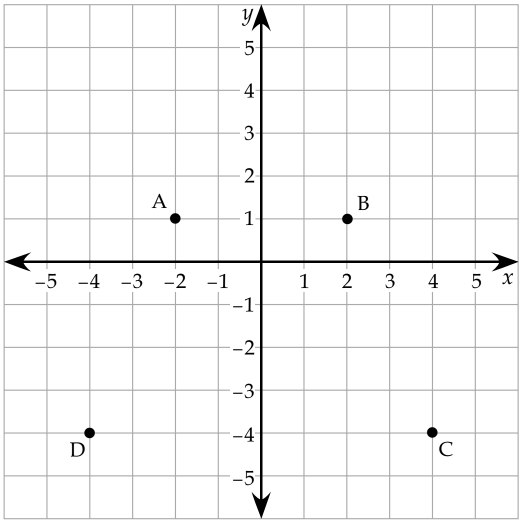
Lesson Introduction

Just as reflections in the x-axis are present throughout nature and in architecture, so are reflections in the y-axis. Consider the image on the right of the Baha'i Temple in India. If you imagine a vertical line going through the centre of the image, this line would represent the y-axis. Everything on either side of the y-axis is identical.

You will learn how these horizontal reflections affect functions.

Reflections in the *y*-axis

Consider the following graph.



How do you get from point A to point B?

You reflect the point A through the y-axis (to the right) to arrive at point B.

How do you get from point C to point D?

You reflect the point C through the y-axis (to the left) to arrive at point D.

In both cases, the first point is flipped over the y-axis, or through the line $x = 0$, to obtain the second point.

Also, points A and B, and points C and D, are equidistant from the y-axis. This means that points A and B are the same distance (2 units) away from the y-axis, and points C and D are also the same distance (4 units) away from the y-axis.

Consider the following chart.

Point	Coordinates	
A	(−2, 1)	
B	(2, 1)	Reflection in the y-axis
C	(4, −4)	
D	(−4, −4)	Reflection in the y-axis

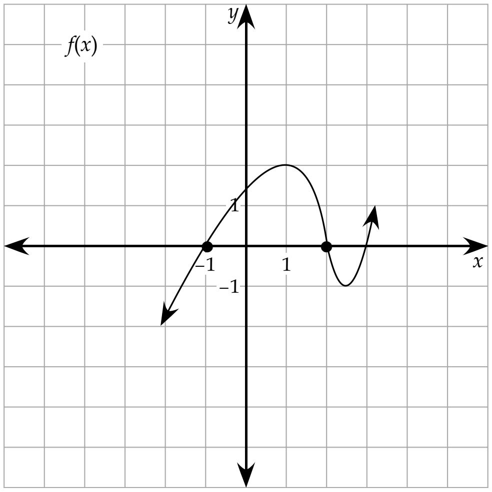
What do you notice about the coordinates of the points that are reflections in the y-axis?

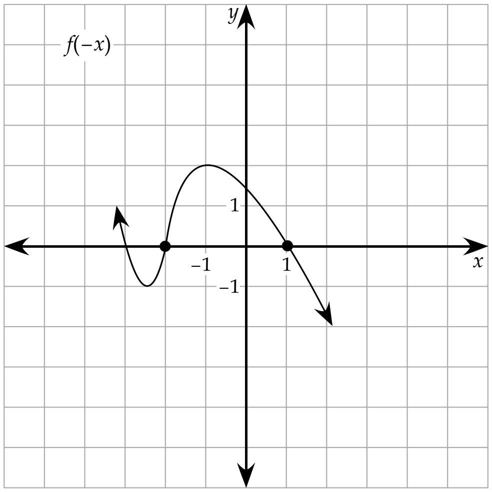
The x-coordinates of these points are opposite. In other words, a reflection through the y-axis produces a change in the sign of the x-coordinate. The y-coordinate stays the same.

This is a type of transformation called a **horizontal reflection.** This can be expressed as $(x, y) \rightarrow (-x, y)$, or $f(x) \rightarrow f(-x)$.

Similar to what you learned in Module 2, coefficients of the function that are inside the function itself, such as the negative sign, affect the x-values. Therefore, $y = f(-x)$ represents a transformation where all of the x-values are divided by negative 1.

Consider the following graphs and the corresponding table of values.





$y = f(x)$	$y = -f(x)$
(−1, 0)	(1, 0)
(1, 2)	(−1, 2)
(2, 0)	(−2, 0)
(3, 0)	(−3, 0)

What do you notice about the x-coordinates?

Each of the x-coordinates on $y = f(-x)$ is the opposite sign of the x-coordinate on $y = f(x)$.

Note: A reflection in the y-axis affects the x-values. The y-values remain the same.

What do you notice about the y-intercept?

The y-intercept stays the same because the x-coordinate of the y-intercept is zero. When you multiply zero by negative one, you still get an answer of zero.

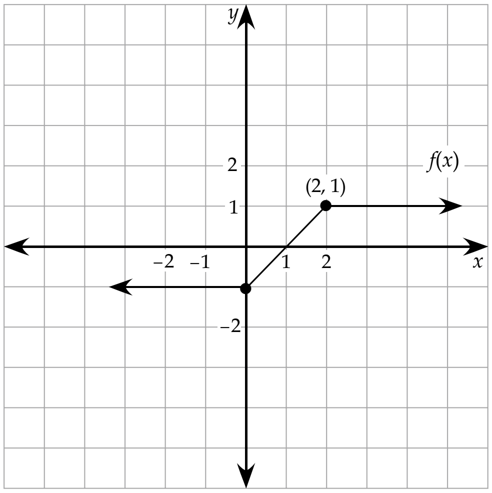
In general, any y-intercepts on the graph of $y = f(x)$ will stay the same on the graph of $y = f(-x)$. The y-intercepts don't change as a result of a horizontal reflection.

Include the above information on your resource sheet. Make sure you understand the connection between a reflection through the y-axis, or the line $x = 0$, the transformation of the functions $y = f(x) \rightarrow y = f(-x)$, and the transformation of the coordinates $(x, y) \rightarrow (-x, y)$.

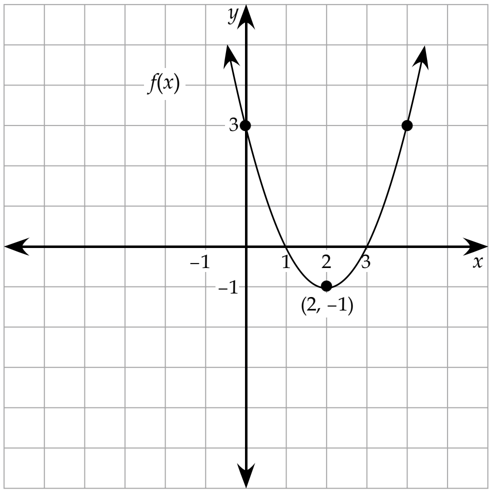
Example 1

Use the sketch of each graph of $f(x)$ to sketch $f(-x)$.

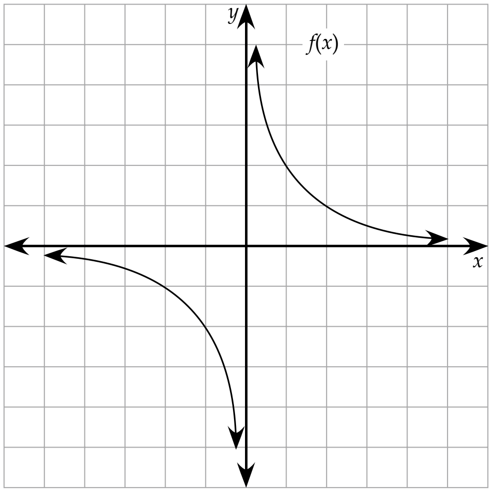
a)



b)



c)

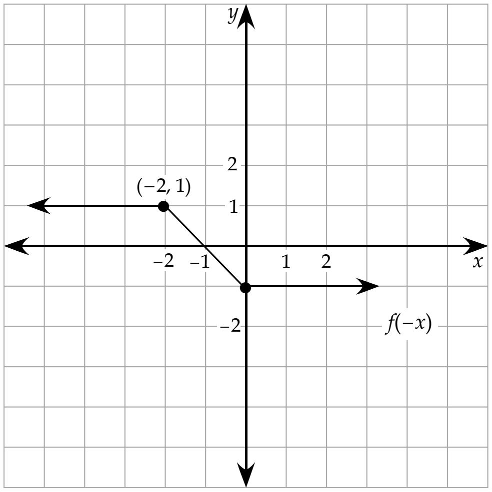


Solutions

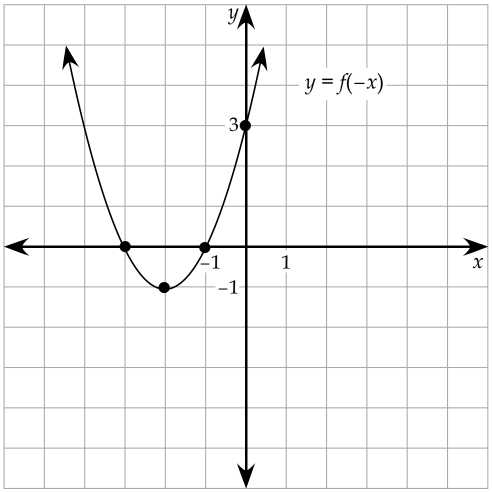
a) The y-intercept stays the same, $(0, -1)$.

The points $(-1, -1)$, $(1, 0)$, and $(2, 1)$ are reflected through the y-axis, which results in their x-coordinates being multiplied by negative one. The corresponding points on the graph of $y = f(-x)$ are $(1, -1)$, $(-1, 0)$, and $(-2, 1)$.

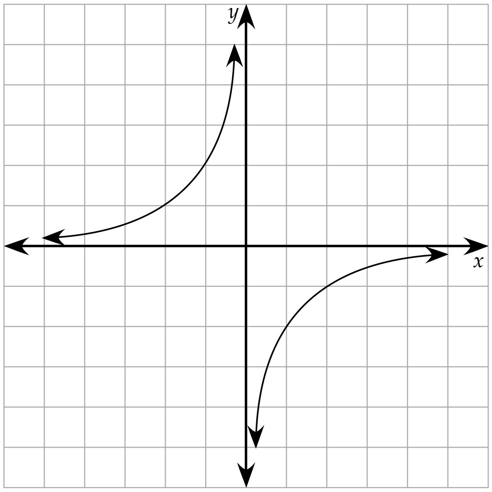
This can be done algebraically as $(-x, y)$.



b) The y-intercept stays the same, $(0, 3)$. The points $(1, 0)$, $(2, -1)$, and $(3, 0)$ on the graph of $y = f(x)$ correspond to the points $(-1, 0)$, $(-2, -1)$, and $(-3, 0)$ on the graph of $y = f(-x)$.



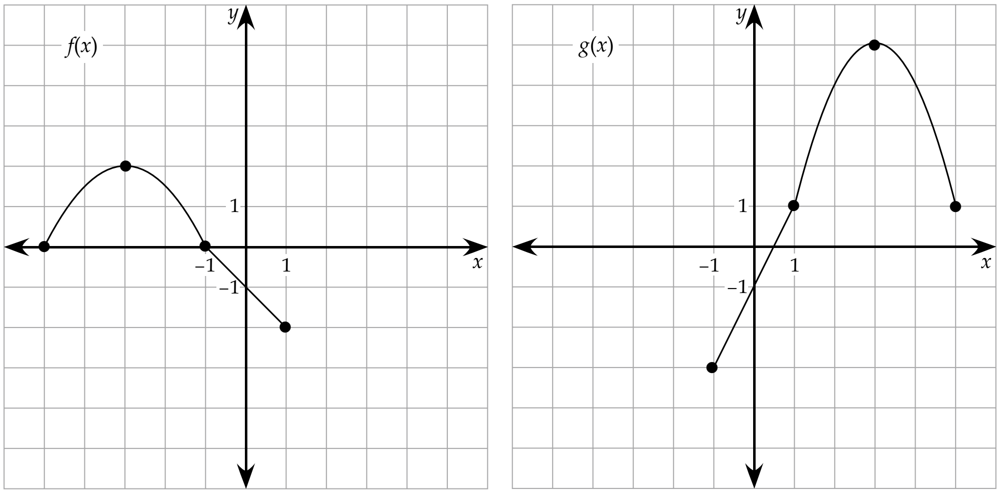
c) The easiest way to complete this transformation is to choose a couple of key points and then flip both portions of the graph horizontally through the y-axis.



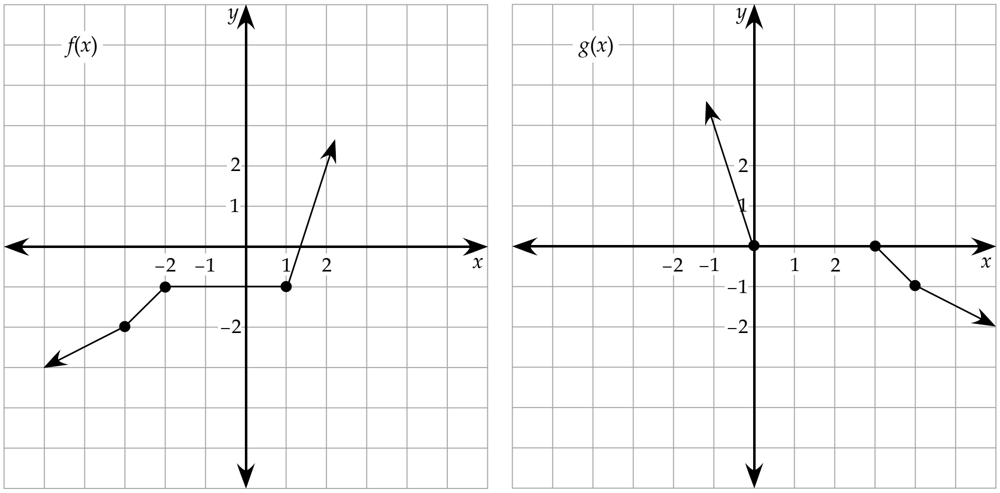
Example 2

For each of the graphs of $f(x)$ below, write the equation of the transformed function, $g(x)$, in terms of $f(x)$.

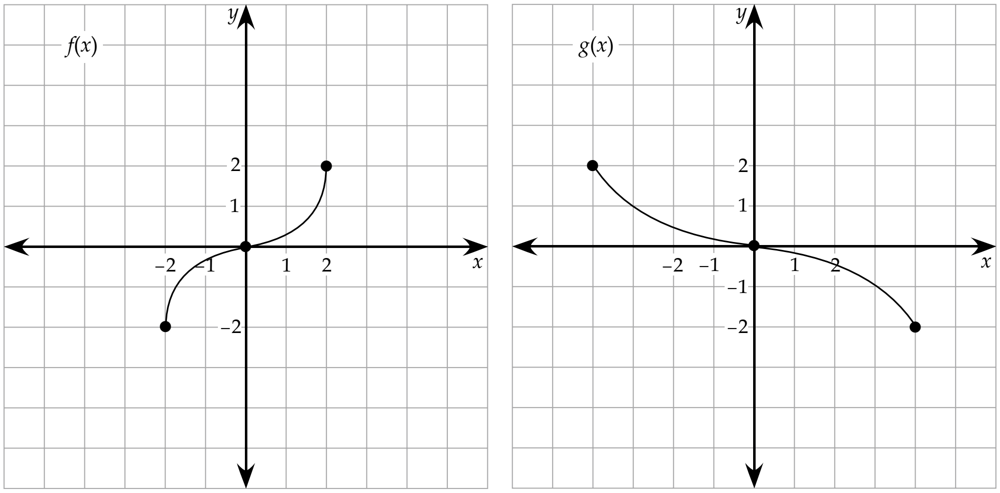
a)



b)



c)



Solutions

a) $g(x)$ is a reflection of $f(x)$ in the y-axis. However, this graph was also shifted one unit vertically and stretched vertically as well by a factor of two.

Note: Stretches and compressions, including reflections, are generally performed before horizontal and vertical shifts.

Therefore, the $f(x)$ graph was first stretched vertically, reflected horizontally over the y-axis, and then shifted vertically. The resulting function is $g(x) = 2f(-x) + 1$.

b) $g(x)$ is a horizontal reflection of $f(x)$ in the y-axis. However, this graph was also shifted one unit up and one unit to the right (after the reflection).

The resulting function is $g(x) = f(-(x - 1)) + 1$.

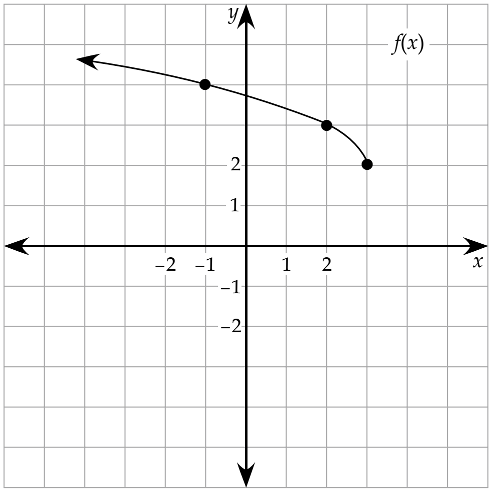
c) $g(x)$ is a horizontal reflection of $f(x)$ in the y-axis. However, this graph was also stretched horizontally by a factor of two.

The resulting function is $g(x) = f\left(-\frac{1}{2}x\right)$.

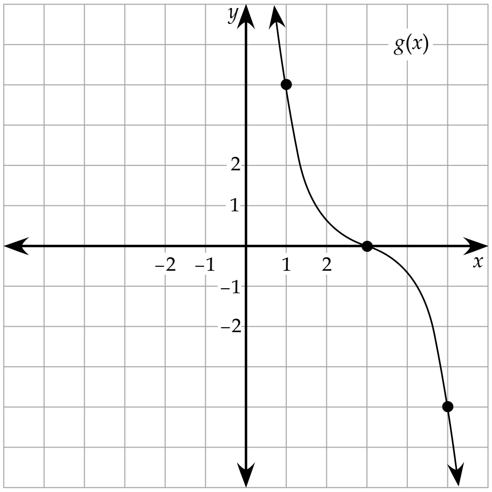
Example 3

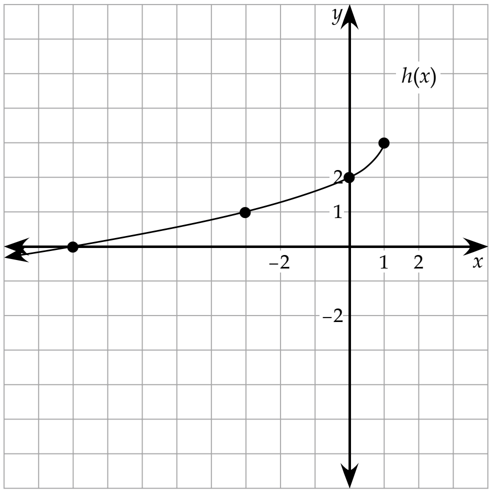
For each of the graphs below, write the equation of the transformed function in terms of the basic function of the same type.

a)



b)



c) 

Solutions

a) This graph is a transformation of the basic square root function $y = \sqrt{x}$.

However, this graph has undergone the following transformations:

- A horizontal reflection in the line $x = 0$
- A vertical shift 2 units up
- A horizontal shift 3 units to the right

The resulting function is $f(x) = \sqrt{-(x - 3)} + 2$.

b) This graph is a transformation of the basic cubic function $y = x^3$.

However, this graph has undergone the following transformations:

- A horizontal reflection in the line $x = 0$ or a vertical reflection in the line $y = 0$

  Note: If you reflect a cubic function through the line $x = 0$, you will arrive at the same function as if you were to reflect the function through the line $y = 0$. This is because $y = -(x^3)$ and $y = (-x)^3$ are equivalent.
- A vertical compression by a factor of 2
- A horizontal shift of 3 units to the right

The resulting function is $g(x) = -\frac{1}{2}(x - 3)^3$ or $g(x) = \frac{1}{2}(-(x - 3))^3$.

c) This graph is a transformation of the basic square root function $y = \sqrt{x}$.

However, this graph has undergone the following transformations:

- A horizontal reflection in y-axis
- A vertical reflection in the x-axis
- A vertical shift 3 units up
- A horizontal shift 1 unit to the right

The resulting function is $h(x) = -\sqrt{-(x-1)} + 3$.

The horizontal reflection through the y-axis can be summarized in the following table. Make sure this information is included on your resource sheet.

Transformation	Horizontal Reflection	Reflection through Line	Effect on (x, y)
$f(-x)$	Reflect through y-axis	Reflect through $x = 0$	$(-x, y)$

Learning Activity 3.2

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Evaluate: $| 4.72 - 6.91 |$
2. Solve for x: $(x + 3)^2 = 4$
3. Multiply: $(2x - 1)(y + 6)$
4. Evaluate: $\frac{3}{5} + \frac{17}{30}$
5. Determine the axis of symmetry of the function $f(x) = \frac{1}{3}(x - 2)^2 + 4$.

continued

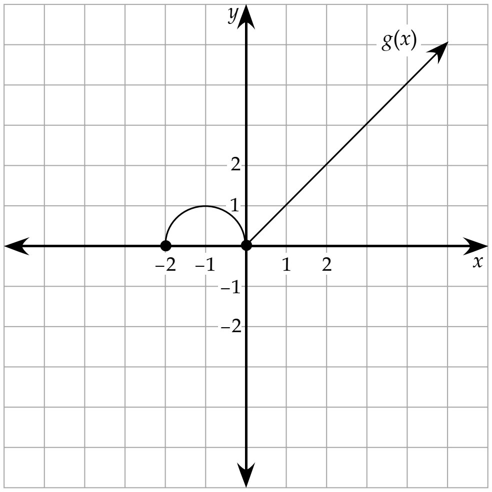
Learning Activity 3.2 (continued)

6. Factor: $2x^2 + 9x + 9$
7. Which is the better deal, a 24 pack of water for \$4.99 or a 36 pack of water for \$6.49?
8. What is the length of the remaining leg of a right-angled triangle if the hypotenuse measures 25 m and one leg measures 15 m?

Part B: Reflections in the *y*-axis

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Using the sketch of $g(x)$, sketch the following.
   a) $y = g(-x)$
   b) $y = g(-x) - 1$
   c) $y = 2g(-(x + 3))$
   d) $y = -g(-x)$



2. If the x-intercepts of a function, $f(x)$, are 3, −5, and 0, find the x-intercepts after the transformation (if possible).
   a) $y = f(-x)$?
   b) $y = f(-x) + 1$?

continued

Learning Activity 3.2 (continued)

3. If the y-intercepts of a function, $f(x)$, are 3, −5, and 0, find the y-intercepts after the transformation (if possible).

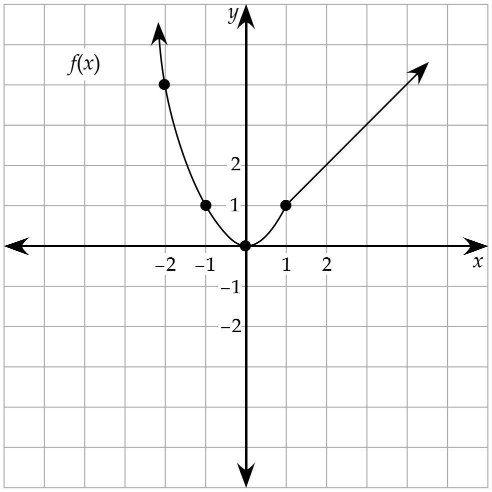
   a) $y = f(-x)$?

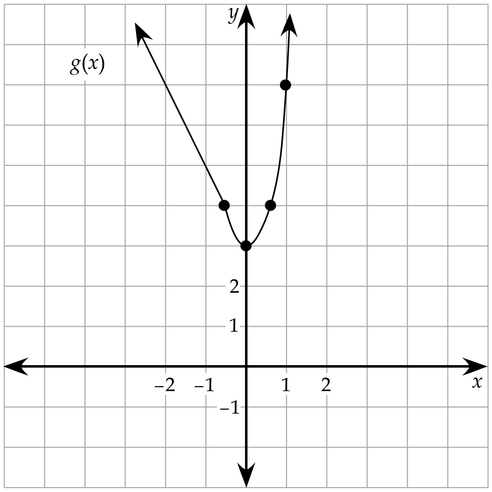
   b) $y = f(-(x + 2))$?

   c) $y = f(-x) + 1$?

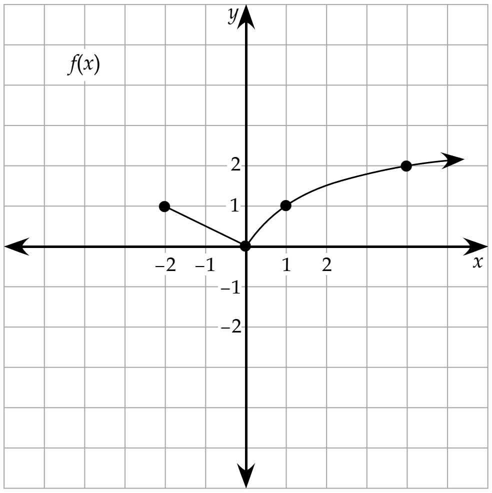
4. For each of the graphs of $f(x)$ below, write the equation of the transformed function, $g(x)$, in terms of $f(x)$.

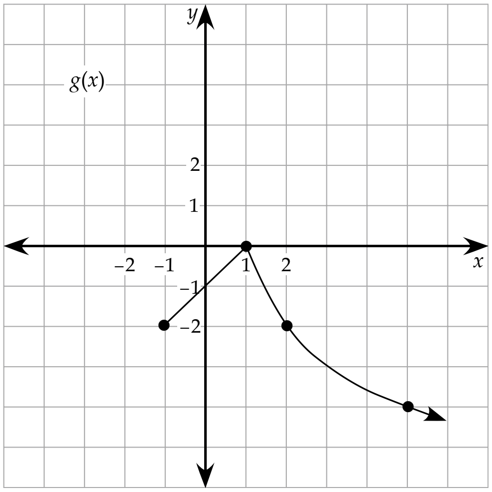
   a)





   b)

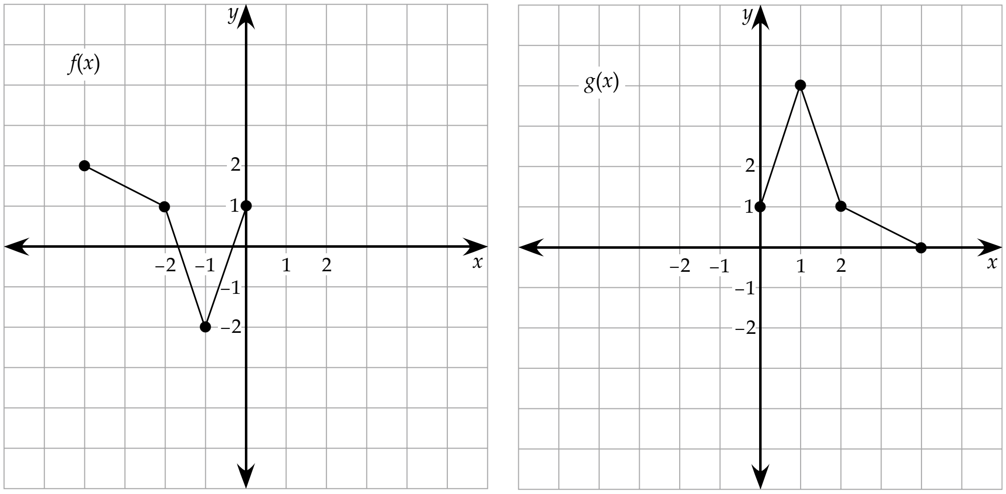




continued

Learning Activity 3.2 (continued)

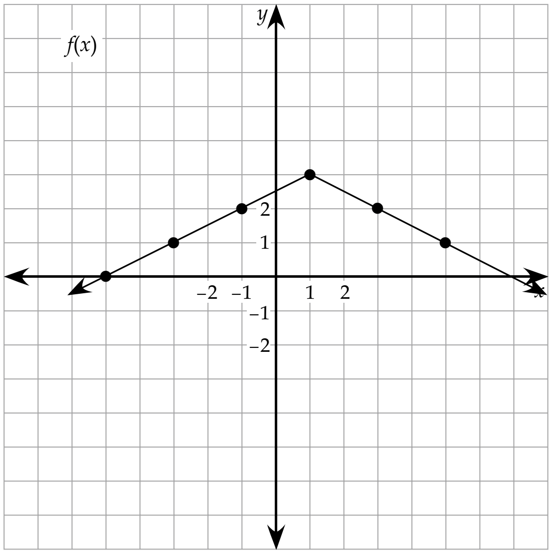
c)



5. Sketch the graph of $y = -2\sqrt{-(x+1)}$. State the domain and range.

6. For each of the graphs below, write the equation of the transformed function in terms of the basic function of the same type.

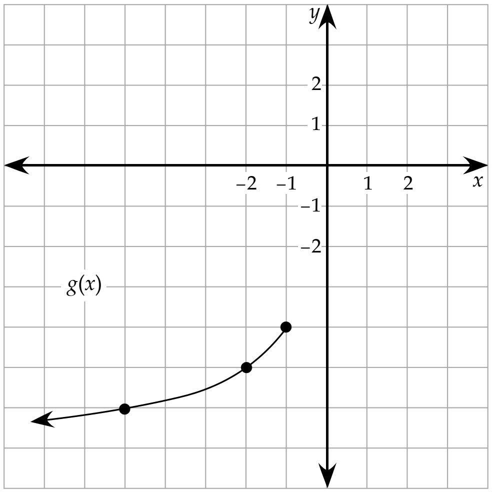
a)



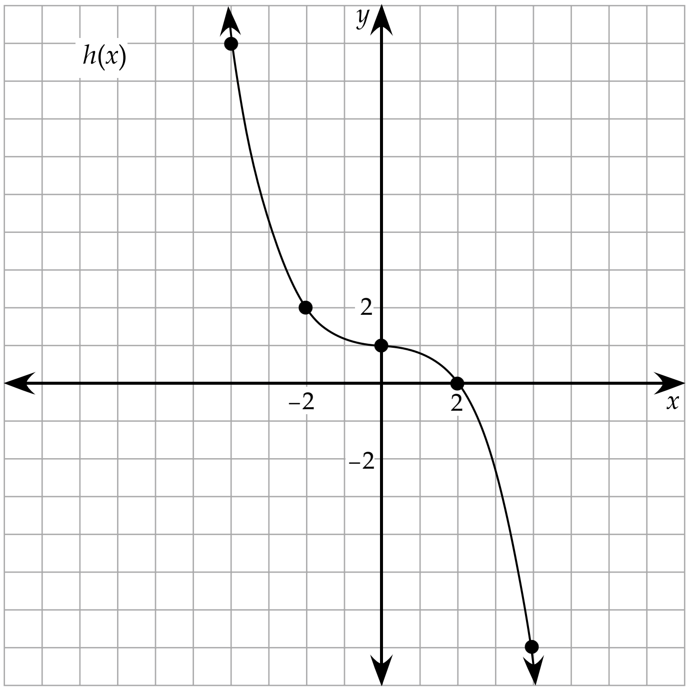
continued

Learning Activity 3.2 (continued)

b)



c)

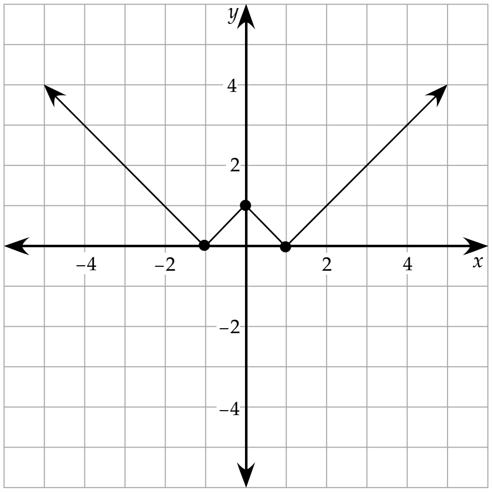


7. Sketch the graph of $f(x) = x^2$ and the graph of $y = f(-x)$. What is unusual about these graphs? Why does this occur?

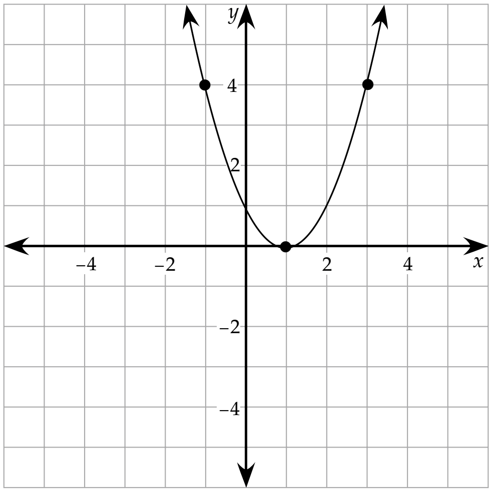
continued

8. If the graph of a function is reflected in the y-axis and the result is the same as the original graph, then the function is referred to as an **even function**. Geometrically, even functions are symmetrical in the y-axis. It can be said that an even function can be flipped in the y-axis and the result is the same graph as the original. Which of the following are graphs of an even function?

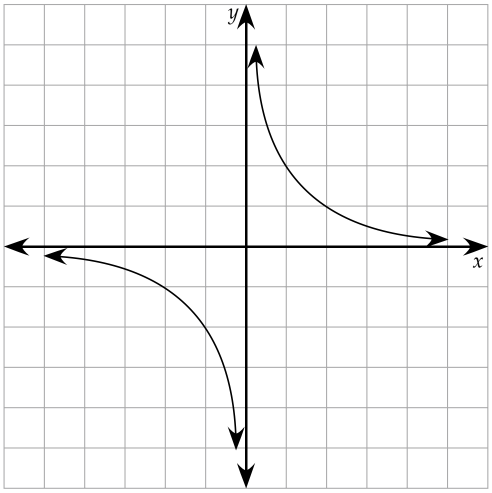
a)



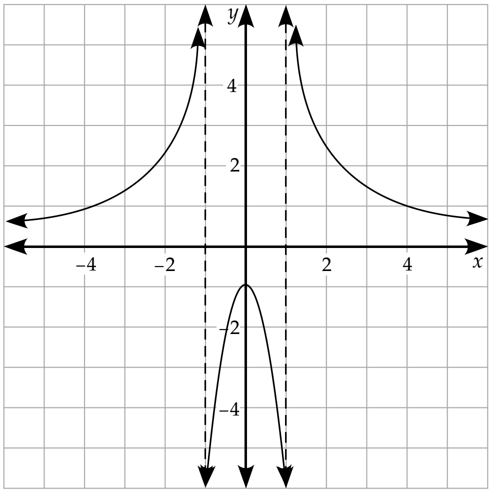
b)



c)



d)

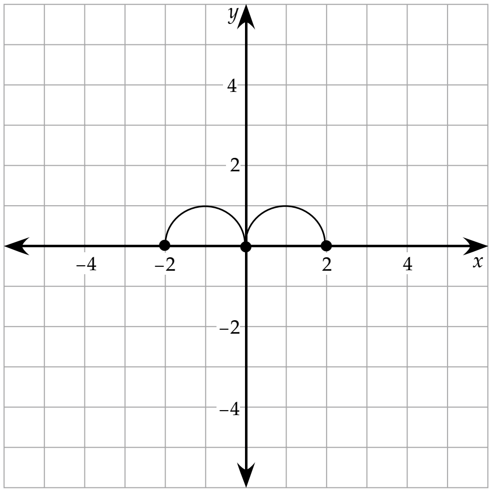


continued

Learning Activity 3.2 (continued)

9. Since there are even functions, it seems only natural that there should be **odd functions**. Odd function is the name given to the functions that can be flipped in the *x*-axis and flipped in the *y*-axis, and the graph will be the same as the original. Determine whether the given function represents an odd function.

a)



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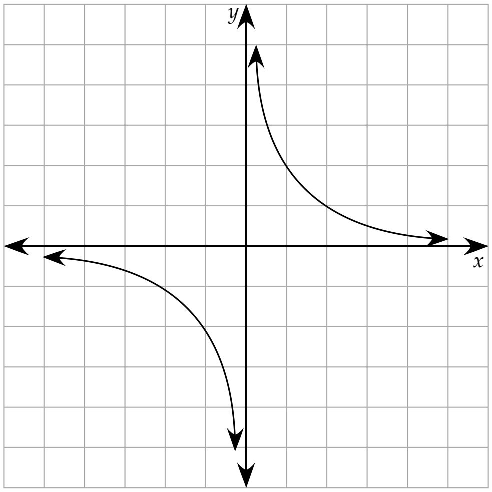
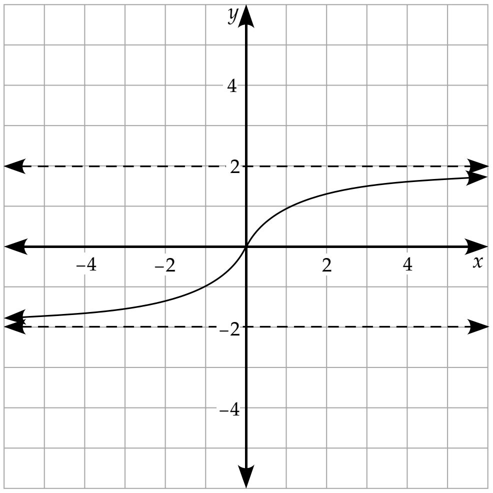


c)



Lesson Summary

In this lesson, you learned about another kind of reflection—a horizontal reflection in the *y*-axis. This is the same as a reflection through the line $x = 0$ or the transformation $y = f(-x)$. In the next lesson, you will learn about another type of reflection through the line $y = x$.

Notes

Assignment 3.1

Reflections in the *x*-axis and in the *y*-axis

Total: 34 marks

You need to show all your steps as you work out your solutions, and make sure your answers are clear (include units, where appropriate). Check the Introduction for the Marking Guidelines that will be used for all assignments and examinations.

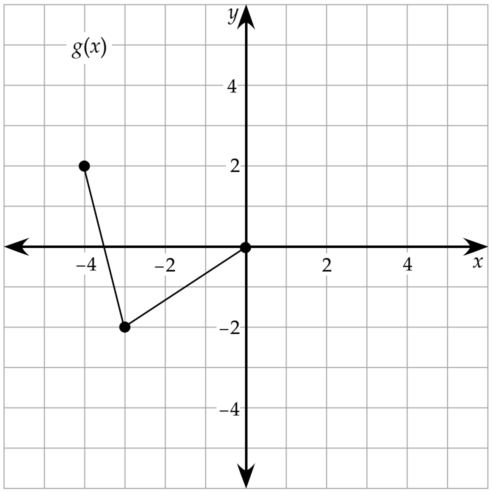
1. A function contains the ordered pairs (1, 0), (4, −3), (−6, 1), and (0, −5).
   a) What are the corresponding ordered pairs if this function is reflected through the *x*-axis (vertically)? *(1 mark)*

   b) What are the corresponding ordered pairs if this function is reflected through the *y*-axis (horizontally)? *(1 mark)*

continued

Assignment 3.1: Reflections in the x-axis and in the y-axis (continued)

2. Using the sketch of $g(x)$, sketch the following. Explain the transformations, in words or algebraically.



a) $y = g(-x) + 3$ *(2 marks)*

b) $y = -g(-(x - 1))$ *(3 marks)*

continued

Assignment 3.1: Reflections in the x-axis and in the y-axis (continued)

c) $y = -\frac{1}{2}g(x) + 3$ *(3 marks)*

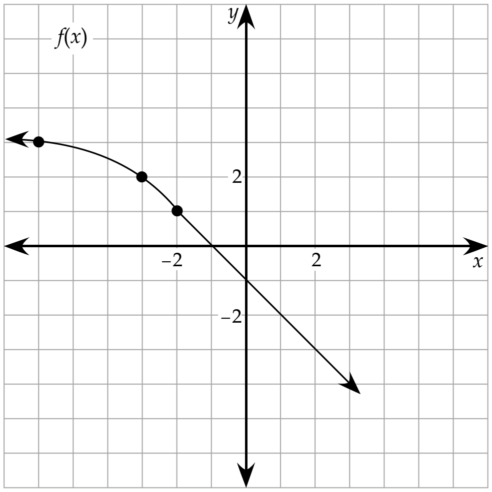
d) $y = -g(-2(x + 2))$ *(3 marks)*

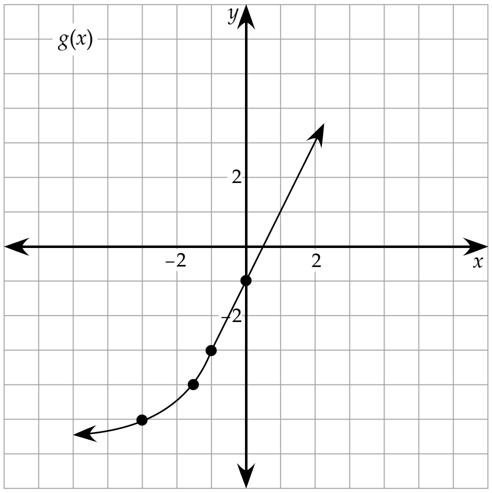
continued

Assignment 3.1: Reflections in the x-axis and in the y-axis (continued)

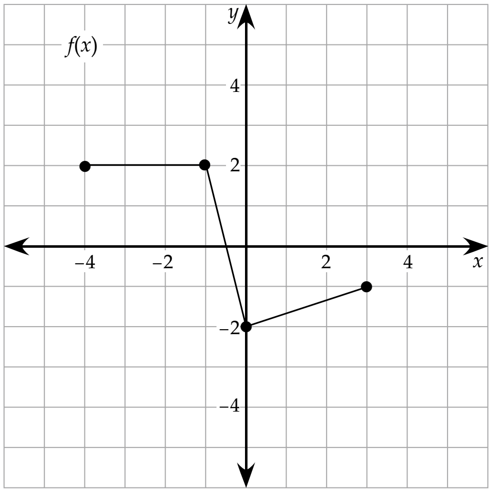
3. For each of the graphs of $f(x)$ below, write the equation of the transformed function, $g(x)$, in terms of $f(x)$.

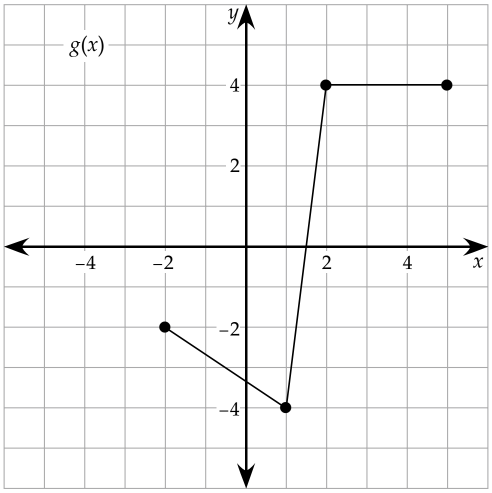
   a) *(2 marks)*





   b) *(2 marks)*





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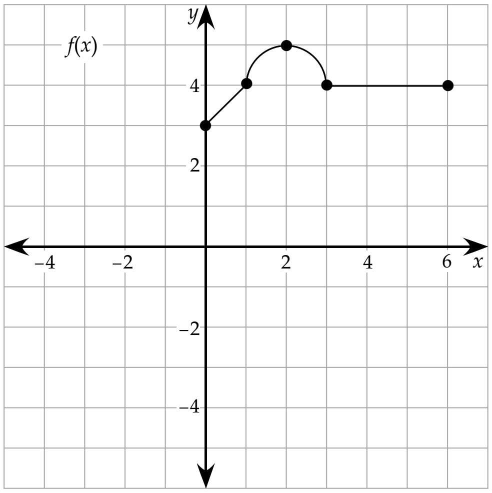
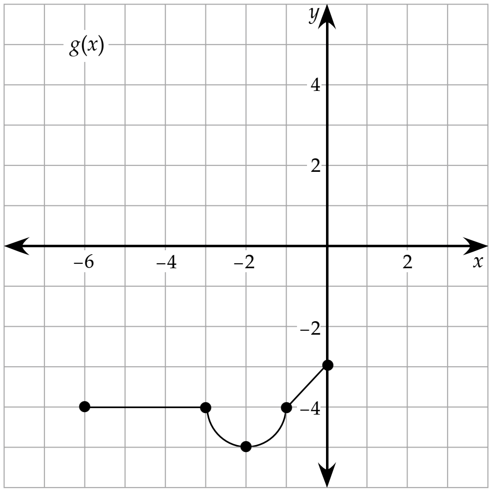
c) *(1 mark)*





continued

Assignment 3.1: Reflections in the x-axis and in the y-axis (continued)

4. Sketch the following functions.

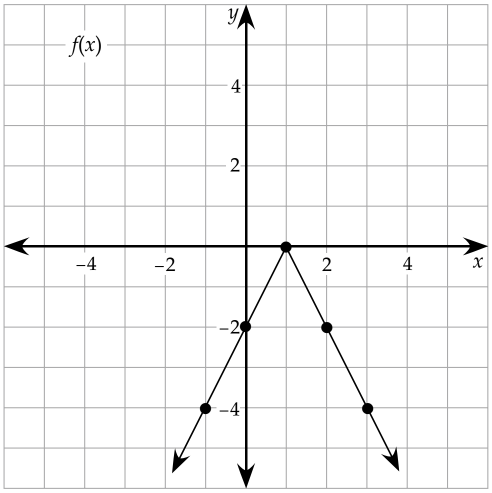
   a) $f(x) = -\left| -(x + 1) \right| - 3$ *(3 marks)*

   b) $g(x) = -3\sqrt{-\frac{1}{2}x}$ *(4 marks)*

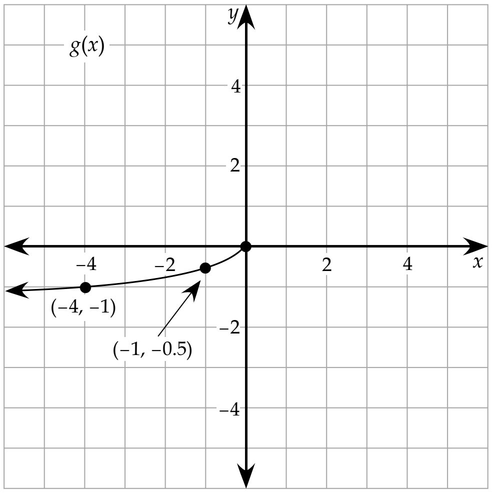
continued

5. For each of the graphs below, write the equation of the transformed function in terms of the basic function of the same type.

   a) *(3 marks)*



   b) *(3 marks)*



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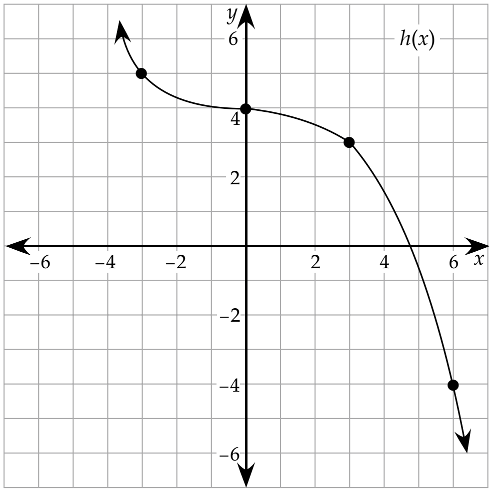
Assignment 3.1: Reflections in the x-axis and in the y-axis (continued)

c) *(3 marks)*



LESSON 3: REFLECTIONS IN THE LINE $y = x$

Lesson Focus

In this lesson, you will

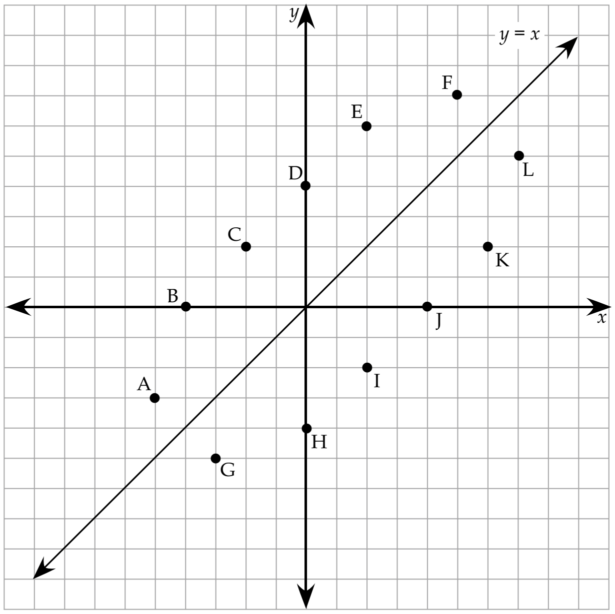
- ☐ learn how the coordinates of an ordered pair are affected when the ordered pair is reflected through the line $y = x$
- ☐ learn how to sketch the graph of a function that is reflected through the line $y = x$
- ☐ learn how to sketch the graph of $y = f^{-1}(x)$
- ☐ learn how to write the equation of a function that has been reflected through the line $y = x$

Lesson Introduction

So far, you have reflected functions vertically in the x-axis, $y = -f(x)$, and horizontally in the y-axis, $y = f(-x)$. It is also possible to reflect functions in the line $y = x$. This type of reflection is related to the inverse of a function. Multiplying a value by 2 and dividing a value by 2 are inverse operations.

Examples of functions that are inverses are $y = 2x$ and $y = \frac{1}{2}x$.

Reflections in the Line $y = x$



Examine the following graph. Consider the relationship of coordinates of the set of points A to F to the set of points G to L and the given line, $y = x$.

When the set of points A to F is reflected over the line $y = x$, you get the set of points G to L. Look at their coordinates.

Points A to F	Points G to L
A(−5, −3)	G(−3, −5)
B(−4, 0)	H(0, −4)
C(−2, 2)	I(2, −2)
D(0, 4)	J(4, 0)
E(2, 6)	K(6, 2)
F(5, 7)	L(7, 5)

As you can see, the coordinates of the two sets are in reverse order. This can be expressed using the transformation $(x, y) \rightarrow (y, x)$.

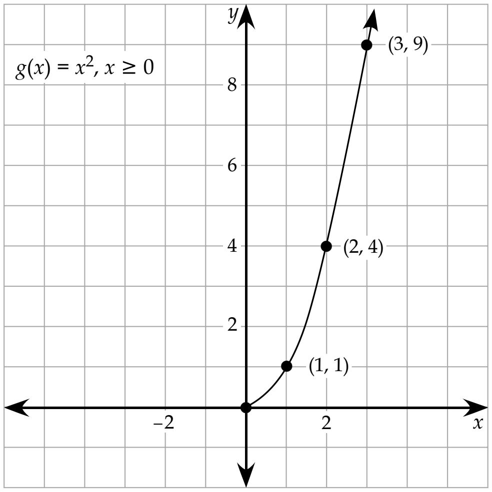
If $f(x) = \{(-5, -3), (-4, 0), (-2, 2), (0, 4), (2, 6), (5, 7)\}$ is the original set of ordered pairs, when these points are reflected over the line $y = x$, the new set of coordinates becomes $\{(-3, -5), (0, -4), (2, -2), (4, 0), (6, 2), (7, 5)\}$.

This type of reflection creates an inverse relation. If the set of points from A to F is called $f(x)$, then the set of points from G to L is called $f^{-1}(x)$. (**Note:** This is read as '*f* inverse of *x*' or 'the inverse of $f(x)$.

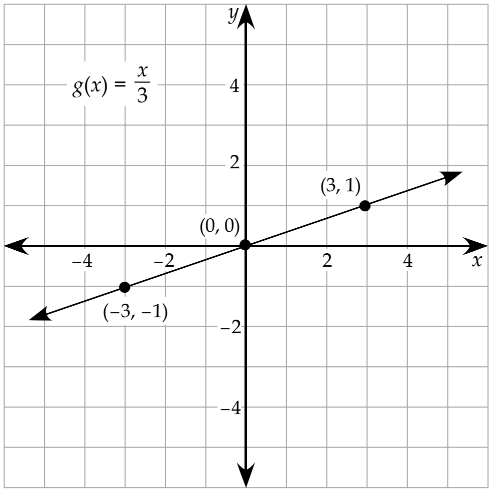
Example 1

Use the graphs drawn below of $g(x)$ to sketch the graph of $g^{-1}(x)$.

a)

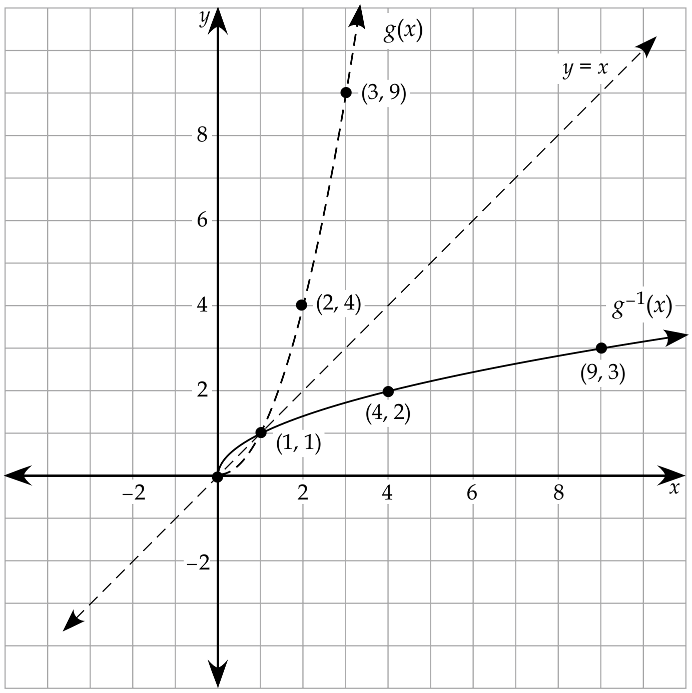


b)



Solutions

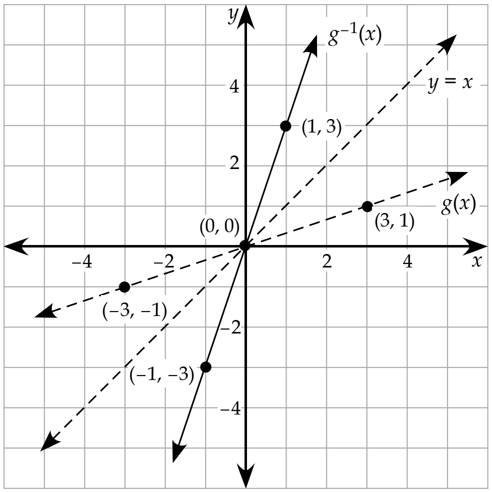
a) To graph the inverse of $g(x)$, you need to switch the order of the coordinates in each ordered pair. Therefore, the points (2, 4) and (3, 9) become (4, 2) and (9, 3) respectively. The points (0, 0) and (1, 1) stay the same.



Algebraically:
$(x, y) \rightarrow (y, x)$

Notice that the inverse of the given function is a reflection in the line $y = x$. Also notice that the inverse of the right half of a parabola looks like a radical function. That is because $y = \sqrt{x}$ is the inverse of $y = x^2$ for non-negative x-values.

b)



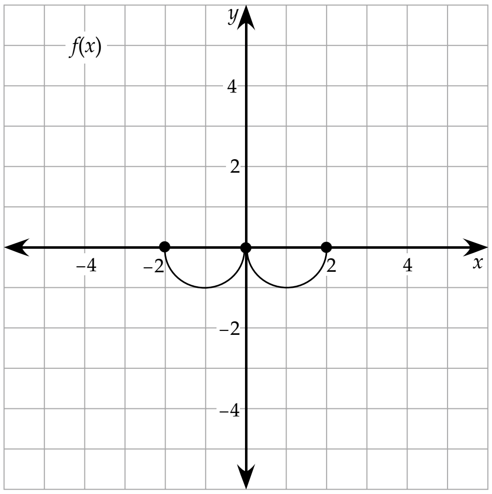
Switch the order of the coordinates in each ordered pair of $g(x)$.

Notice the inverse of the function $g(x) = \dfrac{x}{3}$ is the function $y = 3x$. This is consistent since multiplication is the inverse of division.

Example 2

Draw the inverse of each function, which is a reflection of each function through the line $y = x$.

a)



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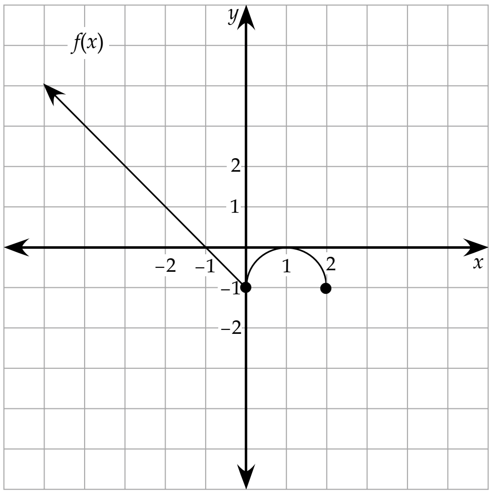
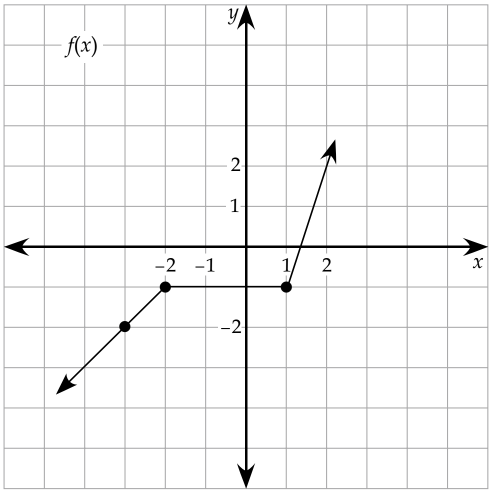


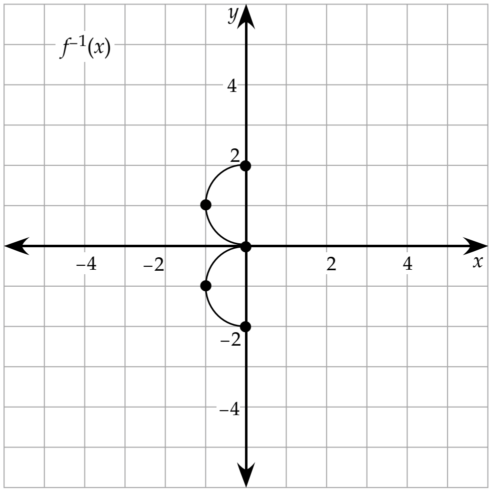
c)



Solutions

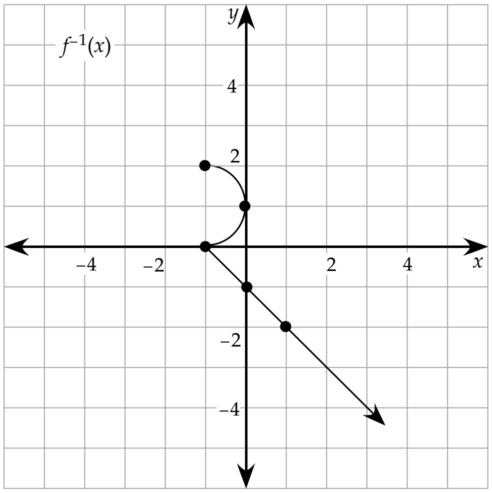
a) To reflect functions through the line $y = x$, it is sometimes easiest to create a table of values. Once you have the coordinates of some points on $f(x)$, it is possible to transform these coordinates using the transformation $(x, y) \rightarrow (y, x)$ to find points on the graph of $f^{-1}(x)$.

$f(x)$	$f^{-1}(x)$
$(-2, 0)$	$(0, -2)$
$(-1, -1)$	$(-1, -1)$
$(0, 0)$	$(0, 0)$
$(1, -1)$	$(-1, 1)$
$(2, 0)$	$(0, 2)$



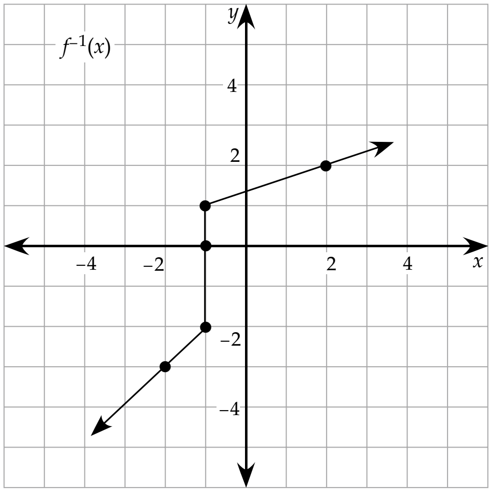
b)

$f(x)$	$f^{-1}(x)$
(−2, 1)	(1, −2)
(0, −1)	(−1, 0)
(1, 0)	(0, 1)
(2, −1)	(−1, 2)



c)

$f(x)$	$f^{-1}(x)$
(−3, −2)	(−2, −3)
(−2, −1)	(−1, −2)
(1, −1)	(−1, 1)
(2, 2)	(2, 2)



Example 3

Sketch the reflection through the line $y = x$ for each of the following functions.

a) $f(x) = x^2$

b) $g(x) = x^3$

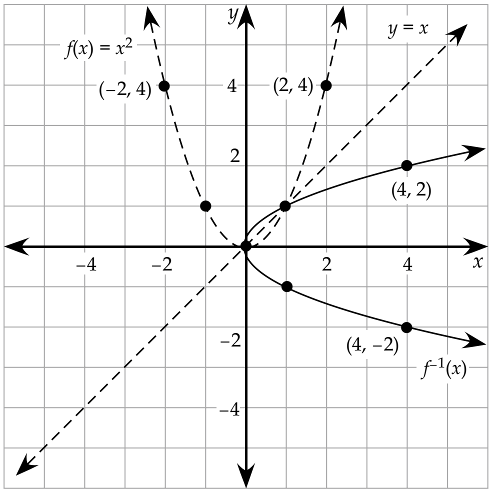
c) $h(x) = \sqrt{x}$

d) $j(x) = |\,x\,|$

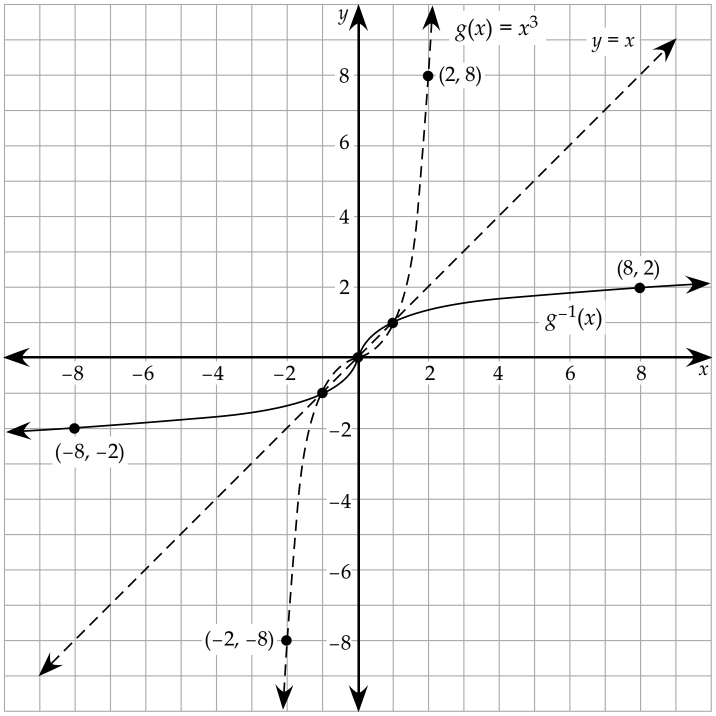
Solutions

Switch the roles of x and y for key points to sketch the inverse, which is the reflection through the line $y = x$.

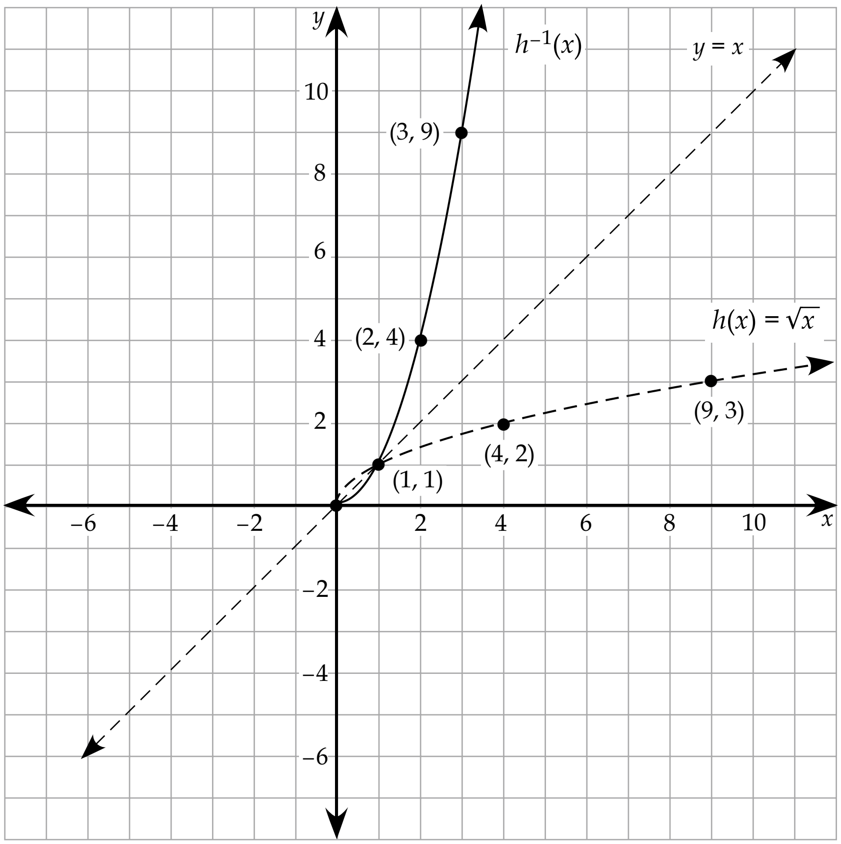
a)



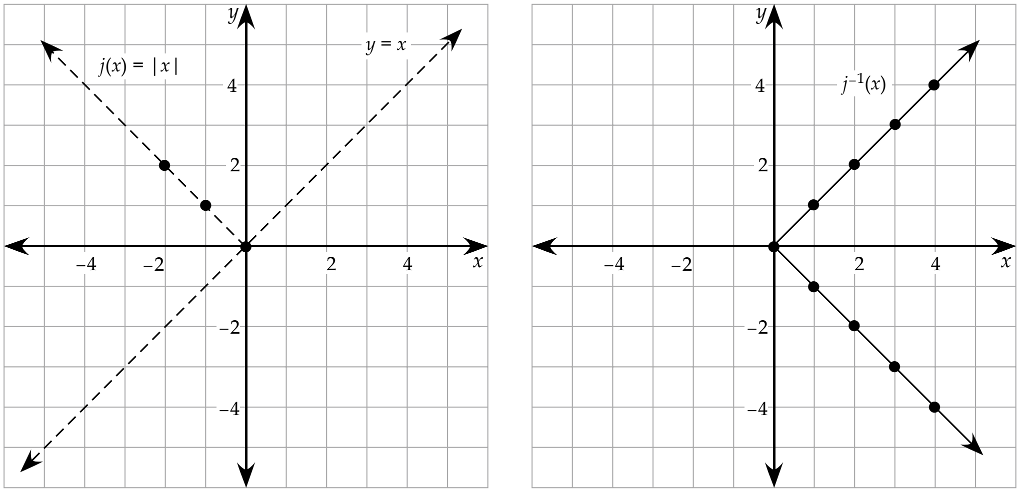
b)



c)



d)



The reflection through the line $y = x$ can be summarized in the following table. Make sure this information is included on your resource sheet.

Transformation	**Reflection through Line**	**Effect on (x, y)**
$f^{-1}(x)$	Reflect through $y = x$	(y, x)

Learning Activity 3.3

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

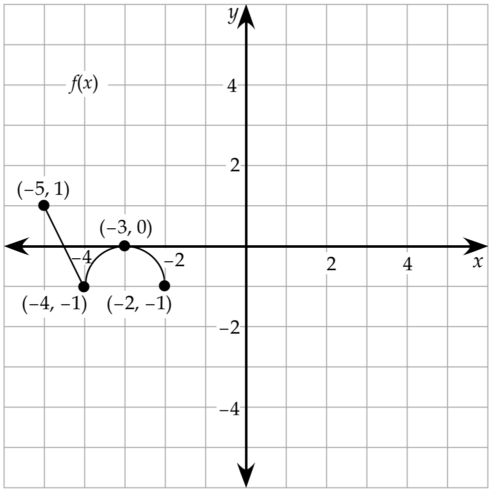
1. How many different permutations are possible of the word PHONE?
2. Evaluate: ${}_4P_4$
3. Estimate the surface area of a sphere with a radius of 5 m.
4. Is the point (2, −5) a solution to the inequality $y > x^2 + 3x - 1$?
5. Factor: $x^2 + 3x - 10$
6. Simplify: $\sqrt[4]{162y^7z^9}$
7. Simplify: $\sqrt{3}\left(3\sqrt{2} - \sqrt{6}\right)$
8. What is 70% of 260?

continued

Part B: Reflections through the Line *y* = *x*

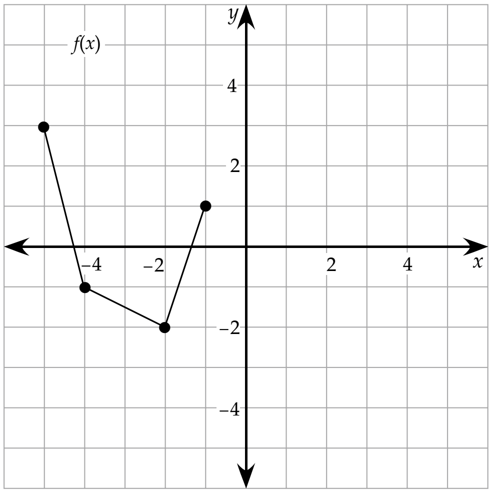
Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Use the graph drawn below of $f(x)$ to sketch the graph of $f^{-1}(x)$.

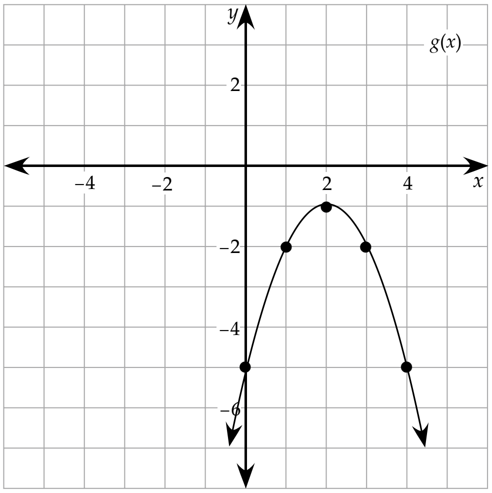


2. Reflect each of the functions below through the line $y = x$.

a)



b)



continued

Learning Activity 3.3 (continued)

c) $j(x) = -\sqrt{\frac{1}{2}(x-1)}$

d) $m(x) = -2x^3 + 1$

3. The following points are on the graph of $h(x)$:

   (−1, −4), (0, −2), (1, 4), (3, 6), and (4, −3)

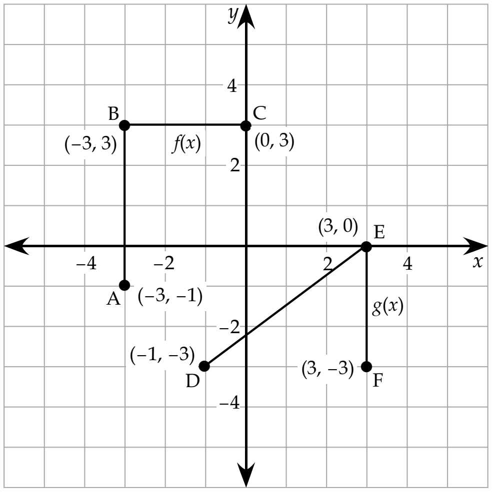
   Determine the corresponding points on the graph of $h^{-1}(x)$.

4. Write the coordinates of the inverse of each of the following functions.

   a) $f(x) = \{(3, 7), (2, 3), (4, 4), (-3, 2), (-7, -9)\}$

   b) $g(x) = \{(-1, 2), (6, 2), (-4, -7), (1, 5), (3, 2)\}$

5. Are the two relations shown inverses of one another?



6. Is the line $y = 0.5x + 2$ the inverse of the line $y = 2x - 4$? Give a graphical solution.

Lesson Summary

Now that you have learned how to reflect functions through the line $y = x$, you can learn more about properties of inverse functions. In the next lesson, you will learn how to determine if two functions are inverses of each other algebraically. You will also learn how to determine if the inverse of a function is a function itself.

Notes

Lesson 4: Inverse Functions

Lesson Focus

In this lesson, you will

☐ learn how to find the inverse of a function algebraically

☐ learn how to determine if a function is one-to-one

☐ learn how to determine restrictions on the domain of a function to ensure its inverse is also a function

☐ learn about the relationship between the domain and range of a function and its inverse

Lesson Introduction

When you reflect a function through the x-axis or the y-axis, the result is always a function. However, when you reflect a function through the line $y = x$, the result is not always a function. To determine whether the inverse is a function, certain tests can be done, such as checking whether the function is one-to-one. You can also restrict the domain of a function to ensure its inverse is a function. It is possible to find the inverse function or relation algebraically as well as graphically. You will learn all of these skills in this lesson.

Inverse Functions

In Lesson 3, you learned that the inverse of a function $f(x)$ is denoted as $f^{-1}(x)$ and read as "f inverse of x" or "the inverse of $f(x)$."

Three facts you should note:

1. The graph of the inverse relation is the reflection of $f(x)$ over the line $y = x$. That is, each corresponding point is the same perpendicular distance away from the line and on opposite sides.
2. The domain of $f(x)$ becomes the range of $f^{-1}(x)$.

   The range of $f(x)$ becomes the domain of $f^{-1}(x)$.
3. The inverse relation of $f(x)$ may or may not be a function.

In Module 2, you learned about composition of functions. You will use that concept to verify that two functions are inverses.

Note: Many students confuse $f^{-1}(x)$ with $\dfrac{1}{f(x)}$. They are not the same:

f^{-1} is a notation that means "the inverse of $f(x)$" whereas $\dfrac{1}{f(x)}$ means "the reciprocal of $f(x)$."

Example 1

Find the inverse of $f(x) = 3x$.

Solution

The given function multiplies the input, x, by 3. To "undo" this function, divide each input by 3.

It may be intuitive for you that if $f(x) = 3x$, then $f^{-1}(x) = \dfrac{x}{3}$.

The value of $f(x)$ when $x = 4$ is 12. Now, use 12 as the input for the inverse function. The value of $f^{-1}(x)$ when $x = 12$ is 4. The inverse function will "undo" the effect of the function.

To show that this holds true for all values of x, look at the compositions of $f(f^{-1}(x))$ and $f^{-1}(f(x))$.

$$\begin{aligned} f\left(f^{-1}(x)\right) &= f\left(\frac{x}{3}\right) \\ &= \frac{3 \cdot x}{3} = x \end{aligned}$$

$$\begin{aligned} f^{-1}\left(f(x)\right) &= f^{-1}(3x) \\ &= \frac{3x}{3} = x \end{aligned}$$

When you begin with any value of x and the result of these two compositions is x, then the two functions are inverses.

Inverse Function: Let f and g be two functions such that $f(g(x)) = x$ for every x in the domain of g and $g(f(x)) = x$ for every x in the domain of f. Then the function g is the inverse of the function f, and is denoted by f^{-1}. That is, $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. The domain of f will be equal to the range of f^{-1}. It also means the range of f will be equal to the domain of f^{-1}. This is consistent since the roles of x and y have been switched for the inverse function.

As you learned in the last lesson, the graphs of f and f^{-1} are related to each other in this way. If the point (a, b) lies on the graph f, then the point (b, a) lies on the graph of f^{-1} and vice versa. This means that the graph of f is a reflection of the graph of f^{-1} in the line $y = x$.

The inverse of a function is a relation but need not be a function. The function $f(x) = x^2$ has an inverse that is not a function. A function is said to be **one-to-one** if its inverse is also a function. This means that not only does each x-value correspond to exactly one y-value, but also each y-value corresponds to exactly one x-value. It can be determined whether a function, $f(x)$, is one-to-one by applying a vertical line test to its inverse, $f^{-1}(x)$ or, more simply, by applying the **horizontal line test** to $f(x)$.

As you know, for a relation to be a function, it must pass the vertical line test. For a relation to be one-to-one, it must pass both the vertical line test and the horizontal line test.

If the graph of the function $y = f(x)$ is such that no horizontal line intersects the graph in more than one point, then f is one-to one since it has an inverse that is a function. Unless a function is one-to-one, then $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ may not hold true for all x in the domain.

Make sure you include the above definitions on your resource sheet.

Example 2

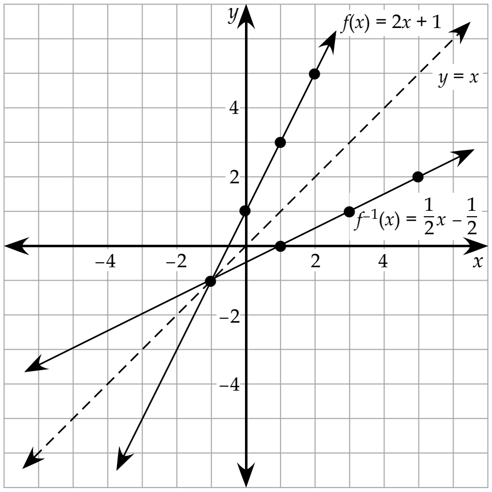
Sketch the following functions and their inverses. Determine if each function is a one-to-one function.

a) $f(x) = 2x + 1$

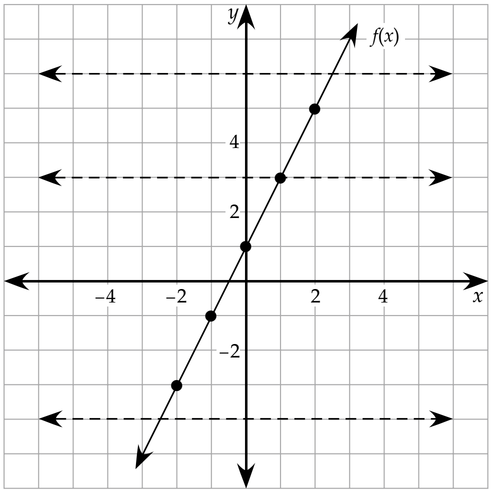
b) $g(x) = x^2$

Solutions

a) Sketch $f(x) = 2x + 1$ and then reflect $f(x)$ over the line $y = x$ to graph $f^{-1}(x)$.

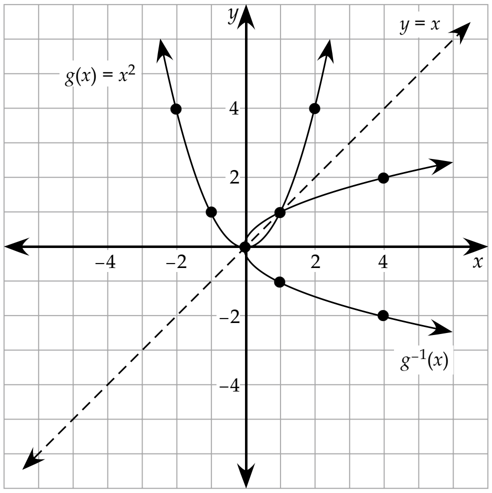


To determine if $f(x)$ is a one-to-one function, you can either do a vertical line test on the inverse function or use the horizontal line test on the graph of $f(x)$.

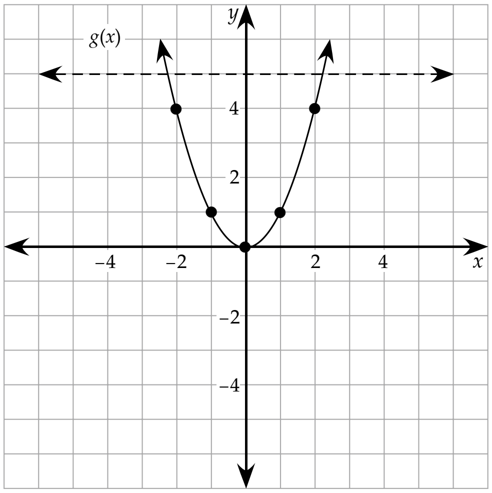


As you can see, no matter where you draw a horizontal line, it will never cross the function more than once. Therefore, $f(x)$ is a one-to-one function.

b) Sketch $g(x) = x^2$ and then reflect $g(x)$ over the line $y = x$ to graph $g^{-1}(x)$.

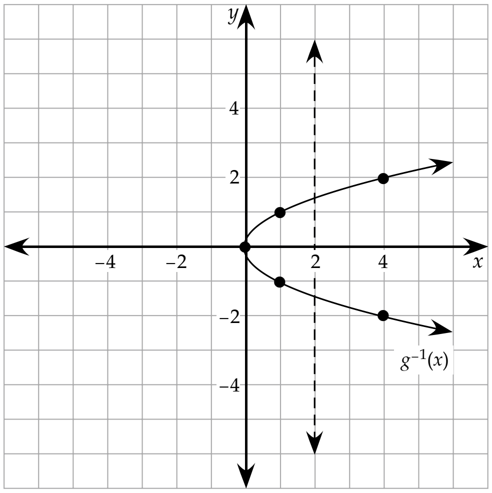


To determine if $g(x)$ is a one-to-one function, use the horizontal line test on the graph of $g(x)$.



$g(x)$ is not a one-to-one function because it is possible to draw a horizontal line that crosses the function more than once.

Alternatively, you can determine if $g(x)$ is a one-to-one function by using the **vertical line test** on the inverse relation $g^{-1}(x)$.



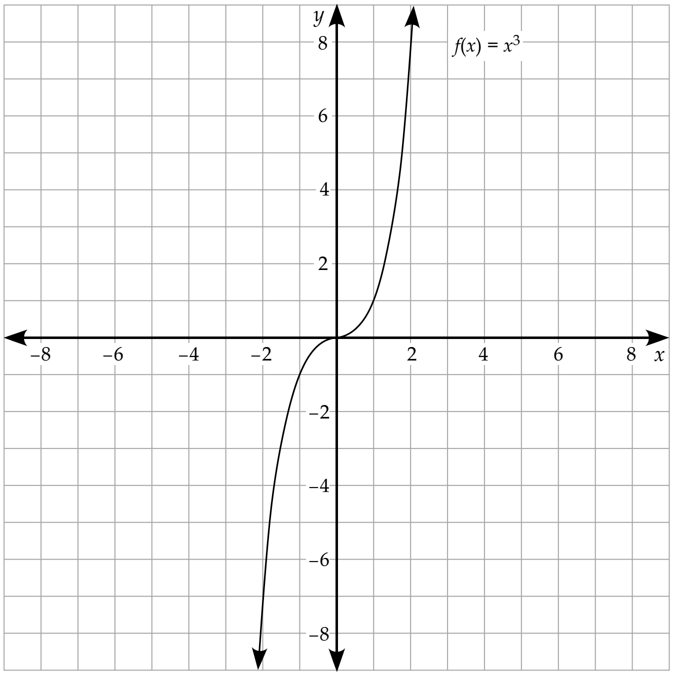
If you do the vertical line test on the graph of $g^{-1}(x)$, it is possible to draw a vertical line that crosses the relation more than once. Therefore, $g^{-1}(x)$ is not a function and it follows that $g(x)$ is not one-to-one.

From this example, you can see how the horizontal line test on the graph of a function is the same as the vertical line test on the graph of the inverse of a function.

Example 3

Is $f(x) = x^3$ a one-to-one function?

Solution

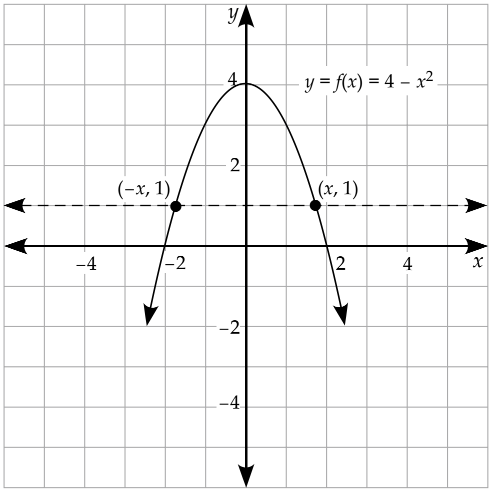


The function $f(x) = x^3$ is a one-to-one function since it passes both the vertical line test and the horizontal line test. Its inverse is also a function.

Example 4

Is $f(x) = 4 - x^2$ a one-to-one function?

Solution



The function, $f(x) = 4 - x^2$, is not one-to-one because it has two points that have the same y-value on the horizontal line, so the inverse will not be a function.

For simple linear functions, as in Example 2(a), you can find the inverse by inspection. For instance, if $f(x) = x - 4$ then $f^{-1}(x) = x + 4$. However, for more complicated functions, it is best to use the following algebraic method for finding the inverse of a given function.

1. Replace $f(x)$ with y and interchange the x- and y-values.
2. Solve for y in terms of x and then replace y by $f^{-1}(x)$.
3. Confirm to see that the domain of f is the range of f^{-1} and that the domain of f^{-1} is the range of f.

Example 5

Find the inverse function for $f(x) = 2x - 1$. Write the domain and range for both.

Solution

$f(x) = 2x - 1$

$y = 2x - 1$ Replace $f(x)$ by y to represent $f(x)$.

So, the inverse is represented by the equation:

$x = 2y - 1$ Interchange x- and y-values to represent $f^{-1}(x)$.

$x + 1 = 2y$ Solve for y.

$$y = \frac{x+1}{2}$$

$f^{-1}(x) = \dfrac{x+1}{2}$ Replace y by $f^{-1}(x)$.

This function can also be written as $f^{-1}(x) = \dfrac{x}{2} + \dfrac{1}{2}$, which is a linear function. In general, the inverse of any linear function is another linear function.

You can check your work to see if $f^{-1}(x)$ is correct by using the composition of functions.

If $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$, then $f^{-1}(x)$ is correct.

$$\begin{aligned} f\left(f^{-1}(x)\right) &= f\left(\frac{x+2}{2}\right) \\ &= \frac{2(x+1)}{2} - 1 \\ &= x + 1 - 1 \\ &= x \end{aligned}$$

$$\begin{aligned} f^{-1}\left(f(x)\right) &= f^{-1}(2x-1) \\ &= \frac{2x-1+1}{2} \\ &= \frac{2x}{2} \\ &= x \end{aligned}$$

$\therefore\ f^{-1}(x) = \dfrac{x+1}{2}$ is the inverse of $f(x) = 2x - 1$.

The domain and range of both f and f^{-1} are the real numbers.

Notice the functions intersect at the point (1, 1), which would lie on the reflection line, $y = x$. All points that lie on the line $y = x$ are invariant points when a function is reflected through the line $y = x$.

Example 6

a) Find the inverse of the function $f(x) = \sqrt{2x-3}$ and sketch the graph.

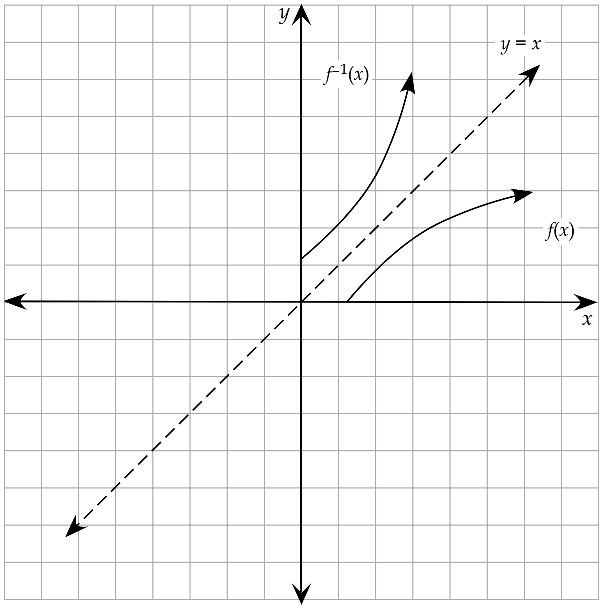
b) State the domain and range of $f(x)$ and $f^{-1}(x)$.

Solutions

a) $f(x) = \sqrt{2x-3},\ x \geq \frac{3}{2},\ y \geq$	Notice the domain and range of $f(x)$.
$y = \sqrt{2x-3}$	Write y instead of $f(x)$ to represent $f(x)$.
The inverse is represented by:	
$x = \sqrt{2y-3}$	Interchange x and y to represent $f^{-1}(x)$.
$x^2 = 2y - 3$	Square both sides.
$x^2 + 3 = 2y$	Solve for y.
$\frac{x^2+3}{2} = y$	
$f^{-1}(x) = \frac{x^2+3}{2},\ x \geq 0,\ y \geq \frac{3}{2}$	Replace y by $f^{-1}(x)$. The domain of the inverse is the range of $f(x)$ and the range of the inverse is the domain of $f(x)$.

b) Domain: $f(x): \left[\frac{3}{2}, \infty\right)$ Domain: $f^{-1}(x)$: $[0, \infty)$

Range: $f(x)$: $[0, \infty)$ Range: $f^{-1}(x): \left[\frac{3}{2}, \infty\right)$



Note that the domain and range for $f(x)$ are interchanged for the domain and range of $f^{-1}(x)$.

Example 7

a) Graph the function $f(x) = (x - 2)^2 - 4$.

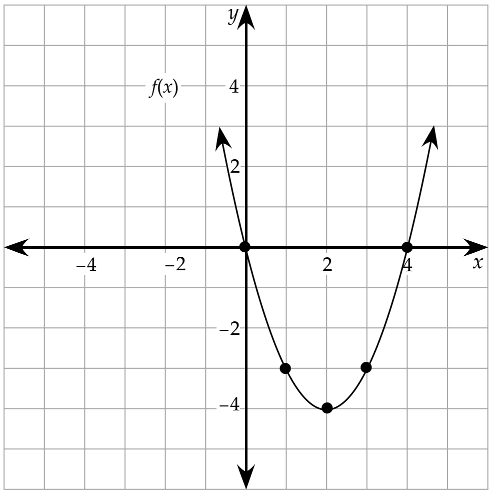
b) Graph the inverse of $f(x)$.

c) Is the inverse of $f(x)$ a function? Explain.

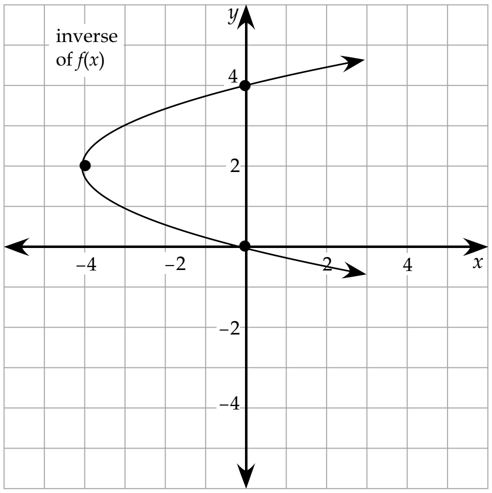
d) Determine restrictions on the domain of $f(x)$ in order for its inverse $f^{-1}(x)$ to be a function.

Solutions

a) This graph is a transformation of the standard quadratic function. The graph of $y = x^2$ has been translated 2 units to the right and 4 units down.



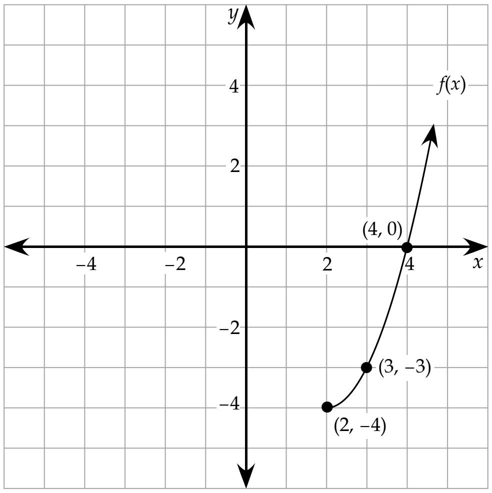
b)



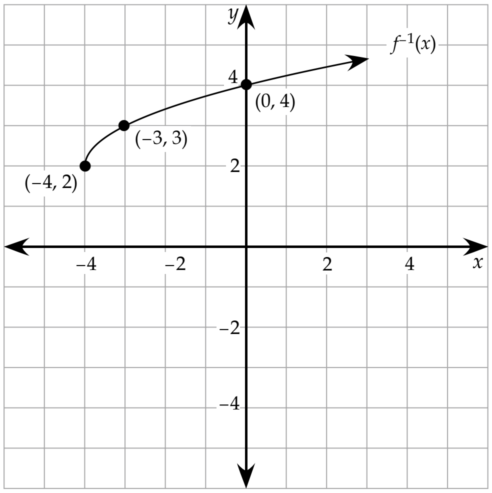
c) The inverse of $f(x)$ is not a function because it does not pass the vertical line test. Another way to determine that the inverse of $f(x)$ is not a function is to see that $f(x)$ does not pass the horizontal line test. $f(x)$ is not one-to-one and the inverse of $f(x)$ is not a function over the domain $(-\infty, \infty)$.

d) In order for the inverse of $f(x)$ to be a function, $f(x)$ must pass the horizontal line test. If you restrict the domain of $f(x)$ to only include a segment of the function, it will pass the horizontal line test. It is a good idea to restrict the domain of the parabola to either the part left of the axis of symmetry ($x = 2$) or the right of the axis of symmetry.

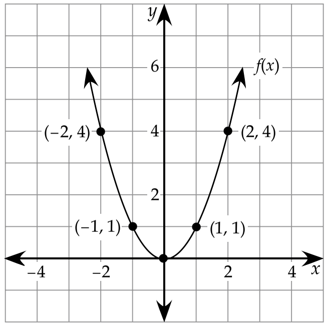
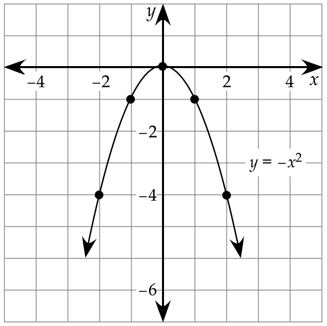
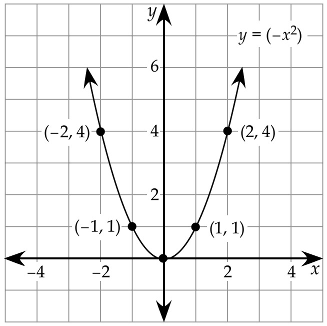
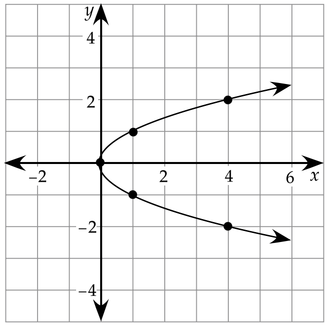
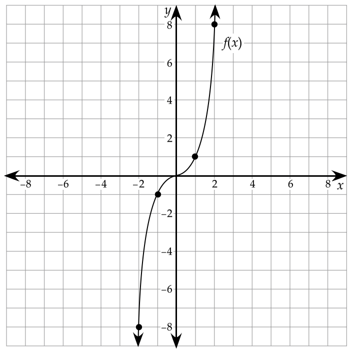
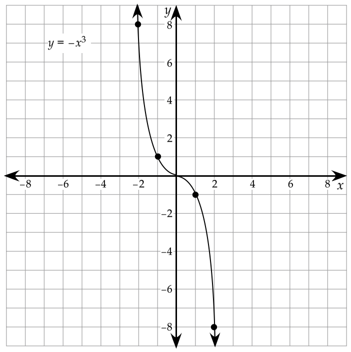
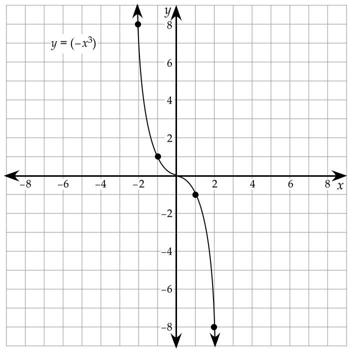
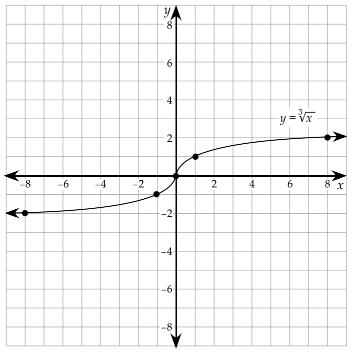
If you restrict the domain of $f(x)$ to be $[2, \infty)$, then only half of the parabola will be graphed.



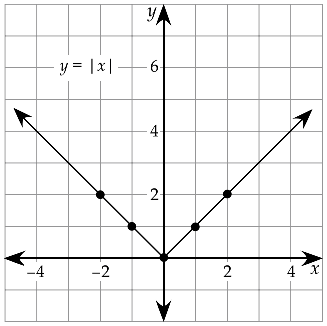
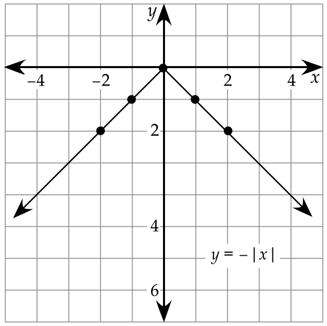
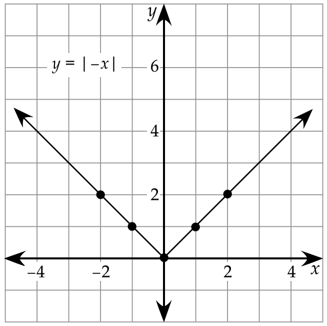
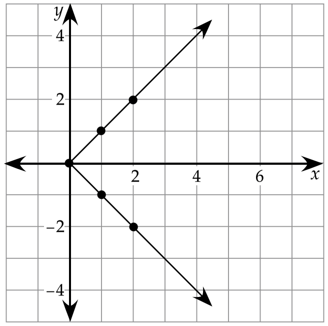
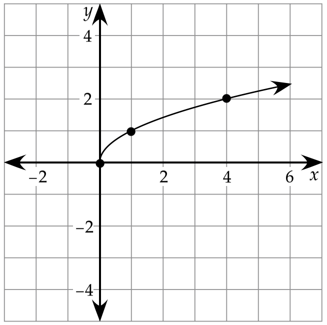
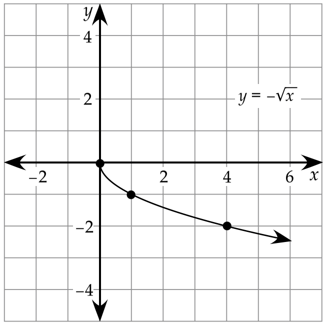
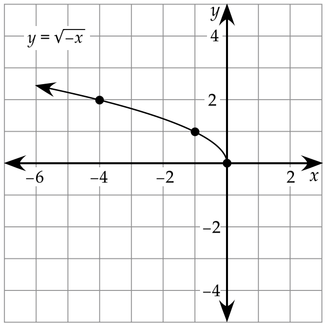
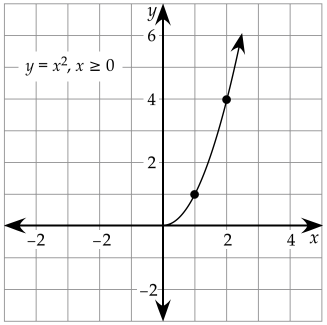
This half of the parabola passes the horizontal line test. Therefore, the inverse will be a function. If the domain is restricted as $f(x) = (x - 2)^2 - 4$, $x \geq 2$.



The following chart summarizes the different reflections of the four main functions. Make sure you are familiar with these reflections. You should note the similarity of the process of reflection regardless of the function type.

Graph Inventory				
Function	**Original Graph**	**Reflection in the *x*-axis**	**Reflection in the *y*-axis**	**Reflection in the line $y = x$**
$y = x^2$				
$y = x^3$				

Graph Inventory (continued)

Function	Original Graph	Reflection in the x-axis	Reflection in the y-axis	Reflection in the line $y = x$
$y = \lvert x \rvert$				
$y = \sqrt{x}$				

Learning Activity 3.4

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Simplify: $\frac{7!}{9!}$.

2. Rationalize the denominator: $\frac{4}{\sqrt{3}-\sqrt{2}}$

3. Your restaurant bill came to \$124.52. If you wish to leave a 15% tip, estimate how much you should leave.

4. If $f(x) = \frac{x^2}{5} + 4,$ evaluate $f(10)$.

5. List all the factors of 48.

6. Simplify: $\sqrt{\frac{27y^3}{3x^2}}$

7. Write as a mixed fraction: $\frac{190}{15}$

8. What is 30% of \$95?

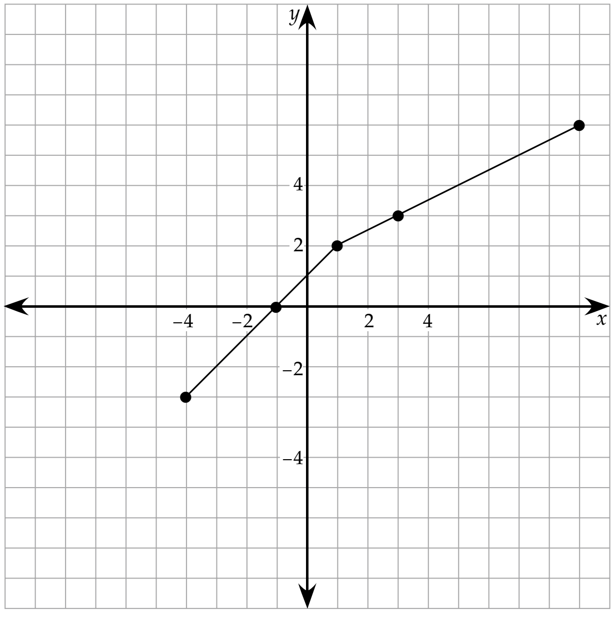
continued

Learning Activity 3.4 (continued)

Part B: Inverse Functions

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Which of the functions listed below have an inverse that is a function? Write the inverse relation in the same format as the original function.
   a) $f(x) = \{(3, 7), (2, 3), (4, 4)\}$
   b) $f(x) = \{(6, 2), (3, 2), (1, 5)\}$
   c) $f(x) = \frac{1}{2}x + 3$
   d) $g(x) = x^2 + 1$
   e) $f(x) = \sqrt{x - 3}$
   f)



continued

2. For each of the following functions $f(x)$, define and graph its inverse $f^{-1}(x)$.

   a) $f(x) = \frac{x}{2}$

   b) $f(x) = x - 8$

   c) $f(x) = \frac{x+2}{3}$

   d) $f(x) = x^2 + 1$, where $x \geq 0$

   e) $f(x) = 2x^2 - 4$, where $x \geq 0$

   f) $f(x) = x^2 - 6x + 9$, where $x \geq 3$

3. Show algebraically that the functions f and g are inverses of each other.

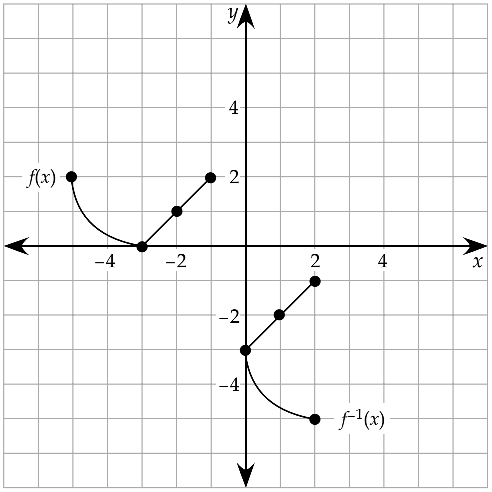
   a) $f(x) = 2x - 3$ $\qquad g(x) = \frac{x+3}{2}$

   b) $f(x) = 4 - x$ $\qquad g(x) = 4 - x$

   c) $f(x) = \sqrt[3]{x+8}$ $\qquad g(x) = x^3 - 8$

4. Determine graphically if the following sets of relations are inverses of each other.

   a)



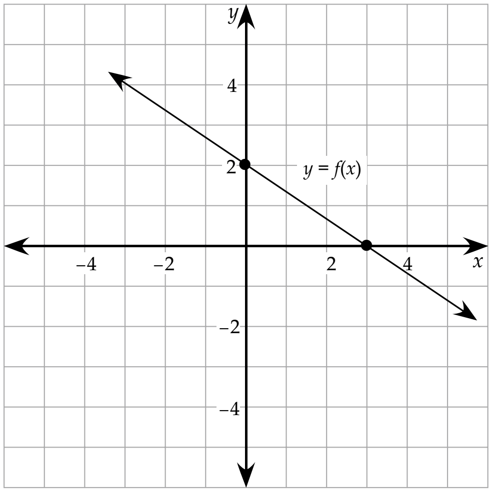
   b) $f(x) = \sqrt{\frac{1}{3}(x-6)}$ $\qquad g(x) = 3x^2 + 6,\ x \geq 0$

continued

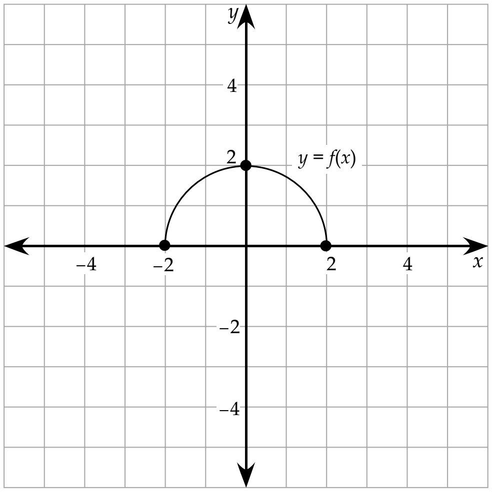
Learning Activity 3.4 (continued)

5. Determine whether each function is one-to-one.

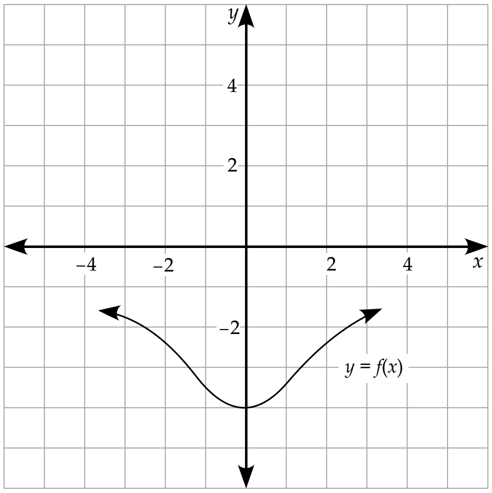
a)



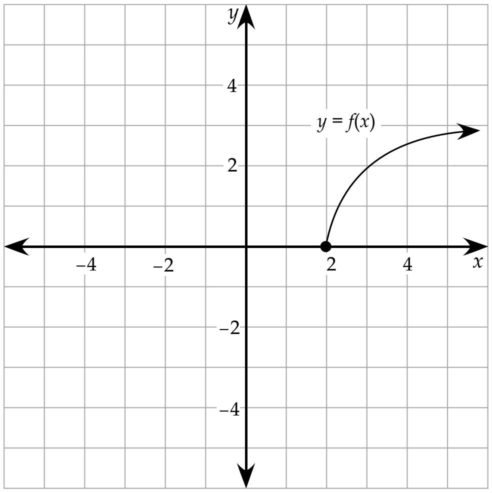
b)



c)



d)



6. Find $f^{-1}(x)$ algebraically.

a) $f(x) = \dfrac{x}{x+2}$

b) $f(x) = \sqrt{x} + 2$

c) $f(x) = \dfrac{x}{x-3}$

d) $f(x) = \dfrac{x-3}{x}$

continued

Learning Activity 3.4 (continued)

7. Find the inverse function equations of the following functions and sketch their graphs. State the domain and range of each function and its inverse.

   a) $f(x) = 2\sqrt{\frac{1}{2}x} - 3$

   b) $g(x) = \frac{1}{2}(x - 3)^3 + 1$

8. If $f(x) = 2x - 3$, find $f^{-1}(x)$ and $\frac{1}{f(x)}$, and ,show that they are not equal.

Lesson Summary

In this lesson, you learned how reflecting a function through the line $y = x$ is related to finding the inverse of a function. You also learned how to find the inverse of a function algebraically. You learned how to determine whether a function is one-to-one, which would indicate the function's inverse is a function itself. You also learned how to restrict the domain of a function to ensure its inverse would also be a function.

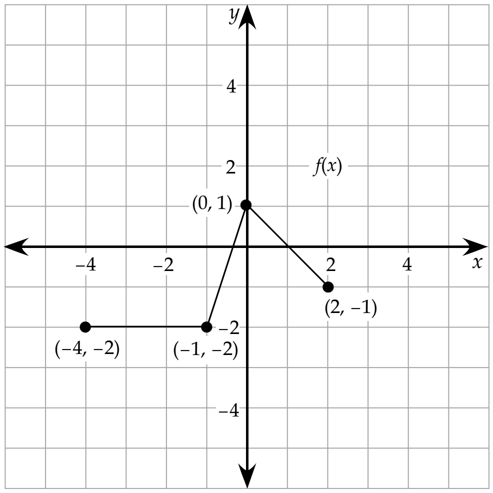
Assignment 3.2

Inverse Functions and Relations

Total: 31 marks

You need to show all your steps as you work out your solutions, and make sure your answers are clear (include units, where appropriate). Check the Introduction for the Marking Guidelines that will be used for all assignments and examinations.

1. Use the graph of $f(x)$, shown below, to sketch the graph of $y = f^{-1}(x)$. *(1 mark)*

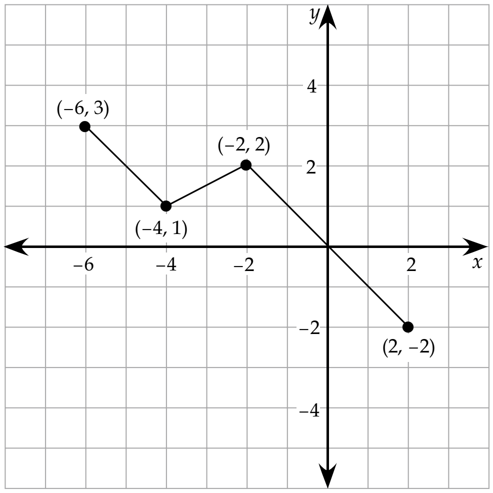


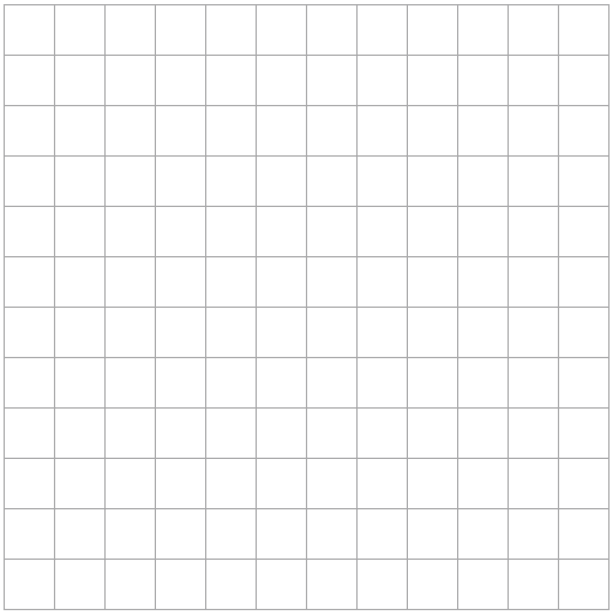
continued

Assignment 3.2: Inverse Functions and Relations (continued)

2. Reflect each of the functions below through the line $y = x$.

   a) *(1 mark)*





continued

Assignment 3.2: Inverse Functions and Relations (continued)

b) $g(x) = 2\sqrt{(x+1)} - 3$ *(2 marks)*

continued

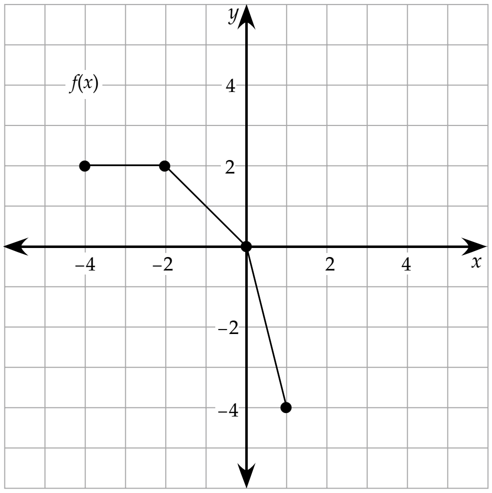
Assignment 3.2: Inverse Functions and Relations (continued)

3. Write the coordinates of the inverse of the following function. *(1 mark)*

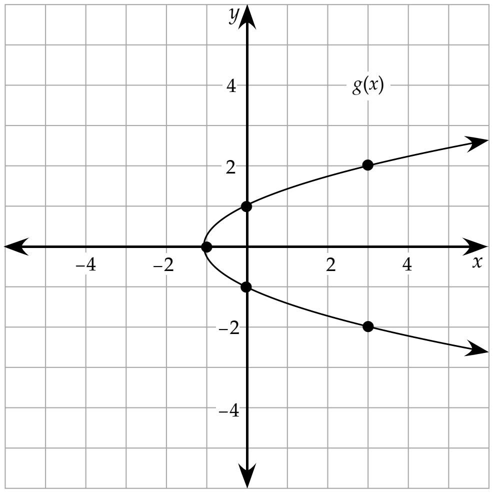
$$f(x) = \{(-3, -4), (-2, 5), (0, 0), (2, -5), (5, 7), (6, -1)\}$$

4. For each of the following relations, determine if they are one-to-one functions. *(3 × 1 mark each = 3 marks)*

a)

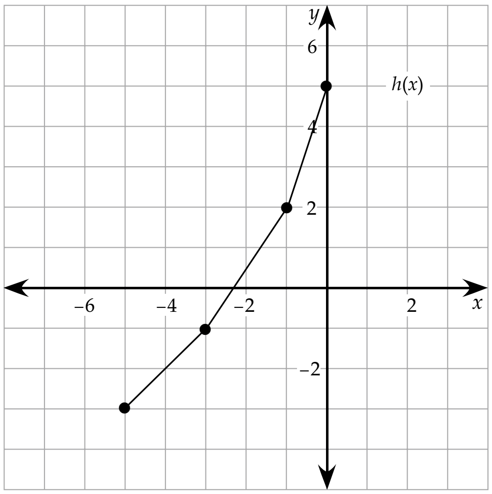


b)



continued

c)



continued

5. Show algebraically that the functions *f* and *g* are inverses of each other.
(2 × 2 marks each = 4 marks)

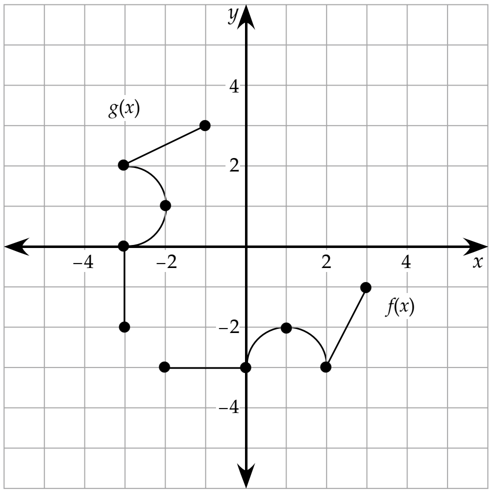
a) $f(x) = \frac{1}{2}x + 1$ $g(x) = 2(x - 1)$

b) $f(x) = \frac{1}{3}(x - 1)^3$ $g(x) = \sqrt[3]{3x} + 1$

continued

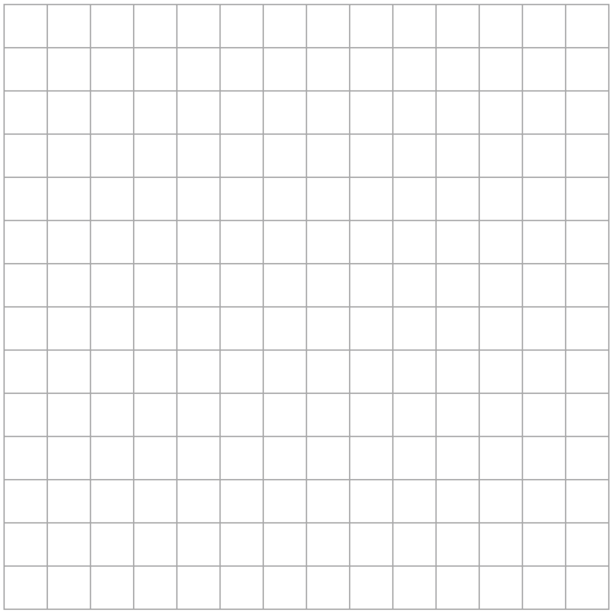
Assignment 3.2: Inverse Functions and Relations (continued)

6. Determine graphically if the following sets of relations are inverses of each other. *(1 mark)*



7. Find $f^{-1}(x)$ algebraically. Graph $f^{-1}(x)$. If necessary, write restrictions on the domain of $f(x)$ to ensure $f^{-1}(x)$ is a function.

   a) $f(x) = 3x + 5$ *(2 marks)*



continued

Assignment 3.2: Inverse Functions and Relations (continued)

b) $f(x) = \frac{1}{2}x - 4$ *(2 marks)*

c) $f(x) = 2x^2 - 6$ *(3 marks)*

continued

Assignment 3.2: Inverse Functions and Relations (continued)

d) $f(x) = x^2 - 8x + 16$ *(3 marks)*

continued

Assignment 3.2: Inverse Functions and Relations (continued)

8. Find the inverses of the following functions and sketch the graph

   a) $f(x) = \sqrt{2(x-1)} + 5$ *(3 marks)*

   b) Explain the connection between the domain and range of $f(x)$ and $f^{-1}(x)$. *(1 mark)*

continued

Assignment 3.2: Inverse Functions and Relations (continued)

9. Given: $g(x) = (x - 4)^2 - 2$

   a) Graph the function $g(x)$. *(1 mark)*

   b) Graph the inverse of $g(x)$. *(1 mark)*

continued

Assignment 3.2: Inverse Functions and Relations (continued)

c) Is the inverse of $g(x)$ a function? Explain. *(1 mark)*

d) Determine restrictions on the domain of $g(x)$ in order for its inverse $g^{-1}(x)$ to be a function. *(1 mark)*

MODULE 3 SUMMARY

In this module, you learned all about reflections. You learned about reflections through the x-axis, the y-axis, and the line $y = x$. A reflection is a different type of transformation and thus you learned about the effect various reflections have on the function $y = f(x)$.

You also learned how to find inverse functions, which are related to reflections through the line $y = x$. You were able to find inverse functions both algebraically and graphically. However, sometimes inverse relations are not functions. If a function is one-to-one, its inverse is a function. If not, you can often restrict the domain of the function to ensure its inverse is a function.

In the next module, you will be learning about polynomial functions. Linear, quadratic, and cubic functions are examples of polynomial functions. You will find patterns in the next module that extend to polynomial functions of any degree.

Submitting Your Assignments

It is now time for you to submit Assignments 3.1 and 3.2 to the Distance Learning Unit so that you can receive some feedback on how you are doing in this course. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 3 assignments and organize your material in the following order:

- ☐ Module 3 Cover Sheet (found at the end of the course Introduction)
- ☐ Assignment 3.1: Reflections in the x-axis and in the y-axis
- ☐ Assignment 3.2: Inverse Functions and Relations

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Notes

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Module 3
Reflections

Learning Activity Answer Keys

MODULE 3:
REFLECTIONS

Learning Activity 3.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. What is the first term in the expansion of $(2x^2 - 3y)^6$?
2. If the graph of $y = -\frac{1}{2}x^3 - 2$ is translated 3 units down, what is an equation of the translated cubic?
3. What is the reciprocal of $x - 3$?
4. Simplify: $((7x^3y^2)^4)^0$
5. A 171-page book contains 9 chapters. Assuming each chapter has the same number of pages, how many pages are there in each chapter?
6. Estimate the taxes, 13%, on a $1050 item.
7. If $f(x) = -x^3 - 2x$, find $f(-2)$.
8. Factor: $81y^4 - 100z^4$.

Answers:

1. $64x^{12}$ $({}_6C_0 \cdot (2x^2)^6 \cdot (-3y)^0 = (2x^2)^6)$
2. $y = -\frac{1}{2}x^3 - 5$ $\left(y = -\frac{1}{2}x^3 - 2 - 3\right)$
3. $\dfrac{1}{x - 3}$
4. 1 (any expression with an exponent of zero is 1)
5. 19 $(9 \times 9 = 81$; this leaves $171 - 81 = 90$; therefore, $171 = 81 + 90 = 9 \times 9 + 9 \times 10 = 9 \times 19)$
6. $135 (10% of 1050 is 105; 1% of 1050 is 10.5; add 105 + 10 + 10 + 10)
7. $f(-2) = 12$ $(-(-2)^3 - 2(-2) = -(-8) + 4 = 8 + 4)$
8. $(9y^2 - 10z^2)(9y^2 + 10z^2)$ (difference of squares pattern)

Part B: Reflections in the *x*-axis

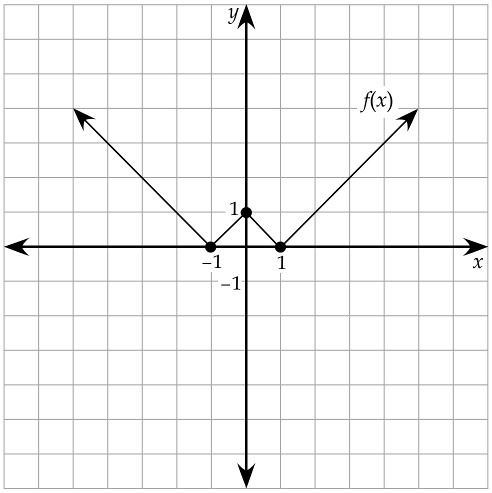
Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Using the sketch of $f(x)$, sketch the following.

   a) $y = -f(x)$

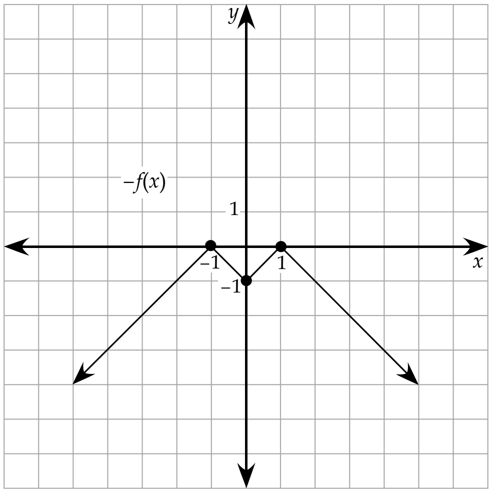
   b) $y = -f(x) + 2$

   c) $y = -f(x - 3) + 1$

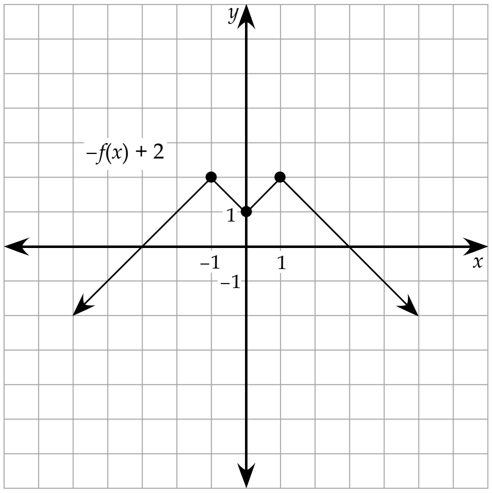


Answers:

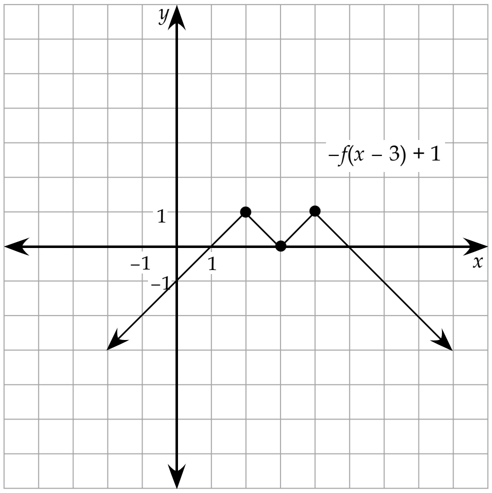
a) This function is reflected in the line $y = 0$.



b) This function is reflected in the line $y = 0$ and then shifted up 2 units.



c) This function is reflected in the line $y = 0$, shifted 3 units to the right, and then shifted 1 unit up.



2. If the x-intercepts of a function, $f(x)$, are 3, -5, and 0, find the x-intercepts after each transformation (if possible).

   a) $y = -f(x)$?

   Answer:

   The x-intercepts of a function that is reflected through the x-axis, such as $-f(x)$, do not change. Therefore, the x-intercepts of $f(x)$ are the same as the x-intercepts of $-f(x)$ which are 3, -5, and 0.

b) $y = -f(x) + 1$?

Answer:

This is impossible to solve since the translation will shift every point one unit up. The new x-intercepts are the points of the curve that originally intersected the line $y = -1$. You do not have enough information to determine these points.

c) $y = -f(x - 2)$?

Answer:

The reflection through the x-axis will have no effect on the location of the x-intercepts. However, the horizontal shift of 2 units to the right will affect the x-intercepts: each x-intercept will increase by a value of 2, or move 2 units to the right. The x-intercepts would be 5, −3, and 2.

3. If the y-intercepts of a relation, $f(x)$, are 3, −5, and 0, find the y-intercepts after each transformation (if possible).

a) $y = -f(x)$?

Answer:

When coordinates are reflected through the x-axis, the y-coordinate of each point is multiplied by negative one.

The original coordinates are (0, 3), (0, −5), and (0, 0).

When these points are reflected in the x-axis, they become (0, −3), (0, 5), and (0, 0). The new y-intercepts are −3, 5, and 0.

b) $y = -f(x) + 1$?

Answer:

When these points are reflected in the x-axis, they become (0, −3), (0, 5), and (0, 0).

The vertical shift of 1 unit up increases each of the y-coordinates by 1. The corresponding coordinates on $-f(x) + 1$ are (0, −2), (0, 6), and (0, 1). The new y-intercepts are −2, −6, and 1.

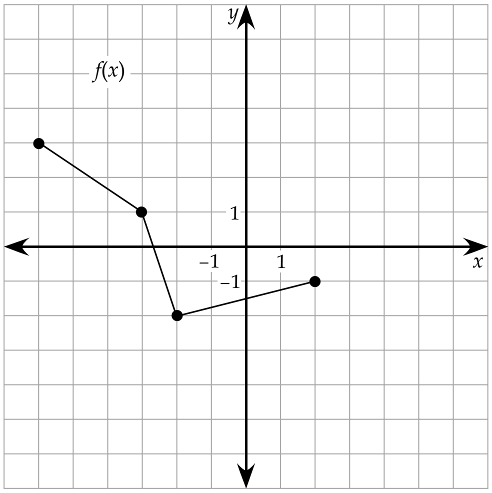
c) $y = -f(x - 2)$?

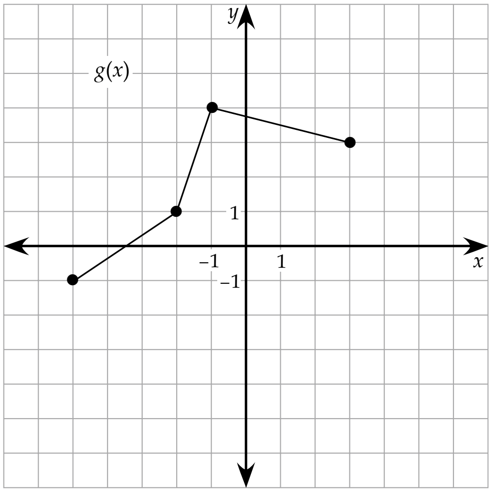
Answer:

This is impossible to determine since the curve is moved 2 units to the right. The original y-intercepts at (0, 3), (0, −5), and (0, 0) become the points (0, −3), (0, 5), and (0, 0) when they are reflected in the x-axis. When these points are shifted 2 units to the right, they become (2, −3), (2, 5), and (2, 0). These points are no longer y-intercepts. You do not have enough information to determine the new y-intercepts.

4. For each of the graphs of $f(x)$ below, write the equation of the transformed function, $g(x)$, in terms of $f(x)$.

a)



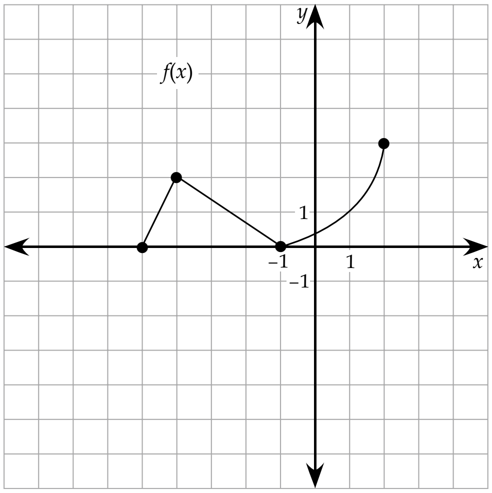


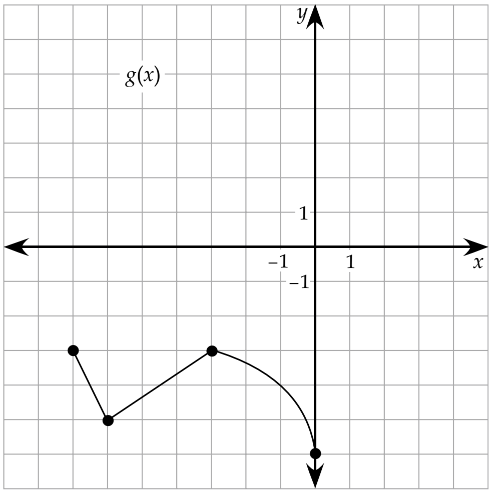
Answer:

This graph has been flipped through the x-axis, moved 1 unit to the right, and moved 2 units up.

$g(x) = -f(x - 1) + 2$

b)

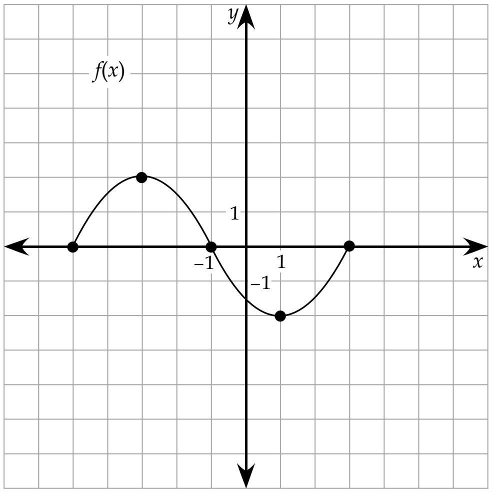


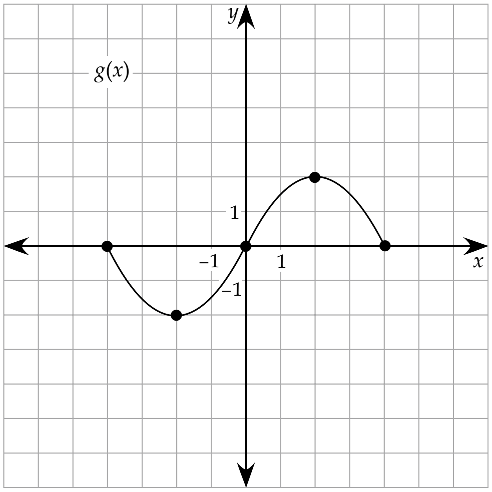


Answer:

This graph has been flipped through the x-axis, moved 2 units to the left, and moved 3 units down.

$g(x) = -f(x + 2) - 3$

c) 



Answer:

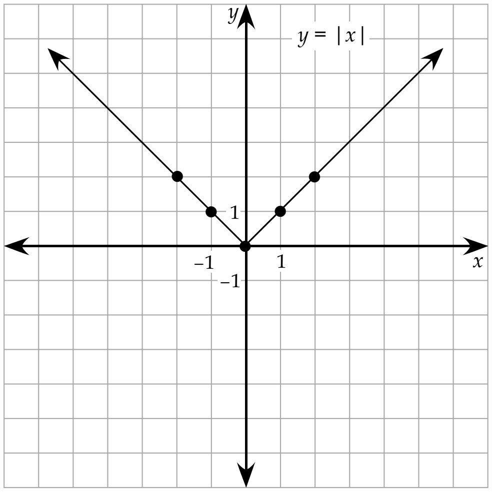
This graph has been flipped through the x-axis and moved 1 unit to the right.

$g(x) = -f(x - 1)$

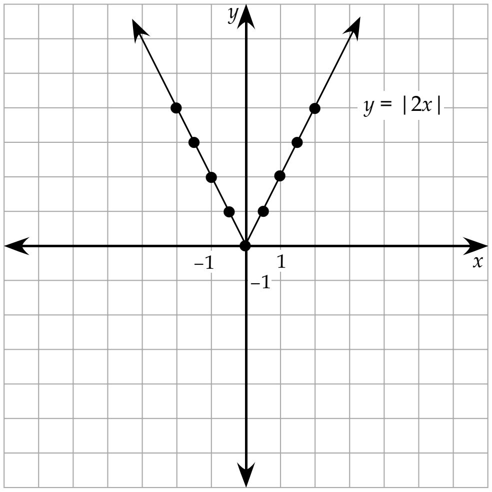
5. Sketch the graph of $y = -| 2(x + 1) |$. State the domain and range.

Answer:

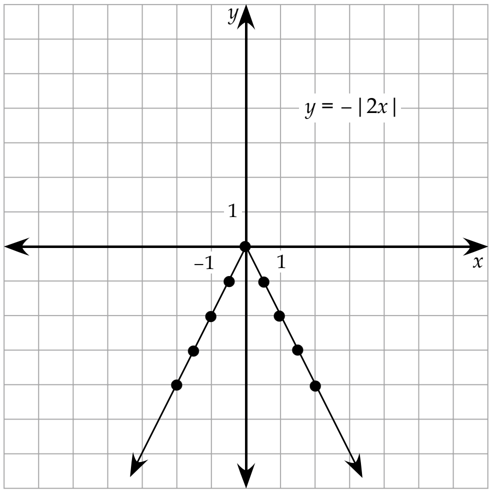
To sketch the graph of $y = -| 2(x + 1) |$, first sketch the graph of $y = | x |$.



Then, compress the graph horizontally by a factor of 2.



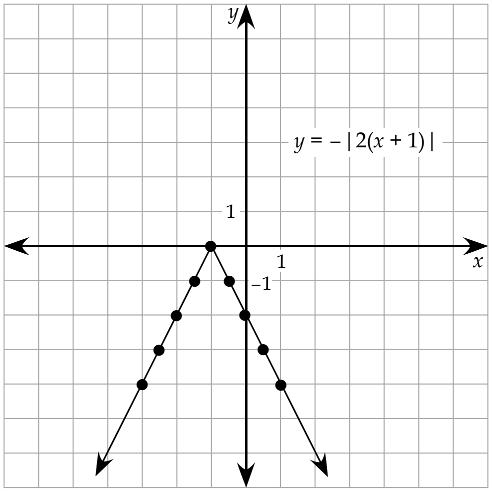
Recall: Stretches, compressions, and reflections are completed before translations (you need to multiply before adding). Therefore, after the compression, reflect the graph in the x-axis.



Then move the graph 1 unit to the left.

Recall: The horizontal shift must be done after the horizontal compression (multiply/divide before add/subtract).

This is the graph of $y = -|\,2(x + 1)\,|$.

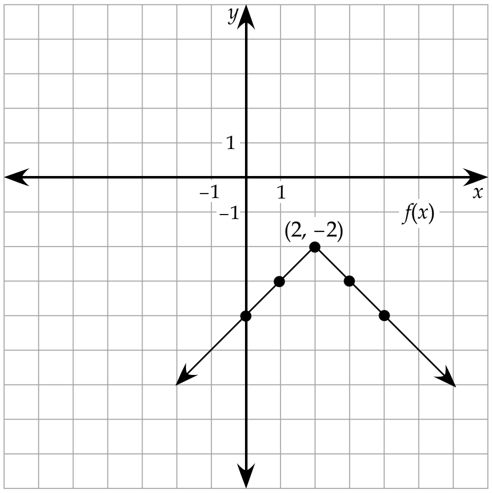


Domain: $\{x \mid x \in \Re\}$

Range: $\{y \mid y \leq 0\}$

6. For each of the graphs below, write the equation of the transformed function in terms of the basic function of the same type.

a)



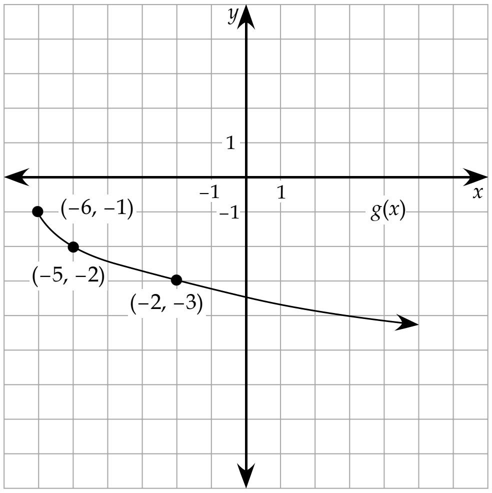
Answer:

This graph is a transformation of the absolute value function $y = |\,x\,|$.

The following transformations have occurred:

- Vertical reflection in the line $y = 0$
- Shift of 2 units to the right
- Shift of 2 units down

Therefore, the function is $f(x) = -|\,x - 2\,| - 2$.

b) 

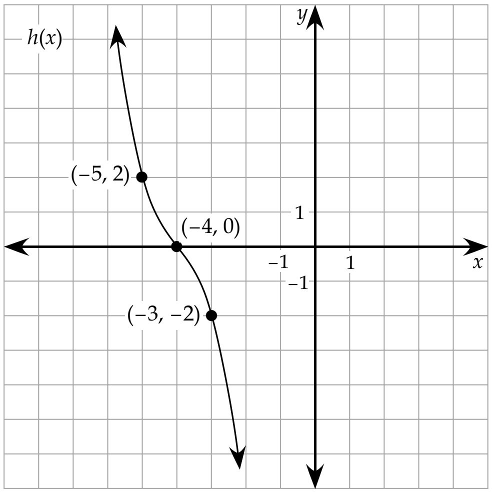
Answer:

This graph is a transformation of the square root function $y = \sqrt{x}$.

The following transformations have occurred:

- Vertical reflection in the line $y = 0$
- Shift of 6 units to the left
- Shift of 1 unit down

Therefore, the resulting function is $g(x) = -\sqrt{x+6} - 1$.

c) 

Answer:

This graph is a transformation of the cubic function $y = x^3$.

The following transformations have occurred:

- Vertical reflection in the line $y = 0$
- Vertical stretch by a factor of 2
- Shift of 4 units to the left

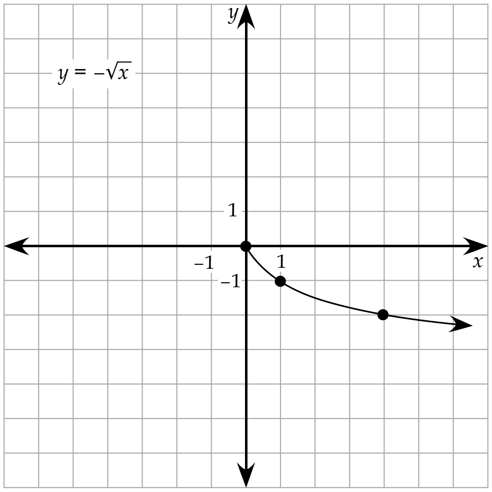
Therefore, the resulting function is $h(x) = -2(x + 4)^3$.

7. Maya and Rayne are debating how to graph the function, $y = -2\sqrt{x}$. Maya believes that the vertical reflection through the x-axis should be graphed first, followed by the vertical stretch. Rayne believes that the vertical stretch should be graphed first, followed by the reflection through the x-axis. Graph the above function using both of their methods. Which method is correct?

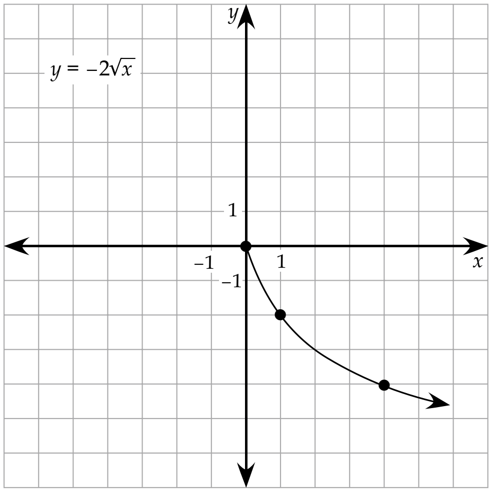
Answer:

First, graph the function using Maya's method.

Reflection through the x-axis:

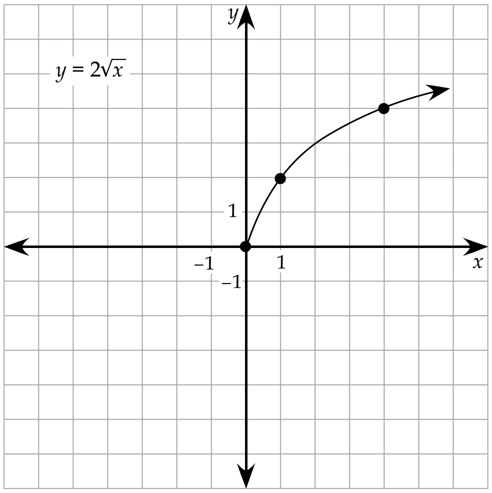


Then vertical stretch:

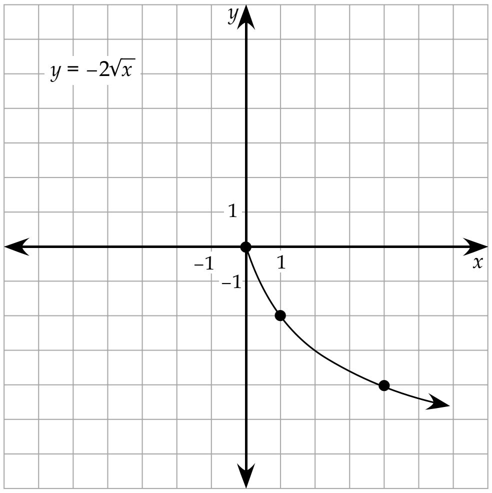


Then, graph the function using Rayne's method.

Vertical stretch:



Then reflection through the x-axis:



As you can see, both Maya and Rayne would have arrived at the same answer. Therefore, in general, it does not matter whether you graph vertical stretches first or vertical reflections first. This is because when you are multiplying by two different values, the order of the multiplication doesn't matter.

Learning Activity 3.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Evaluate: $|\, 4.72 - 6.91 \,|$
2. Solve for x: $(x + 3)^2 = 4$
3. Multiply: $(2x - 1)(y + 6)$
4. Evaluate: $\frac{3}{5} + \frac{17}{30}$
5. Determine the axis of symmetry of the function $f(x) = \frac{1}{3}(x - 2)^2 + 4$.
6. Factor: $2x^2 + 9x + 9$
7. Which is the better deal, a 24 pack of water for $4.99 or a 36 pack of water for $6.49?
8. What is the length of the remaining leg of a right-angled triangle if the hypotenuse measures 25 m and one leg measures 15 m?

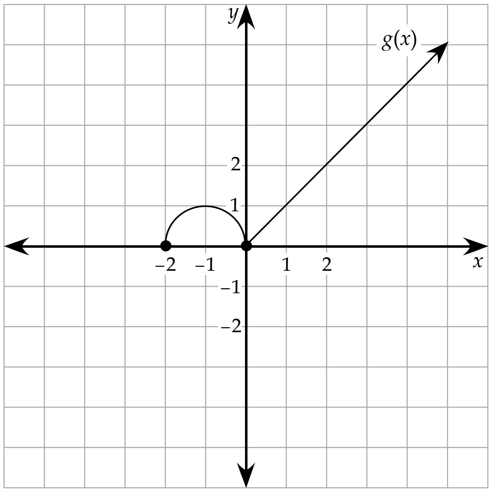
Answers:

1. 2.19 (6.91 − 4.72 is (6 − 4 = 2) and (0.91 − 0.72 = 0.19))
2. $x = -1, -5$ $\left(x + 3 = \pm\sqrt{4} \rightarrow x = 2 - 3 \text{ or } x = -2 - 3\right)$
3. $2xy + 12x - y - 6$
4. $\frac{35}{30} = \frac{7}{6}$ $\left(\frac{3 \times 6}{5 \times 6} + \frac{17}{30}\right)$
5. $x = 2$ (the axis of symmetry goes through the vertex at (2, 4))
6. $(2x + 3)(x + 3)$ (the first coefficients must be 2 and 1, the last coefficient can be 1, 9 or 3, 3)
7. 36 pack of water for $6.49 (half of a 24-pack is about $2.50 for 12; so 24 + 12 = 36 would cost 4.99 + 2.50)
8. 20 m (similar triangle to 3–4–5 is 15–20–25)

Part B: Reflections in the y-axis

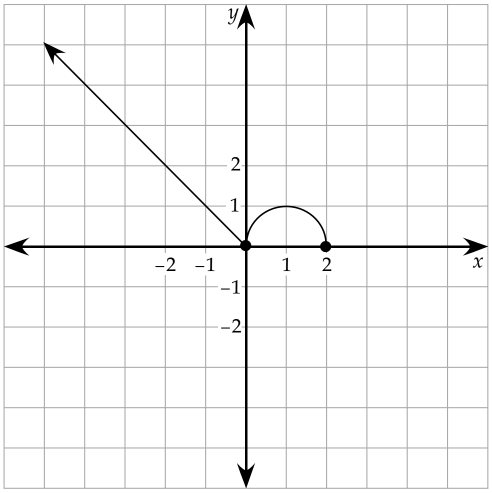
Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Using the sketch of $g(x)$, sketch the following.
   a) $y = g(-x)$
   b) $y = g(-x) - 1$
   c) $y = 2g(-(x + 3))$
   d) $y = -g(-x)$

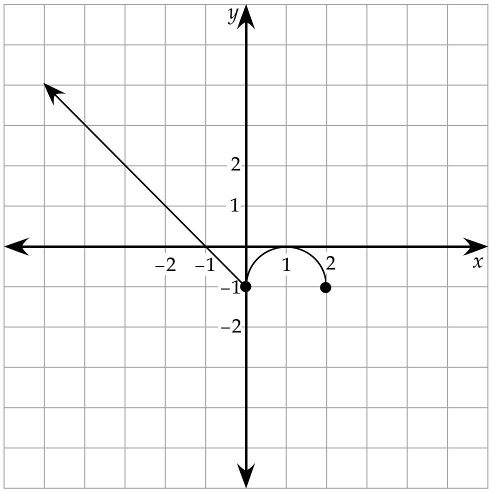


Answers:

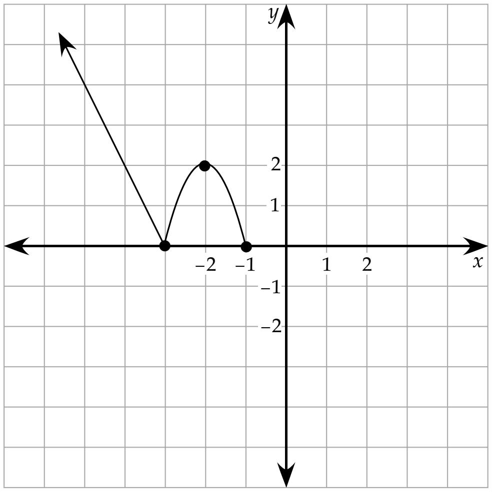
a) $g(-x)$



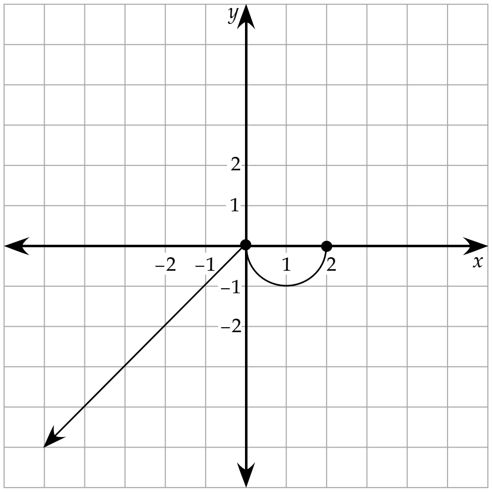
b) $g(-x) - 1$



c) $2g(-(x + 3))$



d) $-g(-x)$



2. If the x-intercepts of a function, $f(x)$, are 3, −5, and 0, find the x-intercepts after the transformation (if possible).

   a) $y = f(-x)$?

      Answer:

      −3, 5, and 0, since a horizontal reflection in the y-axis will reflect any point on the right of the y-axis the same distance from the y-axis but to the left, and vice versa.

   b) $y = f(-x) + 1$?

      Answer:

      Impossible to solve since the translation will shift every point one unit up. The new x-intercepts are the points of the curve that originally intersected the line $y = -1$. You do not have enough information to determine these points.

3. If the y-intercepts of a function, $f(x)$, are 3, −5, and 0, find the y-intercepts after the transformation (if possible).

   a) $y = f(-x)$?

      Answer:

      The y-intercepts are still 3, −5, and 0 since a horizontal reflection in the y-axis does not move any points on the y-axis.

b) $y = f(-(x + 2))$?

Answer:

Impossible to solve since the translation will shift every point two units to the left. You do not have enough information to determine these points.

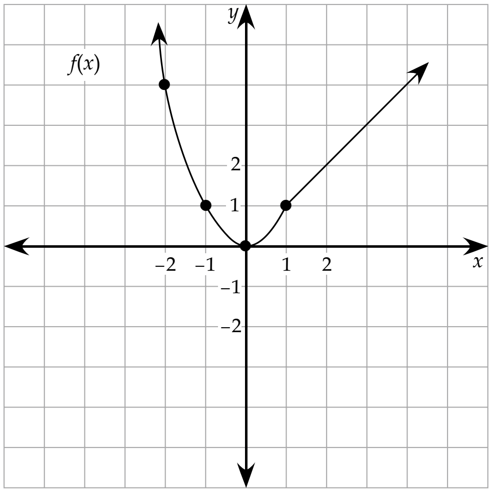
c) $y = f(-x) + 1$?

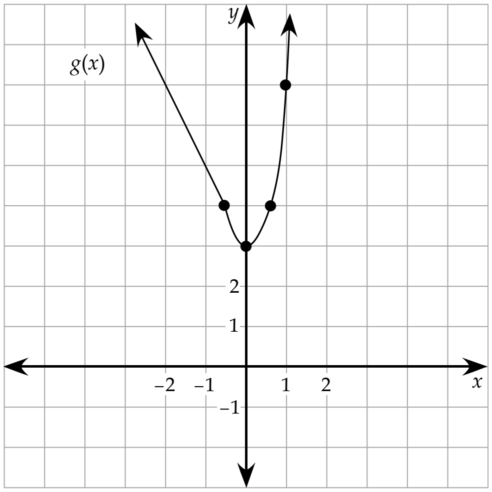
Answer:

The y-intercepts are 4, −4, and 1, since the graph is the same as the graph in part (a), except moved one unit up.

4. For each of the graphs of $f(x)$ below, write the equation of the transformed function, $g(x)$, in terms of $f(x)$.

a)

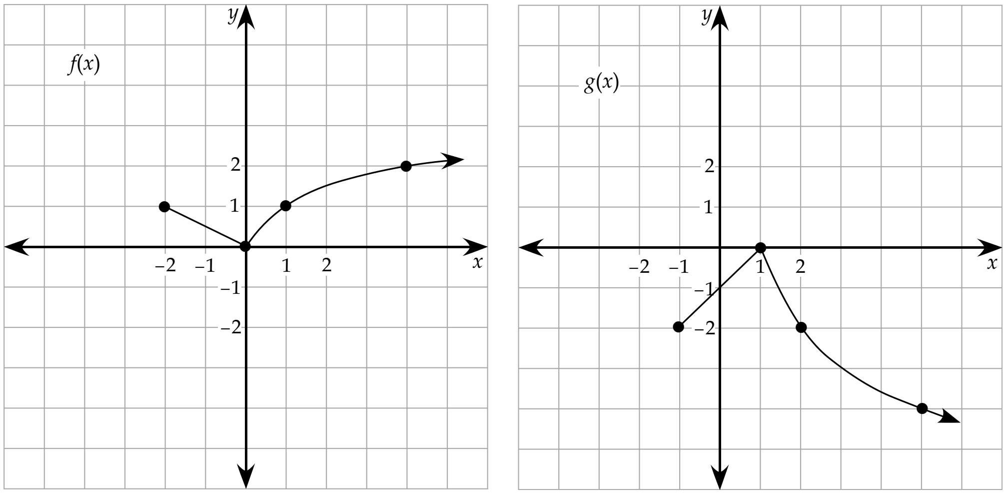




Answer:

This graph has been flipped through the y-axis, compressed horizontally by a factor of 2, and moved 3 units up. The resulting function is $g(x) = f(-2x) + 3$.

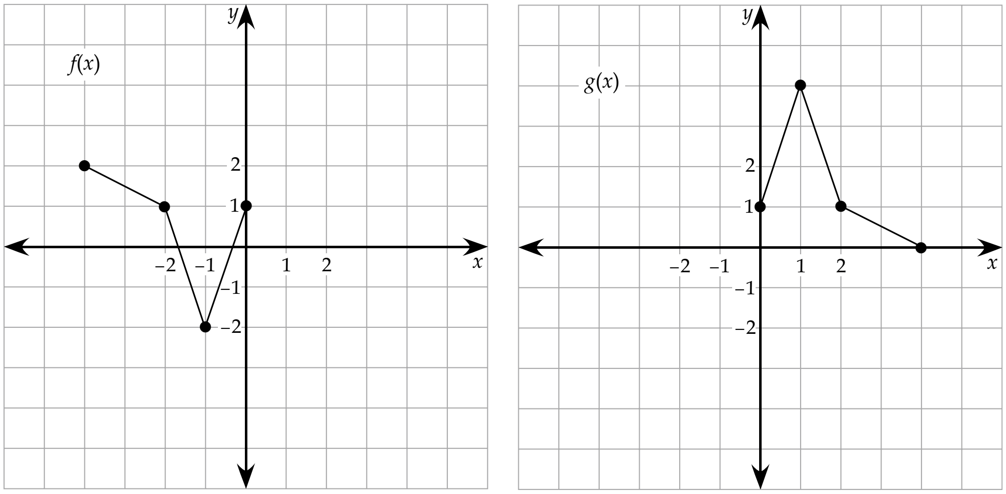
b)



Answer:

This graph has been flipped through the x-axis, stretched vertically by a factor of 2, and moved 1 unit to the right. The resulting function is $g(x) = -2f(x - 1)$.

c)



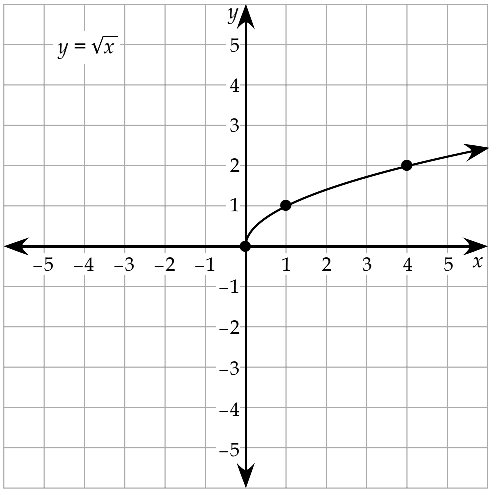
Answer:

This graph has been flipped through the x-axis, flipped through the y-axis, and then shifted vertically 2 units up. The resulting function is $g(x) = -f(-x) + 2$.

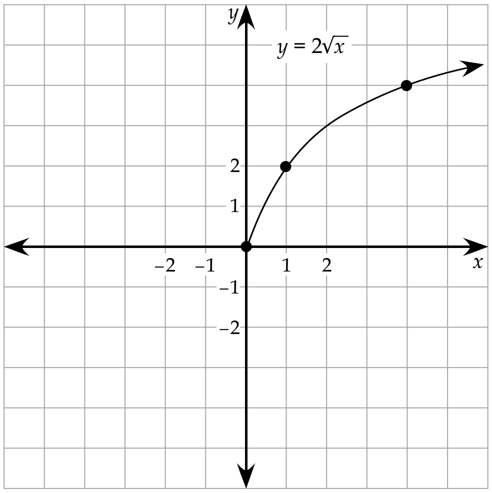
5. Sketch the graph of $y = -2\sqrt{-(x+1)}$. State the domain and range.

Answer:

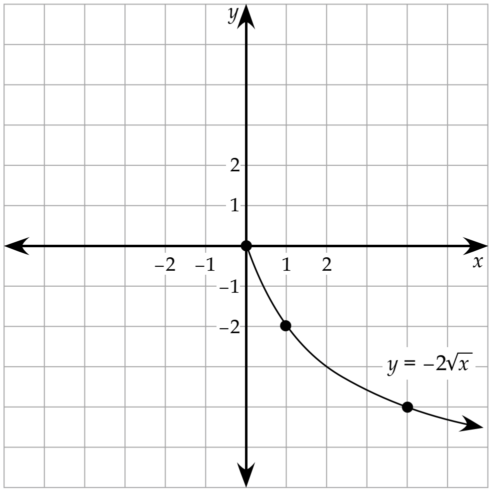
To sketch the graph of $y = -2\sqrt{-(x+1)}$, first sketch the basic graph $y = \sqrt{x}$.



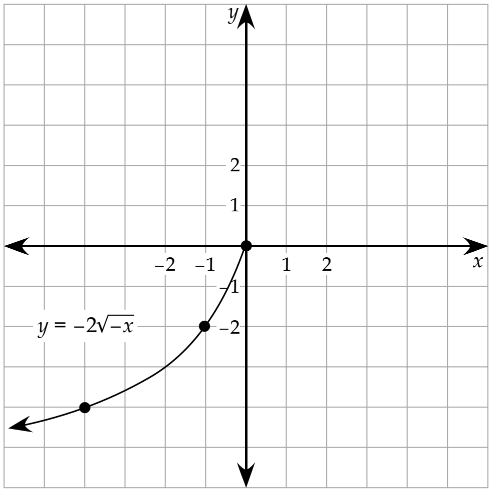
Then, stretch the graph vertically by a factor of 2.



After the stretch, reflect the graph in the x-axis.



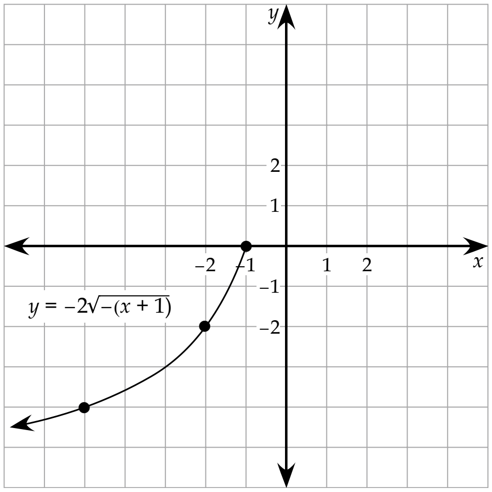
Then, reflect the graph in the y-axis.



Finally, move the graph 1 unit to the left. This is the graph of $y = -2\sqrt{-(x+1)}$.

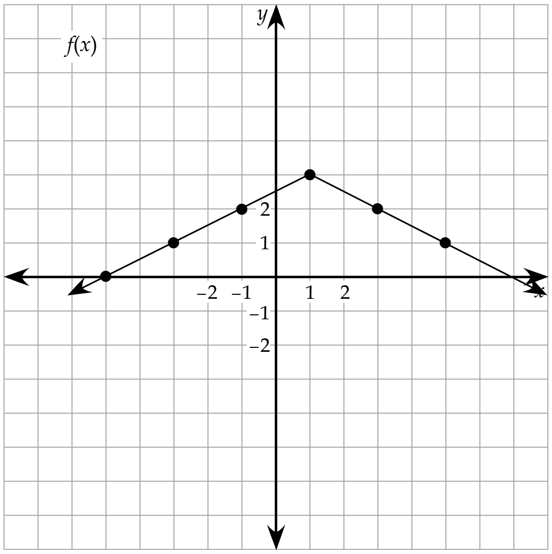
Domain: $\{x \mid x \leq -1\}$

Range: $\{y \mid y \leq 0\}$



6. For each of the graphs below, write the equation of the transformed function in terms of the basic function of the same type.

a)



Answer:

This graph is a transformation of the absolute value function $y = |x|$.

The following transformations have occurred:

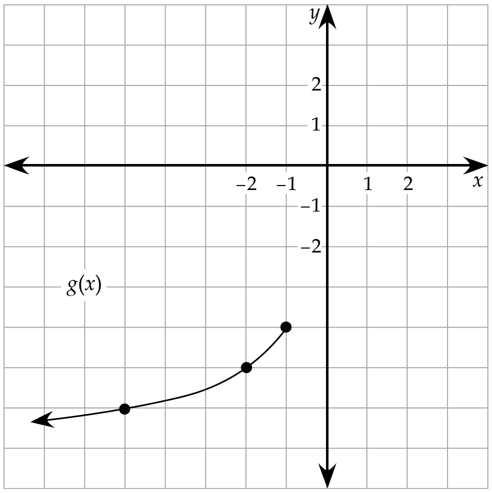
- A reflection in the x-axis
- A vertical compression by a factor of 2
- A vertical shift 3 units up
- A horizontal shift 1 unit to the right

Therefore, the function is $f(x) = -\frac{1}{2}|x - 1| + 3$.

Note: Rather than the vertical compression, the graph can be obtained by a horizontal stretch and then the function could be written as

$f(x) = -\left|\frac{1}{2}(x - 1)\right| + 3$.

b)



Answer:

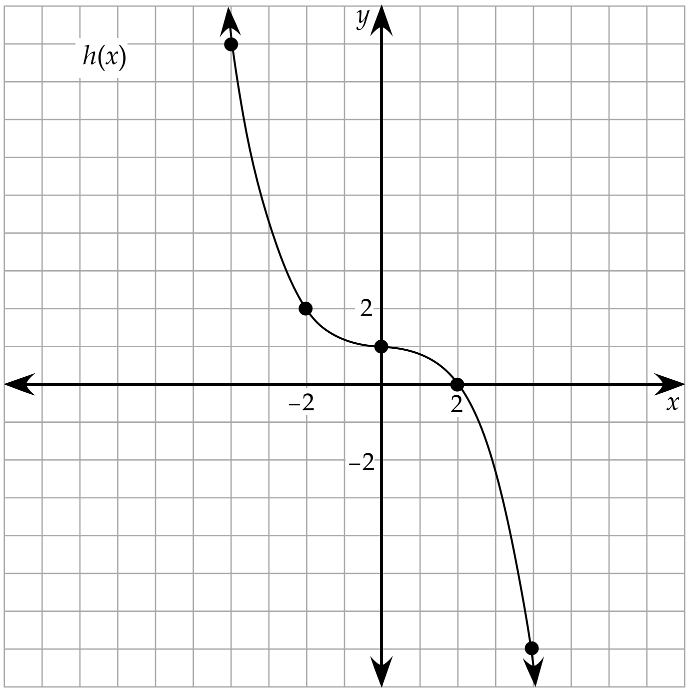
This graph is a transformation of the square root function $y = \sqrt{x}$.

The following transformations have occurred:

- A reflection in the x-axis
- A reflection in the y-axis
- A horizontal shift 1 unit to the left
- A vertical shift 4 units down

Therefore, the function is $g(x) = -\sqrt{-(x+1)} - 4$.

c)



Answer:

This graph is a transformation of the cubic function $y = x^3$.

The following transformations have occurred:

- A reflection in the x-axis
- A horizontal stretch by a factor of 2
- A vertical shift 1 unit up

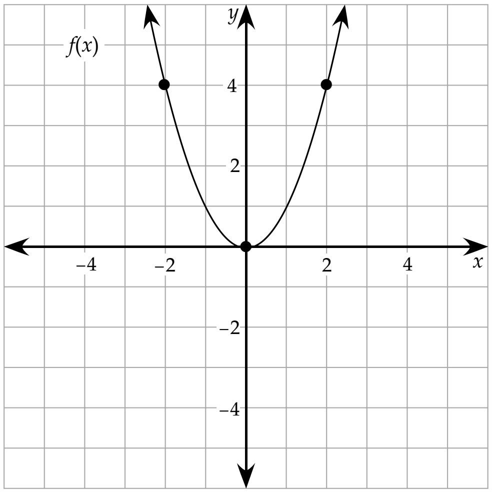
Therefore, the function is $h(x) = -\left(\frac{1}{2}x\right)^3 + 1$.

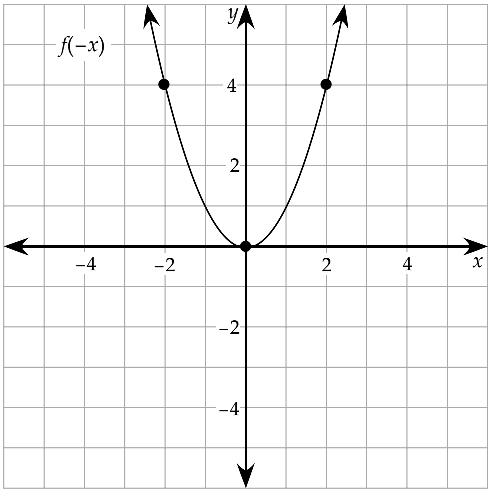
Note: Rather than the vertical compression, the graph can be obtained by a horizontal reflection and then the function could be written as

$f(x) = \left(-\frac{1}{2}x\right)^3 + 1$.

7. Sketch the graph of $f(x) = x^2$ and the graph of $y = f(-x)$. What is unusual about these graphs? Why does this occur?

Answer:

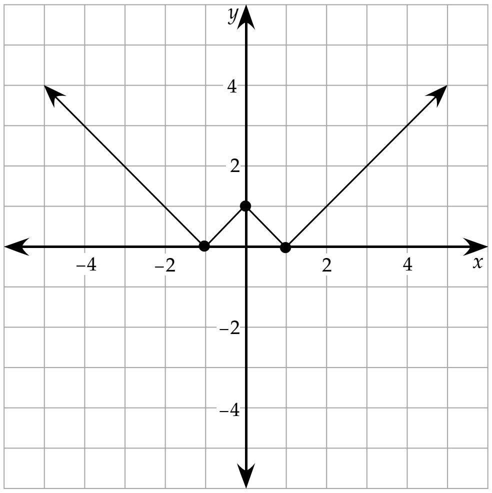




It is unusual that the two functions are exactly the same. This occurs because the graph of $f(x) = x^2$ is symmetrical in the y-axis.

8. If the graph of a function is reflected in the y-axis and the result is the same as the original graph, then the function is referred to as an **even function**. Geometrically, even functions are symmetrical in the y-axis. It can be said that an even function can be flipped in the y-axis and the result is the same graph as the original. Which of the following are graphs of an even function?

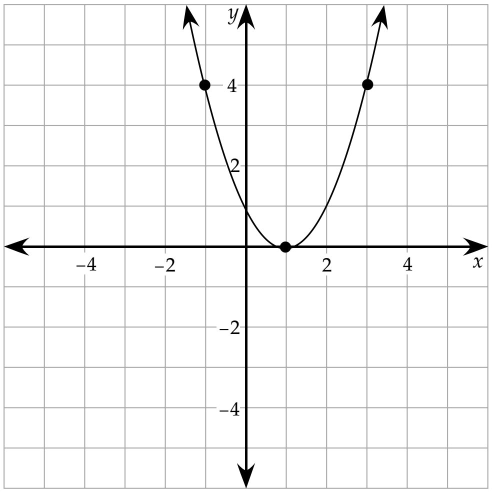
a)



Answer:

This function is an even function because this function is symmetrical in the y-axis.

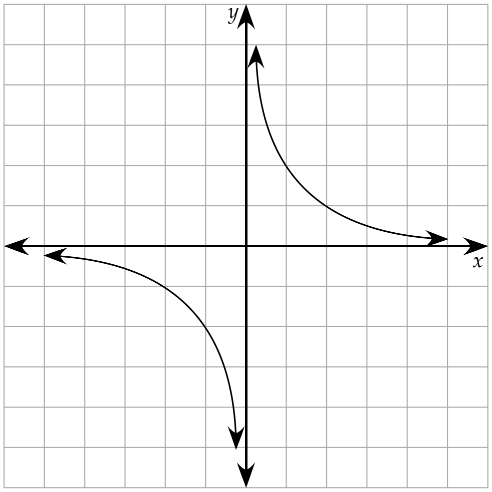
b)



Answer:

This function is not an even function because this function is not symmetrical in the y-axis.

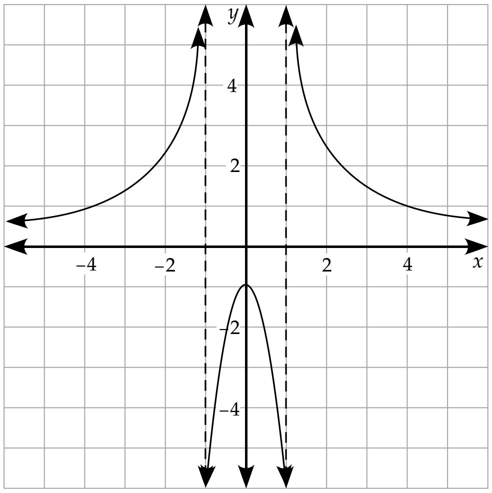
c)



Answer:

This function is not an even function because this function is not symmetrical in the y-axis.

d)

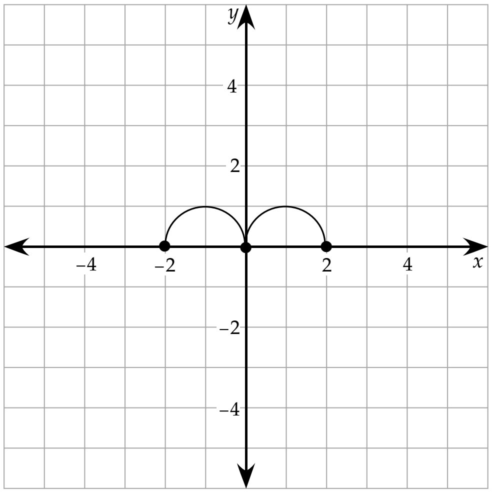


Answer:

This function is an even function because this function is symmetrical in the y-axis.

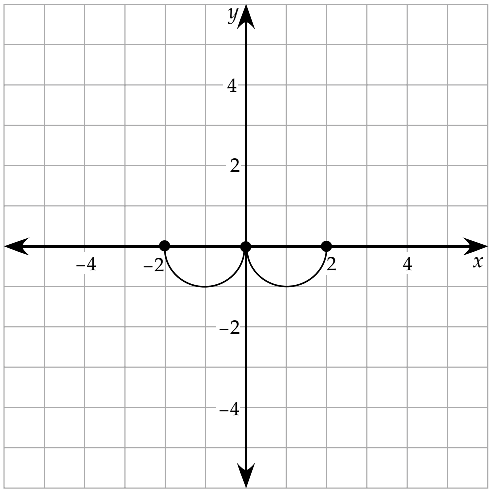
9. Since there are even functions, it seems only natural that there should be **odd functions**. Odd function is the name given to functions that can be flipped in the x-axis and flipped in the y-axis, and the graph will be the same as the original. Determine whether the given function represents an odd function.

a)

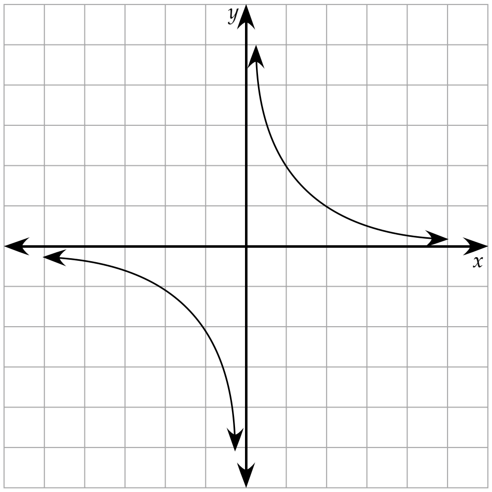


Answer:

When this function is flipped in the x-axis and flipped in the y-axis, the resulting function is shown below.

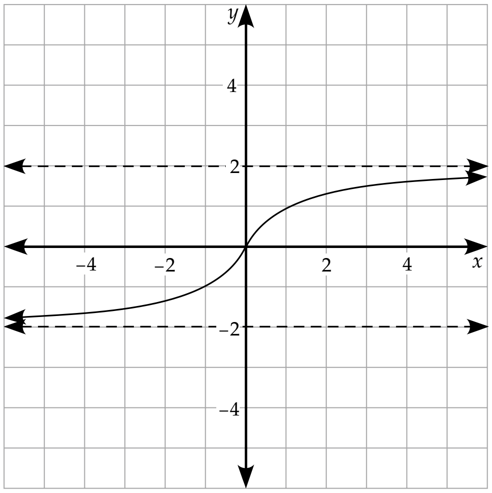


This function is not an odd function. This graph is not the same as the original.

b) 

Answer:

This function is an odd function. When this function is flipped in the x-axis and flipped in the y-axis, the resulting function is the same as the original.

c) 

Answer:

This function is an odd function. When this function is flipped in the x-axis and flipped in the y-axis, the graph is the same as the original.

Learning Activity 3.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. How many different permutations are possible of the word PHONE?
2. Evaluate: ${}_4P_4$
3. Estimate the surface area of a sphere with a radius of 5 m.
4. Is the point (2, −5) a solution to the inequality $y > x^2 + 3x - 1$?
5. Factor: $x^2 + 3x - 10$
6. Simplify: $\sqrt[4]{162y^7z^9}$
7. Simplify: $\sqrt{3}\left(3\sqrt{2} - \sqrt{6}\right)$
8. What is 70% of 260?

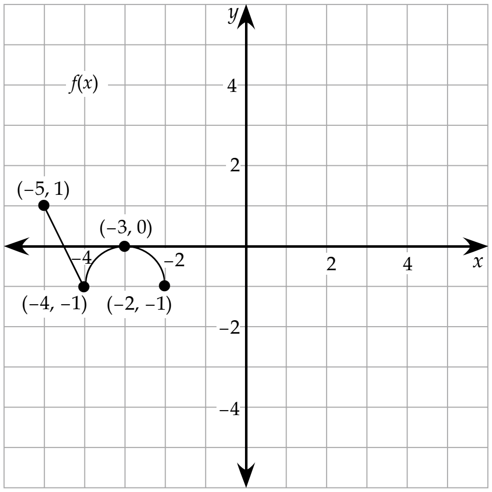
Answers:

1. 120 (5 × 4 × 3 × 2 × 1)
2. 24 (4 × 3 × 2 × 1)
3. 300 m^2 ($\text{SA} = 4\pi r^2$; $\text{SA} = 4 \cdot \pi \cdot 5 \cdot 5 = 100\pi$)
4. No ($-5 > 2^2 + 3(2) - 1$ is false)
5. $(x + 5)(x - 2)$
6. $3yz^2\sqrt[4]{2y^3z}$ $\left(\sqrt[4]{81y^4z^8} \cdot \sqrt[4]{2y^2z}\right)$
7. $3\sqrt{6} - 3\sqrt{2}$ $\left(3\sqrt{6} - \sqrt{18} \rightarrow 3\sqrt{6} - \sqrt{9}\sqrt{2}\right)$
8. 182 (10% of 260 is 26; 7 × 26 is 7 × 20 + 7 × 6)

Part B: Reflections in the line *y* = *x*

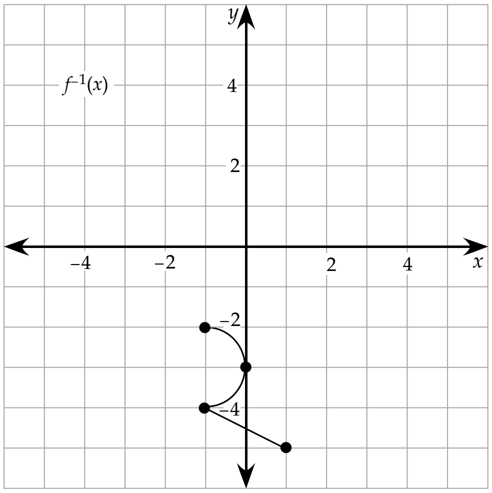
Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Use the graph drawn below of $f(x)$ to sketch the graph of $f^{-1}(x)$.

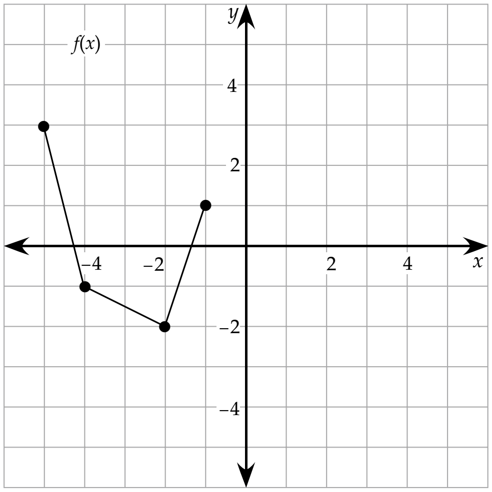


Answer:

$f^{-1}(x)$



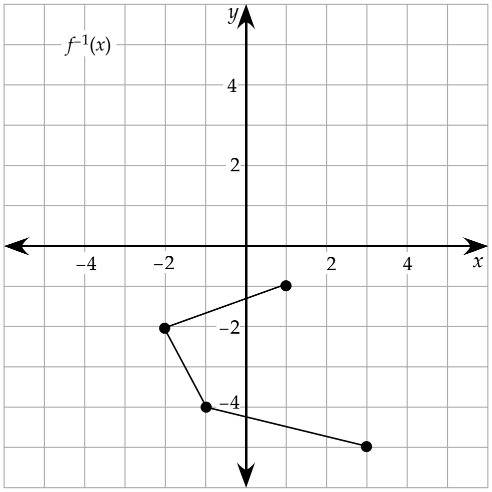
2. Reflect each of the functions below through the line $y = x$.

a) 

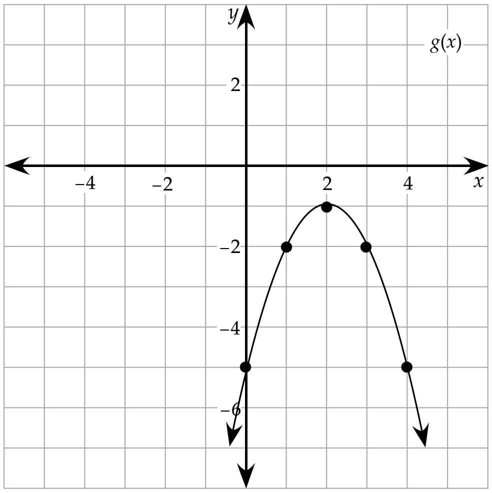
Answer:

First, create a table of values for the key points on $f(x)$. Then, use the transformation $(x, y) \rightarrow (y, x)$ to find the corresponding points on the graph of $f^{-1}(x)$.

$f(x)$	$f^{-1}(x)$
$(-5, 3)$	$(3, -5)$
$(-4, -1)$	$(-1, -4)$
$(-2, -2)$	$(-2, -2)$
$(-1, 1)$	$(1, -1)$

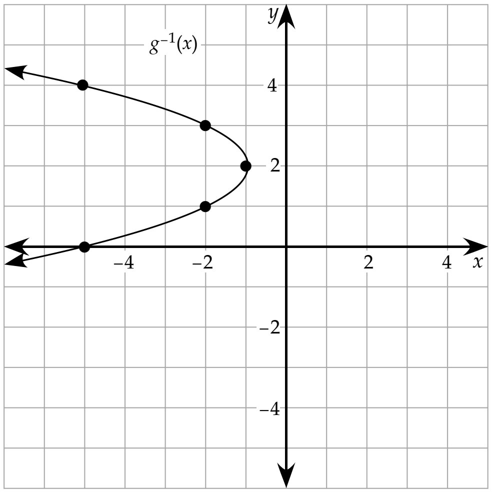


b)



Answer:

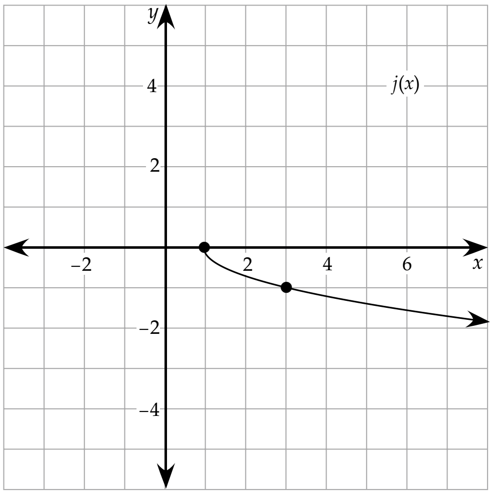
***g*(*x*)**	***g*$^{-1}$(*x*)**
(0, −5)	(−5, 0)
(1, −2)	(−2, 1)
(2, −1)	(−1, 2)
(3, −2)	(−2, 3)
(4, −5)	(−5, 4)



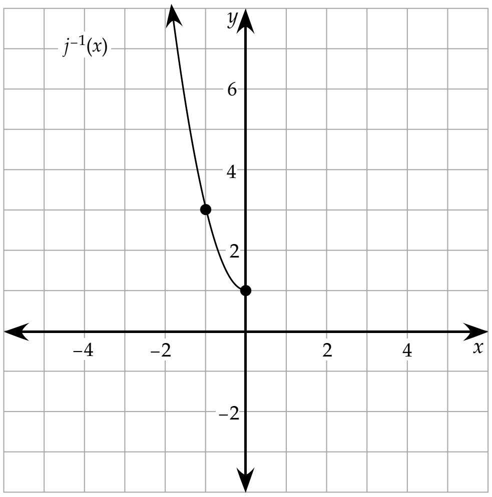
c) $j(x) = -\sqrt{\frac{1}{2}(x-1)}$

Answer:

To graph the reflection of this function through the line $y = x$, first graph $j(x)$ using transformations of the basic function $y = \sqrt{x}$.



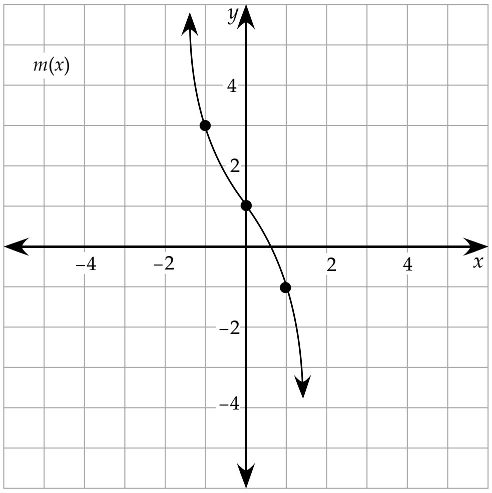
Then, use key points to reflect the function through the line $y = x$.



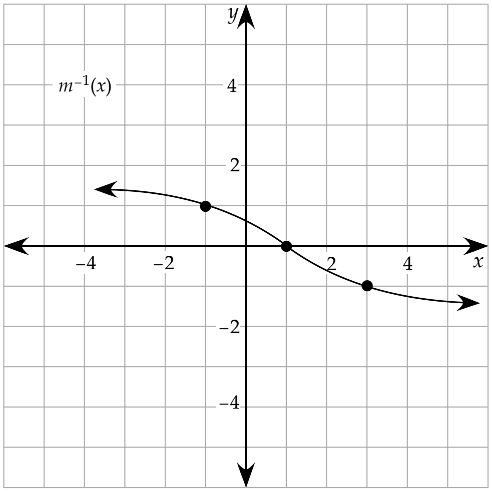
d) $m(x) = -2x^3 + 1$

Answer:

To graph the reflection of this function through the line $y = x$, first graph $m(x)$ using transformations of the basic function $y = x^3$.



Then, use key points to reflect the function through the line $y = x$.



3. The following points are on the graph of $h(x)$:

   (−1, −4), (0, −2), (1, 4), (3, 6), and (4, −3)

   Determine the corresponding points on the graph of $h^{-1}(x)$.

   Answer:

   The corresponding points on the graph of $h^{-1}(x)$ are (−4, −1), (−2, 0), (4, 1), (6, 3), and (−3, 4).

4. Write the coordinates of the inverse of each of the following functions.

   a) $f(x) = \{(3, 7), (2, 3), (4, 4), (-3, 2), (-7, -9)\}$

      Answer:

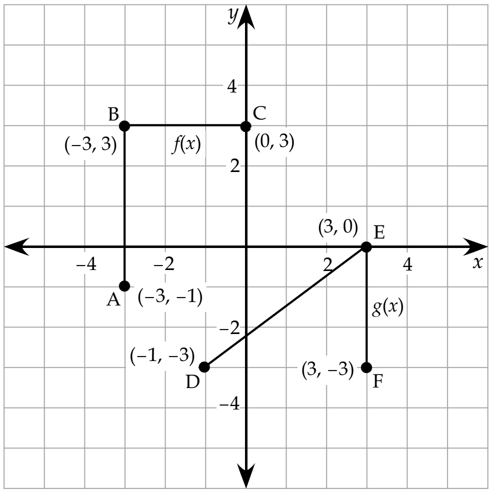
      $f^{-1}(x) = \{(7, 3), (3, 2), (4, 4), (2, -3), (-9, -7)\}$

   b) $g(x) = \{(-1, 2), (6, 2), (-4, -7), (1, 5), (3, 2)\}$

      Answer:

      $g^{-1}(x) = \{(2, -1), (2, 6), (-7, -4), (5, 1), (2, 3)\}$

5. Are the two relations shown below inverses of one another?



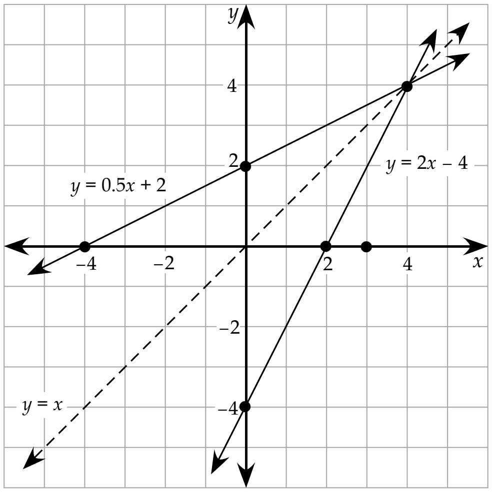
Answer:

No, functions that are inverses will have the same shape since they are reflections over the line y = x. Although the points on these two graphs satisfy the condition $(x, y) \rightarrow (y, x)$, the lines are not drawn connecting the corresponding points. The points A and D, B and F, and C and E are the correct coordinates to be inverses, the points on the line segments between these points do not satisfy the condition and so the lines are not inverses of each other.

6. Is the line $y = 0.5x + 2$ the inverse of the line $y = 2x - 4$? Give a graphical solution.

   Answer:

   Since the graphs of these two lines are symmetric with respect to the line $y = x$, then the two functions are inverses of each other.



Learning Activity 3.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Simplify: $\frac{7!}{9!}$.
2. Rationalize the denominator: $\frac{4}{\sqrt{3}-\sqrt{2}}$
3. Your restaurant bill came to \$124.52. If you wish to leave a 15% tip, estimate how much you should leave.
4. If $f(x) = \frac{x^2}{5} + 4$, evaluate $f(10)$.
5. List all the factors of 48.
6. Simplify: $\sqrt{\frac{27y^3}{3x^2}}$
7. Write as a mixed fraction: $\frac{190}{15}$
8. What is 30% of \$95?

Answers:

1. $\frac{1}{72} \left(\frac{7!}{9 \times 8 \times 7!} = \frac{1}{9 \times 8} \right)$
2. $4\sqrt{3} + 4\sqrt{2} \quad \left(\frac{4}{\sqrt{3}-\sqrt{2}} \cdot \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}} \rightarrow \frac{4\sqrt{3}+4\sqrt{2}}{\sqrt{3^2} + \cancel{\sqrt{6}} - \cancel{\sqrt{6}} - \sqrt{2^2}} = \frac{4\sqrt{3}+4\sqrt{2}}{3-2} \right)$
3. \$18.60 (10% of 124.52 is 12.4; so 5% of 124.52 is 6.2; add 12.4 + 6.2)
4. $f(10) = 24 \left(\frac{100}{5} + 4 \right)$
5. 1, 2, 3, 4, 6, 8, 12, 16, 24, 48
6. $\frac{3y}{x}\sqrt{y} \quad \left(\sqrt{\frac{9y^2}{x^2}} \cdot \sqrt{\frac{y}{1}} \right)$

7. $12\frac{10}{15} = 12\frac{2}{3}\ \left(150 \div 15 + 40 \div 15 \rightarrow 10 + 2\frac{10}{15}\right)$

8. 28.5 (10% of 95 is 9.5; so 30% of 95 is $3 \times 9.5 = 27 + 1.5$)

Part B: Inverse Functions

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Which of the functions listed below have an inverse that is a function? Write the inverse relation in the same format as the original function.

   a) $f(x) = \{(3, 7), (2, 3), (4, 4)\}$

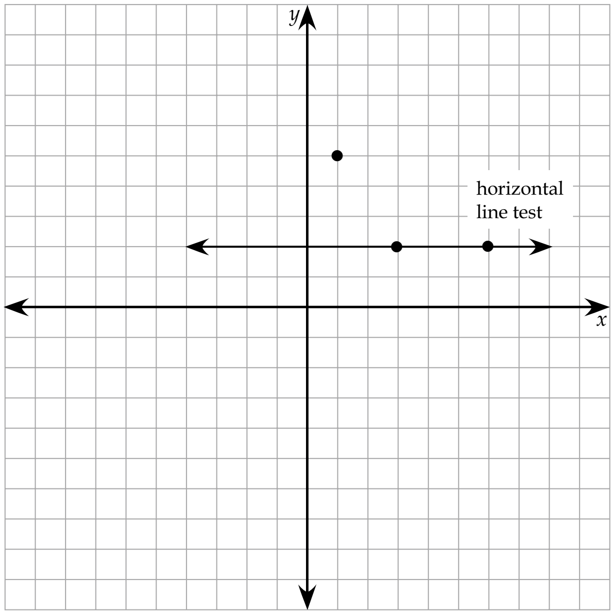
   Answer:

   $f(x)$ has the inverse function $\{(7, 3), (3, 2), (4, 4)\}$.

   b) $f(x) = \{(6, 2), (3, 2), (1, 5)\}$

   Answer:

   $f(x)$ fails the horizontal line test. The inverse is a relation but it is not a function.



   $f(x)$ has the inverse relation $\{(2, 6), (2, 3), (5, 1)\}$.

c) $f(x) = \frac{1}{2}x + 3$

Answer:

$f(x)$ has an inverse function because $f(x)$ is an oblique line; it passes the horizontal line test.

$$y = \frac{1}{2}x + 3$$

Represent the inverse as:

$$x = \frac{1}{2}y + 3$$

$$x - 3 = \frac{1}{2}y$$

$$2x - 6 = y$$

$$f^{-1}(x) = 2x - 6$$

d) $g(x) = x^2 + 1$

Answer:

$g(x)$ is not one-to-one because $g(x)$ is a parabola. It will not pass the horizontal line test. Its inverse will be a relation but not a function.

$$y = x^2 + 1$$

Represent the inverse as:

$$x = y^2 + 1$$

$$x - 1 = y^2$$

$$\pm\sqrt{x - 1} = y$$

$$f^{-1}(x) = \pm\sqrt{x - 1}$$

Note: The "±" indicates there are two inverse values for each x, which confirms the inverse is not a function.

e) $f(x) = \sqrt{x-3}$

Answer:

$f(x)$ is one-to-one since it is a radical function. Thus, the inverse of $f(x)$ is a function. We can state the restricted domain and range for $f(x)$ and find $f^{-1}(x)$.

$$y = \sqrt{x-3},\ x \geq 3,\ y \geq 0$$

Represent the inverse as:

$$x = \sqrt{y-3}$$

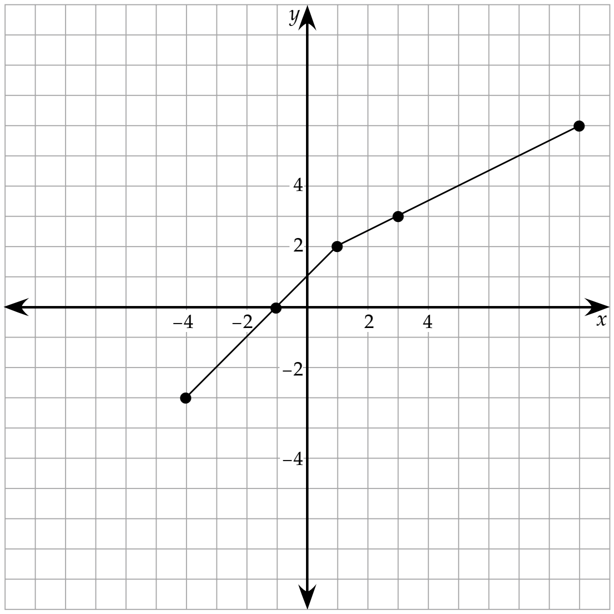
$$x^2 = y - 3$$

$$y = x^2 + 3$$

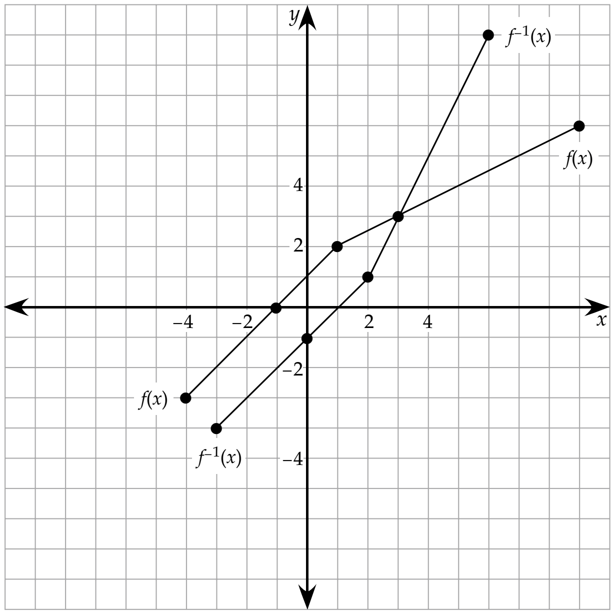
$$f^{-1}(x) = x^2 + 3,\ x \geq 0,\ y \geq 3$$

The inverse is a quadratic function with a restricted domain corresponding to the restricted range of $f(x)$.

f)



Answer:



The inverse, $f^{-1}(x)$, is shown and it is a function because it passes the vertical line test or because $f(x)$ passes the horizontal line test.

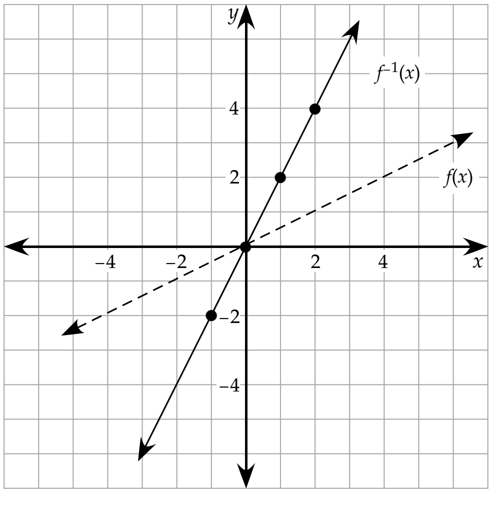
2. For each of the following functions $f(x)$, define and graph its inverse $f^{-1}(x)$.

   a) $f(x) = \dfrac{x}{2}$

   Answer:

   $f(x)$ is an oblique line. It is one-to-one. The inverse of dividing x by 2 is multiplying x by 2.

   $f(x) = \dfrac{x}{2} \qquad \therefore f^{-1}(x) = 2x$

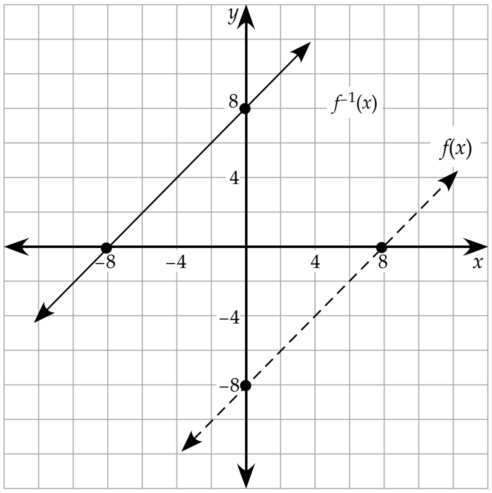


b) $f(x) = x - 8$

Answer:

$f(x)$ is an oblique line. It is one-to-one. The inverse of subtracting 8 from x is adding 8 to x.

$f(x) = x - 8 \quad \therefore f^{-1}(x) = x + 8$



c) $f(x) = \dfrac{x+2}{3}$

Answer:

$f(x)$ is an oblique line. It is one-to-one.

$$f(x) = \frac{x+2}{3}$$

$$y = \frac{x+2}{3}$$ Write y instead of $f(x)$.

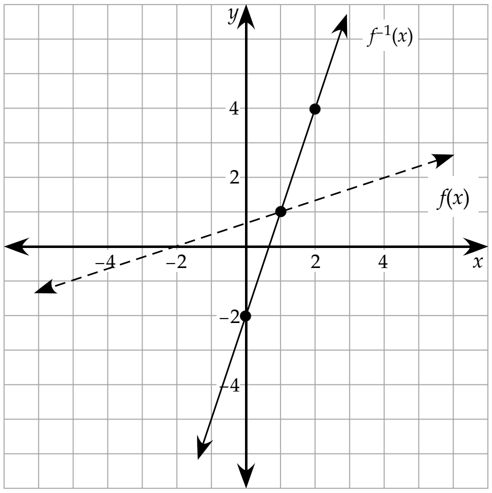
Represent the inverse as:

$$x = \frac{y+2}{3}$$ Interchange x and y.

$$3x = y + 2$$ Simplify.

$$3x - 2 = y$$

$$f^{-1}(x) = 3x - 2$$ Replace y by $f^{-1}(x)$.



d) $f(x) = x^2 + 1$, where $x \geq 0$

Answer:

$f(x)$ is a parabola with its domain restricted, so the inverse will also be a function since $f(x)$ will pass the horizontal line test. It becomes a half-parabola. Notice the domain of f becomes the range of f^{-1} and vice versa.

$f(x) = x^2 + 1,\ x \geq 0,\ y \geq 1$

$y = x^2 + 1$

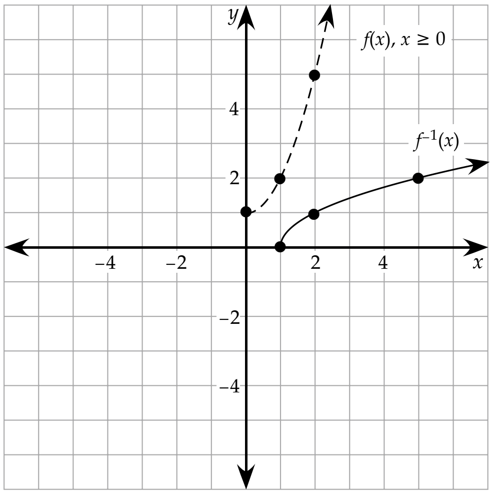
Represent the inverse as:

$x = y^2 + 1$

$x - 1 = y^2$

$\pm\sqrt{x-1} = y$

$f^{-1}(x) = \sqrt{x-1},\ x \geq 1,\ y \geq 0$



Note: When taking the square root, it can be + or −. However, the restriction indicates $y \geq 0$ so only the + is used.

e) $f(x) = 2x^2 - 4$, where $x \geq 0$

Answer:

$f(x)$ is a parabola with its domain restricted, so the inverse will also be a function since $f(x)$ will pass the horizontal line test. It becomes a half-parabola. Notice the domain of f becomes the range of f^{-1} and vice versa.

$$f(x) = 2x^2 - 4,\ x \geq 0,\ y \geq -4$$

$$y = 2x^2 - 4$$

Represent the inverse as:

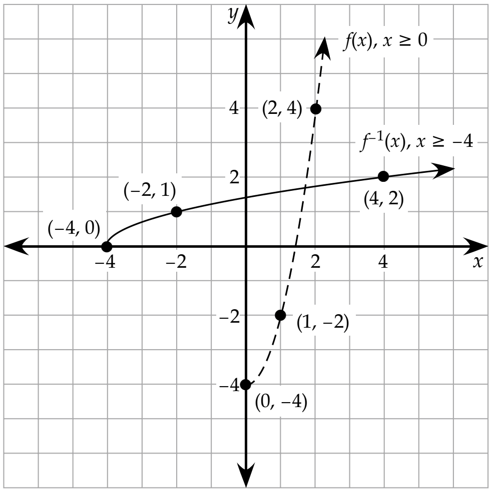
$$x = 2y^2 - 4$$

$$x + 4 = 2y^2$$

$$\frac{x+4}{2} = y^2$$

$$\pm\sqrt{\frac{x+4}{2}} = y$$

$$f^{-1}(x) = \sqrt{\frac{x+4}{2}},\ x \geq -4,\ y \geq 0$$



Note: When taking the square root, include ±. Then use the restriction to determine that $y \geq 0$, so only the + is used.

f) $f(x) = x^2 - 6x + 9$, where $x \geq 3$

Answer:

$f(x)$ is a parabola with its domain restricted to ensure its inverse will also be a function.

$$f(x) = x^2 - 6x + 9,\ x \geq 3,\ y \geq 0$$

$$y = x^2 - 6x + 9$$

Represent the inverse as:

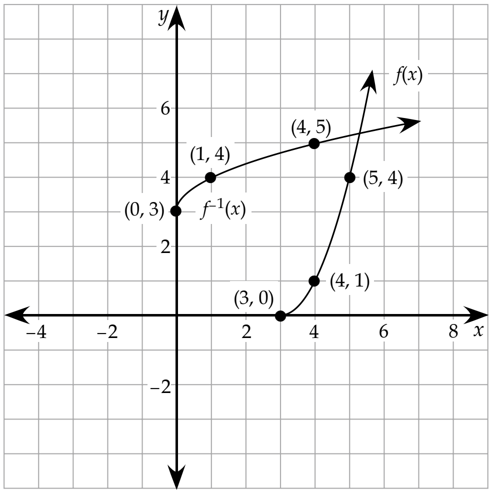
$$y = (x-3)^2$$

$$x = (y-3)^2$$

$$\pm\sqrt{x} = y - 3$$

$$\pm\sqrt{x} + 3 = y$$

$$f^{-1}(x) = \sqrt{x} + 3,\ x \geq 0,\ y \geq 3$$



Note: When the radical symbol is added, you need to consider ±. Since the restriction is $y \geq 3$, then only the + is used.

3. Show algebraically that the functions f and g are inverses of each other.

a) $f(x) = 2x - 3$ $\qquad g(x) = \dfrac{x+3}{2}$

Answer:

$$f(x) = 2x - 3$$

$$\begin{aligned} f(g(x)) &= f\left(\frac{x+3}{2}\right) \\ &= 2\left(\frac{x+3}{2}\right) - 3 \\ &= x + 3 - 3 \\ &= x \end{aligned}$$

$$g(x) = \frac{x+3}{2}$$

$$\begin{aligned} g(f(x)) &= g(2x-3) \\ &= \frac{(2x-3)+3}{2} \\ &= \frac{2x}{2} \\ &= x \end{aligned}$$

∴ Because $f(g(x)) = g(f(x)) = x$, then $f(x)$ and $g(x)$ are inverses.

b) $f(x) = 4 - x$ $\qquad$ $g(x) = 4 - x$

Answer:

$$\begin{aligned} f(x) &= 4 - x \\ f(g(x)) &= f(4 - x) \\ &= 4 - (4 - x) \\ &= x \end{aligned} \qquad \begin{aligned} g(x) &= 4 - x \\ g(f(x)) &= g(4 - x) \\ &= 4 - (4 - x) \\ &= x \end{aligned}$$

∴ Because $f(g(x)) = g(f(x)) = x$, $f(x)$ and $g(x)$ are inverses. Note that the function $y = 4 - x$ is its own inverse. Try graphing it and see if you can determine why that makes sense.

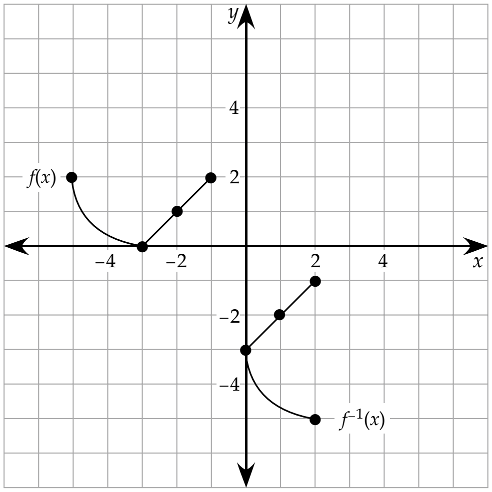
c) $f(x) = \sqrt[3]{x + 8}$ $\qquad$ $g(x) = x^3 - 8$

Answer:

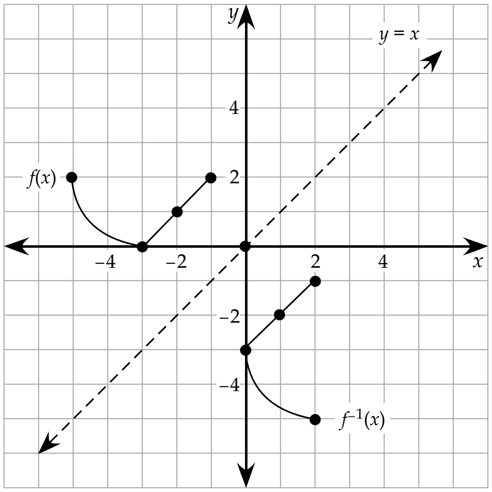
$$\begin{aligned} f(x) &= \sqrt[3]{x + 8} \\ f(g(x)) &= f\left(x^3 - 8\right) \\ &= \sqrt[3]{\left(x^3 - 8\right) + 8} \\ &= \sqrt[3]{x^3} \\ &= x \end{aligned} \qquad \begin{aligned} g(x) &= x^3 - 8 \\ g(f(x)) &= g\left(\sqrt[3]{x + 8}\right) \\ &= \left(\sqrt[3]{x + 8}\right)^3 - 8 \\ &= x + 8 - 8 \\ &= x \end{aligned}$$

∴ Because $f(g(x)) = g(f(x)) = x$, $f(x)$ and $g(x)$ are inverses.

4. Determine graphically if the following sets of relations are inverses of each other.

a) 

Answer:

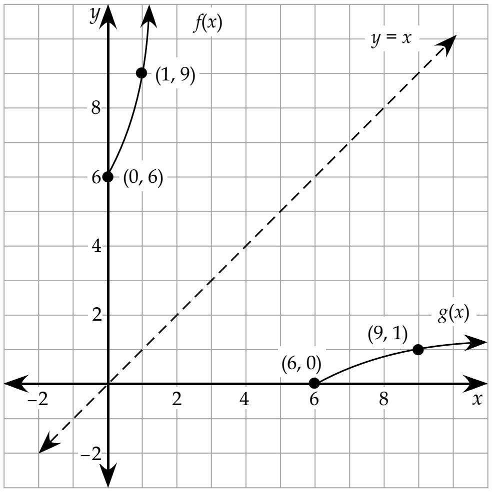


If you look on either side of the line $y = x$, these relations are symmetrical. Therefore, these relations are inverses of each other.

b) $f(x) = \sqrt{\frac{1}{3}(x-6)}$ $\qquad g(x) = 3x^2 + 6,\ x \geq 0$

Answer:

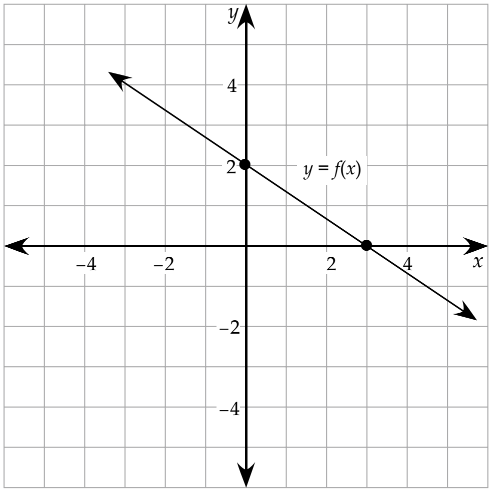
Sketch by transformations of the basic function. The radical graph is stretched horizontally by a factor of 3 and shifted right 6. The quadratic graph is stretched vertically by a factor of 3 and shifted up 6. The parabola is restricted to $x \geq 0$.



If you look on either side of the line $y = x$, these relations are symmetrical. Therefore, these relations are inverses of each other.

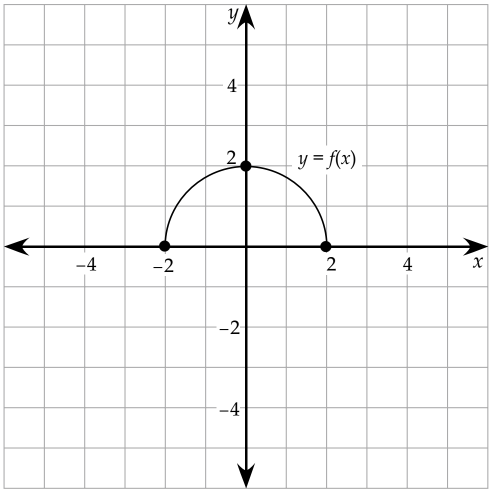
5. Determine whether each function is one-to-one.

a)



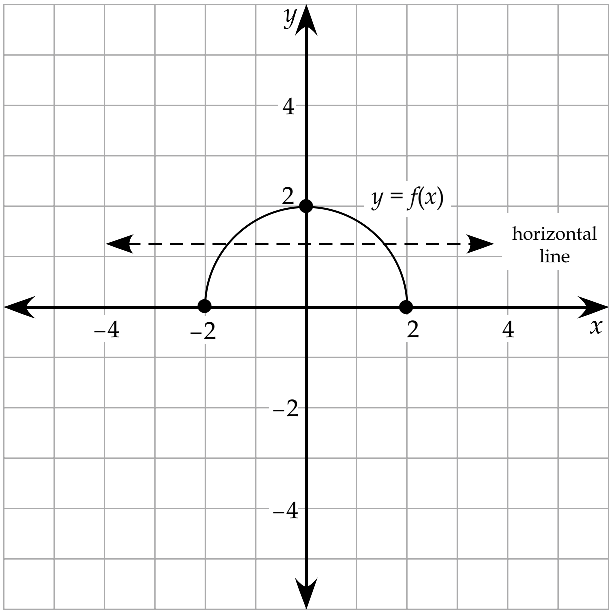
Answer:

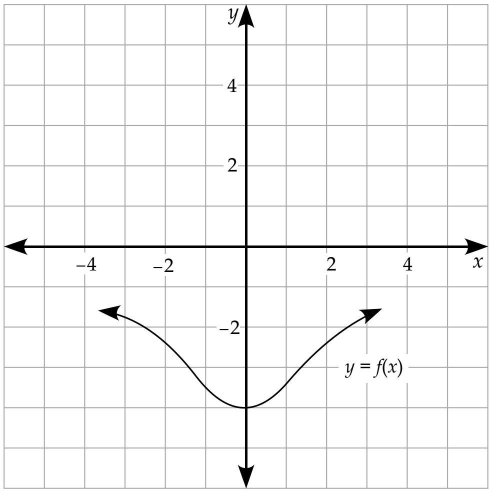
Yes, it is one-to-one as each x-value corresponds to exactly one y-value and each y-value corresponds to exactly one x-value. Furthermore, the graph passes the horizontal line test and the vertical line test.

b) 

Answer:

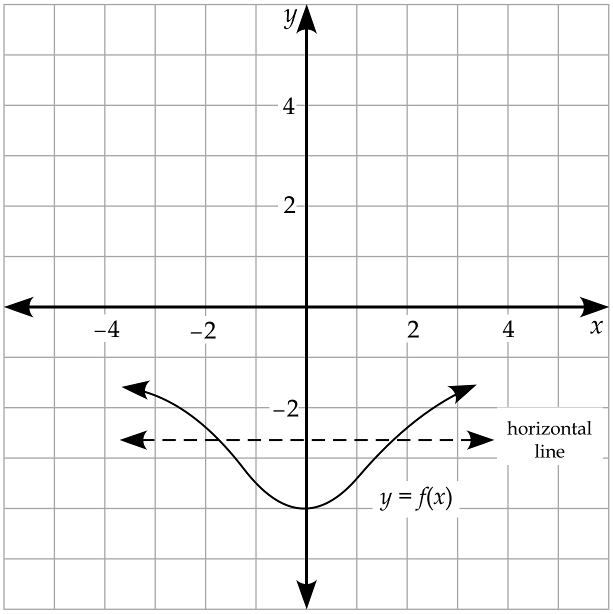
No, it is not one-to-one as it fails the horizontal line test, which indicates that at least two different x-values have the same y-value in their ordered pairs.

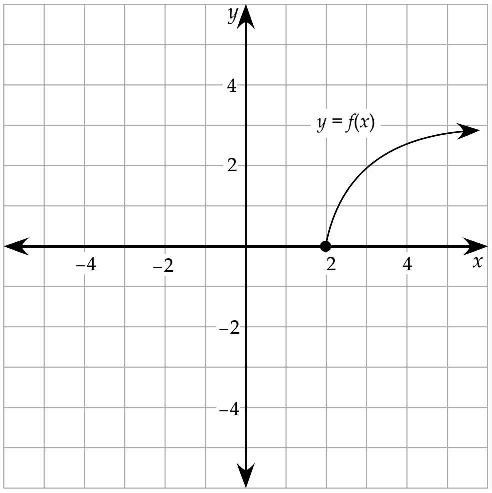


c) 

Answer:

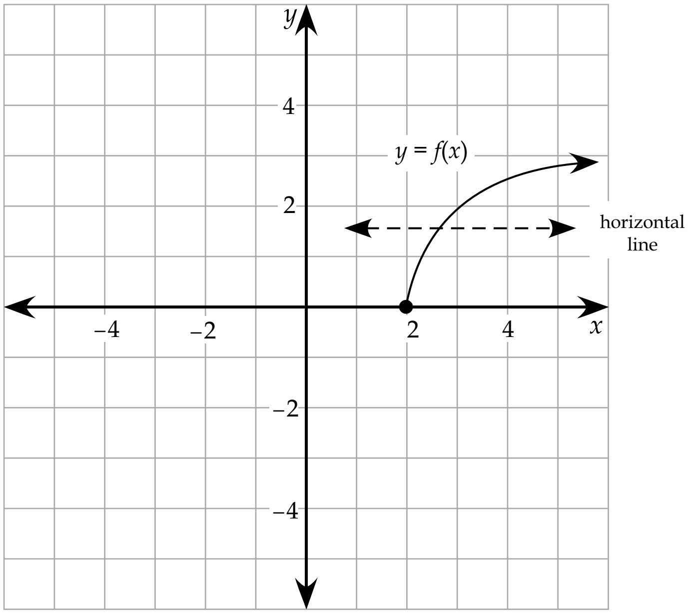
No, for the same reason as given in (b).



d) 

Answer:

Yes, it is one-to-one for the same reason as given in (a).



6. Find $f^{-1}(x)$ algebraically.

a) $f(x) = \dfrac{x}{x+2}$

Answer:

$$f(x) = \frac{x}{x+2}$$

Note: $x \neq -2$ and $f(x)$ will never be 1 for any value of x.

$$y = \frac{x}{x+2}$$

Let $y = f(x)$.

Represent the inverse as:

$$x = \frac{y}{y+2}$$

Interchange x- and y-values for $y = f^{-1}(x)$.

$$x(y+2) = y$$

Simplify and solve for y.

$$xy + 2x = y$$

$$xy - y = -2x$$

Isolate y-terms on one side.

$$y(x-1) = -2x$$

Common factor (to isolate y)

$$y = \frac{-2x}{x-1}$$

$$f^{-1}(x) = \frac{-2x}{x-1}$$

Replace y by $f^{-1}(x)$.
Note: $x \neq 1$ and $f^{-1}(x)$ will never be equal to -2 for any value of x.

b) $f(x) = \sqrt{x} + 2$

Answer:

$$f(x) = \sqrt{x} + 2,\ x \geq 0,\ y \geq 2$$

Write the restrictions.

$$y = \sqrt{x} + 2$$

Let $y = f(x)$.

Represent the inverse as:

$$x = \sqrt{y} + 2$$

Interchange x- and y-values for $y = f^{-1}(x)$.

$$x - 2 = \sqrt{y}$$

Simplify and solve for y.

$$(x-2)^2 = y$$

$$f^{-1}(x) = (x-2)^2,\ x \geq 2,\ y \geq 0$$

Replace y by $f^{-1}(x)$ and write the restriction on the domain.

c) $f(x) = \dfrac{x}{x-3}$

Answer:

$f(x) = \dfrac{x}{x-3}$ **Note:** $x \neq 3$ and $y \neq 1$ for any value of x.

$y = \dfrac{x}{x-3}$ Let $y = f(x)$.

Represent the inverse as:

$x = \dfrac{y}{y-3}$ Interchange x- and y-values for $y = f^{-1}(x)$.

$x(y-3) = y$ Simplify.

$xy - 3x = y$

$xy - y = 3x$ Isolate y-terms on one side.

$y(x-1) = 3x$ Common factor (to isolate y).

$y = \dfrac{3x}{x-1}$

$f^{-1}(x) = \dfrac{3x}{x-1}$ Replace y by $f^{-1}(x)$.
Note: $x \neq 1$ and $y \neq 3$ for any value of x.

d) $f(x) = \dfrac{x-3}{x}$

Answer:

$f(x) = \dfrac{x-3}{x}$ **Note:** $x \neq 0$ and $y \neq 1$ for any value of x.

$y = \dfrac{x-3}{x}$ Let $y = f(x)$.

Represent the inverse as:

$x = \dfrac{y-3}{y}$ Interchange x- and y-values for $y = f^{-1}(x)$.

$xy = y - 3$ Simplify.

$xy - y = -3$ Isolate y-terms on one side.

$y(x-1) = -3$ Common factor.

$y = \dfrac{-3}{x-1}$

$f^{-1}(x) = \dfrac{-3}{x-1}$ Replace y by $f^{-1}(x)$.
Note: $x \neq 1$ and $y \neq 0$ for any value of x.

7. Find the inverse function equations of the following functions and sketch their graphs. State the domain and range of each function and its inverse.

a) $f(x) = 2\sqrt{\frac{1}{2}x} - 3$

Answer:

This function is one-to-one because all transformations of the square root function are one-to-one.

$f(x) = 2\sqrt{\frac{1}{2}x} - 3,\ x \geq 0,\ y \geq -3$ Identify domain and range.

$$y = 2\sqrt{\frac{1}{2}x} - 3$$

Represent the inverse as:

$$x = 2\sqrt{\frac{1}{2}x} - 3$$

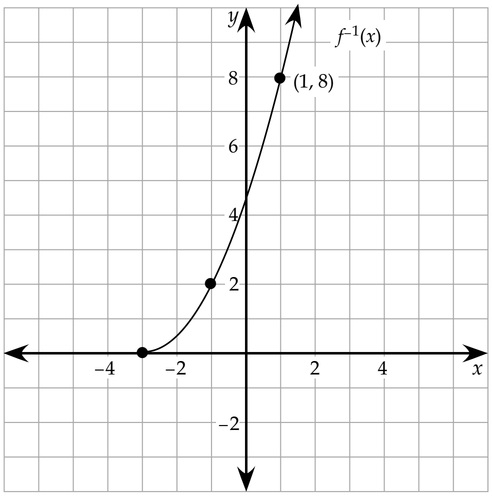
$$x + 3 = 2\sqrt{\frac{1}{2}x}$$

$$\frac{x+3}{2} = \sqrt{\frac{1}{2}y}$$

$$\left(\frac{x+3}{2}\right)^2 = \frac{1}{2}y$$

$$2\left(\frac{x+3}{2}\right)^2 = y$$

$$f^{-1}(x) = 2\left(\frac{x+3}{2}\right)^2,\ x \geq -3,\ y \geq 0$$



Domain: $f(x)$: $[0, \infty)$
Range: $f(x)$: $[-3, \infty)$

Domain: $f^{-1}(x)$: $[-3, \infty)$
Range: $f^{-1}(x)$: $[0, \infty)$

b) $g(x) = \frac{1}{2}(x-3)^3 + 1$

Answer

This function is one-to-one because all transformations of the cubic function are one-to-one.

$$g(x) = \frac{1}{2}(x-3)^3 + 1,\ x \in \Re,\ y \in \Re$$

$$y = \frac{1}{2}(x-3)^3 + 1$$

Represent the inverse as:

$$x = \frac{1}{2}(y-3)^3 + 1$$

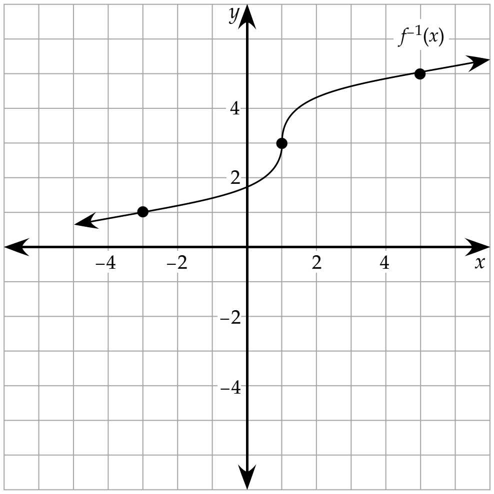
$$x - 1 = \frac{1}{2}(y-3)^3$$

$$2(x-1) = (y-3)^3$$

$$\sqrt[3]{2(x-1)} = y - 3$$

$$\sqrt[3]{2(x-1)} + 3 = y$$

$$g^{-1}(x) = \sqrt[3]{2(x-1)} + 3,\ x \in \Re,\ y \in \Re$$



Domain: $f(x)$: $(-\infty, \infty)$ Domain: $f^{-1}(x)$: $(-\infty, \infty)$

Range: $f(x)$: $(-\infty, \infty)$ Range: $f^{-1}(x)$: $(-\infty, \infty)$

8. If $f(x) = 2x - 3$, find $f^{-1}(x)$ and $\frac{1}{f(x)}$, and show that they are not equal.

Answer:

$f(x) = 2x - 3$

The inverse of $f(x) = f^{-1}(x) = \frac{x+3}{2}$, whereas the reciprocal of

$f(x) = \frac{1}{f(x)} = \frac{1}{2x-3}$.

Thus, $f^{-1}(x) \neq \frac{1}{f(x)}$.

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Module 4
Polynomials

MODULE 4:
POLYNOMIALS

Introduction

Functions are a fundamental topic of mathematics. They describe many relationships in mathematics and science. In this module, you will learn all about polynomial functions up to a degree of five. The concepts of factoring that you have studied in previous modules and in previous pre-calculus courses will be extended to include the remainder theorem and factor theorem. This gives the basis for the study of polynomials, their zeros, and their graphs.

Solving algebraic equations by determining the roots of a polynomial is a very old problem in mathematics. However, these equations used to be written out in words. When the current variable notation was introduced many years ago, it made the solution of the roots of polynomial equations much easier. With the onset of computer technology, the solutions are easier still. When you learn how to find the zeros of a polynomial in this module, you will have learned how to solve one of the problems that stumped ancient mathematicians for years!

Assignments in Module 4

When you have completed the assignments for Module 4, submit your completed assignments to the Distance Learning Unit either by mail or electronically through the learning management system (LMS). The staff will forward your work to your tutor/marker.

Lesson	Assignment Number	Assignment Title
2	Assignment 4.1	Polynomial Functions
4	Assignment 4.2	Factoring and Graphing Polynomials

Resource Sheet

When you write your midterm examination, you are encouraged to take a Midterm Examination Resource Sheet with you into the examination. This sheet will be one letter-sized page, 8½ ″ by 11″, with both sides in your handwriting or typewritten. You will submit it with your examination, but you do not receive any marks for it.

Many students have found that preparing a resource sheet is an excellent way to review. It provides you with a summary of the important facts of each module. You should complete a resource sheet for each module to help with your studying and reviewing. Lesson summaries and module summaries are included for you to use as a guide.

You may use the list of instructions provided below to help you with preparing your resource sheet for the material in Module 4. On this sheet, you should record math terms and definitions, formulas, sample questions, or a list of places where you often make mistakes. You should also identify special areas that require extra attention or review by writing the page numbers.

After you have completed each module's resource sheet, you may summarize the sheets from Modules 1 to 4 to prepare your Midterm Examination Resource Sheet. The final examination for this course is based on Modules 1 to 4.

Resource Sheet for Module 4

As you go through the lessons of this module, you may want to consider the following suggestions regarding the creation of a resource sheet.

1. List all the important math terms, and define them if necessary.
2. List all the formulas and perhaps a sample problem that shows how the formula is used.
3. If necessary, write the solutions to some problems, showing in detail how you did the calculations.
4. Copy any questions that represent the key points of the lesson, and perhaps include the solutions as well.
5. Identify the problems you found most difficult, and copy the page numbers onto the resource sheet so that you can review them before writing the examination. You may also copy the problems and the solutions onto your resource sheet, and later write them onto your Midterm Examination Resource Sheet.
6. Write any comments, ideas, shortcuts, or other reminders that may be helpful during an examination.

Writing Your Midterm Examination

You will write the midterm examination when you have completed Module 4 of this course. The midterm examination is based on Modules 1 to 4, and is worth 20 percent of your final mark in the course. To do well on the midterm examination, you should review all the work you complete in Modules 1 to 4, including all the learning activities and assignments. You will write the midterm examination under supervision.

Notes

LESSON 1: POLYNOMIAL FUNCTIONS

Lesson Focus

In this lesson, you will

- ☐ learn how to identify a polynomial function
- ☐ learn about the features of cubic, quartic, and quintic polynomial functions

Lesson Introduction

Thus far, you have learned about three types of polynomial functions, including constant, linear, and quadratic functions. You have also been introduced briefly to cubic functions throughout this course. In this lesson, you are going to learn more about the properties of cubic, quartic, and quintic polynomial functions.

Characteristics of Polynomial Function Graphs

A polynomial function is a function equation that can be written in the form $f(x) = a_nx^n + a_{(n-1)}x^{(n-1)} + \ldots + a_2x^2 + a_1x + a_0$, where n is a non-negative integer and the coefficients, $a_n, a_{(n-1)}, \ldots, a_2$, a_1, and a_0, are real numbers. That equation looks really confusing, but it just means a function such as $f(x) = 3x^4 - 2x^3 + 6x^2 - 2x + 1$.

Each part of the equation is called a term and each term has a coefficient associated with it. The terms are usually written in descending powers of x. The term containing the highest power of x is called the leading term and its coefficient is referred to as the **leading coefficient.** The power of x contained in the leading term is called the degree of the polynomial. Polynomials of the first few degrees have special names, as listed in the table below.

Degree	Name	Example of Function
0	constant	$f(x) = 2$ or $f(x) = 2x^0$
1	linear	$f(x) = x + 2$ or $f(x) = x^1 + 2$
2	quadratic	$f(x) = x^2 - 5x + 6$
3	cubic	$f(x) = x^3 - 4x^2 + 2x - 6$
4	quartic	$f(x) = -3x^4 + x$
5	quintic	$f(x) = 5x^5 + 4x^3 - 6$

Polynomial functions with higher degrees are named by their degree. For example, $f(x) = x^8 + 6x^7 + 4x^2$ is called an eighth degree polynomial.

Include a summary of the above information on your resource sheet.

Example 1

Label the leading term, leading coefficient, and degree of each of the following polynomials.

a) $y = -3x^5 + 2x^4 + 5x - 12$

b) $y = 2x^3 - 6x^2 + 2x^4 - x + 1$

c) $y = 3 - x^4 + 3x^2 - 7x^3$

Solutions

a) Leading term: $-3x^5$

Leading coefficient: -3

Degree: 5

b) In order to correctly label the parts of this polynomial, you first need to order the terms in descending powers of x.

$$\begin{aligned} y &= 2x^3 - 6x^2 + 2x^4 - x + 1 \\ &= 2x^4 + 2x^3 - 6x^2 - x + 1 \end{aligned}$$

Leading term: $2x^4$

Leading coefficient: 2

Degree: 4

c) First, order the terms in descending powers of x.

$$\begin{aligned} y &= 3 - x^4 + 3x^2 - 7x^3 \\ &= -x^4 - 7x^3 + 3x^2 + 3 \end{aligned}$$

Leading term: $-x^4$

Leading coefficient: -1

Degree: 4

Example 2

Identify which of the following functions are polynomial functions.

a) $y = \sqrt{3x - 4}$

b) $y = 5x^5 - 2x + x^{\frac{3}{2}}$

c) $y = |x - 1|$

d) $y = \dfrac{1}{x^2}$

e) $y = (x - 3)(x + 2)(x - 7)$

Solutions

A polynomial is a series of terms where each term is the product of a constant and a variable. This variable is an exponent that is a non-negative integer.

a) This is a radical function, not a polynomial function, since the exponent would be written as $\frac{1}{2}$.

b) This is not a polynomial function because all of the terms do not have *integral* exponents: $x^{\frac{3}{2}}$ has an exponent equal to $\frac{3}{2}$, ,which is not an integer.

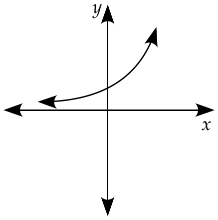
c) This is an absolute value function, not a polynomial function.

d) This is a rational function, not a polynomial function. You can rewrite this function as $x^{(-2)}$, in which case the exponent of x is negative. As polynomials do not contain negative exponents, this is not a polynomial function.

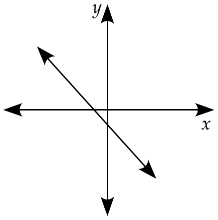
e) This is a polynomial function as it can be multiplied out and rewritten as $y = x^3 - 8x^2 + x + 42$.

$f(x)$ can represent a polynomial function. Any value of x for which $f(x) = 0$ is a **root** of the equation and a **zero** of the function. You have encountered this concept many times before when dealing with rational functions, radical functions, and quadratic functions.

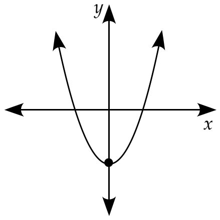
A function, f, is **increasing** in an interval if for any x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$. All this means is that it is increasing if it is always going up as you move from left to right.



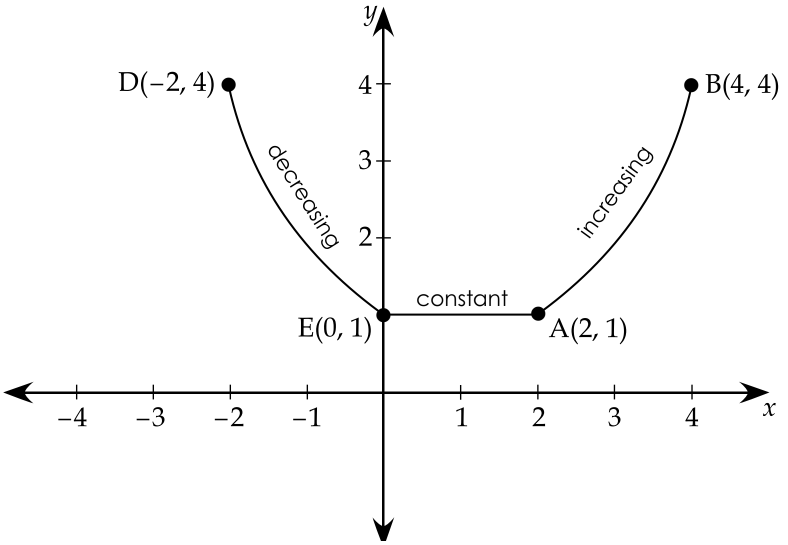
increasing



decreasing



increasing when x > 0



For all points in the interval between A(2, 1) and B(4, 4), the function is increasing.

A function, *f*, is **decreasing** in an interval if, for any x_1 and x_2 in the interval, $x_1 < x_2$, implies that $f(x_1) > f(x_2)$. For points in the interval between D(−2, 4) and E(0, 1), the function is decreasing. That is, it is going down as you move from left to right.

A function, *f*, is **constant** in an interval if, for any x_1 and x_2 in the interval, $f(x_1) = f(x_2)$.

For all points between E(0, 1) and A(2, 1), the *y*-coordinates are the same, so the function between E and A is constant. A constant function is neither increasing nor decreasing.

Example 3

Study the graphs of the following third degree polynomial functions. Describe the similarities and differences in these graphs regarding number of turns of the function curve, number of *x*-intercepts, shape, sign of the leading coefficient, and end behaviour of the function. The graphs are given to you on the following pages, but you could use graphing technology if you want to graph them yourself first.

a) $y = x^3$

b) $y = -x^3$

c) $y = x^3 - 2x^2 - x - 2$

d) $y = -x^3 + 2x^2 + x + 2$

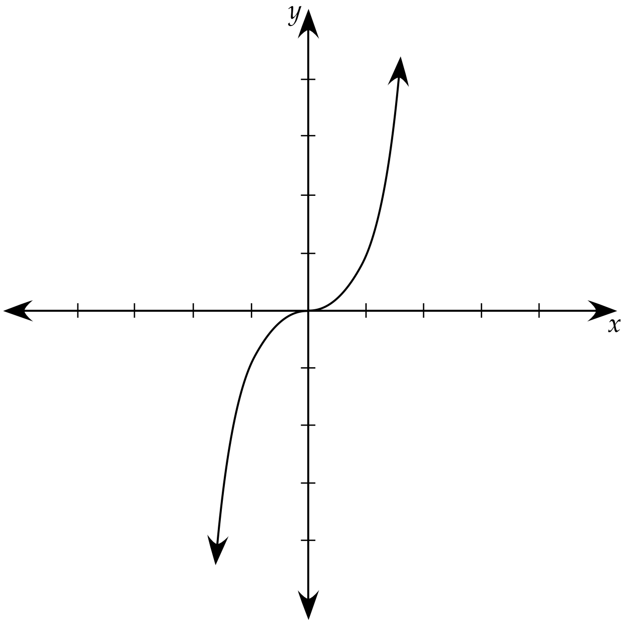
e) $y = (x + 2)(x - 1)^2$

f) $y = -(x + 2)(x - 1)^2$

Solutions

a) $y = x^3$

Graph

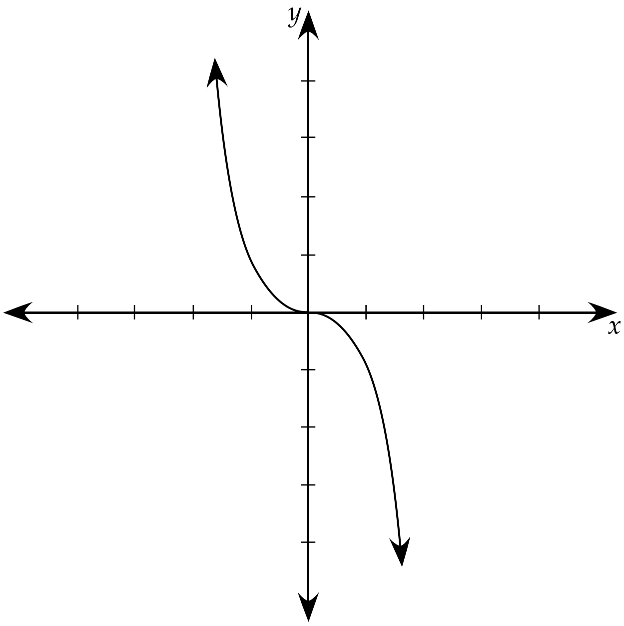


Observations

- No turns since it is always increasing
- Curve flattens out at its zero
- One x-intercept
- Shape of a sideways S
- Leading coefficient is positive
- As x moves to the left, this function approaches negative infinity
- As x moves to the right, this function approaches positive infinity

b) $y = -x^3$

Graph



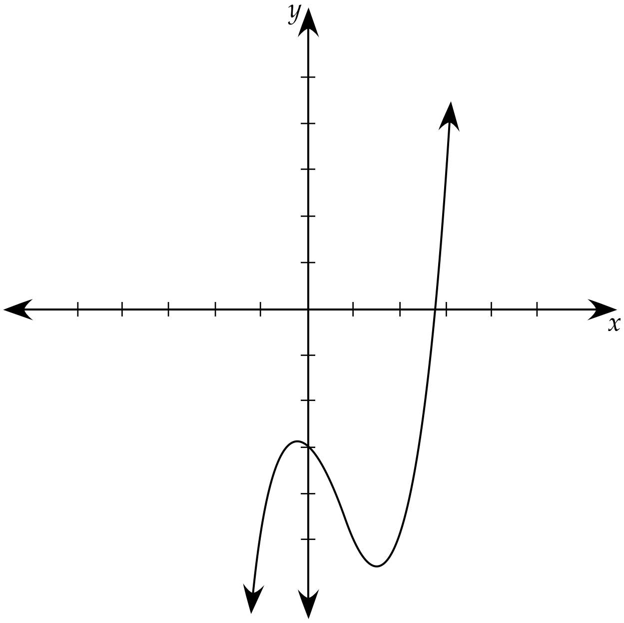
Observations

- No turns since it is always decreasing
- Curve flattens out at its zero
- One x-intercept
- Shape of a sideways S

  Reflection over the x-axis of $y = x^3$
- Leading coefficient is negative
- As x moves to the left, function approaches positive infinity
- As x moves to the right, this function approaches negative infinity

c) $y = x^3 - 2x^2 - x - 2$

Graph

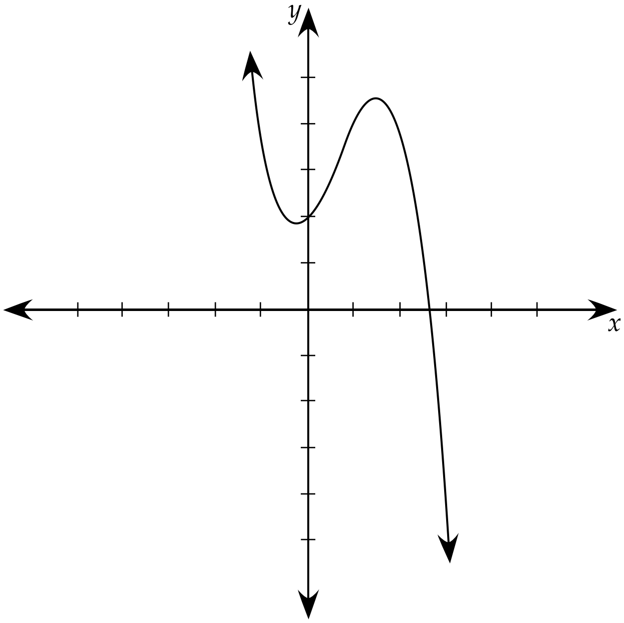


Observations

- Two turns, since it goes from increasing to decreasing and back to increasing
- One x-intercept
- S-shape
- Leading coefficient is positive
- As x moves to the left, this function approaches negative infinity
- As x moves to the right, this function approaches positive infinity

d) $y = -x^3 + 2x^2 + x + 2$

Graph

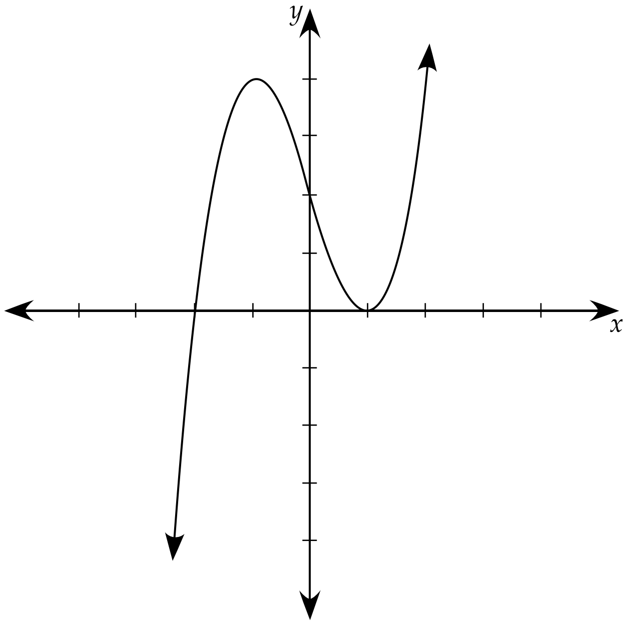


Observations

- Two turns
- One x-intercept
- S-shape
- Reflection over x-axis of the graph in (c)
- Leading coefficient is negative
- As x moves to the left, this function approaches positive infinity
- As x moves to the right, this function approaches negative infinity

e) $y = (x + 2)(x - 1)^2$

Graph

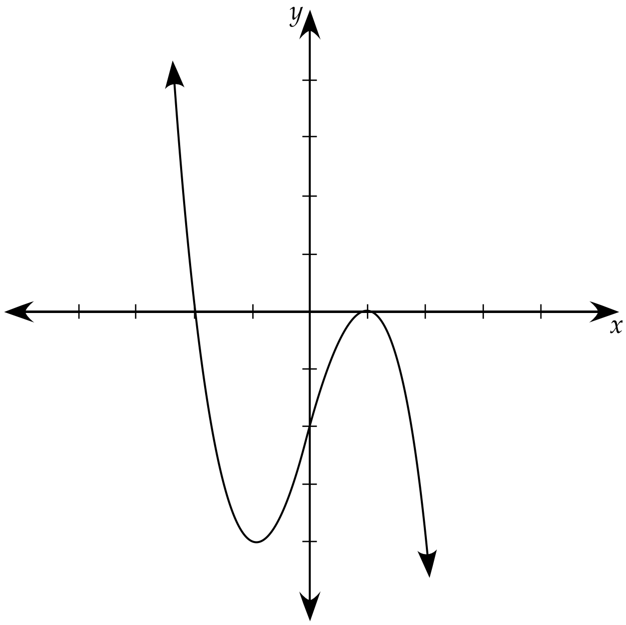


Observations

- Cubic function in factored form equivalent to $y = x^3 - 3x + 2$
- Two turns
- Two x-intercepts at -2 and 1
- Curve flattens out and is tangent at $x = 1$
- S-shape
- Leading coefficient is positive
- As x moves to the left, this function approaches negative infinity
- As x moves to the right, this function approaches positive infinity

f) $y = -(x + 2)(x - 1)^2$

Graph



Observations

- Cubic function in factored form equivalent to $y = -x^3 + 3x - 2$
- Two turns
- Two x-intercepts at -2 and 1
- Curve flattens out and is tangent at $x = 1$
- S-shape
- Leading coefficient is negative
- As x moves to the left, this function approaches positive infinity
- As x moves to the right, this function approaches negative infinity

If you have a graphing calculator or a graphing program on your computer, you should try graphing some other cubic functions. See if you can predict the graph characteristics based on the equation you enter.

Example 4

Study the graphs of the following fourth degree polynomial functions. Describe the same characteristics as in Example 3.

a) $y = x^4$

b) $y = -x^4$

c) $y = (x + 2)(x + 1)(x - 1)(x - 2)$

d) $y = -(x + 2)(x + 1)(x - 1)(x - 2)$

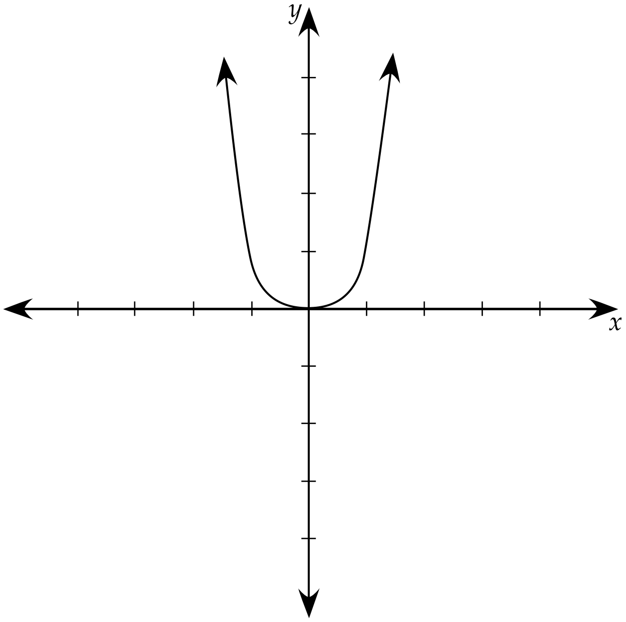
e) $y = x^4 - 3x^3 - x^2 + 3x + 3$

f) $y = -x^4 + 3x^3 + x^2 - 3x - 3$

Solutions

a) $y = x^4$

Graph

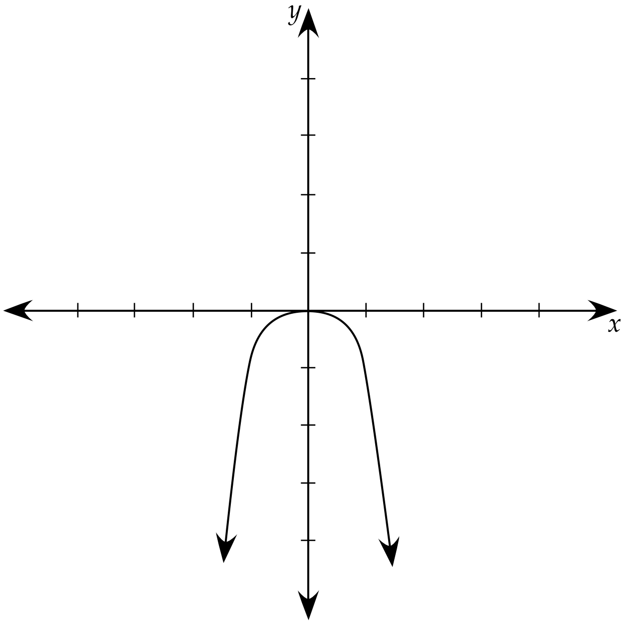


Observations

- One turn
- Curve flattens out at its x-intercept
- One x-intercept
- Opens up
- U-shape
- Leading coefficient is positive
- As x moves to the left, this function approaches positive infinity
- As x moves to the right, this function approaches positive infinity

b) $y = -x^4$

Graph

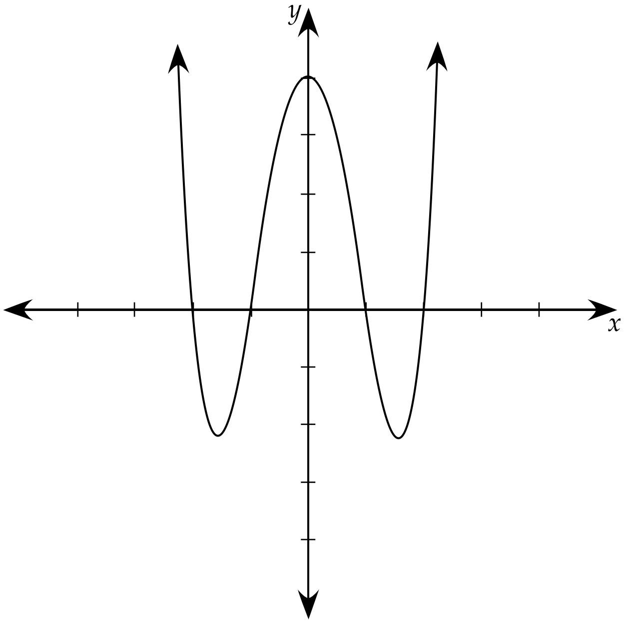


Observations

- One turn
- Curve flattens out at its x-intercept
- One x-intercept
- Opens down
- U-shape
- Leading coefficient is negative
- As x moves to the left, this function approaches negative infinity
- As x moves to the right, this function approaches negative infinity

c) $y = (x + 2)(x + 1)(x - 1)(x - 2)$

Graph

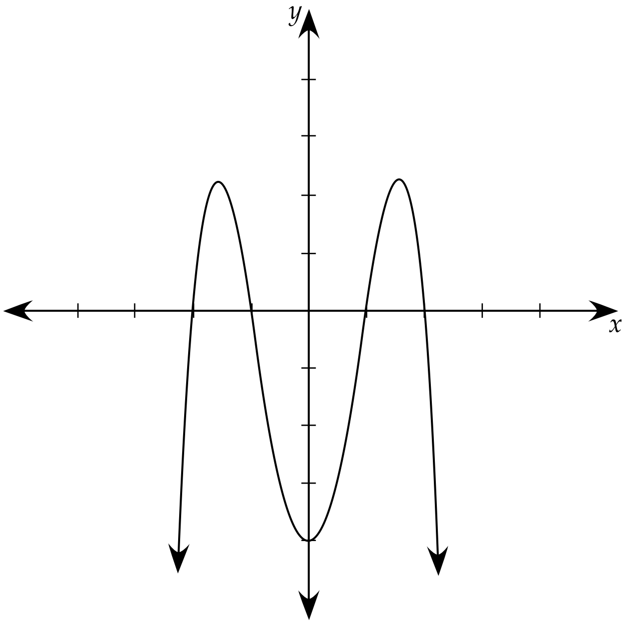


Observations

- Fourth degree function in factored form equivalent to $y = x^4 - 5x^2 + 4$
- Three turns, goes from decreasing to increasing to decreasing and back to increasing
- Four x-intercepts at −2, −1, 1, and 2
- Opens up
- W-shape
- Leading coefficient is positive
- As x moves to the left, this function approaches positive infinity
- As x moves to the right, this function approaches positive infinity

d) $y = -(x + 2)(x + 1)(x - 1)(x - 2)$

Graph

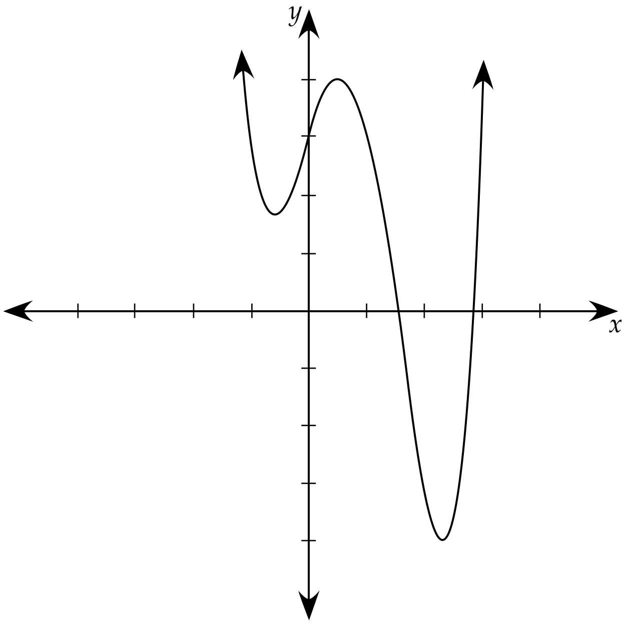


Observations

- Fourth degree function in factored form equivalent to $y = -x^4 + 5x^2 - 4$
- Three turns
- Four x-intercepts at −2, −1, 1, and 2
- Opens down
- M-shape
- Leading coefficient is negative
- As x moves to the left, this function approaches negative infinity
- As x moves to the right, this function approaches negative infinity

e) $y = x^4 - 3x^3 - x^2 + 3x + 3$

Graph

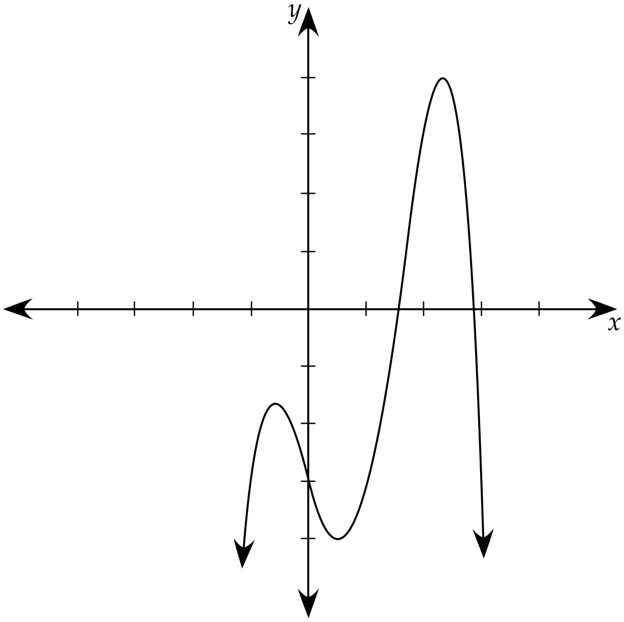


Observations

- Three turns
- Two x-intercepts
- Opens up
- Distorted W-shape
- Leading coefficient is positive
- As x moves to the left, this function approaches positive infinity
- As x moves to the right, this function approaches positive infinity

f) $y = -x^4 + 3x^3 + x^2 - 3x - 3$

Graph



Observations

- Three turns
- Two x-intercepts
- Opens down
- Distorted M-shape
- Leading coefficient is negative
- As x moves to the left, this function approaches negative infinity
- As x moves to the right, this function approaches negative infinity

If you have a graphing calculator or a computer graphing program, you should try graphing some other fourth degree polynomial functions. See if you can predict the graph characteristics based on the equation you enter.

Example 5

Study the graphs of the following fifth degree polynomial functions. Describe the same characteristics as in Example 3 and Example 4.

a) $y = x^5$

b) $y = -x^5$

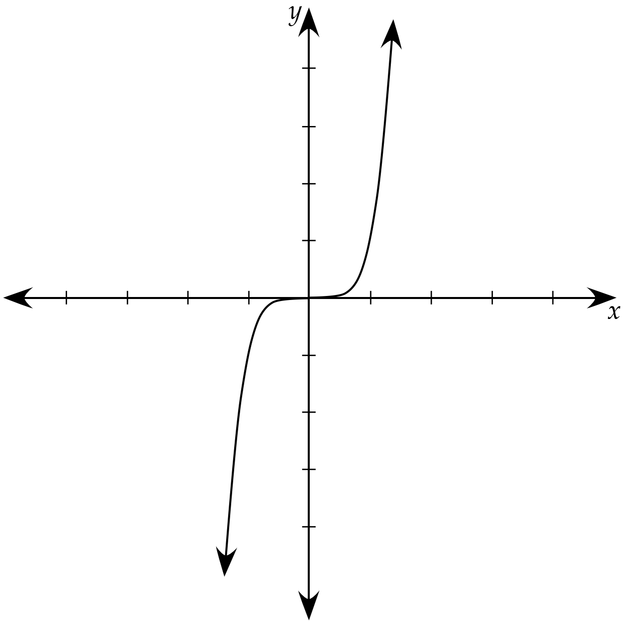
c) $y = (x + 1)(x)(x - 1)(x - 2)(x - 3)$

d) $y = -(x + 1)(x)(x - 1)(x - 2)(x - 3)$

Solutions

a) $y = x^5$

Graph

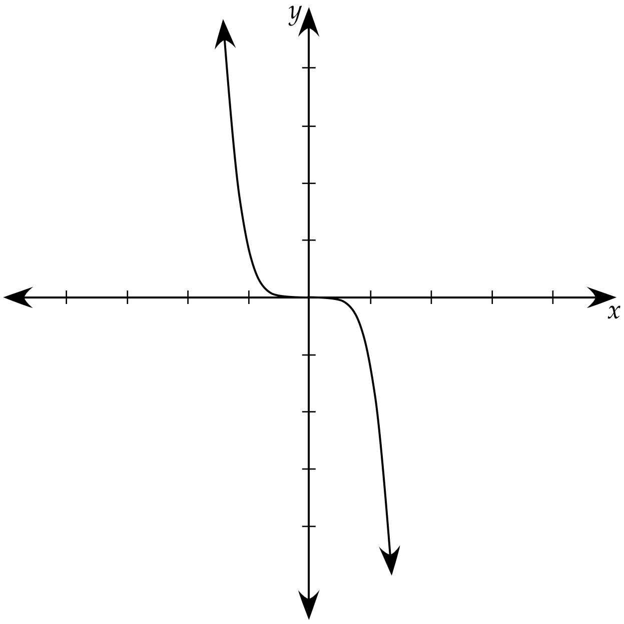


Observations

- No turns since it's always increasing
- Curve is very flat at the x-intercept
- One x-intercept
- Flattened S-shape
- Leading coefficient is positive
- As x moves to the left, this function approaches negative infinity
- As x moves to the right, this function approaches positive infinity

b) $y = -x^5$

Graph

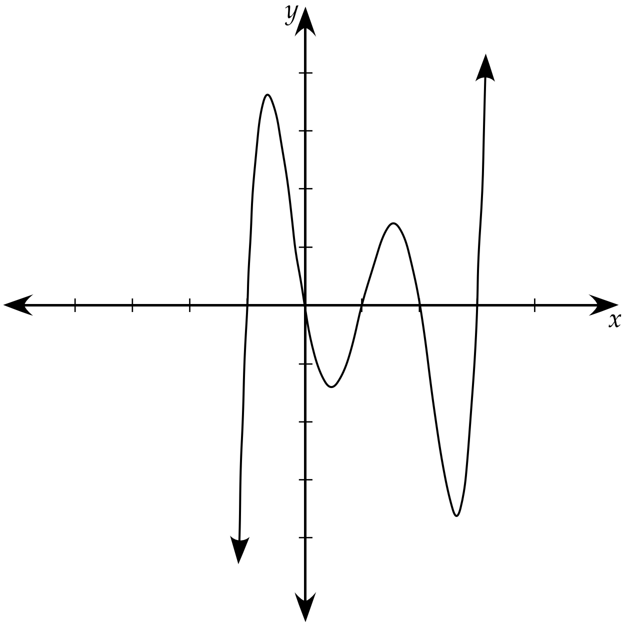


Observations

- No turns since it is always decreasing
- Curve is very flat at the x-intercept
- One x-intercept
- Flattened S-shape
- Leading coefficient is negative
- As x moves to the left, this function approaches positive infinity
- As x moves to the right, this function approaches negative infinity

c) $y = (x + 1)(x)(x - 1)(x - 2)(x - 3)$

Graph

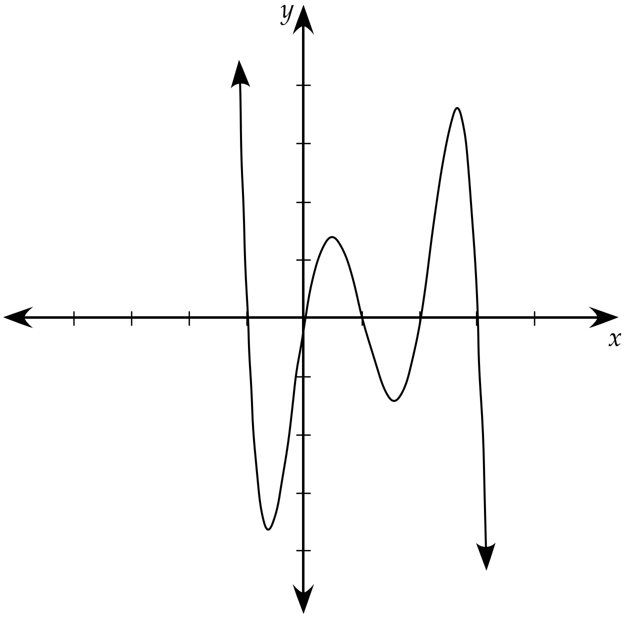


Observations

- Fifth degree function in factored form equivalent to $y = x^5 - 5x^4 + 5x^3 + 5x^2 - 6x$
- Four turns
- Five x-intercepts at −1, 0, 1, 2, 3
- Leading coefficient is positive
- As x moves to the left, this function approaches negative infinity
- As x moves to the right, this function approaches positive infinity

d) $y = -(x + 1)(x)(x - 1)(x - 2)(x - 3)$

Graph



Observations

- Fifth degree function in factored form equivalent to $y = -x^5 + 5x^4 - 5x^3 - 5x^2 + 6x$
- Four turns
- Five x-intercepts at −1, 0, 1, 2, 3
- Leading coefficient is negative
- As x moves to the left, this function approaches positive infinity
- As x moves to the right, this function approaches negative infinity

Based on the observations of the previous graphs, the following features of polynomial functions, *f*, of degree, *n*, should be noted.

1. The graph of a polynomial function is continuous. This means the graph has no breaks—you could sketch the complete curve from left to right without lifting your pencil from the paper.

2. The graph of a polynomial function of degree n has only smooth turns with at most $(n - 1)$ turning points. Turning points are points at which the graph changes from increasing to decreasing or vice versa.

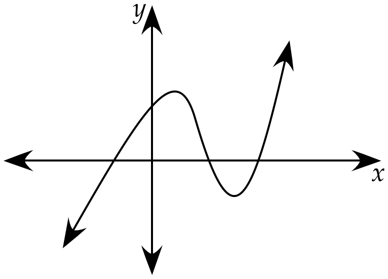
   For the graphs that you investigated in Example 3, the cubic equation has degree 3 so it has at most two turns. For the graphs that you investigated in Example 4, the quartic equations have degree 4 so they have at most three turns. For the graphs that you investigated in Example 5, the quintic equation has degree 5 so it has at most four turns.

3. Note the end behaviour:

   - When the degree, n, of a polynomial is **odd**:

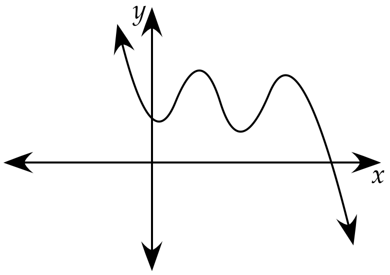
     if the leading coefficient is positive (> 0), then the graph falls to the left and rises to the right just like the linear function $y = x$.

     In other words, this graph goes from Quadrant III to Quadrant I as you move from left to right.



     If the leading coefficient is negative (< 0), then the graph rises to the left and falls to the right just like the linear function, $y = -x$.

     In other words, this graph goes from Quadrant II to Quadrant IV as you move from left to right.

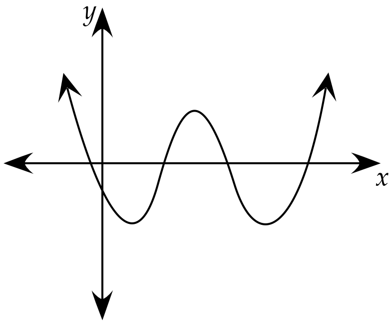


     It may be easier to remember the end behaviour if you think of a linear function. A linear function has an odd degree since $y = x$ is degree 1. All odd degree polynomials have end behaviour like a line: they go down to the left and up to the right if the leading coefficient is positive (like $y = x$), and they go up to the left and down to the right if the leading coefficient is negative (like $y = -x$).

- When the degree, n, of a polynomial is **even**:

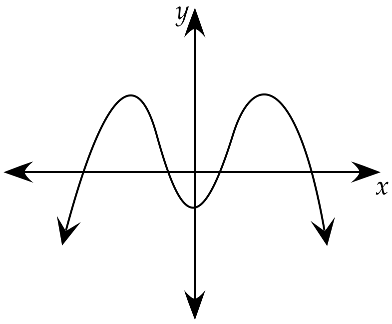
  if the leading coefficient is positive (> 0), then the graph rises to the left and right (opens up) just like the quadratic function $y = x^2$.

  In other words, the graph goes from Quadrant II to Quadrant I as you move from left to right.



  If the leading coefficient is negative (< 0), then the graph falls to the left and right (opens down) just like the quadratic function $y = -x^2$.

  In other words, the graph goes from Quadrant III to Quadrant IV as you move from left to right.



  It may be easier to remember the end behaviour if you think of a quadratic function. A quadratic function has an even degree since $y = x^2$ is degree 2. All even degree polynomials have end behaviour like a parabola: they go up to the left and up to the right if the leading coefficient is positive (like $y = x^2$), and they go down to the left and down to the right if the leading coefficient is negative (like $y = -x^2$).

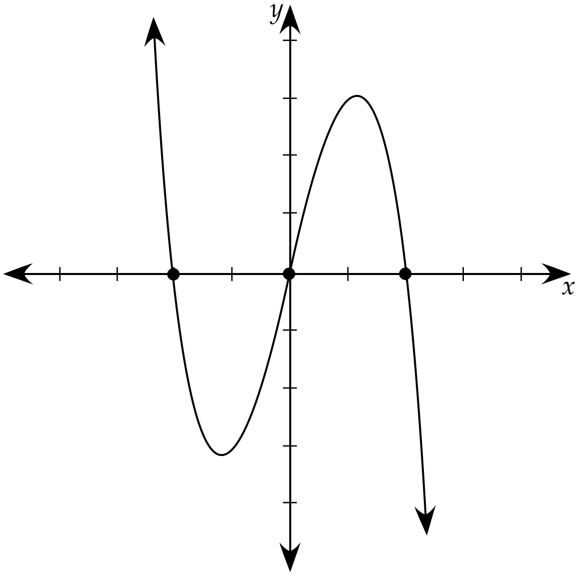
4. The function, f, with degree n has at most n real roots. If you have a cubic function you can expect at most three real roots.

   When you have a quartic function, you can expect it to have at most four real roots.

   When you have a quintic function, you can expect it to have at most five real roots, and so on.

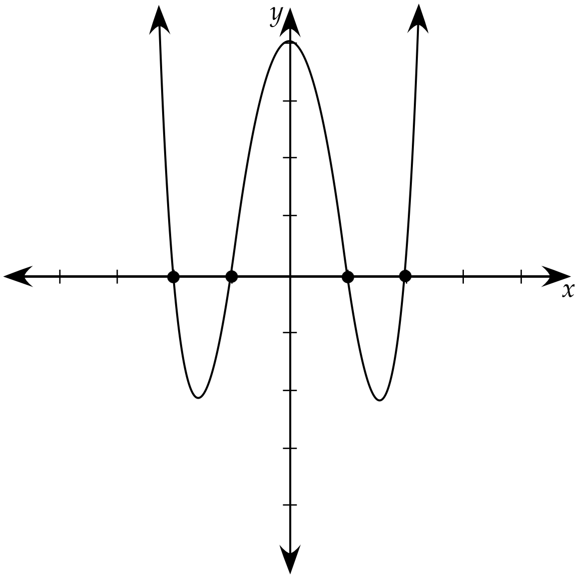
   $f(x)= -x^3 + 4x$

   cubic → at most three roots



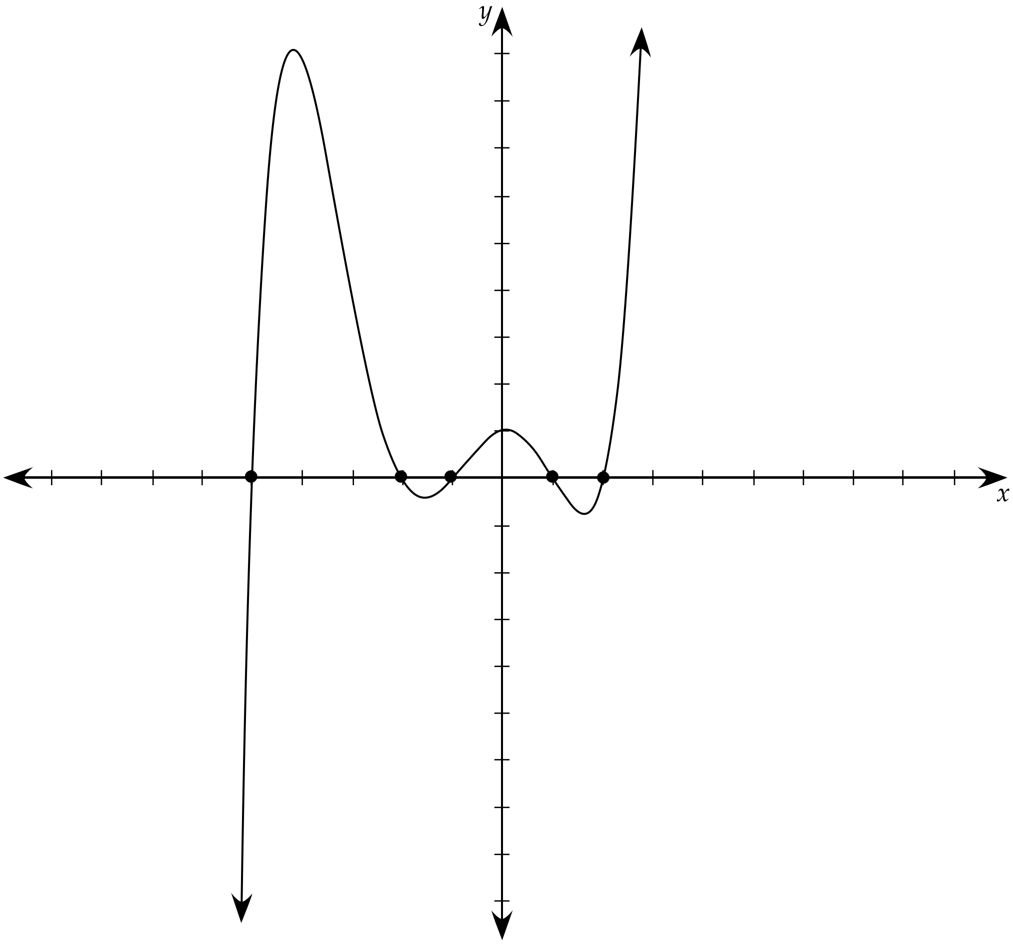
   $f(x)= x^4 - 5x^2 + 4$

   quartic → at most four roots



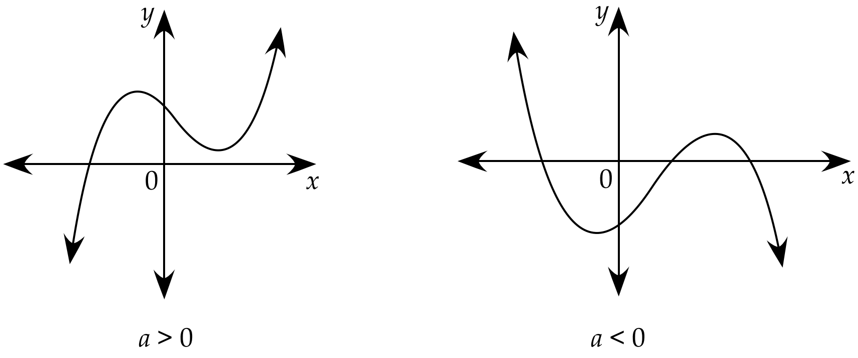
$$f(x) = \frac{1}{20}x^5 + \frac{1}{4}x^4 - \frac{1}{4}x^3 + \frac{5}{4}x^2 + \frac{1}{5}x + 1$$

quintic → at most five roots



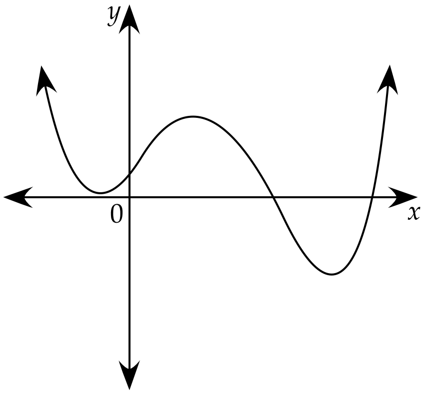
5. In general, the graph of a cubic function is shaped like a “sideways S” as shown.

   Graph of $f(x) = ax^3 + bx^2 + cx + d$:

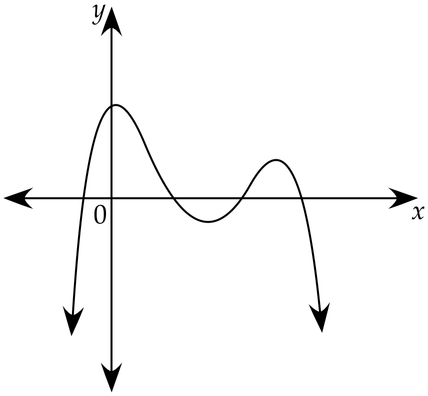


In general, the graph of a quartic equation has a "W-shape" or an "M-shape."

Graph of $f(x) = ax^4 + bx^3 + cx^2 + dx + e$:



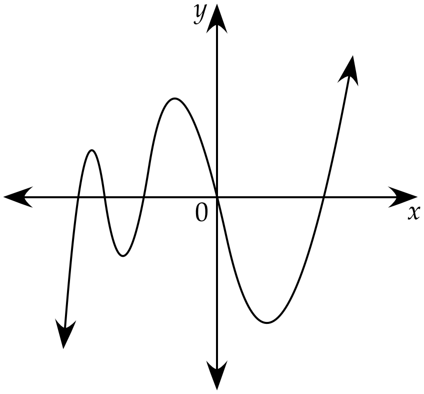
$a > 0$



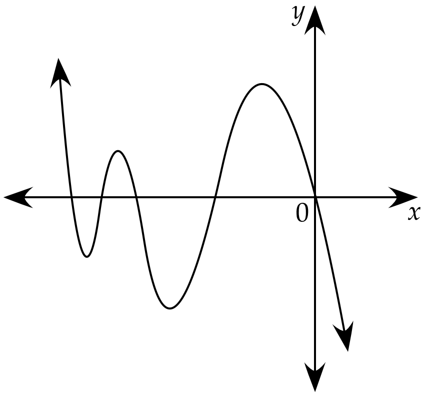
$a < 0$

The general shape of a quintic equation is shown below.

Graph of $f(x)= ax^5 + bx^4 + cx^3 + dx^2 + ex + f$:



$a > 0$



$a < 0$

6. You may have noticed when making observations that when a polynomial is written in factored form, the x-intercepts can be determined easily. Sometimes one factor may occur multiple times. As the curve approaches an x-intercept, which is a multiple root, the graph flattens as it approaches the x-axis. The higher the degree of **multiplicity**, the more the graph flattens.

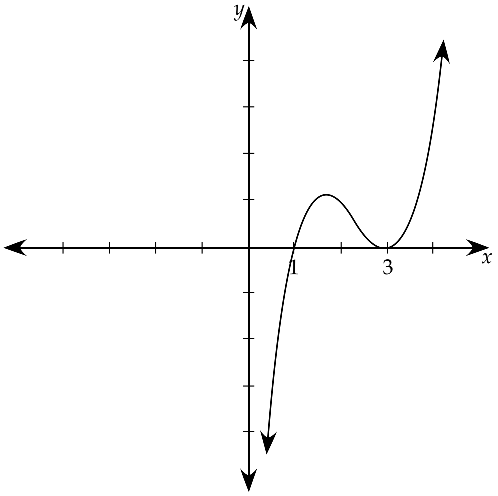
   If a polynomial $f(x)$ has a squared factor such as $(x - c)^2$, then $x = c$ is a double root of $f(x) = 0$. The graph of $y = f(x)$ flattens and is tangent to the x-axis at $x = c$, as shown in Figures 1, 2, and 3. The graph $y = f(x)$ does not cross at $x = c$, but is tangent and turns away from the x-axis.

   Figure 1

   Cubic

   $y = (x - 1)(x - 3)^2$

   There is a double root at 3, since it can be written as $y = (x - 1)(x - 3)(x - 3)$ and the graph flattens out and is tangent to the x-axis at $x = 3$. The graph passes through the x-axis at $x = 1$ without flattening.



   Note that this tangent characteristic is true for all even-powered factors, or even values of n for $(x - c)^n$. The higher the value of n, the more the curve flattens at the x-intercept. For example, in Figure 2, the value of $n = 4$. Therefore, this curve is flatter at its quadruple root than the graphs in Figures 1 and 3 at their double root.

   Figure 2

   Quartic

   $y = (x - 1)^4$

   There is a quadruple root at $x = 1$ and the graph is tangent to the x-axis at $x = 1$.

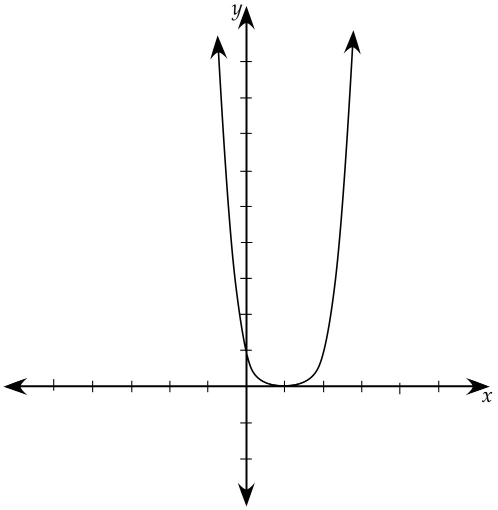
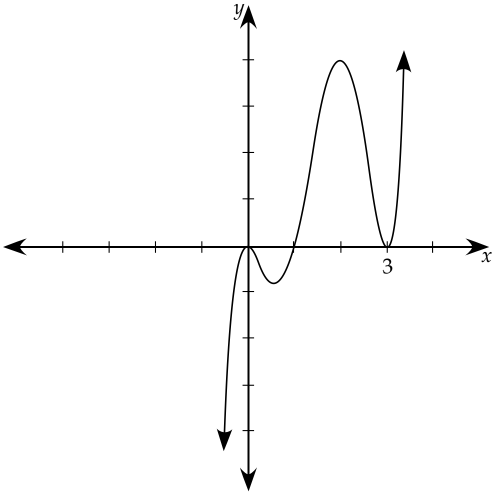


Figure 3

Quartic (5th degree)

$y = x^2(x - 1)(x - 3)^2$

There are double roots at $x = 0$ and $x = 3$ and the graph flattens and is tangent to the x-axis at $x = 0$ and $x = 3$. The curve crosses the x-axis at $x = 1$ without flattening.



If a polynomial $P(x)$ has a cubed factor such as $(x - c)^3$, then $x = c$ is a triple root of $P(x) = 0$. Again, the graph of $y = P(x)$ flattens out around $(c, 0)$ but for odd-powered roots the curve crosses the x-axis at this point, as shown in Figures 4, 5, and 6.

Figure 4

Cubic

$y = (x - 2)^3$

There is a triple root at $x = 2$, so the curve flattens out and crosses the x-axis at $x = 2$.

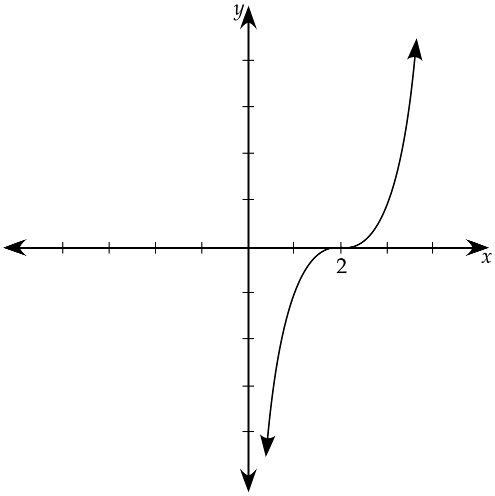


Figure 5

Quartic

$y = (x - 1)^3 (x - 3)$

There is a triple root at $x = 1$, so the curve flattens out and crosses the x-axis at $x = 1$. The curve crosses the x-axis at 3 without flattening.

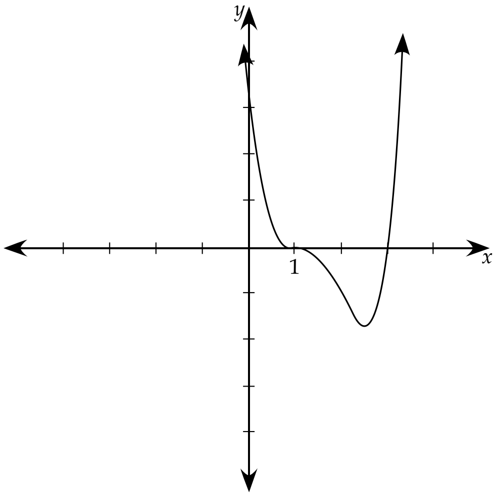
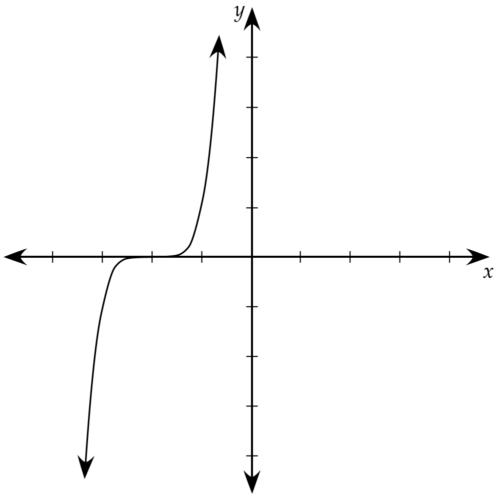


Figure 6

Quintic

$y = (x + 2)^5$

There is a quintuple root at $x = -2$, so the curve flattens out substantially and crosses the x-axis at $x = -2$. Notice how this curve flattens out more than the previous two curves.



Note that the same characteristics are true for any odd-powered factors or for any odd power of n, for $(x - c)^n$. The higher the value of n, the flatter the curve. For example, in Figure 6, $n = 5$; therefore, this curve is flatter at its quintuple root than the graphs in Figures 4 and 5 at their triple root.

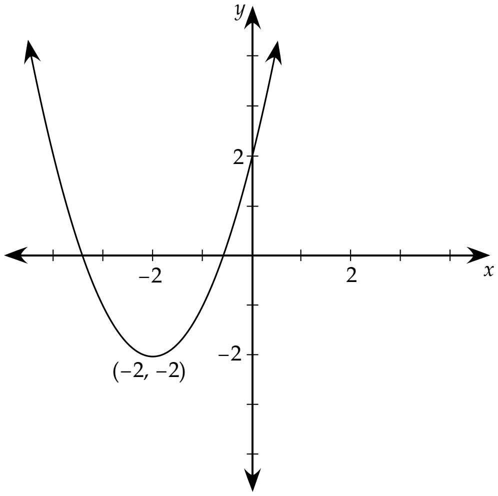
For both even- and odd-powered factors, the higher the degree of multiplicity, the flatter the curve is at the x-intercepts.

7. When you studied quadratic functions, you were able to determine exactly one maximum or minimum value by determining the y-coordinate of the vertex. If the graph of the function opened upward, the function had a minimum value. If the graph of the function opened downward, it had a maximum value.

$y = (x + 2)^2 - 2$

Vertex: $(-2, -2)$

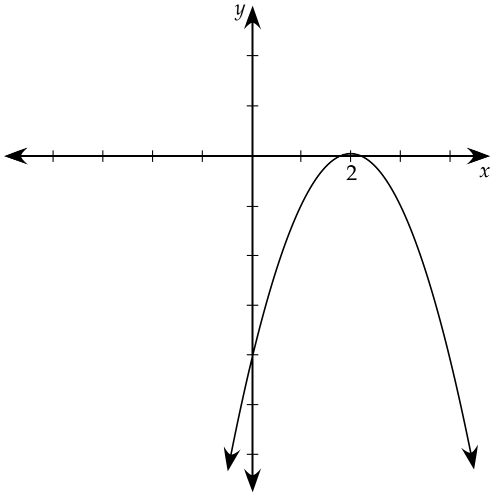
Minimum value of $y = -2$



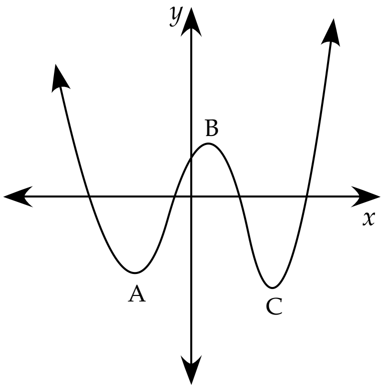
$y = -(x + 2)^2$

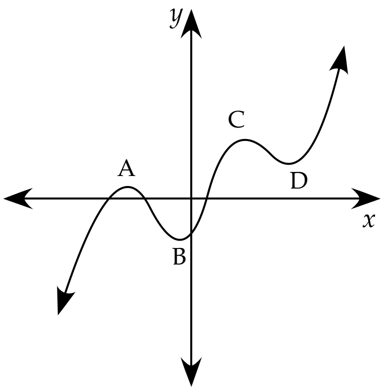
Vertex: $(2, 0)$

Maximum value of $y = 0$



For polynomial functions of higher degree, there are two types of maximum or minimum values: absolute and relative.





A is called a **relative minimum**.

B is called a **relative maximum**.

C is an **absolute minimum**.

There is no **absolute maximum** value for this graph.

A and C are **relative maximums**.

B and D are **relative minimums**.

There is **neither** an absolute maximum nor an absolute minimum for this graph.

In this course, you will not be asked to find the maximum or minimum values of polynomial functions. This can be done using a graphing calculator and algebraically using calculus. So, you can look forward to learning that in another course.

You should summarize these seven points and put them on your resource sheet.

The following Learning Activity will allow you to practice recognizing the properties of cubic, quartic, and quintic polynomial functions. You will also be given the opportunity to summarize the information about the characteristics of polynomial functions that you learned about in this lesson.

Learning Activity 4.1

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

For Questions 1 to 3, decide whether the following situations involve a permutation or a combination.

1. Selecting three students to attend a play.
2. Picking a captain and an assistant captain of a hockey team.
3. Ordering five books on a shelf.
4. What is the reciprocal of $\frac{1}{6x^2}$?
5. Does the following set of points represent a function?

   $(1, 1),(2, 4),(1, 5),(4, 2)$
6. Simplify: $\frac{y^{-4}}{y^3}$
7. How much money will you make, before taxes, if you work 40 hours a week for $14 an hour?
8. If $f(x) = -\frac{x}{x^3 + 1}$, evaluate $f(x)$ at $x = 2$.

continued

Learning Activity 4.1 (continued)

Part B: Polynomial Functions

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Throughout this lesson, you have been presented with a lot of information regarding the characteristics of polynomial functions. Complete the following chart with the information you have learned about polynomials. You may wish to also include a similar chart on your resource sheet.

Polynomials		
	Odd Degree	**Even Degree**
Positive Leading Coefficient		
Negative Leading Coefficient		
***y*-intercept**		
Number of *x*-intercepts		
Number of Turns		
Domain		
Range		
Maximum or Minimum		

continued

2. Identify the polynomial functions in the following set of functions. Justify your answers.
   a) $y = \left| x^2 \right| - 3$
   b) $y = 3 - x^5$
   c) $y = \dfrac{x^2}{x^3 - 8}$
   d) $y = 4^x - 1$
   e) $y = 1$
3. Explain what is meant by a continuous graph.
4. Graph $y = |x|$. Name a feature of the graph of $f(x) = |x|$ that is not shared by the graphs of polynomial functions.
5. Does the graph of $f(x) = 2x^4 - 3x$ rise or fall to the right? How can you tell? What happens to the left?
6. State the maximum number of turns in the following graphs:
   a) $f(x) = x^3 - 4x$
   b) $g(x) = x^6 - 4x^2$
   c) $f(x) = -x^2 - 5x + 6$
   d) $g(x) = x^5 - 4x^3 + 6$
   e) $f(x) = -3x^4 - 5x + 6$
7. Determine the right and left end behaviours of the graphs of the following functions by examining the sign of the leading coefficient and the degree of the polynomial.
   a) $f(x) = -x^3 + 3x$
   b) $f(x) = 2x^4 - 5x^2 + 4$
   c) $f(x) = (x - 1)(x + 3)(x - 1)$
   d) $f(x) = -x^4 + x^2$
   e) $f(x) = -2x^5 + x^4 - 2x$
   f) $f(x) = 3x^5 + x^3 - 2$

continued

Learning Activity 4.1 (continued)

8. Match the polynomial function with the correct graph:

a) $f(x) = -3x + 5$

b) $f(x) = x^2 - 2x$

c) $f(x) = -2x^2 - 9x - 9$

d) $f(x) = 3x^3 - 9x + 1$

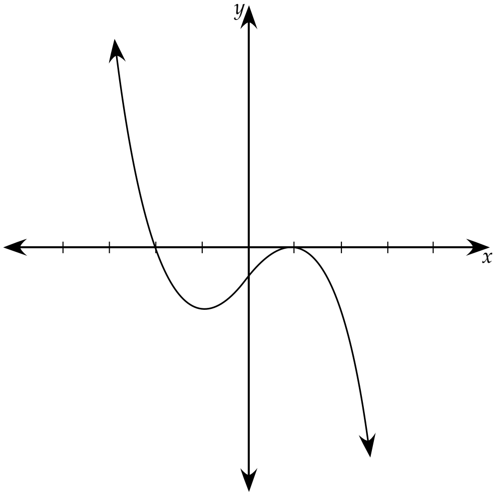
e) $f(x) = -\frac{1}{3}x^3 + x - \frac{2}{3}$

f) $f(x) = -\frac{1}{4}x^4 + 4x^2$

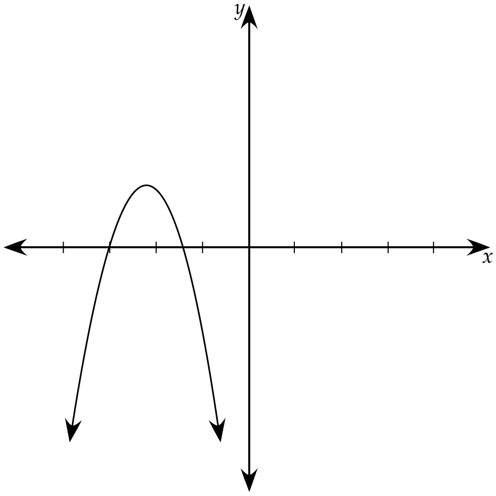
g) $f(x) = 3x^4 + 4x^3$

h) $f(x) = x^5 - 5x^3 + 4x$

i)



ii)



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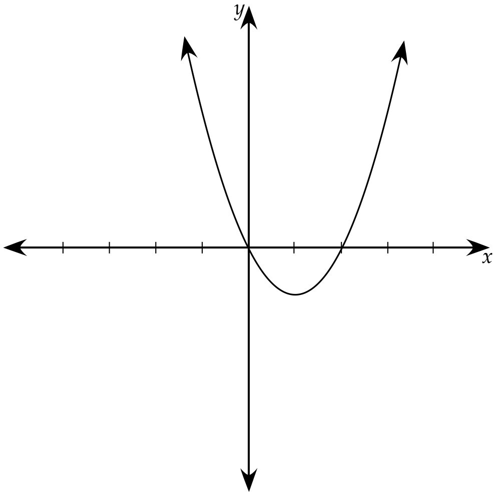

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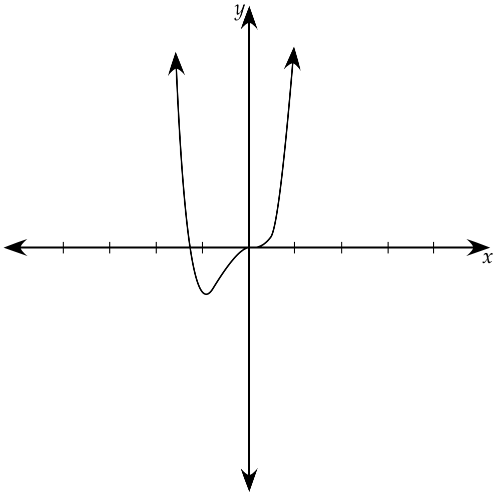

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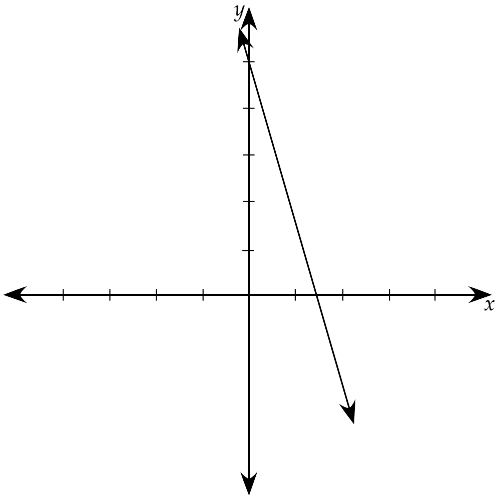
Learning Activity 4.1 (continued)

v)



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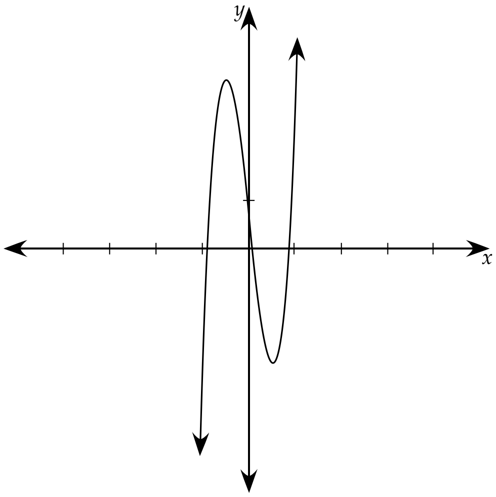
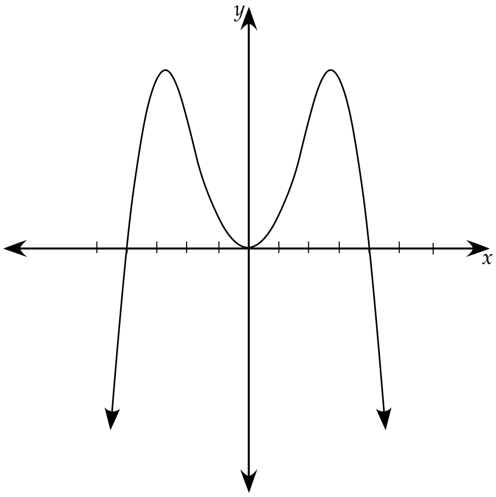
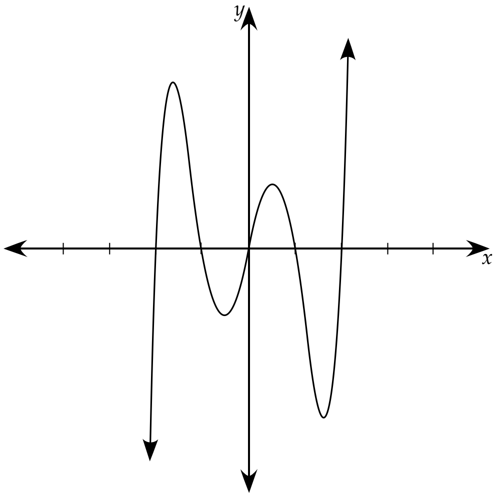


vii)



viii)



continued

Learning Activity 4.1 (continued)

9. Without the aid of graphing technology, sketch the graphs of each of the following polynomials, and label all y-intercepts and x-intercepts. Remember, you do not need to find the coordinates of relative and absolute maximums or minimums. You do need to clearly indicate the end behaviour and the behaviour near the x-intercepts.

   a) $y = (x - 2)(x + 1)^3(x - 3)$
   b) $y = (x + 1)^2(x - 3)$
   c) $y = (x + 2)^2(x - 1)(x - 3)$
   d) $y = x(x - 1)^2(x + 2)(x - 3)$
   e) $y = -2(x - 1)(x + 2)^2(x - 2)$
   f) $y = -3x(x - 1)^2(x + 2)(x + 1)$
   g) Graph a quartic that has roots of −1 and +2 and a multiplicity of 2 at +1, with a leading coefficient of −3.
   h) Graph a quintic with multiplicity of 2 at both $x = 3$ and $x = -1$, with another root at $x = 1$. The leading coefficient is −2.
   i) Sketch a quintic with a multiplicity of 3 at $x = 1$, a multiplicity of 2 at $x = -2$, and a leading coefficient of −3.

Lesson Summary

In this lesson, you were introduced to quartic and quintic polynomials. You also learned about the properties of polynomial functions including the effect the sign of the leading coefficient has on the graph, the effect the degree of the polynomial has on the number of turns in the graph of the function, and how the multiplicity of a zero or an x-intercept affects the shape of the graph at that point. In the next lesson, you will learn how to determine the x-intercepts of a polynomial by division.

Notes

LESSON 2: DIVISION OF POLYNOMIALS

Lesson Focus

In this lesson, you will

- ☐ learn how to divide polynomials using both long division and synthetic division
- ☐ learn how to apply the Remainder Theorem

Lesson Introduction

You have had practice in adding, subtracting, and multiplying polynomials. In this lesson, you will learn how to divide a polynomial by a binomial. Dividing polynomials is especially valuable in factoring and finding the zeros of polynomial functions of higher degree.

There are two algorithms for polynomial division—long division and synthetic division. You will examine long division first and then relate it to synthetic division in this lesson.

Dividing Polynomials by Binomials

The division of polynomials by binomials is a skill that is closely related to the long division algorithm for numbers that you have learned in previous courses. The following example reviews the process of dividing a three-digit number by a one-digit number.

Example 1

Divide 426 by 8.

Solution

$$\begin{array}{r} 053 \\ 8\overline{)426} \\ \underline{-0}\downarrow\; \\ 42\; \\ \underline{-40}\downarrow \\ 26 \\ \underline{-24} \\ 2 \end{array}$$

Note that 426 is called the dividend and 8 is called the divisor.

$\therefore$ 426 ÷ 8 has a quotient of 53 and a remainder of 2.

The result can be written as $53 + \frac{2}{8}$ or as the mixed number $53\frac{2}{8}$. This calculation could also be expressed as $426 = 8 \times 53 + 2$. Notice the positions of the dividend, divisor, quotient, and remainder.

Long Division of Polynomials

Consider the following example of a trinomial divided by a binomial. Try to see if you can follow along using your knowledge of long division of numbers.

Example 2

Divide $x^2 - 6 - x$ by $x + 2$.

Solution

1. Set up the division. Just like numbers are arranged in order by place value, the polynomial terms must be arranged in order. Arrange the dividend and the divisor in descending powers of the variable. Notice the powers of x go from 2 → 1 → 0.
2. To be sure the like terms are lined up in columns, if there are any missing powers, the terms will need to be inserted with coefficients of zero. You will see this illustrated in the next example.

$$x+2\overline{)x^2-x-6}$$

$$\underset{\text{divisor}}{\nearrow} \qquad \underset{\text{dividend}}{\nwarrow}$$

3. Divide the first term of the divisor, x, into the first term of the dividend, x^2, and place the answer, $\left(\frac{x^2}{x} = x\right)$, over the second term.

$$\begin{array}{rl} & x \\ x+2 & \overline{)x^2-x-6} \end{array}$$

4. Multiply the divisor by x and line up the terms.

$$\begin{array}{rl} & x \\ x+2 & \overline{)x^2-x-6} \\ & x^2+2x \end{array}$$

5. Subtract and bring down the -6.

$$\begin{array}{rr} & x \\ x+2 & \overline{)x^2-\ x-6} \\ & \underline{-(x^2+2x)\downarrow} \\ & -3x-6 \end{array}$$

6. Divide the first term of the divisor, x, into the first term of your new polynomial, $-3x$, and place the answer, -3, over the third term of your dividend.

$$\begin{array}{r} x-3 \\ x+2\overline{)x^2-\ x-6} \\ \underline{-\left(x^2+2x\right)\downarrow} \\ -3x-6 \end{array}$$

7. Multiply the divisor by -3.

$$\begin{array}{r} x-3 \\ x+2\overline{)x^2-\ x-6} \\ \underline{-\left(x^2+2x\right)\ \downarrow} \\ -3x-6 \\ -3x-6 \end{array}$$

8. Subtract to determine your remainder.

$$\begin{array}{rrl} & x-3 & \longleftarrow \text{quotient} \\ \text{divisor} \longrightarrow & x+2\overline{)x^2-\ x-6} & \longleftarrow \text{dividend} \\ & \underline{-\left(x^2+2x\right)\downarrow} & \\ & -3x-6 & \\ & \underline{-(-3x-6)} & \\ & 0 & \longleftarrow \text{remainder} \end{array}$$

Note: In this example, there is no remainder.

Write the result as:

$$\frac{x^2-x-6}{x+2}=x-3$$

Note: You can confirm this division is correct by looking at the factored form of x^2-x-6 as $(x-3)(x+2)$. Therefore, by your knowledge of rational expressions, you can reduce $\frac{x^2-x-6}{x+2}=\frac{(x+2)(x-3)}{x+2}$ to $x-3$, as long as $x \neq -2$.

Example 3

Find an expression for $(4x^3 - 3x + 5) \div (x - 3)$.

Solution

1. Rearrange the dividend in descending powers and insert $0x^2$ for the missing term in the sequence. The term is a place holder for the x^2-term so that like-terms line up when doing the long division.

$$x - 3\overline{\big)4x^3 + 0x^2 - 3x + 5}$$

2. To determine what to write in the quotient line, divide the first term of the dividend by the first term of the divisor $\left(\dfrac{4x^3}{x} = 4x^2\right)$ and multiply $4x^2$ by the divisor, line up terms, subtract, and bring down $-3x$.

$$\begin{array}{r l}
 & 4x^2 \\
x - 3 & \overline{\big)4x^3 + 0x^2 - 3x + 5} \\
 & \underline{-\left(4x^3 - 12x^2\right)} \ \downarrow \\
 & \qquad\quad 12x^2 - 3x
\end{array}$$

3. Continue the division of the polynomial until the degree of the remainder is less than the degree of the divisor.

$$\begin{array}{r l}
 & 4x^2 + 12x \\
x - 3 & \overline{\big)4x^3 + 0x^2 - 3x + 5} \\
 & \underline{-\left(4x^3 - 12x^2\right)} \ \downarrow \\
 & \qquad\quad 12x^2 - 3x \longleftarrow \\
 & \qquad\quad \underline{-\left(12x^2 - 36x\right)} \downarrow \\
 & \qquad\qquad\qquad 33x + 5
\end{array}$$

Divide the first term, $12x^2$, by the first term of the divisor to find the next term to write in the quotient, $12x$. Multiply the divisor by $12x$, subtract, and bring down 5.

$$\begin{array}{rll}
 & \;\;4x^2 + 12x + 33 & \longleftarrow \text{quotient} \\
\text{divisor} \longrightarrow x - 3 & \overline{)\,4x^3 + 0x^2 - 3x + 5} & \longleftarrow \text{dividend} \\
 & \underline{-\left(4x^3 - 12x^2\right)} \;\downarrow & \\
 & 12x^2 - 3x & \\
 & \underline{-\left(12x^2 - 36x\right)} \;\downarrow & \\
 & 33x + 5 & \longleftarrow \text{Divide first term, } 33x\text{, by the first term of the divisor and proceed as above.} \\
 & \underline{-\left(33x - 99\right)} & \\
 & 104 & \longleftarrow \text{remainder } [33x - 33x = 0 \text{ and } 5 - (-99) = 5 + 99 = 104]
\end{array}$$

4. Write your solution.

$$\frac{4x^3 - 3x + 5}{x - 3} = 4x^3 + 12x + 33 + \frac{104}{x - 3}$$

Or, after multiplying all terms by (x − 3), you can write:

$$4x^3 - 3x + 5 = (x - 3)(4x^2 + 12x + 22) + 104$$

(dividend) = (divisor × quotient) + (remainder)

Note: Compare this solution with polynomials to the numeric solution in Example 1.

Example 4

Divide $(6x^3 + 16x - 19x^2 - 4) \div (x - 2)$ using long division.

Solution

First, arrange the dividend in descending powers and insert terms with a coefficient of zero as needed.

$$\begin{array}{rl}
 & \;\;6x^2 - 7x + 2 \\
x - 2 & \overline{)\,6x^3 - 19x^2 + 16x - 4} \\
 & \underline{-\left(6x^3 - 12x^2\right)} \;\downarrow \\
 & -7x^2 + 16x \\
 & \underline{-\left(-7x^2 + 14x\right)} \;\downarrow \\
 & 2x - 4 \\
 & \underline{-(2x - 4)} \\
 & 0
\end{array}$$

$\therefore\ 6x^3 + 16x - 19x^2 - 4 = (x - 2)(6x^2 - 7x + 2) + 0$

(dividend) = (divisor × quotient) + (remainder)

Example 5

Divide $3x^3 - x^2 - 6x - 2 \div x^2 + 2$ using long division.

Solution

You need to arrange terms in descending order and insert missing terms with a coefficient of zero in both the divisor and the dividend.

$$\begin{array}{r} 3x-1 \\ x^2+0x+2\overline{)3x^3-x^2-6x-2} \\ \underline{-\left(3x^3+0x^2+6x\right)}\ \downarrow \\ -x^2+0x-2 \\ \underline{-x^2-0x-2} \\ 0 \end{array}$$

$\therefore\ 3x^3 - x^2 - 6x - 2 = (3x - 1)(x^2 + 2)$

Notice how you can divide polynomials by binomials or by trinomials using long division. The divisor can be any polynomial.

The above examples illustrate the division algorithm.

Relating Dividend and Divisor: If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, then there are polynomials $q(x)$ and $r(x)$ such that:

dividend = (divisor)(quotient) + remainder

$\boldsymbol{f(x) = d(x)q(x) + r(x)}$

If the remainder $r(x) = 0$, then $d(x)$ divides evenly into $f(x)$.

A similar but alternative method, called synthetic division, can be used when the divisor is of the form $(x + a)$ or $(x - a)$. It is a bit of a shortcut because only the coefficients of the polynomial need to be written.

Note: Synthetic division should not be used if the divisor is of the form $(ax + b)$ or $(ax - b)$.

Synthetic Division

Similar to polynomial long division, when doing synthetic division, the terms of the dividend must be arranged in descending order of powers and any missing terms replaced with a zero. You will use only the coefficients of the terms to do the division. The divisor has to be of the form $x \pm a$. If $x - a$ is the divisor, a is positive. If $x + a$ is the divisor, treat it as $x - (-a)$ and use as a negative value.

Example 6

Divide $(6x^3 + 16x - 19x^2 - 4) \div (x - 2)$ using synthetic division. Note that this is the same question as in Example 4.

Solution

1. Set up.

$$\begin{array}{l} \text{represents divisor} \\ (x-2) \end{array} \longrightarrow \begin{array}{r|rrr|r} 2 & 6 & -19 & 16 & -4 \\ & & & & \\ \hline & & & & \end{array} \longleftarrow \begin{array}{l} \text{represents dividend} \\ 6x^3 - 19x^2 + 16x - 4 \end{array}$$

2. Bring down the 6 as first coefficient of quotient.

$$\begin{array}{r|rrr|r} 2 & 6 & -19 & 16 & -4 \\ & \downarrow & & & \\ \hline & 6 & & & \end{array}$$

3. Multiply the divisor coefficient, 2 times the 6, and write 12 in the next column.

$$\begin{array}{r|rrr|r} 2 & 6 & -19 & 16 & -4 \\ & & 12 & & \\ \hline & 6 & & & \end{array}$$

4. Add the values −19 and 12 and write the new coefficient of the quotient.

$$\begin{array}{r|rrr|r} 2 & 6 & -19 & 16 & -4 \\ & & 12 & & \\ \hline & 6 & -7 & & \end{array}$$

5. Repeat process by multiplying the divisor coefficient, 2, times the −7, and write −14 in the next column. Then, add to get the next coefficient of the quotient.

$$\begin{array}{r|rrr|r} 2 & 6 & -19 & 16 & -4 \\ & & 12 & -14 & \\ \hline & 6 & -7 & 2 & \end{array}$$

6. Repeat process one more time. Multiply the divisor coefficient, 2, times the 2 and write 4 in the next column. When you add the last column, the result is the remainder and is separated from the quotient by a vertical line.

$$\begin{array}{l|rrrrl}
 & 6 & -19 & 16 & -4 & \\
\text{this represents the quotient} & & 12 & -14 & & \\ \cline{2-5}
6x^2 - 7x + 2\,x \longrightarrow & 6 & -7 & \multicolumn{1}{r|}{2} & 0 & \longleftarrow \text{this represents the remainder 0}
\end{array}$$

$\therefore\ 6x^3 - 19x^2 + 16x - 4 = (x - 2)(6x^2 - 7x + 2)$

The key steps of synthetic division are as follows:

1. Arrange the coefficients of $f(x)$ in order of descending powers of x (write 0 as the coefficient for each missing power).
2. After writing the divisor in the form $x - a$, use a as your divisor to generate the second and third rows of numbers as follows: Bring down the first coefficient of the dividend and multiply it by a; then add the product to the second coefficient of the dividend. Multiply this sum by a, and add the product to the third coefficient of the dividend. Repeat the process until a product is added to the constant term of $f(x)$.
3. The last number in the third row of numbers is the remainder; the other numbers in the third row are the coefficients of the quotient, which is of degree 1 less than $f(x)$.

Example 7

Use synthetic division to divide $x^4 - 10x^2 - 2x + 4$ by $x + 3$.

Solution

Divisor: $x + 3 = x - (-3)$. Notice the sign change!

Note: Since there is no term with x^3, you must insert a 0 as a placeholder to represent $0x^3$.

$$\begin{array}{lr|rrrrrl}
\text{divisor} \longrightarrow & -3 & 1 & 0 & -10 & -2 & 4 & \longleftarrow \text{dividend coefficients} \\
 & & \downarrow & -3 & +9 & 3 & -3 & \\ \cline{3-7}
\text{quotient coefficients} \longrightarrow & & 1 & -3 & -1 & \multicolumn{1}{r|}{1} & 1 & \longleftarrow \text{remainder}
\end{array}$$

$$\frac{x^4 - 10x^2 - 2x + 4}{x + 3} = x^3 - 3x^2 - 1x + 1 + \frac{1}{x + 3}$$

or

$$x^4 - 10x^2 - 2x + 4 = (x + 3)\left(x^3 - 3x^2 - 1x + 1\right) + 1$$

The remainder obtained in the synthetic division or polynomial long division process has an interesting connection to factoring polynomial functions. You will consider this concept in the next lesson.

You need to practise both long division and synthetic division to get good at both. You'll get some practice in the following learning activity.

Learning Activity 4.2

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. How many millimetres are in one kilometre?
2. Is the point (–1, 5) a solution to the inequality $y \leq x^2 - 5x + 6$?
3. Rationalize the denominator: $\dfrac{1}{5 - \sqrt{8}}$
4. Which is the better buy, 3 candy bars for $1.99 or 5 candy bars for $2.50?
5. In which quadrants is $y = \sin \theta$ positive?
6. Factor: $49x^2 - 36y^4$
7. Evaluate: $36^{0.5}$
8. How many terms are there in the expansion of $(x - 4)^7$?

continued

Learning Activity 4.2 (continued)

Part B: Dividing Polynomials

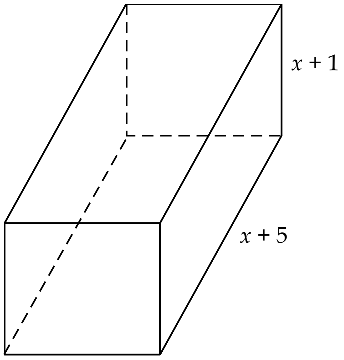
Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Divide using long division and write in the form $f(x) = d(x)q(x) + R$.
   a) $(2x^3 - 3x + 1) \div (x - 2)$
   b) $(2x^2 + x^3 - 3x - 4) \div (x + 2)$
   c) $(6x^3 - 16x^2 + 17x - 6) \div (3x - 2)$
   d) $(2x^3 + 3x^2 - 7 - 4x) \div (x^2 - 2)$
   e) $(x^4 - 2x^3 - 7x^2 + 8x + 12) \div (x + 1)$

2. Use synthetic division to find the quotient and the remainder. Write your answer in the form: $f(x) = d(x)q(x) + R$.
   a) $\left(x^3 - 7x + 6\right) \div (x - 2)$
   b) $\left(3x^4 - x - 4\right) \div (x - 2)$
   c) $\left(x^4 - 2x^3 - 70x + 20\right) \div (x - 5)$
   d) $\left(2x^3 - 5x^2 + 6x + 3\right) \div \left(x - \frac{1}{2}\right)$
   e) $\left(4x^3 + 4x^2 - 7x - 6\right) \div \left(x + \frac{3}{2}\right)$

3. When a polynomial $f(x)$ is divided by $2x + 1$, the quotient is $x^2 - x + 4$ and the remainder is 3. Find $f(x)$.

4. The volume of the following rectangle prism is $V = 3x^3 + 8x^2 - 45x - 50$. Expressions for the length and width are shown. Find an expression for the missing dimension ($V = lwh$).



Lesson Summary

In this lesson, you learned how to divide polynomials by a binomial. This technique is often used to aid in the factoring of polynomials. You will learn how to factor polynomials using this technique in the next lesson.

Notes

Assignment 4.1

Polynomial Functions

Total: 40 marks

You need to show all your steps as you work out your solutions, and make sure your answers are clear (include units, where appropriate). Check the Introduction for the Marking Guidelines that will be used for all assignments and examinations.

1. Identify the polynomial functions in the following set of functions. Justify your answers. *(2 × 1 mark each = 2 marks)*

   a) $y = x^3 + 2x - \sqrt{3}$

   b) $y = \dfrac{1}{x^3} - 4$

2. State the maximum number of turns, as well as the right and left end behaviours for each of the following graphs. *(3 × 2 marks each = 6 marks)*

   a) $f(x) = -2x^3 + 3x^2 - 4x + 1$

continued

b) $g(x) = x^4 - 5x^2 + x - 7$

c) $f(x) = x^5 - 3x^4 - x^3 - 7$

3. Match the polynomial function with the correct graph. *(6 ×0.5 mark each = 3 marks)*

a) $y = \frac{1}{4}x^3 + x^2 - \frac{7}{4}x - \frac{5}{2}$

b) $y = -\frac{1}{4}x^3 + x^2 - \frac{7}{4}x - \frac{5}{2}$

c) $y = x^4 - 3x^3 + 2$

d) $y = -x^4 + 4x^3 - 3x^2 + 1$

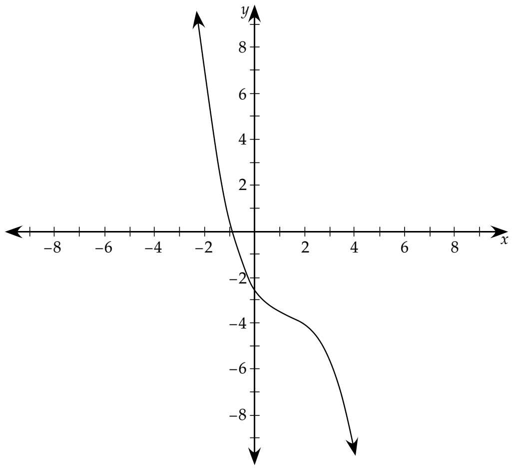
e) $y = x^5 - 2x^4 - 3x^3 + 5x^2 + 4x - 1$

f) $y = -x^5$

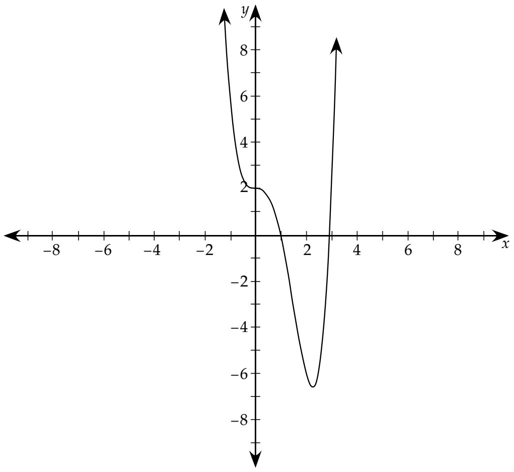
continued

Assignment 4.1: Polynomial Functions (continued)

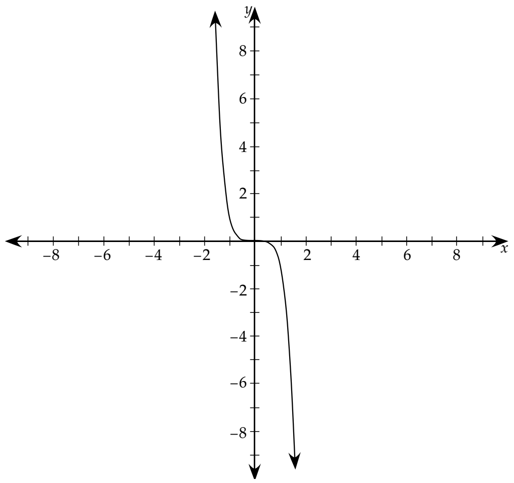
i)



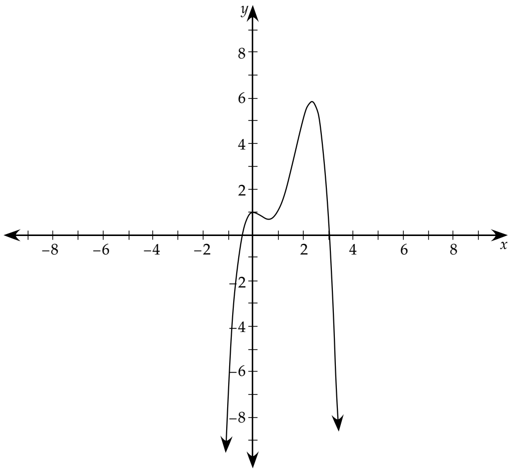
ii)



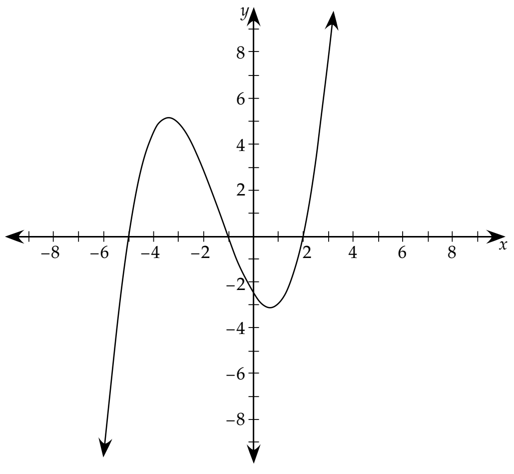
iii)



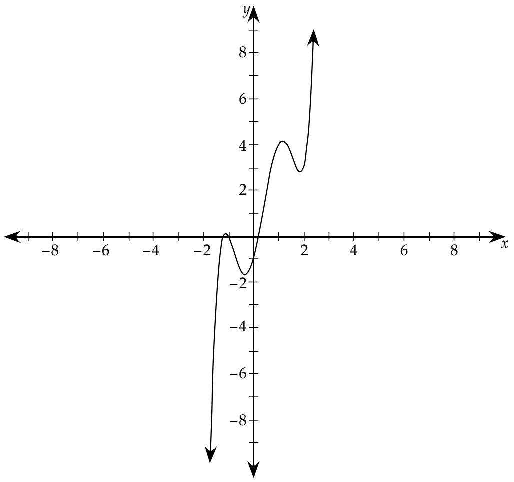
iv)



v)



vi)



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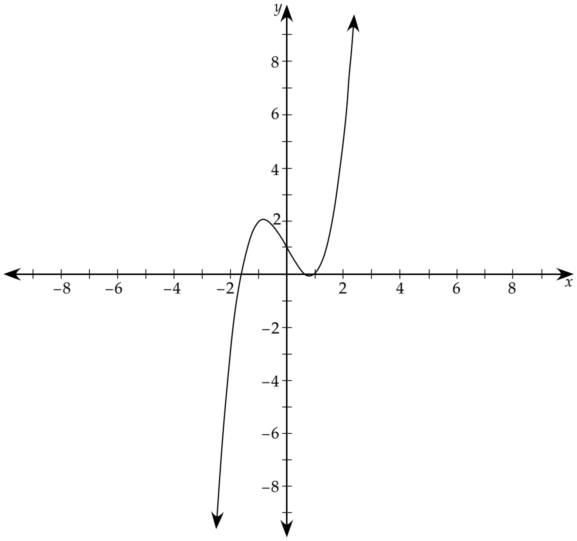
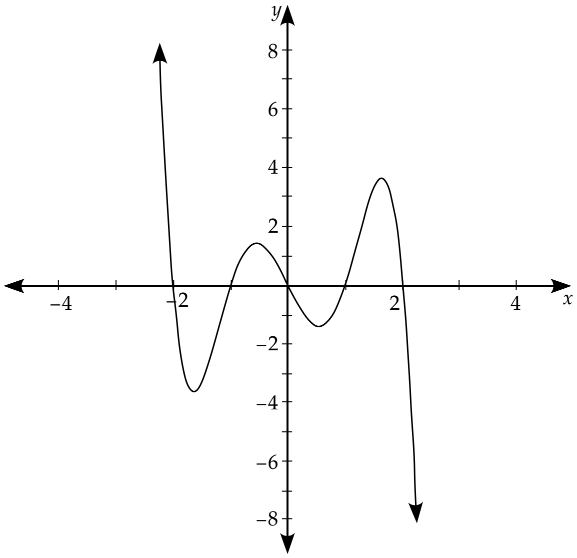
4. Analyze the following graphs to determine the degree and the sign of the leading coefficient of each function. *(6 × 1 mark each = 6 marks)*

a)



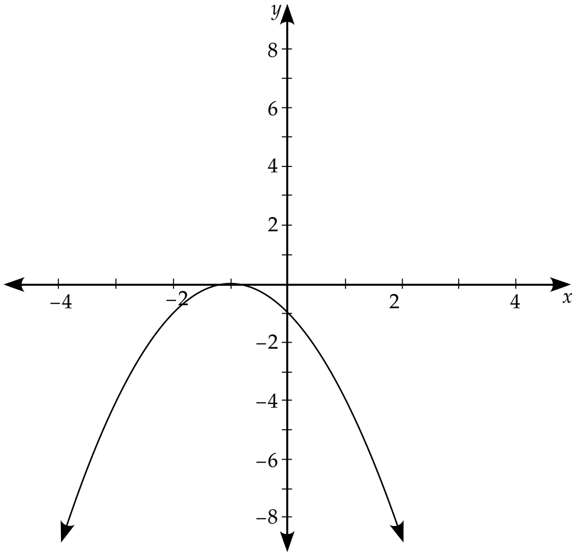
b)



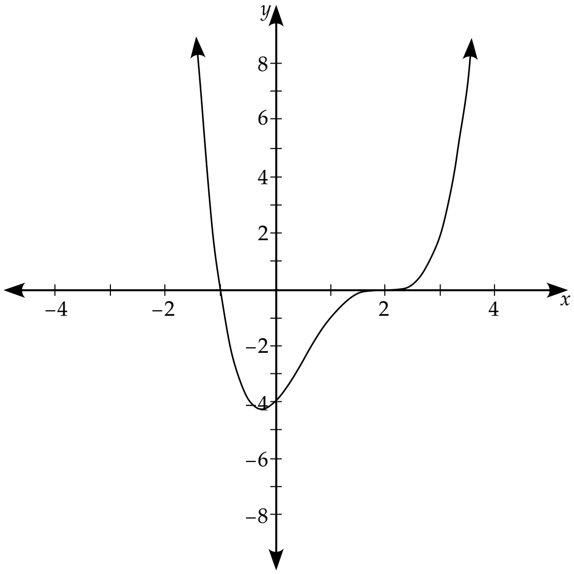
continued

Assignment 4.1: Polynomial Functions (continued)

c)



d)



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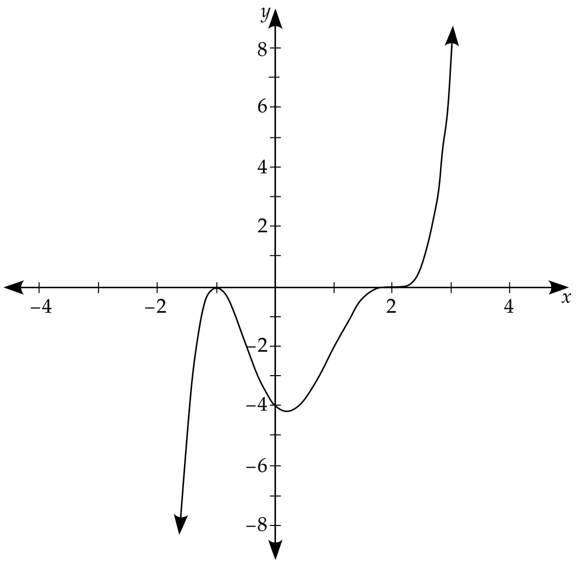
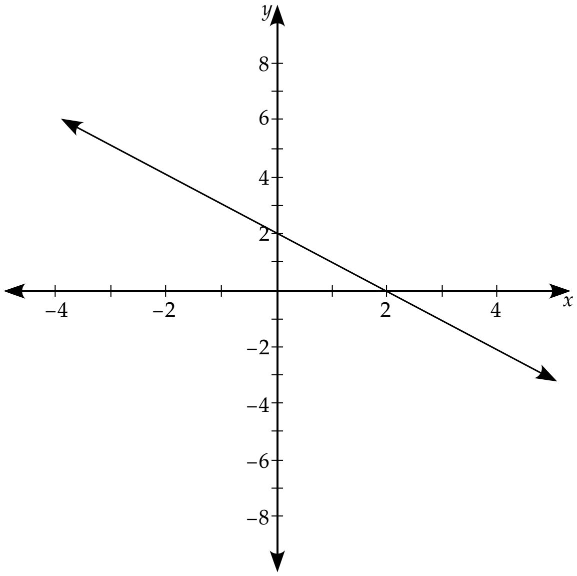
e)



f)



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Assignment 4.1: Polynomial Functions (continued)

5. Divide using long division and write your answer in the form dividend = (divisor)(quotient) + remainder.

   a) $(4x^3 - 2x^2 + 3x - 1) \div (x - 1)$ *(3 marks)*

   b) $(8x^4 - 2x^2 + 2x + 4) \div (2x - 3)$ *(4 marks)*

   c) $-35 - 8x - 22x^2 + 3x^4 - 2x^5 \div (x + 2)$ *(4 marks)*

continued

Assignment 4.1: Polynomial Functions (continued)

6. Use synthetic division to find the quotient and the remainder. Write your answer in the form dividend = (divisor)(quotient) + remainder. *(3 × 4 marks each = 12 marks)*

   a) $(x^3 - x^2 + 2) \div (x + 1)$

   b) $(2x^4 + 4x^3 - x^2 - 49) \div (x - 2)$

   c) $(-x^4 - 15x + 20) \div (x + 3)$

Lesson 3: The Remainder Theorem and the Factor Theorem

Lesson Focus

In this lesson, you will

☐ learn how to apply the Remainder Theorem

☐ learn how to apply the Factor Theorem

Lesson Introduction

In this lesson, you will be introduced to two theorems that will allow you to completely factor a polynomial, assuming the polynomial can be factored.

Factoring Polynomials

In the last example of Lesson 2 you used synthetic division to divide $x^4 - 10x^2 - 2x + 4$ by $x + 3$. The remainder when you did this division was 1. The remainder obtained in the synthetic division process has an interesting connection to polynomial functions. Keep this in mind as you complete the following example.

Example 1

Evaluate $f(x) = x^4 - 10x^2 - 2x + 4$ at $f(-3)$.

Solution

$$\begin{aligned} f(-3) &= (-3)^4 - 10(-3)^2 - 2(-3) + 4 \\ &= 81 - 90 + 6 + 4 \\ &= 1 \end{aligned}$$

Notice $f(-3) = 1$, which is the same as the remainder when you divide $f(x)$ by $x + 3$ (the last example of Lesson 2). This is not a coincidence, as you will see. This relationship is described as the Remainder Theorem.

The Remainder Theorem

The **Remainder Theorem** states that if a polynomial $f(x)$ is divided by $x - a$, then the remainder is $f(a)$.

For dividend, $f(x)$, and divisor, $(x - a)$, you have $f(x) = (x - a)q(x) + R$, where R is the remainder and $q(x)$ is the quotient.

To prove the Remainder Theorem, simply evaluate this function when $x = a$. Find $f(a) = (a - a)q(a) + R$, then simplify to get $f(a) = R$.

The theorem is proved! It says the remainder is $f(a)$ when you divide $f(x)$ by $x - a$.

Use the Remainder Theorem to obtain the value of the remainders in the following example.

Example 2

Use the Remainder Theorem to find the remainder when $(x^4 - 2x^3 + 5x + 2)$ is divided by the following binomials.

a) $x + 1$

b) $x - 1$

c) $x + 2$

d) $x - 2$

Solutions

By the Remainder Theorem, $f(a) = R$.

a) $x + 1$ is rewritten as $x - (-1)$ to determine that $a = -1$

$$\begin{aligned} f(x) &= x^4 - 2x^3 + 5x + 2 \\ f(-1) &= (-1)^4 - 2(-1)^3 + 5(-1) + 2 \\ &= 1 + 2 - 5 + 2 \\ &= 0 \end{aligned}$$

Thus, the remainder when $f(x)$ is divided by $x + 1$ is 0.

You can check this by synthetic or long division.

b) To find the remainder when dividing by $(x - 1)$, evaluate $f(x)$ when $x = 1$.

$$\begin{aligned} f(x) &= x^4 - 2x^3 + 5x + 2 \\ f(1) &= (1)^4 - 2(1)^3 + 5(1) + 2 \\ &= 1 - 2 + 5 + 2 \\ &= 6 \end{aligned}$$

The remainder is 6.

You can check this by synthetic or long division.

c) Evaluate $f(x)$ when $x = -2$.

$$\begin{aligned} f(x) &= x^4 - 2x^3 + 5x + 2 \\ f(-2) &= (-2)^4 - 2(-2)^3 + 5(-2) + 2 \\ &= 16 + 16 - 10 + 2 \\ &= 24 \end{aligned}$$

The remainder is 24.

d) Evaluate $f(x)$ when $x = 2$.

$$\begin{aligned} f(x) &= x^4 - 2x^3 + 5x + 2 \\ f(2) &= (2)^4 - 2(2)^3 + 5(2) + 2 \\ &= 16 - 16 + 10 + 2 \\ &= 12 \end{aligned}$$

The remainder is 12.

When the remainder is 0, such as in (a), what relation must exist between the binomial divisor and the polynomial dividend?

If the remainder is 0, it means that the polynomial is evenly divisible by the binomial. Taken one step further, it means that $x + 1$ must be a factor of $f(x) = x^4 - 2x^3 + 5x + 2$. This idea leads to a corollary of the remainder theorem called the Factor Theorem.

Note: A **corollary** is a name given to a theorem that readily follows another theorem. The corollary statement is self-evident from the original theorem so it doesn't usually require a detailed proof.

The Factor Theorem

The **Factor Theorem** states that a polynomial $f(x)$ has a factor $(x - a)$ if and only if $f(a) = 0$.

This theorem has the statement "if and only if" in it. Therefore, the statement can be made in both directions.

- If $f(a) = 0$, then the polynomial, $f(x)$, has a factor of $(x - a)$.
- If $f(x)$ has a factor of $(x - a)$, then $f(a) = 0$.

Statement 1: If $f(a) = 0$, then the polynomial, $f(x)$, has a factor of $(x - a)$.

By the Remainder Theorem, if $f(a) = 0$ then the remainder is zero when $f(x)$ is divided by $(x - a)$. By definition, that is what it means for $(x - a)$ to be a factor.

Statement 2: If $f(x)$ has a factor of $(x - a)$, then $f(a) = 0$.

Since $(x - a)$ is a factor of $f(x)$, when $f(x)$ is divided by $(x - a)$ the remainder will be 0. By the Remainder Theorem $f(a) = R$ (the remainder). Therefore, $f(a) = 0$.

Example 3

a) Determine whether $x + 2$ is a factor of $f(x) = x^3 - 2x + 4$.

b) Determine the other factors of $f(x)$.

Solutions

a) $x + 2$ will be a factor of $f(x)$ if $f(-2) = 0$, by the Factor Theorem.

$$\begin{aligned} f(x) &= x^3 - 2x + 4 \\ f(-2) &= (-2)^3 - 2(-2) + 4 \\ &= -8 + 4 + 4 \\ &= 0 \\ &= R \end{aligned}$$

Because the remainder is 0, $x + 2$ must be a factor of $f(x)$.

b) To continue factoring, do synthetic division to write $f(x)$ as a product of $(x + 2)$ and another factor.

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -2 & 4 \\ & & -2 & 4 & -4 \\ \hline & 1 & -2 & 2 & \multicolumn{1}{|r}{0} \end{array}$$

The quotient is $x^2 - 2x + 2$.

$\therefore f(x) = (x + 2)(x^2 - 2x + 2)$. These are the only factors of $f(x)$.

If $x^2 - 2x + 2$ was factorable, then you would have listed its factors along with $x + 2$ in your answer.

Finding Factors of Polynomials

	Polynomial	Constant	Leading Coefficient	Factored Form	Zeros
1.	$y = x^3 - 7x + 6$	6	1	$y = (x - 2)(x - 1)(x + 3)$	2, 1, −3
2.	$y = x^3 - 19x - 30$	−30	1	$y = (x - 5)(x + 2)(x + 3)$	5, −2, −3
3.	$y = x^3 - 7x^2 + 16x - 12$	−12	1	$y = (x - 2)(x - 2)(x - 3)$	2, 3
4.	$y = x^3 - 2x^2 - x + 2$	2	1	$y = (x - 2)(x - 1)(x + 1)$	2, 1, −1
5.	$y = 2x^3 - 5x^2 - 4x + 3$	3	2	$y = (2x - 1)(x + 1)(x - 3)$	$\frac{1}{2}$, −1, 3

You will be given polynomials in general form (like the second column) and asked to write them in factored form (like the fifth column), using the Factor Theorem. When you use the Factor Theorem, you will try substituting a variety of values for x to see if the resulting value is zero. You can narrow down the numbers you need to try based on the value of the last term of the polynomial (the constant).

The zeros in the last column are the values of a in the binomial factors $(x - a)$. Notice, in the first 4 examples, that the zeros are always integer factors of the constant term. The zeros, 2, 1, and −3 are all factors of 6. The zeros, 5, −2, and −3 are all factors of −30. This means that if you were asked to find the factors of the polynomial, $x^3 - 6x^2 + 3x + 10$, using the Factor Theorem, the only values you would need to try are the factors of 10. That is, only check whether $f(a) = 0$ for $a = \pm 1, \pm 2, \pm 5, \pm 10$. Looking at the constant term will help you to narrow down the possible integer zeros of a polynomial function.

The last example (5) goes one step further—one of its roots is not an integer. The constant term is 3 so the integer zeros can be found by checking for $f(a) = 0$ when $a = \pm 1, \pm 3$. As you can see in the factored form, that would get you only two of the three factors. Since the leading coefficient in the last example is not 1, it is possible that there are rational roots in addition to the integer roots.

To find rational roots using the Factor Theorem, you need to also test values of a that have any factor of the constant in the numerator and any factor of the leading coefficient in the denominator. Considering the last example, the factors of 3 (±1, ±3) are in the numerator and the factors of 2 (±1, ±2) are in the denominator in all possible combinations. That is, the only possible rational roots that need to be tested are $f(a) = \pm\frac{1}{1}, \pm\frac{3}{1}, \pm\frac{1}{2}, \pm\frac{3}{2}$.

In this course, you will focus on finding integer roots rather than rational roots.

Example 4

Factor completely: $f(x) = x^3 - 2x^2 - 5x + 6$.

Solution

To solve this problem, you first need to list possible a-values using the strategy outlined above. Once you have listed the possible a-values, you can use the Factor Theorem and synthetic division or long division to try the possible a-values as you look for factors.

Note: When the leading coefficient is 1, then the possible a-values for the binomial $(x - a)$ are simply the factors of the constant term.

$f(x) = x^3 - 2x^2 - 5x + 6$

Because the coefficient of x^3 is 1, the possible a-values are the factors of 6.

$$a = \pm 1, \pm 2, \pm 3, \pm 6$$

Try $a = 1$.

Note: It is usually a good idea to check $a = 1$ and $a = -1$ as possible a-values first. Numerically, these are the easiest values to check.

From the Factor Theorem, if $f(a) = 0$, then $x - a$ is a factor.

$$\begin{aligned} f(1) &= 1^3 - 2(1)^2 - 5(1) + 6 \\ &= 1 - 2 - 5 + 6 \\ &= 0 \end{aligned}$$

$\therefore x - 1$ is a factor.

Use synthetic division to get the second factor:

$$\begin{array}{r|rrrr}
1 & 1 & -2 & -5 & 6 \\
 & & 1 & -1 & -6 \\
\hline
 & 1 & -1 & -6 & \multicolumn{1}{|r}{0}
\end{array}$$

Notice the remainder will be zero every time since that is what you found with the Factor Theorem. You can use this fact to check your work.

$\therefore$ Quotient is $x^2 - x - 6$.

$\therefore f(x) = (x - 1)(x^2 - x - 6)$

Notice that the quotient is a quadratic that will factor.

$x^2 - x - 6 = (x - 3)(x + 2)$

$\therefore f(x) = x^3 - 2x^2 - 5x + 6 = (x - 1)(x - 3)(x + 2)$

Example 5

Factor: $2x^3 + 3x^2 - 8x + 3$.

Solution

Because the leading coefficient is 2 and the constant term is 3, the possible a-values are:

$$a = \frac{\text{factors of } 3}{\text{factors of } 2} = \frac{\pm 1, \pm 3}{\pm 1, \pm 2} = \pm\frac{1}{1}, \pm\frac{3}{1}, \pm\frac{1}{2}, \pm\frac{3}{2}$$

Try $a = 1$:

$$\begin{aligned} f(1) &= 2(1^3) + 3(1^2) - 8(1) + 3 \\ &= 2 + 3 - 8 + 3 \\ &= 0 \end{aligned}$$

$\therefore$ $x - 1$ is a factor.

Use synthetic division to get a second factor:

$$\begin{array}{r|rrr|r} 1 & 2 & 3 & -8 & 3 \\ & & 2 & 5 & -3 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

$\therefore$ Quotient is: $2x^2 + 5x - 3$

This will factor: $2x^2 + 5x - 3 = (2x - 1)(x + 3)$

$\therefore$ Factors are: $(x - 1)(2x - 1)(x + 3)$

The Zeros of a Polynomial Function

The factored form of a polynomial will allow you to determine the zeros, or the x-intercepts, of the function. Recall that the zeros of a function occur when the function is set equal to zero.

Example 6

Given $f(x) = 2x^3 + 3x^2 - 8x + 3$, find its zeros.

Solution

From Example 5, you know that the completely factored form of this function is $f(x) = (x - 1)(2x - 1)(x + 3)$.

To find its zeros, let $f(x) = 0$.

$(x - 1)(2x - 1)(x + 3) = 0$

$$\therefore \quad x - 1 = 0 \qquad 2x - 1 = 0 \qquad x + 3 = 0$$

$$x = 1 \qquad x = \frac{1}{2} \qquad x = -3$$

Therefore, this polynomial has three zeros, which are the three x-intercepts at 1, $\frac{1}{2}$, and −3. Finding the zeros of polynomial functions will be useful in the next lesson when you begin to sketch graphs of polynomial functions.

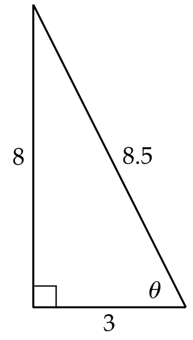
Learning Activity 4.3

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

Use the triangle below to answer Questions 1 to 3.



1. Determine the ratio for sin θ.
2. Determine the ratio for tan θ.
3. Determine the ratio for $\dfrac{1}{\cos\theta}$.
4. In which direction does the parabola $y = -(x + 2)^2$ open?
5. Evaluate: ${}_4P_1$
6. Write as an entire radical: $2x^2y\sqrt[3]{y}$
7. Simplify: $\sqrt{3}\left(\sqrt{3} - \sqrt{2}\right)$
8. Your restaurant bill came to $119.21. If you wish to leave a 20% tip, estimate how much you should leave.

continued

Learning Activity 4.3 (continued)

Part B: Factoring Polynomials

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Determine the remainder when each of the following polynomials are divided by $x + 3$ without using division.
    a) $f(x) = x^3 + 2x^2 - x - 6$
    b) $f(x) = -x^4 - 3x^3 + x + 3$
    c) $f(x) = x^3 - 5$
    d) $f(x)) = x^5 + 2x^4 - 4x^3 - 2x^2 - x + 7$
2. Determine whether $x - 2$ is a factor of each of the following polynomials using the Factor Theorem.
    a) $f(x) = x^3 + 4x^2 - x - 5$
    b) $g(x) = x^3 - x^2 - 8x + 12$
    c) $h(x) = x^4 + 2x^3 - 9x^2 - 2x + 8$
    d) $j(x) = -x^5 + 6x^4 - 2x^3 - x^2 - 7x + 10$.
3. Verify whether or not the following binomials are factors of $f(x) = x^3 + 3x^2 - 4$.
    a) $x + 2$
    b) $x + 1$
    c) $x - 2$
    d) $x - 1$
4. If $P(5) = 0$, what binomial must be a factor of $P(x)$?
5. Factor completely:
    a) $f(x) = x^3 - 8x^2 + 4x + 48$
    b) $f(x) = x^4 - 2x^3 - 17x^2 + 18x + 72$
    c) $f(x) = x^3 - 27$
    d) $f(x) = x^3 - 13x + 12$
    e) $f(x) = x^5 + 5x^4 - 15x^3 - 85x^2 - 26x + 120$ knowing that $(x - 1)$ and $(x + 2)$ are factors.

continued

Learning Activity 4.3 (continued)

6. The polynomial $p(x) = 4x^3 + bx^2 + cx + 11$ has a remainder of -7 when divided by $(x + 2)$ and a remainder of 14 when divided by $x - 1$. Find $p(x)$.
7. For what value of m will the function $f(x) = 3x^3 + mx^2 - 6$ have a remainder of 6 when divided by the factor $x + 2$?
8. Determine the value of k so that the binomial $x + 5$ is a factor of $y = x^3 + kx^2 - x - 30$.
9. The volume of a rectangular swimming pool can be represented by the function $V(d) = d^3 + 73d^2 + 276d$, where d represents depth.
   a) What are the expressions for the possible dimensions of the pool in terms of d?
   b) If the depth of the pool is 6 feet, what are the dimensions of the length and width of the swimming pool?

Lesson Summary

In this lesson, you learned how to apply the Remainder Theorem and the Factor Theorem. Using both of these theorems, you were able to factor polynomials beyond quadratics. It is possible to factor polynomials of higher degrees using these theorems. However, this can be extremely time consuming and you do not need to know how to do that for this course.

In the next lesson, you will be graphing polynomial functions using all of the skills you have learned so far in this module.

LESSON 4: GRAPHING POLYNOMIAL FUNCTIONS

Lesson Focus

In this lesson, you will

☐ learn about the relationship between the factors of a function and the x-intercepts of the corresponding graph

☐ learn how to sketch graphs of polynomial functions

☐ learn how to solve problems by modelling the situation with a polynomial function

Lesson Introduction

Throughout this module, you have learned about polynomial functions up to a degree of 5. You have learned about the properties of polynomial functions and how to completely factor these functions. Using all of this information, you will now be able to sketch graphs of polynomial functions.

Graphing Polynomial Functions

You can use the following information to help you graph polynomial functions.

- the y-intercept
- end behaviour of the graph (left and right)
- the x-intercepts from the roots of the polynomial function
- multiplicity of the roots

The Zeros of a Polynomial Function

The zeros of any function are the x-values that make the function zero. The easiest way to find the zeros of a polynomial function is to first factor the polynomial function completely. Then, determine the values of x that make each of the factors equal to zero. Since you multiply factors, if one of the factors of a polynomial equals zero, the entire polynomial equals zero. **The zeros are the x-intercepts.**

Example 1

Find the x-intercepts and y-intercepts of $f(x) = x^3 - 4x^2 - 12x$.

Solution

Find the y-intercept by substituting $x = 0$ and evaluating $f(0)$.

Using $f(x) = x^3 - 4x^2 - 12x$, then $f(0) = 0^3 - 4(0)^2 - 12(0) = 0$.

Therefore, y-intercept is 0.

Factor $f(x)$:

$f(x) = x(x^2 - 4x - 12)$

$f(x) = x(x - 6)(x + 2)$

Find x-intercepts by substituting $f(x) = 0$ and solving $0 = x(x - 6)(x + 2)$.

Therefore, x-intercepts are 0, 6, and -2.

Example 2

Find the x-intercepts and y-intercepts of $f(x) = x^3 - 7x^2 - 4x + 28$ given that $(x - 2)$ is a factor.

Solution

Factor $f(x)$ by first dividing $f(x)$ by $(x - 2)$. You could use long division but synthetic division is shown here.

$$\begin{array}{r|rrr|r} +2 & 1 & -7 & -4 & 28 \\ & & 2 & -10 & -28 \\ \hline & 1 & -5 & -14 & 0 \end{array}$$

Therefore, $f(x) = (x - 2)(x^2 - 5x - 14)$.

Factor the quadratic, $f(x) = (x - 2)(x - 7)(x + 2)$.

Find x-intercepts by substituting $f(x) = 0$ and solving $0 = (x - 2)(x - 7)(x + 2)$.

Therefore, x-intercepts are 2, 7, and -2.

Find the y-intercept by substituting $x = 0$ and evaluating $f(0)$.

Using $f(x) = x^3 - 7x^2 - 4x + 28$, then $\mathrm{f}(0) = 28$.

Or using the factored form:

$f(x) = (x - 2)(x - 7)(x + 2)$

$f(0) = (-2)(-7)(2) = 28$

Therefore, y-intercept is 28.

Finding the x-intercepts and y-intercepts are required when sketching the graph of a polynomial function. If you were sketching the function, you would also want to know the end behaviour. Since this is a polynomial of odd degree (cubic has degree 3) with a positive leading coefficient (the coefficient of x^3 is 1), then the end behaviour will be just like the line $y = x$. That is, it goes down to the left into Quadrant III and up to the right into Quadrant I.

Example 3

Sketch the graph of $f(x) = x^4 - x^3 - 5x^2 - 3x$ given that $f(-1) = 0$.

Solution

Since $f(-1) = 0$, then according to the Factor Theorem $(x + 1)$ is a factor of $f(x)$.

Factor $f(x)$ by first dividing $f(x)$ by $(x + 1)$. You could use long division but synthetic division is shown here.

$$\begin{array}{r|rrrr|r} -1 & 1 & -1 & -5 & -3 & 0 \\ & & -1 & 2 & 3 & 0 \\ \hline & 1 & -2 & -3 & 0 & 0 \end{array}$$

Therefore, $f(x) = (x + 1)(x^3 - 2x^2 - 3x)$

Factor the cubic, $f(x) = (x + 1)(x)(x^2 - 2x - 3)$

$f(x) = (x + 1)(x)(x - 3)(x + 1)$

Find x-intercepts by substituting $f(x) = 0$ and solving $0 = (x)(x - 3)(x + 1)^2$.

Therefore, x-intercepts are 0, 3, and -1. Notice -1 is a double root.

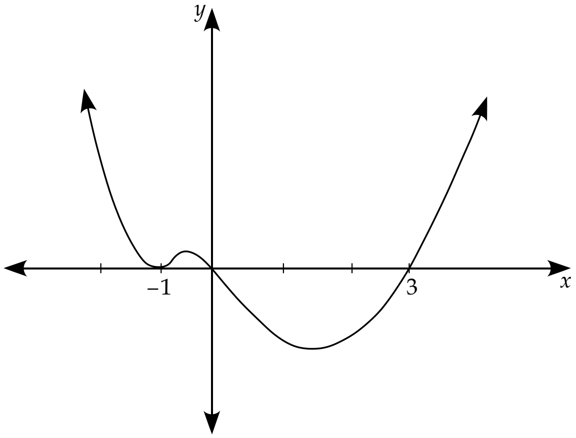
Find the y-intercept by substituting $x = 0$ and evaluating $f(0)$.

Using $f(x) = x^4 - x^3 - 5x^2 - 3x$, then $f(0) = 0$.

Therefore, y-intercept is 0.

The end behaviour needs to be considered when sketching the graph. The leading coefficient of $f(x) = x^4 - x^3 - 5x^2 - 3x$ is positive and a fourth degree polynomial is an even degree. So, the end behaviour is like the parabola $y = x^2$. That is, it goes up to the left into Quadrant II and goes up to the right into Quadrant I.

Be sure to draw the flattening and tangent behaviour of the curve as it crosses the x-axis at the double root where x is -1.



Note: You are not expected to plot the locations of the relative maxima or minima. When you take calculus, you will learn an algebraic way to find those points. Until then, be content to show the x- and y-intercepts, the end behaviour, and the behaviour of the curve near the x-axis.

Example 4

Graph the following polynomial:

$$f(x) = x^3 + 6x^2 + 5x - 12$$

Solution

Before you begin sketching the graph of the polynomial, you need to determine the following characteristics of the function:

- the end behaviour (to the left side and the right side)
- the y-intercept
- the x-intercept(s) (including existence of multiple roots)

End behaviour: As the degree of this polynomial is odd and the leading coefficient is positive, the end behaviour is like the line $y = x$. That is, it goes down as you move to the left and goes up as you move to the right.

y-intercept: The y-intercept of this function occurs at $y = -12$.

x-intercepts: To determine the x-intercepts of the function, you first need to factor the function.

Possible integer roots: ±1, ±2 , ±3, ±4, ±6, ±12

Use the Factor Theorem and test $a = 1$:

$$\begin{aligned} f(1) &= 1^3 + 6(1)^2 + 5(1) - 12 \\ &= 1 + 6 + 5 - 12 \\ &= 0 \end{aligned}$$

$\therefore x - 1$ is a factor of $f(x)$.

Use synthetic division to determine the quotient.

$$\begin{array}{r|rrrr} 1 & 1 & 6 & 5 & -12 \\ & & 1 & 7 & 12 \\ \hline & 1 & 7 & 12 & \multicolumn{1}{|r}{0} \end{array}$$

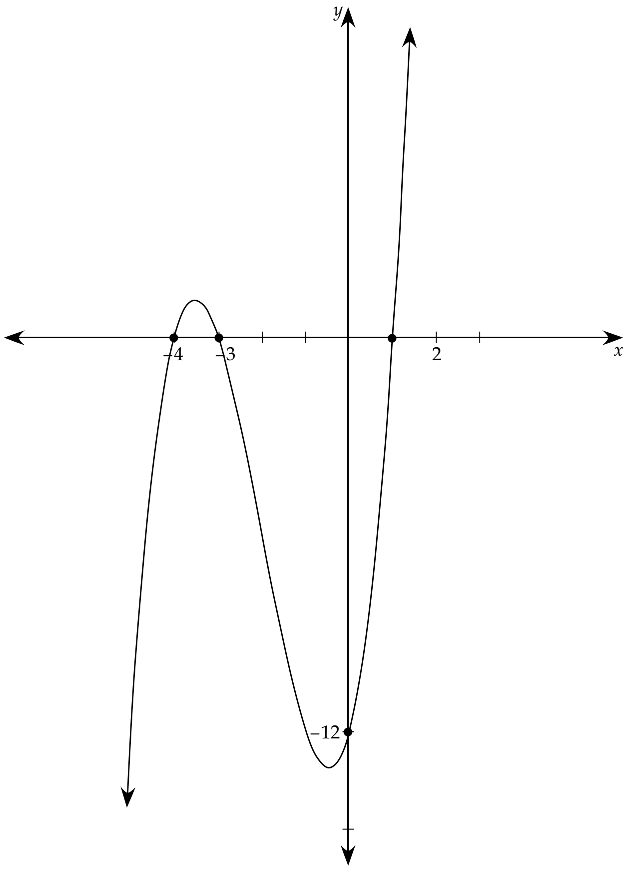
$\therefore f(x) = (x - 1)(x^2 + 7x + 12)$

Factor the quadratic quotient.

$f(x) = (x - 1)(x + 3)(x + 4)$

The x-intercepts of $f(x)$ are thus $x = 1, -3$, and -4.

Using all of the above information, sketch $f(x)$.



Example 5

Graph the following polynomial:

$$g(x) = \left(\frac{1}{20}\right)(x-1)(x-1)(x+5)(x+2)$$

Solution

End behaviour: As the degree of this polynomial is even (fourth degree) and the leading coefficient is positive, the end behaviour is like a parabola. In Quadrant II, the graph rises as you move left and in Quadrant I, the graph rises as you move right.

y-intercept: To determine the y-intercept, substitute $x = 0$ into the factors.

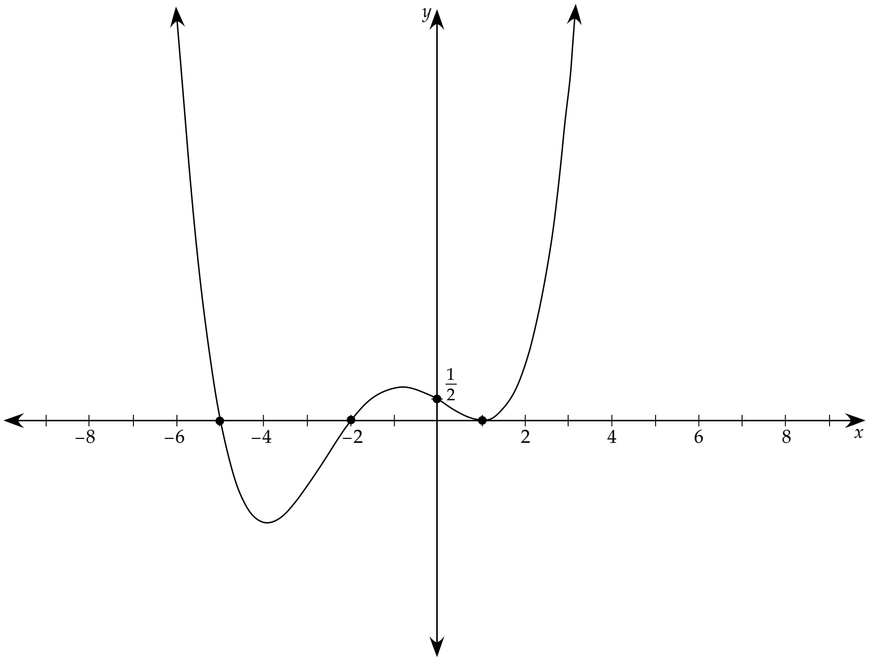
$$y = \left(\frac{1}{20}\right)(0-1)(0-1)(0+5)(0+2)$$

$$y = \left(\frac{1}{20}\right)(-1)(-1)(5)(2)$$

$$y = \frac{1}{2}$$

x-intercepts: The x-intercepts of $g(x)$ occur when the factors of $g(x)$ equal zero and are thus $x = 1$, -2, and -5. There is a double root at $x = 1$. Therefore, the graph of the function flattens and, since the multiplicity is even, it is tangent to the x-axis at this point.

Using all of the above information, sketch $g(x)$.



Example 6

Sketch the graph of $f(x) = -x^5 + 4x^4 - x^3 - 10x^2 + 4x + 8$, knowing $(x - 2)$ is a factor and $(x + 1)$ is a factor.

Solution

Since $(x - 2)$ is a factor of $f(x)$, factor $f(x)$ by first dividing $f(x)$ by $(x - 2)$. You could use long division but synthetic division is shown here.

$$\begin{array}{r|rrrrrr} +2 & -1 & 4 & -1 & -10 & 4 & 8 \\ & & -2 & 4 & 6 & -8 & -8 \\ \hline & -1 & 2 & 3 & -4 & -4 & \big|\; 0 \end{array}$$

Therefore, $f(x) = (x - 2)(-x^4 + 2x^3 + 3x^2 - 4x - 4)$.

Continue factoring $f(x)$ by dividing the remaining quartic by $(x + 1)$.

$$\begin{array}{r|rrrrr} -1 & -1 & 2 & 3 & -4 & -4 \\ & & 1 & -3 & 0 & 4 \\ \hline & -1 & 3 & 0 & -4 & \big|\; 0 \end{array}$$

Therefore, $f(x) = (x - 2)(x + 1)(-x^3 + 3x^2 + 0x - 4)$.

You need to factor the remaining cubic. You could factor out −1, since its leading coefficient is negative. The result is $f(x) = -(x - 2)(x + 1)(x^3 - 3x^2 + 4)$. To continue factoring the cubic, you need to consider possible a-values for a binomial $(x - a)$ to use with the Factor Theorem on the polynomial, $p(x) = x^3 - 3x^2 + 4$. The possible a-values are the factors of 4: ±1, ±2, ±4.

Try $a = 1$:

$p(1) = 1^3 - 3(1)^2 + 4$

$p(1) = 1 - 3 + 4$ Since the remainder is not zero, $(x - 1)$ is not a factor.

$p(1) \neq 0$

Try $a = -1$:

$p(-1) = (-1)^3 - 3(-1)^2 + 4$

$p(-1) = -1 - 3 + 4$ Since the remainder is zero, $(x + 1)$ is a factor.

$p(-1) = 0$

Continue factoring $f(x)$ by dividing the cubic polynomial, $p(x)$, by $(x + 1)$.

$$\begin{array}{r|rrrr} -1 & -1 & -3 & 0 & 4 \\ & & -1 & 4 & -4 \\ \hline & 1 & -4 & 4 & \multicolumn{1}{|r}{0} \end{array}$$

Therefore, $f(x) = -(x - 2)(x + 1)(x + 1)(x^2 - 4x + 4)$.

Finally, factor the remaining quadratic:

$f(x) = -(x - 2)(x + 1)(x + 1)(x^2 - 4x + 4)$
$f(x) = -(x - 2)(x + 1)(x + 1)(x - 2)(x - 2)$
$f(x) = -(x - 2)^3(x + 1)^2$

Find x-intercepts by substituting $f(x) = 0$ and solving $0 = -(x - 2)^3(x + 1)^2$.

Therefore, x-intercepts are 2 and -1.

Notice 2 is a triple root and -1 is a double root.

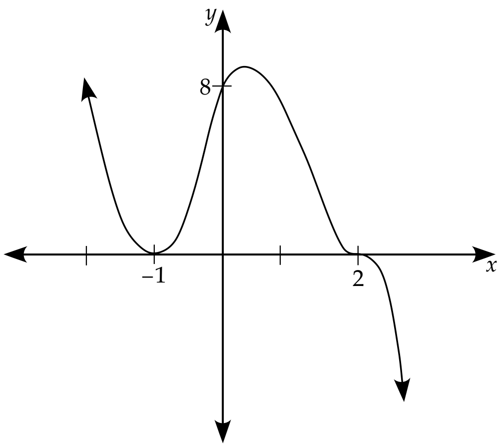
Find the y-intercept by substituting $x = 0$ and evaluating $f(0)$.

Using $f(x) = -x^5 + 4x^4 - x^3 - 10x^2 + 4x + 8$, then $f(0) = 8$.

Therefore, y-intercept is 8.

The end behaviour needs to be considered when sketching the graph. The leading coefficient of $f(x) = -x^5 + 4x^4 - x^3 - 10x^2 + 4x + 8$ is negative and a fifth degree polynomial is an odd degree. So, the end behaviour is like the line $y = -x$. That is, it goes up to the left into Quadrant II and goes down to the right into Quadrant IV.

Be sure to draw the flattening behaviour of the curve as it crosses the x-axis at the triple root where x is 2. Also draw the flattening behaviour of the curve and draw the curve tangent to the x-axis at the double root where x is -1.



Note: Remember, you are not expected to plot the locations of the relative maxima or minima. When you take calculus, you will learn an algebraic way to find those points. Until then, be content to show the x- and y-intercepts, the end behaviour, and the behaviour of the curve near the x-axis.

Applications of Polynomial Functions

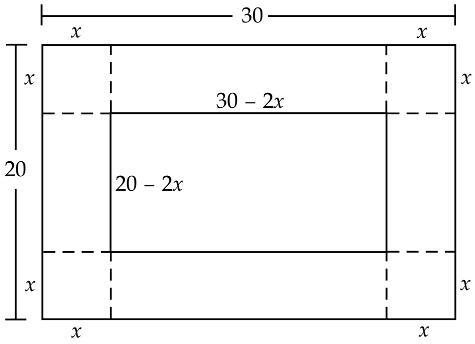
Polynomials can be applied to everyday problems in much the same way, as quadratic functions can be applied to everyday problems.

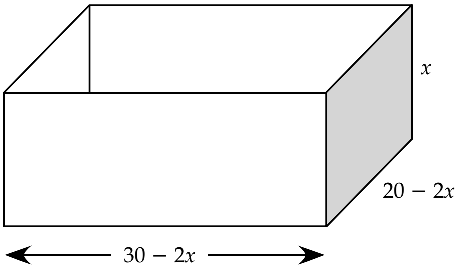
To solve a polynomial function application, you can use the *data collection approach* or the *algebraic approach*.

Consider the following example.

Example 7

A variety of open boxes are made from a 20 cm by 30 cm rectangular flat sheet of material by cutting equal-sized squares from each corner of the sheet and folding up the four sides. Estimate the size of the squares to produce a box of maximum volume.





Each cut-out square has sides that are x cm long.

Solution

Let x cm = side of square to be cut out

$(20 - 2x)$ cm = width of box

$(30 - 2x)$ cm = length of box

Let V cm^3 be the volume

Volume = lwh

$V = (30 - 2x)(20 - 2x)x$

$V = 4x^3 - 100x^2 + 600x$

Graph the function by analyzing important properties of the graph including end behaviour, x-intercepts, and the y-intercept.

This is a third degree polynomial with a positive leading coefficient. Therefore, the polynomial will approach negative infinity as x gets very small and positive infinity as x gets very large.

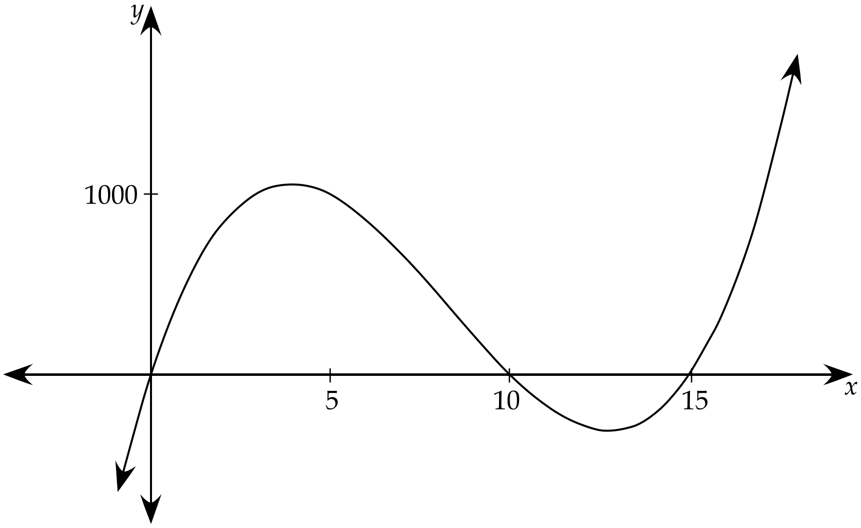
This polynomial is already in factored form and the x-intercepts can be easily found.

In this example, the x-intercepts are $x = 0$, $x = 10$, and $x = 15$. If any one of the dimensions of the box is 0, the volume is 0.

$x = 0$ makes the height zero.
$x = 10$ makes the width zero.
$x = 15$ makes the length zero.

The y-intercept is also 0.

Sketch a graph of the function.



From the graph above, you know that the function representing the volume will be negative between the values of 10 and 15. However, the function cannot be negative in this example. Therefore, we want to restrict the domain of x to be between 0 and 10.

It is difficult to determine what value of x will produce the maximum value simply from a sketch. Therefore, creating a table of values may help you estimate an answer. If you are graphing this function using technology, you can simply find the relative maximum of the function.

x	**Volume**
1	$(30 - 2)(20 - 2)(1) = (28)(18)(1) = 504$
2	$(30 - 4)(20 - 4)(2) = (26)(16)(2) = 832$
3	$(30 - 6)(20 - 6)(3) = (24)(14)(3) = 1008$
4	$(30 - 8)(20 - 8)(4) = (22)(12)(4) = 1056$
5	$(30 - 10)(20 - 10)(5) = (20)(10)(5) = 1000$

According to the table shown, the maximum value of this box will occur near an x-value of 4. This means that by cutting out a square with a side of length 4 cm from each corner, a maximum value can be achieved. Note that this answer is approximate. If you had technology, you could determine a more precise answer of $x = 3.92$.

An estimate of the maximum volume could be calculated by substituting $x = 4$ into the equation.

$V = (30 - 2(4))(20 - 2(4))4$
$V = 22 \times 12 \times 4$
$V = 1056$ is an estimate of the maximum volume when the squares are 4 cm on a side

Learning Activity 4.4

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. If $5^{18} + 5^{18} + 5^{18} + 5^{18} + 5^{18} = 5^x$, find the value of x.
2. Which is the better deal, purchasing a 6-pack of movie tickets for $61.99 or purchasing an 8-pack of movie tickets for $79?
3. Reduce to lowest terms: $\frac{44}{121}$
4. Does the following set of points represent a function?

   (1, 3), (3, 3), (4, 6), (−1, 4)
5. What is the length of the remaining leg of a right-angled triangle if the hypotenuse measures 13 m and one leg measures 12 m?
6. Determine the vertex of the function $f(x) = 2(x + 6)^2 - 1$.
7. Simplify: $\sqrt{128}$
8. Factor: $x^2 - 21x - 72$

continued

Learning Activity 4.4 (continued)

Part B: Polynomial Functions

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. For each of the following functions, find (i) the zeros, (ii) the y-intercept, (iii) left-right end behaviour, and (iv) the sketch of the graph.
   a) $f(x) = x^3 - x^2 - 6x$
   b) $f(x) = -x^3 + 4x$
   c) $f(x) = x^4 - 2x^3 - 3x^2 + 4x + 4$
   d) $f(x) = -x^5$
   e) $f(x) = (x - 4)^3$
   f) $f(x) = x^3 + 5x^2 + 2x - 8$
2. Determine the equation of each of the following polynomials using the given characteristics. Sketch a graph of each function.
   a) A quartic function with two zeros (each with a multiplicity of 2) at -2 and 3, and a y-intercept at -36.
   b) A cubic function with a triple root at 4 that passes through the point (3, 4).
3. From a 9 inch by 12 inch piece of material, small squares are to be cut from two corners so that three edges can be bent up and fastened to manufacture a scoop (a rectangular box with no top and no front). Using graphing technology, find the approximate length of the sides of the small squares that are cut out to give the greatest volume. Find an estimate of the maximum volume.

Lesson Summary

In this lesson, you learned how to graph polynomial functions. To do this, you had to use your knowledge of factoring polynomials and all the characteristics of polynomials that you learned in Lesson 1. Congratulations for finishing the last lesson in Module 4 of this course!

Assignment 4.2

Factoring and Graphing Polynomials

Total: 42 marks

You need to show all your steps as you work out your solutions, and make sure your answers are clear (include units, where appropriate). Check the Introduction for the Marking Guidelines that will be used for all assignments and examinations.

1. Use the Remainder Theorem to determine the remainder when each of the following polynomials are divided by $x - 6$. *(2 × 1 mark each = 2 marks)*
   a) $f(x) = x^3 - 2x^2 - 21x - 18$

   b) $g(x) = x^4 - 8x^3 + 5x^2 + 18x + 114$

2. Use the Factor Theorem to determine whether $x + 4$ is a factor of each of the following polynomials. *(2 × 2 marks each = 4 marks)*
   a) $f(x) = x^3 + 10x^2 + 32x + 32$

   b) $g(x) = x^5 + 7x^4 - 17x^3 - 119x^2 + 16x + 108$

continued

Assignment 4.2: Factoring and Graphing Polynomials (continued)

3. Factor completely:

   a) $f(x) = x^5 + 5x^4 - 21x^3 - 137x^2 - 88x + 240$, knowing that $f(5) = 0$ and $f(-3) = 0$. *(5 marks)*

continued

b) $f(x) = x^3 + 8x^2 + 5x - 50$ *(3 marks)*

continued

c) $f(x) = -x^4 - 4x^3 + 19x^2 + 46x - 120$, knowing that $(x + 5)$ and $(x - 2)$ are factors. *(5 marks)*

continued

Assignment 4.2: Factoring and Graphing Polynomials (continued)

4. a) For the function $f(x) = x^3 - 9x^2 + 23x - 15$, find:

   i) the zeros *(3 marks)*

   ii) the y-intercept *(1 mark)*

   iii) left-right end behaviour *(1 mark)*

   iv) the sketch of the graph *(2 marks)*

continued

Assignment 4.2: Factoring and Graphing Polynomials (continued)

b) For the function $f(x) = (x + 3)^2(2x + 1)(x - 1)$, find:

   i) the zeros *(2 marks)*

   ii) the y-intercept *(1 mark)*

   iii) left-right end behaviour *(1 mark)*

   iv) the sketch of the graph *(2 marks)*

continued

Assignment 4.2: Factoring and Graphing Polynomials (continued)

c) For the function $f(x) = -(x^2 - 6x + 9)(x^2 - x - 6)$, find:

i) the zeros *(2 marks)*

ii) the y-intercept *(1 mark)*

iii) left-right end behaviour *(1 mark)*

iv) the sketch of the graph *(2 marks)*

continued

Assignment 4.2: Factoring and Graphing Polynomials (continued)

5. Three consecutive integers multiply together to give −120.

   a) Determine a polynomial equation to represent this situation. *(1 mark)*

   b) Using your equation from part (a), determine the three consecutive integers that multiply together to give −120. *(3 marks)*

Module 4 Summary

In this module, you learned more about cubic, quartic, and quintic polynomial functions. You learned about the characteristics of each of these functions, as well as techniques you can use to help graph these functions. In order to determine the zeros of each of these functions, you needed to first learn how to divide polynomials by a binomial using long division or synthetic division. Then, using the factor theorem and the remainder theorem, you were able to completely factor polynomial functions.

After you write your midterm examination, you will start the second half of the course. In Module 5, you will be learning about trigonometric functions and the unit circle.

Submitting Your Assignments

It is now time for you to submit Assignments 4.1 and 4.2 to the Distance Learning Unit so that you can receive some feedback on how you are doing in this course. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 4 assignments and organize your material in the following order:

☐ Module 4 Cover Sheet (found at the end of the course Introduction)
☐ Assignment 4.1: Polynomial Functions
☐ Assignment 4.2: Factoring and Graphing Polynomials

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Midterm Examination

Congratulations, you have finished Module 4 in the course. The midterm examination is out of 100 marks and worth 20% of your final mark. In order to do well on this examination, you should review all of your learning activities and assignments from Modules 1 to 4.

You will complete this examination while being supervised by a proctor. You should already have made arrangements to have the examination sent to the proctor from the Distance Learning Unit. If you have not yet made arrangements to write it, then do so now. The instructions for doing so are provided in the Introduction to this module.

You will need to bring the following items to the examination: pens/pencils (2 or 3 of each), blank paper, a ruler, a scientific calculator, and your Examination Resource Sheet. A maximum of 3 hours is available to complete your midterm examination. When you have completed it, the proctor will then forward it for assessment. Good luck!

At this point you will also have to combine your resource sheets from Modules 1 to 4 onto one 8½″ × 11″ paper (you may use both sides). Be sure you have all the formulas, definitions, and strategies that you think you will need. This paper can be brought into the examination with you. We suggest that you divide your paper into two quadrants on each side so that each quadrant contains information from one module.

Examination Review

You are now ready to begin preparing for your midterm examination. Please review the content, learning activities, and assignments from Modules 1 to 4.

The midterm practice examination is also an excellent study aid for reviewing Modules 1 to 4.

You will learn what types of questions will appear on the examination and what material will be assessed. Remember, your mark on the midterm examination determines 20% of your final mark in this course and you will have 3 hours to complete the examination.

Midterm Practice Examination and Answer Key

To help you succeed in your examination, a practice examination can be found in the learning management system (LMS). The midterm practice examination is very similar to the actual examination that you will be writing. The answer key is also included so that, when you have finished writing the practice examination, you can check your answers. This will give you the confidence that you need to do well on your examination. If you do not have access to the Internet, contact the Distance Learning Unit at 1-800-465-9915 to get a copy of the practice examination and the answer key.

To get the most out of your midterm practice examination, follow these steps:

1. Study for the midterm practice examination as if it were an actual examination.
2. Review those learning activities and assignments from Modules 1 to 4 that you found the most challenging. Reread those lessons carefully and learn the concepts.
3. Contact your learning partner and your tutor/marker if you need help.
4. Review your lessons from Modules 1 to 4, including all of your notes, learning activities, and assignments.
5. Use your module resource sheets to make a draft of your Midterm Examination Resource Sheet. You can use both sides of an 8½″ by 11″ piece of paper.
6. Bring the following to the midterm practice examination: pens/pencils (2 or 3 of each), blank paper, a ruler, a scientific calculator, and your Midterm Examination Resource Sheet.
7. Write your midterm practice examination as if it were an actual examination. In other words, write the entire examination in one sitting, and don’t check your answers until you have completed the entire examination. Remember that the time allowed for writing the midterm examination is 3 hours.
8. Once you have completed the entire practice examination, check your answers against the answer key. Review the questions that you got wrong. For each of those questions, you will need to go back into the course and learn the things that you have missed.
9. Go over your resource sheet. Was anything missing or is there anything that you didn’t need to have on it? Make adjustments to your Midterm Examination Resource Sheet. Once you are happy with it, make a photocopy that you can keep.

Notes

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Module 4
Polynomials

Learning Activity Answer Keys

MODULE 4:
POLYNOMIALS

Learning Activity 4.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

For Questions 1 to 3, decide whether the following situations involve a permutation or a combination.

1. Selecting three students to attend a play.
2. Picking a captain and an assistant captain of a hockey team.
3. Ordering five books on a shelf.
4. What is the reciprocal of $\frac{1}{6x^2}$?
5. Does the following set of points represent a function?

   $(1, 1),(2, 4),(1, 5),(4, 2)$
6. Simplify: $\frac{y^{-4}}{y^3}$
7. How much money will you make, before taxes, if you work 40 hours a week for $14 an hour?
8. If $f(x) = -\frac{x}{x^3+1}$, evaluate $f(x)$ at $x = 2$.

Answers:

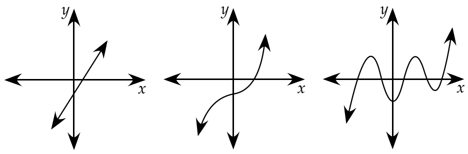
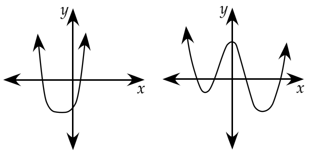
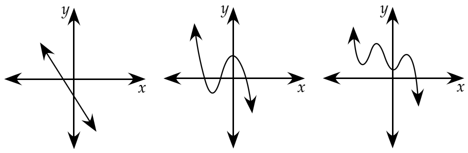
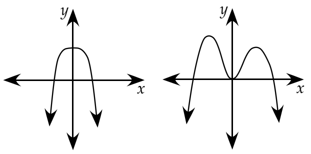
1. Combination (order doesn't matter)
2. Permutation (order makes a difference)
3. Permutation (order makes a difference)
4. $6x^2$
5. No (the input value "1" corresponds to more than one output value)
6. $\frac{1}{y^7}\ \left(y^{-4-3} \rightarrow y^{-7}\right)$
7. $560 ($10 \times 14 = 140 \rightarrow 4 \times 140 = 4 \times 100 + 4 \times 40 = 40 + 160$)
8. $f(2) = -\frac{2}{9}$

Part B: Polynomial Functions

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Throughout this lesson, you have been presented with a lot of information regarding the characteristics of polynomial functions. Complete the following chart with the information you have learned about polynomials. You may wish to also include a similar chart on your resource sheet.

   Answer:

Polynomials		
	Odd Degree	**Even Degree**
Positive Leading Coefficient	Affects end-behaviour 	Affects end-behaviour 
Negative Leading Coefficient	Affects end-behaviour 	Affects end-behaviour 
***y*-intercept**	The y-intercept is the constant in the polynomial function.	The y-intercept is the constant in the polynomial function.
Number of *x*-intercepts	1 to n x-intercepts (where n is the degree of the polynomial)	0 to n x-intercepts (where n is the degree of the polynomial)
Number of Turns	$n - 1$ turns (where n is the degree of the polynomial)	$n - 1$ turns (where n is the degree of the polynomial)
Domain	$(-\infty, \infty)$	$(-\infty, \infty)$
Range	$(-\infty, \infty)$	Depends on the maximum or minimum value of the function
Maximum or Minimum	No absolute maximum or minimum Odd degree polynomials with a degree ≥ 3 may have a relative maximum and/or minimum	Has an absolute maximum or a minimum May also have a relative maximum or minimum

2. Identify the polynomial functions in the following set of functions. Justify your answers.

   a) $y = \left| x^2 \right| - 3$

      Answer:

      This is an absolute value function, not a polynomial function.

   b) $y = 3 - x^5$

      Answer:

      This is a polynomial function with two terms, having a degree of 5 and a leading coefficient of −1.

   c) $y = \dfrac{x^2}{x^3 - 8}$

      Answer:

      This is a rational function, not a polynomial function.

   d) $y = 4^x - 1$

      Answer:

      This is an exponential function, not a polynomial function. Polynomial functions do not contain variables in the exponents.

   e) $y = 1$

      Answer:

      This is a constant function but is a special case of a polynomial function with one term containing a variable with a non-negative integral exponent (x^0).

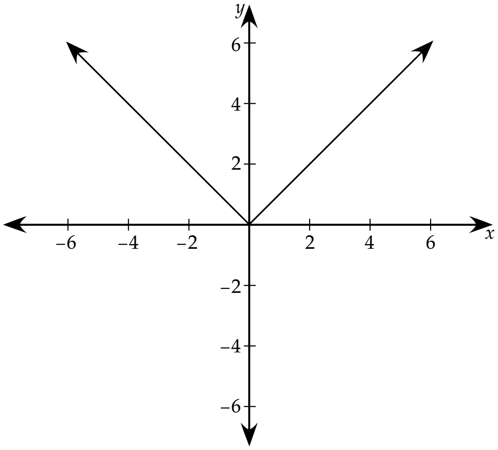
3. Explain what is meant by a continuous graph.

   Answer:

   A continuous graph is a graph that you can sketch without raising your pencil from the paper.

4. Graph $y = | x |$. Name a feature of the graph of $f(x) = | x |$ that is not shared by the graphs of polynomial functions.

   Answer:

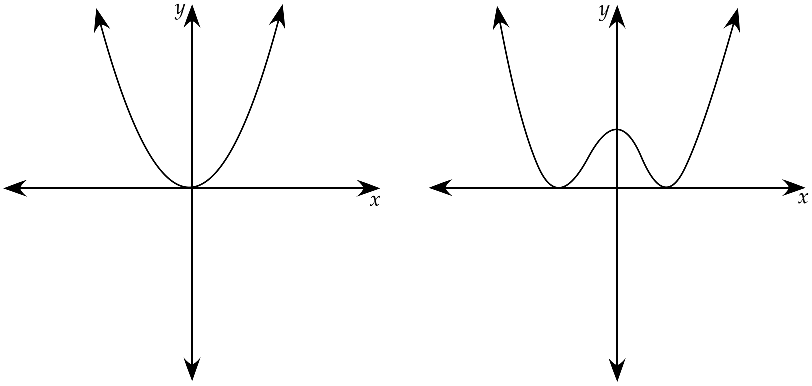


   The graph of $f(x) = | x |$ is a V-shaped curve, so the turn is sharp and not smooth. A sharp turn is called a cusp. Polynomial functions have smooth turns and do not have cusps.

5. Does the graph of $f(x) = 2x^4 - 3x$ rise or fall to the right? How can you tell? What happens to the left?

   Answer:

   Since the degree is an even number (4) and the leading coefficient (2) is positive, the polynomial behaves just like a parabola. It rises to the right and it rises to the left. Two basic shapes for a quartic are shown below, both of which rise to the left and right when the leading coefficient is positive.



6. State the maximum number of turns in the following graphs:
   a) $f(x) = x^3 - 4x$
   b) $g(x) = x^6 - 4x^2$
   c) $f(x) = -x^2 - 5x + 6$
   d) $g(x) = x^5 - 4x^3 + 6$
   e) $f(x) = -3x^4 - 5x + 6$

   Answer:

   The maximum number of turns is $(n - 1)$ where n is the degree of the polynomial function.
   a) Because the degree, n, is 3, then there are $(3 - 1)$ or 2 turns possible.
   b) Because $n = 6$, there are $6 - 1$ or 5 turns possible.
   c) Because $n = 2$, there is $2 - 1$ or 1 turn possible.
   d) Because $n = 5$, there are $5 - 1$ or 4 turns possible.
   e) Because $n = 4$, there are $4 - 1$ or 3 turns possible.

7. Determine the right and left end behaviours of the graphs of the following functions by examining the sign of the leading coefficient and the degree of the polynomial.
   a) $f(x) = -x^3 + 3x$

      Answer:

      Because the degree is odd and the leading coefficient is negative, the graph rises to the left and falls to the right just like the line $y = x$.

   b) $f(x) = 2x^4 - 5x^2 + 4$

      Answer:

      Because the degree is even and the leading coefficient is positive, the graph rises to the left and right just like the line $y = x^2$.

   c) $f(x) = (x - 1)(x + 3)(x - 1)$

      Answer:

      You can determine that the degree is 3 if you multiply the three factors together. Because the degree is odd and the leading coefficient will be positive 1, the graph rises to the right but falls to the left just like the line $y = x$.

d) $f(x) = -x^4 + x^2$

Answer:

Because the degree is even and the leading coefficient is negative, the graph falls to the left and right just like the line $y = x^2$.

e) $f(x) = -2x^5 + x^4 - 2x$

Answer:

Because the degree is odd and the leading coefficient is negative, the graph falls to the right and rises to the left just like the line $y = x$.

f) $f(x) = 3x^5 + x^3 - 2$

Answer:

Because the degree is odd and the leading coefficient is positive, the graph rises to the right and falls to the left just like the line $y = x$.

8. Match the polynomial function with the correct graph:

a) $f(x) = -3x + 5$

b) $f(x) = x^2 - 2x$

c) $f(x) = -2x^2 - 9x - 9$

d) $f(x) = 3x^3 - 9x + 1$

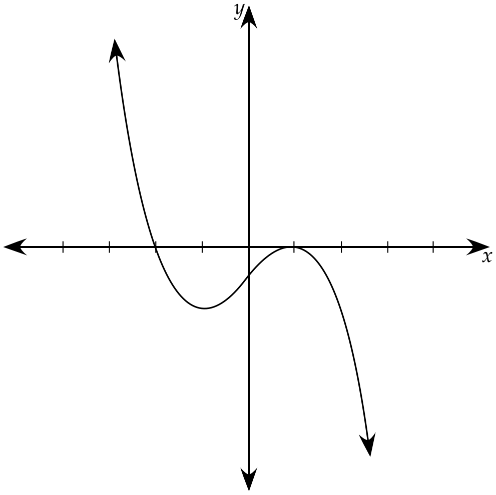
e) $f(x) = -\frac{1}{3}x^3 + x - \frac{2}{3}$

f) $f(x) = -\frac{1}{4}x^4 + 4x^2$

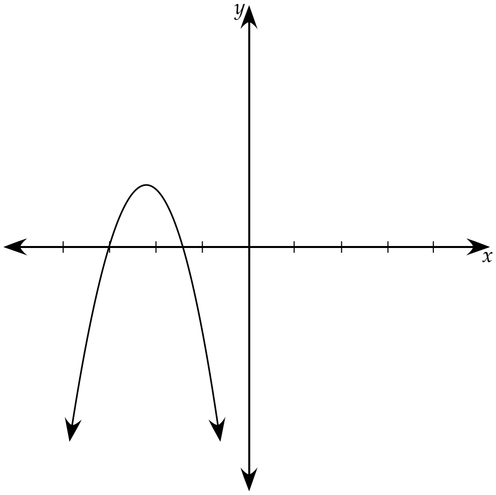
g) $f(x) = 3x^4 + 4x^3$

h) $f(x) = x^5 - 5x^3 + 4x$

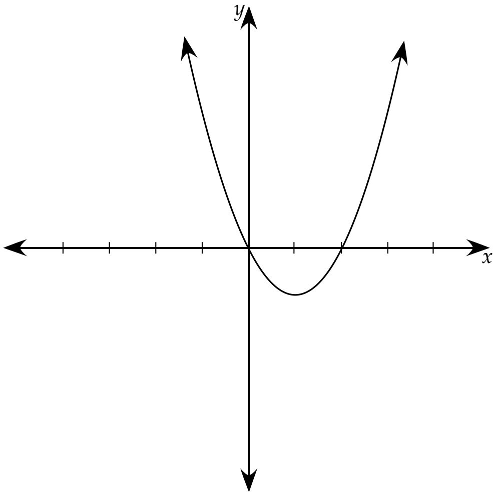
i)



ii)



iii)



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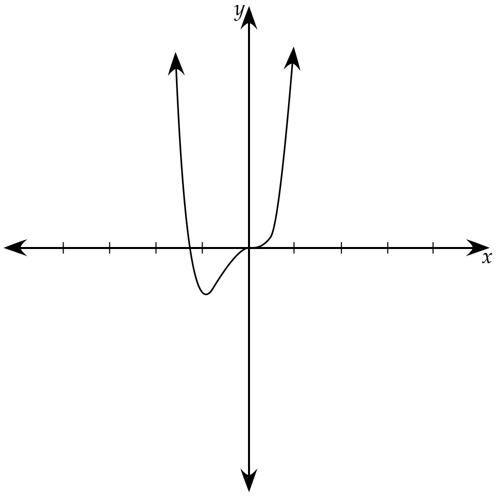
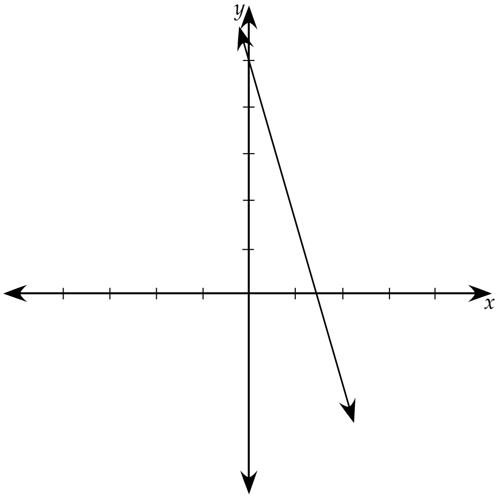
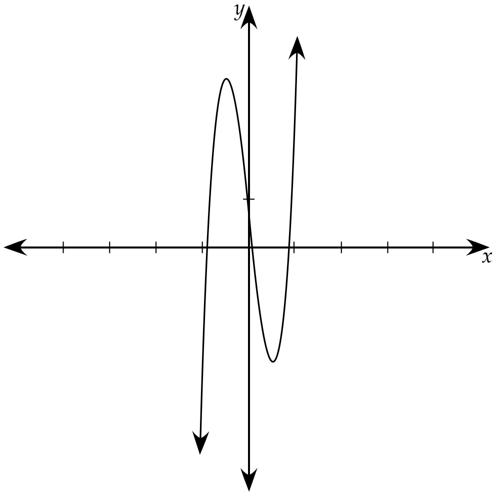


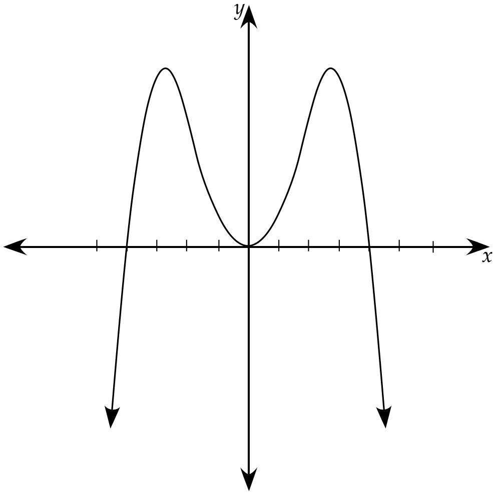
v)



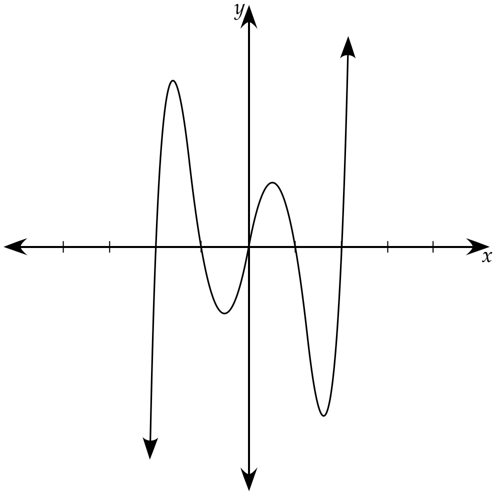
vi)



vii)



viii)



Answer:

a) — v) a line

b) — iii) parabola opening up; y-intercept at 0

c) — ii) parabola opening down; y-intercept at -9

d) — vi) cubic, rising to the right; y-intercept at 1

e) — i) cubic, falling to the right; y-intercept at $-\dfrac{2}{3}$

f) — vii) quartic, opening down; x-intercepts at 0, 4, -4,
since $f(x) = -\dfrac{1}{4}x^2\left(x^2 - 16\right)$; y-intercept at 0

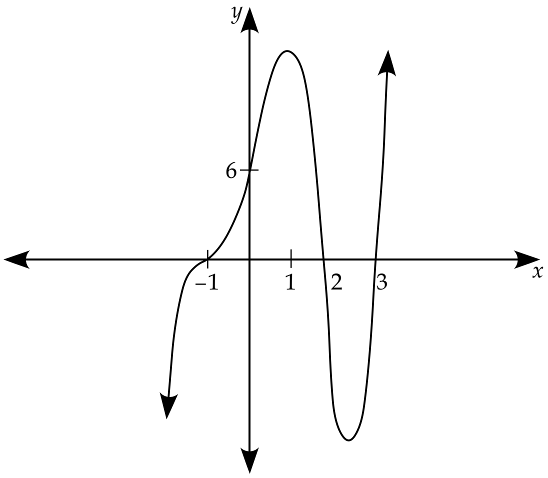
g) — iv) quartic, opening up, triple root at the origin, since
$f(x) = x^3(3x + 4)$; x-intercepts at $0, -\dfrac{4}{3}$; y-intercept at 0

h) — viii) quintic, rising to the right; y-intercept at 0

9. Without the aid of graphing technology, sketch the graphs of each of the following polynomials, and label all y-intercepts and x-intercepts. Remember, you do not need to find the coordinates of relative and absolute maximums or minimums. You do need to clearly indicate the end behaviour and the behaviour near the x-intercepts.

   a) $y = (x - 2)(x + 1)^3(x - 3)$

   Answer:



Degree is 5

x-intercepts at -1, 2, and 3

y-intercept occurs when $x = 0$

y-intercept

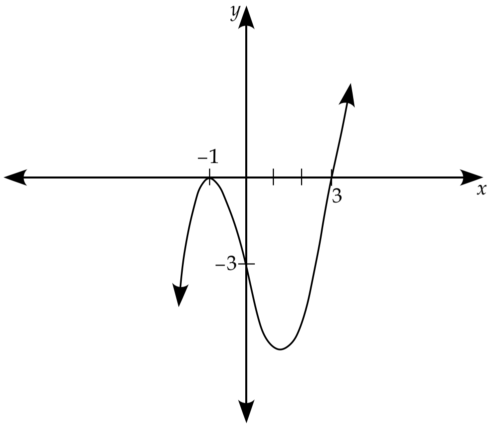
$= (0 - 2)(0 + 1)^2(0 - 3)$

$= 6$

Notice how the graph flattens at $x = -1$ as it crosses because the exponent of $(x + 1)$ is odd (that is, the multiplicity of the root is 3).

b) $y = (x + 1)^2(x - 3)$

Answer:



Degree is 3

x-intercepts at −1 and 3

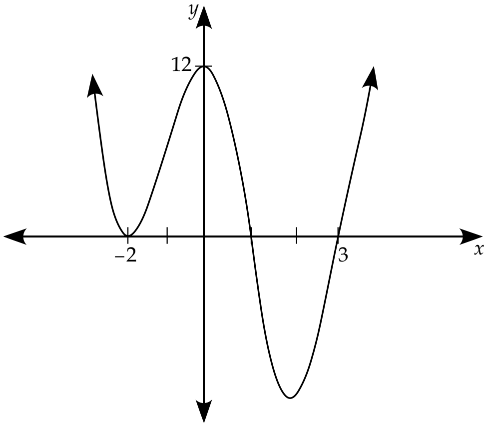
y-intercept = $(0 + 1)^2(0 - 3)$

$= -3$

Notice how the graph flattens at $x = -1$ and is tangent to the x-axis since the exponent of $(x + 1)$ is even (that is, the multiplicity of the root is 2).

c) $y = (x + 2)^2(x - 1)(x - 3)$

Answer:



Degree is 4

x-intercepts at −2, 1, and 3

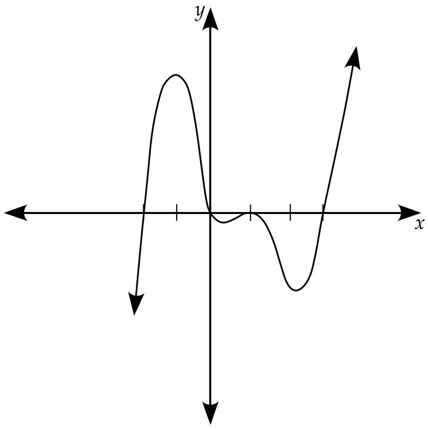
y-intercept = $(0 + 2)^2(0 - 1)(0 - 3)$

$= 12$

Notice how the graph flattens at $x = -2$ and is tangent to the x-axis, since the exponent of $(x + 2)$ is even.

d) $y = x(x - 1)^2(x + 2)(x - 3)$

Answer:



Degree is 5

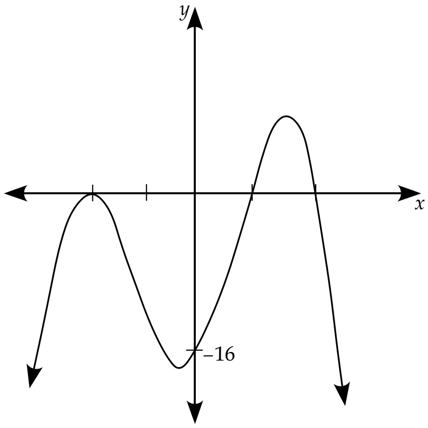
x-intercepts at $-2, 0, 1, 3$

y-intercept = 0

Notice how the graph flattens at $x = 1$ and is tangent because the exponent of $(x - 1)$ is even.

e) $y = -2(x - 1)(x + 2)^2(x - 2)$

Answer:



Degree is 4

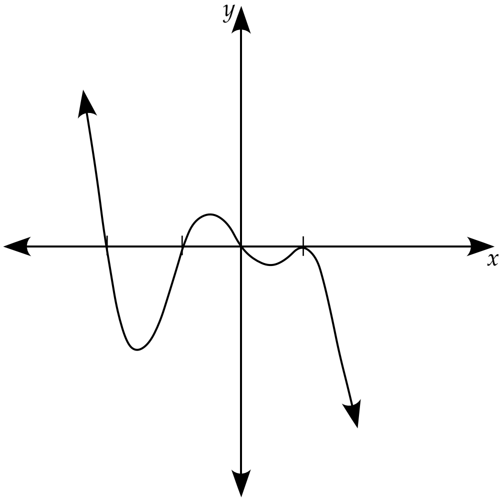
x-intercepts at $-2, 1, 2$

y-intercept = -16

Notice how the graph flattens at $x = -2$ and is tangent because the exponent of $(x + 2)$ is even. Also, the leading coefficient is negative, so it opens down to the left and right for this even-degree polynomial function.

f) $y = -3x(x - 1)^2(x + 2)(x + 1)$

Answer:



Degree is 5

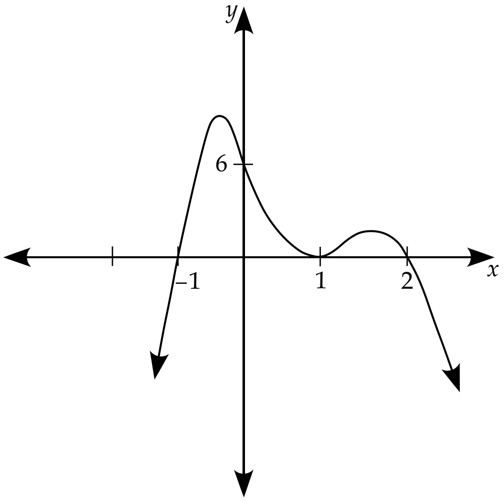
x-intercepts at −2, −1, 0, and 1

y-intercept = 0

Notice how the graph flattens at $x = 1$ and is tangent because the exponent of $(x - 1)$ is even. Also, the leading coefficient is negative so it goes up to the left and down to the right for this odd-degree polynomial function.

g) Graph a quartic with a leading coefficient of −3 that has roots of −1, 1, and +2. The multiplicity of the root equal to 1 is 2.

Answer:



Graph: $y = -3(x + 1)(x - 2)(x - 1)^2$

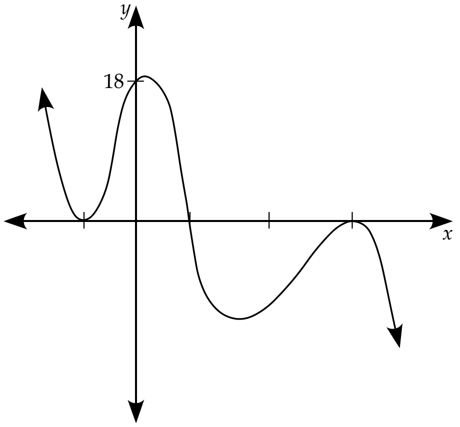
x-intercepts at −1, 1, and 2

y-intercept $= -3(0 + 1)(0-2)(0-1)^2$
$= 6$

Notice how the graph flattens at $x = 1$ and is tangent because the exponent of $(x - 1)$ is even. Also, the leading coefficient is negative, so it opens down to the left and right for this even-degree polynomial function.

h) Graph a quintic with multiplicity of 2 at both $x = 3$ and $x = -1$, with another root at $x = 1$. The leading coefficient is -2.

Answer:



Graph: $y = -2(x - 3)^2(x + 1)^2(x - 1)$

x-intercepts at $-1, 1, 3$

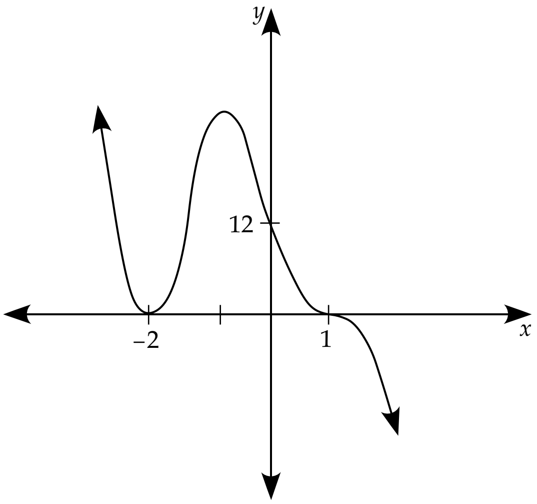
y-intercept = 18

y-intercept = $-2(0 - 3)^2(0 + 1)^2(0 - 1) = -2(9)(1)(-1) = 18$

Notice the graph flattens at $x = -1$ and $x = 3$ and is tangent at both x-intercepts, since both exponents are even. Also, the leading coefficient is negative so it opens up to the left and down to the right for this odd-degree polynomial function.

i) Sketch a quintic with a multiplicity of 3 at $x = 1$, a multiplicity of 2 at $x = -2$, and a leading coefficient of -3.

Answer:



Graph: $y = -3(x - 1)^3(x + 2)^2$

x-intercepts at $-2, 1$

y-intercept = 12

Notice how the graph flattens at $x = -2$ and $x = 1$ and it is tangent at $x = -2$, since the exponent of $(x + 2)$ is even. It crosses at $x = 1$, since the exponent of $(x - 1)$ is odd. Also, the leading coefficient is negative so it opens up to the left and down to the right for this odd-degree polynomial function.

Learning Activity 4.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. How many millimetres are in one kilometre?
2. Is the point (–1, 5) a solution to the inequality $y \leq x^2 - 5x + 6$?
3. Rationalize the denominator: $\dfrac{1}{5 - \sqrt{8}}$
4. Which is the better buy, 3 candy bars for $1.99 or 5 candy bars for $2.50?
5. In which quadrants is $y = \sin\theta$ positive?
6. Factor: $49x^2 - 36y^4$
7. Evaluate: $36^{0.5}$
8. How many terms are there in the expansion of $(x - 4)^7$?

Answers:

1. 1,000,000 mm = 1 km (1000 m = 1 km, 100 cm = 1 m, 10 mm = 1 cm)
2. Yes ($5 \leq (-1)^2 - 5(-1) + 6$ is $5 \leq 12$, which is true)
3. $\dfrac{5 + \sqrt{8}}{17}$ $\left(\dfrac{1\left(5 + \sqrt{8}\right)}{\left(5 - \sqrt{8}\right)\left(5 + \sqrt{8}\right)} = \dfrac{5 + \sqrt{8}}{25 - 8}\right)$
4. 5 candy bars for $2.50 (5 for 2.50 is 50 cents each, so 3 cost $1.50)
5. Quadrant I and Quadrant II
6. $(7x - 6y^2)(7x + 6y^2)$ (difference of squares pattern)
7. 6 $\left(36^{0.5} = \sqrt{36}\right)$
8. 8 (there is one more term than the exponent of the binomial)

Part B: Dividing Polynomials

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Divide using long division and write in the form $f(x) = d(x)q(x) + R$.

   a) $(2x^3 - 3x + 1) \div (x - 2)$

   Answer:

$$\begin{array}{l}
\qquad\qquad\quad 2x^2 + 4x + 5 + \dfrac{11}{x-2} \\
x - 2\overline{)2x^3 + 0x^2 - 3x + 1} \\
\qquad -\underline{(2x^3 - 4x^2)} \\
\qquad\qquad\quad 4x^2 - 3x \\
\qquad\qquad -\underline{(4x^2 - 8x)} \\
\qquad\qquad\qquad\quad 5x + 1 \\
\qquad\qquad\qquad -(\underline{5x - 10}) \\
\qquad\qquad\qquad\qquad\quad 11
\end{array}$$

$$2x^3 - 3x + 1 = (x - 2)(2x^2 + 4x + 5) + 11$$

b) $(2x^2 + x^3 - 3x - 4) \div (x + 2)$

Answer:

$$\begin{array}{rl}
 & x^2+0x-3+\dfrac{2}{x+2} \\
x+2\,\big) & \overline{x^3+2x^2-3x-4} \\
 & \underline{-\left(x^3+2x^2\right)} \\
 & \qquad 0x^2-3x \\
 & \qquad \underline{-\left(0x^2+0x\right)} \\
 & \qquad\qquad -3x-4 \\
 & \qquad\qquad \underline{-\left(-3x-6\right)} \\
 & \qquad\qquad\qquad 2
\end{array}$$

$$x^3+2x^2-3x-4=(x+2)\left(x^2-3\right)+2$$

c) $(6x^3 - 16x^2 + 17x - 6) \div (3x - 2)$

Answer:

$$\begin{array}{rl}
 & \qquad 2x^2-4x+3 \\
3x-2\,\big) & \overline{6x^3-16x^2+17x-6} \\
 & \underline{-\left(6x^3-4x^2\right)} \\
 & \qquad -12x^2+17x \\
 & \qquad \underline{-\left(-12x^2+8x\right)} \\
 & \qquad\qquad 9x-6 \\
 & \qquad\qquad \underline{-\left(9x-6\right)} \\
 & \qquad\qquad\qquad 0
\end{array}$$

$$6x^3-16x^2+17x-6=(3x-2)\left(2x^2-4x+3\right)$$

d) $(2x^3 + 3x^2 - 7 - 4x) \div (x^2 - 2)$

Answer:

$$\begin{array}{r}
2x+3-\dfrac{1}{x^2-2} \\
x^2+0x-2\overline{\big)2x^3+3x^2-4x-7} \\
-\underline{\left(2x^3+0x^2-4x\right)} \\
3x^2+0x-7 \\
-\underline{\left(3x^2+0x-6\right)} \\
-1
\end{array}$$

$$2x^3+3x^2-4x-7=\left(x^2-2\right)(2x+3)-1$$

e) $(x^4 - 2x^3 - 7x^2 + 8x + 12) \div (x + 1)$

Answer:

$$\begin{array}{r}
x^3-3x^2-4x+12 \\
x+1\overline{\big)x^4-2x^3-7x^2+8x+12} \\
-\underline{\left(x^4+x^3\right)} \\
-3x^3-7x^2 \\
-\underline{\left(-3x^3-3x^2\right)} \\
-4x^2+8x \\
-\underline{\left(-4x^2-4x\right)} \\
12x+12 \\
-\underline{(12x+12)} \\
0
\end{array}$$

$$x^4-2x^3-7x^2+8x+12=(x+1)\left(x^3-3x^2-4x+12\right)$$

2. Use synthetic division to find the quotient and the remainder. Write your answer in the form: $f(x) = d(x)q(x) + R$.

a) $\left(x^3 - 7x + 6\right) \div (x - 2)$

Answer:

$$\begin{array}{r|rrrr} 2 & 1 & 0 & -7 & 6 \\ & & 2 & 4 & -6 \\ \hline & 1 & 2 & -3 & 0 \end{array}$$

Quotient: $x^2 + 2x - 3$

Remainder: None

$$x^3 - 7x + 6 = (x - 2)\left(x^2 + 2x - 3\right)$$

b) $\left(3x^4 - x - 4\right) \div (x - 2)$

Answer:

$$\begin{array}{r|rrrrr} 2 & 3 & 0 & 0 & -1 & -4 \\ & & 6 & 12 & 24 & 46 \\ \hline & 3 & 6 & 12 & 23 & 42 \end{array}$$

Quotient: $3x^3 + 6x^2 + 12x + 23$

Remainder: 42

$$3x^4 - x - 4 = (x - 2)\left(3x^3 + 6x^2 + 12x + 23\right) + 42$$

c) $\left(x^4 - 2x^3 - 70x + 20\right) \div (x - 5)$

Answer:

$$\begin{array}{r|rrrrr} 5 & 1 & -2 & 0 & -70 & 20 \\ & & 5 & 15 & 75 & 25 \\ \hline & 1 & 3 & 15 & 5 & 45 \end{array}$$

Quotient: $x^3 + 3x^2 + 15x + 5$

Remainder: 45

$$x^4 - 2x^3 - 70x + 20 = (x - 5)\left(x^3 + 3x^2 + 15x + 5\right) + 45$$

d) $\left(2x^3 - 5x^2 + 6x + 3\right) \div \left(x - \dfrac{1}{2}\right)$

Answer:

$$\begin{array}{c|cccc} \frac{1}{2} & 2 & -5 & 6 & 3 \\ & & 1 & -2 & 2 \\ \hline & 2 & -4 & 4 & \big| \; 5 \end{array}$$

Quotient: $2x^2 - 4x + 4$

Remainder: 5

$$2x^3 - 5x^2 + 6x + 3 = \left(x - \frac{1}{2}\right)\left(2x^2 - 4x + 4\right) + 5$$

e) $\left(4x^3 + 4x^2 - 7x - 6\right) \div \left(x + \dfrac{3}{2}\right)$

Answer:

$$\begin{array}{c|cccc} -\frac{3}{2} & 4 & 4 & -7 & -6 \\ & & -6 & 3 & 6 \\ \hline & 4 & -2 & -4 & \big| \; 0 \end{array}$$

Quotient: $4x^2 - 2x - 4$

Remainder: None

$$4x^3 + 4x^2 - 7x - 6 = \left(x + \frac{3}{2}\right)\left(4x^2 - 2x - 4\right)$$

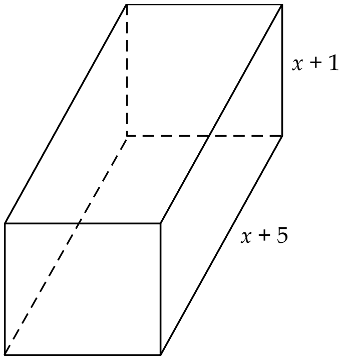
3. When a polynomial $f(x)$ is divided by $2x + 1$, the quotient is $x^2 - x + 4$ and the remainder is 3. Find $f(x)$.

   Answer:

   Using the Division Algorithm, we can write:

   $$\begin{aligned} f(x) &= (2x + 1)(x^2 - x + 4) + 3 \\ &= 2x^3 - 2x^2 + 8x + x^2 - x + 4 + 3 \\ &= 2x^3 - x^2 + 7x + 7 \end{aligned}$$

4. The volume of the following rectangle prism is $V = 3x^3 + 8x^2 - 45x - 50$. Expressions for the length and height are shown. Find an expression for the missing dimension ($V = lwh$).



Answer:

The volume of the rectangular prism is

$$V = 3x^3 + 8x^2 - 45x - 50 = (x + 5)(x + 1)(w).$$

Use synthetic division with V and $x + 1$.

$$\begin{array}{r|rrrr}
-1 & 3 & 8 & -45 & -50 \\
 & & -3 & -5 & +50 \\
\hline
 & 3 & 5 & -50 & \multicolumn{1}{|r}{0}
\end{array}$$

$\therefore V = (x + 1)(3x^2 + 5x - 50)$.

Now factor the quadratic or divide again using $(3x^2 + 5x - 50) \div (x + 5)$

$\therefore V = (x + 1)(3x - 10)(x + 5)$.

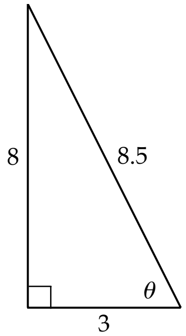
$\therefore$ An expression for the missing dimension is $3x - 10$.

Learning Activity 4.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

Use the triangle below to answer Questions 1 to 3.



1. Determine the ratio for $\sin\theta$.
2. Determine the ratio for $\tan\theta$.
3. Determine the ratio for $\dfrac{1}{\cos\theta}$.
4. In which direction does the parabola $y = -(x + 2)^2$ open?
5. Evaluate: ${}_4P_1$
6. Write as an entire radical: $2x^2y\sqrt[3]{y}$
7. Simplify: $\sqrt{3}\left(\sqrt{3} - \sqrt{2}\right)$
8. Your restaurant bill came to \$119.21. If you wish to leave a 20% tip, estimate how much you should leave.

Answers:

1. $\dfrac{8}{8.5}\left(\sin\theta = \dfrac{\text{opp}}{\text{hyp}}\right)$
2. $\dfrac{8}{3}\left(\tan\theta = \dfrac{\text{opp}}{\text{adj}}\right)$
3. $\dfrac{8.5}{3}\left(\dfrac{1}{\cos\theta}\text{ is the reciprocal; }\cos\theta = \dfrac{\text{adj}}{\text{hyp}}\right)$
4. Down
5. 4
6. $\sqrt{8x^6y^4}\ \left(2x^2y\sqrt[3]{y} = \sqrt[3]{8}\sqrt[3]{x^6}\sqrt[3]{y^3}\sqrt[3]{y}\right)$
7. $3 - \sqrt{6}\ \left(\sqrt{3}\left(\sqrt{3} - \sqrt{2}\right) = \sqrt{3}\sqrt{3} - \sqrt{3}\sqrt{2}\right)$
8. \$24 (10% of 119.21 is about \$12, so 20% is about 24)

Part B: Factoring Polynomials

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Determine the remainder when each of the following polynomials are divided by $x + 3$ without using division.

   Use the Remainder Theorem to answer this question: If a polynomial is divided by $x - a$, then the remainder is $R = f(a)$.

   a) $f(x) = x^3 + 2x^2 - x - 6$

   Answer:

   $$\begin{aligned} f(-3) &= (-3)^3 + 2(-3)^2 - (-3) - 6 \\ &= -27 + 18 + 3 - 6 \\ &= -12 \end{aligned}$$

   b) $f(x) = -x^4 - 3x^3 + x + 3$

   Answer:

   $$\begin{aligned} f(-3) &= -(-3)^4 - 3(-3)^3 - 3 + 3 \\ &= -81 + 81 - 3 + 3 \\ &= 0 \end{aligned}$$

   c) $f(x) = x^3 - 5$

   Answer:

   $$\begin{aligned} f(-3) &= (-3)^3 - 5 \\ &= -\ 27 - 5 \\ &= -32 \end{aligned}$$

   d) $f(x)) = x^5 + 2x^4 - 4x^3 - 2x^2 - x + 7$

   Answer:

   $$\begin{aligned} f(-3) &= (-3)^5 + 2(-3)^4 - 4(-3)^3 - 2(-3)^2 - (-3) + 7 \\ &= -243 + 162 + 108 - 18 + 3 + 7 \\ &= 19 \end{aligned}$$

2. Determine whether $x - 2$ is a factor of each of the following polynomials using the Factor Theorem.

   a) $f(x) = x^3 + 4x^2 - x - 5$

      Answer:

      $f(2) = 2^3 + 4(2)^2 - 2 - 5$

      $= 8 + 16 - 2 - 5$

      $= 17$

      $\therefore$ $x - 2$ is not a factor of this polynomial as the remainder is not equal to zero.

      Note: The constant term is -5. Since 2 is not a factor of 5, $x - 2$ could not be a factor of $x^3 + 4x^2 - x - 5$.

   b) $g(x) = x^3 - x^2 - 8x + 12$

      Answer:

      $g(2) = 2^3 - 2^2 - 8(2) + 12$

      $= 8 - 4 - 16 + 12$

      $= 0$

      $\therefore$ $x - 2$ is a factor of this polynomial as the remainder is equal to zero.

   c) $h(x) = x^4 + 2x^3 - 9x^2 - 2x + 8$

      Answer:

      $h(2) = 2^4 + 2(2)^3 - 9(2)^2 - 2(2) + 8$

      $= 16 + 16 - 36 - 4 + 8$

      $= 0$

      $\therefore$ $x - 2$ is a factor of this polynomial as the remainder is equal to zero.

   d) $j(x) = -x^5 + 6x^4 - 2x^3 - x^2 - 7x + 10.$

      Answer:

      $j(2) = -2^5 + 6(2)^4 - 2(2)^3 - 2^2 - 7(2) + 10$

      $= -32 + 96 - 16 - 4 - 14 + 10$

      $= 40$

      $\therefore$ $x - 2$ is not a factor of this polynomial as the remainder is not equal to zero.

3. Verify whether or not the following binomials are factors of $f(x) = x^3 + 3x^2 - 4$.

   a) $x + 2$

   Answer:

   $f(-2) = (-2)^3 + 3(-2)^2 - 4$
   $= -8 + 12 - 4$
   $= 0$

   Yes, $(x + 2)$ is a factor.

   b) $x + 1$

   Answer:

   $f(-1) = (-1)^3 + 3(-1)^2 - 4$
   $= -1 + 3 - 4$
   $= -2$

   No, $(x + 1)$ is not a factor.

   c) $x - 2$

   Answer:

   $f(2) = (2)^3 + 3(2)^2 - 4$
   $= 8 + 12 - 4$
   $= 16$

   No, $(x - 2)$ is not a factor.

   d) $x - 1$

   Answer:

   $f(1) = (1)^3 + 3(1)^2 - 4$
   $= 1 + 3 - 4$
   $= 0$

   Yes, $(x - 1)$ is a factor.

4. If $P(5) = 0$, what binomial must be a factor of $P(x)$?

   Answer:

   If $P(5) = 0$, then the binomial $x - 5$ must be a factor of $P(x)$.

5. Factor completely:

   a) $f(x) = x^3 - 8x^2 + 4x + 48$

   Answer:

   Possible integer roots = factors of constant term

   $= \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 16, \pm 24, \pm 48$

   Note: Since the leading coefficient is 1, then there will be no other rational roots.

   To find a factor, find $f(a)$. If $f(a) = 0$, then $(x - a)$ is a factor.

$$\begin{aligned} f(x) &= x^3 - 8x^2 + 4x + 48 \\ f(1) &= 1^3 - 8(1)^2 + 4(1) + 48 \\ &= 1 - 8 + 4 + 48 \\ &= 45 \\ &\neq 0 \therefore x - 1 \text{ is not a factor} \end{aligned}$$

$$\begin{aligned} f(2) &= 2^3 - 8(2)^2 + 4(2) + 48 \\ &= 8 - 32 + 8 + 48 \\ &= 32 \\ &\neq 0 \therefore x - 2 \text{ is not a factor} \end{aligned}$$

$$\begin{aligned} f(3) &= 3^3 - 8(3)^2 + 4(3) + 48 \\ &= 27 - 72 + 12 + 48 \\ &= 15 \\ &\neq 0 \therefore x - 3 \text{ is not a factor} \end{aligned}$$

$$\begin{aligned} f(4) &= 4^3 - 8(4)^2 + 4(4) + 48 \\ &= 64 - 128 + 16 + 48 \\ &= 0 \therefore x - 4 \text{ is a factor} \end{aligned}$$

   Use synthetic division to find the quotient.

$$\begin{array}{r|rrr|r} 4 & 1 & -8 & 4 & 48 \\ & & 4 & -16 & -48 \\ \hline & 1 & -4 & -12 & 0 \end{array}$$

   $\therefore f(x) = (x - 4)(x^2 - 4x - 12)$

   Factor the quadratic to get $f(x) = (x - 4)(x - 6)(x + 2)$.

b) $f(x) = x^4 - 2x^3 - 17x^2 + 18x + 72$

Answer:

Possible integer roots = factors of constant term

$= \pm 1, \pm 2, \pm 3, \pm 6, \pm 8, \pm 9, \pm 12, \pm 24, \pm 36, \pm 72$

$$\begin{aligned} f(1) &= 1^4 - 2(1)^3 - 17(1)^2 + 18(1) + 72 \\ &= 1 - 2 - 17 + 18 + 72 \\ &= 72 \\ &\neq 0 \therefore x - 1 \text{ is not a factor} \end{aligned}$$

$$\begin{aligned} f(-1) &= (-1)^4 - 2(-1)^3 - 17(-1)^2 + 18(-1) + 72 \\ &= 1 + 2 - 17 - 18 + 72 \\ &= 40 \\ &\neq 0 \therefore x + 1 \text{ is not a factor} \end{aligned}$$

$$\begin{aligned} f(2) &= (2)^4 - 2(2)^3 - 17(2)^2 + 18(2) + 72 \\ &= 16 - 16 - 68 + 36 + 72 \\ &= 40 \\ &\neq 0 \therefore x - 2 \text{ is not a factor} \end{aligned}$$

$$\begin{aligned} f(-2) &= (-2)^4 - 2(-2)^3 - 17(-2)^2 + 18(-2) + 72 \\ &= 16 + 16 - 68 - 36 + 72 \\ &= 0 \\ &= 0 \therefore x + 2 \text{ is a factor} \end{aligned}$$

Use synthetic division to find the quotient.

$$\begin{array}{r|rrrr|r} -2 & 1 & -2 & -17 & 18 & 72 \\ & & -2 & 8 & 18 & -72 \\ \hline & 1 & -4 & -9 & 36 & 0 \end{array}$$

$f(x) = (x + 2)(x^3 - 4x^2 - 9x + 36)$

$q(x) = x^3 - 4x^2 - 9x + 36$ can still factor

Note that we need not test any values of a that have already failed. However, we need to test $x = -2$ again to check for multiple roots.

$$\begin{aligned} q(-2) &= (-2)^3 - 4(-2)^2 - 9(-2) + 36 \\ &= -8 - 16 + 18 + 36 = 30 \neq 0 \end{aligned}$$

$\therefore$ No multiple root.

$\therefore$ Test $x = 3$ in $q(x)$.

$q(3) = 33 - 4(3)^2 - 9(3) + 36$

$= 27 - 36 - 27 + 36$

$= 0$

$\therefore$ $x - 3$ is a factor.

Use synthetic division to get the new quotient.

$$\begin{array}{r|rrr|r} 3 & 1 & -4 & -9 & 36 \\ & & 3 & -3 & -36 \\ \hline & 1 & -1 & -12 & 0 \end{array}$$

$\therefore f(x) = (x + 2)(x - 3)(x^2 - x - 12)$.

Now we can factor the quadratic.

$\therefore f(x)$ in factored form is $f(x) = (x + 2)(x - 3)(x + 3)\ (x - 4)$.

c) $f(x) = x^3 - 27$

Answer:

Possible integer roots: ±1, ±3, ±9, ±27

$f(3) = 3^3 - 27$

$= 27 - 27$

$= 0$

$\therefore$ $x - 3$ is a factor

Do synthetic division:

$$\begin{array}{r|rrr|r} 3 & 1 & 0 & 0 & -27 \\ & & 3 & 9 & 27 \\ \hline & 1 & 3 & 9 & 0 \end{array}$$

$\therefore f(x) = (x - 3)(x^2 + 3x + 9)$.

Notice $(x^2 + 3x + 9)$ does not factor further.

$\therefore f(x) = (x - 3)(x^2 + 3x + 9)$.

d) $f(x) = x^3 - 13x + 12$

Answer:

Possible integer roots: ±1, ±2, ±3, ±4, ±6, ±12

Test $a = 1$:

$f(1) = 1^3 - 13(1) + 12$

$f(1) = 1 - 13 + 12$

$f(1) = 0$

$\therefore$ $x - 1$ is a factor.

Use synthetic division to get the quotient.

$$\begin{array}{r|rrr|r} 1 & 1 & 0 & -13 & 12 \\ & & 1 & 1 & -12 \\ \hline & 1 & 1 & -12 & 0 \end{array}$$

$\therefore f(x) = (x - 1)(x^2 + x - 12)$

Continue factoring the remaining quadratic to get:

$f(x) = (x - 1)(x + 4)(x - 3)$

e) $f(x) = x^5 + 5x^4 - 15x^3 - 85x^2 - 26x + 120$ knowing that $(x - 1)$ and $(x + 2)$ are factors.

Answer:

As $x - 1$ is a factor, one root is already provided for you, $x = 1$. Use synthetic division to get the quotient.

$$\begin{array}{r|rrrrr|r} 1 & 1 & 5 & -15 & -85 & -26 & 120 \\ & & 1 & 6 & -9 & -94 & -120 \\ \hline & 1 & 6 & -9 & -94 & -120 & 0 \end{array}$$

$\therefore f(x) = (x - 1)(x^4 + 6x^3 - 9x^2 - 94x - 120)$

$q(x) = x^4 + 6x^3 - 9x^2 - 94x - 120$ is still factorable.

You also know that $x + 2$ is a factor. Therefore, use $x = -2$ and synthetic division to get the quotient.

$$\begin{array}{r|rrrr|r} -2 & 1 & 6 & -9 & -94 & -120 \\ & & -2 & -8 & 34 & 120 \\ \hline & 1 & 4 & -17 & -60 & 0 \end{array}$$

$\therefore q(x) = (x + 2)(x^3 + 4x^2 - 17x - 60)$ and

$f(x) = (x - 1)(x + 2)(x^3 + 4x^2 - 17x - 60)$

$r(x) = x^3 + 4x^2 - 7x - 60$ is still factorable.

Possible integer roots of $r(x)$ are ±1, ±2, ±3, ±4, ±5, ±6, ±10, ±12, ±15, ±20, ±30, ±60

Test $a = -2$ for a possible double root.

$$\begin{aligned} r(-2) &= (-2)^3 + 4(-2)^2 - 17(-2) - 60 \\ &= -8 + 16 + 34 - 60 \\ &= -8 \end{aligned}$$

$\therefore x + 2$ is not a double root.

Continue testing a-values until you arrive at a remainder of 0.

$$\begin{aligned} r(3) &= (3)^3 + 4(3)^2 - 17(3) - 60 \\ &= 27 + 36 - 51 - 60 \\ &= -48 \end{aligned}$$

$$\begin{aligned} r(-3) &= (-3)^3 + 4(-3)^2 - 17(-3) - 60 \\ &= -27 + 36 + 51 - 60 \\ &= 0 \end{aligned}$$

$\therefore x + 3$ is a factor.

Use synthetic division to get the quotient.

$$\begin{array}{r|rrr|r} -3 & 1 & 4 & -17 & -60 \\ & & -3 & -3 & 60 \\ \hline & 1 & 1 & -20 & 0 \end{array}$$

$\therefore r(x) = (x + 3)(x^2 + x - 20)$ and

$f(x) = (x - 1)(x + 2)(x + 3)(x^2 + x - 20)$

But $s(x) = x^2 + x - 20$ is still factorable and $s(x) = (x + 5)(x - 4)$.

$\therefore$ The completely factored form of $f(x) = (x - 1)(x + 2)(x + 3)(x - 4)(x + 5)$.

6. The polynomial $p(x) = 4x^3 + bx^2 + cx + 11$ has a remainder of -7 when divided by $(x + 2)$ and a remainder of 14 when divided by $x - 1$. Find $p(x)$.

Answer:

Recall: The Remainder Theorem states that if a polynomial $f(x)$ is divided by $x - a$, then the remainder is $R = f(a)$.

$\therefore p(-2) = -7$ and $p(1) = 14$.

$p(-2) = 4(-2)^3 + b(-2)^2 + c(-2) + 11$

$= -32 + 4b - 2c + 11$

$= 4b - 2c - 21$

But $p(-2) = -7$.

$\therefore 4b - 2c - 21 = -7$

$4b - 2c = 14$

$p(1) = 4(1)^3 + b(1)^2 + c(1) + 11$

$= 4 + b + c + 11$

$= b + c + 15$

But $p(1) = 14$.

$\therefore b + c + 15 = 14$

$b + c = -1$

Use the system of two equations with two unknowns that you have discovered to solve for b and c.

$4b - 2c = 14$ and $b + c = -1$

Solve by substitution.

Substitute $b = -1 - c$ in for b in $4b - 2c = 14$.

$4(-1 - c) - 2c = 14$

$-4 - 4c - 2c = 14$

$-6c = 18$

$c = -3$

Now find b using $c = -3$ in one of the equations.

$\therefore 4b - 2(-3) = 14$

$4b + 6 = 14$

$4b = 8$

$b = 2$

$\therefore p(x) = 4x^3 + 2x^2 - 3x + 11$

You could check the remainders for the dividends given.

7. For what value of m will the function $f(x) = 3x^3 + mx^2 - 6$ have a remainder of 6 when divided by the factor $x + 2$?

   Answer:

   If the function has a remainder of 6 when divided by $x + 2$, then $f(-2) = 6$.

   $$\begin{aligned} f(-2) = 6 &= 3(-2)^3 + m(-2)^2 - 6 \\ 6 &= -24 + 4m - 6 \\ 6 &= -30 + 4m \\ 36 &= 4m \\ m &= 9 \end{aligned}$$

8. Determine the value of k so that the binomial $x + 5$ is a factor of $y = x^3 + kx^2 - x - 30$.

   Answer:

   By the Factor Theorem, if $x + 5$ is a factor of $y = x^3 + kx^2 - x - 30$, then the function must equal zero when $x = -5$.

   $$\begin{aligned} 0 &= (-5)^3 + k(-5)^2 - (-5) - 30 \\ 0 &= -125 + 25k + 5 - 30 \\ 0 &= 25k - 150 \\ 150 &= 25k \\ k &= 6 \end{aligned}$$

9. The volume of a rectangular swimming pool can be represented by the function $V(d) = d^3 + 73d^2 + 276d$, where d represents depth.

   a) What are the expressions for the possible dimensions of the pool in terms of d?

      Answer:

      The possible dimensions of the pool are the factors of $V(d)$.

      $V(d) = d(d^2 + 73d + 276) = d(d + 69)(d + 4)$

      Therefore, the expressions for the possible dimensions of the pool are d, $d + 69$, and $d + 4$.

   b) If the depth of the pool is 6 feet, what are the dimensions of the length and width of the swimming pool?

      Answer:

      If the depth is 6 feet, then the length of the pool is $d + 69 = 6 + 69 = 75$ feet and the width of the pool is $d + 4 = 6 + 4 = 10$ feet.

Learning Activity 4.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. If $5^{18} + 5^{18} + 5^{18} + 5^{18} + 5^{18} = 5^x$, find the value of x.
2. Which is the better deal, purchasing a 6-pack of movie tickets for \$61.99 or purchasing an 8-pack of movie tickets for \$79?
3. Reduce to lowest terms: $\frac{44}{121}$
4. Does the following set of points represent a function?

   (1, 3), (3, 3), (4, 6), (−1, 4)
5. What is the length of the remaining leg of a right-angled triangle if the hypotenuse measures 13 m and one leg measures 12 m?
6. Determine the vertex of the function $f(x) = 2(x + 6)^2 - 1$.
7. Simplify: $\sqrt{128}$
8. Factor: $x^2 - 21x - 72$

Answers:

1. $x = 19$ $(5^{18} + 5^{18} + 5^{18} + 5^{18} + 5^{18} = 5(5^{18}) = 5^{19})$
2. 8-pack of movie tickets for \$79 (since 6 for 61.99 is a little more than 10 each)
3. $\frac{4}{11}\left(\text{divide by } \frac{11}{11}\right)$
4. Yes (Each input value corresponds to only one output value.)
5. 5 m $\left(\sqrt{13^2 - 12^2} = \sqrt{25}\right)$
6. (−6, −1) (from (0, 0) for $f(x) = x^2$, shift left 6 and down 1)
7. $8\sqrt{2}$ $\left(\sqrt{128} = \sqrt{64}\sqrt{2}\right)$
8. $(x - 24)(x + 3)$

Part B: Polynomial Functions

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. For each of the following functions, find (i) the zeros, (ii) the y-intercept, (iii) left-right end behaviour, and (iv) the sketch of the graph.

   a) $f(x) = x^3 - x^2 - 6x$

   Answer:

   i) $f(x) = x^3 - x^2 - 6x$

   $f(x) = x(x^2 - x - 6)$ Common factor.

   $f(x) = x(x - 3)(x + 2)$ Complete factorization.

   To find zeros:

   Let $f(x) = 0$

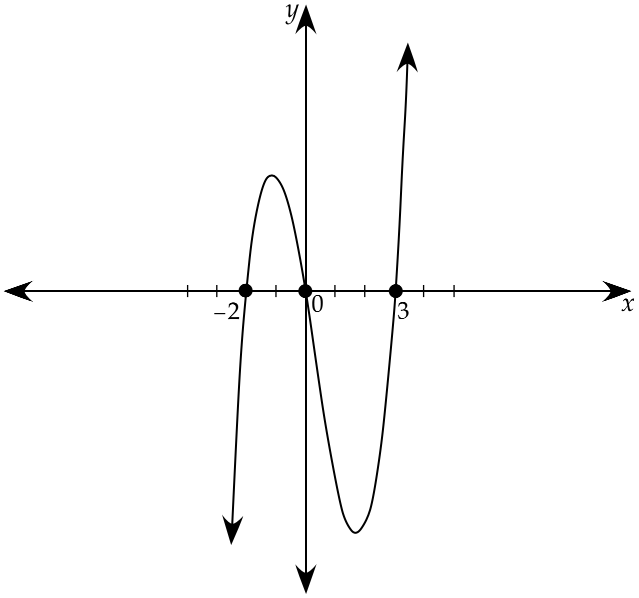
   $\therefore x(x - 3)(x + 2) = 0$

   $x = 0, x = 3, \text{ or } x = -2$

   ii) y-intercept is zero

   iii) Because the degree of the function is odd and the leading coefficient is positive, the graph falls to the left and rises to the right.

   iv) Graph



b) $f(x) = -x^3 + 4x$

Answer:

i) $f(x) = -x^3 + 4x$

$f(x) = -x(x^2 - 4)$

$f(x) = -x(x - 2)(x + 2)$

Let $f(x) = 0$ to find the zeros.

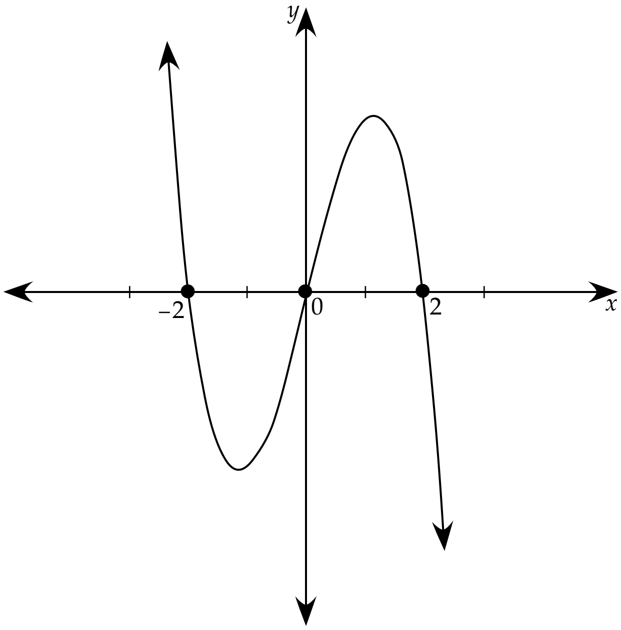
$-x(x - 2)(x + 2) = 0$

$x = 0$, $x = 2$, or $x = -2$

ii) y-intercept is zero

iii) Because the degree of the function is odd and the leading coefficient is negative, the graph falls to the right and rises to the left.

iv) Graph



c) $f(x) = x^4 - 2x^3 - 3x^2 + 4x + 4$

Answer:

i) $f(x) = x^4 - 2x^3 - 3x^2 + 4x + 4$

Use the Factor Theorem to find the factors.

Possible zeros are the factors of 4: ±1, ±2, ±4

$$\begin{aligned} f(x) &= x^4 - 2x^3 - 3x^2 + 4x + 4 \\ f(-1) &= (-1)^4 - 2(-1)^3 - 3(-1)^2 + 4(-1) + 4 \\ &= 1 + 2 - 3 - 4 + 4 = 0 \end{aligned}$$

Since $f(-1) = 0$, then $(x + 1)$ is a factor and -1 is a zero.

Use synthetic division to get a new quotient to factor:

$$\begin{array}{r|rrrrr} -1 & 1 & -2 & -3 & +4 & +4 \\ & & -1 & 3 & 0 & -4 \\ \hline & 1 & -3 & 0 & 4 & \multicolumn{1}{|r}{0} \end{array}$$

$f(x) = (x + 1)(x^3 - 3x^2 + 4)$

New quotient: $x^3 - 3x^2 + 4 = q(x)$

Test $x = -1$ again to check for a multiple root.

$$\begin{aligned} f(-1) &= (-1)^3 - 3(-1)^2 + 4 \\ &= -1 - 3 + 4 \\ &= 0 \end{aligned}$$

$\therefore$ $(x + 1)$ is a factor and $x = -1$ is a double root.

Use synthetic division to get a new quotient:

$$\begin{array}{r|rrrr} -1 & 1 & -3 & +0 & 4 \\ & & -1 & 4 & -4 \\ \hline & 1 & -4 & 4 & \multicolumn{1}{|r}{0} \end{array}$$

$f(x) = (x + 1)(x + 1)(x^2 - 4x + 4)$

We now have a quotient which is a quadratic that factors.

$\therefore f(x) = (x + 1)^2(x - 2)(x - 2)$

$f(x) = (x + 1)^2(x - 2)^2$

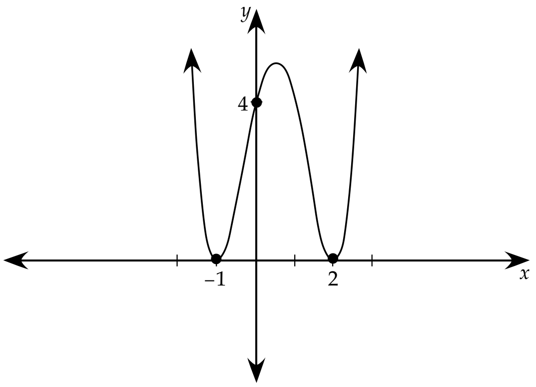
There are two multiple roots at $x = 2$ and $x = -1$.

Thus, the curve is tangent to the x-axis at $x = 2$ and $x = -1$.

ii) y-intercept is 4

iii) Because the degree of the function is even and the leading coefficient is positive, the graph rises to the right and to the left.

iv) Graph



d) $f(x) = -x^5$

Answer:

i) Zeros

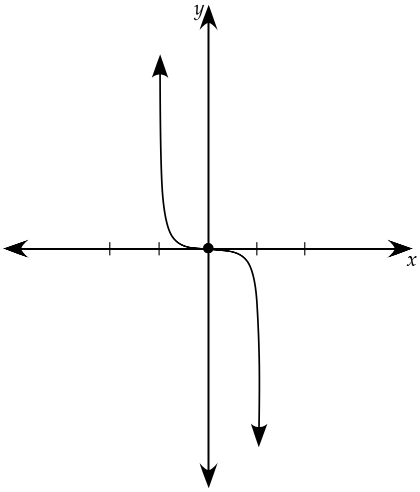
Let $f(x) = 0$

$$-x^5 = 0$$
$$x^5 = 0$$
$$x = 0$$

ii) y-intercept is 0

iii) Because the degree of the function is odd and the leading coefficient is negative, the graph rises to the left and falls to the right.

iv) Graph



Notice that the graph really flattens out at the x-intercept, since it is a quintuple root.

e) $f(x) = (x - 4)^3$

Answer:

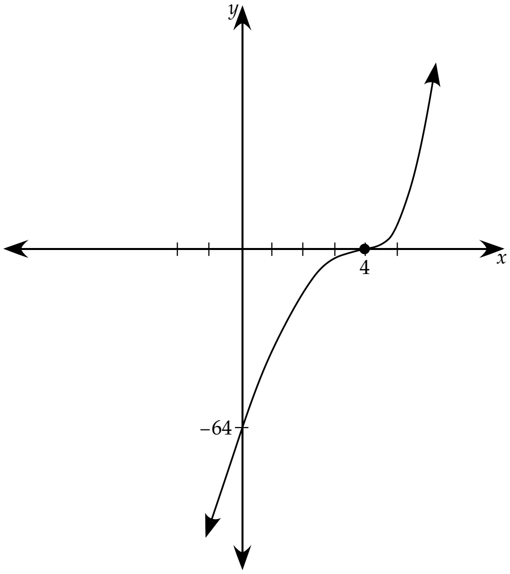
i) Let $f(x) = 0$

$$(x - 4)^3 = 0$$
$$x - 4 = 0$$
$$x = 4$$

ii) y-intercept is -64

iii) Because the degree of the function is odd and the leading coefficient is positive, the graph of the function rises to the right and falls to the left.

iv) Graph



Note: $f(x)$ is a horizontal transformation of $y = x^3$ shifted 4 units to the right.

Notice that the graph flattens out at the x-intercept because it is a triple root.

f) $f(x) = x^3 + 5x^2 + 2x - 8$

Answer:

i) $f(x) = x^3 + 5x^2 + 2x - 8$

Possible zeros are the factors of 8: ±1, ±2, ±4, ±8

$$\begin{aligned} f(1) &= 1^3 + 5(1)^2 + 2(1) - 8 \\ &= 1 + 5 + 2 - 8 \\ &= 0 \end{aligned}$$

$\therefore x - 1$ is a factor and $x = 1$ is a zero.

Use synthetic division to get a new quotient:

$$\begin{array}{r|rrrr} 1 & 1 & 5 & 2 & -8 \\ & & 1 & 6 & 8 \\ \hline & 1 & 6 & 8 & \multicolumn{1}{|r}{0} \end{array}$$

$\therefore f(x) = (x - 1)(x^2 + 6x + 8)$

We can now factor the quadratic to get:

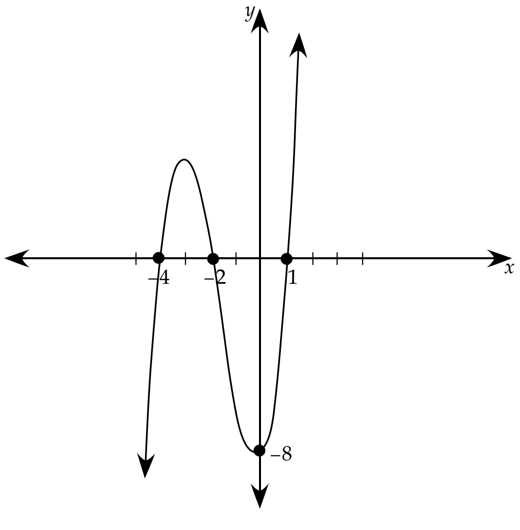
$f(x) = (x - 1)(x + 4)(x + 2)$

Zeros = 1, −4, −2

ii) y-intercept is −8

iii) Because the degree of the function is odd and the leading coefficient is positive, the graph of the function rises to the right and falls to the left.

iv) Graph



2. Determine the equation of each of the following polynomials using the given characteristics. Sketch a graph of each function.

   a) A quartic function with two zeros (each with a multiplicity of 2) at −2 and 3, and a y-intercept at −36.

   Answer:

   You know that $(x + 2)^2$ and $(x - 3)^2$ must be factors of this function. Therefore, the function must be a multiple of these factors.

   $y = a(x + 2)^2(x - 3)^2$

   Using the above equation, and the fact that the y-intercept is −36, you can determine the function.

   The y-intercept is located at the point (0, −36). Substitute this point into the above equation.

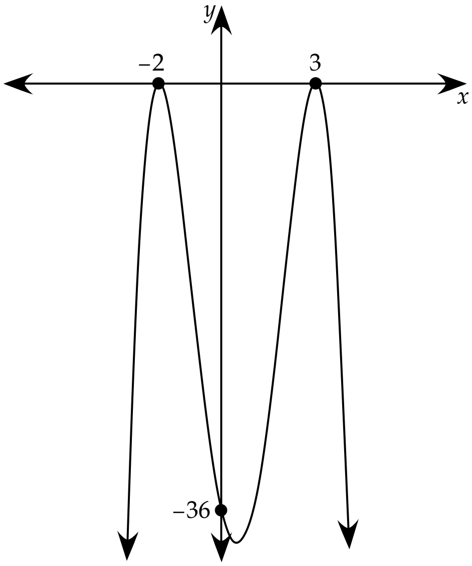
   $$-36 = a(0 + 2)^2(0 - 3)^2$$

   $$-36 = a(4)(9)$$

   $$-36 = a(36)$$

   $$a = -1$$

   This function equation is thus, $y = -(x + 2)^2(x - 3)^2$.



b) A cubic function with a triple root at 4 that passes through the point (3, 4).

Answer:

As this function has a triple root at 4, this function must be a multiple of $(x - 4)^3$ or $y = a(x - 4)^3$. You can substitute the point (3, 4) into this equation to determine the a-value and the resulting equation of the function.

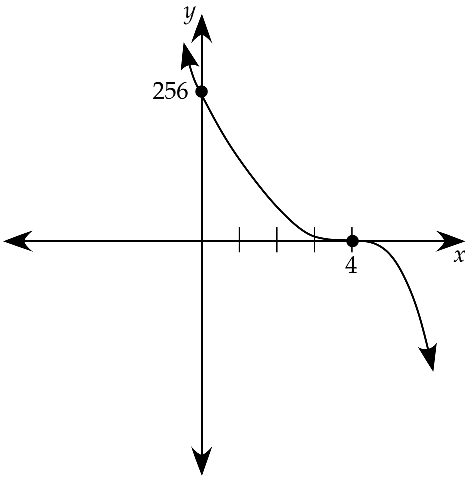
$$4 = a(3 - 4)^3$$
$$4 = a(-1)^3$$
$$-4 = a$$

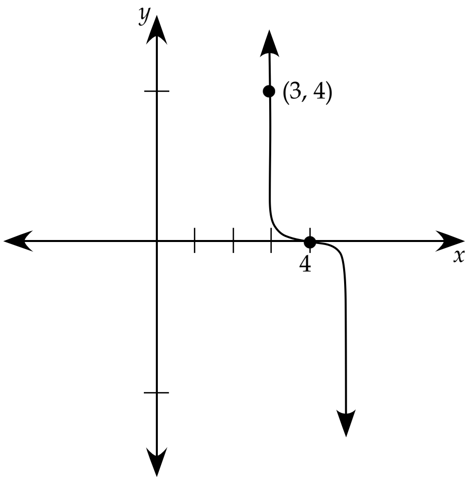
$\therefore$ This function equation is $y = -4(x - 4)^3$.

x-intercept: 4

y-intercept: 256

Your graph should resemble one of the sketches below.





3. From a 9 inch by 12 inch piece of material, small squares are to be cut from two corners so that three edges can be bent up and fastened to manufacture a scoop (a rectangular box with no top and no front). Find the approximate length of the sides of the small squares that are cut out to give the greatest volume. Find an estimate of the maximum volume.

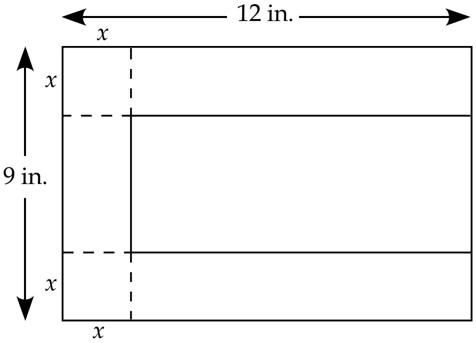
   Answer:

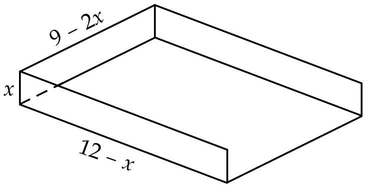
   Let the length of square to be cut out be x cm.

   $V = lwh$

   $V = (12 - x)(9 - 2x)(x)$

   $V = 2x^3 - 33x^2 + 108x$





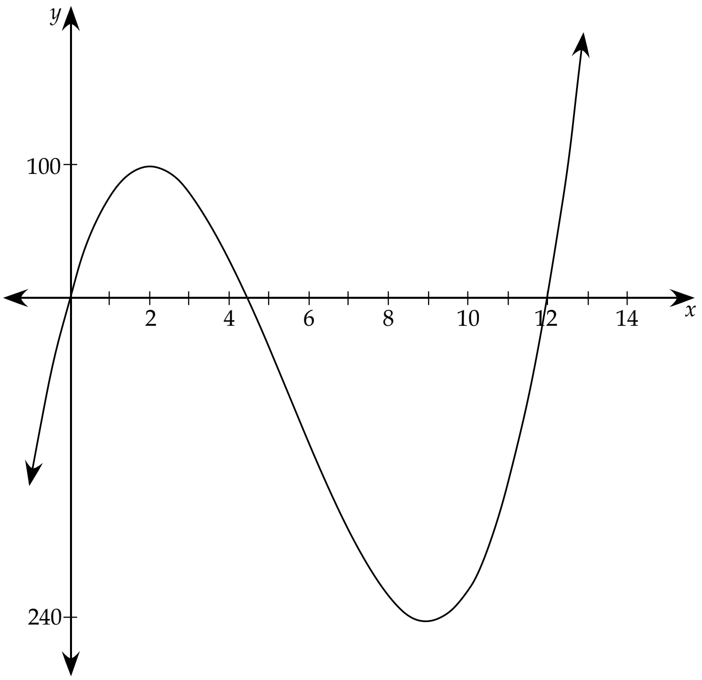
   Graph this function to approximate the location of the relative maximum value.

   Zeros: The zeros of the function $2x^3 - 33x^2 + 108x$ occur when $12 - x$, $9 - 2x$, and x are set equal to zero. The zeros are 12, $\frac{9}{2}$, and 0.

   x-intercepts at 12, 4.5, and 0

   y-intercept at 0

   End behaviour: cubic function with positive leading coefficient so the end behaviour is like the line $y = x$ and goes down to the left and up to the right.



Note: The domain for the context of this problem is only the part where $x > 0$ and $x < 4.5$, since x-values outside this domain would not result in scoops that have a valid volume or side length.

Create a table of values to estimate the side length for maximum volume.

x	Volume = $(12 - x)(9 - 2x)(x)$
1.5	$(12 - 1.5)(9 - 2(1.5))(1.5) = 94.5$
2	$(12 - 2)(9 - 2(2))(2) = 100$
2.5	$(12 - 2.5)(9 - 2(2.5))(2.5) = 95$

From the table of values, you can confirm that the maximum volume occurs approximately when $x = 2$. An estimate of the maximum volume when $x = 2$ is 100 inches^3.

Since $V = (12 - 2)(9 - 2(2))(2)$

$V = (10)(5)(2)$

$V = 100$

You can only approximate the maximum function value using your best estimate of x from your graph and then substituting that value of x into the function equation.

Notes

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Module 5
Trigonometric Functions and the Unit Circle

MODULE 5:
TRIGONOMETRIC FUNCTIONS AND THE UNIT CIRCLE

Introduction

There were 15:58 hours of daylight in Winnipeg on July 13th, 2012. Would you expect that this amount would have been any different on July 13th, 2000, or will be any different from the amount on July 13th, 2040? The answer is no. The reason is that this number of hours of daylight on any given day in any specific place will be approximately the same or repeat itself over the years. This information or data will be different for each place in the world, but it will remain repetitive for that place over the years. This is an example of a function (number of daylight hours over time) that is called **periodic**. It can be modelled by the sine function or the cosine function and is called a **sinusoidal**. There are many similar such occurrences in the world (for example, the amount of air in your lungs over time, the average daily temperatures during the year, and many other examples of this repetitive phenomenon).

In this module, you will examine **trigonometric functions**, which are a type of periodic function, and more of the properties and applications associated with them. First, you will study trigonometric ratios in more detail. You have been studying trigonometric ratios since Grade 10 mathematics. In this module, you will study the graphs of the trigonometric functions. You will look at tables of values of these functions, generate their graphs, and list their properties. Using techniques you have been developing since Module 2, you will also study transformations of these periodic functions.

Remember, it is important that you do all the questions in the learning activities because new concepts are sometimes developed within these learning activities.

Assignments in Module 5

When you have completed the assignments for Module 5, submit your completed assignments to the Distance Learning Unit either by mail or electronically through the learning management system (LMS). The staff will forward your work to your tutor/marker.

Lesson	Assignment Number	Assignment Title
2	Assignment 5.1	Degrees, Radians, and the Unit Circle
4	Assignment 5.2	The Six Trigonometric Ratios
6	Assignment 5.3	Trigonometric Functions and Their Graphs

Resource Sheet

When you write your final examination, you are encouraged to take a Final Examination Resource Sheet with you into the examination. This sheet will be one letter-sized page, 8½ ″ by 11″, with both sides in your handwriting or typewritten. You will submit it with your examination, but you do not receive any marks for it.

Many students have found that preparing a resource sheet is an excellent way to review. It provides you with a summary of the important facts of each module. You should complete a resource sheet for each module to help with your studying and reviewing. Lesson summaries and module summaries are included for you to use as a guide.

You may use the list of instructions that follows to help you with preparing your resource sheet for the material in Module 5. On this sheet, you should record math terms and definitions, formulas, sample questions, or a list of places where you often make mistakes. You should also identify special areas that require extra attention or review by writing the page numbers.

After you have completed each module's resource sheet, you may summarize the sheets from Modules 1 to 8 to prepare your Final Examination Resource Sheet. The final examination for this course is based on Modules 1 to 8.

Resource Sheet for Module 5

As you go through the lessons of this module, you may want to consider the following suggestions regarding the creation of a resource sheet.

1. List all the important math terms, and define them if necessary.
2. List all the formulas and perhaps a sample problem that shows how the formula is used.
3. If necessary, write the solutions to some problems, showing in detail how you did the calculations.
4. Copy any questions that represent the key points of the lesson, and perhaps include the solutions as well.
5. Identify the problems you found most difficult, and copy the page numbers onto the resource sheet so that you can review them before writing the examination. You may also copy the problems and the solutions onto your resource sheet, and later write them onto your Final Examination Resource Sheet.
6. Write any comments, ideas, shortcuts, or other reminders that may be helpful during an examination.

Notes

LESSON 1: RADIAN MEASURES

Lesson Focus

In this lesson, you will

☐ learn about the relationship between radians and degrees

☐ learn about the relationship between radians and arc length

☐ learn how to convert an angle from degrees to radians and vice versa

☐ learn how to determine the measures of angles that are coterminal with a given angle in standard position

☐ learn how to sketch angles (both positive and negative) in standard position

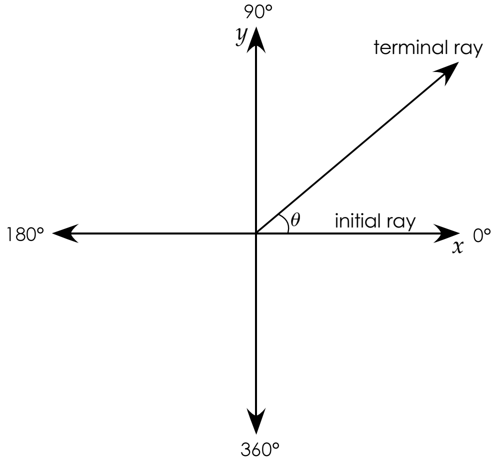
Lesson Introduction

In Grade 11 Pre-Calculus Mathematics, you learned how to measure angles between 0° and 360° in standard position. In this lesson, you will learn a new unit for measuring angles of rotation that is related to the radius of a circle. This unit of measure, called **radians**, is based on the properties of the circle itself. When you relate the measure of an angle to the radius of a circle, many mathematical formulas and functions become simpler and easier to use.

Measurement of Angles

Angles can be measured in many ways. For example, sometimes people measure angles in revolutions, or how many times a certain object turns in a complete circle or 360°. In this course, you will be measuring angles in both units of radians and degrees. You have studied the measurement of angles using degrees in previous courses; the following is a brief review.

Review of Angles and Degrees



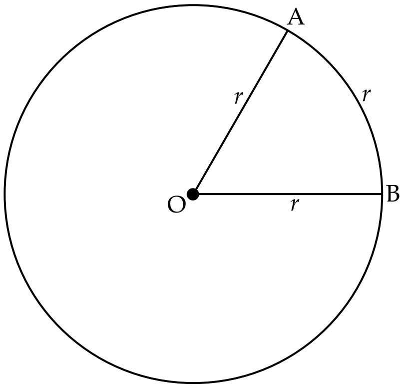
To measure angles in standard position, start measuring the angle along the **initial ray**, or the positive x-axis. If the angle is positive, measure counter-clockwise. If the angle is negative, measure clockwise. The **terminal ray** of the angle is the ray at which the angle terminates.

Radians—Where Do They Come From?

You may be wondering why you need to learn a new unit for measuring angles when you already know how to measure angles in degrees. To answer this, consider the question as to why there are 360 degrees in a circle. The Babylonians first started measuring angles in this way over three thousand years ago. Many historians debate about why the Babylonians chose 360 degrees. It may have to do with the fact that there are approximately 360 days in a year or that 360 is divisible by many numbers. However, choosing 360° is not based on any property of circles. This is why an alternate unit of measure called radians was developed.

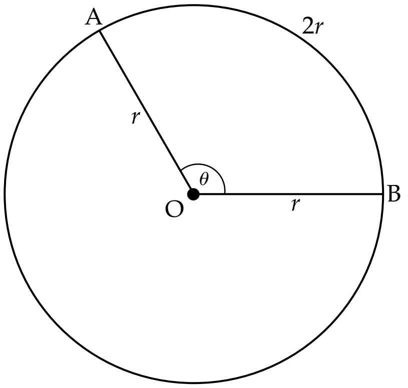
The number of radians is determined by how far a point on the terminal ray has travelled around the arc of a circle . This distance is measured in terms of the radius of the circle. To fully understand the concept of a radian, consider the following diagram.

The arc length is the length or distance measured along the circumference of a circle. Draw a circle with radius r and centre O. Draw an arc AB so that the length of the arc is the same length as the radius of the circle. The measure of the central angle, $\angle AOB$, is **1 radian**. The **arc length** equals the radius and the measure of $\angle AOB$ is said to be 1 radian.



As you can see, 1 radian is much larger than 1° since there are 360° in a full circle or 180° in a half circle.

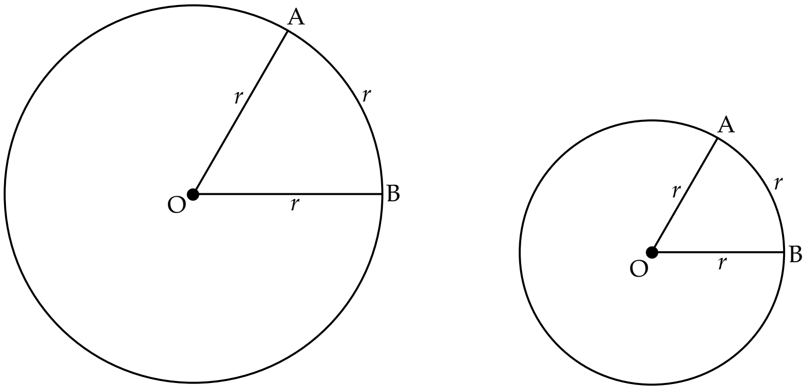
If the arc length AB were twice the size of the radius, then the measure of ∠AOB equals **2 radians**.



Note: In order to measure the length of an arc, you need a flexible object that can wrap around the circle. A piece of string works well. Try it. See that the arc AB has the same length as OB in the first diagram. In the second diagram, the arc AB is twice the length as OB.

Using this string, estimate the number of radians in a half circle. Then, estimate the number of radians in one entire circle. You should find that there are a little more than 3 radians in a half circle, and a little more than 6 radians in a full circle. You will determine these exact answers later in the lesson.

It may not be obvious to you, but the size of an angle measured in this way does not depend on the size of the circle. The central angle will be the same size of an arc length of one radius, regardless of the length of the radius. You can see this demonstrated in the two circles below.



One **radian** is the measure of a central angle θ that intercepts an arc s equal in length to the radius r of the circle.

Include the above definition and any images that you think are helpful on your resource sheet.

Radians is a unit of measure that is an alternative to degrees. Unlike degrees, there is no shorthand symbol for radians. Even though radians may be new to you, they are the preferred unit of measure in mathematics. If no symbol appears after an angle measure, the measure of the angle is assumed to be in radians. There will always be a degree symbol (°) after an angle measured in degrees.

Radians can be used instead of degrees, similar to how inches can be used instead of centimetres.

Example 1

Use the definition of radian measure to fill in the following table.

Arc Length	Radius	Central Angle (Radians)
2 cm	2 cm	
6 cm	3 cm	
20 cm	5 cm	
2 cm	4 cm	
16 cm	4 cm	

Solution

To determine the measure of the central angle in radians, you need to determine how many times the radius goes into the arc length.

Arc Length	Radius	Central Angle (Radians)
2 cm	2 cm	$\frac{2}{2} = 1$
6 cm	3 cm	$\frac{6}{3} = 2$
20 cm	5 cm	$\frac{20}{5} = 4$
2 cm	4 cm	$\frac{2}{4} = \frac{1}{2} = 0.5$
16 cm	4 cm	$\frac{16}{4} = 4$

From the chart above, you can determine the relationship among these three quantities—the arc length, the radius, and the central angle. The measure of the central angle in radians is equal to the ratio of the arc length to the radius.

$$\text{central angle} = \frac{\text{arc length}}{\text{radius}}$$

This can also be stated as:

arc length = central angle in radian measure × radius

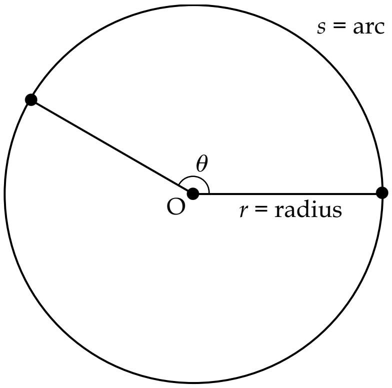
This formula can also be written as:

$s = \theta r$, where

s is the arc length

θ is the measure of the central angle in radians

r is the radius



Example 2

a) Find the length of the arc intercepted by a central angle of 3 radians if the radius of the circle is 20 cm.

b) Find the length of the radius of a circle if a central angle of 2 radians intercepts a 15 m arc.

Solutions

a) Since the angle is 3 radians, the arc length must be three times as long as the radius.

$$\begin{aligned} s &= \theta r \\ &= 3(20) \\ &= 60 \text{ cm} \end{aligned}$$

b) Since $s = \theta r$, it follows that $15 = 2(r)$ so that $r = 7.5$ m.

Converting between Radians and Degrees

Now that you know the difference between radians and degrees, it will be useful to know how to convert between them.

Since the formula for circumference of a circle is $2\pi r$, the arc length for one complete revolution is $2\pi r$.

central angle = $\frac{s}{r}$

central angle = $\frac{\text{arc length}}{\text{radius}}$

central angle = $\frac{2\pi r}{r}$

central angle = 2π radians

Measured in degrees, one revolution is 360°. Measured in radians, one revolution is 2π radians. Therefore, 2π radians = 360° or π radians = 180°.

On a previous page of this lesson, you were asked to determine how many radians were in a complete circle using a piece of string. You found an answer of a little more than 6 or 2π, radians. Similarly, you found that there were a little more than 3 or π radians in a half circle.

Example 3

a) Convert 1° into radians.

b) Convert 1 radian into degrees.

Solutions

a) Since 360° = 2π radians, dividing both sides of the equation by 360 and simplifying, you get:

$$\frac{360°}{360} = \frac{2\pi \text{ radians}}{360}$$

$$1° = \frac{\pi}{180} \text{ radians} = 0.017 \text{ radians}$$

As you know, 1° is a very small angle.

In general, you can write the following rule since π radians is the same angle of rotation as 180°

To convert θ degrees to radians, multiply θ by $\frac{\pi}{180}$.

b) Since 360° = 2π radians, dividing both sides of the equation by 2π and simplifying, you get:

$$\frac{360^\circ}{2\pi} = \frac{2\pi}{2\pi}\text{ radians}$$

$$\left(\frac{180}{\pi}\right)^\circ = 1\text{ radian}$$

$$57.3^\circ = 1\text{ radian}$$

The rule for converting any angle of rotation in radians to degrees can be written as follows.

> To convert θ radians to degrees, multiply θ by $\frac{180}{\pi}$.

You may want to include this conversion information on your resource sheet. Find a way to remember when to use each conversion factor. For example, when converting from degrees, divide by 180 (and multiply by π). Similarly, when converting from radians, divide by π (and multiply by 180).

Example 4

a) Convert 45° into radians.

b) Convert 3 radians into degrees.

Solutions

a) **Method 1:**

To convert degrees into radians, multiply the value in degrees by $\frac{\pi}{180}$.

$$45^\circ = (45^\circ)\left(\frac{\pi}{180}\right) = \frac{\pi}{4}\text{ radians} = 0.78\text{ radians}$$

The exact answer is $\frac{\pi}{4}$ radians and an approximate answer is 0.78 radians.

Method 2:

Alternatively, you can use unit analysis to convert between degrees and radians.

You know that π radians is equivalent to 180°. To convert 45° into radians, you want to eliminate degrees and only be left with radians. This can be done using the following calculation.

$$45\text{ degrees} = 45\ \cancel{\text{degrees}} \times \left(\frac{\pi\ \text{radians}}{180\ \cancel{\text{degrees}}}\right) = \frac{\pi}{4}\text{ radians} = 0.78\text{ radians}$$

b) **Method 1:**

To convert radians to degrees, multiply the value in radians by $\frac{180}{\pi}$.

$$3 = 3\left(\frac{180}{\pi}\right) = 171.9^\circ$$

Usually, an exact answer is not written for degrees as $\frac{540^\circ}{\pi}$.

Method 2:

In this example, you want to convert radians to degrees. This can be done using the following calculation, which involves unit analysis.

$$3\text{ radians} = 3\ \cancel{\text{radians}} \times \left(\frac{180\text{ degrees}}{\pi\ \cancel{\text{radians}}}\right) = 171.9\text{ degrees}$$

Special Triangles/Special Angles

In Grade 11 Pre-Calculus Mathematics, you learned about special angles that produce exact values when you calculate any trigonometric ratios. These angles were all multiples of 30°, 45°, 60°, and 90°. The unit circle, which you will be learning about in the next lesson, includes these angles in radians. Therefore, you need to be able to convert all of these special angles from degrees to radians.

Example 5

Fill in the following table by converting each of the angles to radians.

Note: Radians are sometimes written as a multiple of π, so an exact value can be written rather than a decimal approximation. The first two conversions are done for you.

Degrees	Radians	Degrees	Radians
0°	0	210°	
30°	$30^\circ = (30)\left(\frac{\pi}{180}\right) = \frac{\pi}{6}$	225°	
45°		240°	
60°		270°	
90°		300°	
120°		315°	
135°		330°	
150°		360°	
180°			

Solution

Degrees	Radians	Degrees	Radians
0°	0	210°	$210° = (210°)\left(\frac{\pi}{180}\right) = \frac{7\pi}{6}$
30°	$30° = (30)\left(\frac{\pi}{180}\right) = \frac{\pi}{6}$	225°	$225° = (225°)\left(\frac{\pi}{180}\right) = \frac{5\pi}{4}$
45°	$45° = (45°)\left(\frac{\pi}{180}\right) = \frac{\pi}{4}$	240°	$240° = (240°)\left(\frac{\pi}{180}\right) = \frac{4\pi}{3}$
60°	$60° = (60°)\left(\frac{\pi}{180}\right) = \frac{\pi}{3}$	270°	$270° = (270°)\left(\frac{\pi}{180}\right) = \frac{3\pi}{2}$
90°	$90° = (90°)\left(\frac{\pi}{180}\right) = \frac{\pi}{2}$	300°	$300° = (300°)\left(\frac{\pi}{180}\right) = \frac{5\pi}{3}$
120°	$120° = (120°)\left(\frac{\pi}{180}\right) = \frac{2\pi}{3}$	315°	$315° = (315°)\left(\frac{\pi}{180}\right) = \frac{7\pi}{4}$
135°	$135° = (135°)\left(\frac{\pi}{180}\right) = \frac{3\pi}{4}$	330°	$330° = (330°)\left(\frac{\pi}{180}\right) = \frac{11\pi}{6}$
150°	$150° = (150°)\left(\frac{\pi}{180}\right) = \frac{5\pi}{6}$	360°	$360° = (360°)\left(\frac{\pi}{180}\right) = 2\pi$
180°	$180° = (180°)\left(\frac{\pi}{180}\right) = \pi$		

Learning Activity 5.1

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. What is the domain of the function $f(x) = \sqrt{x - 3}$?
2. Solve for x: $x! = 3(x - 1)!$
3. State the non-permissible values of the function $f(x) = \dfrac{6x}{x^2 - 1}$.
4. Simplify: $\dfrac{(3x^3)(6x^2)}{x^4}$
5. How much will 12 litres of milk cost if a 4-litre jug costs $5.29?
6. Simplify: $(2x^3 - 4x^2 + 6x - 2) - (-3x^3 + 4x^2 - 3x - 1)$
7. Factor: $10x^2 - 7x - 12$
8. Evaluate: $\dfrac{7}{81} - \dfrac{2}{9}$

continued

Learning Activity 5.1 (continued)

Part B: Radians and Degrees

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Convert each of the following angles to radians. Write the exact answer.

   a) 25° b) −125°

   c) 460° d) −51°

2. Convert each of the following angles measured in radians to degrees. Round to the nearest tenth of a degree in parts (c) and (d).

   a) $-\frac{7\pi}{6}$ b) $\frac{11\pi}{12}$

   c) 2.634 d) −0.9825

3. When the angles of a triangle are measured in radians, what is the sum of the angles?

4. Determine the complement and supplement of $\frac{5\pi}{12}$.

   (Recall: Two angles are complementary if the sum of their measures is 90°. Two angles are supplementary if the sum of their measures is 180°. Note that if a question is given in radians, the answer should be in radians.)

5. Express the supplement of 130° in radian measure. Write an exact answer.

6. An isosceles triangle has a base angle with measure $\frac{2\pi}{7}$. What are the measures of the other two angles? Express the values as exact answers in radians.

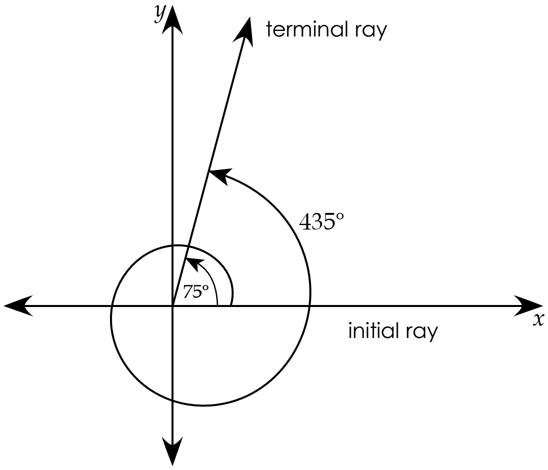
7. If a wheel having a circumference of 30 cm rolls 5 cm, how many radians has it turned? How many degrees has it turned? Write exact answers.

8. Draw a circle of any size. Using this circle, draw an angle of approximately 4 radians.

9. Draw a circle with a radius of 6 cm. Draw an arc length of 18 cm. What is the measure of the central angle subtended by this arc?

Coterminal Angles

It is possible to draw multiple angles in the same position with different measurements. These angles are called coterminal angles. Coterminal angles are two angles drawn in standard position that share a terminal side. For example, 75° and 435° are coterminal angles.



75° and 435° have the same terminal side. In other words, they are drawn in the same position. Since 360° brings an angle all the way around the circle, angles are coterminal if they differ by 360°.

$435° - 75° = 360°$

Another angle that is coterminal with these angles would be $75° - 360° = -285°$ since you can go all the way around in a clockwise direction, which is $-360°$.

There are an infinite number of angles that are coterminal with any given angle. You can simply rotate all the way around the origin either counterclockwise or clockwise. You will keep adding or keep subtracting 360° from the given angle to determine the measures of more coterminal angles.

Example 6

Determine all angles that are coterminal with the given angle over the domain $[-720°, 720°]$.

a) 25°

b) −283°

Solutions

a) To find all the angles that are coterminal with 25° over the domain [−720°, 720°], keep adding 360° until you calculate an angle that is greater than 720°. All of the angles you find that are greater than 720° are still coterminal with 25° but lie outside the domain of the question. Then, keep subtracting 360° from 25° until you calculate an angle that is less than −720°. All of the angles you find that are less than −720° are still coterminal with 25° but lie outside the domain of the question.

$25° + 360° = 385°$

$385° + 360° = 745° > 720°$

$25° - 360° = -335°$

$-335° - 360° = -695°$

$-695° - 360° = -1055° < -720°$

Therefore, the angles that are coterminal with 25° in the interval [−720°, 720°] are 385°, − 335°, and −695°.

b) $-283° + 360° = 77°$

$77° + 360° = 437°$

You can simplify this process by noticing that if you add 360° to this value, the resulting angle will not be in the required domain and, therefore, you can omit the next step.

$437° + 360° = 797° > 720°$

Now you need to subtract 360° to get the negative coterminal angles.

$-283° - 360° = -643°$

You can also omit the next step by noticing that the next coterminal angle will not be in the required domain.

$-643° - 360° = -1003° < -720°$

The angles that are coterminal with −283° in the interval [−720°, 720°] are 437°, 77°, and −643°.

It is also possible to calculate coterminal angles in radians. Remember, there are 2π radians in a circle and $360° = 2\pi$ radians. Therefore, when you are given an angle in radians and asked to find coterminal angles, you can add or subtract 2π from the angle. Consider the following example

Example 7

Determine all angles that are coterminal with the given angle over the domain $[-4\pi, 4\pi]$.

a) $\dfrac{\pi}{2}$

b) $\dfrac{5\pi}{3}$

Solutions

a) To find all angles that are coterminal with $\dfrac{\pi}{2}$ over the domain $[-4\pi, 4\pi]$, add 2π to $\dfrac{\pi}{2}$ until you calculate a value that is greater than 4π $\left(\text{which is } \dfrac{8\pi}{2}\right)$. All of the values you calculate that are less than or equal to 4π are coterminal with $\dfrac{\pi}{2}$.

$$\frac{\pi}{2} + 2\pi = \frac{\pi}{2} + \frac{4\pi}{2} = \frac{5\pi}{2}$$

$$\frac{5\pi}{2} + \frac{4\pi}{2} = \frac{9\pi}{2} > 4\pi$$

Now, subtract 2π from $\dfrac{\pi}{2}$ until you calculate a value that is less than -4π. All of the values you calculate that are greater than -4π are coterminal with $\dfrac{\pi}{2}$.

$$\frac{\pi}{2} - 2\pi = \frac{\pi}{2} - \frac{4\pi}{2} = -\frac{3\pi}{2}$$

$$-\frac{3\pi}{2} - \frac{4\pi}{2} = -\frac{7\pi}{2}$$

$$-\frac{7\pi}{2} - \frac{4\pi}{2} = -\frac{11\pi}{2} < -4\pi$$

The angles that are coterminal with $\dfrac{\pi}{2}$ over the domain $[-4\pi, 4\pi]$ are $\dfrac{5\pi}{2}$, $-\dfrac{3\pi}{2}$, and $-\dfrac{7\pi}{2}$.

b) Add $2\pi\left(\text{which is } \frac{6\pi}{3}\right)$ to $\frac{5\pi}{3}$ until you reach a value that is greater than $4\pi\left(\text{which is } \frac{12\pi}{3}\right)$.

$$\frac{5\pi}{3}+\frac{6\pi}{3}=\frac{11\pi}{3}$$

$$\frac{11\pi}{3}+\frac{6\pi}{3}=\frac{17\pi}{3}>4\pi$$

Subtract 2π from $\frac{5\pi}{3}$ until you reach a value that is less than -4π.

$$\frac{5\pi}{3}-\frac{6\pi}{3}=-\frac{\pi}{3}$$

$$-\frac{\pi}{3}-\frac{6\pi}{3}=-\frac{7\pi}{3}$$

$$-\frac{7\pi}{3}-\frac{6\pi}{3}=-\frac{13\pi}{3}<-4\pi$$

All of the angles you calculated that are in the domain $[-4\pi, 4\pi]$ are coterminal with the angle $\frac{5\pi}{3}$. These angles are $\frac{11\pi}{3}$, $-\frac{\pi}{3}$, and $-\frac{7\pi}{3}$.

Coterminal Angles in General Form

Sometimes you may be asked to list *all* the angles that are coterminal with a given angle. As it is impossible to write an infinite list of values, writing coterminal angles using a mathematical notation called general form can be quite useful.

You have already discovered that you need to add 360° to or subtract 360° from angles that are measured in degrees in order to find coterminal angles. In general form, all coterminal angles can be written as:

$\theta + (360°)n$ where $n \in I$

θ is the given angle

n is an integer $(\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots)$

You have also discovered that you need to add 2π to or subtract 2π from angles that are measured in radians in order to find coterminal angles. In general form, all coterminal angles can be written as:

$\theta + (2\pi)n$ where $n \in I$

θ is the given angle
n is an integer (. . ., −3, −2, −1, 0, 1, 2, 3, . . .)

Example 8

Express the angles that are coterminal with the following angles in general form. Then, find one angle that is coterminal with the given angle using the coterminal angles in general form formula.

a) 124°

b) $-\dfrac{2\pi}{3}$

Solutions

a) The angles that are coterminal with 124° are $124° + (360°)n$, where $n \in I$.

To determine one coterminal angle, you can choose any value for n (except $n = 0$ as that will result in the original angle).

Let $n = 2$:

$\theta = 124° + (360°)(2)$ Substituting $n = 2$.

$\theta = 124° + 720°$ Simplifying.

$\theta = 844°$

b) The angles that are coterminal with $-\dfrac{2\pi}{3}$ are $-\dfrac{2\pi}{3} + 2\pi n$, where $n \in I$.

Let $n = -1$:

$\theta_c = -\dfrac{2\pi}{3} + 2\pi\,(-1)$ Substituting $n = -1$.

$\theta_c = -\dfrac{2\pi}{3} - 2\pi$

$\theta_c = -\dfrac{2\pi}{3} - \dfrac{6\pi}{3}$ Putting each term over the same denominator.

$\theta_c = -\dfrac{8\pi}{3}$

Sketching Angles

In order to draw angles that are over 360° or 2π radians and negative angles that are under 0° or 0 radians, you can use a corresponding coterminal angle that is located in the domain $0° \leq \theta \leq 360°$ or $0 \leq \theta \leq 2\pi$.

Example 9

Sketch the following angles.

a) $528°$

b) $-1237°$

c) -7π

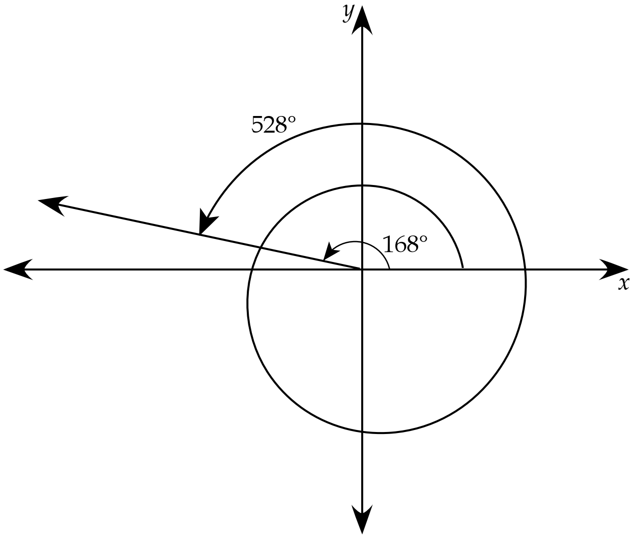
d) $\dfrac{11\pi}{2}$

Solutions

a) You can use a coterminal angle of 528° that is between 0° and 360°. To do this, subtract 360° from 528° until you arrive at an angle that satisfies those conditions.

$$528° - 360° = 168°$$

To sketch 528°, simply sketch an angle of 168°.



Note: You must show the path of the entire angle (528°). This illustrates both the direction of the angle and the number of revolutions.

b) To determine a coterminal angle of −1237° that is between 0° and 360°, keep adding 360° to −1237° until you arrive at an angle that satisfies these conditions.

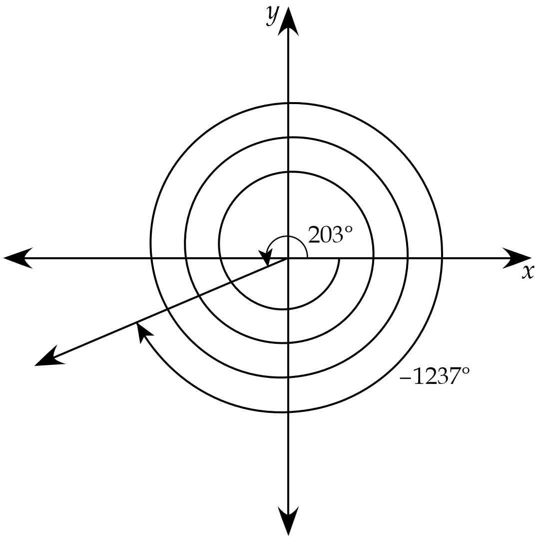
$-1237° + 360° = -877°$

$-877° + 360° = -517°$

$-517° + 360° = -157°$

$-157° + 360° = 203°$

Thus, 203° and −1237° are coterminal angles and share the same terminal side.



Note: You could have drawn a clockwise angle to represent −157° instead of using 203°. To draw a negative angle, you measure clockwise through the quadrants. Therefore, to draw an angle of −1237°, use the following steps:

- Start at 0°
- Head clockwise
- Make three complete 360° circles through the quadrants
- Continue your fourth circle through the quadrant until you reach the terminal arm of −157° or 203° (located in Quadrant III)
- This is your angle of −1237°

c) To determine a coterminal angle of -7π that is between 0 and 2π, keep adding 2π to -7π until you arrive at an angle that is between 0 and 2π.

$-7\pi + 2\pi = -5\pi$

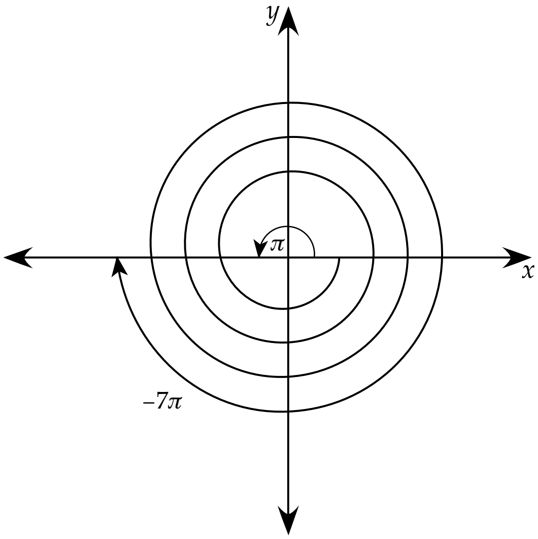
$-5\pi + 2\pi = -3\pi$

$-3\pi + 2\pi = -\pi$

$-\pi + 2\pi = \pi$

Recall: 2π radians represents a complete circle, or 360°, and thus π radians represents half a circle, or 180°.

As you can see, -7π radians represents travelling clockwise 3 and a half times.

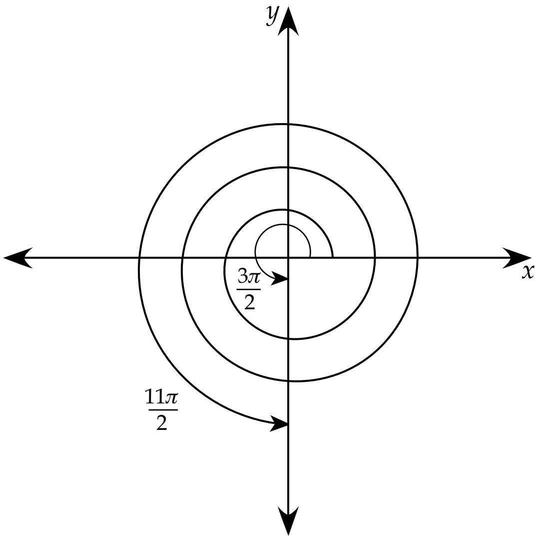


d) To determine a coterminal angle of $\frac{11\pi}{2}$ that is between 0 and 2π, keep subtracting 2π from $\frac{11\pi}{2}$ until you arrive at an angle that is between 0 and 2π.

$$\frac{11\pi}{2} - 2\pi = \frac{11\pi}{2} - \frac{4\pi}{2} = \frac{7\pi}{2}$$

$$\frac{7\pi}{2} - \frac{4\pi}{2} = \frac{3\pi}{2}$$

To draw an angle of $\frac{3\pi}{2}$ radians, you need to go three-quarters of the way through the quadrants. This is the same as an angle of 270°.



Make sure you complete the following learning activity, as it will allow you to practice the skills you just learned in this lesson.

Learning Activity 5.2

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Rationalize the denominator: $\frac{1}{\sqrt{2}}$
2. Determine if the following set of points represents a one-to-one function: $(1, 5), (-6, 3), (-4, 6), (7, -5)$
3. How many ways can you arrange 4 pictures on a shelf?
4. Estimate the taxes, 13%, on a \$2450 item.
5. If $f(x) = 2x^3 - 7$, evaluate $f(x)$ at $x = 4$.
6. Simplify: $| -2 + 5(-4) |$.
7. Factor: $4x^2 - 64$
8. If you have cycled through 18 out of the 45 km of a bicycle route, what fraction of the route have you completed?

continued

Part B: Coterminal Angles

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Determine all of the angles that are coterminal with the given angle over the domain [−1080°, 360°].
   a) 258°
   b) 613°
   c) −142°
   d) −631°
2. Determine all of the angles that are coterminal with the given angle over the domain $[-6\pi, 4\pi]$.
   a) $-\frac{9\pi}{4}$
   b) $-\frac{13\pi}{6}$
   c) $-\frac{\pi}{14}$
   d) $\frac{2\pi}{7}$
3. Write all the angles that are coterminal with the following angles.
   a) $-\frac{\pi}{8}$
   b) 416°
   c) −17°
   d) $\frac{3}{7}$

continued

Learning Activity 5.2 (continued)

4. Determine whether the following sets of angles are coterminal.
   a) $-365°$ and $715°$
   b) $\frac{\pi}{4}$ and $-\frac{21\pi}{4}$

5. Sketch the following angles.
   a) $-672°$
   b) $714°$
   c) 9π
   d) $-\frac{5\pi}{2}$

6. Your younger brother is playing in a revolving door at a department store. Before you can stop him, he has rotated the door through an angle of 2700°. If he was originally inside the store, where is he now? Explain.

Lesson Summary

In this lesson, you were introduced to radians. You learned what 1 radian represents, as well as how to convert between degrees and radians. You also learned how to calculate coterminal angles or angles that have the same terminal side. You will be using all of this information to develop the concept of the unit circle in the next lesson.

LESSON 2: THE UNIT CIRCLE

Lesson Focus

In this lesson, you will

- ☐ learn how to locate the position of the terminal point P(θ) on the unit circle for special values of the real number θ
- ☐ learn how to state a value of θ, which would produce a given position of a terminal point P(θ)
- ☐ learn how to state the coordinates of P(θ) if θ has a special value
- ☐ learn how to find the equation of the unit circle using the Pythagorean theorem
- ☐ learn how to find the general equation of a circle centred on the origin

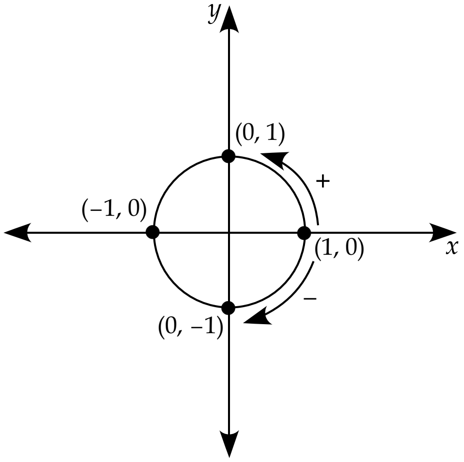
Lesson Introduction

In the last lesson, you learned how to convert between radians and degrees, as well as how to find coterminal angles. You also learned that a relationship exists between the length of an arc on a circle and the size of the corresponding angle at the centre of the circle. These concepts will be further developed in this lesson, as you will be learning how to locate special points on the unit circle.

The Unit Circle

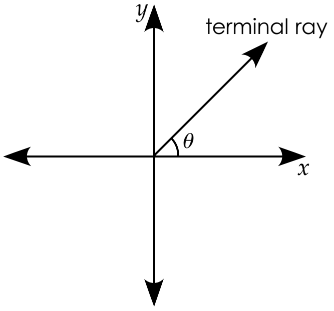
The **unit circle** is the name given to a circle with centre at the origin and with a radius of one unit.

The formula for finding the circumference of a circle is $C = 2\pi r$. The circumference of this circle is $2\pi r = 2\pi(1) = 2\pi$.

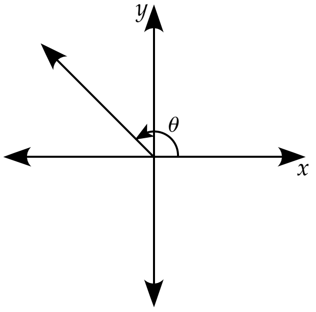


From Grade 11 Pre-Calculus Mathematics, you may remember how to draw angles in standard position. Four sketches of angles in standard position, each demonstrating an angle in a different quadrant, are shown below. Throughout this lesson, you are going to be examining angles when the terminal ray has a length of 1, so that the endpoint of the ray will be on the unit circle.

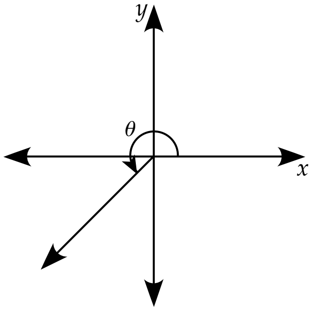
Quadrant I



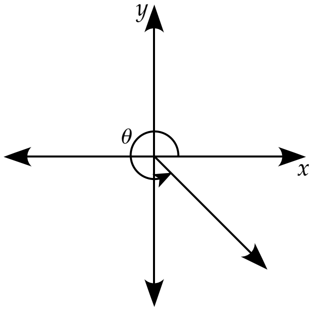
Quadrant II



Quadrant III



Quadrant IV



Special Values on the Unit Circle

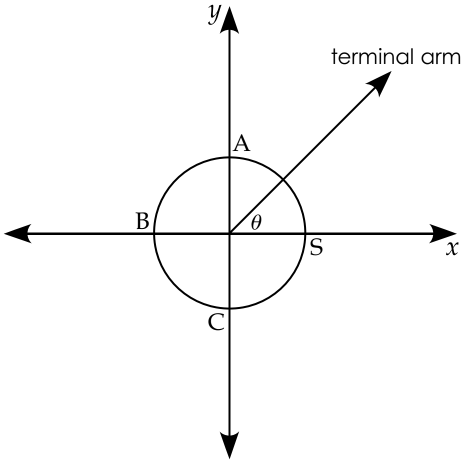
To develop the unit circle, you need to move certain distances around this circle, governed by the following rules.

1. Consistent with angles of rotation, on the unit circle the point with coordinates (1, 0) is the starting point. The angle is said to be in **standard position** when the starting point of the arc is (1, 0).
2. If the angle to be moved is positive, you will move in a counter-clockwise direction along the unit circle.
3. If the angle to be moved is negative, you will move in a clockwise direction along the unit circle.
4. The angle to be moved is called θ. It is the angle of rotation from the starting point (1, 0). It is a real number.
5. The notation, P(θ), will be used to indicate the point on the unit circle where the angle of rotation ends after you have rotated through an angle of θ units. P(θ) will be called the **terminal point**.

Note: θ is an angle of rotation. When using radians, it is the same as the arc length on the unit circle. That is the advantage of using radians over degrees.

The following diagram will be used to label some special points on the unit circle.

Diagram 1



Where is P(0)? This notation asks, "What are the coordinates of the terminal point if you do not move from the starting point?" The answer is S, since you always begin measuring the angle of rotation from the point with coordinates (1, 0).

Where is P(π)? This asks for the terminal point of a counter-clockwise trip halfway around the circle. Thus, the answer is the point B with coordinates (−1, 0).

Recall: Once around a circle is 2π radians and halfway around a circle is π radians.

It follows that $P\left(\frac{\pi}{2}\right)$ is located at A with coordinates (0, 1), as $\frac{\pi}{2}$ radians is a quarter of the way around a circle. Also, $P\left(\frac{3\pi}{2}\right)$ is located at C with coordinates (0, −1), as $\frac{3\pi}{2}$ radians is three-quarters of the way around a circle.

Example 1

Use Diagram 1 to fill in the following table with a letter corresponding to the position of the terminal arm after rotating θ radians around the unit circle.

θ	$-\pi$	$\frac{\pi}{2}$	$-\frac{\pi}{2}$	$\frac{3\pi}{2}$	$-\frac{3\pi}{2}$	-2π	$\frac{5\pi}{2}$	$-\frac{7\pi}{2}$	2π	101π	100π
P(θ)											

Solution

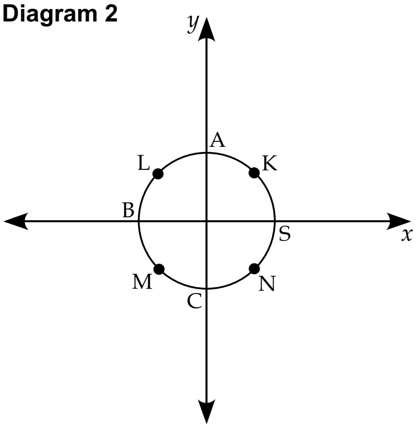
θ	$-\pi$	$\frac{\pi}{2}$	$-\frac{\pi}{2}$	$\frac{3\pi}{2}$	$-\frac{3\pi}{2}$	-2π	$\frac{5\pi}{2}$	$-\frac{7\pi}{2}$	2π	101π	100π
P(θ)	B	A	C	C	A	S	A	A	S	B	S

Note: To find P(101π), notice that when you travel halfway around the circle, you are travelling π radians, or an odd multiple of radians. Therefore, every time you travel to point B, you will have travelled an odd multiple of radians.

To find P(100π), notice that when you travel completely around the circle, you are traveling 2π radians, or an even multiple of radians. Therefore, every time you travel to point S, you will have travelled an even multiple of radians.

To further develop the unit circle, four additional points are inserted at the midpoints of each quarter arc.

Diagram 2



$P\left(\frac{\pi}{4}\right)$ is located at K. In other words, an angle of rotation of $\frac{\pi}{4}$ radians terminates at point K. Each successive point indicates an angle that is an additional $\frac{\pi}{4}$ radians larger than the previous angle. Therefore, $P\left(\frac{2\pi}{4}\right) = P\left(\frac{\pi}{2}\right)$ is located at A. This follows what you learned in Example 1.

Example 2

Use Diagram 2 to fill in the corresponding letter for each angle of rotation in the following table.

θ	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$-\frac{\pi}{4}$	$-\frac{3\pi}{4}$	$-\frac{5\pi}{4}$	$\frac{7\pi}{4}$	$\frac{5\pi}{4}$	$-\frac{7\pi}{4}$	$\frac{11\pi}{4}$	$-\frac{57\pi}{4}$	$\frac{83\pi}{4}$
$P(\theta)$											

Solution

θ	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$-\frac{\pi}{4}$	$-\frac{3\pi}{4}$	$-\frac{5\pi}{4}$	$\frac{7\pi}{4}$	$\frac{5\pi}{4}$	$-\frac{7\pi}{4}$	$\frac{11\pi}{4}$	$-\frac{57\pi}{4}$	$\frac{83\pi}{4}$
$P(\theta)$	K	L	N	M	L	N	M	K	L	N	L

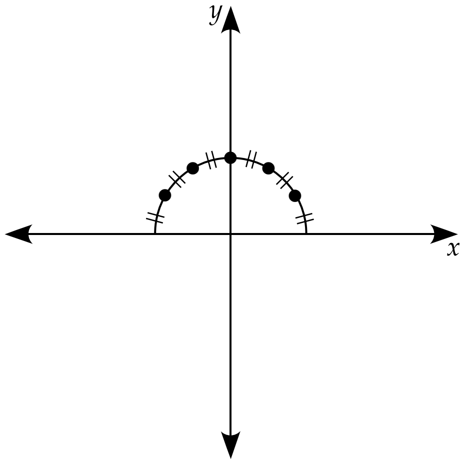
Note: To find $P\left(-\frac{57\pi}{4}\right)$, first find an appropriate coterminal angle.

$-\frac{57\pi}{4}$ is coterminal with $-\frac{\pi}{4}$. That is, you can rotate 7 times around $\left(\text{since } 7 \times 2\pi = 7 \times \frac{8\pi}{4} = \frac{56}{4}\right)$ in a clockwise direction. Therefore, $P\left(-\frac{56\pi}{4}\right) = S$. However, you still need to travel $-0.25\pi = -\frac{\pi}{4}$ additional radians $\left(P\left(-\frac{57\pi}{4}\right) = N\right)$.

To find $P\left(\frac{83\pi}{4}\right)$, you can find a coterminal angle. $\frac{83\pi}{4}$ is coterminal with $\frac{3\pi}{4}$. That is, you can rotate 10 times around $\left(\text{since } 10 \times 2\pi = 10 \times \frac{8\pi}{4} = \frac{80\pi}{4}\right)$ in a counter-clockwise direction. Therefore $P\left(\frac{80\pi}{4}\right) = S$. An additional $\frac{3\pi}{4}$ radians in the positive direction will lead you to the point L, and thus $P\left(\frac{83\pi}{4}\right) = L$.

To complete the list of special angles of rotation that you need to know on the unit circle, the semicircle is subdivided into six angles of rotation of equal radians.

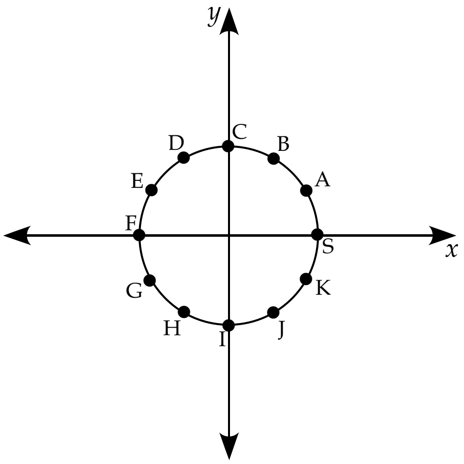
Diagram 3



Each small arc has a length of $\frac{\pi}{6}$. Since $\frac{2\pi}{6}$ reduces to $\frac{\pi}{3}$, it follows that you will have multiples of $\frac{\pi}{6}$, some of which are also multiples of $\frac{\pi}{3}$, as you travel around the circle.

Example 3

Use the diagram below to fill in the table, using the fact that $\mathrm{P}\left(\frac{\pi}{6}\right) = \mathrm{A}$. Each successive point indicates an angle that is an additional $\frac{\pi}{6}$ radians larger than the previous angle.



θ	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$-\frac{\pi}{3}$	$-\frac{2\pi}{3}$	$-\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{5\pi}{3}$	$-\frac{7\pi}{6}$	$\frac{11\pi}{6}$	$\frac{7\pi}{3}$	$-\frac{13\pi}{6}$
P(θ)											

Solution

θ	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$-\frac{\pi}{3}$	$-\frac{2\pi}{3}$	$-\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{5\pi}{3}$	$-\frac{7\pi}{6}$	$\frac{11\pi}{6}$	$\frac{7\pi}{3}$	$-\frac{13\pi}{6}$
P(θ)	A	B	J	H	G	G	J	E	K	B	K

In summary, the special values of θ for angles of rotation on the unit circle in one counter-clockwise revolution are:

Radians	which corresponds to	**Degrees**
$0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$		0°, 90°, 180°, 360°
$\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$		30°, 150°, 210°, 330°
$\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$		45°, 135°, 225°, 315°
$\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$		60°, 120°, 240°, 300°

Other special values are multiples of these basic ones.

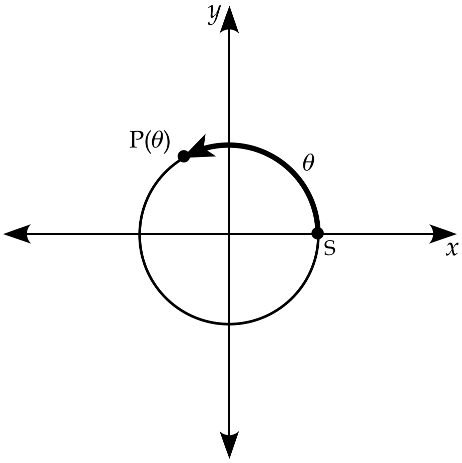
Now that you know these special values of θ, you can determine the coordinates of P(θ). Notice that the radian measures with a denominator of 2 have reference angles of 90°. Radian measures with a denominator of 6 have reference angles equal to 30°. Radian measures with a denominator of 4 have reference angles of 45°. Finally, radian measures with a denominator of 3 have reference angles of 60°.

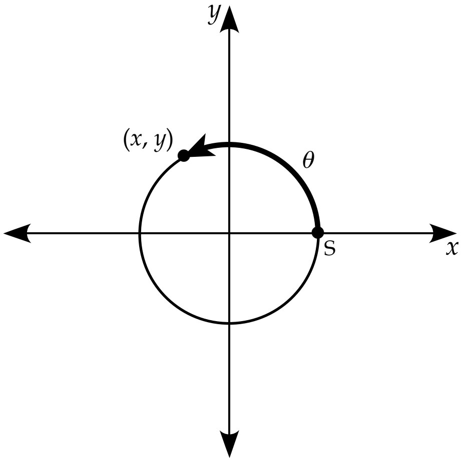
The Coordinates of P(θ)

For every angle of rotation θ on the unit circle there is only one terminal point, P(θ), determined by this angle.

For example, when $\theta = 0$, $P(0) = (1, 0)$. This is the starting point on the unit circle. Also, when $\theta = \frac{\pi}{2}$, $P\left(\frac{\pi}{2}\right) = (0, 1)$. These are the coordinates at the top of the unit circle.

This relationship is illustrated in the diagrams below.



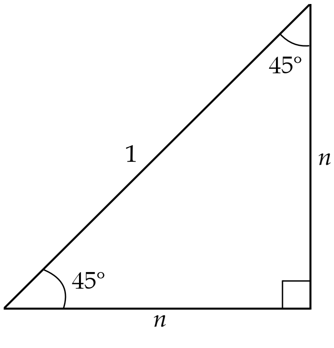


The above two examples were rather easy because the terminal point was on an axis. What happens when the terminal point is not on an axis? Consider the first quadrant of the unit circle.

Remember, on the unit circle, the radian measure of an angle and the arc-length spanned by that angle are the same value. This will always happen, since the radius of the unit circle is 1 and the conversion formula $s = \theta r$ always becomes $s = \theta(1) = \theta$. But be careful. The arc length is a real number representing length, while the radian measure is the same real number but represents the measure of the corresponding central angle in radians. For example, if the radius of a circle is 1 cm and the arc length is 3 cm, then the central angle is 3 radians. This is one reason why it is useful to measure angles in radians.

Angle of Rotation $\theta = \frac{\pi}{4}$ and $P\left(\frac{\pi}{4}\right)$

Recall what you learned about the 45–45–90 triangle in Grade 11 Pre-Calculus Mathematics. You will use these values to find coordinates on the unit circle.

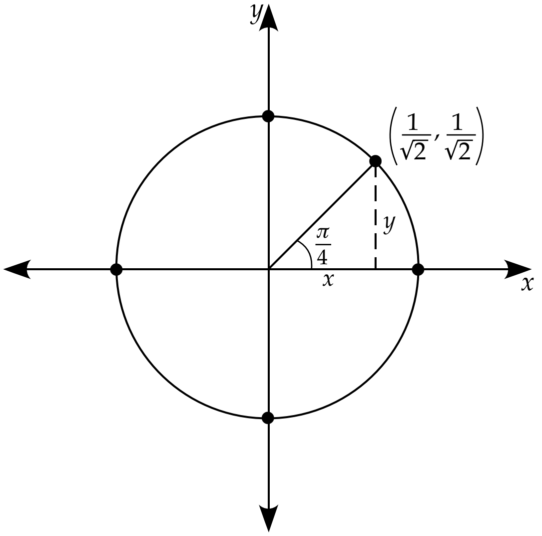


$$\begin{aligned} n^2 + n^2 &= 1 \\ 2n^2 &= 1 \\ n^2 &= \frac{1}{2} \\ n &= \frac{\sqrt{1}}{\sqrt{2}} \\ n &= \frac{1}{\sqrt{2}} \end{aligned}$$

Note: $\frac{1}{\sqrt{2}}$ is equivalent to $\frac{\sqrt{2}}{2}$.

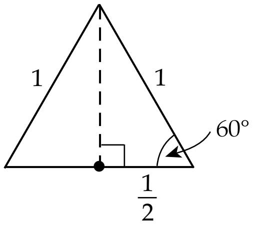
This is an isosceles triangle because it has two identical angles and the sides opposite these angles are of equal length.

You can draw the same triangle on the unit circle in Quadrant I. The angle of 45° corresponds to an angle of $\frac{\pi}{4}$ radians. For the x-coordinate and the y-coordinate on the unit circle, you need to find the horizontal length and the vertical length of the triangle. In this example, both of these lengths are $\frac{1}{\sqrt{2}}$ as calculated above, since the radius is 1 unit.



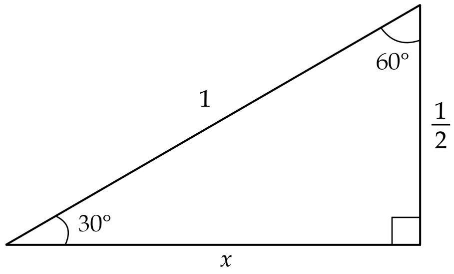
Coordinates for $P\left(\frac{\pi}{6}\right)$ and $P\left(\frac{\pi}{3}\right)$

Recall the 30–60–90 triangle you discovered in Grade 11 Pre-Calculus Mathematics. In a 30–60–90 triangle, the length of the shortest side is always half the length of the hypotenuse. This can be shown by drawing an equilateral triangle with side lengths of 1 unit and all angles of 60°. Draw an altitude from one vertex to the midpoint of the opposite side.



When the hypotenuse is 1, the length of the shortest side of the triangle is $\frac{1}{2}$.

You can find the length of the third side of the triangle using the Pythagorean theorem, as shown below.



$$\left(\frac{1}{2}\right)^2 + x^2 = 1^1$$

$$\frac{1}{4} + x^2 = 1$$

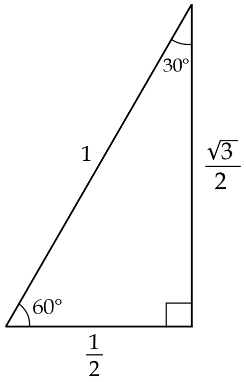
$$x^2 = 1 - \frac{1}{4}$$

$$x^2 = \frac{3}{4}$$

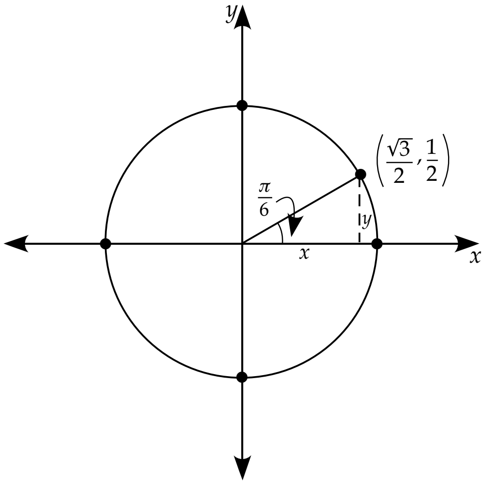
$$x = \frac{\sqrt{3}}{\sqrt{4}}$$

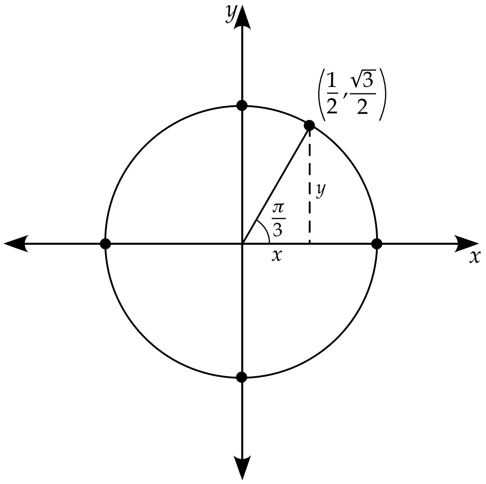
$$x = \frac{\sqrt{3}}{2}$$

You can also rotate this triangle to arrive at the 60–30–90 triangle.



Both of these triangles can be drawn on the unit circle. Recall that 30° is equivalent to $\frac{\pi}{6}$ radians and 60° is equivalent to $\frac{\pi}{3}$ radians.



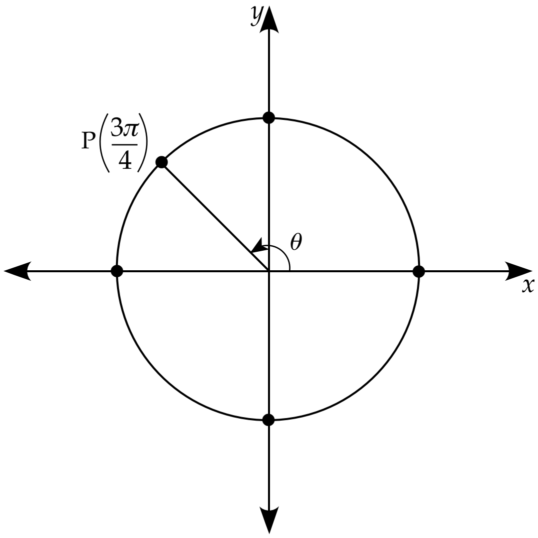


Using the symmetry of these triangles drawn in the other quadrants, you can now find the values for the coordinates for all the special points on the unit circle.

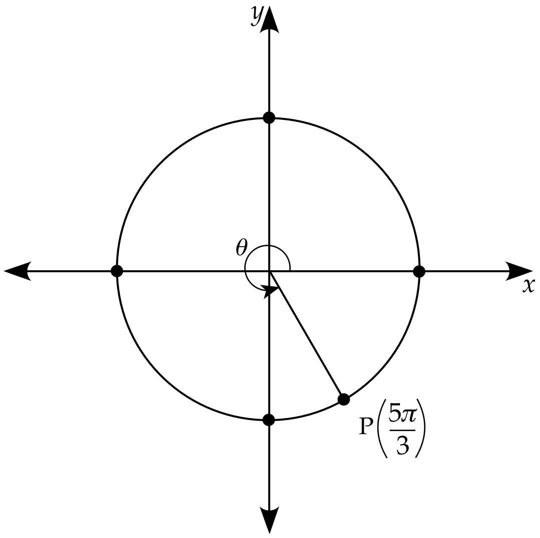
Example 4

Determine the coordinates of each of the following points on the unit circle.

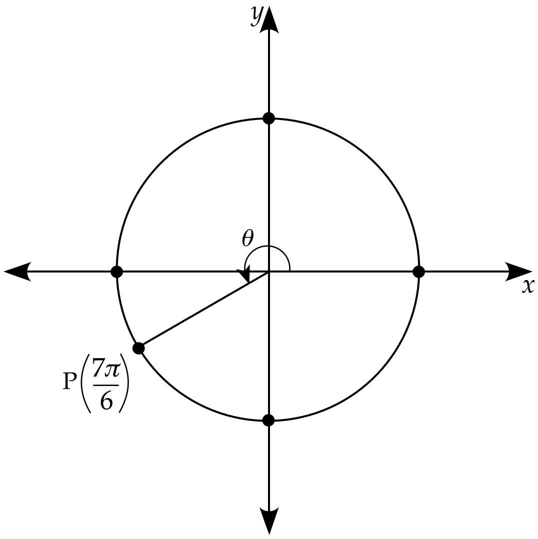
a)



b)

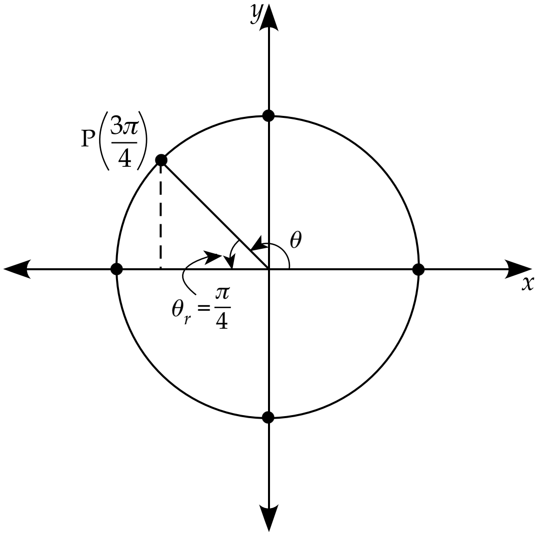


c)



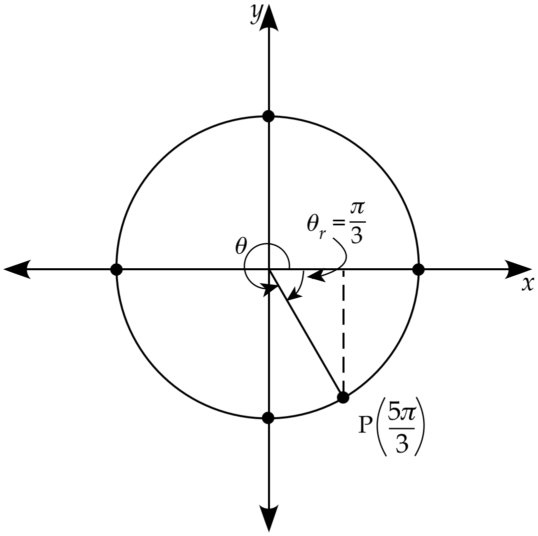
Solutions

a) To determine the coordinates on the unit circle, you need to determine which triangle the point corresponds to.



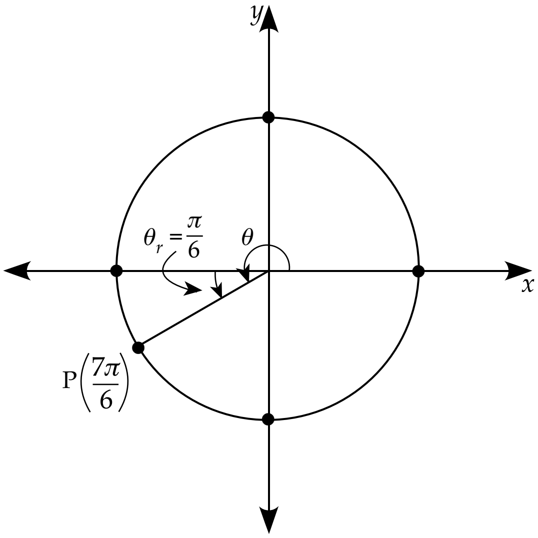
The reference angle is $\frac{\pi}{4}$. Recall the reference angle is the angle made between the terminal arm of the angle and the closest x-axis. Therefore, you are looking at the 45–45–90 triangle. This triangle has an x-coordinate of $\frac{1}{\sqrt{2}}$ and a y-coordinate of $\frac{1}{\sqrt{2}}$. However, in Quadrant II, x is negative. The coordinates of this point are $P\left(\frac{3\pi}{4}\right) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

b)



The reference angle is $\frac{\pi}{3}$. Therefore, you are looking at the 30–60–90 triangle. From this triangle, you can tell that the distance travelled in the x-direction will be smaller than the distance travelled in the y-direction. Thus, the x-coordinate will be $\frac{1}{2}$ and the y-coordinate will be $\frac{\sqrt{3}}{2}$. In Quadrant IV, y is negative. The coordinates of this point are $P\left(\frac{5\pi}{3}\right) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

c)



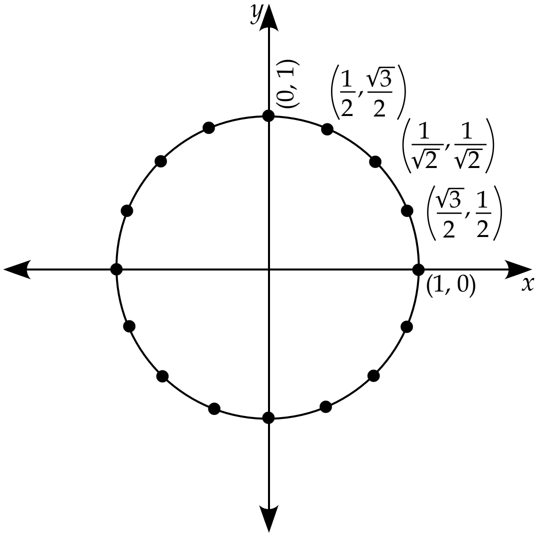
The reference angle is $\frac{\pi}{6}$.

Therefore, you are looking at the 30–60–90 triangle. This triangle is in the opposite orientation from (b). Thus, the *x*-coordinate will be larger than the *y*-coordinate. In Quadrant III, both the *x*-coordinate and the *y*-coordinate are negative. The coordinates of this point are $P\left(\frac{\pi}{6}\right) = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

The chart on the following page summarizes all of the special values on the unit circle, including the angles in both degrees and radians. Use this chart to complete the unit circle.

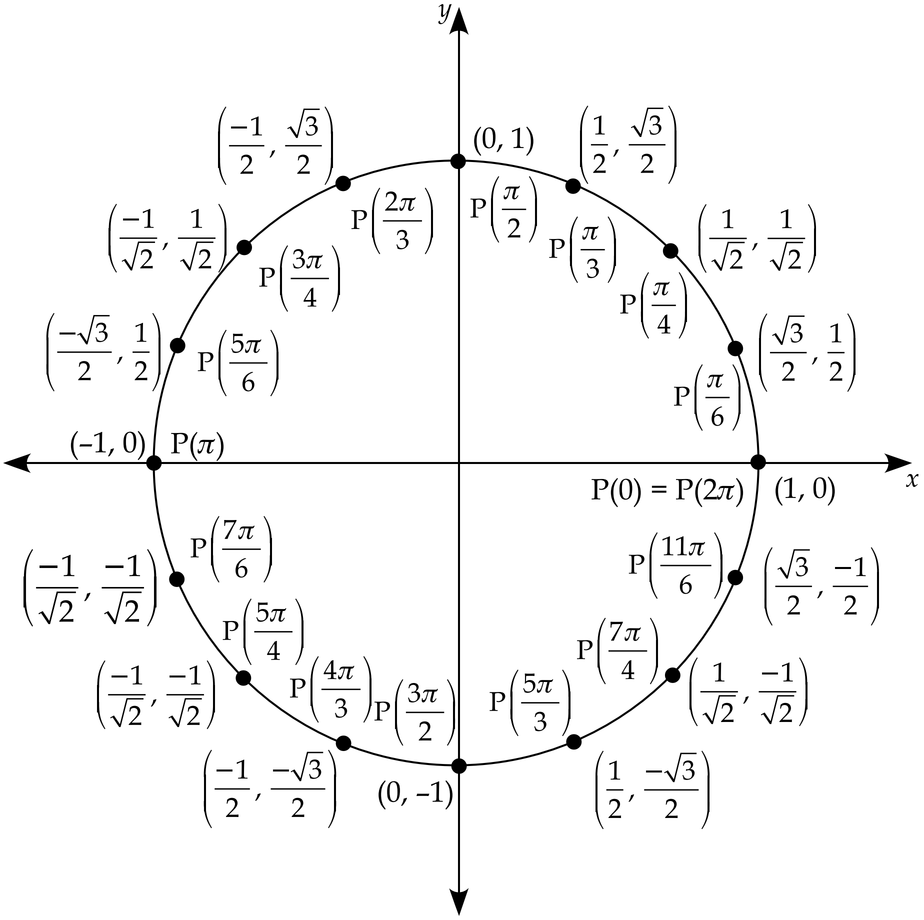
Example 5

Use the given coordinates to help you find the coordinates of every other labelled special point. Include the θ-value in radians for each point indicated.



Solution

These are all the special values on the unit circle. Using the previous chart, you can also add in the degree values for each of the angles in radians.



Notice the symmetry of coordinate values between Quadrant I and the other quadrants.

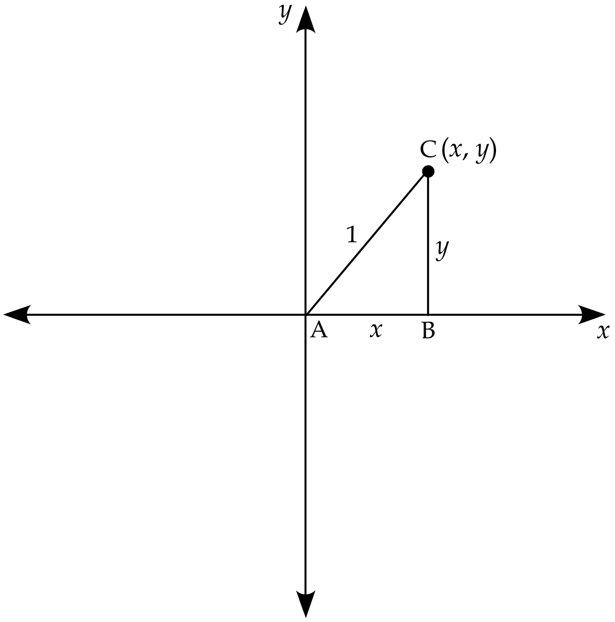
Practise copying this diagram until you have it memorized. Then, always have a copy of it handy, but only use it if you forget values you need to use when solving trigonometric questions that use the special values. Once you have memorized the diagram, it should not take you more than a few minutes to make your own copy when you need it, such as during an examination. In calculus courses, students are expected to have memorized the coordinates of each of these values on the unit circle associated with specific angles of rotation.

To practice writing the angles and coordinates on the unit circle, two blank unit circles are included at the end of this lesson.

The Equation of the Unit Circle

Using the Pythagorean theorem and what you know about the unit circle so far, you can develop the equation of the unit circle.

Consider a point (x, y) on the unit circle.



The distance from A to B, or AB, is the x-coordinate of the point (x, y).

The distance from B to C, or BC, is the y-coordinate of the point (x, y).

You also know that the radius of the unit circle is always 1.

You can use the Pythagorean theorem to develop the equation of the unit circle:

$$|AB|^2 + |BC|^2 = 1^2$$
$$|x|^2 + |y|^2 = 1$$
$$x^2 + y^2 = 1$$

This is the equation of the unit circle. Every point on the unit circle will satisfy this equation.

You may want to add the unit circle equation to your resource sheet.

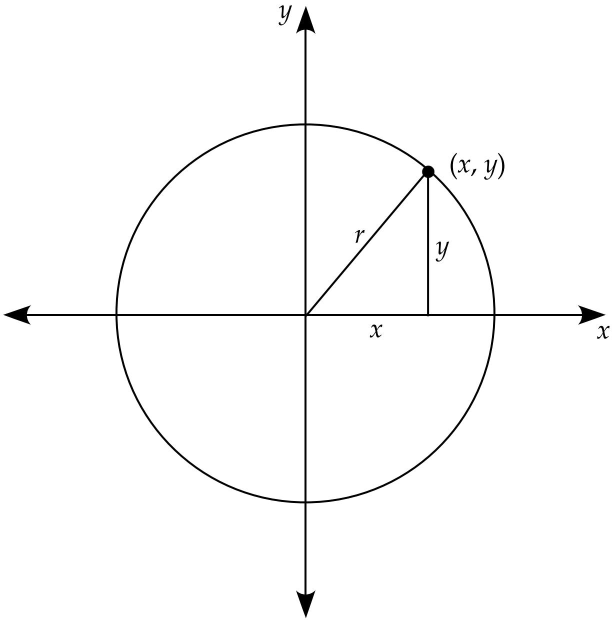
The Equation of a Circle Centred on the Origin

What do you think happens if the radius of a circle changes?

If you let the radius of a circle be represented by r, the equation of any circle, centred on the origin, becomes:

$$x^2 + y^2 = r^2$$

The x-component is the distance moved along the x-axis while the y-component is the distance moved along the y-axis. The radius of the circle forms the hypotenuse of the triangle these line segments create. Consider the following image.



From the diagram, you can see how you can use the Pythagorean theorem to develop the general equation of a circle.

Learning Activity 5.3

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. State the non-permissible values of the function $f(x) = \dfrac{x-5}{x^2}$.
2. Find the positive coterminal angle for 502° in the interval $0° \le \theta \le 360°$.
3. What is the domain of the function $f(x) = \dfrac{1}{\sqrt{x}}$?
4. Rationalize the denominator: $\dfrac{1}{\sqrt{3}}$
5. Simplify: $\sqrt{32x^4y^6z^3}$
6. Multiply: $\left(\sqrt{3} - 2\sqrt{6}\right)\left(\sqrt{3} + 2\sqrt{6}\right)$
7. List all the factors of 81.
8. Evaluate: $\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4}$

continued

Learning Activity 5.3 (continued)

Part B: The Unit Circle

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Use the equation of the unit circle to show that the following coordinates are points on the unit circle.

   a) $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

   b) $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

2. Are the following points on the unit circle? Explain.

   a) $\left(\frac{1}{3}, \frac{\sqrt{8}}{3}\right)$

   b) $\left(\frac{1}{4}, \frac{\sqrt{5}}{2}\right)$

   c) $\left(\frac{5}{6}, \frac{\sqrt{8}}{6}\right)$

   d) $\left(\frac{\sqrt{23}}{5}, \frac{\sqrt{2}}{5}\right)$

3. Determine the missing coordinates of each of the following points that are located on the unit circle.

   a) $\left(x, \frac{\sqrt{6}}{4}\right)$

   b) $\left(x, \frac{\sqrt{2}}{\sqrt{3}}\right)$

   c) $\left(\frac{\sqrt{6}}{\sqrt{7}}, y\right)$

   d) $\left(\frac{\sqrt{15}}{4}, y\right)$

4. Determine the exact coordinates corresponding to the following angles of rotation on the unit circle. Try to do this from memory using the lengths of sides of the 45–45–90 or the 30–60–90 triangles.

   a) $-\frac{\pi}{4}$

   b) $\frac{25\pi}{6}$

   c) $\frac{13\pi}{2}$

   d) $-\frac{11\pi}{3}$

continued

Learning Activity 5.3 (continued)

5. Given the following points on the unit circle, find *two* angles of rotation that correspond to each point.

   a) $(0, 1)$

   b) $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

   c) $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

   d) $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

6. Determine the equation of a circle centred at the origin with a radius of 7 cm.

7. If you haven't already, fill out one of the empty unit circles that are located at the end of this lesson. Some students find the unit circle difficult to memorize. However, if you look for patterns, memorizing the unit circle can be as easy as memorizing the values in only one quadrant.

   a) What patterns do you notice in the denominators of the coordinates on the unit circle?

   b) What patterns do you notice in the numerators of the coordinates on the unit circle?

   c) Where are the x-coordinates positive? Negative?

   d) Where are the y-coordinates positive? Negative?

   e) Colour each terminal arm according to the largest angle for which it is a multiple, using the following key. (For example, $\frac{\pi}{3} = \frac{2\pi}{6}$ is a multiple of $\frac{\pi}{3}$ and $\frac{\pi}{6}$. Therefore, the terminal arm is coloured blue because $\frac{\pi}{3}$ is the largest angle for which it is a multiple.)

      Multiples of $\frac{\pi}{3}$ $\longrightarrow$ blue

      Multiples of $\frac{\pi}{4}$ $\longrightarrow$ red

      Multiples of $\frac{\pi}{6}$ $\longrightarrow$ green

      Multiples of $\frac{\pi}{2}$ $\longrightarrow$ orange

continued

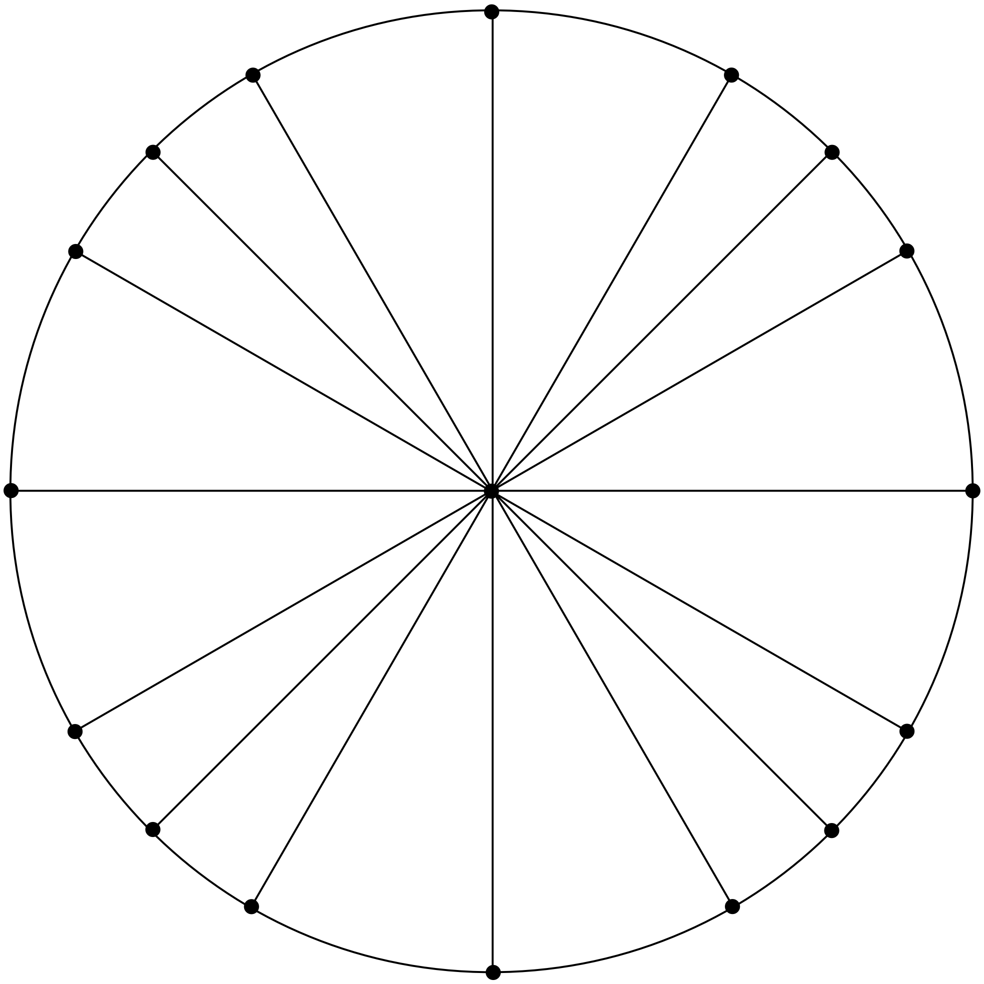
Learning Activity 5.3 (continued)

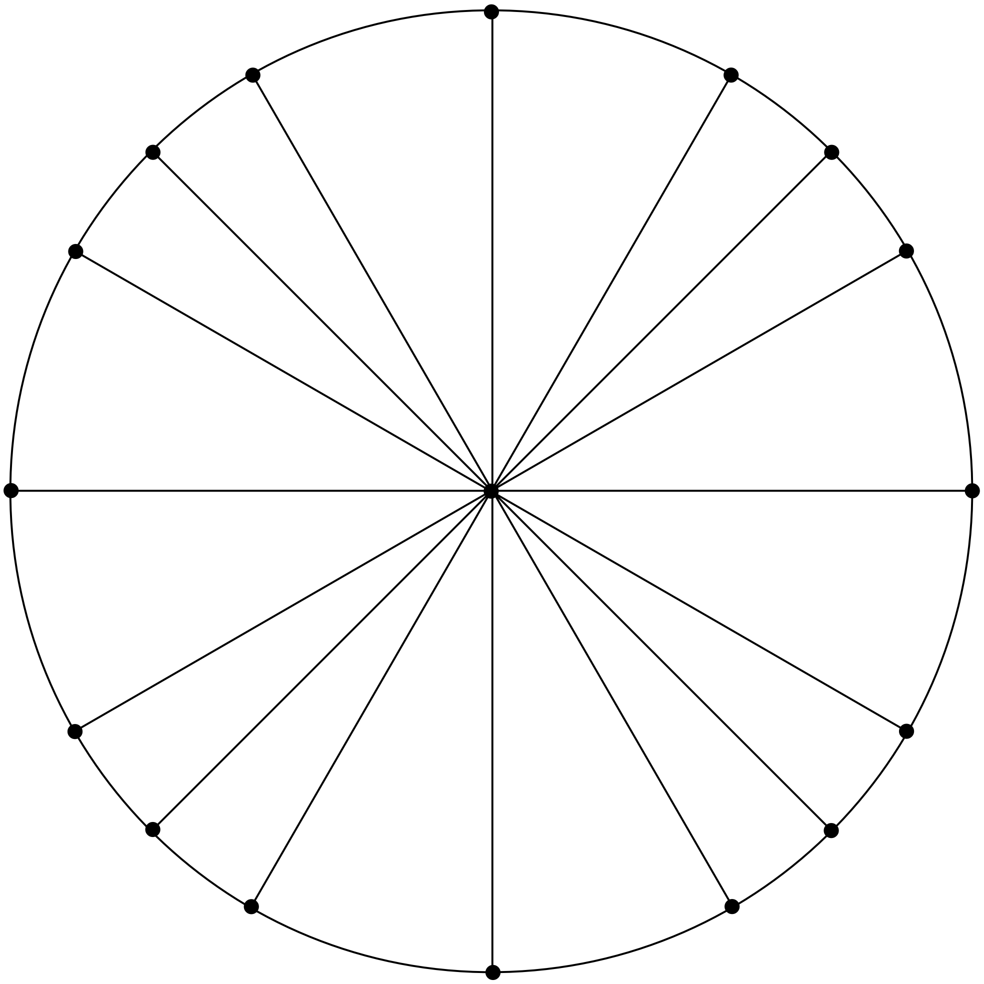
f) What patterns do you notice in the colours of the terminal arms of the special angles located on the unit circle?

8. Explain one property of the unit circle.

Lesson Summary

In this lesson, you learned the equation of and particular coordinates on the unit circle. You calculated all of the special values on the unit circle and looked for patterns in these values. In the next lesson, you will learn how the unit circle coordinates are related to trigonometry. The value of the unit circle is to use the special values to answer questions about exact trigonometric ratios of sine, cosine, and tangent. It is important that you know these special values on the unit circle.





Assignment 5.1

Degrees, Radians, and the Unit Circle

Total: 31 marks

You need to show all your steps as you work out your solutions, and make sure your answers are clear (include units, where appropriate). Check the Introduction for the Marking Guidelines that will be used for all assignments and examinations.

1. Convert each of the following angles to radians. Simplify and write the exact answer. *(2×1 mark each = 2 marks)*
   a) $-572°$

   b) $724°$

2. Convert each of the following angles to degrees. *(2×1 mark each = 2 marks)*
   a) $-\frac{4\pi}{9}$

   b) $\frac{7\pi}{3}$

3. Explain how you would draw an angle of 3 radians on any given circle. *(1 mark)*

continued

4. Determine all of the angles that are coterminal with the given angle over the domain [−720°, 720°]. *(2 × 3 marks each = 6 marks)*
   a) 310°

   b) −681°

5. Determine all of the angles that are coterminal with the given angle over the domain $[-4\pi, 4\pi]$. *(2 × 3 marks each = 6 marks)*
   a) $-\dfrac{\pi}{6}$

continued

Assignment 5.1: Degrees, Radians, and the Unit Circle (continued)

b) $\frac{3\pi}{4}$

6. Express the angles that are coterminal with the following angles in general form. *(2 × 1 mark each = 2 marks)*

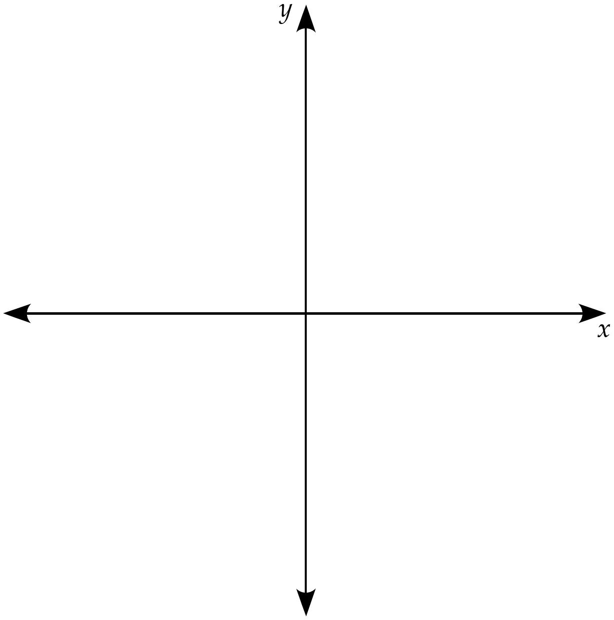
a) $\frac{11\pi}{3}$

b) 276°

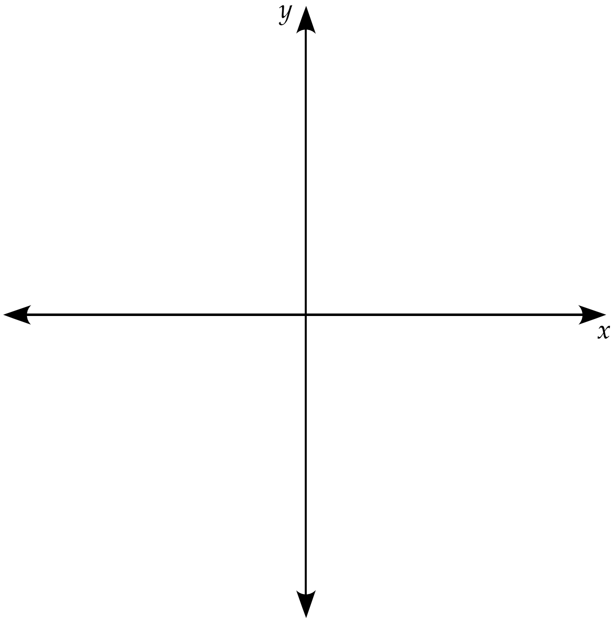
continued

7. Sketch the following angles. Be sure to state the coterminal angle. Determine the exact coordinates on the unit circle corresponding to each of the following angles. *(4 × 2 marks each = 8 marks)*

   a) 600°



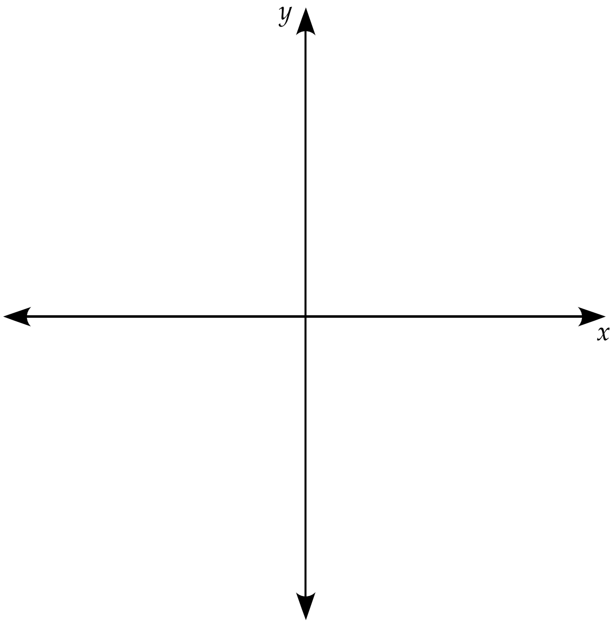
   b) $-\dfrac{17\pi}{6}$



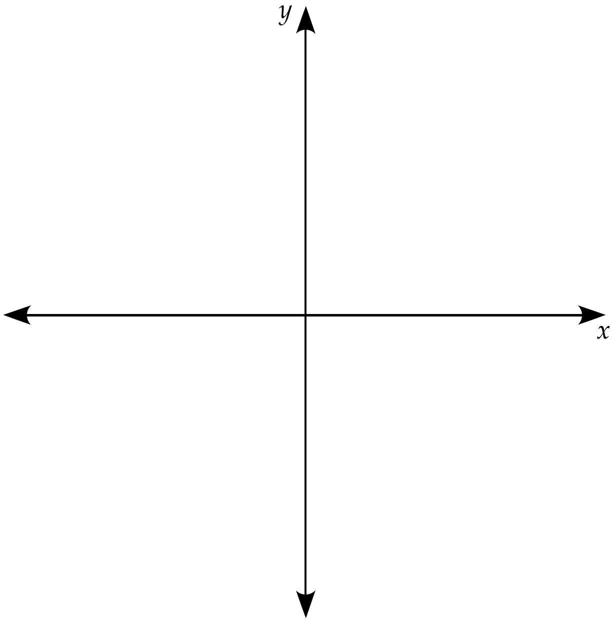
continued

Assignment 5.1: Degrees, Radians, and the Unit Circle (continued)

c) 12π



d) $-225°$



continued

Assignment 5.1: Degrees, Radians, and the Unit Circle (continued)

8. Show that the point with coordinates $\left(\frac{\sqrt{11}}{6}, \frac{5}{6}\right)$ is on the unit circle. *(2 marks)*

9. Is the point $\left(\frac{5}{4}, \frac{1}{4}\right)$ on the unit circle? Explain without doing any calculations.

   (1 mark)

10. Determine the equation of a circle centred at the origin with a radius of 5 cm. *(1 mark)*

LESSON 3: THE TRIGONOMETRIC RATIOS

Lesson Focus

In this lesson, you will

☐ learn how to define the trigonometric ratios as circular functions

☐ learn how the trigonometric ratios relate to the unit circle

☐ learn how to determine the exact value of a trigonometric ratio using the unit circle

☐ learn how to determine an approximate value of a trigonometric ratio

☐ learn how to determine the measures of angles in a specified domain that produce a given trigonometric ratio

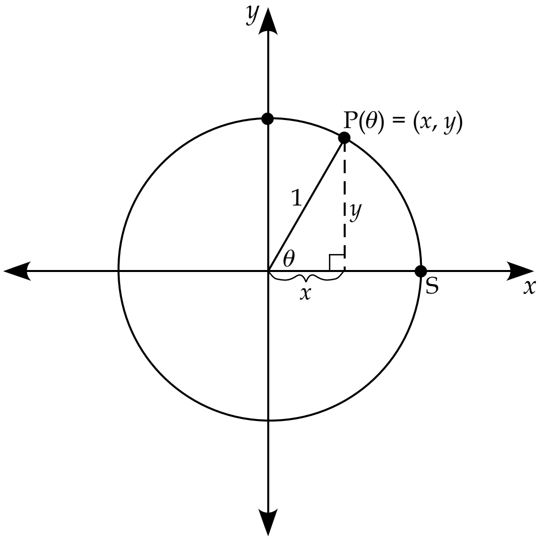
Lesson Introduction

In the last lesson, you focused on the coordinates of special points on the unit circle. In this lesson and the next lesson, you are going to be defining circular functions, also called trigonometric functions, in terms of the coordinates of these points. In this lesson you will be concentrating on the three primary trigonometric ratios of cosine, sine, and tangent.

Definition of Circular Functions: Cosine, Sine, and Tangent

The special triangles were used to calculate the coordinates of the points on the unit circle. You will see how the trigonometric values of special angles can be determined from these coordinates.

Consider the following triangle. This triangle represents any triangle that you can draw on the unit circle. Because this triangle is drawn on the unit circle, the hypotenuse represents the radius of the unit circle, which is always 1.



To determine expressions for sine, cosine, and tangent, you can use the three primary trigonometric ratios for right triangles.

Recall: $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\begin{aligned}\sin\theta &= \frac{\text{opposite}}{\text{hypotenuse}} \\ &= \frac{y}{1} \\ &= y\end{aligned} \qquad \begin{aligned}\cos\theta &= \frac{\text{adjacent}}{\text{hypotenuse}} \\ &= \frac{x}{1} \\ &= x\end{aligned} \qquad \begin{aligned}\tan\theta &= \frac{\text{opposite}}{\text{adjacent}} \\ &= \frac{y}{x}\end{aligned}$$

If θ is an angle of rotation in standard position (that is, with its starting point at (1, 0)), and if its terminal point is the point, P(θ), with coordinates (x, y), then

$$\begin{aligned}\mathrm{P}(\theta) &= (x, y) \\ &= (\cos\theta, \sin\theta)\end{aligned}$$

with $x = \cos\theta$

and $y = \sin\theta$

This is a wonderful connection between trigonometry and the coordinates of points on the unit circle. The coordinates of the points on the unit circle can be used to generate the trigonometric function values for sin θ, cos θ, and tan θ. This is a really important point. This means that the unit circle coordinate values you determined (and memorized) in the previous lesson are actually the cos θ or sin θ values related to the special angles of rotation.

$$\cos\theta = x$$

$$\sin\theta = y$$

$$\tan\theta = \frac{y}{x}$$

It may be helpful to include the above statements about coordinates on the unit circle on your resource sheet.

Note: The trigonometric functions, including $f(\theta) = \sin\theta$, $f(\theta) = \cos\theta$, and $f(\theta) = \tan\theta$, are sometimes called circular functions due to their connection to coordinates on a unit circle.

Exact Values

Using what you know about these three trigonometric ratios, you will be able to write exact values of trigonometric functions for any angles of rotation that are found on the unit circle. Whenever a problem asks for an exact value or tells you not to use your calculator, you should always refer to your unit circle to determine the exact value of the sine, cosine, or tangent functions.

Example 1

Use the coordinates of the special points on the unit circle and the definitions of the circular functions to find

a) $\sin\frac{\pi}{6}$

b) $\cos\pi$

c) $\tan\left(-\frac{\pi}{3}\right)$

Solutions

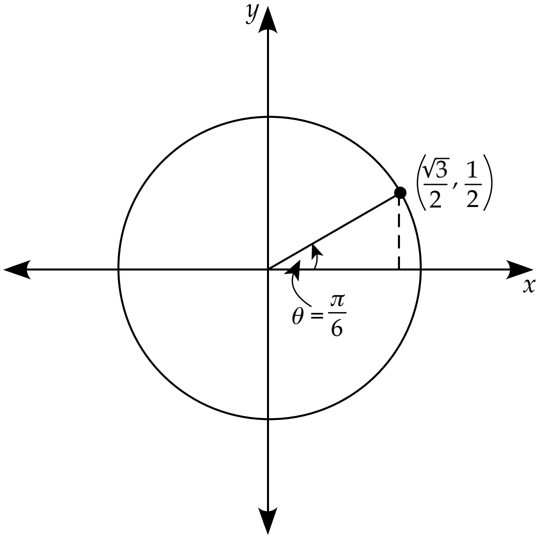
a) You might use the following steps.

   i) Locate where the angle of rotation $\frac{\pi}{6}$ terminates on the unit circle.

   ii) Determine the coordinates of this point, $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, from memory or by using your unit circle diagram. You could draw in the 30–60–90 triangle on a quick sketch, as shown below.

   iii) Since $\sin \theta = y$, select the y-coordinate, $y = \frac{1}{2}$.

   Thus, $\sin \frac{\pi}{6} = \frac{1}{2}$.



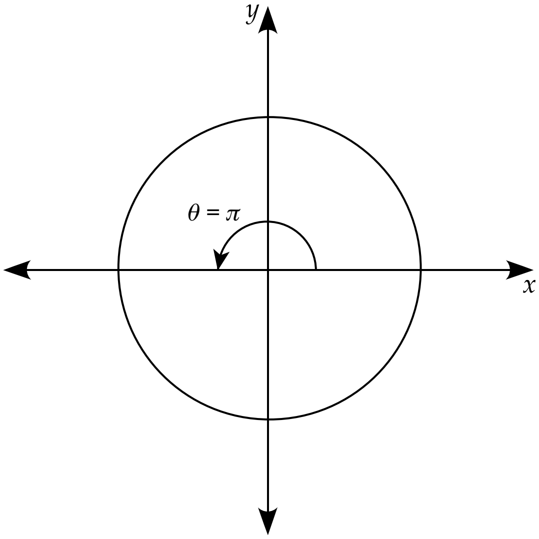
b) The steps are.

i) Locate where the point terminates on the unit circle when $\theta = \pi$.

ii) Find the coordinates of this point, $(-1, 0)$, using your unit circle diagram.

iii) Since $\cos\theta = x$, select the x-coordinate as your answer.

Thus, $\cos\pi = -1$.



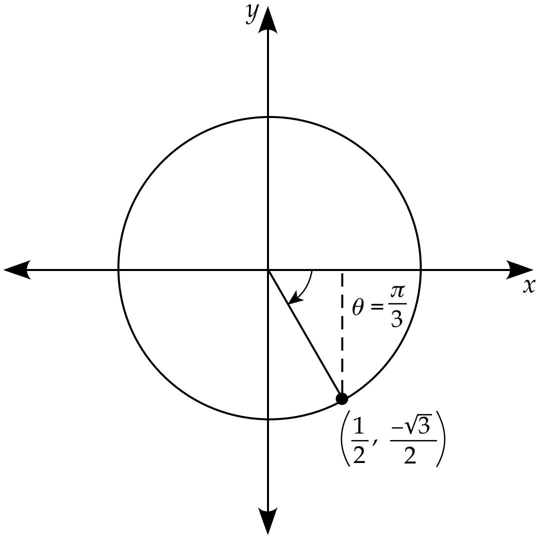
c) The steps are:

i) Locate where the point terminates on the unit circle if $\theta = -\frac{\pi}{3}$.

ii) Find the coordinates of this point, $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, using your unit circle diagram. You could find the coordinates from memory or by drawing a 30–60–90 triangle, as shown below.

iii) Find $\tan\left(-\frac{\pi}{3}\right) = \frac{y}{x} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\frac{\sqrt{3}}{2}\cdot\frac{2}{1} = -\sqrt{3}$.

Thus, $\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$.



Example 2

Without the use of a calculator find the exact value of:

$2 \cos 30° + \sin 60° \tan 30°$.

Solution

First, you need to determine the exact value of each trigonometric ratio using your unit circle values.

$$\cos 30° = \cos \frac{\pi}{6} = x\text{-coordinate of P} \quad \left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$\tan 30° = \frac{y\text{-coordinate}}{x\text{-coordinate}} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \left(\frac{1}{2}\right)\cdot\left(\frac{2}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}$$

$$\sin 60° = \sin \frac{\pi}{3} = y\text{-coordinate of P}\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

Using all of this information, you can combine these values to calculate the answer to the question.

$$2 \cos 30° + \sin 60° \tan 30° = 2\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{3}}\right) = \sqrt{3} + \frac{1}{2} = \frac{2\sqrt{3}+1}{2}$$

Decimal Approximations

Not every angle is included as a special value on the unit circle. Therefore, sometimes calculators are necessary to find decimal approximations of trigonometric ratios. However, you need to be careful to ensure your calculator is in the correct mode. Calculators have a “degree” and a “radian” mode. When the angle is given in degrees, make sure your calculator is in degree mode before you calculate the trigonometric ratio. When the angle is given in radians, make sure your calculator is in radian mode before you calculate the trigonometric ratio. Note that the mode (degree or radian) of your calculator only matters when you are using trigonometric functions such as sine, cosine, and tangent. Consider the following example.

Example 3

Find sin 3.

Solution

Since degrees are not specified, this angle is given in radians. To calculate this trigonometric value, ensure your calculator is in radian mode.

$$\sin 3 = 0.1411$$

To check your answer, change 3 radians into degrees:

$$3\left(\frac{180^\circ}{\pi}\right) = \frac{540^\circ}{\pi}$$

$$\sin 3 = \sin\left(\frac{540^\circ}{\pi}\right) = \sin\left(171.887^\circ\right)$$

To calculate this ratio, ensure your calculator is in degree mode.

$$\sin(171.887^\circ) = 0.1411$$

As you can see, it is possible to calculate trigonometric ratios when you are given angles in degrees or radians, as long as your calculator is in the right mode.

Note: If the angle of rotation is not one of the special angles that you memorized (one of the values on the unit circle), you will need to use your calculator to find the answer. You need to switch your calculator to degree mode or to radian mode, as appropriate, and enter the central angle measure to calculate the answer.

Example 4

Find the exact value, if possible. Otherwise, use a calculator to find the approximate value, rounded to 5 decimal places.

a) $\cos \pi$

b) $\sin 20°$

c) $\tan \dfrac{3\pi}{4}$

d) $\tan 1$

Solutions

a) π is a special unit circle angle. Use your unit circle to find $\cos \pi$. Remember, the cosine of an angle is the corresponding x-coordinate on the terminal arm of the angle, which is at $(-1, 0)$ for π radians.

$$\cos \pi = -1$$

b) Since 20° is not a special unit circle value, use a calculator, in degree mode, to find the answer.

$$\sin 20° = 0.34202$$

Note that the y-coordinate on the unit circle on the terminal arm of an angle of 20° will be 0.34202.

c) $\dfrac{3\pi}{4}$ is a special unit circle value with coordinates of $\left(-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$ or $\left(-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$.

To calculate $\tan \dfrac{3\pi}{4}$, use $\dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{y}{x}$.

$$\tan \frac{3\pi}{4} = \frac{y}{x} = \frac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{2} \cdot -\frac{2}{\sqrt{2}} = -1$$

d) 1 radian is not a special unit circle value. Use your calculator, in radian mode, to find the answer.

$$\tan 1 = 1.55741$$

Determining Angles

Suppose you are given the value of a trigonometric function. How would you determine the angle, or arc length, that corresponds to that value?

This is where the inverse trigonometric functions, $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$, come into play. These functions are also sometimes denoted as follows:

$\sin^{(-1)} x = \arcsin x$

$\cos^{(-1)} x = \arccos x$

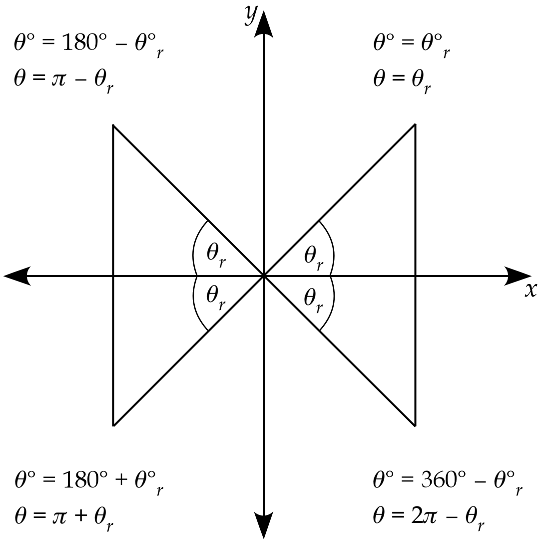
$\tan^{(-1)} x = \arctan x$

Writing the functions in this way emphasizes the fact that an angle in radians is the same as an arc length. It also decreases the confusion over the −1 symbol.

Note: $\sin^{(-1)} x$ is not the same as $(\sin \theta)^{(-1)}$. $(\sin \theta)^{-1} = \dfrac{1}{\sin \theta}$, which is a reciprocal function. Reciprocal functions and inverse functions are different, as you have discovered in previous modules. Make sure you are not confused by this notation.

The sine function gives you a unit circle y-coordinate value that corresponds to a specific angle. The arcsine (or sine inverse) function gives you an angle that corresponds to a specific unit circle y-coordinate value.

In order to use the inverse trigonometric functions to find an angle, you need to recall the concept of reference angles. You learned about reference angles, in degrees, in Grade 11 Pre-Calculus Mathematics. Reference angles can also be calculated in radians. Recall that θ_r is a notation meaning reference angle.



If you look at your unit circle, you can verify that all angles with a $\frac{\pi}{6}$ reference angle all have the same x- and y-coordinates of $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Similarly, all angles with a $\frac{\pi}{4}$ reference angle all have coordinates of $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. All angles with a $\frac{\pi}{3}$ reference angle all have coordinates of $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. The only thing about these points that changes is the sign of each coordinate depending upon which quadrant the angle is located in.

You will also need to recall where each trigonometric ratio is positive or negative. Now that you know $x = \cos \theta$ and $y = \sin \theta$ on a unit circle, you know that $\cos \theta$ is positive where x is positive (in Quadrant I and Quadrant IV). Also, $\sin \theta$ is positive where y is positive (in Quadrant I and Quadrant II). Knowing the relationship between y and $\sin \theta$ and between x and $\cos \theta$ may be an easy way for you to remember the signs of the trigonometric functions in each quadrant. Alternatively, you may also use the CAST acronym to help you remember.

sin θ is positive (**cos** θ and **tan** θ are negative)	S	A	All of **sin** θ, **cos** θ, and **tan** θ are positive
tan θ is positive (**sin** θ and **cos** θ are negative)	T	C	**cos** θ is positive (**sin** θ and **tan** θ are negative)

Use the concept of reference angles and where each trigonometric ratio is positive or negative to complete the following example.

Example 5

Determine the measures of all the angles that satisfy the following conditions.

a) $\sin \theta = 0.88295,\ 0° \leq \theta < 360°$

b) $\cos \theta = -\frac{1}{2},\ 0° \leq \theta < 2\pi$

c) $\tan \theta = 0.44523,\ -360° \leq \theta < 0°$

d) $\tan \theta = \frac{\sqrt{3}}{3},\ -2\pi \leq \theta < 0$

Solutions

a) To determine all of the angles that satisfy these conditions, you first need to take the inverse sine of this value. To do that, make sure your calculator is in degrees (this is because you want to find an angle in degrees as specified by the domain for θ). Use the "$\sin^{-1}$" function to calculate θ.

$\sin^{-1}(0.88295) = 62°$

Now, you need to determine which quadrants sine is positive in. This is because you were given a positive sine ratio value of 0.88295. Sine is positive in Quadrants I and II.

Use what you know about reference angles or use the reference angle formulas to determine the measures of the angles that satisfy these conditions.

In Quadrant I, $\theta_r = \theta = 62°$.

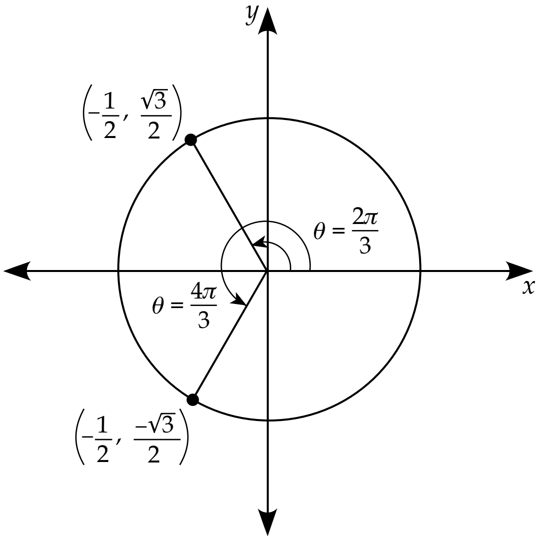
In Quadrant II, $\theta = 180° - \theta_r = 180° - 62° = 118°$.

Therefore, the two angles that satisfy these conditions are 62° and 118°.

b) It is possible to answer this question using exact values.

You can use the unit circle to find solutions to the equation $\cos\theta = -\frac{1}{2}$.

In other words, you are looking for the angle in radians (as specified by the domain for θ) whose cosine produces a value of $-\frac{1}{2}$. This is the same as looking for x-coordinates on the unit circle that are equal to $-\frac{1}{2}$ and determining the corresponding angle.



As you are looking for angles between 0 and 2π, you can read the angles in radians directly off the unit circle.

The two angles in which $\cos\theta = -\frac{1}{2}$ are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.

c) First, take the inverse tangent of this value (in degrees).

$\tan^{(-1)} 0.44523 = 24°$

Tangent is positive in Quadrants I and III.

Therefore, you need to find angles in degrees between −360° and 0°, one in Quadrant I, and one in Quadrant III.

To determine a coterminal angle in Quadrant I, you go all the way around the circle or simply subtract 360° from 24°.

Quadrant I: $24° - 360° = -336°$.

Now determine the angle located in Quadrant III.

Quadrant III: $\theta = 180° + \theta_r = 180° + 24° = 204°$

You need to find a coterminal angle to 204° located between −360° and 0°. To find this angle, subtract 360° from 204°.

$\theta = 204° - 360° = -156°$

The two angles in which $\tan\theta = 0.44523$, when $-360° \leq \theta < 0°$, are −336° and −156°.

d) You may not recognize this value right away as a special value on the unit circle. However, it is.

If you calculate the tangent value of $\frac{\pi}{6}$ you will get:

$$\tan\frac{\pi}{6} = \frac{y}{x} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{2}\cdot\frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

If you rationalize the denominator of this fraction, you get:

$$\frac{1}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Thus, $\tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$ and $\frac{\pi}{6}$ is your reference angle for this question.

However, you need to find negative angles. Therefore, subtract 2π from this angle to find a negative coterminal angle in the required domain.

$$\frac{\pi}{6} - 2\pi = \frac{\pi}{6} - \frac{12\pi}{6} = -\frac{11\pi}{6}$$

Tangent is positive in Quadrants I and III. Therefore, there is another angle at $\pi + \frac{\pi}{6} = \frac{7\pi}{6}$. Now find a coterminal angle to match $\frac{7\pi}{6}$ in the required domain.

$$\theta = \frac{7\pi}{6} - 2\pi = \frac{7\pi}{6} - \frac{12\pi}{6} = -\frac{5\pi}{6}$$

The two solutions to this equation are:

Quadrant I: $\theta = -\frac{11\pi}{6}$

Quadrant III: $\theta = -\frac{5\pi}{6}$

On your unit circle diagram, you may find it helpful to note that the tangent values for $f(\theta)$ can be found for each coordinate using $\frac{y}{x}$.

Make sure you complete the following learning activity, as it will allow you to practice what you have just learned.

Learning Activity 5.4

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. In which quadrant is $\theta = \frac{5\pi}{8}$ located?

2. Solve for x: $\frac{2x+1}{x-2} - \frac{x-1}{x-2} = 0$

continued

Learning Activity 5.4 (continued)

3. Is $x - 2$ a factor of $p(x) = 4x^3 - 8x - 16$?
4. Evaluate: ${}_5C_4$
5. Factor: $121x^2 - 36y^{10}$.
6. Evaluate: $\dfrac{7}{6} \div \dfrac{8}{3}$
7. Evaluate: $\left|-2\dfrac{1}{3} - 4\dfrac{2}{5}\right|$
8. Simplify: $\sqrt{18} - \sqrt{50}$

Part B: Sine, Cosine, Tangent, and the Unit Circle

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Without your calculator, find exact values of cos θ, sin θ, and tan θ for each of the following values of θ. For each question, write the coordinates of P(θ) from the unit circle and then find the three circular functions.

a) $\dfrac{2\pi}{3}$

b) $\dfrac{7\pi}{6}$

c) 27π

d) $\dfrac{5\pi}{4}$

e) $-\dfrac{\pi}{6}$

f) 14π

g) $-\dfrac{5\pi}{2}$

h) 53π

i) $-\dfrac{11\pi}{6}$

j) $\dfrac{17\pi}{3}$

k) $\dfrac{\pi}{2}$

l) $-\pi$

m) $\dfrac{4\pi}{3}$

n) $\dfrac{3\pi}{2}$

o) $\dfrac{5\pi}{6}$

p) $\dfrac{15\pi}{4}$

q) $-\dfrac{7\pi}{3}$

r) 4π

continued

Learning Activity 5.4 (continued)

2. Determine the exact value of each expression (that is, do not use a calculator).

   Note: Although each expression involves functions and should contain notation such as sin (θ), mathematicians always like shortcuts. As long as there is no ambiguity, write these circular function statements as sin θ, etc., without the brackets. This notation is used throughout this course.

   a) $2\cos\frac{\pi}{3} + \tan\frac{\pi}{4}$

   b) $\sin\left(-\frac{15\pi}{2}\right)\cos 20\pi\ \tan\frac{13\pi}{6}$

   c) $\tan^2\frac{\pi}{6}$

   d) $\cos(\sin\pi)$

   e) $\tan\frac{2\pi}{3}\cos\left(-\frac{5\pi}{4}\right) + \sin\frac{3\pi}{2}\tan\frac{5\pi}{6}$

   f) $\cos\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$

   g) $\sin\frac{2\pi}{3}\cos\frac{7\pi}{6}\tan\left(-\frac{3\pi}{4}\right)$

   h) $\sin\left(-\frac{47\pi}{2}\right)\cos(-47\pi)$

3. Find an approximate value, rounded to 5 decimal places (that is, use a calculator).

   a) sin 62°

   b) tan 129°

   c) $\cos\left(-\frac{\pi}{8}\right)$

   d) tan 5

4. Solve the following equations over the interval $0 \le \theta < 2\pi$. Do these questions without the use of a calculator. All answers must be exact values.

   a) $\sin\theta = \frac{\sqrt{2}}{2}$

   b) $\tan\theta = \sqrt{3}$

   c) $\cos\theta = \frac{\sqrt{3}}{2}$

   d) $\cos\theta = -\frac{1}{2}$ and $\tan\theta > 0$

   e) $\sin\theta = \frac{\sqrt{3}}{2}$ and $\cos\theta < 0$

   f) $2\cos\theta = 2$

continued

Learning Activity 5.4 (continued)

5. Determine the measures of *all* the angles that satisfy the following conditions over the interval $-180° \le \theta < 180°$.

   a) $\sin\theta = -\dfrac{1}{\sqrt{2}}$

   b) $\cos\theta = 0.6819$

   c) $\tan\theta = -1$

   d) $\tan\theta = 0.2123$

6. Can you determine which of the following is the largest without using a calculator? Use a calculator to check your work.

   $\cos\dfrac{\pi}{6}$ or $\cos\dfrac{\pi}{10}$

7. Determine the measures of all the angles of rotation in standard position, in the specified domain, that correspond to the following coordinates on the unit circle. Round to 3 decimal places where necessary.

   a) $\left(-\dfrac{1}{2}, -\dfrac{\sqrt{3}}{2}\right),\ 0 \le \theta < 2\pi$

   b) $(0, 1),\ 0 \le \theta < 4\pi$

   c) $\left(\dfrac{4}{5}, -\dfrac{3}{5}\right),\ 0 \le \theta < 2\pi$

   d) $\left(\dfrac{5}{13}, \dfrac{12}{13}\right),\ -2\pi \le \theta < 2\pi$

Lesson Summary

In this lesson, you learned how the trigonometric functions of sine, cosine, and tangent are related to the coordinates of points on the unit circle. You also learned how to answer questions based on these trigonometric functions. In the next lesson, you are going to learn about three more trigonometric functions—secant, cosecant, and cotangent.

Lesson 4: The Reciprocal Trigonometric Ratios

Lesson Focus

In this lesson, you will

- ☐ learn how to define the reciprocal trigonometric ratios as circular functions
- ☐ learn how the reciprocal trigonometric ratios relate to the unit circle
- ☐ learn how to determine the exact value of a reciprocal trigonometric ratio using the unit circle
- ☐ learn how to determine an approximate value of a reciprocal trigonometric ratio
- ☐ learn how to determine the measures of angles in a specified domain that satisfy a given reciprocal trigonometric ratio

Lesson Introduction

In the last lesson, you learned about some exact values of the three primary trigonometric ratios: sin θ, cos θ, and tan θ. In this lesson, you are going to learn about an additional three trigonometric ratios that are reciprocals of sine, cosine, and tangent.

The Reciprocal Trigonometric Ratios: Secant, Cosecant, and Cotangent

There are three more circular functions, which are defined as reciprocals of the basic three trigonometric functions. They are called the secant (abbreviated as sec), cosecant (abbreviated as csc), and cotangent (abbreviated as cot) functions.

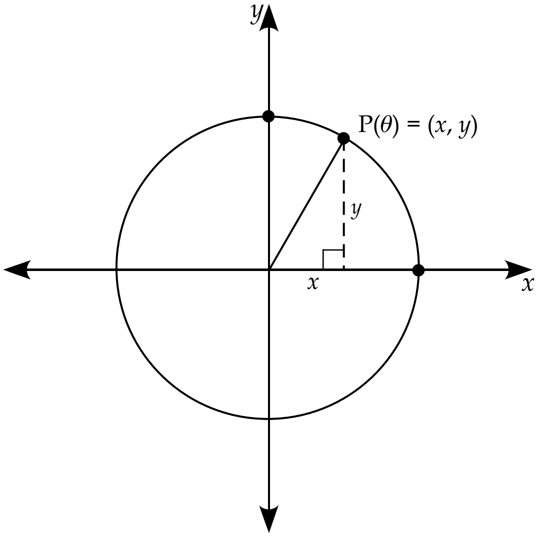
They are defined as follows.

Provided that the denominators are not equal to zero, if θ is an angle of rotation in standard position, with terminal point $P(\theta) = (x, y)$, then

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{x}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{y}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{x}{y}$$



The relationship for $\cot\theta$ can also be written as $\cot\theta = \dfrac{1}{\tan\theta}$ but this representation introduces a new restriction: $\dfrac{1}{\tan\theta}$ is not defined where $\tan\theta$ is zero or where $\tan\theta$ is not defined.

You may want to include these definitions of the reciprocal trigonometric functions on your resource sheet.

Hint: Because of their names, it is easy to remember that tangent and cotangent are reciprocals of each other. To remember which reciprocal goes with sine and cosine, note that in each pair, only one of the functions has "co" with it:

- Sine and cosecant
- Cosine and secant
- Tangent and cotangent

Using these three new trigonometric functions, it is possible to further develop the relationship between the coordinates of the unit circle to include the six trigonometric functions.

Note: Circular functions and trigonometric functions are synonymous terms and are used as such throughout this course.

Exact Values of Reciprocal Trigonometric Ratios

Using the unit circle, it is possible to determine exact values of trigonometric ratios.

Example 1

Use the coordinates of the special points on the unit circle and the definitions of the circular functions to find

a) $\sec \frac{\pi}{6}$

b) $\csc \pi$

c) $\cot\left(-\frac{\pi}{3}\right)$

Solutions

a) When you are dealing with reciprocal trigonometric ratios, you can rewrite them using the three primary trigonometric ratios: sine, cosine, and tangent.

$\sec \frac{\pi}{6}$ can also be written as $\dfrac{1}{\cos \frac{\pi}{6}}$

$$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\sec \frac{\pi}{6} = \frac{1}{\cos \frac{\pi}{6}} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

Notice that $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $\sec \frac{\pi}{6} = \frac{2}{\sqrt{3}}$.

To find the secant θ value, take the reciprocal of the cos θ value. **Do not** take the reciprocal of the angle θ. For example, $\cos \frac{\pi}{6}$ does not equal $\sec \frac{6}{\pi}$; that is like saying cos 30° = $\sec\left(\frac{1}{30}\right)^{\circ}$. It does not even make sense. It is the ratio that is "flipped." So, if $\cos \theta = \frac{\sqrt{3}}{2}$, $\sec \theta = \frac{2}{\sqrt{3}}$.

b) $\csc \pi = \frac{1}{\sin \pi} = \frac{1}{0}$, which is undefined (or ∞).

c) $\cot\left(-\frac{\pi}{3}\right) = \dfrac{1}{\tan\left(-\frac{\pi}{3}\right)} = \dfrac{1}{-\sqrt{3}} = -\dfrac{1}{\sqrt{3}}$.

Example 2

Simplify the following expressions using the exact value of each trigonometric ratio.

a) $\csc\left(\frac{\pi}{4}\right) + \sec\left(\frac{7\pi}{4}\right)$

b) $\cot\left(\frac{11\pi}{6}\right) + \sec\left(\frac{7\pi}{6}\right) + \csc\left(\frac{2\pi}{3}\right)$

c) $\sec\left(\frac{5\pi}{3}\right)\cos\left(\frac{5\pi}{3}\right)$

d) $\dfrac{\sec\left(\frac{5\pi}{6}\right)}{\cot\left(\frac{4\pi}{3}\right)}$

Solutions

a) $\csc\frac{\pi}{4} = \dfrac{1}{\sin\frac{\pi}{4}} = \dfrac{2}{\sqrt{2}}$

$$\sec\frac{7\pi}{4} = \frac{1}{\cos\frac{7\pi}{4}} = \frac{2}{\sqrt{2}}$$

$$\csc\left(\frac{\pi}{4}\right) + \sec\left(\frac{7\pi}{4}\right) = \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{4}{\sqrt{2}}$$

b) $\cot\frac{11\pi}{6} = \dfrac{1}{\tan\frac{11\pi}{6}} = -\dfrac{3}{\sqrt{3}}$

$$\sec\frac{7\pi}{6} = \frac{1}{\cos\frac{7\pi}{6}} = -\frac{2}{\sqrt{3}}$$

$$\csc\frac{2\pi}{3} = \frac{1}{\sin\frac{2\pi}{3}} = \frac{2}{\sqrt{3}}$$

$$\cot\left(\frac{11\pi}{6}\right) + \sec\left(\frac{7\pi}{6}\right) + \csc\left(\frac{2\pi}{3}\right) = -\frac{3}{\sqrt{3}} - \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}} = -\frac{3}{\sqrt{3}}$$

c) $\sec \frac{5\pi}{3} = \dfrac{1}{\cos \frac{5\pi}{3}} = \frac{2}{1}$

$$\cos \frac{5\pi}{3} = \frac{1}{2}$$

$$\sec\left(\frac{5\pi}{3}\right)\cos\left(\frac{5\pi}{3}\right) = 2\left(\frac{1}{2}\right) = 1$$

Note: As you know, secant and cosine are reciprocal functions. Therefore, when the corresponding reciprocals are evaluated at any angle and then multiplied together, the result will always be 1.

d) $\sec \frac{5\pi}{6} = \dfrac{1}{\cos \frac{5\pi}{6}} = -\frac{2}{\sqrt{3}}$

$$\cot \frac{4\pi}{3} = \dfrac{1}{\tan \frac{4\pi}{3}} = \frac{1}{\sqrt{3}}$$

$$\frac{\sec\left(\frac{5\pi}{6}\right)}{\cot\left(\frac{4\pi}{3}\right)} = \frac{-\frac{2}{\sqrt{3}}}{\frac{1}{\sqrt{3}}} = -\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{1} = -2$$

Decimal Approximations

Sine, cosine, and tangent are all functions that your calculator deals with directly. Cosecant, secant, and cotangent, on the other hand, are not. Therefore, before you find a decimal approximation for the cosecant, secant, or cotangent of an angle, you need to convert the expression into an equivalent expression in terms of sine, cosine, or tangent. Consider the following example.

Example 3

Find an approximate value rounded to 5 decimal places.

a) sec 16°

b) cot 82°

c) $\csc \frac{7\pi}{8}$

d) sec 6

Solutions

a) First, find the value of the associated primary trigonometric function.

$$\cos\theta = 0.96126$$

Then calculate the value of the reciprocal trigonometric function.

$$\sec 16^\circ = \frac{1}{\cos 16^\circ} = \frac{1}{0.96126} = 1.04030$$

b) Convert this expression into an expression involving tangent and use your calculator to compute the answer.

$$\cot 82^\circ = \frac{1}{\tan 82^\circ} = \frac{1}{7.11537} = 0.14054$$

c) Compute this value using your calculator in radian mode.

$$\csc\frac{7\pi}{8} = \frac{1}{\sin\frac{7\pi}{8}} = \frac{1}{0.38268} = 2.61313$$

d) $\sec 6 = \dfrac{1}{\cos 6} = \dfrac{1}{0.96017} = 1.04148$

Determining Angles

Sine, cosine, and tangent all have related inverse functions—arcsine, arccosine, and arctangent. You can use these inverse functions to solve for a missing angle either exactly (by using the unit circle), or approximately (by using your calculator). There are no inverse functions on your calculator associated with secant, cosecant, and cotangent. Instead, you first need to convert each expression into a cosine, sine, or tangent ratio, and then use the primary trigonometric inverse function to find an exact value or a decimal approximation. Consider the following example.

Example 4

Determine the measures of *all* the angles that satisfy the following conditions in the given domain.

a) $\csc \theta = 1.01543,\ 0° \le \theta < 360°$

b) $\sec \theta = 2,\ 0 \le \theta < 2\pi$

c) $\cot \theta = 0.24933,\ -360° \le \theta < 0°$

d) $\csc \theta = -\dfrac{2}{\sqrt{3}},\ -2\pi \le \theta < 0$

Solutions

a) First, convert this expression to one involving sine.

$$\csc \theta = 1.01543$$

$$\therefore \sin \theta = \frac{1}{1.01543}$$

$$\sin^{-1}\left(\frac{1}{1.01543}\right) = \theta$$

$$\theta_r = 80°$$

Note: Do not take the inverse sine of 1.01543. You first need to find the reciprocal. This is a common mistake many students make.

Now, sine, and therefore cosecant, are both positive in Quadrants I and II.

Note: Cosecant is positive wherever sine is positive because the reciprocal of a positive value is still positive.

Therefore, the two angles that satisfy the above conditions are:

Quadrant I: $\theta = 80°$

Quadrant II: $\theta = 180° - \theta_r = 180° - 80° = 100°$

b) Convert this expression to one involving cosine.

$$\sec \theta = 2$$

$$\therefore \cos \theta = \frac{1}{2}$$

You can use your unit circle to answer this question.

$$\theta = \frac{\pi}{3} \text{ and } \theta = \frac{5\pi}{3}$$

As both of these angles are in the required domain, these angles are both solutions to the above equation.

c) $\cot\theta = 0.24933$

$$\tan\theta = \frac{1}{0.24933}$$

$$\tan^{-1}\left(\frac{1}{0.24933}\right) = \theta$$

$$\theta = 76°$$

Tangent and cotangent are positive in Quadrants I and III. The two angles between 0° and 360° that satisfy this equation are:

Quadrant I: $\theta = 76°$

Quadrant III: $\theta = 180° + \theta_r = 180° + 76° = 256°$

However, you are looking for angles between −360° and 0°. To find these angles, simply subtract 360° from each of the above angles to find coterminal angles. This will allow you to calculate angles that are coterminal with the above angles in the correct domain.

Quadrant I: $\theta = 76° - 360° = -284°$

Quadrant III: $\theta = 256° - 360° = -184°$

Thus, the two angles that satisfy the above conditions are $\theta = -284°$ and $\theta = -184°$.

d) $\csc\theta = -\dfrac{2}{\sqrt{3}}$

$$\sin\theta = -\frac{\sqrt{3}}{2}$$

The angles that satisfy this equation can be found by looking at your unit circle.

$$\theta = \frac{4\pi}{3} \text{ and } \theta = \frac{5\pi}{3}$$

However, you are looking for angles that are between -2π and 0. Therefore, in order to find coterminal angles in this domain, subtract 2π from each of the above angles.

Quadrant III: $\theta = \dfrac{4\pi}{3} - 2\pi = \dfrac{4\pi}{3} - \dfrac{6\pi}{3} = -\dfrac{2\pi}{3}$

Quadrant IV: $\theta = \dfrac{5\pi}{3} - \dfrac{6\pi}{3} = -\dfrac{\pi}{3}$

The two angles that satisfy the above conditions are $\theta = -\dfrac{2\pi}{3}$ and $\theta = -\dfrac{\pi}{3}$.

Example 5

Determine the exact values of the other five trigonometric ratios that satisfy the given information.

a) $\sin\theta = \frac{1}{2}, 0 \le \theta < \frac{\pi}{2}$

b) $\cos\theta = \frac{\sqrt{3}}{5}, \frac{3\pi}{2} \le \theta < 2\pi$

c) $\tan\theta = -\frac{1}{\sqrt{7}}, \pi \le \theta < 2\pi$

Solutions

a) The angle that satisfies these conditions is located in Quadrant I and is $\frac{\pi}{6}$.

You can read three trigonometric ratios—sine, cosine, and tangent—directly from the coordinates of $P\left(\frac{\pi}{6}\right)$ on the unit circle.

$$\sin\theta = \frac{1}{2}$$

$$\cos\theta = \frac{\sqrt{3}}{2}$$

$$\tan\theta = \frac{\sqrt{3}}{3}$$

To determine the other three trigonometric ratios, you can find the reciprocal of each of the previous ratios.

$$\csc\theta = 2$$

$$\sec\theta = \frac{2}{\sqrt{3}}$$

$$\cot\theta = \frac{3}{\sqrt{3}}$$

b) This question is slightly more difficult, as this angle is not one of the special angles on the unit circle. One of the ways to answer this question is to determine the sine ratio by using the equation of the unit circle, $x^2 + y^2 = 1$.

In this case, the x-coordinate, which is the cosine ratio, is $\frac{\sqrt{3}}{5}$.

The y-coordinate will be the sine ratio.

$$\begin{aligned} x^2 + y^2 &= 1 \\ \left(\frac{\sqrt{3}}{5}\right)^2 + y^2 &= 1 \\ 1 - \frac{3}{25} &= y^2 \\ \frac{25}{25} - \frac{3}{25} &= y^2 \\ \frac{22}{25} &= y^2 \\ y &= \pm\frac{\sqrt{22}}{5} \end{aligned}$$

In order to determine whether the sine coordinate is positive or negative, you need to look at the domain. As this domain consists of Quadrant IV only $\left(\frac{3\pi}{2} \le \theta \le 2\pi\right)$, sine will be negative.

$$\sin\theta = -\frac{\sqrt{22}}{5}$$

Now, you can determine the tangent ratio and then the other three reciprocal trigonometric ratios.

$$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{\sqrt{22}}{5}}{\frac{\sqrt{3}}{5}} = -\frac{\sqrt{22}}{5}\cdot\frac{5}{\sqrt{3}} = -\frac{\sqrt{22}}{\sqrt{3}}$$

$$\csc\theta = \frac{1}{\sin\theta} = -\frac{5}{\sqrt{22}}$$

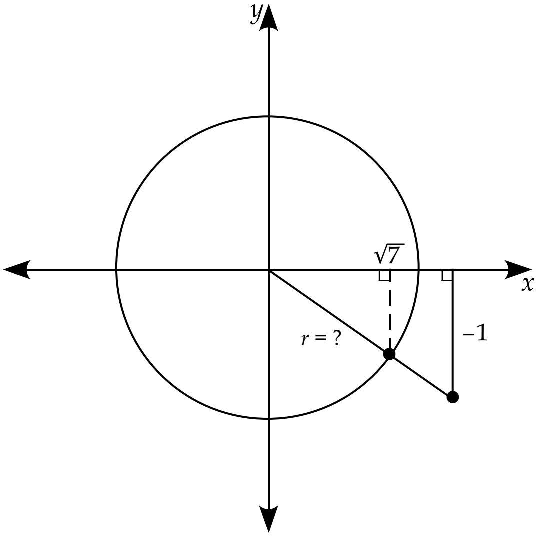
$$\sec\theta = \frac{1}{\cos\theta} = \frac{5}{\sqrt{3}}$$

$$\cot\theta = \frac{1}{\tan\theta} = -\frac{\sqrt{3}}{\sqrt{22}}$$

c) $\tan\theta = -\dfrac{1}{\sqrt{7}}$

You know that $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{y}{x} = -\dfrac{1}{\sqrt{7}}$.

The ratio of y over x is -1 over $\sqrt{7}$. Use these x- and y-values to draw a triangle in Quadrant IV where $x = \sqrt{7}$ and $y = -1$. This is a similar triangle to the desired triangle on the unit circle.



$$r^2 = \left(\sqrt{7}\right)^2 + (-1)^2$$

$$r^2 = 7 + 1$$

$$r^2 = 8$$

$$r = \sqrt{8}$$

Note: ± is not required because r is the radius of a circle and, by definition, is always positive.

Now you can use the right triangle trigonometric ratios (think SOH CAH TOA).

$$\sin\theta = -\frac{1}{\sqrt{8}} \qquad \csc\theta = -\sqrt{8}$$

$$\cos\theta = \frac{\sqrt{7}}{\sqrt{8}} \qquad \sec\theta = \frac{\sqrt{8}}{\sqrt{7}}$$

$$\tan\theta = -\frac{1}{\sqrt{7}} \qquad \cot\theta = -\sqrt{7}$$

Example 6

Determine the exact values of the six trigonometric ratios given that the point $\left(\frac{3}{5}, -\frac{4}{5}\right)$ is a point on the terminal arm of an angle in standard position.

Solution

Given a point on the unit circle, you know that the x-coordinate corresponds to the cosine ratio while the y-coordinate corresponds to the sine ratio.

$$\cos\theta = \frac{3}{5}$$

$$\sin\theta = -\frac{4}{5}$$

Using these two ratios, you can determine the other four trigonometric ratios.

$$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{4}{5}}{\frac{3}{5}} = -\frac{4}{5}\cdot\frac{5}{3} = -\frac{4}{3}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{5}{3}$$

$$\csc\theta = -\frac{5}{4}$$

$$\cot\theta = \frac{1}{\tan\theta} = -\frac{3}{4}$$

Learning Activity 5.5

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Determine the inverse function if $f(x) = x + 4$.
2. State the non-permissible values of the function $f(x) = \dfrac{x+3}{5x^2 - 4x - 1}$.
3. State an angle that is coterminal to −124°.
4. Simplify: $\dfrac{x^2 + 3x + 2}{x + 2}$
5. What is the volume of a cube with a side length of $\sqrt{2}$ cm?
6. Your restaurant bill came to $65.29. If you wish to leave a 15% tip, estimate how much you should leave.
7. Determine the vertex of the function $f(x) = -\dfrac{1}{2}(x - 2)^2 + 5$.
8. Multiply: $\left(\dfrac{1}{x} + 5\right)(x - 2)$

continued

Learning Activity 5.5 (continued)

Part B: The Reciprocal Trigonometric Ratios: Cosecant, Secant, and Cotangent

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Use the coordinates of the special points on the unit circle and the definitions of the circular functions to find the following values. Include a diagram with your answer.

a) $\sec \dfrac{11\pi}{6}$

b) $\sec 150°$

c) $\csc \left(-\dfrac{\pi}{6}\right)$

d) $\csc (-225°)$

e) $\cot \dfrac{5\pi}{4}$

f) $\cot 120°$

2. Determine the *exact* value of each expression.

a) $\sec \dfrac{\pi}{3} \sin \dfrac{\pi}{6} - \cot \dfrac{\pi}{4}$

b) $\sec^2 \dfrac{\pi}{6} + \csc^2 \dfrac{\pi}{6} \left(\text{Note: } \sec^2 \dfrac{\pi}{6} \text{ means } \left(\sec \dfrac{\pi}{6}\right)^2.\right)$

c) $\cot^2 \dfrac{5\pi}{6}$

d) $\tan \dfrac{2\pi}{3} \cot \dfrac{2\pi}{3}$

e) $\cos^2 \dfrac{\pi}{4} \sec^2 \dfrac{\pi}{4}$

f) $4 \csc \left(-\dfrac{7\pi}{6}\right)$

continued

3. Find the *exact* solution of the following equations over the interval $0 \le \theta < 2\pi$.
   a) $\sec\theta = 2$
   b) $\csc\theta = \sqrt{2}$
   c) $\cot\theta = -1$ and $\sin\theta < 0$

4. Find an approximate value rounded to 5 decimal places.
   a) $\sec 82°$
   b) $\cot\frac{2\pi}{9}$
   c) $\csc 193°$
   d) $\csc 1$

5. Determine the measures of *all* the angles that satisfy the following conditions on the domain.
   a) $\csc\theta = -1.04284,\ 0 \le \theta < 2\pi$
   b) $\sec\theta = 1.13257,\ 0° \le \theta < 360°$
   c) $\cot\theta = -1.66428,\ -360° \le \theta < 0°$
   d) $\cot\theta = 2.31187,\ -2\pi \le \theta < 0$

6. Determine the exact values of the other five trigonometric ratios that satisfy the given information.
   a) $\sec\theta = -2,\ \pi \le \theta < \frac{3\pi}{2}$
   b) $\csc\theta = \frac{6}{\sqrt{2}},\ \frac{\pi}{2} \le \theta < \frac{3\pi}{2}$
   c) $\cot\theta = \frac{3}{8},\ 0 \le \theta < \frac{\pi}{2}$

continued

Learning Activity 5.5 (continued)

7. Determine the exact values of the six trigonometric ratios given each of the following points on the terminal arm of an angle in standard position. Explain the strategy you used to determine the six trigonometric ratios.

   a) $\left(-\frac{3}{4}, -\frac{\sqrt{7}}{4}\right)$

   b) $\left(-\frac{1}{3}, \frac{\sqrt{8}}{3}\right)$

Lesson Summary

In this lesson, you learned about the three reciprocal trigonometric ratios—cosecant, secant, and cotangent. All six trigonometric ratios combine to give you a complete picture of the unit circle. In the next two lessons, you are going to be learning more about the six trigonometric functions and their corresponding graphs.

Assignment 5.2

The Six Trigonometric Ratios

Total: 32 marks

You need to show all your steps as you work out your solutions, and make sure your answers are clear (include units, where appropriate). Check the Introduction for the Marking Guidelines that will be used for all assignments and examinations.

1. Use the coordinates of the points on the unit circle and the definitions of the circular functions to find the following values.
   a) $\sin 390°$ *(2 marks)*

   b) $\cos\left(-\frac{5\pi}{6}\right)$ *(2 marks)*

   c) $\tan 0°$ *(1 mark)*

   d) $\csc \frac{\pi}{3}$ *(1 mark)*

   e) $\sec 210°$ *(1 mark)*

   f) $\cot\left(-\frac{\pi}{2}\right)$ *(2 marks)*

continued

Assignment 5.2: The Six Trigonometric Ratios (continued)

2. Determine the *exact* value of the following expression. *(4 marks)*

$$\sin^3\left(-\frac{\pi}{2}\right) + 2\cos^2\left(\frac{\pi}{4}\right)\cot^2\left(-\frac{\pi}{4}\right)$$

3. Find an approximate value, rounded to 5 decimal places. *(6 × 1 mark each = 6 marks)*

a) $\sin \frac{9\pi}{13}$

b) $\cos 582°$

c) $\tan 7$

continued

d) csc 51°

e) sec $\frac{\pi}{10}$

f) cot 81°

4. Find the exact values of the remaining five circular functions if given the following information. *(4 marks)*

$$\sin\theta = -\frac{5}{13} \text{ and } \tan\theta < 0$$

continued

5. Solve the following equations over the indicated intervals. All answers must be *exact* special values. *(2 × 2 marks each = 4 marks)*

   a) $\sec\theta = -\frac{2}{\sqrt{2}}, 0 \le \theta < 2\pi$

   b) $\tan\theta = 0, -720° \le \theta < 0$

6. Determine the measures of all the angles of rotation in the standard position over the interval $-4\pi \le \theta < 0$ that correspond to the coordinates $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. *(2 marks)*

7. Determine the exact values of the six trigonometric ratios, given that the point $\left(\frac{2\sqrt{6}}{5}, -\frac{1}{5}\right)$ is a point on the terminal arm of an angle in standard position. *(3 marks)*

LESSON 5: GRAPHS OF TRIGONOMETRIC FUNCTIONS

Lesson Focus

In this lesson, you will

- ☐ learn how to sketch the basic graphs of $f(\theta) = \sin\theta$, $f(\theta) = \cos\theta$, and $f(\theta) = \tan\theta$
- ☐ learn about the properties of each of the above three functions

Lesson Introduction

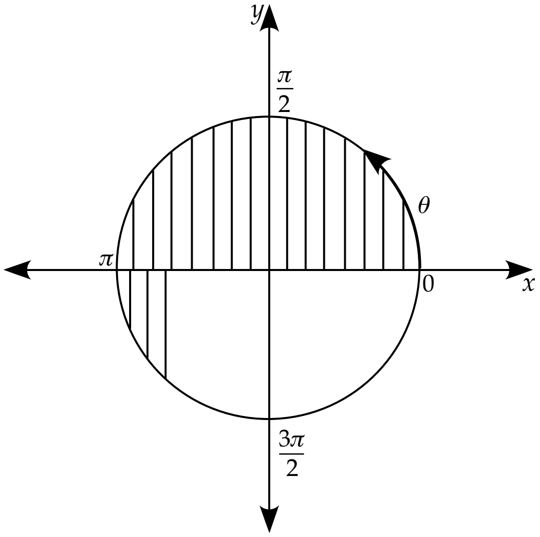
You have discovered how the six trigonometric ratios are related to the (x, y) coordinates of points on the unit circle throughout previous lessons. These trigonometric ratios can also be represented as functions. In this lesson, you are going to learn about the graphs of the three primary trigonometric functions: sine, cosine, and tangent.

Graphs of the Sine, Cosine, and Tangent Functions

The three primary trigonometric functions, $f(\theta) = \sin\theta$, $f(\theta) = \cos\theta$, and $f(\theta) = \tan\theta$ are functions that relate an angle measure, θ, to a coordinate on the unit circle, or the ratio of the coordinates.

The Sine Curve

The sine function relates an angle measure, or arc length, θ, to the y-coordinate of the corresponding point on the terminal arm of the angle. Therefore, it is possible to use the unit circle to help you draw the sine curve. Use the diagram below to make your observations as the angle, θ, increases from quadrant to quadrant. How does the y-value of each point change?



Using the unit circle, you can also create a table of values for the sine function, $f(\theta) = \sin \theta$.

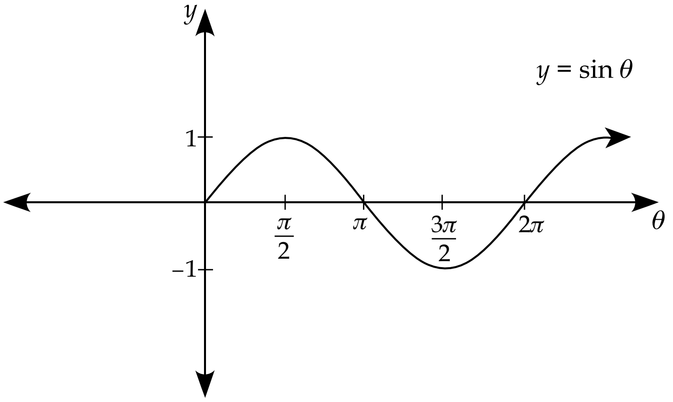
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin \theta$ (exact value)	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\sin \theta$ (decimal approximation)	0	0.5	0.7071	0.866	1	0.866	0.7071	0.5	0

θ	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin \theta$ (exact value)	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
$\sin \theta$ (decimal approximation)	0	-0.5	-0.7071	-0.866	-1	-0.866	-0.7071	-0.5	0

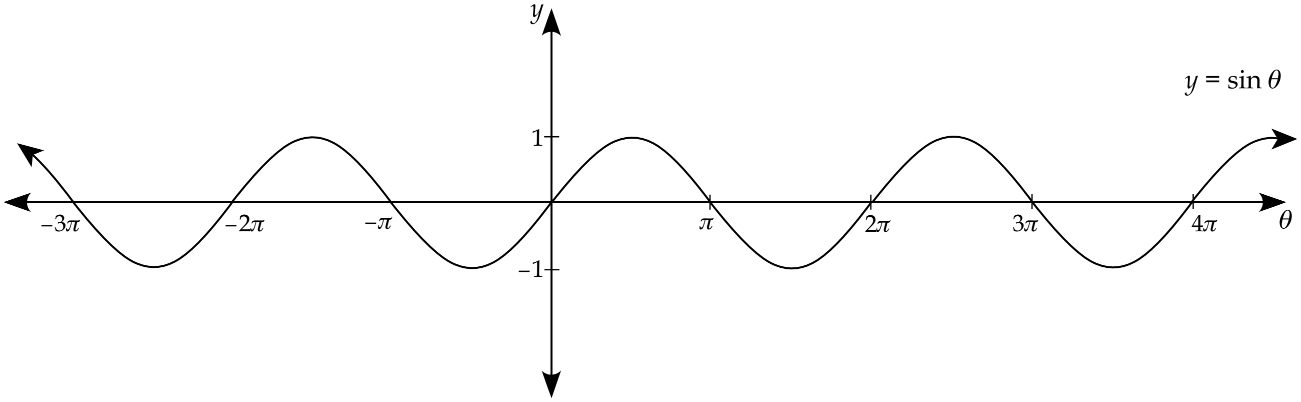
Using the data from the tables above, you can draw the sine curve, $f(\theta) = \sin \theta$. Here are some observations to help you think about patterns in data.

As θ increases	y-values	plotting the $y = \sin \theta$ curve
in Quadrant I, from 0 to $\frac{\pi}{2}$	are positive and increasing from 0 to 1	starting at the origin, the curve rises slowly from 0 to 1
in Quadrant II, from $\frac{\pi}{2}$ to π	are positive and decreasing from 1 to 0	the curve declines slowly from 1 to 0
in Quadrant III, from π to $\frac{3\pi}{2}$	are negative and decreasing from 0 to -1	the curve declines slowly from 0 to -1
in Quadrant IV, from $\frac{3\pi}{2}$ to 2π	are negative and increasing from -1 to 0	the curve rises slowly from -1 to 0 when θ completes one revolution at 2θ

The result is a portion of the sine curve, $y = \sin \theta$, for the values of θ from 0 to 2π; in other words, once around the unit circle. As the choice of notation for the independent variable is arbitrary, you will often be asked to sketch $y = \sin x$ instead of $y = \sin \theta$. It really does not matter; it's simply a choice of names. Label the axes accordingly.



You can continue the curve by choosing values for θ that are less than 0 or larger than 2π, to get the following diagram.



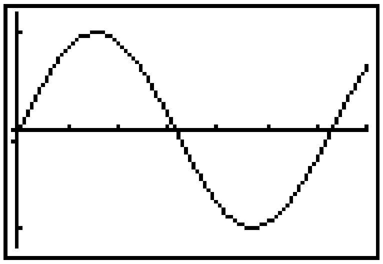
If you have a graphing calculator you can sketch the curve by following these steps.

1. Press MODE and select Radian.
2. Set your viewing WINDOW as follows:

$X_{min} = -0.1$
$X_{max} = 7$
$X_{scl} = 1$
$Y_{min} = -1.2$
$Y_{max} = 1.2$
$Y_{scl} = 1$

3. Press Y= and enter sin(X) as Y_1.
4. Press GRAPH.

   You will get the following graph.



   Alternatively, by pressing ZOOM 7:ZTrig, and then TRACE, you will see the curve begin at $\theta = -6.152286\ldots$ and end at $\theta = 6.152286$. The curve will go through "two cycles."

If you want to see a visual simulation of how the sine curve is related to the unit circle, visit the following websites and try out these applets:

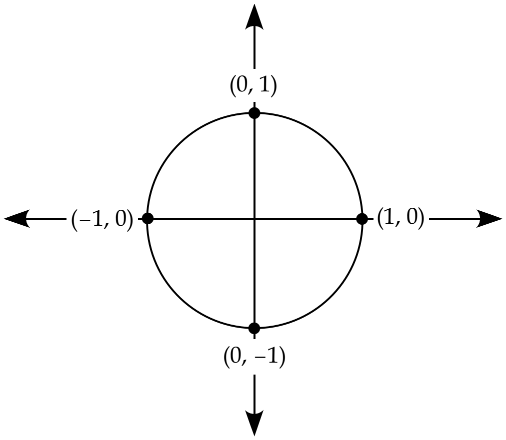
- www.dynamicgeometry.com/JavaSketchpad/Gallery/Trigonometry_and_Analytic_Geometry/Sine_Wave_Geometry.html
- http://mrgarnettmath.wikispaces.com/Sine+Wave+Applet

These applets show how the y-coordinate of the unit circle relating to a specific angle in standard position, relates to the sine curve.

Or, use your favourite search engine and the key words "sine curve and unit circle."

Key Points of the Sine Curve

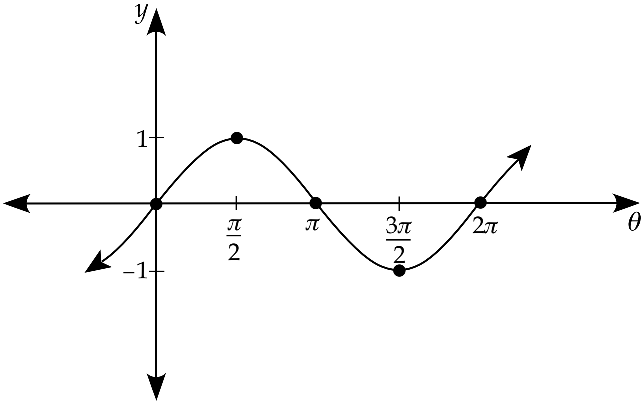
To draw a sketch by hand quickly, you need a few key points. The key points for the sine curve occur at the y-coordinates of the P(θ) values on the x-axis and y-axis of the unit circle where $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π.



The corresponding sine function values from the unit circle are:

$$\sin(0) = 0,\ \sin\frac{\pi}{2} = 1,\ \sin\pi = 0, \sin\frac{3\pi}{2} = -1,\ \sin 2\pi = 0$$

Plot these key points and a smooth curve through them.

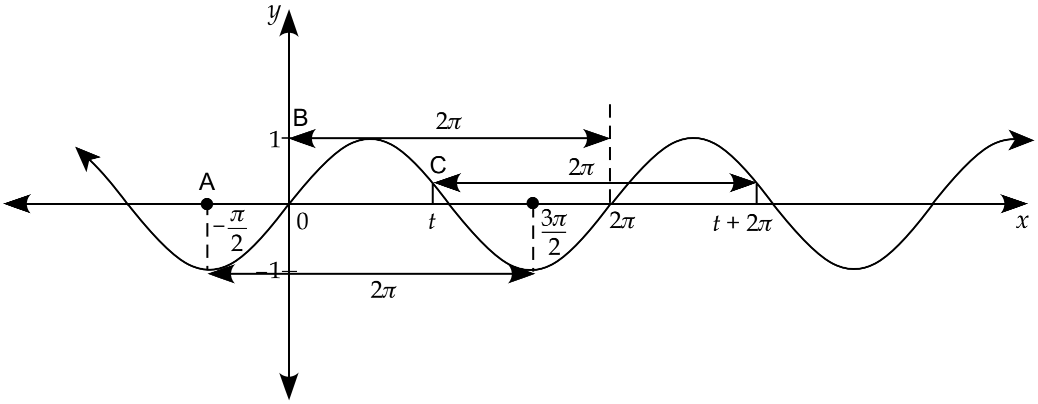


Properties of the Sine Curve

You should be able to draw a sketch of the sine curve $y = \sin\theta$ and list the following properties using your sketch.

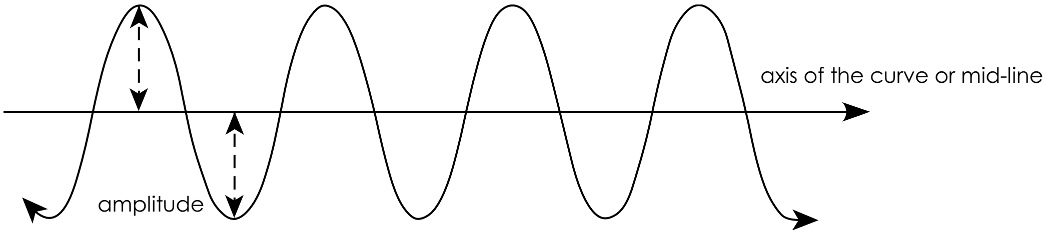
1. The domain is the set of all real numbers, since you can travel around the unit circle as far as you wish in a clockwise or counter-clockwise direction. This means the domain is the angle of rotation from the starting point (1, 0). The angle is a real number.
2. The range is [−1, 1]. The maximum of any y-coordinate at a point on the unit circle is 1. The lowest value of any y-coordinate on the unit circle is −1.
3. The y-intercept is 0. If the angle of rotation is 0, you are at the starting point, (1, 0), and the y-coordinate is 0, so sin (0) = 0.
4. The zeros of the sine function are all integral multiples of π, namely, $\{\ldots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots\}$. This set is usually written in more concise notation as $\{x \mid x = k\pi, k \in I\}$. This set is infinite and therefore the set is given as a formula that generates all the possible zeros.

5. **Definition:** A function $f(x)$ is a **periodic function** if there exists a number $p > 0$, such that for all x in the domain of f, $f(x + p) = f(x)$. The smallest such number p is called the period of f. This means that the function goes through cycles, always repeating itself. The period is the shortest distance you must travel along the x-axis for the function to begin another cycle. For the sine curve, the period is 2π. The θ values where a function repeats is analogous to the coterminal angles.



From the diagram, you can see that it doesn't matter where you start measuring the period, either at A, B, or C, just ensure that you measure one full cycle. The period is always a distance measured along the x-axis. One period is from $-\dfrac{\pi}{2}$ to $\dfrac{3\pi}{2}$ or from 0 to 2π or from t to $(t + 2\pi)$. Each distance is 2π. The sine function repeats every 2π radians.

6. **Definition:** Curves with wavelike forms, such as the sine curve, have a line midway between the high and low points of the curve. This line is called the **axis of the curve** or the **mid-line**. The distance from this axis to the maximum or to the minimum function value is called the **amplitude** of the curve. Since the amplitude is a measure of distance, the value of the amplitude is written without a sign (so it is always positive).



The amplitude of the sine curve is 1.

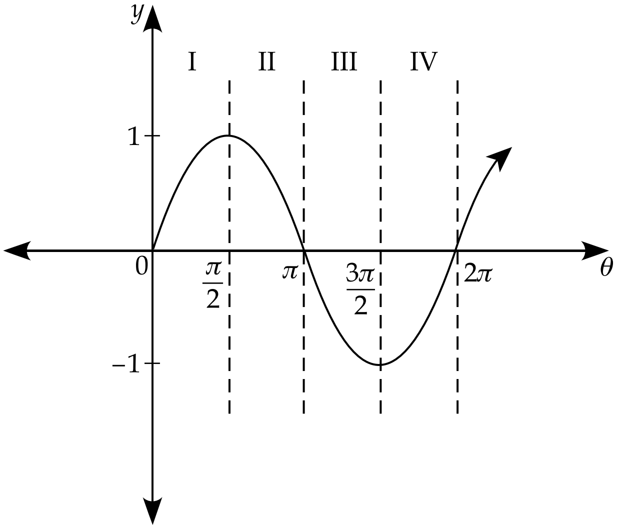
To make the other properties more clear, restrict the domain to one cycle of the sine curve, from 0 to 2π, and subdivide the curve into the four quadrants associated with the points on a unit circle.

Quadrant I on the unit circle is between 0 and $\frac{\pi}{2}$.

Quadrant II on the unit circle is between $\frac{\pi}{2}$ and π.

Quadrant III on the unit circle is between π and $\frac{3\pi}{2}$.

Quadrant IV on the unit circle is between $\frac{3\pi}{2}$ and 2π.



7. Since the curve is above the x-axis in Quadrants I and II, it follows that $\sin \theta > 0$ in these two quadrants. Similarly, since the curve is below the x-axis in Quadrants III and IV, $\sin \theta < 0$ in Quadrants III and IV. This is consistent with the CAST rule.

8. As you move on the curve from left to right the curve is increasing (moving up) in Quadrants I and IV and is decreasing in Quadrants II and III.

The Cosine Curve

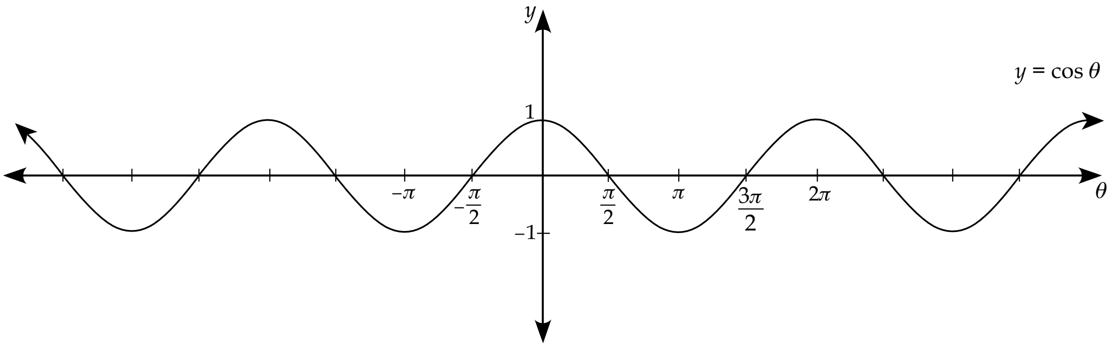
Now try sketching the curve, $y = \cos\theta$, using the x-coordinates from the points on the unit circle as the $\cos\theta$ values and the corresponding angles of rotation as the θ values. Then list properties of the cosine curve.

Again, a table of values for $y = \cos\theta$ relating the angle to the corresponding x-coordinate on the terminal arm of the angle on the unit circle can be created to help you draw the cosine curve.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos\theta$ (exact value)	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\cos\theta$ (decimal approximation)	1	0.866	0.7071	0.5	0	−0.5	−0.7071	−0.866	−1

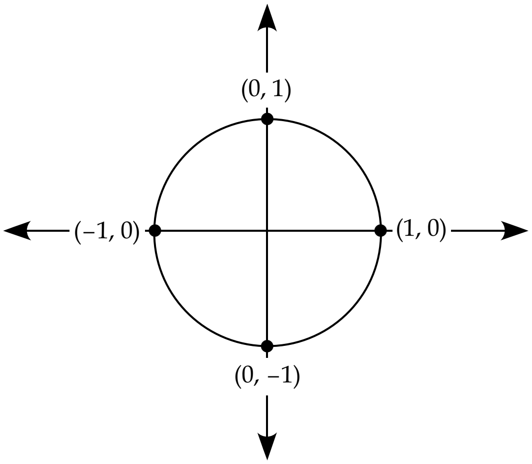
θ	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\cos\theta$ (exact value)	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	0
$\cos\theta$ (decimal approximation)	−1	−0.866	−0.7071	−0.5	0	0.5	0.7071	0.866	0

The cosine curve is shown below.



Key Points of the Cosine Curve

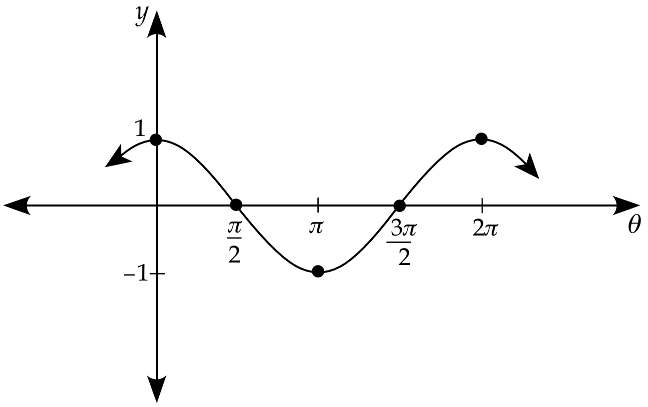
To draw a sketch by hand quickly, you need a few key points. The key points for the cosine curve occur at the x-coordinates of the P(θ) values on the x-axis and y-axis of the unit circle, where $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π.



The corresponding cosine function values from the unit circle are:

$$\cos(0) = 1, \cos\frac{\pi}{2} = 0, \cos\pi = -1, \cos\frac{3\pi}{2} = 0, \text{ and } \cos(2\pi) = 1$$

Plot these five key points and a smooth curve through them.



Properties of the Cosine Curve

The eight properties corresponding to the ones listed for the sine curve are:

1. Domain: $\{x \mid x \in \Re\}$ or $(-\infty, \infty)$
2. Range: $\{y \mid -1 \leq y \leq 1\}$ or $[-1, 1]$
3. y-intercept: 1
4. Zeros: $\theta = \left\{\frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi, k \in I\right\}$
5. Period: 2π.
6. Amplitude: 1.
7. $\cos\theta > 0$ in Quadrants I and IV on the unit circle; $\cos\theta < 0$ in Quadrants II and III on the unit circle.
8. The cosine curve is increasing in Quadrants III and IV and decreasing in Quadrants I and II.

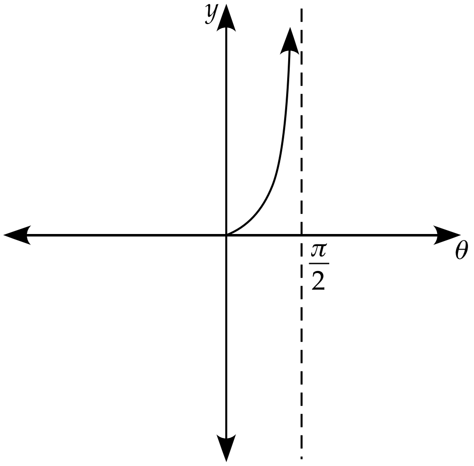
Note: For the zeros of cosine, any of the following notations are correct:

- $\theta = \left\{\frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi, k \in I\right\}$
- $\left\{\theta \mid \theta = \frac{\pi}{2} + k\pi, k \in I\right\}$
- zeros are the odd integral multiples of $\frac{\pi}{2}$
- $\left\{\theta \mid \theta = (2k+1)\frac{\pi}{2},\ k \in I\right\}$

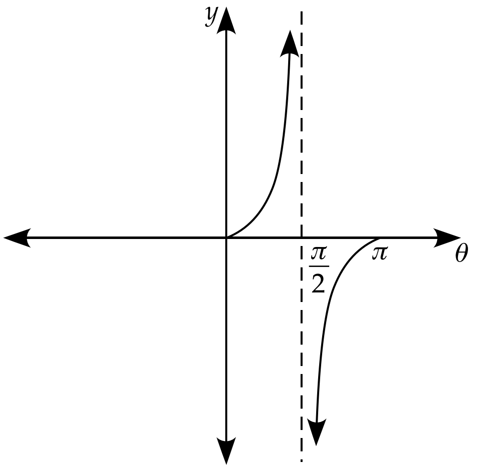
Take a minute to confirm that all four of these notations are equivalent.

The Tangent Curve

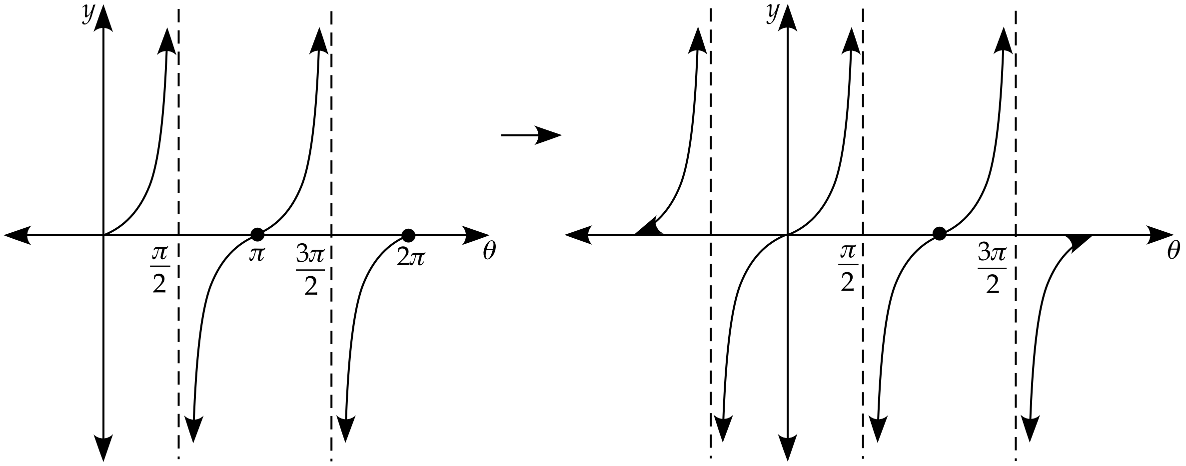
The graph of the tangent function is the graph of the ratio of $\frac{y}{x}$ for every point on the unit circle. It is also the slope of the terminal arm of the angle of rotation, since $\frac{y}{x}$ is the same as $\frac{\text{rise}}{\text{run}}$. The value of this slope begins with $\frac{0}{1} = 0$ at the point (1, 0) on the unit circle, the slope is 1 at $\theta = \frac{\pi}{4}$, and the slope ends with infinitely large values as the slope approaches the undefined value $\frac{1}{0}$ at the point (0, 1) on the unit circle. In other words, in Quadrant I of the unit circle, the tangent curve begins at the origin and increases very rapidly as θ gets closer and closer to $\frac{\pi}{2}$. At $\theta = \frac{\pi}{2}$, the curve has a vertical asymptote whose equation is $\theta = \frac{\pi}{2}$.



In Quadrant II of the unit circle, the values begin as very large negatives and increase to 0 as the x-values increase from $\frac{\pi}{2}$ to π.



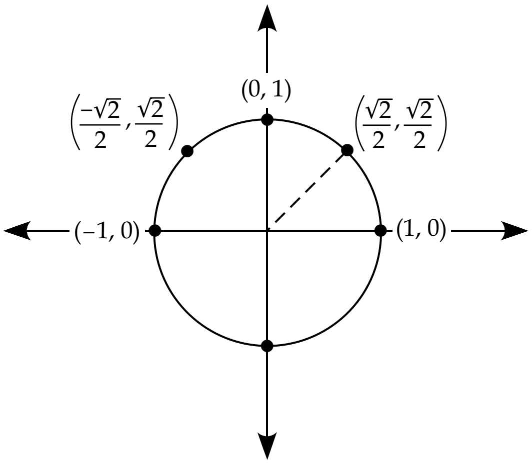
In Quadrants III and IV of the unit circle, the curve repeats its performances in Quadrants I and II, respectively.



Key Points of the Tangent Curve

To draw a sketch by hand quickly, you need a few key points. The key points for the tangent function occur where the slope, $\frac{y}{x}$, can easily be intercepted.

This occurs where $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$, and 2π. Only the first five θ values are really needed as you will see.



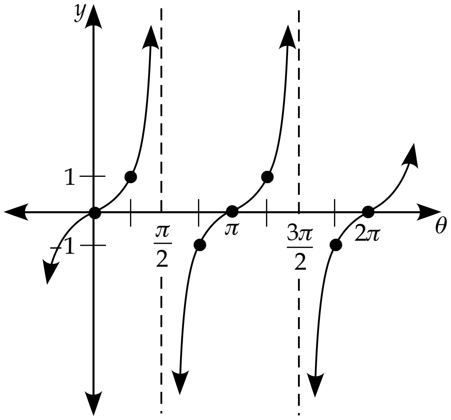
The corresponding tan function values from the unit circle are the slope of the terminal arm of the angle of rotation since the slope is $\frac{y}{x}$.

$$\tan(0) = 0,\ \tan\frac{\pi}{4} = 1,\ \tan\frac{\pi}{2} \text{ is } \infty,\ \tan\frac{3\pi}{4} = -1, \text{ and } \tan(\pi) = 0$$

You will see the next values repeat.

$$\tan\frac{5\pi}{4} = 1,\ \tan\frac{3\pi}{4} \text{ is } \infty,\ \tan\frac{7\pi}{4} = -1, \text{ and } \tan 2\pi = 0$$

Plot these key points and a smooth curve through them. Where tan θ is undefined at ∞, draw a vertical asymptote.



Properties of the Tangent Curve

The eight properties are:

1. Vertical asymptote repeats at odd multiples of $\frac{\pi}{2}$.

   Domain: $\left\{\theta \,\middle|\, \theta \neq \frac{(2k+1)\pi}{2},\ k \in I\right\}$ or $\left\{\theta \,\middle|\, \theta \neq \frac{\pi}{2} + k\pi,\ k \in I\right\}$

   Note: Either of the above answers is correct. If you look closely, you can see that they represent the same restrictions.

2. Range: $\{y \mid y \in \Re\}$
3. y-intercept: 0
4. Zeros: $\{k\pi, k \in I\}$.
5. Period: π (**Note:** tan repeats every π radians rather than every 2π radians).
6. Equations of the Asymptotes:

   $\left\{\theta \,\middle|\, \theta = \frac{(2k+1)\pi}{2},\ k \in I\right\}$ or $\left\{\theta \,\middle|\, \theta = \frac{\pi}{2} + k\pi,\ k \in I\right\}$.

7. $\tan\theta > 0$ in Quadrants I and III; $\tan\theta < 0$ in Quadrants II and IV.
8. The tangent curve is increasing in all four quadrants.

Note: Unlike the sine and cosine curves, the tangent curve does not have an amplitude associated with it.

Learning Activity 5.6

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Does the following set of points represent a one-to-one function?

   $(-3, 5), (7, -2), (6, -2), (-8, 3)$

2. Find the positive coterminal angle for −237° in the interval $0° \leq \theta \leq 360°$.
3. Is $x = -6$ a solution to the inequality $-x^2 - 3x + 5 > 0$?
4. Simplify: $(6x^2y^3)^2$
5. Determine the axis of symmetry of the quadratic function $f(x) = 3(x + 5)^2$.
6. Evaluate: $\sqrt[3]{-729}$.
7. Write as a mixed fraction: $\frac{17}{6}$
8. Convert 0.0125 into a fraction.

continued

Learning Activity 5.6 (continued)

Part B: Graphical Properties of Sine, Cosine, and Tangent

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Complete the following table of the properties of the sine, cosine, and tangent graphs. You may wish to include some, or all, of this information on your resource sheet for future reference.

	$\sin\theta$	$\cos\theta$	$\tan\theta$
Domain			
Range			
Period			
Amplitude			
Equation of Asymptotes			
x-intercepts			
y-intercept			

2. Demonstrate how the unit circle coordinates at the x- and y-axis can help to sketch $y = \cos\theta$ and $y = \sin\theta$.

Lesson Summary

In this lesson, you learned about the graphs of the three main trigonometric functions—sine, cosine, and tangent. You also learned about the properties of each of these graphs. In the next lesson, you will learn how the sine and cosine graphs, as well as their properties, change when they undergo transformations.

Lesson 6: Transformations of Sine and Cosine Graphs

Lesson Focus

In this lesson, you will

- ☐ learn how to sketch transformations of $f(\theta) = \sin \theta$ and $f(\theta) = \cos \theta$ and describe how the basic properties have been altered
- ☐ learn how to determine the equation of a periodic function when given its graph
- ☐ learn about various applications of trigonometric functions

Lesson Introduction

In the last lesson, you learned about the properties of the standard sine, cosine, and tangent graphs. In this lesson, you are going to be examining how these properties are affected when the sine and cosine graphs are transformed. You have already studied transformations of quadratic functions, cubic functions, radical functions, rational functions, and absolute value functions. Transformations of trigonometric functions are very similar.

Transformations of the Sine and Cosine Curves

Now that you know how to draw the basic sine and cosine curves, you can transform these curves using transformations.

The standard form of the sine function is: $y = a \sin [b(x - c)] + d$

The standard form of the cosine function is: $y = a \cos [b(x - c)] + d$

The variables a, b, c, and d have the same effect on sinusoidal functions as they do on other functions, such as quadratic, cubic, or radical functions.

Transformations of the sine curve lead to a family of curves called sinusoidals. In general, any equation that can be written in the form $y = a \sin [b(x - c)] + d$ or $y = a \cos [b(x - c)] + d$ is said to be **sinusoidal** as well as **periodic**. The numbers a, b, c, and d are called **parameters**.

Did you notice that the shape of $y = \sin \theta$ and $y = \cos \theta$ are exactly the same? The cosine curve is just a shifted $y = \sin \theta$ curve.

You may find it helpful to include the above information on your resource sheet.

Transformations Affecting the *y*-values

Example 1

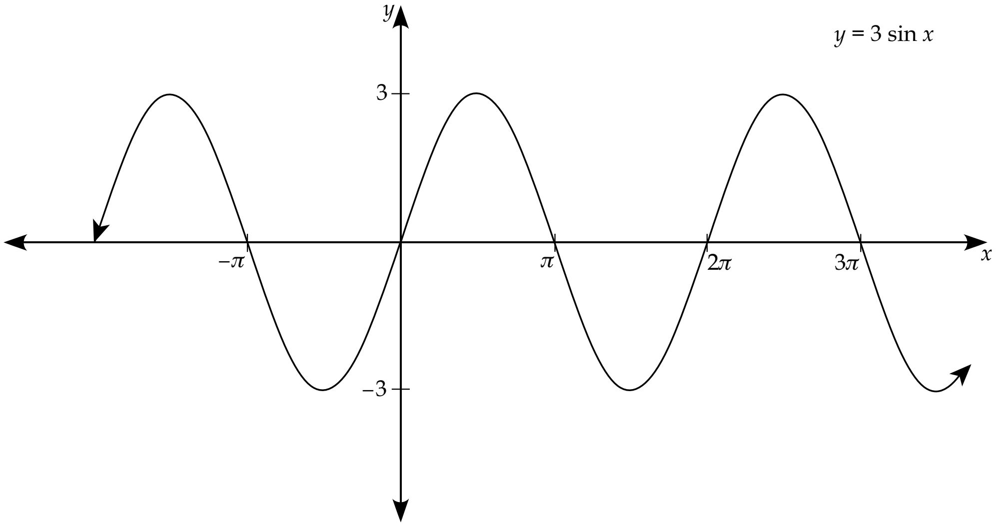
Using technology, sketch the following graphs. If you do not have access to technology, look directly at the solutions.

a) $y = 3 \sin x$ and state its range and period

b) $y = -\cos x$ and state the values of its intercepts

Solutions

a) Notice the variables are x and y. Usually x or θ is used for the domain and y is used for the range. Just do not confuse this independent variable x with the x-coordinate on the unit circle. The x-coordinate on the unit circle is used to help you remember the $\cos \theta$ and $\sin \theta$ values for various angles of rotation.

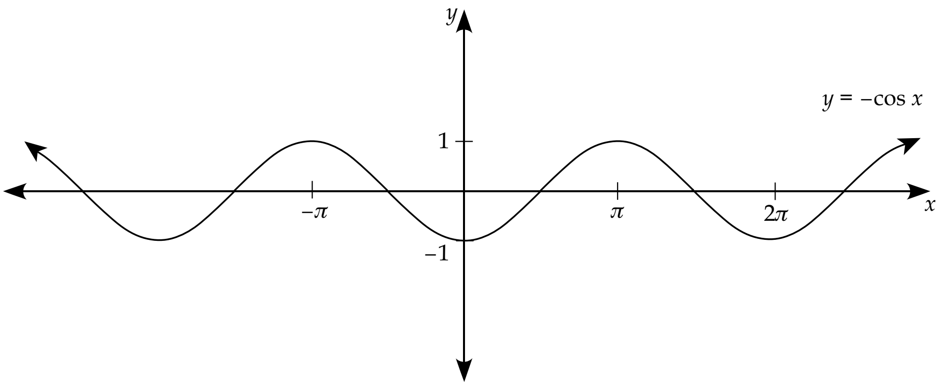


The graph of $y = \sin x$ is stretched vertically by a factor of 3.

Range = $[-3, 3]$.

Period is still 2π because the x-values have not been affected by the transformation.

b)



The graph of $y = \cos x$ is reflected over the x-axis (a flip).

y-intercept = −1.

x-intercepts are still $\left\{x \middle| x = \dfrac{(2k+1)\pi}{2},\ k \in I\right\}$ or $\left\{x \middle| x = \dfrac{\pi}{2} + \pi k,\ k \in I\right\}$.

In the Example 1(a), the highest and lowest points on the wave are affected. In other words, the amplitude of the graph is affected. You learned the meaning of the term **amplitude** in the last lesson. It is the distance from the sinusoidal axis of the curve or midline to the maximum. The amplitude can be calculated by taking the largest value of y, subtracting the smallest value of y, and dividing the result by 2. From this example, you can see that the amplitude is the same as the vertical stretch factor. The stretch factor is a.

Example 2

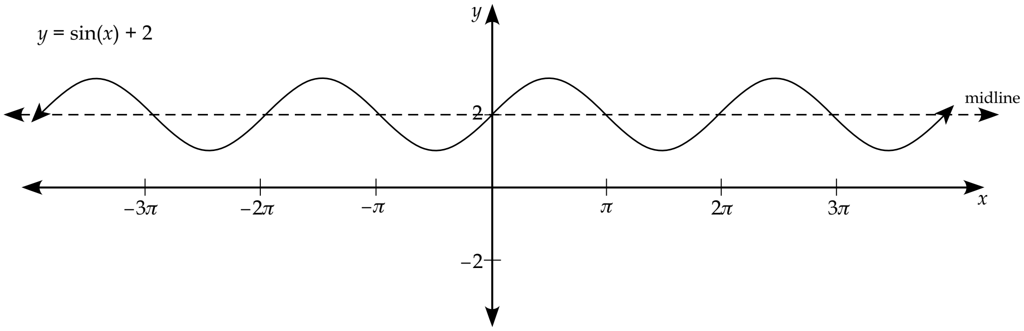
Use technology to sketch the graph of

a) $y = \sin x + 2$ and state its range and period

b) $y = \cos x - 3$ and state the values of its intercepts.

Solutions

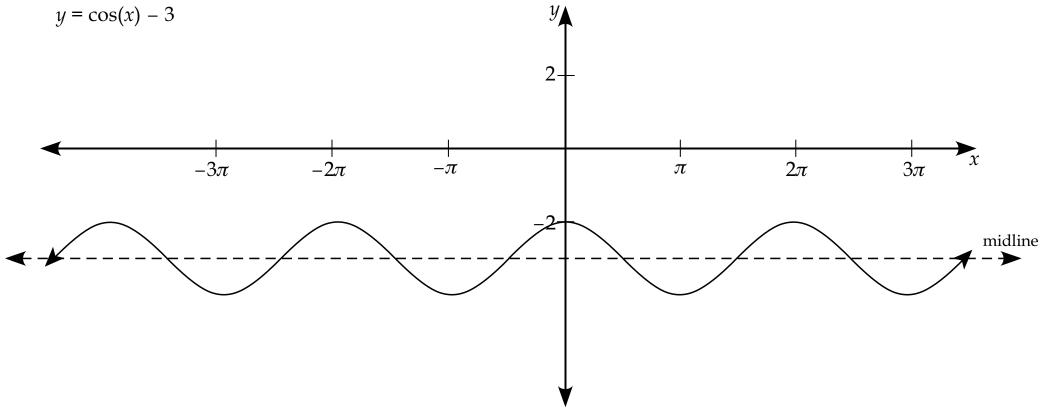
a)



The graph of $y = \sin x$ is shifted vertically 2 units up from the standard $y = \sin x$ curve.

Range = [1, 3].

Period is still 2π because the x-values have not been affected by the transformation.

b) 

The graph of $y = \cos x$ is shifted vertically 3 units down from the standard $y = \cos x$ curve.

This function has no intercepts as it has been shifted completely below the x-axis.

In Example 2, both of the graphs kept their same shape but they were shifted up or down d units. If d is positive, the graph moves up, if d is negative, the graph moves down.

Transformations Affecting the *x*-values

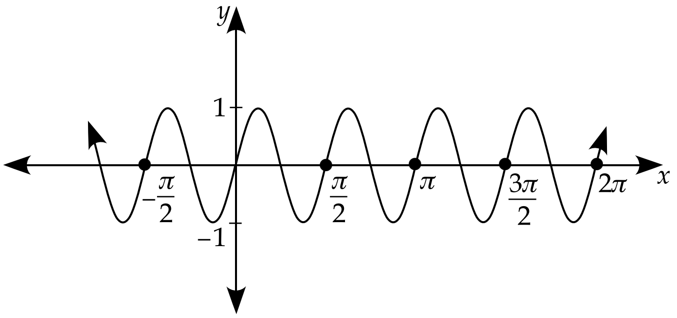
Example 3

Use technology to sketch and state the period of

a) $y = \sin 4x$

b) $y = \cos\left(\dfrac{1}{2}x\right)$

Solutions

a) 

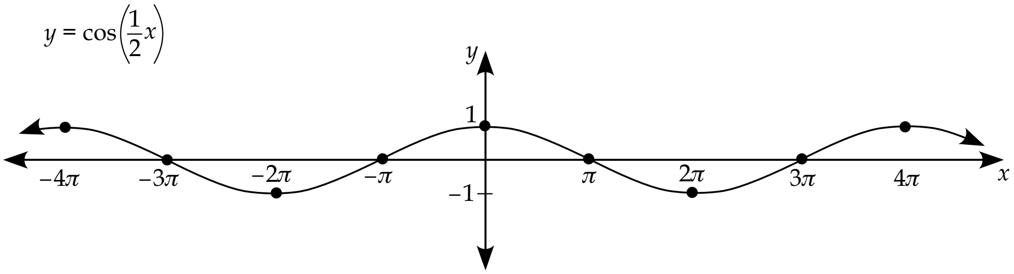
The graph of $y = \sin x$ is compressed horizontally by a factor of 4.

The period is now reduced by a factor of 4 because of the compression. Thus, since the period was 2π, now the period is $\frac{2\pi}{4} = \frac{\pi}{2}$.

A useful hint: Many students find it easier to sketch circular function curves by following the period of the function to determine where the transformed curve begins and ends one cycle. For example, the period of a basic sine curve is 2π; therefore, find the starting and ending point of one cycle as follows:

Curve	Start One Cycle At	End One Cycle At	Period
basic $y = \sin x$	0	2π	2π
transformed function $y = \sin 4x$	When does $4x = 0$? When $x = 0$.	When does $4x = 2\pi$? When $x = \frac{\pi}{2}$.	$\frac{\pi}{2}$

b)



The graph of $y = \cos x$ is compressed horizontally by a factor of $\frac{1}{2}$ or stretched horizontally by 2.

Since the period was 2π, now the period is $2\pi \div \frac{1}{2} = 4\pi$.

When the constant b is changed, the period of the graph changes. The period can be calculated by using the formula $\frac{2\pi}{b}$. This is because the basic functions $y = \sin x$ and $y = \cos x$ have period = 2π. The functions $y = \sin bx$ and $y = \cos bx$ have the period, compressed by a factor of b so the period = $\frac{2\pi}{b}$.

Example 4

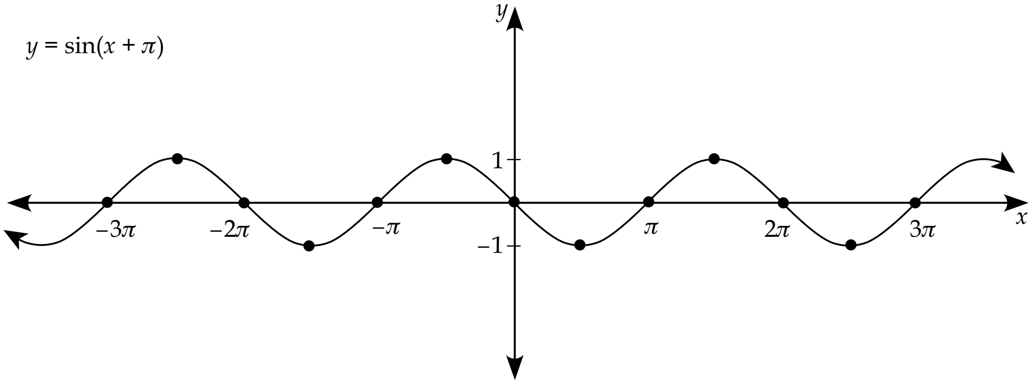
Sketch and state the period of

a) $y = \sin\left(x + \pi\right)$

b) $y = \cos\left(x - \dfrac{\pi}{2}\right)$

Solutions

a)



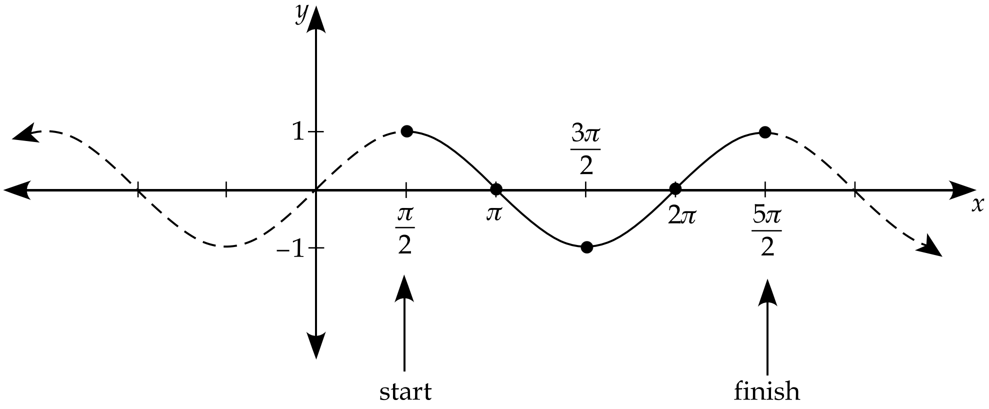
This function has been moved π units to the left.

The period of this function is still 2π since the coefficient of x is 1.

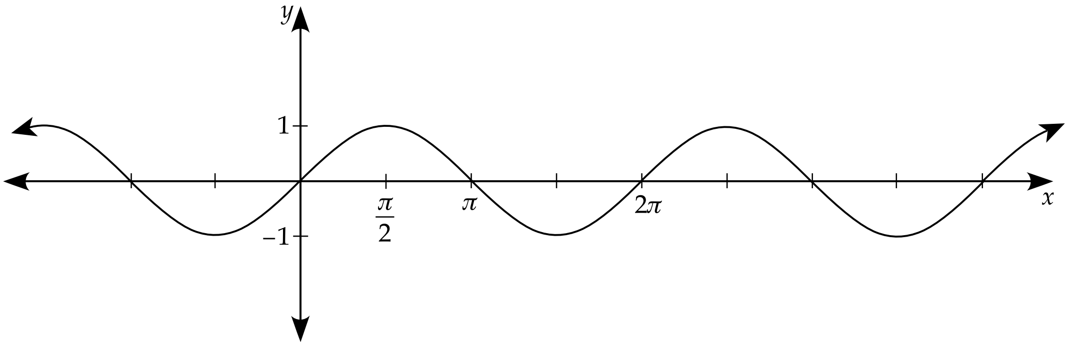
Note: $y = \sin\,(x - \pi)$ would look exactly the same as $y = \sin\,(x + \pi)$. Can you think of why? This concept will be further explored in the following learning activity.

b) To graph $y = \cos\left(x - \dfrac{\pi}{2}\right)$ start with the graph of $y = \cos x$ and then shift it $\dfrac{\pi}{2}$ units to the right.

Thus, begin drawing the basic cycle (with 5 key points) of the cosine curve at $x = \frac{\pi}{2}$ and finish the cycle at $x = \frac{5\pi}{2}$.



Then fill in the rest of the curve to get the final solution shown below.



Note: You will notice that the period is only affected if you change the numerical coefficient before the x (that is, when you make a horizontal stretch or compression).

In summary, the transformed cosine and sine graphs can be written in the forms

$$y = a \sin b(x - c) + d$$

or $y = a \cos b(x - c) + d$

where $|a|$ = amplitude (a vertical stretch, multiply y-values by "a")

$\frac{2\pi}{|b|}$ = period (a horizontal compression, divide x-values by "b")

c = horizontal translation (also called a phase shift for sinusoidals)

if $c > 0$ shift c units right

if $c < 0$ shift c units left

d = vertical translation

if $d > 0$ shift up d units

if $d < 0$ shift down d units

Note: A **phase shift** is the scientific name for a horizontal translation for sinusoidal curves.

You also have two more handy formulas to use for graphing cosine and sine curves.

$$|a| = \text{amplitude} = \left|\frac{\max - \min}{2}\right|$$

(This is half the distance between the minimum and maximum value.)

$$d = \text{vertical shift} = \text{midline} = \frac{\max + \min}{2}$$

(The midline is at the average of the maximum and minimum values.)

Include a summary of how the transformations of trigonometric functions are related to their properties on your resource sheet.

Example 5

Graph the following functions using transformations. State the amplitude, domain, period, and range of each function.

a) $y = 3\sin(2x)$

b) $y = 2\cos(x) + 5$

c) $y = 2\sin\frac{1}{2}(x + \pi) - 1$

d) $y = \frac{1}{3}\cos\left(x - \frac{3\pi}{2}\right) + 4$

Solutions

a) This is a transformation of the basic sine graph, which has undergone the following transformations:

- Vertical stretch by a factor of 3
- Horizontal compression by a factor of 2

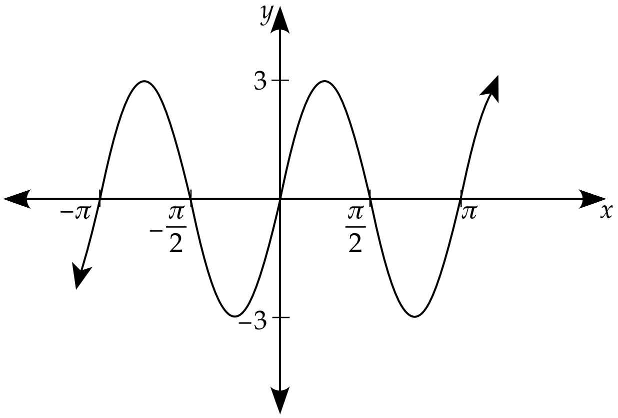
Before you sketch the graph of the function, it may be helpful for you to state the important properties of the function. However, the range and the zeros of a function are often easier to determine after you graph the function.

Amplitude: $|a| = |3| = 3$

Domain: $\Re$

Period: $\frac{2\pi}{2} = \pi$

The key points for the $y = \sin(x)$ function have been stretched vertically 3 times and compressed horizontally 2 times.



b) This is a transformation of the basic cosine graph, which has undergone the following transformations:

- Vertical stretch by a factor of 2
- Vertical shift 5 units up

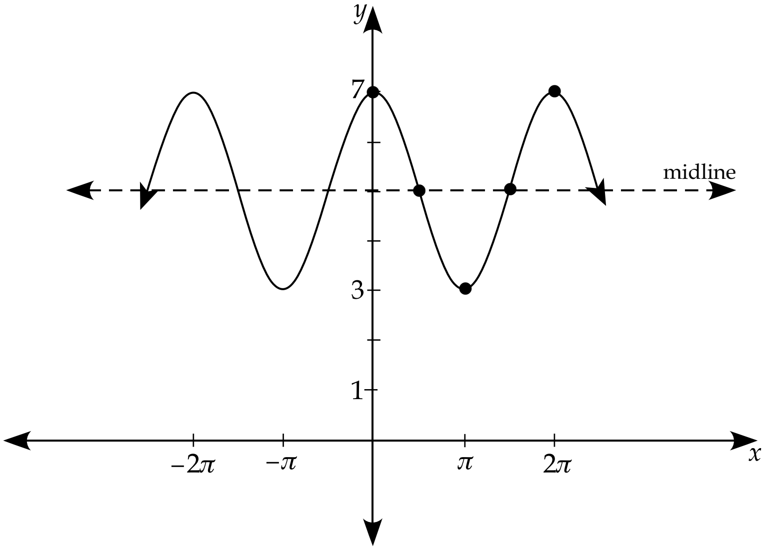
Recall: Stretches and compressions should be performed before shifts or translations.

Amplitude: $|a| = |2| = 2$

Domain: $\Re$

Period: $\frac{2\pi}{1} = 2\pi$

Transform the five key points.



c) This is a transformation of the basic sine graph, which has undergone the following transformations:

Vertical stretch by a factor of 2

Horizontal compression by a factor of $\frac{1}{2}$. (**Note:** This is the same as a horizontal stretch by a factor of 2.)

Horizontal shift π units to the left

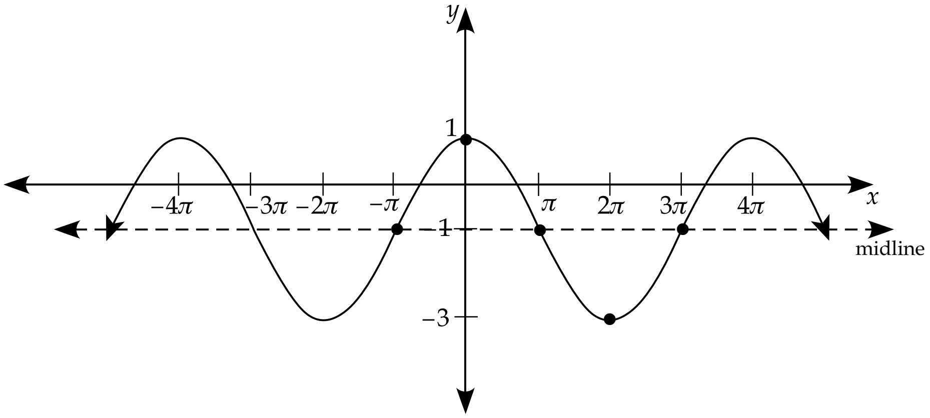
Vertical shift 1 unit down

Amplitude: $|\, a \,| = |\, 2 \,| = 2$

Domain: $\Re$

Period: $\frac{2x}{\frac{1}{2}} = 2\pi\,(2) = 4x$

Transform the five key points.



From the graph, you can easily determine the range.

Range: [−3, 1]

d) This is a transformation of the basic cosine graph, which has undergone the following transformations:

Vertical compression by a factor of 3

Horizontal shift $\frac{3\pi}{2}$ units to the right

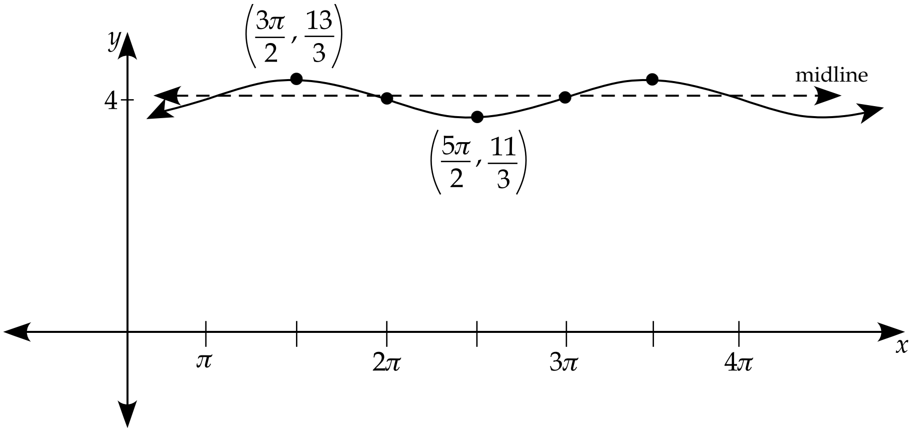
Vertical shift 4 units up

Amplitude: $\frac{1}{3}$

Domain: $\Re$

Period: $\frac{2\pi}{1} = 2\pi$

Transform the five key points.



Range: $\left[\frac{11}{3}, \frac{13}{3}\right]$

Finding an Equation of a Periodic Function Given Its Graph

The cosine curve and the sine curve are the same shape; they are just shifted a bit horizontally. It is possible, therefore, to use either the sine function or the cosine function as a model to find the equation of a periodic function. The basic sine curve passes through the origin (0, 0) on its way up to the maximum that it reaches at $x = \frac{\pi}{2}$ and $y = 1$, while the basic cosine function passes through one of its maximums, (0, 1), on its way down. These two points for these two curves are extremely important when you are finding the equation of a periodic function. To find the equation of a periodic graph, you must find a, b, c, and d for the equation $y = a \sin b(x - c) + d$ or $y = a \cos b(x - c) + d$.

Note: You may also be asked to find the equation of a sinusoidal function. Since they are the same shape, any sine or cosine curve is called a sinusoidal function.

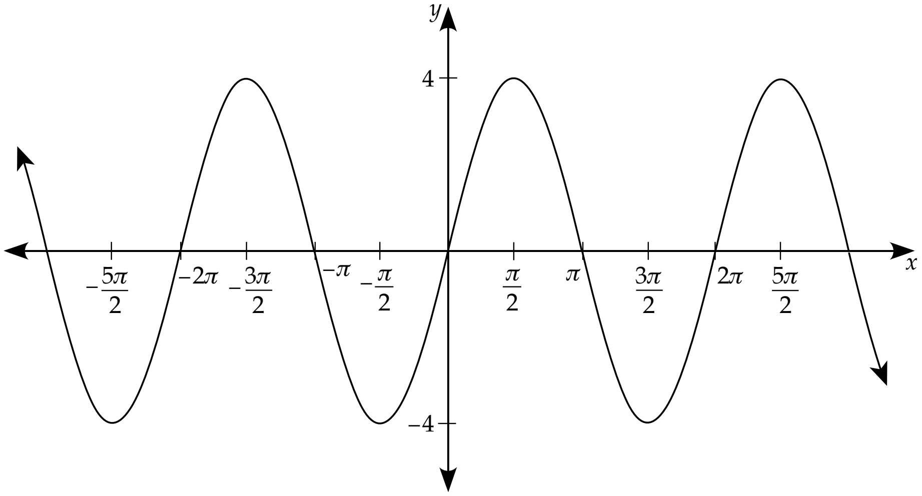
Finding *a*

The value of a is found by subtracting the smallest value of y from the largest value of y, and dividing the result by 2. When you do this, you are finding the amplitude of the curve. This is equivalent to using the following formula:

$$|a| = \text{amplitude} = \left|\frac{\max - \min}{2}\right|$$

Example 6

Find the amplitude of the function graphed below.



Solution

Amplitude is the same whether the sine or cosine function is used as the model. In this example, it is $\frac{(4-(-4))}{2} = 4.$ Therefore, the value of $a = 4$.

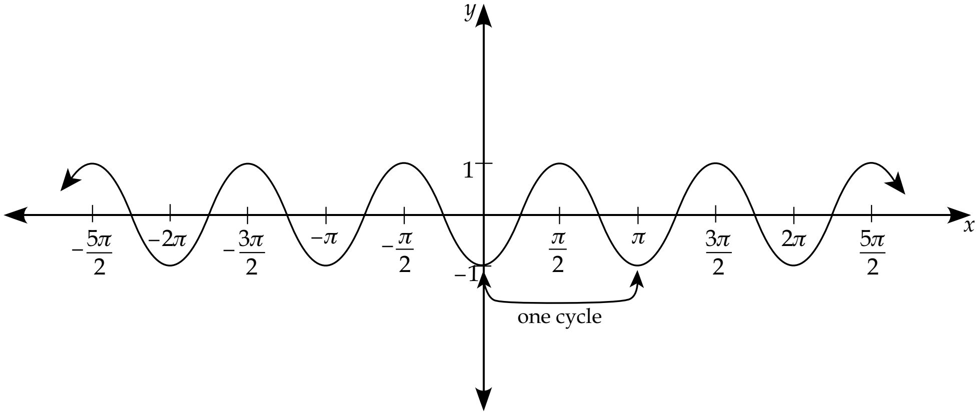
The amplitude is the distance from the midline up to the maximum.

Finding *b*

The value of b is found by finding the period (horizontal length of one cycle of the graph) of the function and dividing 2π by this value. This is the same whether sine or cosine is used. It can be found by counting how far it is on the graph from a high point to the next high point, or from a low point, to the next low point, or from a midway point, to the next midway point. You can use the following formula $b = \frac{2\pi}{\text{period}}$.

Example 7

Find the value of b by determining the period of the function graphed below.



Solution

Because one complete cycle of the graph is π units long horizontally, the value of $b = \dfrac{2\pi}{\pi} = 2$.

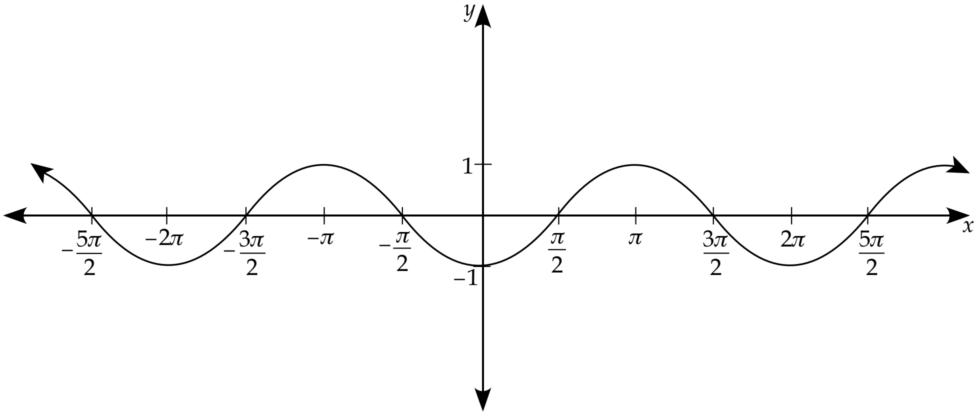
Finding *c*

The value of c is dependent on whether you use the sine function or cosine function. If you use the sine function, you will determine how far the graph is moved horizontally to the left or right using the midway point on the graph as a reference point.

If you use the cosine function, then you will determine c using the maximum point on the graph and determine how far it has moved left or right from the y-axis.

Example 8

Find the value of c if the sine function is used as a model and if the cosine function is used as a model of the function graphed below.



Solution

If the sine function is used, consider the point $\left(\frac{\pi}{2}, 0\right)$ as the starting point.

The standard sine graph has to be moved $\frac{\pi}{2}$ units to the right. Therefore, c would be equal to $\frac{\pi}{2}$. Alternatively, the point at $\left(-\frac{3\pi}{2}, 0\right)$ could be used.

The equation may be given as $y = \sin\left(x - \frac{\pi}{2}\right)$ or as $y = \sin\left(x + \frac{3\pi}{2}\right)$.

If you used the cosine function, consider the point $(\pi, 1)$ as the starting point. You would need to move the standard cosine graph π units to the right to achieve the graph above. Alternatively, the point at $(-\pi, 1)$ could be used. Therefore, the value of c would be $-\pi$ or π. The equation may be given as $y = \cos(x + \pi)$ or as $y = \cos(x - \pi)$.

All four of these equations describe the same function shown.

Finding *d*

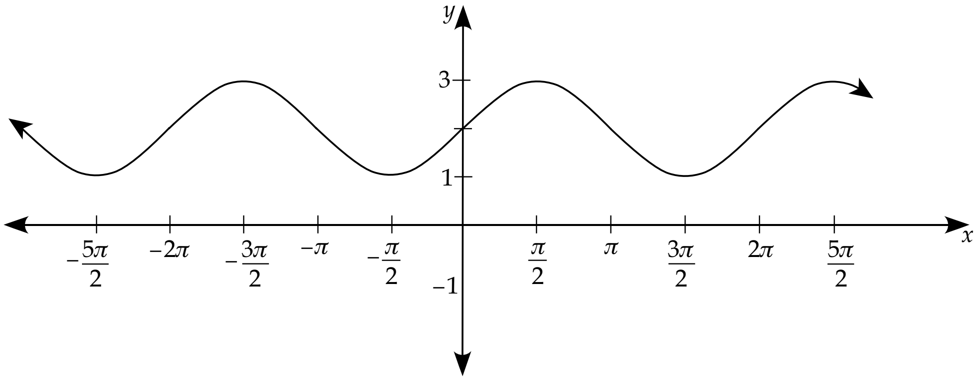
To find d, you must determine how far a midline point has to be moved up or down to lie on the x-axis. There is no difference for sine or cosine functions. This is the same as using the following formula:

$$d = \text{vertical shift} = \text{midline} = \frac{\text{max} + \text{min}}{2}$$

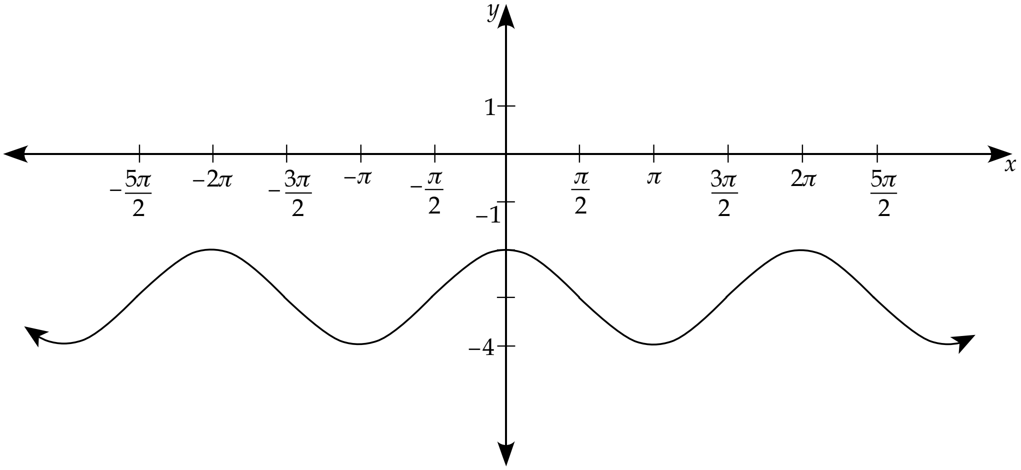
Example 9

Determine the values of d in the functions graphed below.

a)



b)



Solutions

a) Any one of the midline points has been shifted up 2 units from the x-axis. Therefore, $d = 2$.

b) Any one of the midline points has been shifted down 3 units from the x-axis. Therefore, $d = -3$.

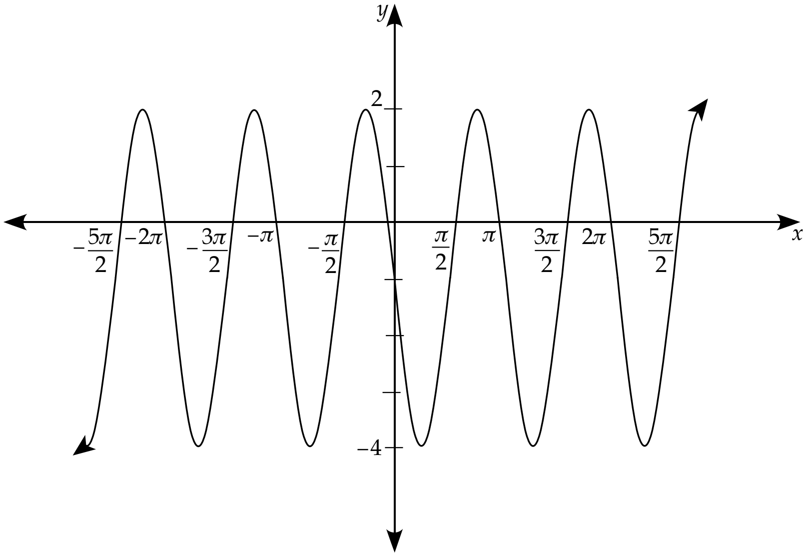
In summary, to determine the equation of a periodic function when given its graph, use the following steps.

1. Determine whether you will use sine or cosine. Sometimes it will be easier to use sine rather than cosine, and sometimes the reverse is true. Generally, it is determined by how easy it is to find c. The a, b, and d values are identical for sine and cosine.
2. Find a, b, c, and d, and state in the form:

$$y = a \sin b(x - c) + d \text{ or } y = a \cos b(x - c) + d$$

Example 10

Determine a periodic function that corresponds to the following graph.



Solution

It is recommended that you draw in the midline as a reference for the *a* and *d* values.

Amplitude $= a = \dfrac{2-(-4)}{2} = 3$

Period $= \pi \qquad \therefore\ b = \dfrac{2\pi}{\pi} = 2$

$c = \dfrac{\pi}{2}$ for sine function considering the point $\left(\dfrac{\pi}{2}, -1\right)$ on the midline.

$c = \dfrac{3\pi}{4}$ for cosine function considering the point $\left(\dfrac{3\pi}{4}, 2\right)$ at a maximum.

This function has been shifted down 1 unit from the *x*-axis, $\therefore\ d = -1$

Therefore, two possible functions are $y = 3\sin 2\left(x - \dfrac{\pi}{2}\right) - 1$ or

$y = 3\cos 2\left(x - \dfrac{3\pi}{4}\right) - 1.$

Note: All parameters are the same for the cosine function and the sine function except for *c*.

You may wish to include a description on how to find the variables a, b, c, and d for a trigonometric function in standard form on your resource sheet.

Trigonometric Function Applications

Trigonometric functions have many applications including modeling waves, temperature, motion, sales of seasonal products, voltage, and even employment.

When solving applications involving periodic functions, you may find it helpful to follow these steps:

1. Draw a rough sketch that highlights the major points stated.
2. Find the algebraic model that fits the sinusoidal graph.
3. Use the model to find any values of variables needed.

Example 11

An electric heater cuts in and out on a cyclical basis as it heats the water in a hot tub. The water temperature (C) in degrees Celsius varies sinusoidally with time (t). The thermostat is set to turn the heater on when the temperature drops to 36°C and off when it reaches 42°C. In order to measure how long one complete cycle of this process took, a timer was started at 0 when the heater cut out. The heater came on again 30 minutes later and turned off again after another 30 minutes.

a) Draw a sketch of this sinusoidal function showing the temperature (C) versus the time (t) for $0 < t < 60$.

b) Write an equation expressing the temperature (C) in terms of the number of minutes elapsed (t).

c) To the nearest tenth, what would the temperature of the water be when the timer reads 10 minutes?

Solutions

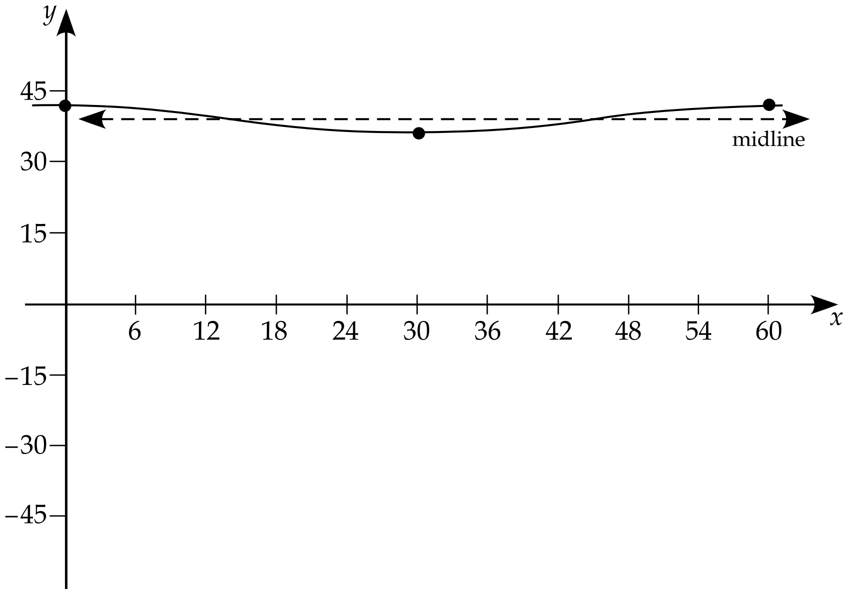
a) From the question, you can determine three points on this graph.

At $t = 0$, you know that the temperature is 42°C because the heater turned off.

At $t = 30$, you know that the temperature is 36°C because the heater turned back on.

At $t = 60$, you know that the temperature is again at its highest point, 42°C, because the heater turned off.

Using these three points and the fact that the electric heater operates on a sinusoidal, you can draw the following graph.



b) $a = \left|\dfrac{\text{max} - \text{min}}{2}\right| = \left|\dfrac{42 - 36}{2}\right| = 3$

$$b = \frac{2\pi}{\text{period}} = \frac{2\pi}{60} = \frac{\pi}{30}$$

$c = 0$ (if you are using the cosine function as a model since maximum is at (0, 45))

$$d = \frac{\text{max} + \text{min}}{2} = \frac{42 + 36}{2} = 39$$

Therefore, $C(t) = 3\cos\dfrac{\pi}{30}(t) + 39$.

c) Use the function from (b) to evaluate the function at the point $t = 10$.

$$C(t) = 3\cos\frac{\pi}{30}(t) + 39$$

$$\begin{aligned} C(10) &= 3\cos\frac{\pi}{30}(10) + 39 \\ &= 3\cos\frac{\pi}{3} + 39 \\ &= 3\left(\frac{1}{2}\right) + 39 \\ &= \frac{3}{2} + 39 \\ &= \frac{81}{2} = 40.5° \end{aligned}$$

Learning Activity 5.7

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Which is the better deal, a 12-pack of macaroni and cheese for \$4.49 or a 36-pack of macaroni and cheese for \$11.99?
2. What is the length of the hypotenuse of a right-angled triangle if the two legs of the triangle measure 3 m and 4 m respectively?
3. Factor: $(x - 1)^2 - (y + 3)^2$
4. Simplify: $\dfrac{\frac{7}{2}}{5}$
5. Reduce to lowest terms: $\dfrac{56}{180}$
6. Express $7 + \dfrac{6}{11}$ as an improper fraction.
7. Is 239 a prime number?
8. What is 25% of \$1250?

continued

Part B: Graphing Trigonometric Functions

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Find the period of each function.
   a) $y = 3 \sin 2x$
   b) $y = \pi \sin (-\pi x)$
   c) $y = \cos (x - \pi)$
   d) $y = -\sin (3x - 4)$
   e) $y = \cos (4x + \pi)$
   f) $y = \sin \left(-\frac{1}{2}x\right)$

2. Find the x-intercepts of each circular function. You are being asked to write a set of all the x-intercepts, not just those on the interval $[0, 2\pi]$.
   a) $y = \sin 3x$
   b) $y = \cos 2x$
   c) $y = \sin \pi x$
   d) $y = -3 \sin (-2\pi x)$

3. Sketch each of the following. State the domain, range, amplitude, y-intercept, and period.
   a) $y = 2 \sin x$
   b) $y = \sin 2x$
   c) $y = \sin (x - \pi)$
   d) $y = 2 \cos x$
   e) $y = -\cos 2x$
   f) $y = \cos (-2x)$
   g) $y = | \sin x |$
   h) $y = \cos x - 1$
   i) $y = \cos \pi x$
   j) $y = -2 \sin \left(\frac{1}{2}x\right)$
   k) $y = | \tan x |$

continued

Learning Activity 5.7 (continued)

4. Sketch the following curves. State their properties.

   a) $y = 3 \sin 2\left(x - \frac{\pi}{2}\right) + 1$

   b) $y = 2 \cos \left(x + \frac{\pi}{2}\right) - 1$

5. $y = \sec \theta$ and $y = \csc \theta$ are reciprocals of one of the basic three circular functions. Use the graphs of the basic functions to sketch their reciprocals. Recall: Plot invariant points where $y = \pm 1$ and vertical asymptotes at x-intercepts.

6. For each of the functions in the previous question, list all the properties of each function (domain, range, y-intercept, zeros, equations of asymptotes, and period).

7. The graphs of $y = \sin (x - \pi)$ and $y = \sin (x + \pi)$ are identical. Explain.

8. Write an equation that describes the sine curve in terms of the cosine curve.

9. Find the amplitude, period, horizontal shift, and vertical shift for each of the following:

   a) $y = 2 \sin 3(x - 4) + 1$

   b) $y = -3 \cos 5x$

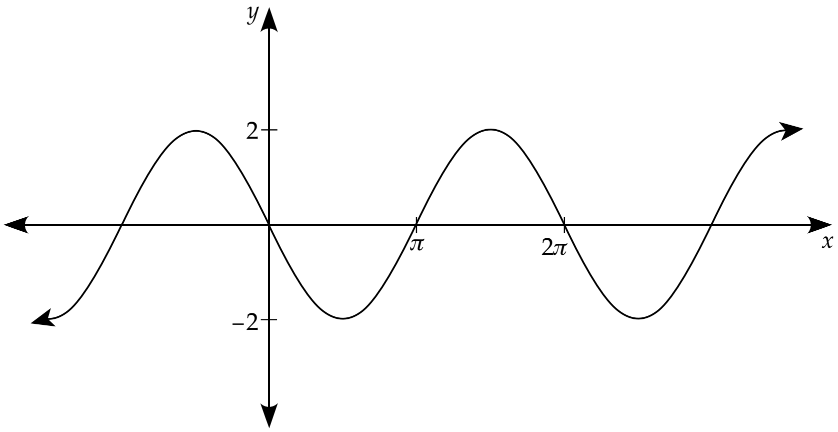
   c) $y = 4 \cos \pi x$

   d) $y = 7 \cos 2\left(x - \frac{\pi}{2}\right) + 3$

   e) $y = -5 \sin 2\pi(x - 1) - 4$

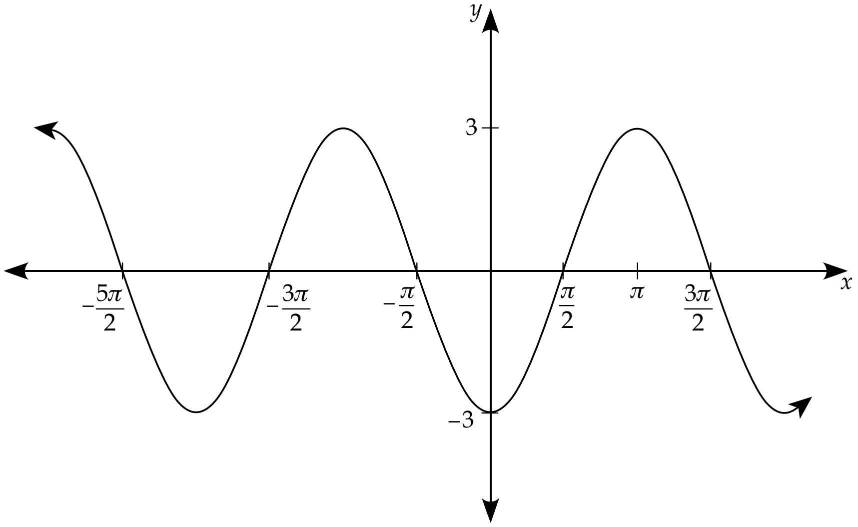
10. For each of the following graphs of $y = a \sin b(x - c) + d$, find the values of a, b, c, and d.

    a)



continued

b)



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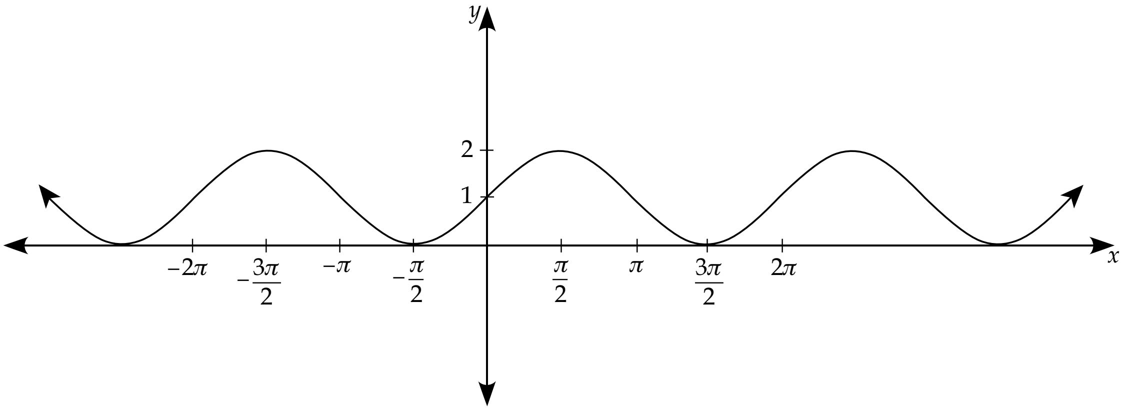
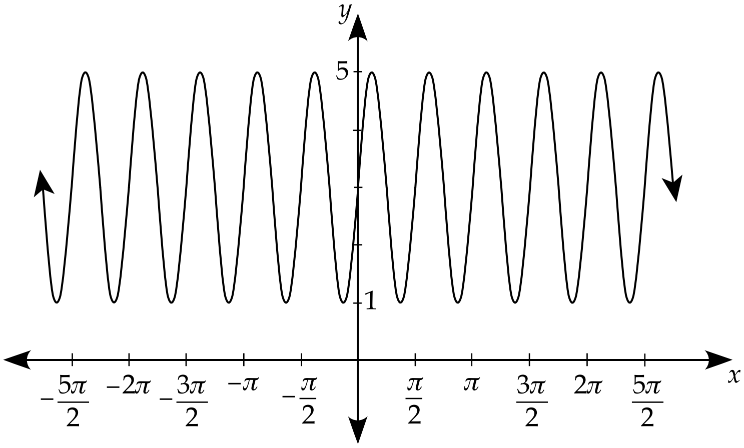


11. Use the same graphs as in the previous question, but change the function to $y = a \cos b(x - c) + d$ and find the values of a, b, c, and d.

continued

12. Find an equation of each of the following graphs as sine functions.

a)



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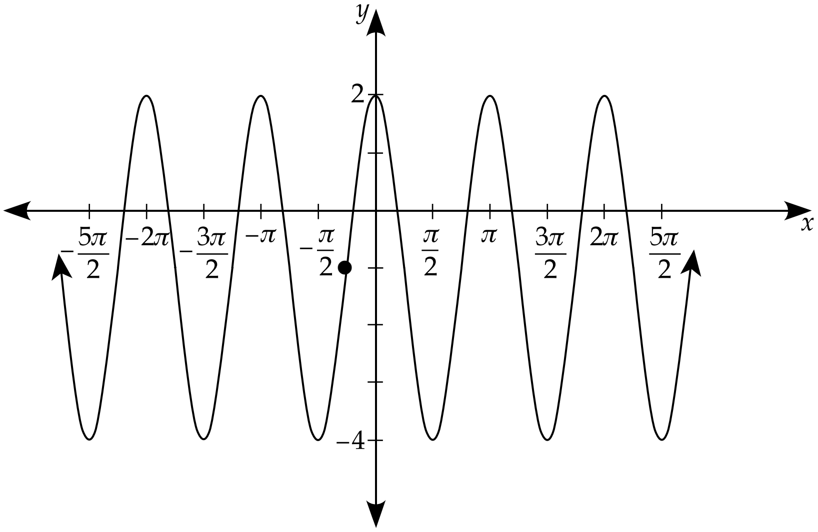

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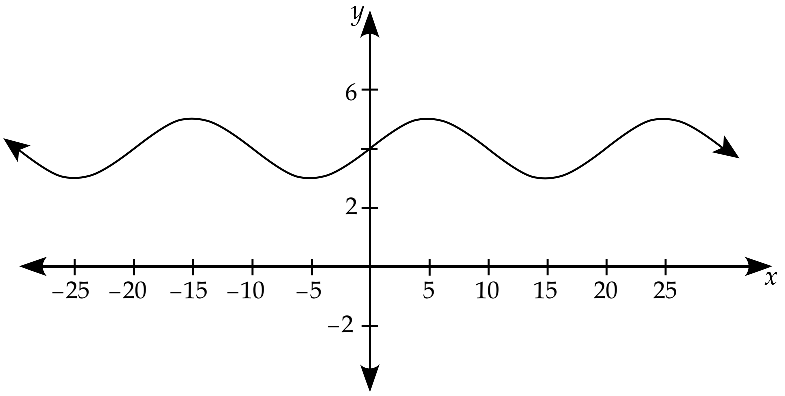
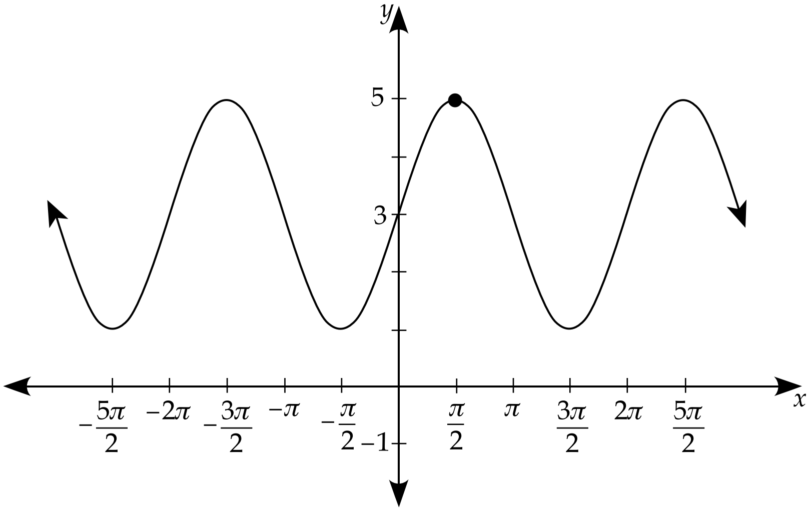


continued

Learning Activity 5.7 (continued)

13. Find an equation of each of the following graphs as cosine functions.

a)



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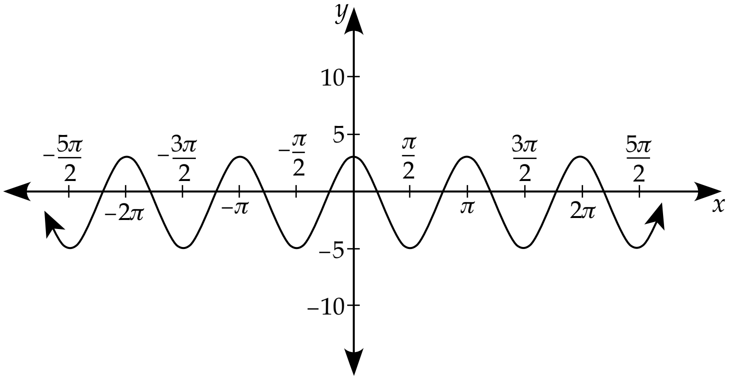
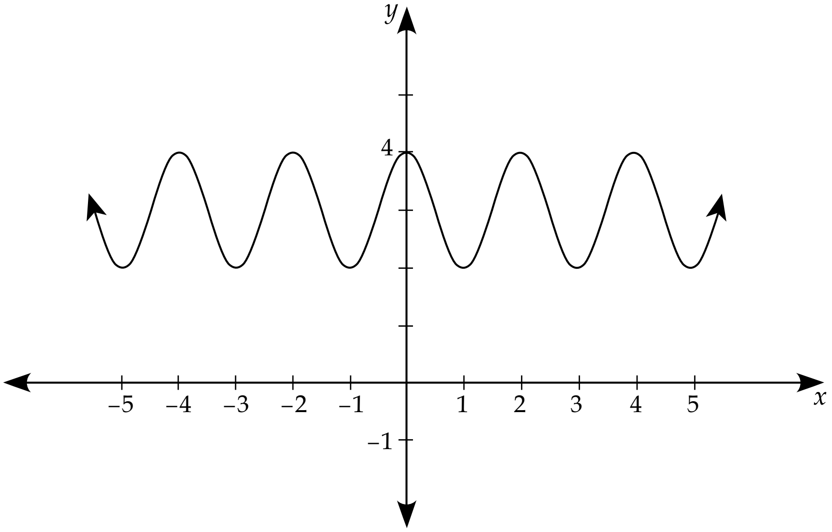
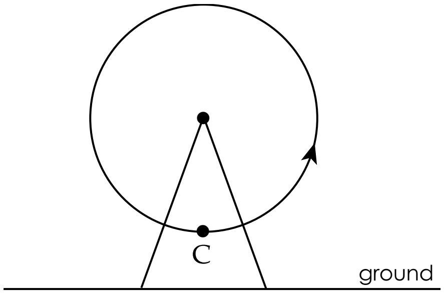


c)



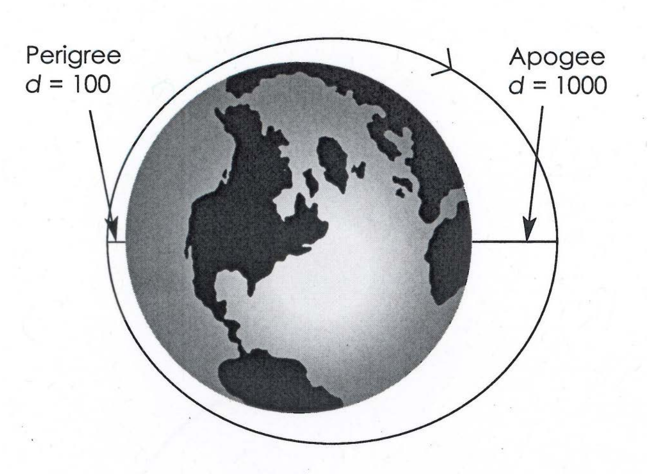
continued

14. A Ferris wheel is 1 metre from the ground. The distance of a chair on the Ferris wheel from the ground varies sinusoidally with time. Suppose the diameter of a Ferris wheel is 10 m and the top point the chair reaches is 11 m. The wheel makes one counter-clockwise revolution every eight seconds. The Ferris wheel starts with chair *C* at the lowest position.



   a) Sketch the function *C*(*t*) and write a formula *C*(*t*), in terms of cosine, which states the height of chair *C* above the ground *t* seconds after the Ferris wheel started.
   b) What is the lowest that any chair is above the ground?
   c) State the height of chair *C* above the ground when $t = 3$, 4, and 10 seconds.

15. A spacecraft is in an elliptical orbit around Earth, as shown in the diagram. At time $t = 0$ hours, it is at its highest point $d = 1000$ kilometres above Earth's surface. Fifty minutes later, it is at its lowest point, $d = 100$ kilometres above the surface.



   a) Assuming that *d* varies sinusoidally with time, draw the sketch of the sinusoidal function.
   b) Write an equation expressing *d* in terms of *t*.
   c) In order to transmit information back to Earth, the spacecraft must be within 700 kilometres of the surface. For how many consecutive minutes will the spacecraft be able to transmit?

Lesson Summary

In this lesson, you learned about transformations of the sine and cosine functions. Using these transformations, you were able to determine the function of a sinusoidal curve when given its graph. You were also able to solve an application question involving a periodic function.

This is the last lesson in Module 5. Make sure you complete and send in the three assignments for this module.

Assignment 5.3

Trigonometric Functions and Their Graphs

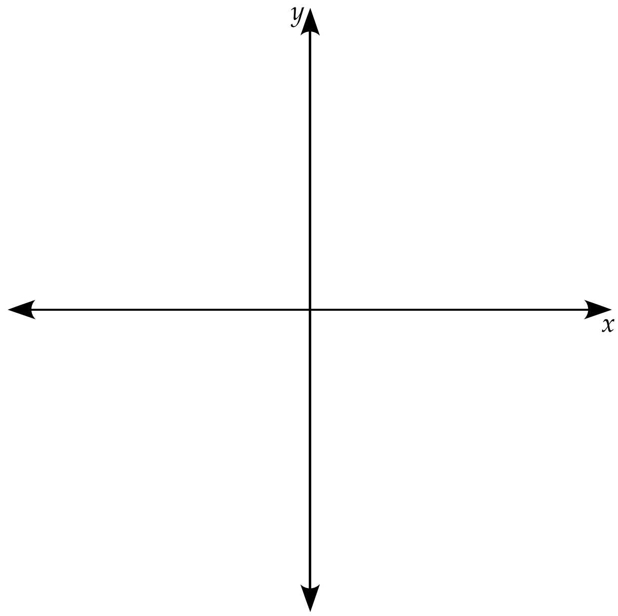
Total: 26 marks

You need to show all your steps as you work out your solutions, and make sure your answers are clear (include units, where appropriate). Check the Introduction for the Marking Guidelines that will be used for all assignments and examinations.

1. Fill in the property chart for each function. Sketch the following curves.

   a) $y = -2\sin\left(\frac{\pi}{2}x\right) + 4$ *(6 marks)*

	$y = -2\sin\left(\frac{\pi}{2}x\right) + 4$
Period	
Amplitude	
Phase Shift (Horizontal)	
Domain	
Range	
y-intercept	

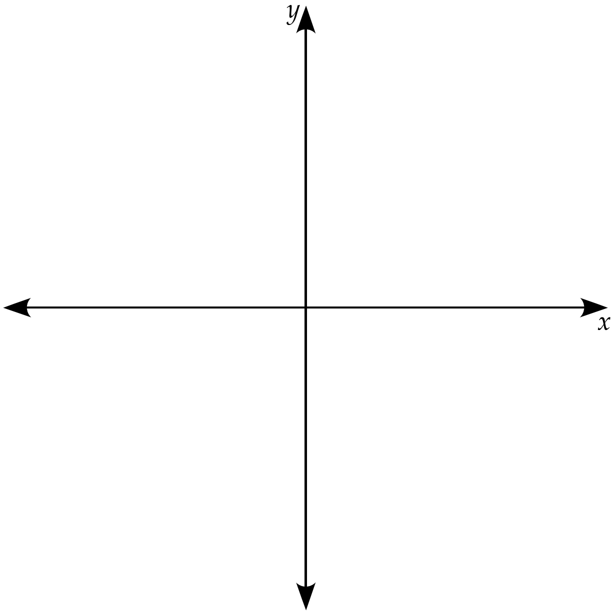


continued

Assignment 5.3: Trigonometric Functions and Their Graphs (continued)

b) $y = \cos\left(\frac{1}{2}(x - \pi)\right) - 3$ *(6 marks)*

	$y = \cos\left(\frac{1}{2}(x - \pi)\right) - 3$
Period	
Amplitude	
Phase Shift (Horizontal)	
Domain	
Range	
y-intercept	



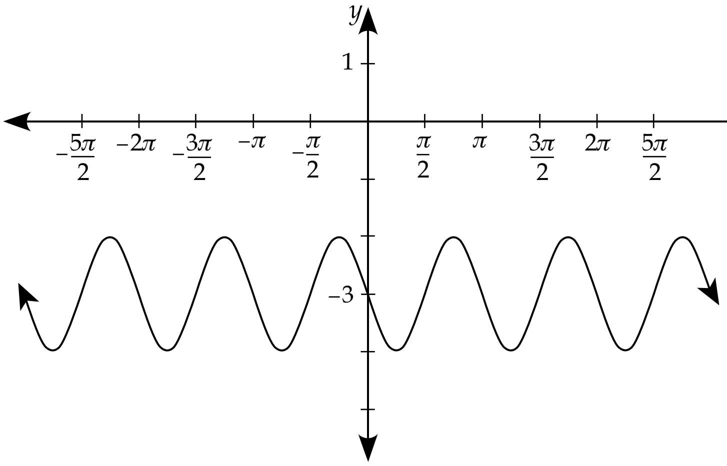
continued

2. Find the amplitude, period, horizontal shift, and vertical shift for the following functions. *(2 × 2 marks each = 4 marks)*

   a) $y = -5\sin\left(\frac{\pi}{4}\left(x + \frac{3\pi}{2}\right)\right) - 2$

   b) $y = 2\cos\left(-2\left(x - \frac{\pi}{6}\right)\right) + 3$

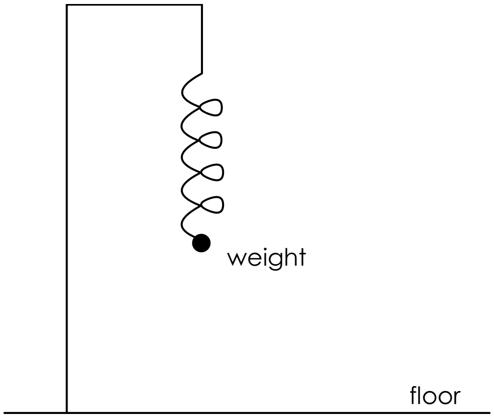
3. Find an equation of the following graph as a sine function, and then as a cosine function. *(4 marks)*



continued

Assignment 5.3: Trigonometric Functions and Their Graphs (continued)

4. A weight attached to the end of a spring is bouncing up and down. As it bounces, the weight's distance from the floor varies sinusoidally. You begin measuring the weight's distance from the floor when it is at its lowest point, 20 cm above the floor. One second after you start timing, the weight reaches its highest point, 50 cm above the floor. You could model the height as a function of time.

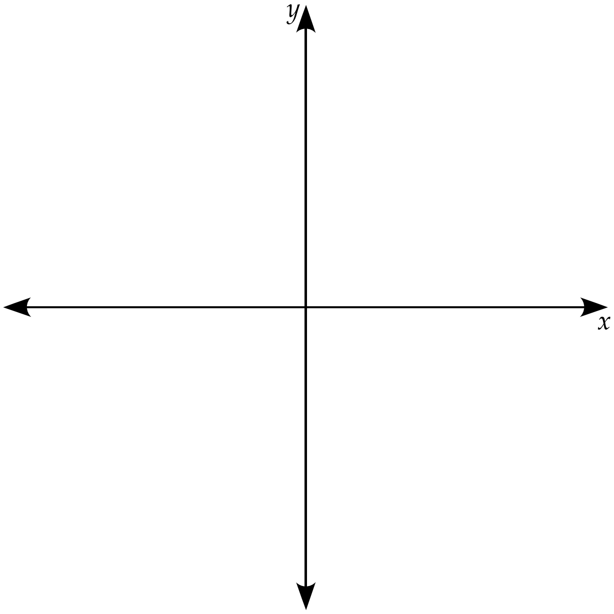


   a) What is the amplitude of this function? *(1 mark)*

   b) What is the period of this function? *(1 mark)*

continued

c) Sketch at least one period of the graph of this function. *(1 mark)*



d) Write a possible formula for the distance of the weight above the ground t seconds after you began your measurements. *(2 marks)*

e) What will be the weight's distance above the floor 1.5 seconds after you began your measurements? *(1 mark)*

Notes

MODULE 5 SUMMARY

Congratulations, you have finished the first module in the second half of this course!

In this module, you learned how the six trigonometric ratios—sine, cosine, tangent, cosecant, secant, and cotangent—were related to the unit circle. You then studied these ratios as functions. With the sine and cosine functions, you performed transformations and analyzed the properties of each function.

In the next module, you will continue to study the topic of trigonometry with a focus on trigonometric identities built from the unit circle.

Submitting Your Assignments

It is now time for you to submit Assignments 5.1 to 5.3 to the Distance Learning Unit so that you can receive some feedback on how you are doing in this course. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 5 assignments and organize your material in the following order:

☐ Module 5 Cover Sheet (found at the end of the course Introduction)
☐ Assignment 5.1: Degrees, Radians, and the Unit Circle
☐ Assignment 5.2: The Six Trigonometric Ratios
☐ Assignment 5.3: Trigonometric Functions and Their Graphs

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Notes

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Module 5
Trigonometric Functions and the Unit Circle

Learning Activity Answer Keys

MODULE 5:
TRIGONOMETRIC FUNCTIONS AND THE UNIT CIRCLE

Learning Activity 5.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. What is the domain of the function $f(x) = \sqrt{x-3}$?
2. Solve for x: $x! = 3(x - 1)!$
3. State the non-permissible values of the function $f(x) = \dfrac{6x}{x^2 - 1}$.
4. Simplify: $\dfrac{(3x^3)(6x^2)}{x^4}$
5. How much will 12 litres of milk cost if a 4-litre jug costs $5.29?
6. Simplify: $(2x^3 - 4x^2 + 6x - 2) - (-3x^3 + 4x^2 - 3x - 1)$
7. Factor: $10x^2 - 7x - 12$
8. Evaluate: $\dfrac{7}{81} - \dfrac{2}{9}$

Answers:

1. $x \geq -3$
2. $x = 3$ ($x(x - 1)! = 3(x - 1)!$; $x = 3$)
3. $x \neq -1$ and $x \neq 1$ ($x^2 - 1 = (x - 1)(x + 1)$ and the denominator cannot be zero)
4. $18x$ $\left(\dfrac{18x^5}{x^4} = 18x\right)$
5. $15.87 ($3 \times 5.29 = 15.87$)
6. $5x^3 - 8x^2 + 9x - 1$ ($2x^3 - 4x^2 + 6x - 2 + 3x^3 - 4x^2 + 3x + 1 = 5x^3 - 8x^2 + 9x - 1$)
7. $(5x + 4)(2x - 3)$
8. $-\dfrac{11}{81}$ $\left(\dfrac{7}{81} - \dfrac{18}{81} = -\dfrac{11}{81}\right)$

Part B: Radians and Degrees

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Convert each of the following angles to radians. Write the exact answer.

   a) 25°

   Answer:

   $$25\left(\frac{\pi}{180}\right) = \frac{5\pi}{36}$$

   b) −125°

   Answer:

   $$-125\left(\frac{\pi}{180}\right) = -\frac{25\pi}{36}$$

   c) 460°

   Answer:

   $$460\left(\frac{\pi}{180}\right) = \frac{23\pi}{9}$$

   d) −51°

   Answer:

   $$-51\left(\frac{\pi}{180}\right) = -\frac{17\pi}{60}$$

2. Convert each of the following angles measured in radians to degrees. Round to the nearest tenth of a degree in parts (c) and (d).

   a) $-\frac{7\pi}{6}$

   Answer:

   $$\frac{-7\pi}{6}\left(\frac{180}{\pi}\right) = -210°$$

   b) $\frac{11\pi}{12}$

   Answer:

   $$\frac{11\pi}{12}\left(\frac{180}{\pi}\right) = 165°$$

   c) 2.634

   Answer:

   $$2.634\left(\frac{180}{\pi}\right) = 150.9°$$

   d) −0.9825

   Answer:

   $$-0.9825\left(\frac{180}{\pi}\right) = -56.3°$$

3. When the angles of a triangle are measured in radians, what is the sum of the angles?

   Answer:

   All 3 interior angles of a triangle sum to 180°, which is equivalent to π radians. Therefore, the sum of the angles in a triangle, when the angles are measured in radians, is π radians.

4. Determine the complement and supplement of $\frac{5\pi}{12}$.

(Recall: Two angles are complementary if the sum of their measures is 90°. Two angles are supplementary if the sum of their measures is 180°. Note that if a question is given in radians, the answer should be in radians.)

Answer:

The complement of $\frac{5\pi}{12}$:

$\frac{5\pi}{12}$ and its complement need to sum to 90°, which is $\frac{\pi}{2}$ radians.

Let the complement of $\frac{5\pi}{12}$ be represented by x.

$$
\begin{aligned}
x + \frac{5\pi}{12} &= \frac{\pi}{2} \\
x &= \frac{\pi}{2} - \frac{5\pi}{12} \\
x &= \frac{6\pi}{12} - \frac{5\pi}{12} = \frac{\pi}{12}
\end{aligned}
$$

The complement of $\frac{5\pi}{12}$ is $\frac{\pi}{12}$.

The supplement of $\frac{5\pi}{12}$:

$\frac{5\pi}{12}$ and its supplement need to sum to 180°, which is π radians.

Let the supplement of $\frac{5\pi}{12}$ be represented by y.

$$
\begin{aligned}
y + \frac{5\pi}{12} &= \pi \\
y &= \pi - \frac{5\pi}{12} \\
y &= \frac{12\pi}{12} - \frac{5\pi}{12} = \frac{7\pi}{12}
\end{aligned}
$$

The supplement of $\frac{5\pi}{12}$ is $\frac{7\pi}{12}$.

5. Express the supplement of 130° in radian measure. Write an exact answer.

   Answer:

   $$180° - 130° = 50° = 50\left(\frac{\pi}{180}\right) = \frac{5\pi}{18}$$

6. An isosceles triangle has a base angle with measure $\frac{2\pi}{7}$. What are the measures of the other two angles? Express the values as exact answers in radians.

   Answer:

   Base angles in an isosceles triangle are equal. Therefore, the other base angle equals $\frac{2\pi}{7}$.

   Also, the sum of the angles in a triangle is π radians (equal to 180°).

   Therefore, the third angle $= \pi - 2\left(\frac{2\pi}{7}\right) = \pi - \frac{4\pi}{7} = \frac{7\pi}{7} - \frac{4\pi}{7} = \frac{3\pi}{7}$.

7. If a wheel having a circumference of 30 cm rolls 5 cm, how many radians has it turned? How many degrees has it turned? Write exact answers.

   Answer:

   Since $C = 2\pi r$

   $$30 = 2\pi r$$

   $$\frac{30}{2\pi} = r$$

   $$\frac{15}{\pi} = r$$

   Since $s = \theta r$, it follows that:

   $$5 = \theta\left(\frac{15}{\pi}\right)$$

   $$5\left(\frac{\pi}{15}\right) = \theta$$

   $$\frac{\pi}{3} = \theta$$

   Thus, $\theta = \frac{\pi}{3}$ or 60°.

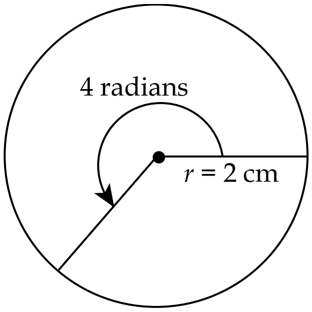
   Note: Alternatively, you could determine the number of degrees by the ratio $\frac{5 \text{ cm}}{30 \text{ cm}}$ to determine the fraction of 360° that the wheel has turned.

   $$\frac{5}{30} = \frac{1}{6} \text{ and } \frac{1}{6} \times 360° = 60°$$

8. Draw a circle of any size. Using this circle, draw an angle of approximately 4 radians.

   Answer:

   Answers may vary. However, the angle of 4 radians needs to correspond to an arc length of approximately 4 times the radius of the circle you drew. For example,



9. Draw a circle with a radius of 6 cm. Draw an arc length of 18 cm. What is the measure of the central angle subtended by this arc?

   Answer:

   The measure of this angle is $\frac{18}{6} = 3$ radians. This is because the arc length is three times the radius. If the arc length was the same length as the radius of the circle, the measure of the angle would be 1 radian.

   You can also use the formula $s = \theta r$ to determine θ in radians.

$$18 = \theta\,(6)$$

$$\frac{18}{6} = \theta$$

$$\theta = 3 \text{ radians}$$

Learning Activity 5.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Rationalize the denominator: $\frac{1}{\sqrt{2}}$
2. Determine if the following set of points represents a one-to-one function: $(1, 5), (-6, 3), (-4, 6), (7, -5)$
3. How many ways can you arrange 4 pictures on a shelf?
4. Estimate the taxes, 13%, on a $2450 item.
5. If $f(x) = 2x^3 - 7$, evaluate $f(x)$ at $x = 4$.
6. Simplify: $| -2 + 5(-4) |$.
7. Factor: $4x^2 - 64$
8. If you have cycled through 18 out of the 45 km of a bicycle route, what fraction of the route have you completed?

Answers:

1. $\frac{\sqrt{2}}{2}\left(\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}\right)$
2. Yes (for each x-value, there is a distinct y-value)
3. 24 ($4! = 24$)
4. $317 (10% of 2450 is 245, 1% of 2450 is 24.5, 13% of 2450 is approximately $245 + 24 + 24 + 24 = 245 + 72 = 317$)
5. $f(4) = 121$ ($f(4) = 2(4)^3 - 7 = 2(64) - 7 = 128 - 7 = 121$)
6. 22 ($| -2 - 20 | = | -22 | = 22$)
7. $4(x + 4)(x - 4)(4(x^2 - 16))$
8. $\frac{18}{45} = \frac{2}{5}$ (divide numerator and denominator by 9)

Part B: Coterminal Angles

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Determine all of the angles that are coterminal with the given angle over the domain [−1080°, 360°].

   a) 258°

   Answer:

   Note: Adding 360° to 258° will give you an angle that is over 360°. Therefore, this angle is not in the required domain.

   $258° - 360° = -102°$

   $-102° - 360° = -462°$

   $-462° - 360° = -822°$

   The angles that are coterminal with 258° over the domain [−1080°, 360°] are −102°, −462°, and −822°.

   b) 613°

   Answer:

   $613° - 360° = 253°$

   $253° - 360° = -107°$

   $-107° - 360° = -467°$

   $-467° - 360° = -827°$

   The angles that are coterminal with 613° over the domain [−1080°, 360°] are 253°, −107°, −467°, and −827°.

   c) −142°

   Answer:

   $-142° + 360° = 218°$

   $-142° - 360° = -502°$

   $-502° - 360° = -862°$

   The angles that are coterminal with −142° over the domain [−1080°, 360°] are 218°, −502°, and −862°.

d) $-631°$

Answer:

$-631° + 360° = -271°$

$-271° + 360° = 89°$

$-631° - 360° = -991°$

The angles that are coterminal with $-631°$ over the domain $[-1080°, 360°]$ are $89°$, $-271°$, and $-991°$.

2. Determine all of the angles that are coterminal with the given angle over the domain $[-6\pi, 4\pi]$.

   a) $-\dfrac{9\pi}{4}$

   Answer:

   The domain can be written with a denominator of 4 as. $\left[-\dfrac{24\pi}{4}, \dfrac{16\pi}{4}\right]$.

$$-\frac{9\pi}{4} + 2\pi = -\frac{9\pi}{4} + \frac{8\pi}{4} = -\frac{\pi}{4}$$

$$-\frac{\pi}{4} + \frac{8\pi}{4} = \frac{7\pi}{4}$$

$$\frac{7\pi}{4} + \frac{8\pi}{4} = \frac{15\pi}{4}$$

$$-\frac{9\pi}{4} - \frac{8\pi}{4} = -\frac{17\pi}{4}$$

   The angles that are coterminal with $-\dfrac{9\pi}{4}$ over the domain $[-6\pi, 4\pi]$ are $\dfrac{15\pi}{4}, \dfrac{7\pi}{4}, -\dfrac{\pi}{4}$, and $-\dfrac{17\pi}{4}$.

b) $-\frac{13\pi}{6}$

Answer:

The domain can be written with a denominator of 6 as $\left[-\frac{36\pi}{6}, \frac{24\pi}{6}\right]$.

$$-\frac{13\pi}{6} + 2\pi = -\frac{13\pi}{6} + \frac{12\pi}{6} = -\frac{\pi}{6}$$

$$-\frac{\pi}{6} + \frac{12\pi}{6} = \frac{11\pi}{6}$$

$$\frac{11\pi}{6} + \frac{12\pi}{6} = \frac{23\pi}{6}$$

$$-\frac{13\pi}{6} - \frac{12\pi}{6} = -\frac{25\pi}{6}$$

The angles that are coterminal to $-\frac{13\pi}{6}$ over the domain $[-6\pi, 4\pi]$ are $\frac{23\pi}{6}$, $\frac{11\pi}{6}$, $-\frac{\pi}{6}$, and $-\frac{25\pi}{6}$.

c) $-\frac{\pi}{14}$

Answer:

The domain can also be written as $\left[-\frac{84\pi}{14}, \frac{56\pi}{14}\right]$.

$$-\frac{\pi}{14} + 2\pi = -\frac{\pi}{14} + \frac{28\pi}{14} = \frac{27\pi}{14}$$

$$\frac{27\pi}{14} + \frac{28\pi}{14} = \frac{55\pi}{14}$$

$$-\frac{\pi}{14} - \frac{28\pi}{14} = -\frac{29\pi}{14}$$

$$-\frac{29\pi}{14} - \frac{28\pi}{14} = -\frac{57\pi}{14}$$

The angles that are coterminal with $-\frac{\pi}{14}$ over the domain $[-6\pi, 4\pi]$ are $\frac{55\pi}{14}$, $\frac{27\pi}{14}$, $-\frac{29\pi}{41}$, and $-\frac{57\pi}{14}$.

d) $\dfrac{2\pi}{7}$

Answer:

The domain can also be written as $\left[-\dfrac{42\pi}{7}, \dfrac{28\pi}{7}\right]$.

$$\frac{2\pi}{7} + 2\pi = \frac{2\pi}{7} + \frac{14\pi}{7} = \frac{16\pi}{7}$$

$$\frac{2\pi}{7} - \frac{14\pi}{7} = -\frac{12\pi}{7}$$

$$-\frac{12\pi}{7} - \frac{14\pi}{7} = -\frac{26\pi}{7}$$

$$-\frac{26\pi}{7} - \frac{14\pi}{7} = -\frac{40\pi}{7}$$

The angles that are coterminal with $\dfrac{2\pi}{7}$ over the domain $[-6\pi, 4\pi]$ are $\dfrac{16\pi}{7}, -\dfrac{12\pi}{7}, -\dfrac{26\pi}{7}$, and $-\dfrac{40\pi}{7}$.

3. Write all the angles that are coterminal with the following angles.

a) $-\dfrac{\pi}{8}$

Answer:

Use general form.

$-\dfrac{\pi}{8} + 2\pi n, n \in I$

b) 416°

Answer:

$416° + (360°)n, n \in I$

c) 17°

Answer:

$-17° + (360°)n, n \in I$

d) $\frac{3}{7}$

Answer:

Note: As there is no degree sign after this angle, this angle is measured in radians.

$$\frac{3}{7} + 2\pi n,\ n \in I$$

4. Determine whether the following sets of angles are coterminal.

a) −365° and 715°

Answer:

There are two methods to determine whether or not two angles are coterminal. First, you could keep adding 360° to −365° to see if you eventually arrive at an answer of 715°. Or, you could determine the formula for calculating all angles that are coterminal with −365°. Using this formula, you can substitute in 715° as your coterminal angle and solve for n. If n is a natural number, you know these angles are coterminal. If n is not a natural number, these angles are not coterminal.

Angles that are coterminal with −365°:

$$\theta = -365^\circ \pm (360^\circ)n,\ n \in I$$

Substitute in 715°:

$$715^\circ = -365^\circ + 360^\circ n$$

Solve for n:

$$715^\circ + 365^\circ = 360^\circ n$$

$$\frac{1080^\circ}{360^\circ} = 3 = n$$

Therefore, these angles are coterminal, as n is an integer.

b) $\frac{\pi}{4}$ and $-\frac{21\pi}{4}$

Answer:

Angles that are coterminal with $\frac{\pi}{4}$:

$$\theta = \frac{\pi}{4} + 2\pi n,\ n \in I$$

Substitute in $-\frac{21\pi}{4}$:

$$-\frac{21\pi}{4} = \frac{\pi}{4} + 2\pi n$$

Solve for n:

$$-\frac{21\pi}{4} - \frac{\pi}{4} = 2\pi n$$

$$-\frac{22\pi}{4} = 2\pi n$$

$$-\frac{\frac{22\pi}{4}}{2\pi} = -\frac{22\pi}{4} \cdot \frac{1}{2\pi} = -\frac{11}{4} = n$$

Therefore, these angles are *not* coterminal, as n is not an integer.

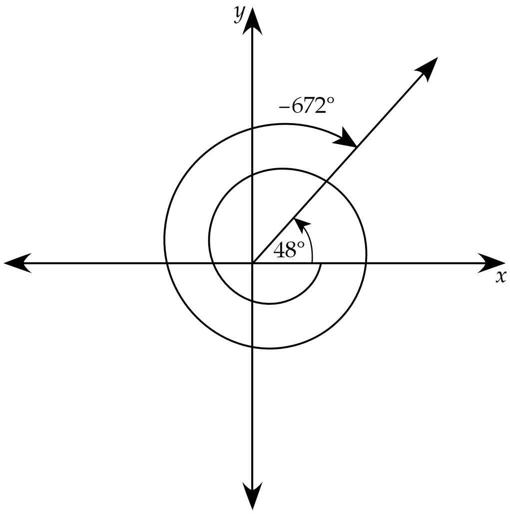
5. Sketch the following angles.

a) −672°

Answer:

First, determine a coterminal angle to −672° that is between 0° and 360°.

−672° + 360°
−312° + 360° = 48°



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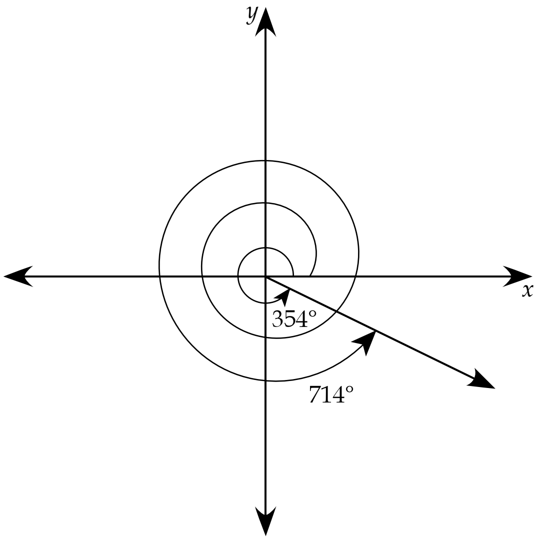
b) 714°

Answer:

Determine a coterminal angle to 714° that is between 0° and 360°.

$714° - 360° = 354°$



c) 9π

Answer:

Determine a coterminal angle to 9π that is between 0 and 2π.

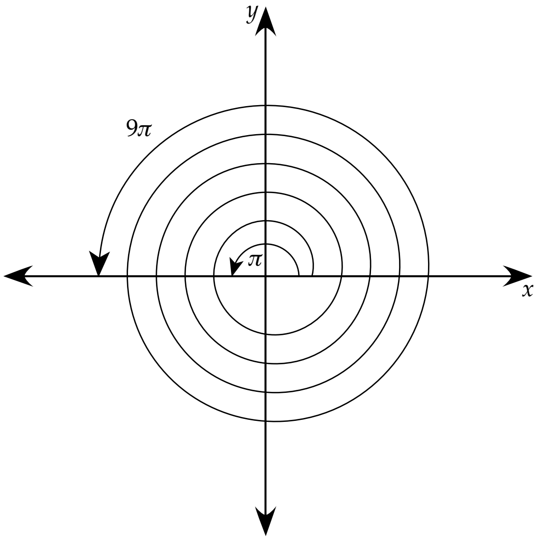
Subtract 2π repeatedly:

$9\pi - 2\pi = 7\pi$

$7\pi - 2\pi = 5\pi$

$5\pi - 2\pi = 3\pi$

$3\pi - 2\pi = \pi$

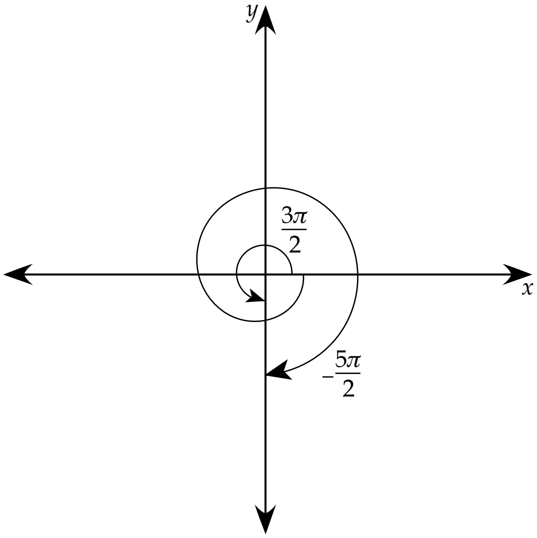


d) $-\frac{5\pi}{2}$

Answer:

Determine a coterminal angle to $-\frac{5\pi}{2}$ that is between 0 and 2π.

$$-\frac{5\pi}{2} + 2\pi = -\frac{5\pi}{2} + \frac{4\pi}{2} = -\frac{\pi}{2} + \frac{4\pi}{2} = \frac{3\pi}{2}$$



6. Your younger brother is playing in a revolving door at a department store. Before you can stop him, he has rotated the door through an angle of 2700°. If he was originally inside the store, where is he now? Explain.

   Answer:

   After every complete rotation, or 360°, your younger brother will end up inside the store.

   2700° − 360° = 2340°

   2340° − 360° = 1980°

   1980° − 360° = 1620°

   1620° − 360° = 1260°

   1260° − 360° = 900°

   900° − 360° = 540°

   540° − 360° = 180°

   7(360°) + 180° = 2700°

   After 7 complete rotations, your younger brother ends up inside the store. He then turns an additional 180° and ends up outside the store.

Learning Activity 5.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. State the non-permissible values of the function $f(x) = \frac{x-5}{x^2}$.
2. Find the positive coterminal angle for 502° in the interval $0° \le \theta \le 360°$.
3. What is the domain of the function $f(x) = \frac{1}{\sqrt{x}}$?
4. Rationalize the denominator: $\frac{1}{\sqrt{3}}$
5. Simplify: $\sqrt{32x^4y^6z^3}$
6. Multiply: $\left(\sqrt{3} - 2\sqrt{6}\right)\left(\sqrt{3} + 2\sqrt{6}\right)$
7. List all the factors of 81.
8. Evaluate: $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

Answers:

1. $x \neq 0$
2. 142° ($502° - 360° = 142°$)
3. $\{x \mid x \in \Re, x > 0\}$ (x cannot be zero or negative)
4. $\frac{\sqrt{3}}{3}$ $\left(\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}\right)$
5. $4x^2y^3z\sqrt{2z}$ $\left(\begin{aligned}\sqrt{32x^4y^6z^3} &= \sqrt{16}\sqrt{2}\sqrt{x^2}\sqrt{x^2}\sqrt{y^2}\sqrt{y^2}\sqrt{y^2}\sqrt{z^2}z \\ &= 4xxyyyz\sqrt{2}\sqrt{z} = 4x^2y^3z\sqrt{2z}\end{aligned}\right)$
6. -21 $\left(\sqrt{3}\sqrt{3} + 2\sqrt{18} - 2\sqrt{18} - 4\sqrt{6}\sqrt{6} = 3 - 4(6) = 3 - 24 = -21\right)$
7. 1, 3, 9, 27, 81
8. $\frac{13}{12} = 1\frac{1}{12}$ $\left(\frac{6}{12} + \frac{4}{12} + \frac{3}{12} = \frac{13}{12}\right)$

Part B: The Unit Circle

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Use the equation of the unit circle to show that the following coordinates are points on the unit circle.

   a) $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

   Answer:

   The equation of the unit circle is $x^2 + y^2 = 1$.

   $$\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = \frac{4}{4} = 1$$

   Therefore, the point is on the unit circle. You may also note that it is in Quadrant II, since x is negative and y is positive.

   b) $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

   Answer:

   The equation of the unit circle is $x^2 + y^2 = 1$.

   $$\left(\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right) = \frac{2}{4} + \frac{2}{4} = \frac{4}{4} = 1$$

   Therefore, the point is on the unit circle. You may also note that it is in Quadrant IV, since x is positive and y is negative.

2. Are the following points on the unit circle? Explain.

   a) $\left(\frac{1}{3}, \frac{\sqrt{8}}{3}\right)$

   Answer:

   If a point is located on the unit circle, then the equation of the unit circle, $x^2 + y^2 = 1$, should hold for each point.

   $$\left(\frac{1}{3}\right)^2 + \left(\frac{\sqrt{8}}{3}\right)^2 = \frac{1}{9} + \frac{8}{9} = \frac{9}{9} = 1$$

   Therefore, this point is on the unit circle.

   b) $\left(\frac{1}{4}, \frac{\sqrt{5}}{2}\right)$

   Answer:

   $$\left(\frac{1}{4}\right)^2 + \left(\frac{\sqrt{5}}{2}\right)^2 = \frac{1}{16} + \frac{5}{4} = \frac{1}{16} + \frac{20}{16} = \frac{21}{16} \neq 1$$

   Therefore, this point is not on the unit circle.

   c) $\left(\frac{5}{6}, \frac{\sqrt{8}}{6}\right)$

   Answer:

   $$\left(\frac{5}{6}\right)^2 + \left(\frac{\sqrt{8}}{6}\right)^2 = \frac{25}{36} + \frac{8}{36} = \frac{33}{36} \neq 1$$

   Therefore, this point is not on the unit circle.

   d) $\left(\frac{\sqrt{23}}{5}, \frac{\sqrt{2}}{5}\right)$

   Answer:

   $$\left(\frac{\sqrt{23}}{5}\right)^2 + \left(\frac{\sqrt{2}}{5}\right)^2 = \frac{23}{25} + \frac{2}{25} = \frac{25}{25} = 1$$

   Therefore, this point is on the unit circle.

3. Determine the missing coordinates of each of the following points that are located on the unit circle.

a) $\left(x, \dfrac{\sqrt{6}}{4}\right)$

Answer:

If each point is on the unit circle, then each point needs to satisfy the unit circle equation, $x^2 + y^2 = 1$.

$$x^2 + \left(\frac{\sqrt{6}}{4}\right)^2 = 1$$

$$x^2 = 1 - \frac{6}{16}$$

$$x^2 = \frac{16}{16} - \frac{6}{16}$$

$$x^2 = \frac{10}{16}$$

$$x = \pm\frac{\sqrt{10}}{4}$$

Notice that x can be positive or negative when $y = \dfrac{\sqrt{6}}{4}$, since the point could be in Quadrant I or Quadrant II.

b) $\left(x, \dfrac{\sqrt{2}}{\sqrt{3}}\right)$

Answer:

$$x^2 + \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2 = 1$$

$$x^2 = 1 - \frac{2}{3}$$

$$x^2 = \frac{3}{3} - \frac{2}{3}$$

$$x^2 = \frac{1}{3}$$

$$x = \pm\frac{1}{\sqrt{3}}$$

c) $\left(\frac{\sqrt{6}}{\sqrt{7}}, y\right)$

Answer:

$$\left(\frac{\sqrt{6}}{\sqrt{7}}\right)^2 + y^2 = 1$$

$$y^2 = 1 - \frac{6}{7}$$

$$y^2 = \frac{7}{7} - \frac{6}{7}$$

$$y^2 = \frac{1}{7}$$

$$y = \pm\frac{1}{\sqrt{7}}$$

d) $\left(\frac{\sqrt{15}}{4}, y\right)$

Answer:

$$\left(\frac{\sqrt{15}}{4}\right)^2 + y^2 = 1$$

$$y^2 = 1 - \frac{15}{16}$$

$$y^2 = \frac{16}{16} - \frac{15}{16}$$

$$y^2 = \frac{1}{16}$$

$$y = \pm\frac{1}{4}$$

Notice that y can be positive or negative for these x-values, since the points could be in Quadrant I or Quadrant IV.

4. Determine the exact coordinates corresponding to the following angles of rotation on the unit circle. Try to do this from memory using the lengths of sides of the 45–45–90 or the 30–60–90 triangles.

a) $-\frac{\pi}{4}$

Answer:

This angle rotates clockwise, terminates in Quadrant IV, and corresponds to the point $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

b) $\frac{25\pi}{6}$

Answer:

This angle is coterminal with $\frac{\pi}{6}\left(\frac{25\pi}{6} - \frac{24\pi}{6}\right)$ so it is in Quadrant I.

It corresponds to the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

c) $\dfrac{13\pi}{2}$

Answer:

This angle is coterminal with $\dfrac{\pi}{2}\left(\dfrac{13\pi}{2} - \dfrac{12\pi}{2}\right)$ so it is on the positive y-axis. It corresponds to the point (0, 1).

d) $-\dfrac{11\pi}{3}$

Answer:

This angle is coterminal with $\dfrac{\pi}{3}\left(-\dfrac{11\pi}{3} + \dfrac{12\pi}{3}\right)$ so it is in Quadrant I.

It corresponds to the point, $\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$.

5. Given the following points on the unit circle, find *two* angles of rotation that correspond to each point.

a) (0, 1)

Answer:

All angles that correspond to this point can be expressed as $\dfrac{\pi}{2} + 2\pi n,\ n \in I$.

For example, . . ., $-\dfrac{3\pi}{2}, \dfrac{\pi}{2}, \dfrac{5\pi}{2}$, . . .

b) $\left(-\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$

Answer:

All angles that correspond to this point can be expressed as $\dfrac{5\pi}{6} + 2\pi n,\ n \in I$.

For example, . . ., $-\dfrac{7\pi}{6}, \dfrac{5\pi}{6}, \dfrac{17\pi}{6}$, . . .

c) $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

Answer:

All angles that correspond to this point can be expressed as $\frac{4\pi}{3} + 2\pi n,\ n \in I.$

For example, . . ., $-\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{10\pi}{3}$, . . .

d) $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

Answer:

All angles that correspond to this point can be expressed as $\frac{\pi}{4} + 2\pi n,\ n \in I.$

For example, . . ., $-\frac{7\pi}{4}, \frac{\pi}{4}, \frac{9\pi}{4}$, . . .

6. Determine the equation of a circle centred at the origin with a radius of 7 cm.

   Answer:

   The equation of a circle centred at the origin with radius r is $x^2 + y^2 = r^2$.

   The equation of a circle centred at the origin with radius 7 is $x^2 + y^2 = 7^2$ or $x^2 + y^2 = 49$.

7. If you haven't already, fill out one of the empty unit circles that are located at the end of this lesson. Some students find the unit circle difficult to memorize. However, if you look for patterns, memorizing the unit circle can be as easy as memorizing the values in only one quadrant.

   a) What patterns do you notice in the denominators of the coordinates on the unit circle?

   Answer:

   The denominators of the coordinates on the unit circle are either 1 or 2. The denominators of the coordinates on the x- and y-axes are 1 while every other coordinate has a denominator of 2.

b) What patterns do you notice in the numerators of the coordinates on the unit circle?

Answer:

The numerators of the coordinates on the unit circle are always one of four values: $\sqrt{0}$, $\sqrt{1}$, $\sqrt{2}$, and $\sqrt{3}$.

c) Where are the x-coordinates positive? Negative?

Answer:

The x-coordinates are positive in Quadrants I and IV, since the positive x-axis is the boundary between them.

The x-coordinates are negative in Quadrants II and III, since the negative x-axis is the boundary between them.

d) Where are the y-coordinates positive? Negative?

Answer:

The y-coordinates are positive in Quadrants I and II, since the positive y-axis is the boundary between them.

The y-coordinates are negative in Quadrants III and IV, since the negative y-axis is the boundary between them.

e) Colour each terminal arm according to the largest angle for which it is a multiple, using the following key. (For example, $\frac{\pi}{3} = \frac{2\pi}{6}$ is a multiple of $\frac{\pi}{3}$ and $\frac{\pi}{6}$. Therefore, the terminal arm is coloured blue because $\frac{\pi}{3}$ is the largest angle for which it is a multiple.)

Multiples of $\frac{\pi}{3}$ $\longrightarrow$ blue

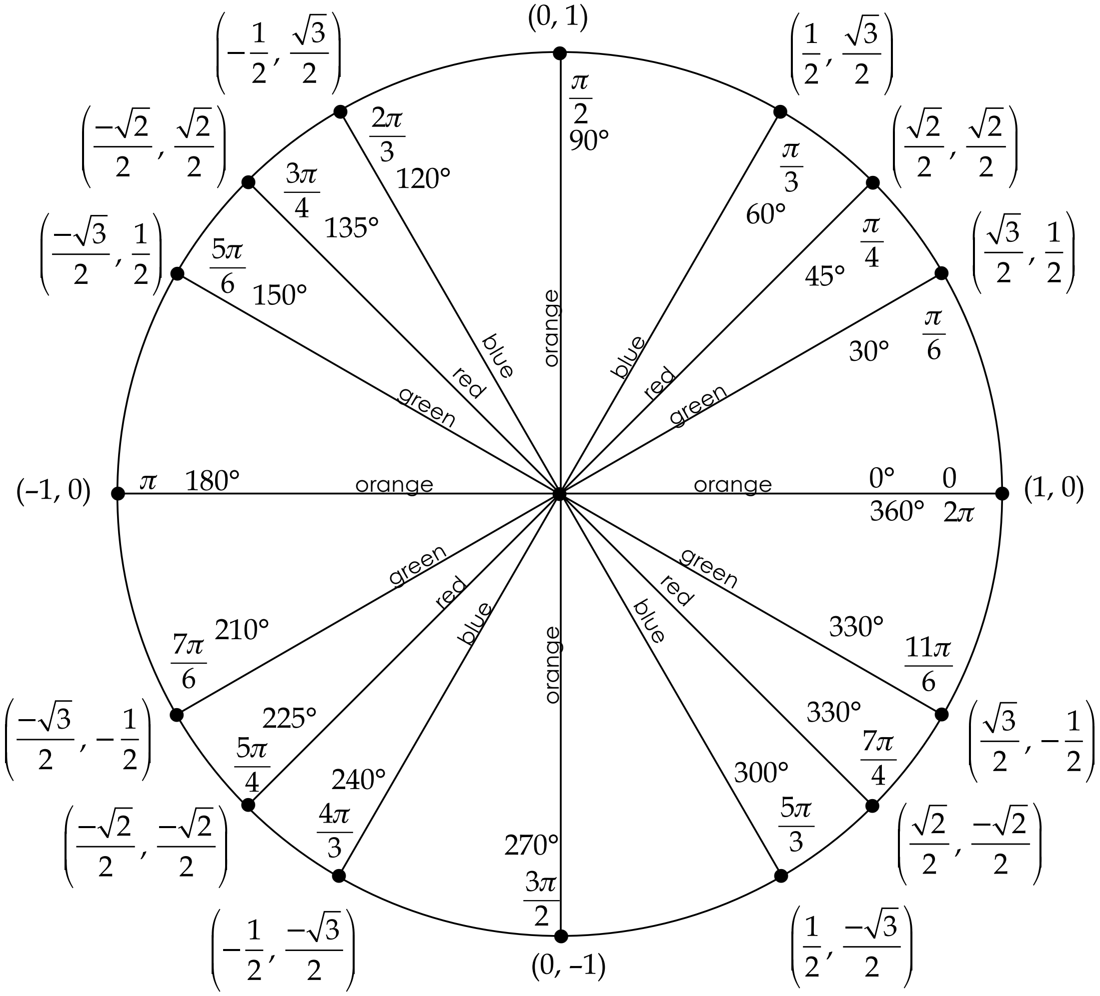
Multiples of $\frac{\pi}{4}$ $\longrightarrow$ red

Multiples of $\frac{\pi}{6}$ $\longrightarrow$ green

Multiples of $\frac{\pi}{2}$ $\longrightarrow$ orange

Answer:

The Unit Circle Key



f) What patterns do you notice in the colours of the terminal arms of the special angles located on the unit circle?

Answer:

Every two terminal arms that form a diameter of the unit circle are the same colour. Also, the colours are the same for angles that have the same reference angle. For example, the reference angle for all green terminal arms is 30° or $\frac{\pi}{6}$ radians. A very useful pattern to notice to help you memorize values is that the terminal arms with the same colour (reference angle) also have the same coordinate values, but the signs are different depending on the quadrant.

8. Explain one property of the unit circle.

   Answer:

   Answers may vary. For example, you might write that, for any of its points, the sum of the square of the x-coordinate and the square of the y-coordinate is equal to 1. Alternatively, you might simply say the radius is 1 and the centre is at the origin. Another property is that exact coordinates can be found using the dimensions of a 45–45–90 triangle and a 30–60–90 triangle.

Learning Activity 5.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. In which quadrant is $\theta = \frac{5\pi}{8}$ located?
2. Solve for x: $\frac{2x+1}{x-2} - \frac{x-1}{x-2} = 0$
3. Is $x - 2$ a factor of $p(x) = 4x^3 - 8x - 16$?
4. Evaluate: ${}_5C_4$
5. Factor: $121x^2 - 36y^{10}$.
6. Evaluate: $\frac{7}{6} \div \frac{8}{3}$
7. Evaluate: $\left| -2\frac{1}{3} - 4\frac{2}{5} \right|$
8. Simplify: $\sqrt{18} - \sqrt{50}$

Answers:

1. Quadrant II (since it is more than $\frac{\pi}{2}$ and less than π)
2. $x = -2$ $(2x + 1 - (x - 1) = 0; 2x + 1 - x + 1 = 0; x + 2 = 0; x = -2)$
3. Yes $(p(2) = 4(2)^3 - 8(2) - 16 = 32 - 16 - 16 = 0$; so the remainder is 0 and $x - 2$ is a factor)
4. 5 $\left(\frac{5 \times 4 \times 3 \times 2}{4 \times 3 \times 2 \times 1}\right)$
5. $(11x - 6y^5)(11x + 6y^5)$ [difference of squares pattern]
6. $\frac{7}{16}$ $\left(\frac{7}{6} \cdot \frac{3}{8} = \frac{21}{48} = \frac{7}{16}\right)$
7. $\frac{101}{15}$ $\left(\left| -\frac{7}{3} - \frac{22}{5} \right| = \left| -\frac{35}{15} - \frac{66}{15} \right| = \left| -\frac{101}{15} \right|\right)$
8. $-2\sqrt{2}$ $\left(\sqrt{18} - \sqrt{50} = 3\sqrt{2} - 5\sqrt{2}\right)$

Part B: Sine, Cosine, Tangent, and the Unit Circle

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Without your calculator, find exact values of cos θ, sin θ, and tan θ for each of the following values of θ. For each question, write the coordinates of P(θ) from the unit circle and then find the three circular functions.

   a) $\frac{2\pi}{3}$
   b) $\frac{7\pi}{6}$
   c) 27π
   d) $\frac{5\pi}{4}$
   e) $-\frac{\pi}{6}$
   f) 14π
   g) $-\frac{5\pi}{2}$
   h) 53π
   i) $-\frac{11\pi}{6}$
   j) $\frac{17\pi}{3}$
   k) $\frac{\pi}{2}$
   l) $-\pi$
   m) $\frac{4\pi}{3}$
   n) $\frac{3\pi}{2}$
   o) $\frac{5\pi}{6}$
   p) $\frac{15\pi}{4}$
   q) $-\frac{7\pi}{3}$
   r) 4π

Answer:

Question	θ	P(θ)	cos θ	sin θ	tan θ
(a)	$\frac{2\pi}{3}$	$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\sqrt{3}$
(b)	$\frac{7\pi}{6}$	$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
(c)	27π	$(-1, 0)$	-1	0	0
(d)	$\frac{5\pi}{4}$	$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1
(e)	$-\frac{\pi}{6}$	$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$
(f)	14π	$(1, 0)$	1	0	0
(g)	$-\frac{5\pi}{2}$	$(0, -1)$	0	-1	undefined
(h)	53π	$(-1, 0)$	-1	0	0
(i)	$-\frac{11\pi}{6}$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
(j)	$\frac{17\pi}{3}$	$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$
(k)	$\frac{\pi}{2}$	$(0, 1)$	0	1	undefined
(l)	$-\pi$	$(-1, 0)$	-1	0	0
(m)	$\frac{4\pi}{3}$	$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\sqrt{3}$
(n)	$\frac{3\pi}{2}$	$(0, -1)$	0	-1	undefined
(o)	$\frac{5\pi}{6}$	$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$
(p)	$\frac{15\pi}{4}$	$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
(q)	$-\frac{7\pi}{3}$	$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$
(r)	4π	$(1, 0)$	1	0	0

2. Determine the exact value of each expression (that is, do not use a calculator).

   Note: Although each expression involves functions and should contain notation such as sin (θ), mathematicians always like shortcuts. As long as there is no ambiguity, write these circular function statements as sin θ, etc., without the brackets. This notation is used throughout this course.

   a) $2\cos\frac{\pi}{3} + \tan\frac{\pi}{4}$

   Answer:

   $$\cos\frac{\pi}{3} = \frac{1}{2}$$

   $$\tan\frac{\pi}{4} = 1$$

   $$2\cos\frac{\pi}{3} + \tan\frac{\pi}{4} = 2\left(\frac{1}{2}\right) + 1 = 1 + 1 = 2$$

   b) $\sin\left(-\frac{15\pi}{2}\right)\cos 20\pi\ \tan\frac{13\pi}{6}$

   Answer:

   $$\sin\left(-\frac{15\pi}{2}\right) = 1$$

   $$\cos 20\pi\ = 1$$

   $$\tan\frac{13\pi}{6} = \frac{\sqrt{3}}{3}$$

   $$\sin\left(-\frac{15\pi}{2}\right)\cos 20\pi\ \tan\frac{13\pi}{6} = (1)(1)\left(\frac{\sqrt{3}}{3}\right) = \frac{\sqrt{3}}{3}$$

   c) $\tan^2\frac{\pi}{6}$

   Answer:

   $$\tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$$

   $$\tan^2\frac{\pi}{6} = \left(\tan\frac{\pi}{6}\right)^2 = \left(\frac{\sqrt{3}}{3}\right)^2 = \frac{3}{9} = \frac{1}{3}$$

The notation $\tan^2\theta$ means the same as $(\tan\theta)^2$. This notation is used so as not to confuse with $\tan(\theta^2)$. Similarly, $\sin^2\theta$ means $(\sin\theta)^2$.

d) $\cos(\sin\pi)$

Note: This is a composition of functions.

$\sin\pi = 0$

$\cos(\sin\pi) = \cos 0 = 1$

e) $\tan\dfrac{2\pi}{3}\cos\left(-\dfrac{5\pi}{4}\right) + \sin\dfrac{3\pi}{2}\tan\dfrac{5\pi}{6}$

Answer:

$$\tan\frac{2\pi}{3} = -\sqrt{3}$$

$$\cos\left(-\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$\sin\frac{3\pi}{2} = -1$$

$$\tan\frac{5\pi}{6} = -\frac{\sqrt{3}}{3}$$

$$\tan\frac{2\pi}{3}\cos\left(-\frac{5\pi}{4}\right) + \sin\frac{3\pi}{2}\tan\frac{5\pi}{6} = \left(-\sqrt{3}\right)\left(-\frac{\sqrt{2}}{2}\right) + (-1)\left(-\frac{\sqrt{3}}{3}\right)$$

$$= \frac{\sqrt{6}}{2} + \frac{\sqrt{3}}{3}$$

$$= \frac{3\sqrt{6}}{6} + \frac{2\sqrt{3}}{6}$$

$$= \frac{3\sqrt{6} + 2\sqrt{3}}{6}$$

f) $\cos\left(\dfrac{\pi}{2} - \dfrac{\pi}{6}\right)$

Answer:

$$\cos\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \cos\left(\frac{3\pi}{6} - \frac{\pi}{6}\right) = \cos\left(\frac{2\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right)$$

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

g) $\sin \frac{2\pi}{3} \cos \frac{7\pi}{6} \tan \left(-\frac{3\pi}{4}\right)$

Answer:

$$\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos \frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$$

$$\tan \left(-\frac{3\pi}{4}\right) = 1$$

$$\sin \frac{2\pi}{3} \cos \frac{7\pi}{6} \tan\left(-\frac{3\pi}{4}\right) = \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right)(1) = -\frac{3}{4}$$

h) $\sin \left(-\frac{47\pi}{2}\right)\cos (-47\pi)$

Answer:

$$\sin \left(-\frac{47\pi}{2}\right) =1$$

$$\cos(-47\pi) = -1$$

$$\sin \left(-\frac{47\pi}{2}\right) \cos (-47\pi) = (1)(-1) = -1$$

3. Find an approximate value, rounded to 5 decimal places (that is, use a calculator).

a) sin 62°

Answer:

sin 62° = 0.88296

b) tan 129°

Answer:

tan (129°) = –1.23490

c) $\cos \left(-\frac{\pi}{8}\right)$

Answer:

$$\cos \left(-\frac{\pi}{8}\right) = 0.92388$$

d) tan 5

Answer:

tan 5 = −3.38052

4. Solve the following equations over the interval $0 \leq \theta < 2\pi$. Do these questions without the use of a calculator. All answers must be exact values.

a) $\sin \theta = \frac{\sqrt{2}}{2}$

Answer:

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

b) $\tan \theta = \sqrt{3}$

Answer:

$$\theta = \frac{\pi}{3}, \frac{4\pi}{3}$$

c) $\cos \theta = \frac{\sqrt{3}}{2}$

Answer:

$$\theta = \frac{\pi}{6}, \frac{11\pi}{6}$$

d) $\cos \theta = -\frac{1}{2}$ and $\tan \theta > 0$

Answer:

This occurs in Quadrant III, as it is the only quadrant where cosine is negative and tangent is positive.

$$\theta = \frac{4\pi}{3}$$

e) $\sin\theta = \frac{\sqrt{3}}{2}$ and $\cos\theta < 0$

Answer:

This occurs in Quadrant II, as it is the only quadrant where sine is positive and cosine is negative.

$\theta = \frac{2\pi}{3}$

f) $2\cos\theta = 2$

Answer:

$\cos\theta = 1$

$\theta = 0$

Note: θ cannot equal 2π, as the interval you are solving this equation over is specified as $0 \le \theta < 2\pi$.

5. Determine the measures of *all* the angles that satisfy the following conditions over the interval $-180° \le \theta < 180°$.

a) $\sin\theta = -\frac{1}{\sqrt{2}}$

Answer:

$-\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$

From the unit circle values, the angles that satisfy this equation are in Quadrants III and IV and are coterminal with $\theta = \frac{5\pi}{4} = 225°$ and $\theta = \frac{7\pi}{4} = 315°$.

As $-180° \le \theta < 180°$ represents one complete rotation around the unit circle, one angle from each of these quadrants will be in the solution to this equation.

Quadrant III: $225° - 360° = -135°$

Quadrant IV: $315° - 360° = -45°$

b) $\cos \theta = 0.6819$

Answer:

$\cos^{(-1)} (0.6819) = 47°$

Cosine is positive in Quadrants I and IV.

Quadrant I: $\theta_r = \theta = 47°$

Quadrant IV: $\theta = 360° - \theta_r = 360° - 47° = 313°$

Now that you have located an angle in Quadrant IV where cosine is positive, subtract 360° from this angle to find a coterminal angle that fits in the restrictions $-180° \leq \theta < 180°$.

Quadrant IV: $313° - 360° = -47°$

c) $\tan \theta = -1$

Answer:

From the unit circle values, the angles that satisfy this equation are $\theta = \frac{3\pi}{4} = 135°$ and $\theta = \frac{7\pi}{4} = 315°$.

These angles are located in Quadrants II and IV.

Quadrant II: $\theta = 135°$

This angle is in the required domain.

Quadrant IV: $\theta = 315° - 360° = -47°$

Subtract 360° from this angle to find a coterminal angle that is in the required domain.

d) $\tan \theta = 0.2123$

Answer:

$\tan^{(-1)} (0.2123) = 11.99°$

Tangent is positive in Quadrants I and IV.

Quadrant I: $\theta = 11.99°$

Quadrant IV: You first need to find the related angle located in Quadrant IV.

$\theta = 360° - \theta_r = 360° - 11.99° = 348.01°$

Now, find a coterminal angle by subtracting 360° from this angle to determine an angle that is in the correct domain.

Quadrant IV: $\theta = 348.01° - 360° = -11.99°$

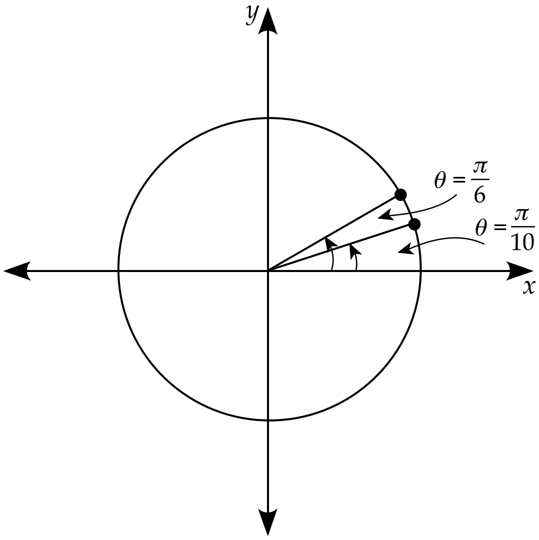
6. Can you determine which of the following is the largest without using a calculator? Use a calculator to check your work.

   $\cos \frac{\pi}{6}$ or $\cos \frac{\pi}{10}$

   Answer:

   Both of these angles are located in Quadrant I. The largest value of cosine in Quadrant I is 1. As you progress through the quadrant or increase the value of the angle from 0 to $\frac{\pi}{2}$, the cosine values, which are the x-coordinates of the points on the unit circle, decrease to 0. Therefore, smaller angles will have a larger cosine value since their x-coordinate is further to the right.

   Thus, $\cos \frac{\pi}{10}$ is larger.



   From the diagram, you can also notice that the x-coordinate of $\theta = \frac{\pi}{10}$ is larger than the x-coordinate of $\theta = \frac{\pi}{6}$.

   Checking with a calculator:

   $\cos \frac{\pi}{6} = 0.866$

   $\cos \frac{\pi}{10} = 0.951$

7. Determine the measures of all the angles of rotation in standard position, in the specified domain, that correspond to the following coordinates on the unit circle. Round to 3 decimal places where necessary.

   a) $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), 0 \le \theta < 2\pi$

   Answer:

   Using the unit circle, you can determine the angle that corresponds to this coordinate over the specified domain is $\frac{4\pi}{3}$.

   b) (0, 1) $0 \le \theta < 4\pi$

   Answer:

   Using the unit circle, the angle that satisfies these conditions between $0 \le \theta < 2\pi$ is $\frac{\pi}{2}$.

   To determine the angle that satisfies these conditions between $2\pi \le \theta < 4\pi$, add 2π to the above angle to find a coterminal angle.

   $$\frac{\pi}{2} + 2\pi = \frac{\pi}{2} + \frac{4\pi}{2} = \frac{5\pi}{2}$$

   Therefore, the two angles that satisfy these conditions are $\frac{\pi}{2}$ and $\frac{5\pi}{2}$.

   c) $\left(\frac{4}{5}, -\frac{3}{5}\right), 0 \le \theta < 2\pi$

   Answer:

   This is not one of the special values located on the unit circle. However, you know that every x-coordinate on the unit circle is equal to $\cos\theta$ while every y-coordinate on the unit circle is equal to $\sin\theta$.

   This gives you the following two equations:

   $$\cos\theta = \frac{4}{5}$$

   $$\sin\theta = -\frac{3}{5}$$

Solve these equations, making sure your calculator is in radian mode.

$$\cos\theta = \frac{4}{5}$$

$$\cos^{-1}\left(\frac{4}{5}\right) = \theta = 0.644$$

θ_r = 0.644 radians and θ is in Quadrant I or Quadrant IV, since $\cos\theta$ is positive

$$\sin\theta = -\frac{3}{5}$$

$$\sin^{-1}\left(-\frac{3}{5}\right) = \theta = -0.644$$

θ_r = 0.644 radians and θ is in Quadrant III or Quadrant IV, since $\sin\theta$ is negative

The angle required has a reference angle of 0.644 radians and is in Quadrant IV.

$\theta = -0.644$ is located in Quadrant IV. However, this angle is not in the domain of the question. You need to add 2π to this angle to find an angle that satisfies the conditions of the question.

$$\theta = -0.644 + 2\pi = 5.639 \text{ radians}$$

This angle is located in Quadrant IV and is between 0 and 2π radians.

Note: You can also manipulate $\theta = 0.644$ to be an angle in Quadrant IV by finding a related angle in Quadrant IV. To do this, subtract θ from 2π.

$$\theta = 2\pi - 0.644 = 5.639$$

As you can see, you will arrive at the same answer by either solving the cosine equation or the sine equation. Therefore, it is only necessary to solve one equation.

d) $\left(\frac{5}{13}, \frac{12}{13}\right), -2\pi \le \theta < 2\pi$

Answer:

$$\cos\theta = \frac{5}{13}$$

$$\theta = \cos^{-1}\left(\frac{5}{13}\right) = 1.176$$

This angle is located in Quadrant I, where cosine and sine are both positive. This is one of the angles you are looking for, as the x-coordinate and the y-coordinate are both positive at the point $\left(\frac{5}{13}, \frac{12}{13}\right)$. This means that this point is located in Quadrant I.

To determine the angle that is located between -2π and 0, subtract 2π from the above angle.

$$\theta = 1.176 - 2\pi = -5.107$$

Therefore, the two angles that satisfy the above conditions are 1.176 and −5.107 radians.

Learning Activity 5.5

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Determine the inverse function if $f(x) = x + 4$.
2. State the non-permissible values of the function $f(x) = \dfrac{x+3}{5x^2 - 4x - 1}$.
3. State an angle that is coterminal to $-124°$.
4. Simplify: $\dfrac{x^2 + 3x + 2}{x + 2}$
5. What is the volume of a cube with a side length of $\sqrt{2}$ cm?
6. Your restaurant bill came to \$65.29. If you wish to leave a 15% tip, estimate how much you should leave.
7. Determine the vertex of the function $f(x) = -\frac{1}{2}(x-2)^2 + 5$.
8. Multiply: $\left(\dfrac{1}{x} + 5\right)(x - 2)$

Answers:

1. $y = x - 4$
2. $x \neq -\frac{1}{5}$ and $x \neq 1$ $\left(5x^2 - 4x - 1 = (5x+1)(x-1)\right)$
3. 236° ($-124° + 360° = 236°$)
4. $(x+1),\ x \neq -2$ $\left(\dfrac{x^2 + 3x + 2}{x+2} = \dfrac{(x+1)(x+2)}{x+2} = x + 1\right)$
5. $2\sqrt{2}$ cm^3 $\left(\left(\sqrt{2}\right)^3 = \sqrt{2}\sqrt{2}\sqrt{2} = 2\sqrt{2}\right)$
6. \$9.75 (10% of 65.29 is approximately 6.50, so 5% is 3.25; add 6.50 + 3.25 = 9.75)
7. (2, 5)
8. $5x - \frac{2}{x} - 9$ $\left(\left(\dfrac{1}{x} + 5\right)(x-2) = 1 - \dfrac{2}{x} + 5x - 10 = 5x - \dfrac{2}{x} - 9\right)$

Part B: The Reciprocal Trigonometric Ratios: Cosecant, Secant, and Cotangent

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Use the coordinates of the special points on the unit circle and the definitions of the circular functions to find the following values. Include a diagram with your answer.

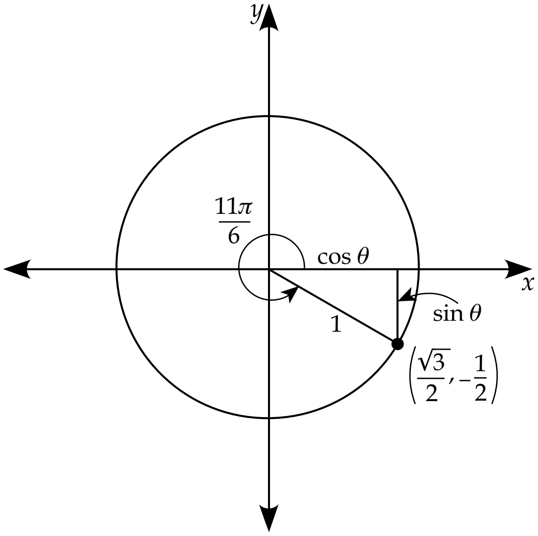
a) $\sec \frac{11\pi}{6}$

Answer:

$$\sec \frac{11\pi}{6} = \frac{1}{\cos \frac{11\pi}{6}}$$

$$\cos \frac{11\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\therefore \sec \frac{11\pi}{6} = \frac{2}{\sqrt{3}}$$



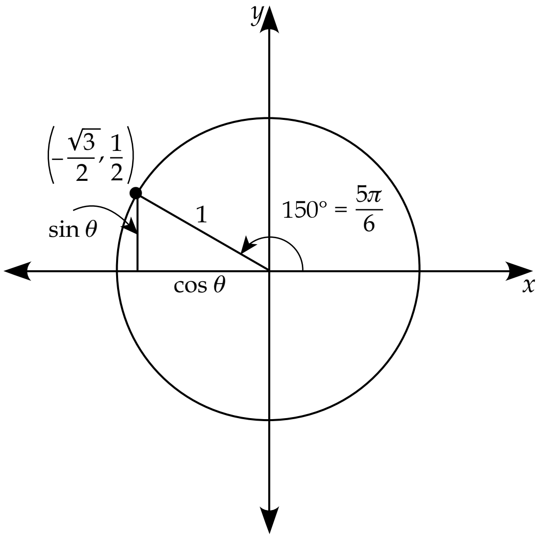
b) sec 150°

Answer:

$$\sec 150° = \frac{1}{\cos 150°}$$

$$\cos 150° = -\frac{\sqrt{3}}{2}$$

$$\therefore \sec 150° = -\frac{2}{\sqrt{3}}$$



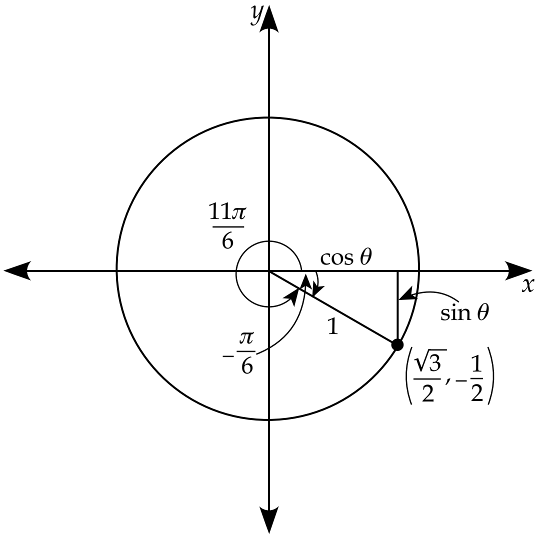
c) $\csc\left(-\frac{\pi}{6}\right)$

Answer:

$$\csc\left(-\frac{\pi}{6}\right) = \csc\left(\frac{11\pi}{6}\right)$$

$$\sin\left(\frac{11\pi}{6}\right) = -\frac{1}{2}$$

$$\therefore \csc\left(-\frac{11\pi}{6}\right) = -2$$



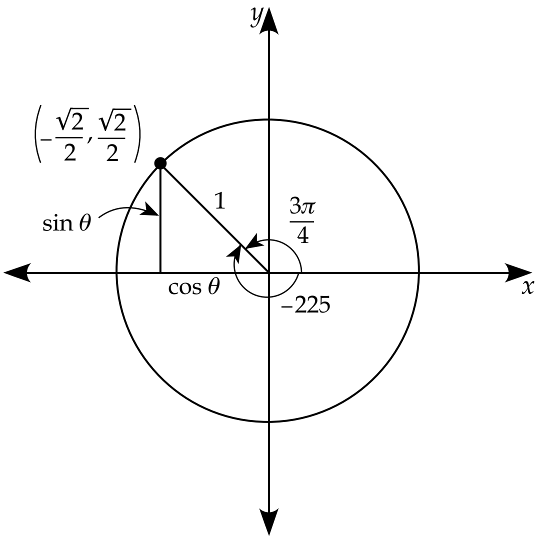
d) csc (−225°)

Answer:

$$\csc(-225°) = \frac{1}{\sin(-225°)}$$

$$\sin(-225°) = \frac{\sqrt{2}}{2}$$

$$\therefore \csc(-225°) = \frac{2}{\sqrt{2}}$$

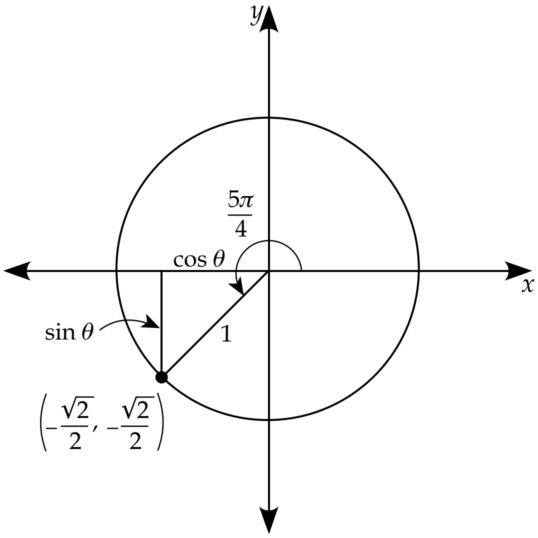


e) $\cot \frac{5\pi}{4}$

Answer:

$$\cot \frac{5\pi}{4} = \frac{\cos\left(\frac{5\pi}{4}\right)}{\sin\left(\frac{5\pi}{4}\right)}$$

$$\therefore \cot \frac{5\pi}{4} = \frac{-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = 1$$

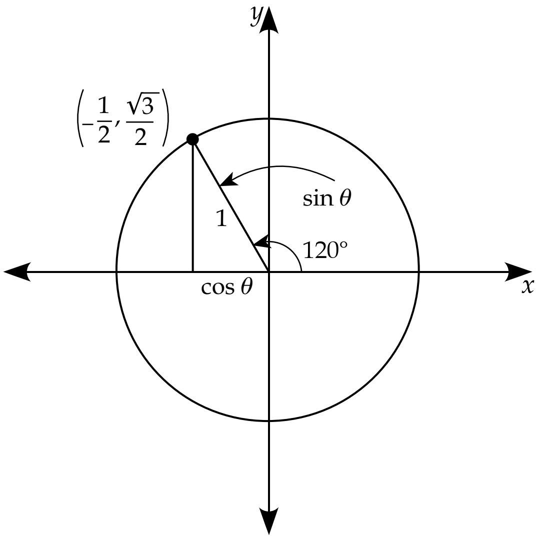


f) cot 120°

Answer:

$$\cot 120° = \frac{\cos 120°}{\sin 120°}$$

$$\cot 120° = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$$



2. Determine the *exact* value of each expression.

a) $\sec\frac{\pi}{3}\sin\frac{\pi}{6} - \cot\frac{\pi}{4}$

Answer:

$$\sec\frac{\pi}{3} = \frac{1}{\cos\frac{\pi}{3}} = \frac{2}{1}$$

$$\sin\frac{\pi}{6} = \frac{1}{2}$$

$$\cot\frac{\pi}{4} = \frac{1}{\tan\frac{\pi}{4}} = \frac{1}{1} = 1$$

$$\sec\frac{\pi}{3}\sin\frac{\pi}{6} - \cot\frac{\pi}{4} = 2\left(\frac{1}{2}\right) - 1 = 0$$

b) $\sec^2\frac{\pi}{6} + \csc^2\frac{\pi}{6}\left(\text{Note: } \sec^2\frac{\pi}{6} \text{ means } \left(\sec\frac{\pi}{6}\right)^2.\right)$

Answer:

$$\sec\frac{\pi}{6} = \frac{1}{\cos\frac{\pi}{6}} = \frac{2}{\sqrt{3}}$$

$$\sec^2\frac{\pi}{6} = \left(\frac{2}{\sqrt{3}}\right)^2 = \frac{4}{3}$$

$$\csc\frac{\pi}{6} = \frac{1}{\sin\frac{\pi}{6}} = 2$$

$$\csc^2\frac{\pi}{6} = 2^2 = 4$$

$$\sec^2\frac{\pi}{6} + \csc^2\frac{\pi}{6} = \frac{4}{3} + 4 = 5\frac{1}{3} \text{ or } \frac{16}{3}$$

c) $\cot^2 \frac{5\pi}{6}$

Answer:

$$\cot \frac{5\pi}{6} = \frac{1}{\tan \frac{5\pi}{6}} = \frac{1}{-\frac{\sqrt{3}}{3}} = -\frac{3}{\sqrt{3}}$$

$$\cot^2 \frac{5\pi}{6} = \left(-\frac{3}{\sqrt{3}}\right)^2 = \frac{9}{3} = 3$$

d) $\tan \frac{2\pi}{3} \cot \frac{2\pi}{3}$

Answer:

$$\tan \frac{2\pi}{3} = -\sqrt{3}$$

$$\cot \frac{2\pi}{3} = \frac{1}{\tan \frac{2\pi}{3}} = \frac{1}{-\sqrt{3}}$$

$$\tan \frac{2\pi}{3} \cot \frac{2\pi}{3} = \left(-\sqrt{3}\right)\left(-\frac{1}{\sqrt{3}}\right) = 1$$

e) $\cos^2 \frac{\pi}{4} \sec^2 \frac{\pi}{4}$

Answer:

$$\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\cos^2 \frac{\pi}{4} = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4} = \frac{1}{2}$$

$$\sec \frac{\pi}{4} = \frac{1}{\cos \frac{\pi}{4}} = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}}$$

$$\sec^2 \frac{\pi}{4} = \left(\frac{2}{\sqrt{2}}\right)^2 = \frac{4}{2} = 2$$

$$\cos^2 \frac{\pi}{4} \sec^2 \frac{\pi}{4} = \left(\frac{1}{2}\right)(2) = 1$$

Note: Both (d) and (e) could have been done quickly by inspection, since reciprocals multiplied together equal 1.

f) $4\csc\left(-\frac{7\pi}{6}\right)$

Answer:

$$\csc\left(-\frac{7\pi}{6}\right) = \csc\frac{5\pi}{6} = \frac{1}{\sin\frac{5\pi}{6}} = 2$$

$$4\csc\left(-\frac{7\pi}{6}\right) = 4(2) = 8$$

3. Find the *exact* solution of the following equations over the interval $0 \le \theta < 2\pi$.

a) $\sec\theta = 2$

Answer:

$$\sec\theta = \frac{1}{\cos\theta} = 2$$

This step need not be shown. If $\sec\theta = \frac{2}{1}$, it follows that $\cos\theta = \frac{1}{2}$.

$$\cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

b) $\csc\theta = \sqrt{2}$

Answer:

$$\csc\theta = \frac{1}{\sin\theta} = \sqrt{2}$$

$$\sin\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

c) $\cot \theta = -1$, and $\sin \theta < 0$

Answer:

$$\cot \theta = \frac{1}{\tan \theta} = -1$$

$$\tan \theta = -1$$

$$\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$$

But $\sin \theta < 0$ occurs in Quadrants III and IV. Therefore, the answer to this question lies in Quadrant IV only and is $\theta = \frac{7\pi}{4}$.

4. Find an approximate value rounded to 5 decimal places.

a) $\sec 82°$

Answer:

In degree mode on your calculator, $\cos 82° = 0.13917$.

$$\sec 82° = \frac{1}{\cos 82°} = \frac{1}{0.13917} = 7.18546$$

b) $\cot \frac{2\pi}{9}$

Answer:

In radian mode on your calculator, $\tan \frac{2\pi}{9} = 0.83910$.

$$\cot \frac{2\pi}{9} = \frac{1}{\tan \frac{2\pi}{9}} = \frac{1}{0.83910} = 1.19175$$

c) $\csc 193°$

Answer:

In degree mode on your calculator, $\sin 193° = -0.22495$.

$$\csc 193° = \frac{1}{\sin 193°} = \frac{1}{-0.22495} = -4.44543$$

d) csc 1

Answer:

In radian mode on your calculator, sin 1 = 0.84147.

$$\csc 1 = \frac{1}{\sin 1} = \frac{1}{0.84147} = 1.18840$$

5. Determine the measures of *all* the angles that satisfy the following conditions on the domain.

a) $\csc \theta = -1.04284$, $0 \le \theta < 2\pi$

Answer:

$$\csc \theta = \frac{1}{\sin \theta} = -1.04284$$

$$\therefore \sin \theta = -0.95892$$

$$\theta = \sin^{-1}(-0.95892) = -1.28317$$

$$\theta_r = 1.28317$$

This angle is located in Quadrant IV. However, you need to find both angles between 0 and 2π that are located in Quadrants III and IV, where sine and cosecant are negative.

Quadrant IV: You can determine the angle in this quadrant by adding 2π to the angle you calculated above.

$$\theta = -1.28317 + 2\pi = 5.00002$$

Quadrant III: Determine the related angle in Quadrant III.

$$\theta = \pi + 1.28317 = 4.42476$$

Therefore, the two angles that satisfy the above equation are $\theta = 5.00002$ and $\theta = 4.42476$.

b) $\sec \theta = 1.13257$, $0° \le \theta < 360°$

Answer:

$$\sec \theta = \frac{1}{\cos \theta} = 1.13257$$

$$\cos \theta = \frac{1}{1.13257} = 0.88295$$

$$\cos^{-1}(0.88295) = \theta = 28.00000$$

Cosine and secant are positive in Quadrants I and IV.

Quadrant I: $\theta = 28°$

Quadrant IV: $\theta = 360° - 28° = 332°$

Therefore, the two angles that satisfy this equation are $\theta = 28°$ and $\theta = 332°$.

c) $\cot \theta = -1.66428$, $-360° \le \theta < 0°$

Answer:

$$\begin{aligned}
\cot \theta &= \frac{1}{\tan \theta} = -1.66428 \\
\tan \theta &= \frac{1}{-1.66428} = -0.60086 \\
\theta &= \tan^{-1}(-0.60086) = -31° \\
\theta_r &= 31°
\end{aligned}$$

Tangent and cotangent are negative in Quadrants II and IV.

Quadrant II: To determine this angle, first find a positive angle in Quadrant II and then subtract 360° from that angle to ensure your angle is between −360° and 0°.

$180° - 31° = 149°$

$\theta = 149° - 360° = -211°$

Quadrant IV: To determine this angle, first find a positive angle in Quadrant IV and then subtract 360° from that angle.

$360° - 31° = 329°$

$329° - 360° = -31°$

Therefore, the two angles that satisfy this equation are $\theta = -211°$ and $\theta = -31°$.

d) $\cot \theta = 2.31187$, $-2\pi \le \theta < 0$

Answer:

$$\begin{aligned}
\cot \theta &= \frac{1}{\tan \theta} = 2.31187 \\
\tan \theta &= \frac{1}{2.31187} = 0.43255 \\
\theta &= \tan^{-1}(0.43255) = 0.40823
\end{aligned}$$

Tangent and cotangent are positive in Quadrants I and III.

Quadrant I: $\theta = 0.40823$ radians

Quadrant III: $\theta = \pi + 0.40823 = 3.54984$ radians

Therefore, the two angles that satisfy this equation are $\theta = 0.40823$ and $\theta = 3.54982$.

6. Determine the exact values of the other five trigonometric ratios that satisfy the given information.

   a) $\sec\theta = -2,\ \pi \le \theta < \dfrac{3\pi}{2}$

   Answer:

   $$\cos\theta = \frac{1}{\sec\theta} = -\frac{1}{2}$$

   This x-coordinate value is on the unit circle and occurs at angles $\dfrac{2\pi}{3}$ and $\dfrac{4\pi}{3}$ of radians. The angle of $\dfrac{4\pi}{3}$ radians is in the correct quadrant, Quadrant III, specified in the domain. Therefore, sine is negative.

   $$\sin\theta = -\frac{\sqrt{3}}{2}$$

   $$\tan\theta = \sqrt{3}$$

   $$\csc\theta = \frac{1}{\sin\theta} = -\frac{2}{\sqrt{3}}$$

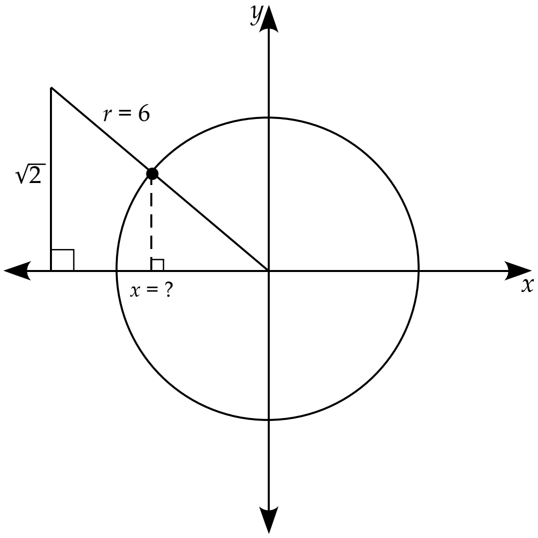
   $$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\sqrt{3}}$$

   b) $\csc\theta = \dfrac{6}{\sqrt{2}},\ \dfrac{\pi}{2} \le \theta < \dfrac{3\pi}{2}$

   Answer:

   $$\sin\theta = \frac{1}{\csc\theta} = \frac{\sqrt{2}}{6}$$

   The ratio of $\sin\theta$ is $\dfrac{y}{r}$. Therefore, $y = \sqrt{2}$ and $r = 6$. Use these values to draw a triangle in Quadrant II, where cosecant is positive and the angle will be between $\dfrac{\pi}{2}$ and $\dfrac{3\pi}{2}$. This triangle is similar to the desired triangle on the unit circle.



Determine x by using the Pythagorean theorem.

$$r^2 = x^2 + y^2$$

$$(6)^2 = x^2 + \left(\sqrt{2}\right)^2$$

$$36 = x^2 + 2$$

$$34 = x^2$$

$$x = -\sqrt{34}$$

Note: x is negative because you are in Quadrant II.

Now determine the four missing ratios using what you know about right triangle trigonometric ratios (think SOH CAH TOA).

$$\cos\theta = -\frac{\sqrt{34}}{6}$$

$$\sec\theta = -\frac{6}{\sqrt{34}}$$

$$\tan\theta = -\frac{\sqrt{2}}{\sqrt{34}} \text{ or } -\frac{1}{\sqrt{17}}$$

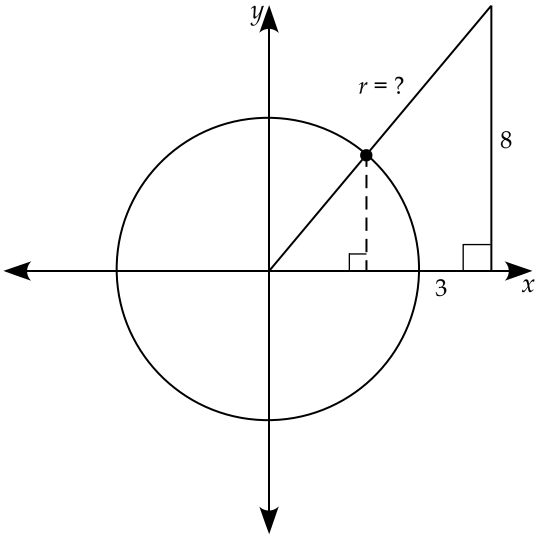
$$\cot\theta = -\frac{\sqrt{34}}{\sqrt{2}} \text{ or } -\sqrt{17}$$

c) $\cot \theta = \frac{3}{8}, 0 \le \theta < \frac{\pi}{2}$

Answer

You are looking for an angle in Quadrant I, where x and y are both positive.

If $\cot \theta = \frac{3}{8}$, then $\tan \theta = \frac{8}{3}$, $y = 8$, and $x = 3$. Use these values to draw a triangle in Quadrant I. This triangle is similar to the desired triangle on the unit circle.



$$r^2 = x^2 + y^2$$
$$r^2 = (3)^2 + (8)^2$$
$$r^2 = 9 + 64$$
$$r^2 = 73$$
$$r = \sqrt{73}$$

You can now find the remaining trigonometric ratios.

$$\sin \theta = \frac{8}{\sqrt{73}}$$

$$\csc \theta = \frac{\sqrt{73}}{8}$$

$$\cos \theta = \frac{3}{\sqrt{73}}$$

$$\sec \theta = \frac{\sqrt{73}}{3}$$

7. Determine the exact values of the six trigonometric ratios, given each of the following points on the terminal arm of an angle in standard position. Explain the strategy you used to determine the six trigonometric ratios.

   a) $\left(-\frac{3}{4}, -\frac{\sqrt{7}}{4}\right)$

   Answer:

   To determine the six trigonometric ratios, first determine the cosine ratio by using the *x*-coordinate and the sine ratio by using the *y*-coordinate on the unit circle.

   The tangent ratio can be determined from the sine and cosine ratios, and then each of the reciprocal ratios can be determined.

$$\sin\theta = -\frac{\sqrt{7}}{4}$$

$$\cos\theta = -\frac{3}{4}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{\sqrt{7}}{4}}{-\frac{3}{4}} = -\frac{\sqrt{7}}{4}\cdot -\frac{4}{3} = \frac{\sqrt{7}}{3}$$

$$\csc\theta = \frac{1}{\sin\theta} = -\frac{4}{\sqrt{7}}$$

$$\sec\theta = \frac{1}{\cos\theta} = -\frac{4}{3}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{3}{\sqrt{7}}$$

b) $\left(-\frac{1}{3}, \frac{\sqrt{8}}{3}\right)$

Answer:

$$\sin\theta = \frac{\sqrt{8}}{3}$$

$$\cos\theta = -\frac{1}{3}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{\sqrt{8}}{3}}{-\frac{1}{3}} = \frac{\sqrt{8}}{3}\cdot -\frac{3}{1} = -\sqrt{8}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{3}{\sqrt{8}}$$

$$\sec\theta = \frac{1}{\cos\theta} = -3$$

$$\cot\theta = \frac{1}{\tan\theta} = -\frac{1}{\sqrt{8}}$$

Learning Activity 5.6

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Does the following set of points represent a one-to-one function?

   $(-3, 5), (7, -2), (6, -2), (-8, 3)$

2. Find the positive coterminal angle for $-237°$ in the interval $0° \leq \theta \leq 360°$.
3. Is $x = -6$ a solution to the inequality $-x^2 - 3x + 5 > 0$?
4. Simplify: $(6x^2y^3)^2$
5. Determine the axis of symmetry of the quadratic function $f(x) = 3(x + 5)^2$.
6. Evaluate: $\sqrt[3]{-729}$.
7. Write as a mixed fraction: $\dfrac{17}{6}$
8. Convert 0.0125 into a fraction.

Answers:

1. No (The y-coordinate, -2, has two x-values that map to it.)
2. $123°$ $(-237° + 360° = 123°)$
3. No $\left(-(-6)^2 - 3(-6) + 5 = -36 + 18 + 5 = -13 \not> 0\right)$
4. $36x^4y^6$
5. $x = -5$
6. -9
7. $2\dfrac{5}{6}$
8. $\dfrac{125}{10\,000}$ or $\dfrac{1}{80}$

Part B: Graphical Properties of Sine, Cosine, and Tangent

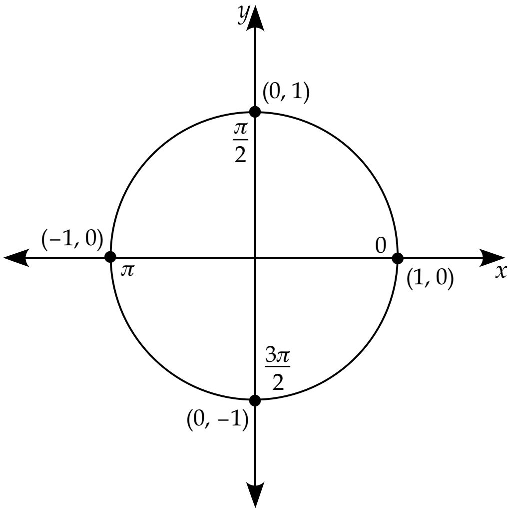
Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Complete the following table of the properties of the sine, cosine, and tangent graphs. You may wish to include some, or all, of this information on your resource sheet for future reference.

	$\sin\theta$	$\cos\theta$	$\tan\theta$
Domain	$\Re$	$\Re$	$\left\{x \mid x \neq \frac{(2k+1)\pi}{2},\ k \in I\right\}$
Range	$[-1, 1]$	$[-1, 1]$	$\Re$
Period	2π	2π	π
Amplitude	1	1	
Equation of Asymptotes			$\left\{x \mid x = \frac{(2k+1)\pi}{2},\ k \in I\right\}$
x-intercepts	$\{x \mid x = k\pi, k \in I\}$	$\left\{x \mid x = \frac{(2k+1)\pi}{2},\ k \in I\right\}$	$\{x \mid x = k\pi, k \in I\}$
y-intercept	0	1	0

2. Demonstrate how the unit circle coordinates at the x- and y-axis can help to sketch $y = \cos\theta$ and $y = \sin\theta$.

Answer:



(1, 0) means $\cos(0) = 1$ and $\sin(0) = 0$

(0, 1) means $\cos\left(\frac{\pi}{2}\right) = 0$ and $\sin\left(\frac{\pi}{2}\right) = 1$

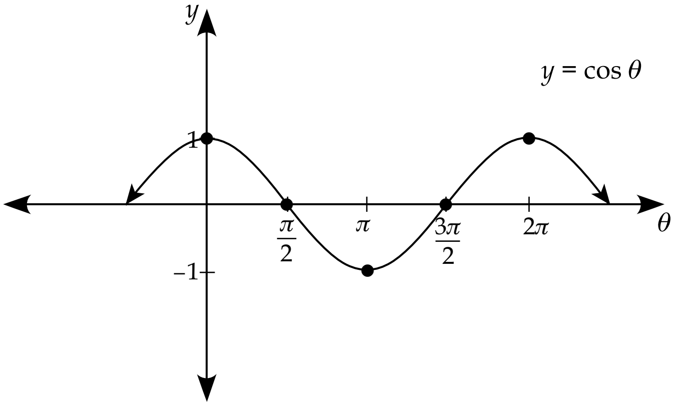
(−1, 0) means $\cos(\pi) = -1$ and $\sin(\pi) = 0$

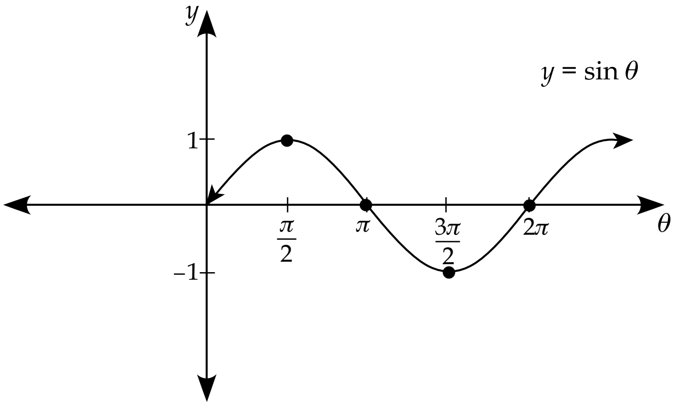
(0, −1) means $\cos\left(\frac{3\pi}{2}\right) = 0$ and

$\sin\left(\frac{3\pi}{2}\right) = -1$

(1, 0) means $\cos(2\pi) = 1$ and $\sin(2\pi) = 0$

Plot these points and connect the points with smooth curves.





Learning Activity 5.7

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Which is the better deal, a 12-pack of macaroni and cheese for \$4.49 or a 36-pack of macaroni and cheese for \$11.99?
2. What is the length of the hypotenuse of a right-angled triangle if the two legs of the triangle measure 3 m and 4 m respectively?
3. Factor: $(x-1)^2-(y+3)^2$
4. Simplify: $\dfrac{\frac{7}{2}}{5}$
5. Reduce to lowest terms: $\dfrac{56}{180}$
6. Express $7+\dfrac{6}{11}$ as an improper fraction.
7. Is 239 a prime number?
8. What is 25% of \$1250?

Answers:

1. 36-pack for \$11.99 (3 times 4.49 is more than \$12)
2. 5 m (3^2+4^2 = hypotenuse squared)
3. $(x-y-4)(x+y+2)$ (Difference of squares: $[(x-1-(y+3))(x-1+y+3)=(x-y-4)(x+y+2)]$)
4. $\dfrac{7}{10}\left(\dfrac{7}{2}\div 5=\dfrac{7}{2}\cdot\dfrac{1}{5}=\dfrac{7}{10}\right)$
5. $\dfrac{14}{45}$ (divide by a common factor of 4)
6. $\dfrac{83}{11}\left(\dfrac{77}{11}+\dfrac{6}{11}=\dfrac{83}{11}\right)$
7. Yes (check for divisibility by 2, 3, 5, 7, 11, 13)
8. \$312.50 (10% of 1250 is 125, so 20% is 250 and 5% is 62.5; add 250 + 62.5)

Part B: Graphing Trigonometric Functions

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Find the period of each function.

   a) $y = 3 \sin 2x$

   Answer:

   $$\frac{2\pi}{2} = \pi$$

   b) $y = \pi \sin(-\pi x)$

   Answer:

   $$\frac{2\pi}{\pi} = 2$$

   c) $y = \cos(x - \pi)$

   Answer:

   2π

   d) $y = -\sin(3x - 4)$

   Answer:

   $$\frac{2\pi}{3}$$

   e) $y = \cos(4x + \pi)$

   Answer:

   $$\frac{2\pi}{4} = \frac{\pi}{2}$$

   f) $y = \sin\left(-\frac{1}{2}x\right)$

   Answer:

   $$\frac{2\pi}{\frac{1}{2}} = 4\pi$$

2. Find the x-intercepts of each circular function. You are being asked to write a set of all the x-intercepts, not just those on the interval $[0, 2\pi]$.

   Note: Be sure you know how to denote various sets of integers.

   - All integers are denoted as $\{k \mid k \in I\}$.
   - Even integers are denoted by $\{2k \mid k \in I\}$.
   - Odd integers are denoted by $2k + 1$ or $2k - 1$, since odd integers are one greater or one less than any even integer denoted by $2k$. Using set notation, we write $\{2k + 1 \mid k \in I\}$.

a) $y = \sin 3x$

Answer:

$\sin 3x = 0$

$\therefore\ 3x = \ldots, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$

$\left(\text{since these are the basic } y = \sin\theta\ \ x\text{-intercepts}\right)$

$\therefore\ x = \ldots, -\dfrac{\pi}{3}, 0, \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \ldots \left(\text{integer multiples of } \dfrac{\pi}{3}\right)$

$\left\{x \,\middle|\, x = k\dfrac{\pi}{3}, k \in \mathrm{I}\right\}$

b) $y = \cos 2x$

Answer:

$\cos 2x = 0$

$\therefore\ 2x = \ldots, -\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \ldots$

$\left(\text{since these are the basic } y = \cos\theta\ \ x\text{-intercepts}\right)$

$\therefore\ x = \ldots, -\dfrac{\pi}{4}, \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \ldots \ \left(\text{odd multiples of } \dfrac{\pi}{4}\right)$

$\left\{x \,\middle|\, x = (2k+1)\dfrac{\pi}{4}, k \in \mathrm{I}\right\}$

c) $y = \sin \pi x$

Answer:

$\sin \pi x = 0$

$\pi x = \ldots, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$

$\left(\text{since these are the basic } y = \sin\theta\ \ x\text{-intercepts}\right)$

$x = \ldots, \dfrac{-\pi}{\pi}, \dfrac{0}{\pi}, \dfrac{\pi}{\pi}, \dfrac{2\pi}{\pi}, \dfrac{3\pi}{\pi}, \ldots$

$x = \ldots, -1, 0, 1, 2, 3, \ldots$

$\therefore$ simply, $\left\{x \,\middle|\, x \in \mathrm{I}\right\}$

d) $y = -3 \sin(-2\pi x)$

Answer:

$$-3 \sin(-2\pi x) = 0$$

$$-2\pi x = \ldots, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$$

$$x = \ldots, \frac{1}{2}, 0, -\frac{1}{2}, -1, -\frac{3}{2}, \ldots$$

$$\left\{x \;\middle|\; x = \frac{k}{2}, k \in \mathrm{I}\right\}$$

3. Sketch each of the following. State the domain, range, amplitude, y-intercept, and period.

a) $y = 2 \sin x$

b) $y = \sin 2x$

c) $y = \sin(x - \pi)$

d) $y = 2 \cos x$

e) $y = -\cos 2x$

f) $y = \cos(-2x)$

g) $y = |\sin x|$

h) $y = \cos x - 1$

i) $y = \cos \pi x$

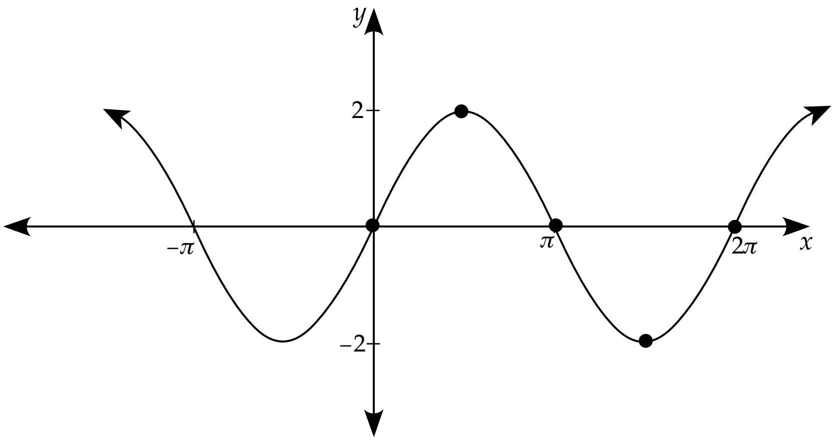
j) $y = -2 \sin\left(\frac{1}{2}x\right)$

k) $y = |\tan x|$

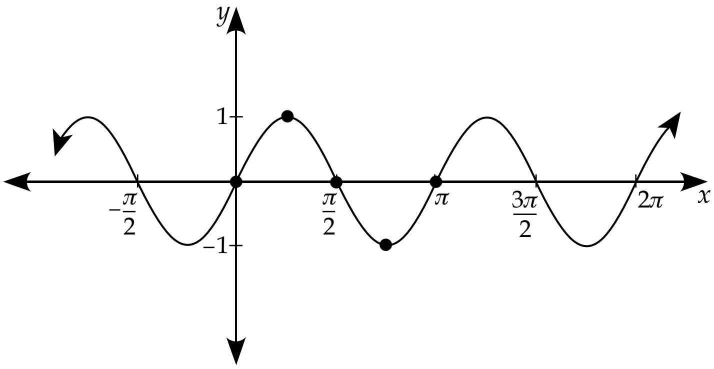
Answers:

Question	Domain	Range	Amplitude	y-intercept	Period
(a)	$\Re$	$[-2, 2]$	2	0	2π
(b)	$\Re$	$[-1, 1]$	1	0	π
(c)	$\Re$	$[-1, 1]$	1	0	2π
(d)	$\Re$	$[-2, 2]$	2	2	2π
(e)	$\Re$	$[-1, 1]$	1	-1	π
(f)	$\Re$	$[-1, 1]$	1	1	π
(g)	$\Re$	$[0, 1]$	not applicable	0	π
(h)	$\Re$	$[-2, 0]$	1	0	2π
(i)	$\Re$	$[-1, 1]$	1	1	2
(j)	$\Re$	$[-2, 2]$	2	0	4π
(k)	$\left\{x \;\middle\|\; x \neq \frac{(2k+1)\pi}{2}, k \in I\right\}$	$[0, \infty)$	not applicable	0	π

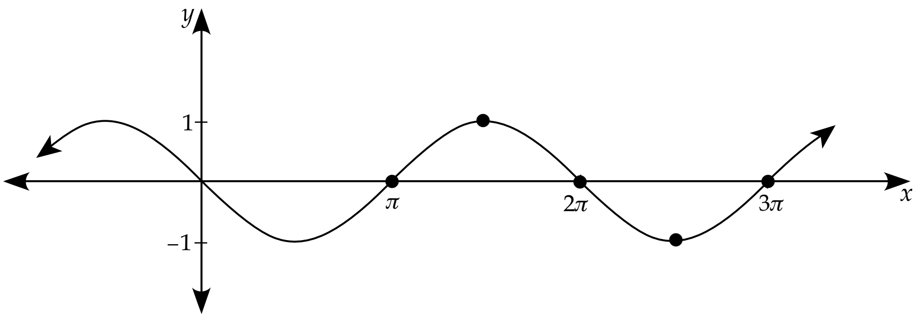
a) $y = 2 \sin x$



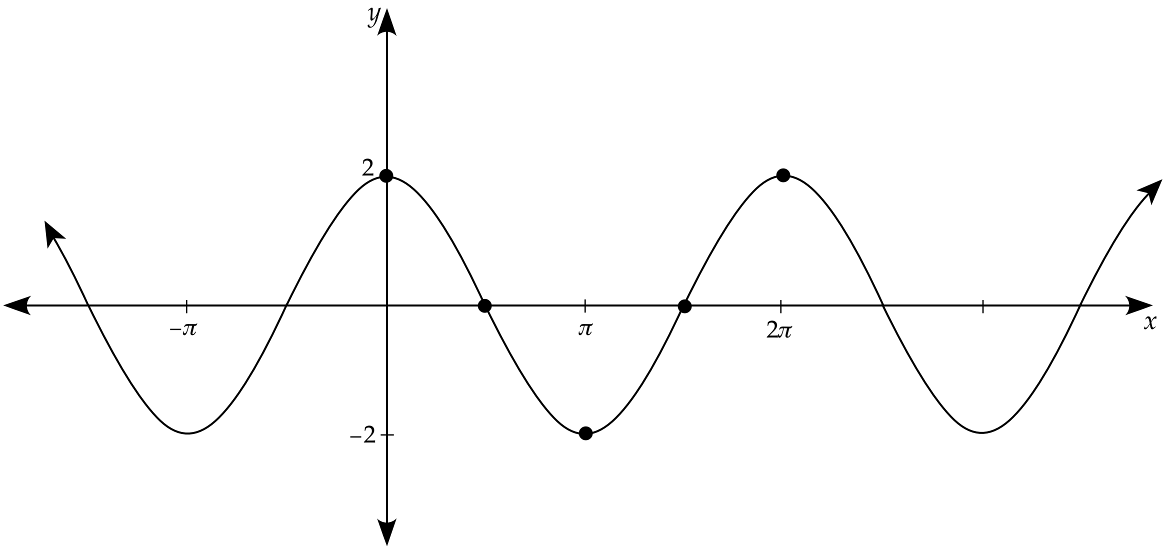
b) $y = \sin 2x$



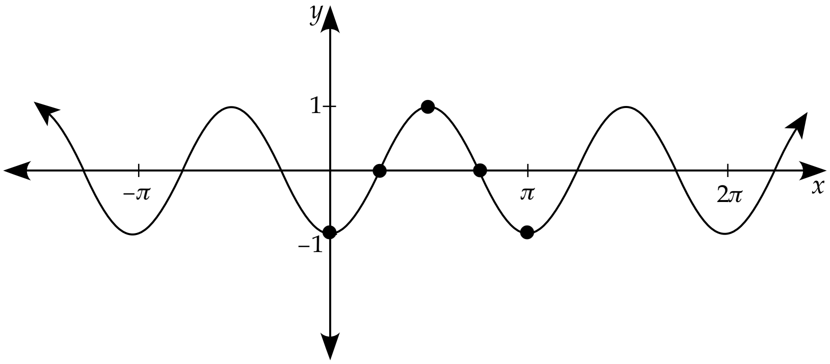
c) $y = \sin (x - \pi)$



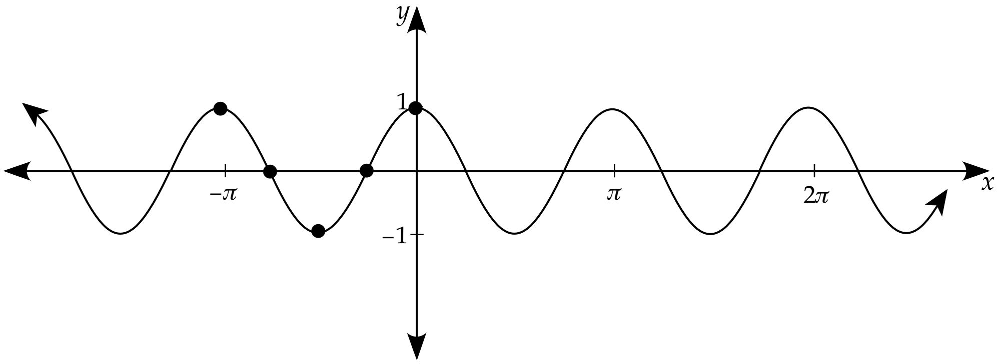
d) $y = 2\cos x$



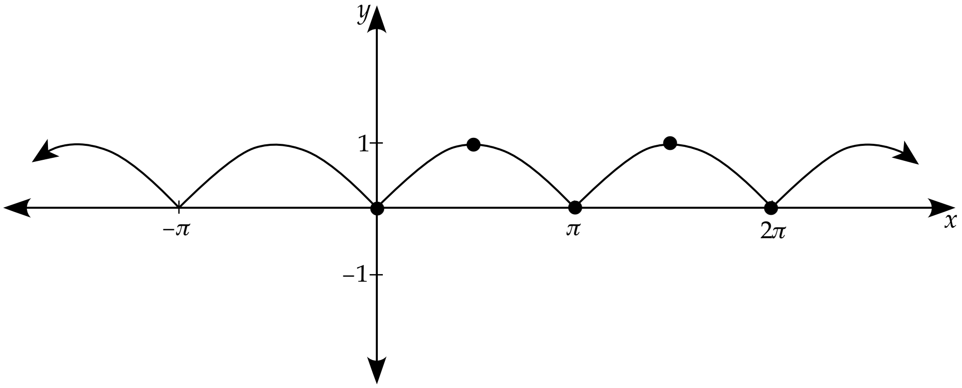
e) $y = -\cos 2x$



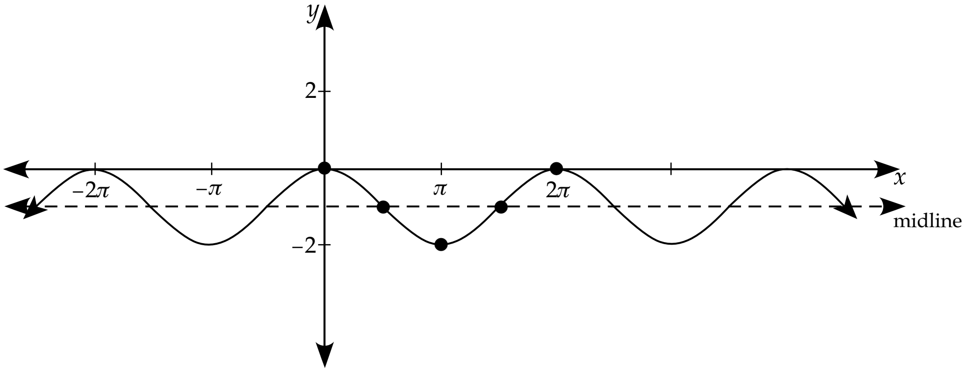
f) $y = \cos(-2x)$



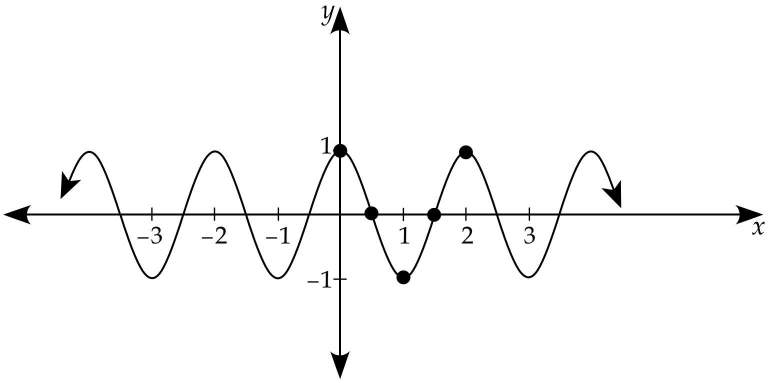
g) $y = |\sin x|$



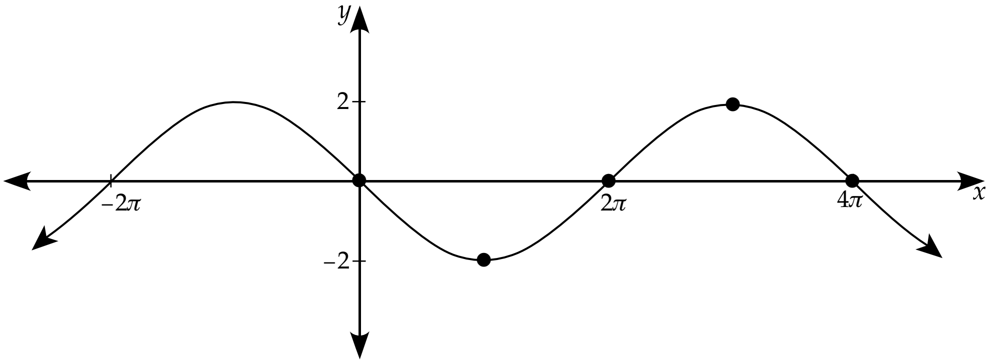
h) $y = \cos x - 1$



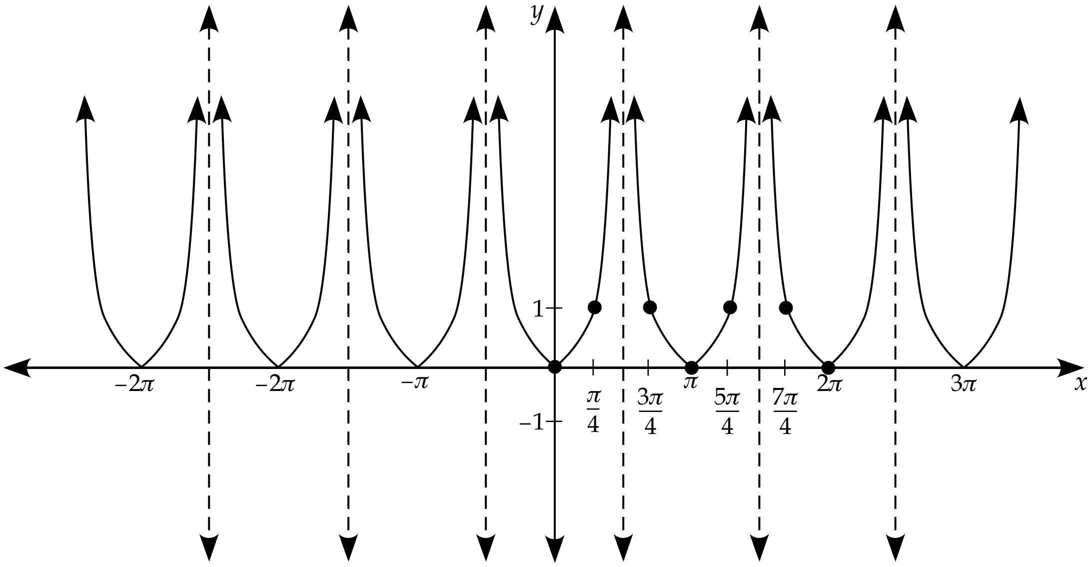
i) $y = \cos \pi x$



j) $y = -2 \sin \left(\frac{1}{2} x \right)$



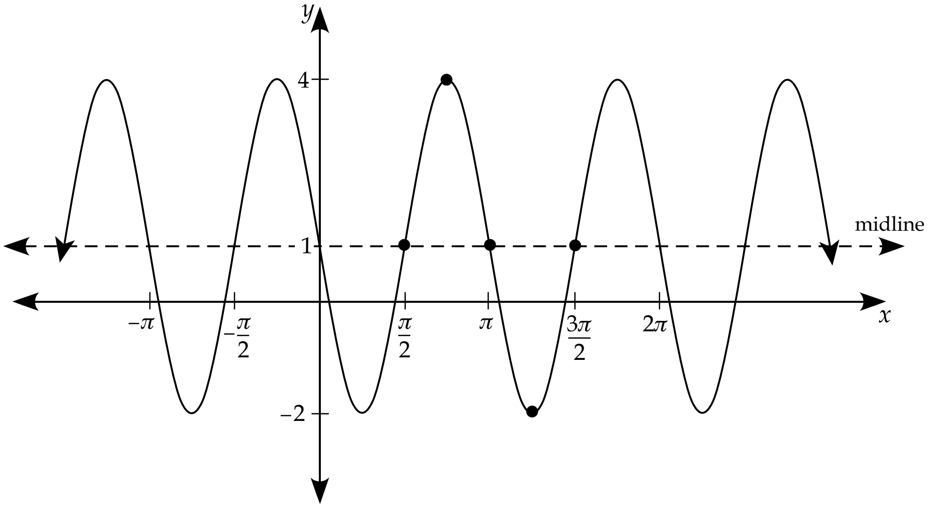
k) $y = | \tan x |$



4. Sketch the following curves. State their properties.

a) $y = 3\sin 2\left(x - \dfrac{\pi}{2}\right) + 1$

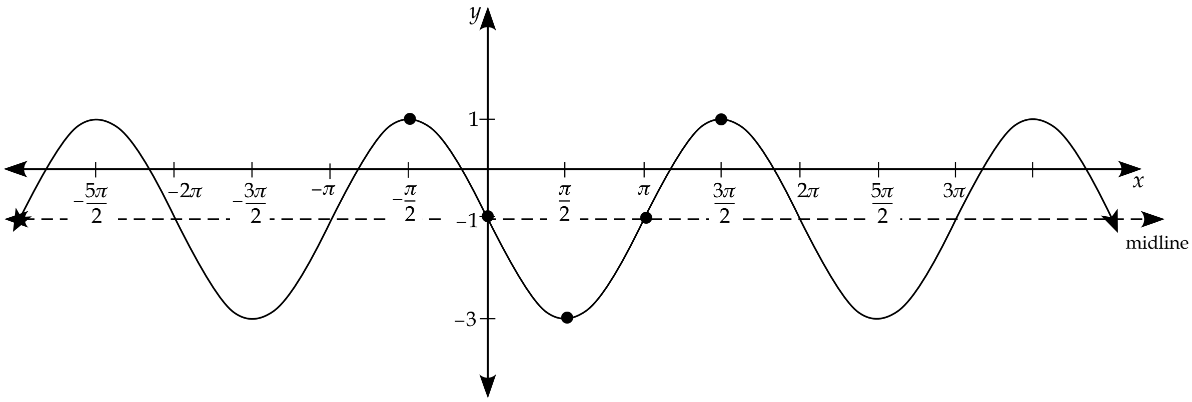
Answer:



Property	$y = 3\sin 2\left(x - \frac{\pi}{2}\right) + 1$	$f(x) = a\sin b(x + c) + d$
Period	$\frac{2\pi}{2} = \pi$	$\frac{2\pi}{\lvert b \rvert}$
Amplitude	3	$\lvert a \rvert$
Phase Shift (Horizontal)	$\frac{\pi}{2}$	$-c$
Maximum	4	$d + \lvert a \rvert$
Minimum	−2	$d - \lvert a \rvert$
Range	[−2, 4]	[min, max]
Domain	$\Re$	$\Re$

b) $y = 2\cos\left(x + \frac{\pi}{2}\right) - 1$

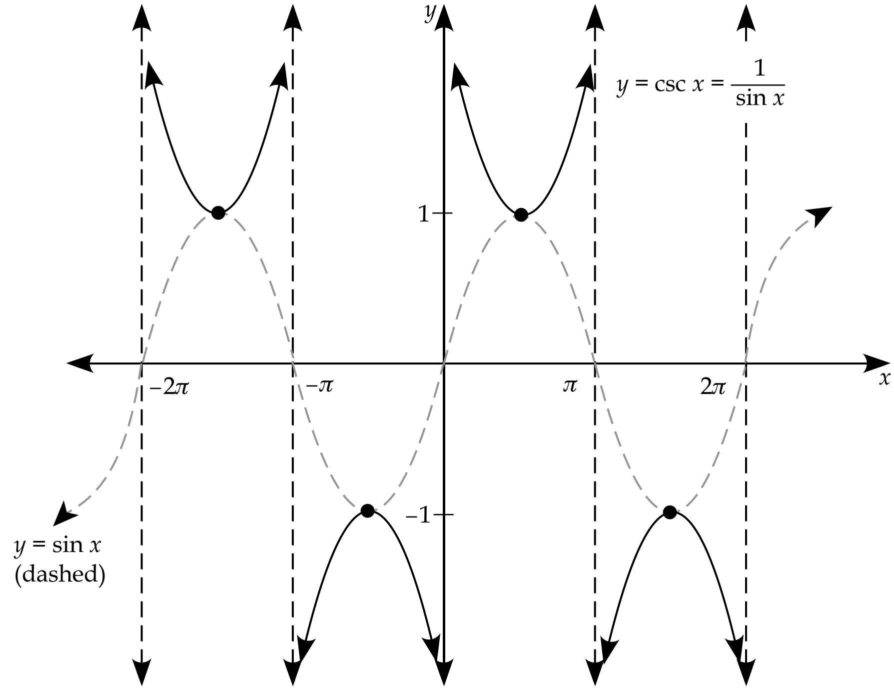
Answer:

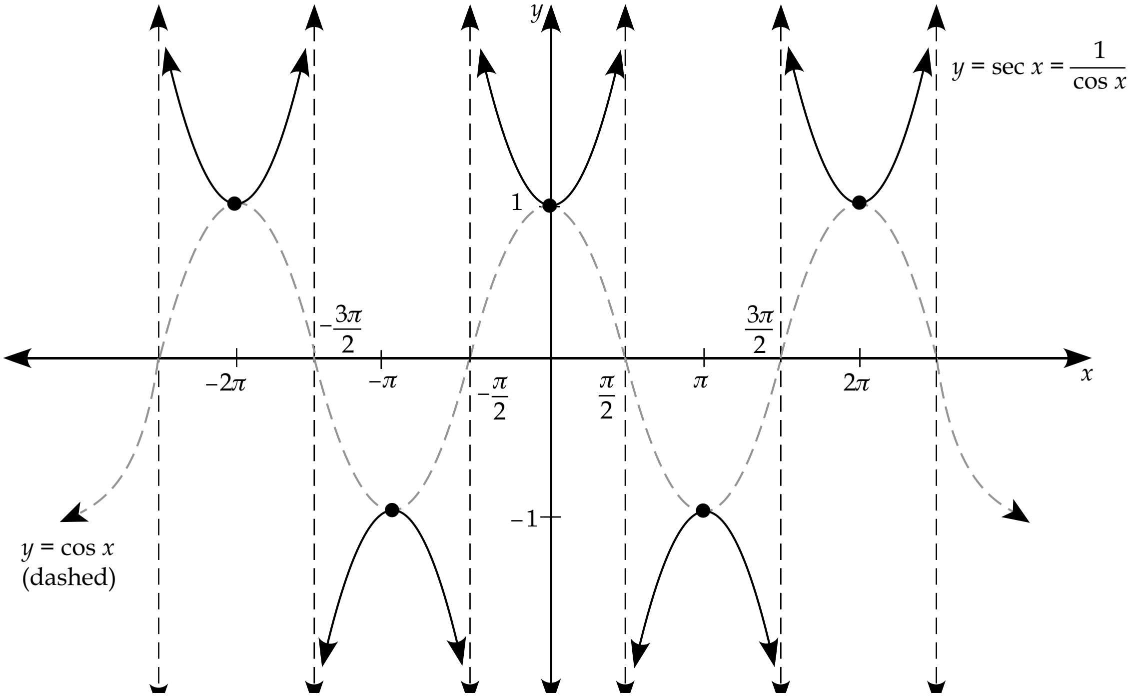


Property	$y = 2\cos\left(x + \frac{\pi}{2}\right) - 1$	$y = a\cos b(x + c) + d$
Period	$\frac{2\pi}{1} = 2\pi$	$\frac{2\pi}{\lvert b \rvert}$
Amplitude	2	$\lvert a \rvert$
Phase Shift (Horizontal)	$-\frac{\pi}{2}$	$-c$
Maximum	1	$d + \lvert a \rvert$
Minimum	−3	$d - \lvert a \rvert$
Range	[−3, 1]	[min, max]
Domain	$\Re$	$\Re$

5. $y = \sec \theta$ and $y = \csc \theta$ are reciprocals of one of the basic three circular functions. Use the graphs of the basic functions to sketch their reciprocals. Recall: Plot invariant points where $y = \pm 1$ and vertical asymptotes at x-intercepts.

Answer:





6. For each of the functions in the previous question, list all the properties of each function (domain, range, *y*-intercept, zeros, equations of asymptotes, and period).

   Answer:

Property	**csc x**	**sec x**
Domain	$\{x \mid x \neq k\pi, k \in \mathrm{I}\}$	$\left\{x \mid x \neq (2k+1)\frac{\pi}{2}, k \in \mathrm{I}\right\}$
Range	$(-\infty, -1] \cup [1, \infty)$	$(-\infty, -1] \cup [1, \infty)$
y-intercept	none	1
Zeros	none	none
Asymptotes	$\{x = k\pi, k \in \mathrm{I}\}$	$\left\{x \mid x = (2k+1)\frac{\pi}{2}, k \in \mathrm{I}\right\}$
Period	2π	2π

7. The graphs of $y = \sin(x - \pi)$ and $y = \sin(x + \pi)$ are identical. Explain.

   Answer:

   The graph of $y = \sin(x - \pi)$ is the standard sine graph moved π units to the right. The graph of $y = \sin(x + \pi)$ is the standard sine graph moved π units to the left. The difference between the starting points of these two graphs is thus $\pi - (-\pi) = 2\pi$. This is the same as the period of the sine function. Therefore, when $y = \sin(x + \pi)$ ends one period, the graph $y = \sin(x - \pi)$ starts one period. These graphs thus become identical.

8. Write an equation that describes the sine curve in terms of the cosine curve.

   Answer:

   It can be said that the cosine curve is the sine curve out of phase by $\frac{\pi}{2}$. For example, the cosine curve $y = \cos\left(x - \frac{\pi}{2}\right)$ is exactly the same as the sine curve $y = \sin x$.

9. Find the amplitude, period, horizontal shift, and vertical shift for each of the following:

   a) $y = 2\sin 3(x - 4) + 1$

   b) $y = -3\cos 5x$

   c) $y = 4\cos \pi x$

   d) $y = 7\cos 2\left(x - \dfrac{\pi}{2}\right) + 3$

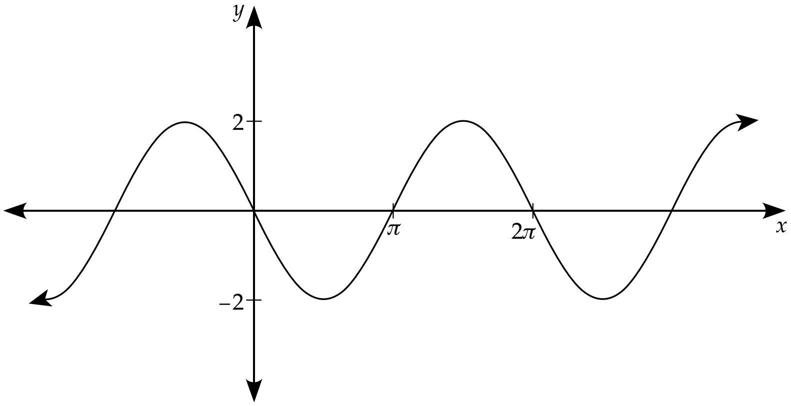
   e) $y = -5\sin 2\pi(x - 1) - 4$

   Answers:

Question	**Amplitude**	**Period**	**Horizontal Shift**	**Vertical Shift**
(a)	2	$\frac{2\pi}{3}$	4	1
(b)	3	$\frac{2\pi}{5}$	0	0
(c)	4	$\frac{2\pi}{\pi} = 2$	0	0
(d)	7	$\frac{2\pi}{2} = \pi$	$\frac{\pi}{2}$	3
(e)	5	$\frac{2\pi}{2\pi} = 1$	1	-4

10. For each of the following graphs of $y = a\sin b(x - c) + d$, find the values of a, b, c, and d.

    a)

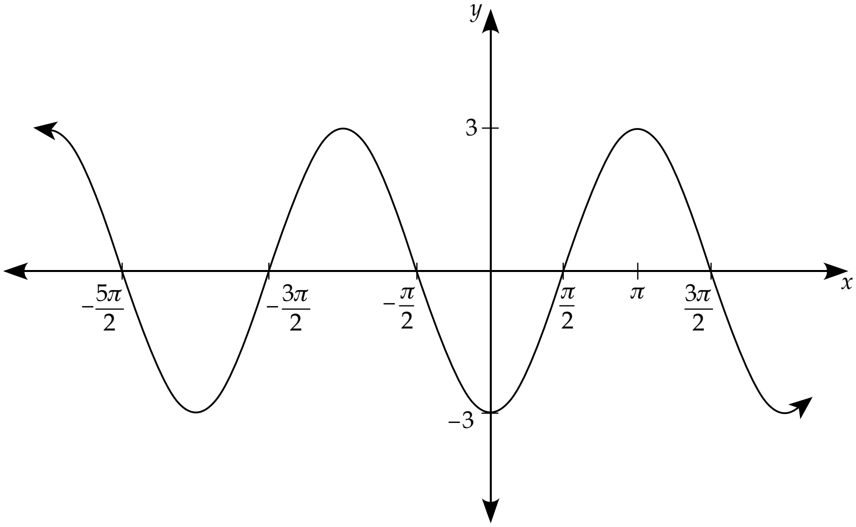


Answer:

Note: The period is 2π and $b = \dfrac{2\pi}{\text{period}}$.

$a = -2 \qquad b = 1 \qquad c = 0 \qquad d = 0$

b)

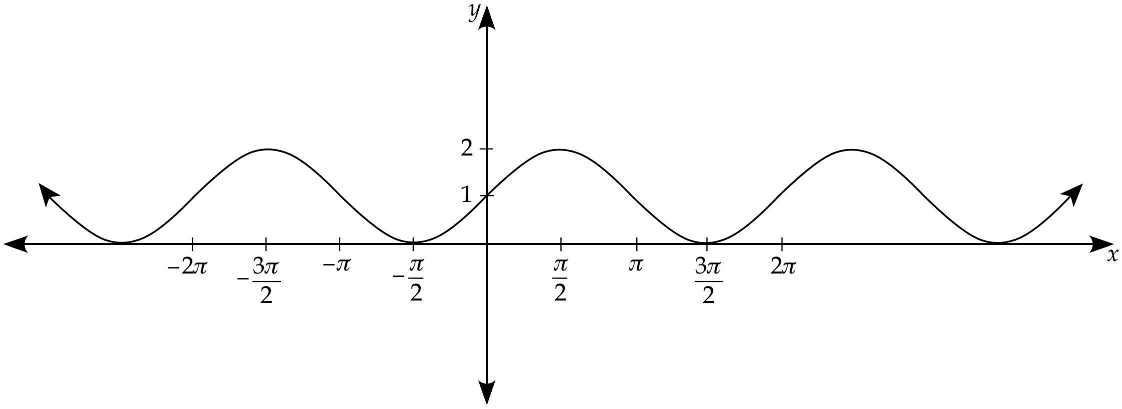


Answer:

Note: The period is 2π and $b = \frac{2\pi}{\text{period}}$.

$a = 3$ $b = 1$ $c = \frac{\pi}{2}$ $d = 0$

c)



Answer:

Note: The period is 2π and $b = \frac{2\pi}{\text{period}}$.

$a = 1$ $b = 1$ $c = 0$ $d = 1$

11. Use the same graphs as in the previous question, but change the function to $y = a \cos b(x - c) + d$ and find the values of a, b, c, and d.

Answers:

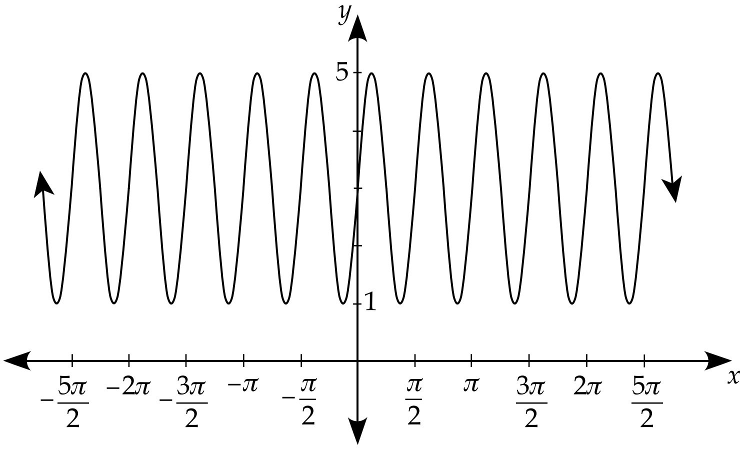
a) $a = 2$ $b = 1$ $c = -\frac{\pi}{2}$ $d = 0$

b) $a = -3$ $b = 1$ $c = 0$ $d = 0$

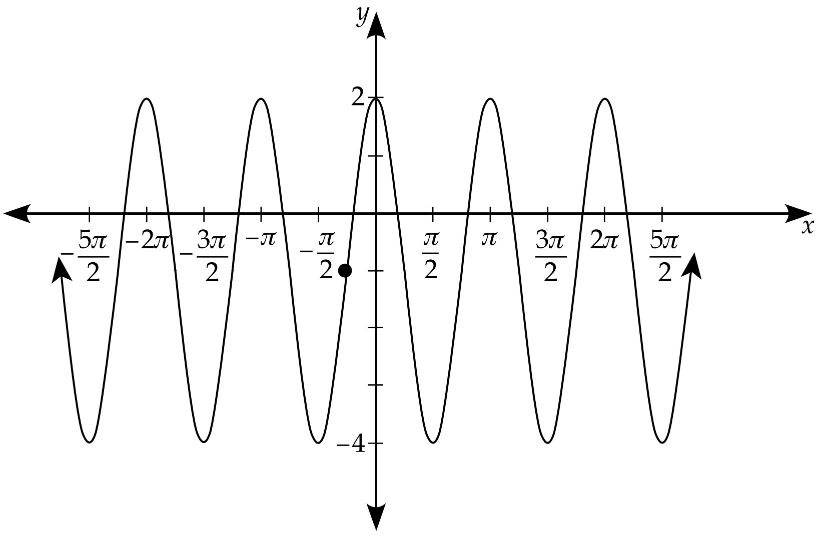
c) $a = 1$ $b = 1$ $c = \frac{\pi}{2}$ $d = 1$

12. Find an equation of each of the following graphs as sine functions.

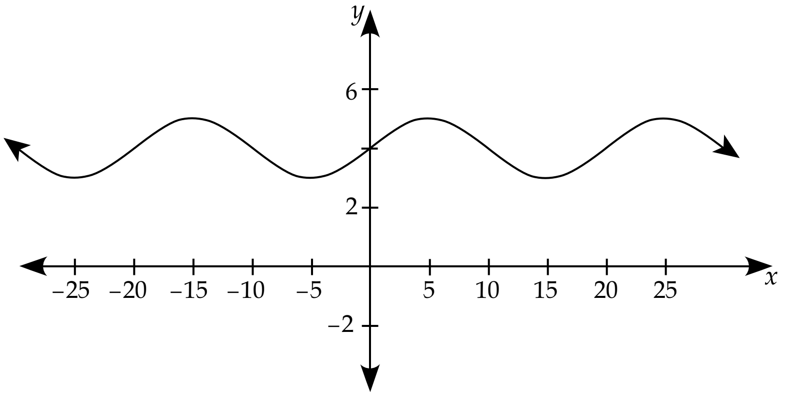
a)



b)



c)

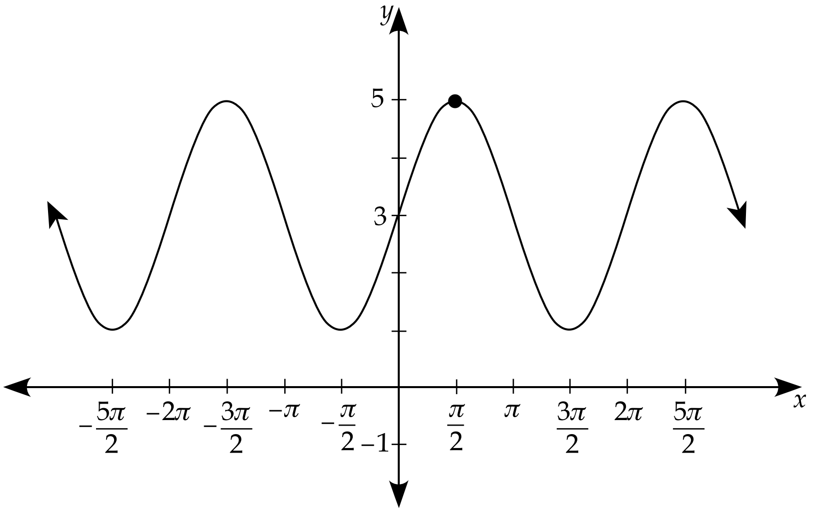


Answers:

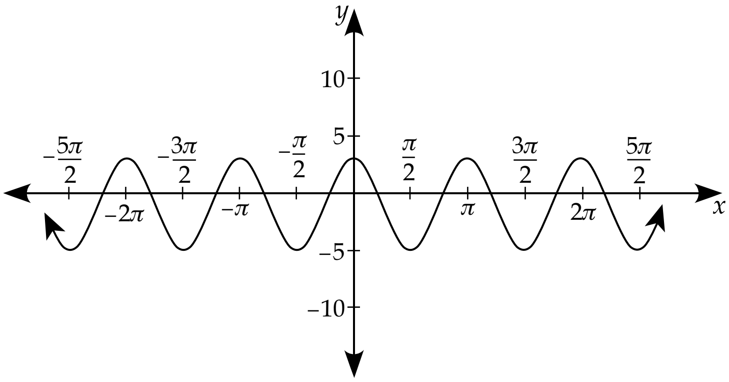
Question	Amplitude	Period	Horizontal Shift	Vertical Shift	Equation
(a)	2	$\frac{\pi}{2}$	0	3	$y = 2\sin 4x + 3$
b)	3	π	$-\frac{\pi}{4}$	−1	$y = 3\sin 2\left(x + \frac{\pi}{4}\right) - 1$
(c)	1	20	0	4	$y = \sin\left(\frac{\pi}{10}x\right) + 4$

13. Find an equation of each of the following graphs as cosine functions.

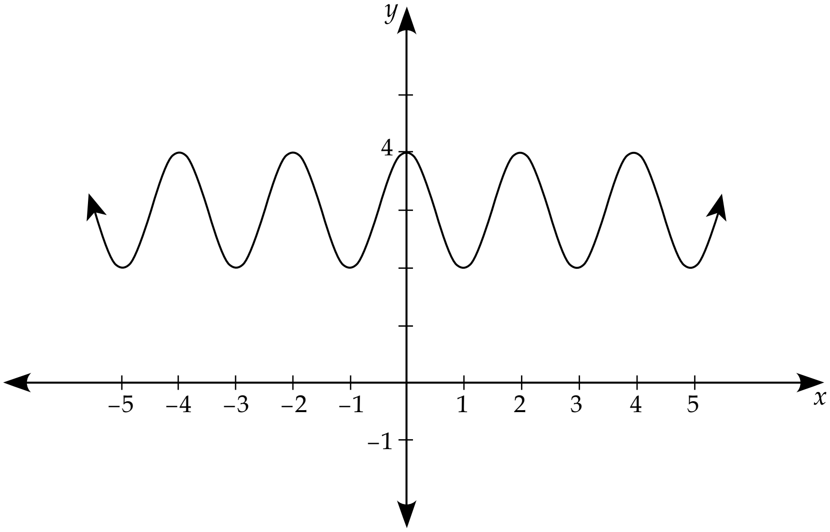
a)



b)



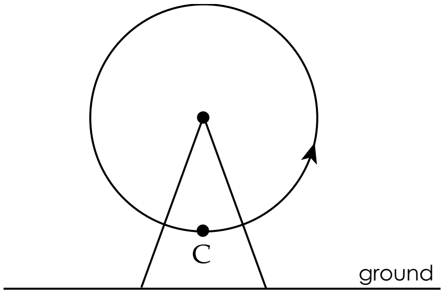
c)



Answers:

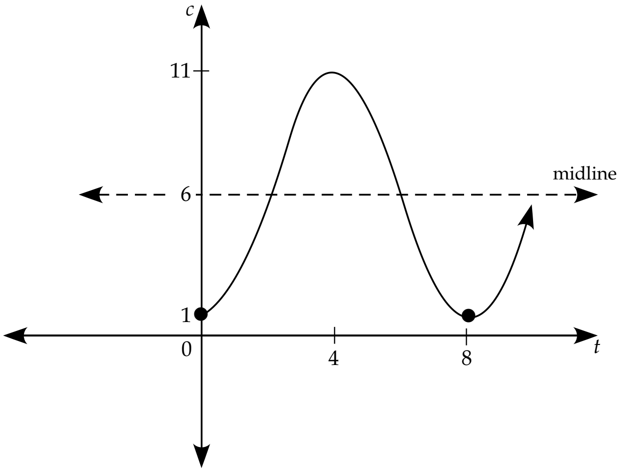
Question	Amplitude	Period	Horizontal Shift	Vertical Shift	Equation
(a)	2	2π	$\frac{\pi}{2}$	3	$y = 2\cos\left(x - \frac{\pi}{2}\right) + 3$
b)	4	π	0	−1	$y = 4\cos 2x - 1$
(c)	1	2	0	3	$y = \cos \pi x + 3$

14. A Ferris wheel is 1 metre from the ground. The distance of a chair on the Ferris wheel from the ground varies sinusoidally with time. Suppose the diameter of a Ferris wheel is 10 m and the top point the chair reaches is 11 m. The wheel makes one counter-clockwise revolution every eight seconds. The Ferris wheel starts with chair *C* at the lowest position.



a) Sketch the function *C*(*t*) and write a formula *C*(*t*), in terms of cosine, which states the height of chair *C* above the ground *t* seconds after the Ferris wheel started.

Answer:



$$C(t) = -5\cos\left(\frac{\pi}{4}t\right) + 6$$

or

$$C(t) = 5\cos\frac{\pi}{4}(t-4) + 6$$

b) What is the lowest that any chair is above the ground?

Answer:

The lowest any chair is above the ground is 1 metre.

c) State the height of chair C above the ground when $t = 3$, 4, and 10 seconds.

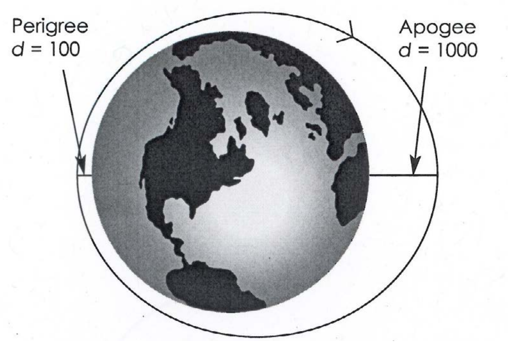
Answer:

$$
\begin{aligned}
C(3) &= -5\cos\left[\left(\frac{\pi}{4}\right)(3)\right] + 6 \\
&= -5\cos\left(\frac{3\pi}{4}\right) + 6 \\
&= -5\left(-\frac{\sqrt{2}}{2}\right) + 6 \\
&= \frac{5\sqrt{2}}{2} + \frac{12}{2} \\
&= \frac{5\sqrt{2} + 12}{2} \\
&\approx 9.54 \text{ m}
\end{aligned}
$$

$$
\begin{aligned}
C(4) &= -5\cos\left[\left(\frac{\pi}{4}\right)(4)\right] + 6 \\
&= -5\cos(\pi) + 6 \\
&= -5(-1) + 6 \\
&= 5 + 6 \\
&= 11 \text{ m}
\end{aligned}
$$

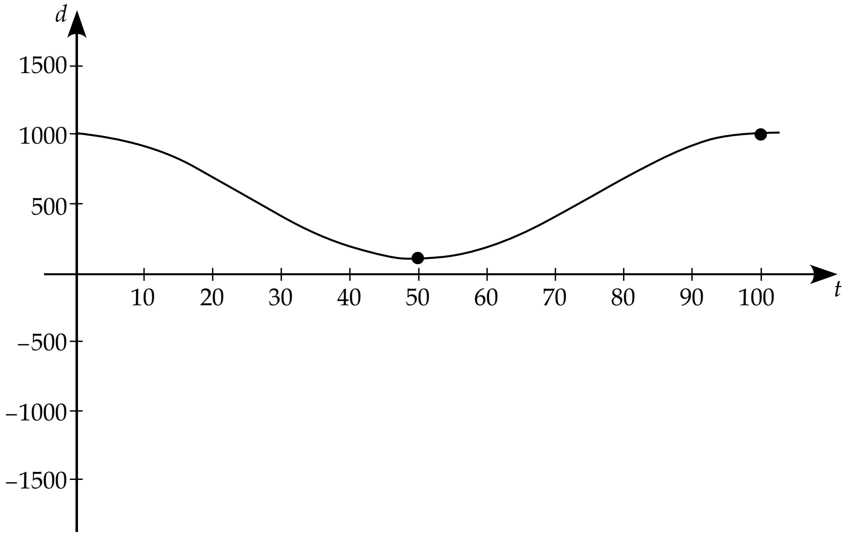
$$
\begin{aligned}
C(10) &= -5\cos\left[\left(\frac{\pi}{4}\right)(10)\right] + 6 \\
&= -5\cos\left(\frac{5\pi}{2}\right) + 6 \\
&= -5(0) + 6 \\
&= 0 + 6 \\
&= 6 \text{ m}
\end{aligned}
$$

15. A spacecraft is in an elliptical orbit around Earth, as shown in the diagram. At time $t = 0$ hours, it is at its highest point $d = 1000$ kilometres above Earth's surface. Fifty minutes later, it is at its lowest point, $d = 100$ kilometres above the surface.



a) Assuming that d varies sinusoidally with time, draw the sketch of the sinusoidal function.

Answer:



b) Write an equation expressing d in terms of t.

Answer:

Using the cosine curve as a model:

$$a = \frac{1000 - 100}{2} = 450$$

$$b = \frac{2\pi}{100} = \frac{\pi}{50}$$

$$c = 0$$

$$d = (100 + 1000) \div 2 = 550$$

The equation in terms of cosine is $d = 450 \cos \frac{\pi}{50}(t) + 550$.

c) In order to transmit information back to Earth, the spacecraft must be within 700 kilometres of the surface. For how many consecutive minutes will the spacecraft be able to transmit?

Answer:

$$700 = 450 \cos \frac{\pi}{50}(t) + 550$$

$$150 = 450 \cos \frac{\pi}{50}(t)$$

$$\frac{1}{3} = \cos \frac{\pi}{50}(t)$$

$$\cos^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{50}(t)$$

$$1.23096 = \frac{\pi}{50}t$$

$$t = 19.59134$$

At $t = 19.59134$ minutes, the spacecraft gets close enough to Earth to begin transmitting information.

Now, you need to determine the minute when the spacecraft gets too far away from Earth to transmit information. When you took the inverse cosine of $\frac{1}{3}$, you calculated a value of 1.23096 radians, which is an angle located in Quadrant I of the unit circle. Cosine is also positive in Quadrant IV of the unit circle. An angle that is related with the above angle in Quadrant IV is $2\pi - 1.23096 = 5.05223$

Now you can solve the following equation to determine when the spacecraft is too far away from Earth to transmit.

$$5.05223 = \frac{\pi}{50}t$$

$$t = 80.40874$$

Subtract the initial time from the final time to discover how many minutes the spacecraft is close enough to Earth to transmit information.

$$80.40874 - 19.59134 = 60.8174$$

Therefore, the spacecraft can transmit information to Earth for approximately 61 consecutive minutes.

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Module 6
Trigonometric Equations and Identities

MODULE 6:
TRIGONOMETRIC EQUATIONS AND IDENTITIES

Introduction

In the last module, you were introduced to the concept of radians and the six trigonometric functions. As well, you were shown how to graph trigonometric functions using transformations. In this module, you will combine all of these concepts in order to solve trigonometric equations and prove various trigonometric identities. In order to find the zeros of various trigonometric functions or to solve the corresponding trigonometric equations, you will develop skills in circular function equation solving. You will also develop relationships amongst these functions called identities.

Identities can be useful as they have the power to reduce what may seem like a very complex expression down to a simpler expression that may involve only one trigonometric ratio. These identities are all constructed from the unit circle. Therefore, you must make sure that you are comfortable dealing with the unit circle and all the concepts discussed in Module 5 before you begin this module!

Assignments in Module 6

When you have completed the assignments for Module 6, submit your completed assignments to the Distance Learning Unit either by mail or electronically through the learning management system (LMS). The staff will forward your work to your tutor/marker.

Lesson	Assignment Number	Assignment Title
1	Assignment 6.1	Solving Trigonometric Equations
3	Assignment 6.2	Using Elementary Identities
5	Assignment 6.3	Sum and Difference and Double Angle Identities

Resource Sheet

When you write your final examination, you are encouraged to take a Final Examination Resource Sheet with you into the examination. This sheet will be one letter-sized page, 8½ ″ by 11″, with both sides in your handwriting or typewritten. You will submit it with your examination, but you do not receive any marks for it.

Many students have found that preparing a resource sheet is an excellent way to review. It provides you with a summary of the important facts of each module. You should complete a resource sheet for each module to help with your studying and reviewing. Lesson summaries and module summaries are included for you to use as a guide.

You may use the list of instructions provided below to help you with preparing your resource sheet for the material in Module 6. On this sheet, you should record math terms and definitions, formulas, sample questions, or a list of places where you often make mistakes. You should also identify special areas that require extra attention or review by writing the page numbers.

After you have completed each module's resource sheet, you may summarize the sheets from Modules 1 to 8 to prepare your Final Examination Resource Sheet. The final examination for this course is based on Modules 1 to 8.

Resource Sheet for Module 6

As you go through the lessons of this module, you may want to consider the following suggestions regarding the creation of a resource sheet.

1. List all the important math terms, and define them if necessary.
2. List all the formulas and perhaps a sample problem that shows how the formula is used.
3. If necessary, write the solutions to some problems, showing in detail how you did the calculations.
4. Copy any questions that represent the key points of the lesson, and perhaps include the solutions as well.
5. Identify the problems you found most difficult, and copy the page numbers onto the resource sheet so that you can review them before writing the examination. You may also copy the problems and the solutions onto your resource sheet, and later write them onto your Final Examination Resource Sheet.
6. Write any comments, ideas, shortcuts, or other reminders that may be helpful during an examination.

LESSON 1: TRIGONOMETRIC EQUATIONS

Lesson Focus

In this lesson, you will

☐ learn how to solve first- and second-degree trigonometric equations

☐ learn how to express the solution of a trigonometric equation in general form

☐ learn about how the solution of a trigonometric equation is related to the zeros of the corresponding trigonometric function

Lesson Introduction

You were introduced to trigonometric equations in Module 5. In Module 5, you were asked to solve linear trigonometric equations where you found an angle that corresponded to a certain trigonometric ratio on the unit circle. In this lesson, you will learn how to solve quadratic trigonometric equations. You will also learn how to put these solutions in general form, as there may be an infinite number of possible solutions for any given equation.

Trigonometric Equations

A **trigonometric equation** is an equation that contains one or more trigonometric functions. In this course, you will be studying linear and quadratic trigonometric equations.

Linear/First-Degree Trigonometric Equations

Linear trigonometric equations are also called first-degree trigonometric equations. These trigonometric equations contain trigonometric functions to the first power only. For example, $\sin x = 2$ is an example of a linear trigonometric equation. However, $\sin^2 x = 2$ is not an example of a linear trigonometric equation.

You solved equations of this type in Module 5, where the solutions were exact answers and where the solutions were decimal approximations.

Example 1

Solve the following equations over the given intervals. Give exact solutions when possible. Round to 3 decimal places when necessary.

a) $\cos\theta = \frac{1}{2}, 0 \le \theta < 2\pi$

b) $2\cot\theta + \frac{2}{\sqrt{3}} = 0, 0 \le \theta < 2\pi$

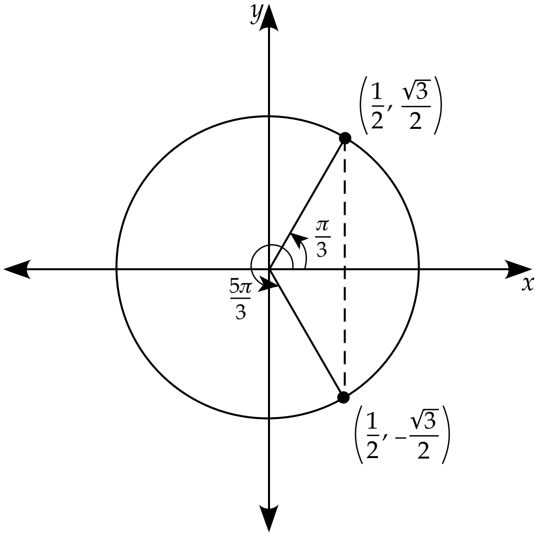
c) $\cos x = 0.675, 0 \le \theta < 2\pi$

d) $\sin x = 0.123, 0° \le \theta < 360°$

Solutions

a) This question is asking, "What is the measure of the angle, in radians, that produces a cosine ratio of $\frac{1}{2}$?"

Looking at your unit circle, you can determine that the possible angle solutions for this question are $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$.



b) To solve this equation, isolate the trigonometric function $\cot\theta$.

$$2\cot\theta = -\frac{2}{\sqrt{3}}$$

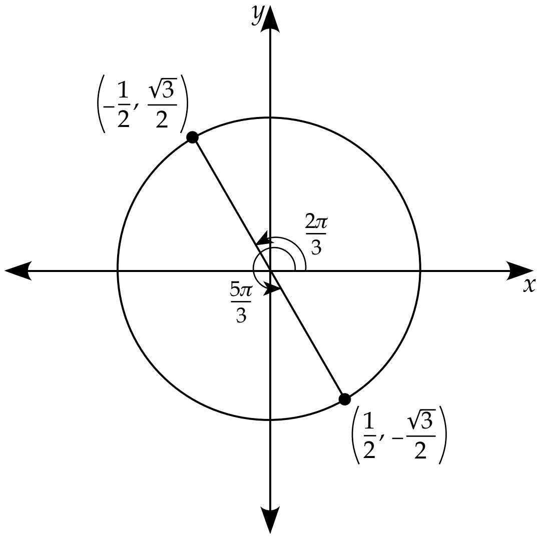
$$\cot\theta = -\frac{1}{\sqrt{3}}$$

Now, use the identity $\cot\theta = \frac{1}{\tan\theta}$ to solve this equation.

$$\frac{1}{\tan\theta} = -\frac{1}{\sqrt{3}}$$

$$\tan\theta = -\sqrt{3}$$

$$\theta = \frac{2\pi}{3}, \frac{5\pi}{3}$$



c) For this question, since 0.675 is not one of the unit circle *x*-coordinates, it is not possible to know an exact answer. Therefore, you are going to need to use the inverse cosine function on your calculator to get an approximate answer. Make sure your calculator is in radians, as you are asked to find an angle between 0 and 2π.

$\cos\theta = 0.675$

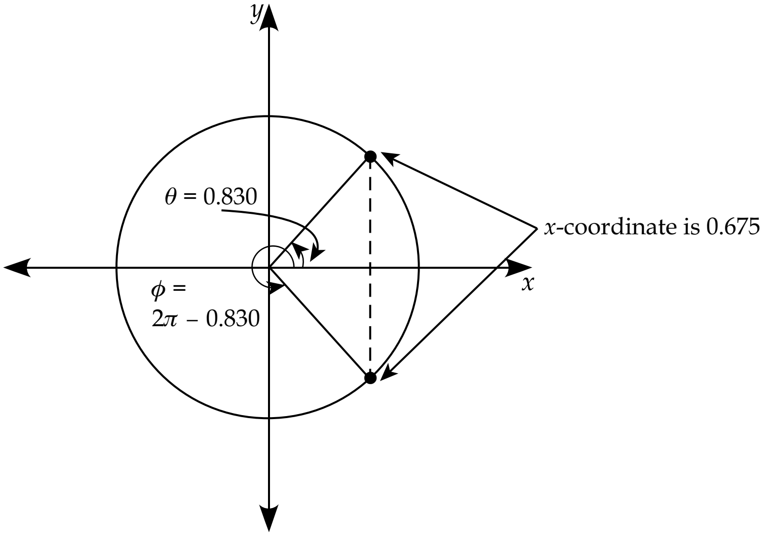
$\theta = \cos^{-1}(0.675)$

$\theta = 0.830$

The cosine ratio is positive in Quadrants I and IV. Thus, you need to find one solution in each of these quadrants.

Quadrant I: $\theta = 0.830$

Quadrant IV: $\phi = 2\pi - 0.830 = 5.432$



d) Make sure your calculator is in degrees for this question, as you need to find an angle between 0° and 360°.

$\sin \theta = 0.123$

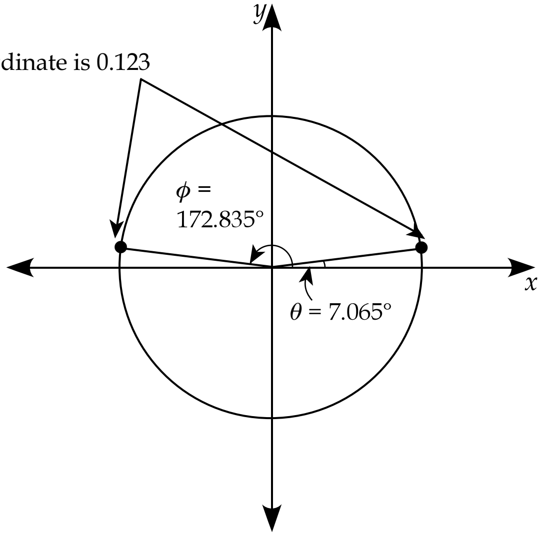
$\theta = \sin^{-1}(0.123)$

$\theta = 7.065°$

The sine ratio is positive in Quadrants I and II.

Quadrant I: $\theta = 7.065°$

Quadrant II: $\phi = 180° - 7.065° = 172.935°$



Example 2

Using reference angles, find solutions for the following angles that are not one of the points on the unit circle.

a) $3 \sin x = -2$

b) $7 \cos \theta + 5 = 3 \cos \theta + 2$

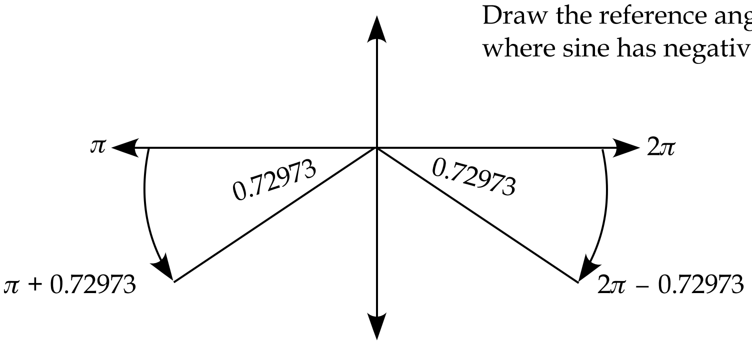
Solutions

a) $3 \sin x = -2$ $\quad$ $0° \leq \theta \leq 2\pi$

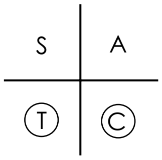
$\sin x = \frac{-2}{3}$ $\quad$ Isolate the trigonometric function ($\sin x$).

$x_r = \sin^{-1}\left(\frac{2}{3}\right)$ $\quad$ Find reference angle (ignore the negative).

$x_r = 0.72973$



Draw the reference angle in the quadrants where sine has negative values.



$x = \pi + 0.72973$

$x = 2\pi - 0.72973$

$x = 2.412$

$x = 5.553$

b) $7 \cos \theta + 5 = 3 \cos \theta + 2$ $\quad 0° \le \theta \le 360°$

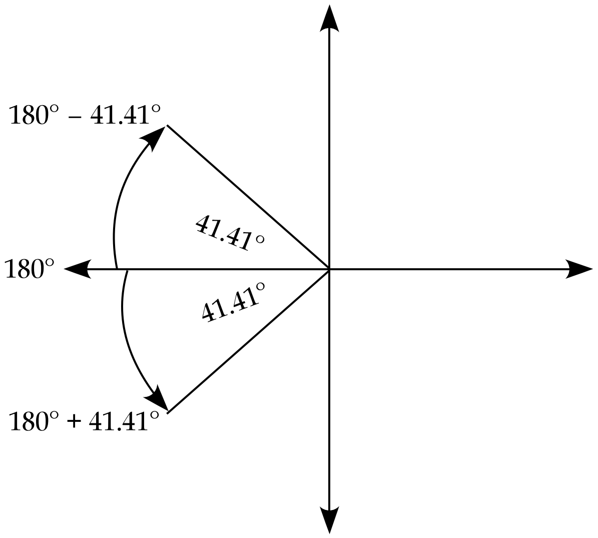
$4 \cos \theta = -3$

$\cos \theta = \dfrac{-3}{4}$ Isolate $\cos \theta$.

$\theta_r = \cos^{-1}\left(\dfrac{3}{4}\right)$

$\theta_r = 41.41°$

Find the reference angle (ignore the negative sign at this stage).



Draw the reference angle in the quadrants where cosine has negative values.

$\theta = 180° - 41.41° = 138.59°$

$\theta = 180° + 41.41° = 221.41°$

General Solution to a Trigonometric Equation

In the previous examples, you solved a trigonometric equation over a restricted interval. If the domain of θ is not restricted, then you are finding a general solution where the domain is $\theta \in \Re$.

What happens if you are asked for all the solutions to a given trigonometric equation? To find a general solution to a trigonometric equation, you need to first determine all the solutions in one rotation of the unit circle, or between 0 and 2π radians or 0° and 360°. Once you have found all of these solutions, using the concept of coterminal angles, you can express the general solution to the equation. Consider the following example.

Example 3

Find the general solution for each of the following trigonometric equations. You can do this without a calculator since these angles of rotation are the values known to you on the unit circle.

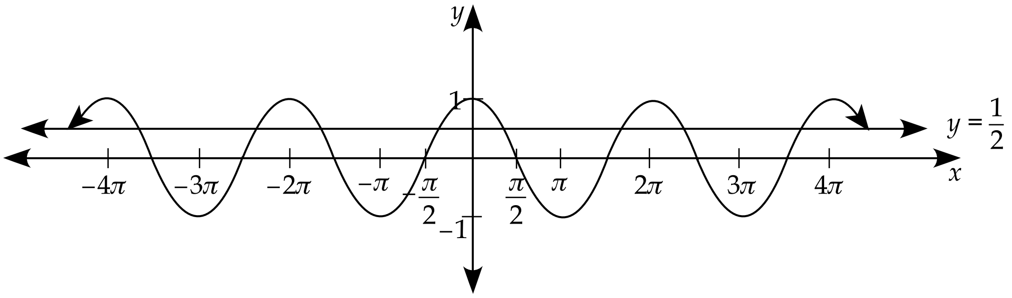
a) $\cos\theta = \frac{1}{2}$

b) $\tan\theta = \frac{\sqrt{3}}{3}$

c) $\csc\theta = -\frac{2}{\sqrt{3}}$

Solutions

a) To think about what is happening, graph $y = \cos\theta$ for $-4\pi \le \theta \le 4\pi$ and graph $y = \frac{1}{2}$ to help solve this graphically.



Note that the line $y = \frac{1}{2}$ crosses the graph of $y = \cos x$ eight times. If the graph of $y = \cos x$ were extended in both directions, there would be an infinite number of solutions. Next, you will see how to represent the infinite solutions that exist for $\theta \in \Re$.

First, determine the solution to this equation in the interval $0 \le \theta < 2\pi$.

$$\cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3} \text{ and } \frac{5\pi}{3}$$

Now, express each of the angles that are coterminal with the above angles in general form.

$$\theta = \frac{\pi}{3}, \text{ or } \frac{\pi}{3} + 2\pi, \text{ or } \frac{\pi}{3} + 4\pi, \text{ or } \frac{\pi}{3} + 6\pi, \ldots$$

$$\text{Also, } \theta = \frac{\pi}{3} - 2\pi, \text{ or } \frac{\pi}{3} - 4\pi, \text{ or } \frac{\pi}{3} - 6\pi, \ldots$$

This can be written more concisely by noticing that you need to start with $\frac{\pi}{3}$ and add or subtract multiples of 2π.

$$\theta = \frac{\pi}{3} + 2\pi n,\ n \in \text{I}$$

A similar statement can be written with the angle $\theta = \frac{5\pi}{3}$.

$$\theta = \frac{5\pi}{3} + 2\pi n,\ n \in \text{I}$$

Since n can be any integer (positive or negative), you have now represented the infinite number of solutions or the "general solution."

b) First, determine the solution to this equation in the interval $0 \le \theta < 2\pi$.

$$\tan\theta = \frac{\sqrt{3}}{3}$$

$$\theta = \frac{\pi}{6} \text{ and } \frac{7\pi}{6}$$

Now, express each of the angles that are coterminal with the above angles in general form.

$$\theta = \frac{\pi}{6}, \text{ or } \frac{\pi}{6} + 2\pi, \text{ or } \frac{\pi}{6} + 4\pi, \text{ or } \frac{\pi}{6} + 6\pi, \ldots$$

$$\text{Also, } \theta = \frac{\pi}{6} - 2\pi, \text{ or } \frac{\pi}{6} - 4\pi, \text{ or } \frac{\pi}{6} - 6\pi, \ldots$$

This can be written more concisely by noticing that you need to start with $\frac{\pi}{6}$ and add multiples of 2π.

$$\theta_c = \frac{\pi}{6} + 2\pi n,\ n \in \mathrm{I}$$

$$\theta_c = \frac{7\pi}{6} + 2\pi n,\ n \in \mathrm{I}$$

You can reduce these two equations down to one equation by noticing that $\frac{\pi}{6}$ and $\frac{7\pi}{6}$ are π radians away from each other. Therefore, you can express all the angles that are coterminal with $\frac{\pi}{6}$ and $\frac{7\pi}{6}$ as:

$$\theta = \frac{\pi}{6} + \pi n,\ n \in \mathrm{I}$$

This equation is thus the general solution to the above trigonometric equation.

Note: When $n = 1,\ \theta = \frac{7\pi}{6}$.

c) Solve $\csc\theta = -\frac{2}{\sqrt{3}}$ over the interval $0 \le \theta < 2\pi$.

$$\csc\theta = -\frac{2}{\sqrt{3}}$$

$$\frac{1}{\sin\theta} = -\frac{2}{\sqrt{3}}$$

$$\sin\theta = -\frac{\sqrt{3}}{2}$$

Without a calculator, using your unit circle values, you can determine the values of θ.

$$\theta = \frac{4\pi}{3} \text{ and } \frac{5\pi}{3}$$

To determine the general solution to the above equation, express each of the angles that are coterminal with the above angles in general form.

$$\theta = \frac{4\pi}{3} + 2\pi n,\ n \in \mathrm{I}$$

$$\theta = \frac{5\pi}{3} + 2\pi n,\ n \in \mathrm{I}$$

Note: These angles are not π radians away from each other. Therefore, the above technique of combining the two equations into one equation does not apply.

Remember, the period of the tangent and cotangent functions is π. The period of the other four functions is 2π. You may want to sketch the graphs of the functions to see what is happening.

Quadratic/Second-Degree Trigonometric Equations

Quadratic trigonometric equations are also called second-degree trigonometric equations. These trigonometric equations contain trigonometric functions to the second power. For example, $\cos^2 x = 0$ and $\sin^2 x + \sin x + 1$ are examples of quadratic trigonometric equations.

It is possible to solve these types of equations by factoring. Consider the following example.

Example 4

Solve the following equations over the given intervals. Express your answer as an exact solution.

a) $\tan^2 \theta - 1 = 0,\ 0 \le \theta < 2\pi$

b) $4 \cos^2 \theta = 1,\ -\pi \le \theta < \pi$

c) $2 \sin^2 \theta + \sin \theta = 0,\ 0 \le \theta < 2\pi$

Solutions

a) To solve this type of equation, it is sometimes easier to substitute a variable in for the trigonometric function.

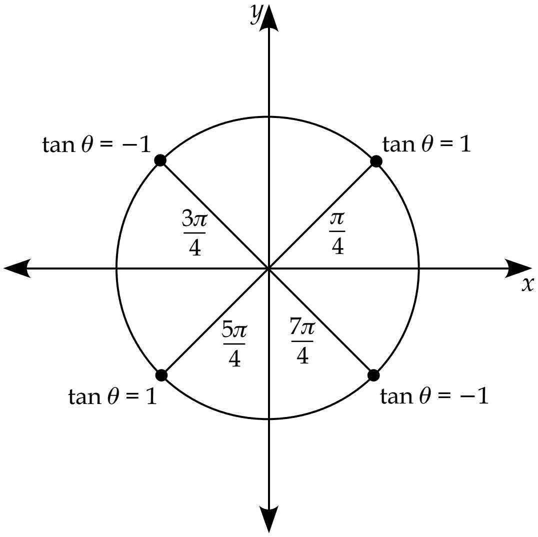
Let $w = \tan \theta$.

$$\begin{aligned} \tan^2 \theta - 1 &= 0 \\ w^2 - 1 &= 0 \\ (w - 1)(w + 1) &= 0 \\ w &= 1, -1 \end{aligned}$$

Substitute $\tan \theta$ back in for w.

$\tan \theta = 1$ and $\tan \theta = -1$

Solve both of the above equations over the interval $0 \le \theta < 2\pi$.



$\tan\theta = 1$ $\qquad$ $\tan\theta = -1$

$\theta = \frac{\pi}{4}$ and $\frac{5\pi}{4}$ $\qquad$ $\theta = \frac{3\pi}{4}$ and $\frac{7\pi}{4}$

b) Let $w = \cos\theta$.

$$4\cos^2\theta = 1$$

$$4w^2 = 1$$

$$4w^2 - 1 = 0$$

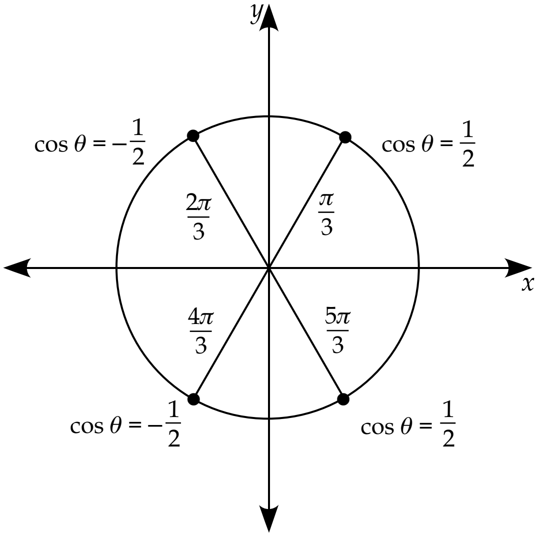
$$(2w-1)(2w+1) = 0$$

$$w = \frac{1}{2} \text{ and } -\frac{1}{2}$$

Substitute $\cos\theta$ back in for w.

$$\cos\theta = \frac{1}{2} \text{ and } \cos\theta = -\frac{1}{2}$$

Solve these equations over the interval $0 \le \theta < 2\pi$.



$$\cos\theta = \frac{1}{2} \qquad\qquad \cos\theta = -\frac{1}{2}$$

$$\theta = \frac{\pi}{3} \text{ and } \frac{5\pi}{3} \qquad\qquad \theta = \frac{2\pi}{3} \text{ and } \frac{4\pi}{3}$$

Checking the domain of the original question, you are asked to solve over the interval $-\pi \le \theta < \pi$

Notice that $\frac{5\pi}{3}$ is not in the interval $-\pi \le \theta < \pi$. Therefore, find a coterminal angle to $\frac{5\pi}{3}$ that is in the correct interval by subtracting 2π.

$$\theta = \frac{5\pi}{3} - 2\pi = \frac{5\pi}{3} - \frac{6\pi}{3} = -\frac{\pi}{3}$$

Similarly, find a coterminal angle to $\theta = \frac{4\pi}{3}$ that is in the correct interval.

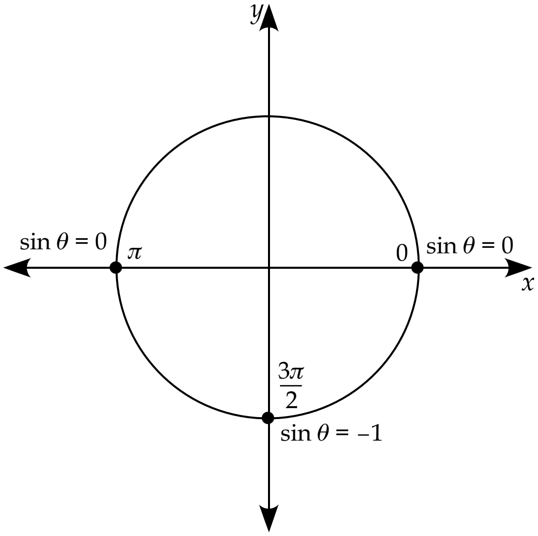
$$\theta = \frac{4\pi}{3} - \frac{6\pi}{3} = -\frac{2\pi}{3}$$

Therefore, the four solutions to this equation in the specified interval are $\theta = \frac{\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}$, and $-\frac{2\pi}{3}$.

c) Let $w = \sin\theta$.

$$\begin{aligned} 2\sin^2\theta + \sin\theta &= 0 \\ 2w^2 + w &= 0 \\ 2w(w+1) &= 0 \\ w &= 0, -1 \end{aligned}$$

$\sin\theta = 0$ and $\sin\theta = -1$



$\sin\theta = 0$	$\sin\theta = -1$
$\theta = 0$ and π	$\theta = \dfrac{3\pi}{2}$

The three solutions to this equation over the interval $0 \le \theta < 2\pi$ are $\theta = 0, \pi$, and $\dfrac{3\pi}{2}$.

Example 5

Solve the equation $\sin x \tan x + \sin x = 0$ over the interval $0 \le \theta \le \pi$.

Solution

This equation may seem difficult to solve. However, if you factor out $\sin x$, the equation reduces down to two equations that are fairly easy to solve.

$$\sin x \tan x + \sin x = 0$$

$$\sin x (\tan x + 1) = 0$$

$$\sin x = 0, \tan x = -1$$

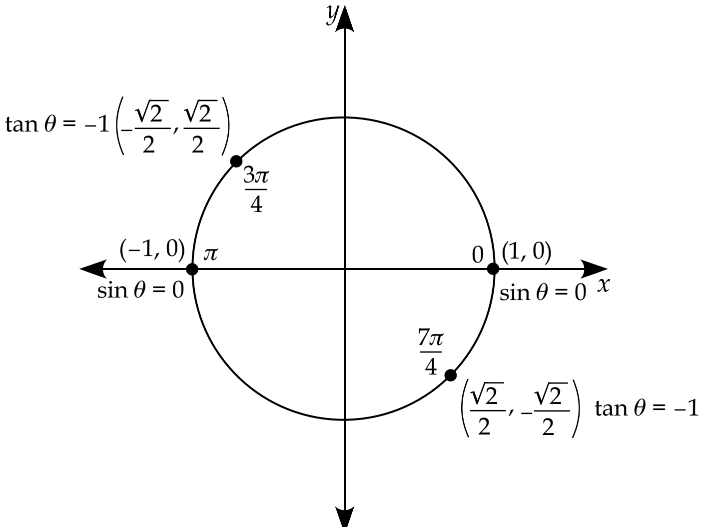
$\sin x = 0$

$x = 0$ and π

$\tan x = -1$

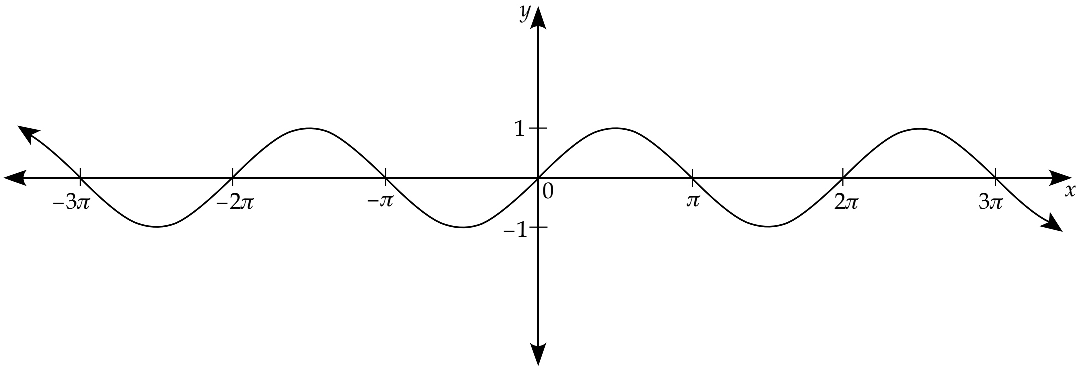
$x = \dfrac{3\pi}{4}$ and $x = \dfrac{7\pi}{4}$

The solutions that are in the correct interval are $x = 0$ and $x = \dfrac{3\pi}{4}$ and $x = \pi$.



Graphical Interpretation of Solutions to Trigonometric Equations

Consider the standard sine graph.



How does this graph relate to the equation $\sin \theta = 0$?

If you were to solve the equation $\sin \theta = 0$, the solutions would be $\theta = 0, \pi, 2\pi, 3\pi, \ldots$ and continue on indefinitely. The general solution would be $\theta = \pi n$, $n \in \mathrm{I}$.

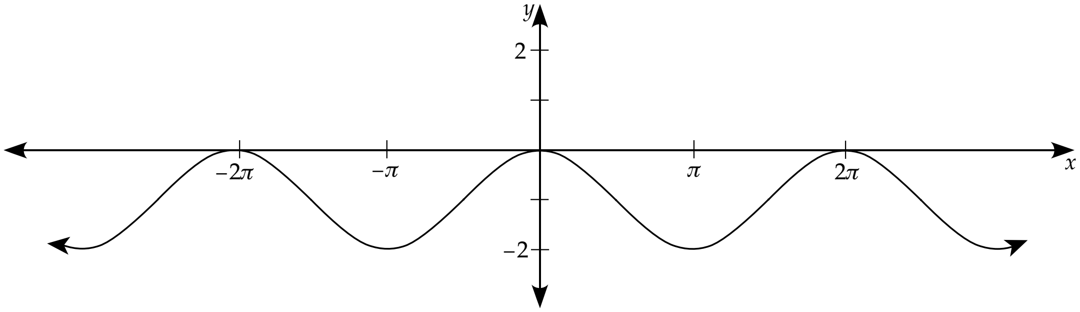
These solutions are the same as the zeros, or x-intercepts, on the graph of the function shown above, $y = \sin \theta$.

Example 6

Determine the zeros of the function $y = \cos\theta - 1$. Find the general solution to the equation $\cos\theta = 1$. Explain how the two solutions are related.

Solution

The function $y = \cos\theta - 1$ is the standard cosine function moved 1 unit down.



From the graph, you can see that the zeros of this function are all going to be multiples of 2π, or $\theta = 2\pi n$, $n \in \mathrm{I}$.

The solution to the equation $\cos\theta = 1$ in the interval $0 < \theta \leq 2\pi$ is 2π.

The general solution to the equation $\cos\theta = 1$ is $\theta = 2\pi n$, $n \in \mathrm{I}$.

The equations of the zeros of the function $y = \cos\theta - 1$ and the general solution to the equation $\cos\theta = 1$ are identical. This is because you are solving the same problem in two different ways. The equation, $\cos\theta = 1$, can be written as $\cos\theta - 1 = 0$. So, the solution to $\cos\theta - 1 = 0$ will be the same as the x-intercepts of the function $y = \cos\theta - 1$.

Thus, if you ever want to check to ensure your answer for the general solution of an equation is correct, you can graph the corresponding function and see where the zeros or x-intercepts of that function are located.

Learning Activity 6.1

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. What is the equation of the vertical asymptote of the function $y = \dfrac{1}{3x + 5}$?
2. What is the amplitude of the function $y = 3 \sin x - 4$?
3. Simplify: $\dfrac{6!}{3!2!}$
4. State an angle that is coterminal to 122°.
5. In which quadrant is $\theta = \dfrac{7\pi}{5}$ located?
6. State the non-permissible values of the function $f(x) = \dfrac{x}{2x - 8}$.
7. Rationalize the denominator: $\dfrac{6}{\sqrt{7}}$
8. Simplify: $\dfrac{12a^5d^3e^2}{8a^6d^4e}$

continued

Learning Activity 6.1 (continued)

Part B: Solving Trigonometric Equations

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Solve the following equations over the indicated intervals. Provide exact answers using your unit circle values wherever possible. Round to 3 decimal places when necessary.

   a) $2 \tan \theta + \frac{2}{\sqrt{3}} = 0, \qquad 0 \leq \theta < 2\pi$

   b) $\cos x = -0.123, \qquad 0 \leq x < 2\pi$

   c) $\sin x = -0.25, \qquad 0° \leq x < 360°$

   d) $2 \csc x = 4, \qquad -2\pi \leq x < 0$

2. Solve the following equations over the interval $0 \leq \theta < 2\pi$. Provide exact answers.

   a) $\csc \theta - \sin \theta = 0$

   b) $\cot \theta - \cot \theta \sin \theta = 0$

3. Find the general solution to each of the following equations.

   a) $\sin \theta = -\frac{1}{2}$, and $\tan \theta > 0$

   b) $2 \csc \theta = -4$

   c) $2 \cos^2 \theta + \cos \theta = 1$

   d) $\tan^2 \theta - \tan \theta = 0$

4. Solve the following equations over the indicated intervals. Provide exact answers.

   a) $\cos^2 x - 2 \cos x = 0, 0 \leq \theta < 2\pi$

   b) $2 \cos^2 \theta + \cos \theta = 0, 0 \leq \theta < 2\pi$

   c) $2 \sin^2 \theta = \sin \theta, -\pi \leq \theta < \pi$

continued

Learning Activity 6.1 (continued)

5. a) Graph the function $y = \sin\left(\theta - \frac{\pi}{2}\right) + 1$.

   b) Determine the equation of the zeros of the function $y = \sin\left(\theta - \frac{\pi}{2}\right) + 1$.

   c) Determine the solution of the equation $\sin\left(\theta - \frac{\pi}{2}\right) = -1$ based on the equation of the zeros of the function $y = \sin\left(\theta - \frac{\pi}{2}\right) + 1$.

Lesson Summary

In this lesson, you expanded on your knowledge of trigonometric equations from Module 5 to learn how to solve both linear and quadratic trigonometric equations. You also learned how to denote the general solution for a trigonometric equation. In the next lesson, you will expand your knowledge about trigonometric identities.

Notes

Assignment 6.1

Solving Trigonometric Equations

Total: 21 marks

You need to show all your steps as you work out your solutions, and make sure your answers are clear (include units, where appropriate). Check the Introduction for the Marking Guidelines that will be used for all assignments and examinations.

1. Solve the following equations over the indicated intervals. Provide exact answers wherever possible. Round to 3 decimal places when necessary.

   a) $\tan x = \sqrt{3}$, $0 \leq x < 2\pi$ *(1 mark)*

   b) $\sec x = 2.5$, $0° \leq x < 180°$ *(2 marks)*

   c) $\csc x = \sec x$, $-\pi \leq \theta < \pi$ *(2 marks)*

continued

Assignment 6.1: Solving Trigonometric Equations (continued)

2. Find the general solution to each of the following equations.

   a) $\sin\theta = -\frac{1}{\sqrt{2}}$ *(2 marks)*

   b) $\cos^2\theta - \cos\theta = 0$ *(3 marks)*

3. Solve the following equations over the indicated intervals. Provide exact answers.

   a) $2\sin^2\theta - \sin\theta = 1, \quad -360° \le \theta < 0°$ *(4 marks)*

   b) $\cos x \tan x + \cos x = 0, \quad 0 \le x < 2\pi$ *(3 marks)*

continued

Assignment 6.1: Solving Trigonometric Equations (continued)

4. Consider the function $y = -\cos\left(x + \frac{3\pi}{2}\right) - 1$.

   a) Determine the x-intercepts of the above function by graphing. *(3 marks)*

   b) Explain how you can determine the general solution of the equation $\cos\left(x + \frac{3\pi}{2}\right) = -1$, using the information you found in (a). *(1 mark)*

Notes

LESSON 2: ELEMENTARY TRIGONOMETRIC IDENTITIES

Lesson Focus

In this lesson, you will

☐ learn the difference between a trigonometric identity and a trigonometric equation

☐ learn about the eight elementary identities

☐ learn how to verify identities

Lesson Introduction

As functions written with various combinations of the circular functions become more and more complicated, it becomes more difficult to sketch them and to solve equations involving these more complicated expressions. However, there are relationships amongst these trigonometric functions that enable you to change them to simpler and more manageable forms. For example, the function $y = \sin^3 x + \sin x \cos^2 x$ can be simplified to become the function $y = \sin x$. The simpler form is much easier to graph. These two forms of equivalent relationships are called identities.

Elementary Trigonometric Identities

You are familiar with some of the elementary identities from Module 5. A list of five of the elementary identities follows. These identities are true for all values of x in their respective domains.

$$\sec x = \frac{1}{\cos x} \qquad \tan x = \frac{\sin x}{\cos x}$$

$$\csc x = \frac{1}{\sin x} \qquad \cot x = \frac{\cos x}{\sin x}$$

$$\cot x = \frac{1}{\tan x}$$

Any expression on one side of the "=" sign can be replaced by the expression on the other side of the sign.

The identities look like equations, but there is a subtle difference. An **equation** usually has specific values that are solutions. For example, the equation $x^2 - 4 = 0$ has the numbers 2 and -2 as its solutions. An **identity** is true for every value in the domain of the expressions.

For example, $x^2 - 4 = (x - 2)(x + 2)$ is true for *every* real number x.

For example, if $x = 5$, then

$$\begin{aligned} \text{Left Side: } x^2 - 4 &= 5^2 - 4 \\ &= 25 - 4 \\ &= 21 \end{aligned}$$

and if $x = 5$, then

$$\begin{aligned} \text{Right Side: } (x - 2)(x + 2) &= (5 - 2)(5 + 2) \\ &= 3(7) \\ &= 21 \end{aligned}$$

Thus, left side = right side.

If $x = 6$, then

$$\begin{aligned} \text{Left Side: } \mathrm{x}^2 - 4 &= 6^2 - 4 \\ &= 36 - 4 \\ &= 32 \end{aligned}$$

and if $x = 6$, then

$$\begin{aligned} \text{Right Side: } (x - 2)(x + 2) &= (6 - 2)(6 + 2) \\ &= 4(8) \\ &= 32 \end{aligned}$$

Thus, left side = right side.

No matter what the value of x, the left and right side will always be the same. In summary, an equation is true for some numbers, while an identity is true for all numbers in the domain.

You may wish to include a summary of the above definitions and explanation on your resource sheet.

Using Identities

The purpose of identities is to simplify your work!

If you are asked to sketch a function such as $y = \cot x \sin x$, you can simply change this function to a simpler identity $y = \frac{\cos x}{\sin x}(\sin x) = \cos x$, which is just the graph of the basic $y = \cos x$ curve.

However, you need to be careful with the domain of each of the functions. If you have $y = \cos x$, then its domain is the real numbers. However, if the function is defined as $y = \cot x \sin x$, then you must remember to eliminate values from the domain where $\cot x$ does not exist $\left(\text{since } \cot x = \frac{\cos x}{\sin x}\right)$.

That is, you need to consider where the denominator, $\sin x$, might be equal to zero. At these points, the function will not be the same as $y = \cos x$, since the points are not in the domain.

Another example would be to solve an equation such as $\sin x \tan x \csc x - 1 = 0$. You can change this equation into the equivalent equation:

$\sin x \tan x \left(\frac{1}{\sin x}\right) - 1 = 0$ or $\tan x - 1 = 0$, if $\sin x \neq 0$, which is much easier to solve.

Example 1

Consider the expression $\frac{\sin x \sec x}{\tan x \cos x}$.

a) Determine the non-permissible values of the expression.

b) Simplify the expression.

Solutions

a) The non-permissible values of this expression exist when $\tan x = 0$, when $\cos x = 0$, and where $\sec x$ is not defined (this is the same as where $\cos x = 0$).

When $\tan x = 0$, $x = k\pi$, $k \in \mathrm{I}$.

When $\cos x = 0$, $x = \frac{\pi}{2} + 2k\pi$, and $x = \frac{3\pi}{2} + 2k\pi$, $k \in \mathrm{I}$.

Note: $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ are π radians apart $\left(\frac{\pi}{2} + \pi = \frac{3\pi}{2}\right)$. They are also odd multiples of $\frac{\pi}{2}$.

$$\frac{\pi}{2} \cdot 1 = \frac{\pi}{2}$$

$$\frac{\pi}{2} \cdot 3 = \frac{3\pi}{2}$$

Another way of expressing an odd integer is the expression $2k + 1$ where $k \in I$. This simply means that you are adding one (+1) to any even number ($2k$) to arrive at an odd number ($2k + 1$).

Therefore, the expression $(2k+1)\left(\frac{\pi}{2}\right)$ or $\frac{(2k+1)\pi}{2}$, $k \in I$, is equivalent to the expressions $x = \frac{\pi}{2} + 2k\pi$, and $x = \frac{3\pi}{2} + 2k\pi$, $k \in I$.

Thus, the non-permissible values when $\cos x = 0$ could also be written as $x = \frac{(2k+1)\pi}{2}$, $k \in I$.

b) $$\frac{\sin x \sec x}{\tan x \cos x} = \frac{\sin x\left(\frac{1}{\cos x}\right)}{\frac{\sin x}{\cos x}(\cos x)}$$

$$= \frac{\frac{\sin x}{\cos x}}{\sin x}$$

$$= \frac{\sin x}{\cos x} \cdot \left(\frac{1}{\sin x}\right)$$

$$= \frac{1}{\cos x}$$

$$= \sec x$$

Pythagorean Identities

There are three additional identities that you will find very useful. They are called the Pythagorean identities because they are derived from and resemble the form of the Pythagorean theorem.

Since the equation of the unit circle is $x^2 + y^2 = 1$ and the x-value on this circle is $\cos\theta$ and the y-value is $\sin\theta$, the sixth elementary identity is

$$\boxed{\cos^2\theta + \sin^2\theta = 1}$$

You discovered this identity in Module 5.

Alternate forms of this same identity will be useful to you. These alternate forms are easily derived using your algebra skills.

$\sin^2\theta = 1 - \cos^2\theta$ and $\cos^2\theta = 1 - \sin^2\theta$ or by factoring:

$\sin^2\theta = (1 - \cos\theta)(1 + \cos\theta)$

and $\cos^2\theta = (1 - \sin\theta)(1 + \sin\theta)$

Note: It is beneficial if you understand and remember all forms of this identity. Therefore, it might be a good idea to add this identity in its various forms to your resource sheet.

In order to help you remember the elementary identities and become more proficient in using them, you will be asked to practice proving certain relationships. When proving identities, it is very important to realize you don't know whether the expressions on either side of the "=" sign are actually identities until after you have proven it. Thus, you cannot assume that they are equal and proceed to transpose terms from one side to the other as you would when solving equations.

Example 2

Prove that $2\sin^2 x - 1 = 1 - 2\cos^2 x$.

Solution

Working on the left-hand side (LHS), you get

$$\begin{aligned}\text{LHS} &= 2\sin^2 x - 1\\ &= 2(1 - \cos^2 x) - 1\\ &= 2 - 2\cos^2 x - 1\\ &= 1 - 2\cos^2 x\\ &= \text{RHS}\end{aligned}$$

Since the RHS has $\cos x$ in it, convert $\sin^2 x$ to its identity, $1 - \cos^2 x$.

Proofs are different from solving equations or verifying values. The setup or format is critical to the validity and clarity of the proof. Often the left-hand side (LHS) and right-hand side (RHS) are written separately in two columns without the equals sign between them. You cannot assume the two sides of the proof are equal (that is, what you are trying to prove), so you may not have an equals sign (=) equating the two sides at any point in the proof. The very last line can state LHS = RHS to show you have proven the identity.

Example 3

Prove that $\sec^2 x \csc^2 x = \sec^2 x + \csc^2 x$.

Solution

A typical strategy used to prove identities is to change each side to expressions involving $\sin x$ and $\cos x$, simplify each side until the sides are exactly the same.

Caution: The two sides are not equal until you have established that both sides are identical, so do not equate the two sides until the very end. Instead, work on the LHS independently from the RHS.

$$\begin{aligned}\text{LHS} &= \sec^2 x \csc^2 x\\ &= \left(\frac{1}{\cos x}\right)^2\left(\frac{1}{\sin x}\right)^2\\ &= \frac{1}{\cos^2 x \sin^2 x}\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \sec^2 x + \csc^2 x\\ &= \left(\frac{1}{\cos x}\right)^2+\left(\frac{1}{\sin x}\right)^2\\ &= \frac{1}{\cos^2 x}+\frac{1}{\sin^2 x}\\ &= \frac{\sin^2 x + \cos^2 x}{\cos^2 x \sin^2 x}\\ &= \frac{1}{\cos^2 x \sin^2 x}\end{aligned}$$

Since LHS = RHS, the identity has been proven.

Note: If you are working on both sides as in the previous example, you should write the LHS and RHS in two separate columns and work on each column separately and independently. You cannot move terms across the equals sign (=) as you would when solving an equation. When both sides are exactly the same expression, then you can state LHS = RHS to show the identity is proven.

Example 4

Verify that the equation $\dfrac{\cos x}{1-\sin x}=\dfrac{1+\sin x}{\cos x}$ is true for $x=\dfrac{\pi}{3}$. Explain why this is insufficient to prove the equation is an identity.

Solution

Let $x=\dfrac{\pi}{3}$

$$\begin{aligned}
\text{LHS} &= \frac{\cos\left(\frac{\pi}{3}\right)}{1-\sin\left(\frac{\pi}{3}\right)}\\
&= \frac{\frac{1}{2}}{1-\frac{\sqrt{3}}{2}}\\
&= \frac{\frac{1}{2}}{\frac{2-\sqrt{3}}{2}}\\
&= \left(\frac{1}{2}\right)\cdot\left(\frac{2}{2-\sqrt{3}}\right)\\
&= \frac{1}{2-\sqrt{3}}\\
&= \frac{\left(2+\sqrt{3}\right)}{\left(2-\sqrt{3}\right)\left(2+\sqrt{3}\right)}\\
&= \frac{\left(2+\sqrt{3}\right)}{4-\sqrt{9}}\\
&= \frac{\left(2+\sqrt{3}\right)}{4-3}\\
&= 2+\sqrt{3}
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= \frac{\left(1+\sin\left(\frac{\pi}{3}\right)\right)}{\cos\left(\frac{\pi}{3}\right)}\\
&= \frac{\left(1+\frac{\sqrt{3}}{2}\right)}{\frac{1}{2}}\\
&= \frac{\left(\frac{2+\sqrt{3}}{2}\right)}{\frac{1}{2}}\\
&= \frac{2+\sqrt{3}}{2}\cdot\frac{2}{1}\\
&= 2+\sqrt{3}
\end{aligned}$$

As the left-hand side equals the right-hand side, this equation is true for $x = \frac{\pi}{3}$. However, just because this equation is true for one value does not mean this equation is true for all values. To prove this equation is true for all values, you need to manipulate one side of the equation to show the left-hand side equals the right-hand side *without* substituting in any values for x.

The following are some common techniques used when proving identities:

- Start working on the more complicated side of the equation.
- Only work with one side of the equation at a time—don't transfer terms over the "=" sign.
- Reduce all expressions to equivalent expressions only involving sine and cosine by using the elementary identities.
- When you see fractions, either reduce them or put them over a common denominator.
- Simplify each side using other identities (e.g., replace $\cos^2\theta + \sin^2\theta$ with 1).
- Factoring (e.g., $\sin^3 x + \sin x \cos^2 x =(\sin x)(\sin^2 x + \cos^2 x)$).

Aside: Notice that the expression in the last bullet eventually simplifies to $\sin x\,(1) = \sin x$.

The remaining two Pythagorean identities are found by dividing the first Pythagorean identity by $\cos^2 x$ or by $\sin^2 x$.

$$\begin{tikzcd}
 & \cos^2 x + \sin^2 x = 1 \arrow[dl, "(\text{dividing by } \cos^2 x)"'] \arrow[dr, "\text{dividing by } (\sin^2 x)"] & \\
\frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} & & \frac{\cos^2 x}{\sin^2 x} + \frac{\sin^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}
\end{tikzcd}$$

$$1 + \tan^2 x = \sec^2 x \qquad\qquad \cot^2 x + 1 = \csc^2 x$$

Thus, the two remaining elementary identities are:

$$\boxed{\begin{array}{c} 1 + \tan^2 x = \sec^2 x \\ \cot^2 x + 1 = \csc^2 x \end{array}}$$

You should add these final two Pythagorean identities to your resource sheet.

Example 5

Find cos x if cot x = 2 and sin x < 0 and x is in the interval $[0, 2\pi]$.

Solution

Method 1: Using Triangles

Since cot $x > 0$ and sin $x < 0$, then $x \in$ Quadrant III.

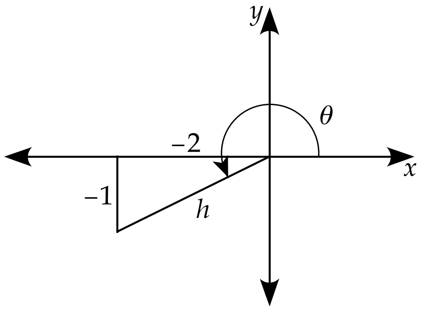
In Quadrant III, you know that cosine is also negative. Therefore, the x- and y-coordinates will both be negative.

Therefore, you can rewrite cot x = 2 as $\cot x = \frac{\cos\theta}{\sin\theta} = \frac{-2}{-1}$. In this ratio, both the cosine and sine components are negative.

Note: Alternatively, you could write cot x = 2 as $\cot x = \frac{-4}{-2}$. This ratio satisfies the requirements of both sine and cosine being negative.

If you rewrite cot x = 2 as $\cot x = \frac{-2}{-1}$, the adjacent and opposite sides of the right triangle are −2 and −1, respectively. You can use the Pythagorean theorem to find the third side.

$$h^2 = (-2)^2 + (-1)^2$$
$$h^2 = 4 + 1$$
$$h = \sqrt{5} \text{ or } \cancel{-\sqrt{5}}$$



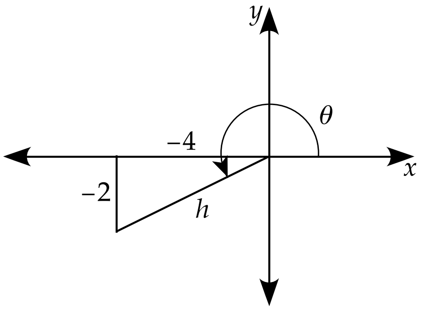
Note: Select $h = \sqrt{5}$ as the answer because h is the hypotenuse, which represents the radius of a circle so it must have a positive measure.

Therefore, $\cos x = \frac{\text{adjacent side}}{\text{hypotenuse}} = -\frac{2}{\sqrt{5}}$.

If you decided to rewrite cot x = 2 as $\cot x = \frac{-4}{-2}$, your solution would be as follows:

$$\cot x = \frac{-4}{-2}$$

The adjacent and opposite sides of the right triangle are −4 and −2, respectively.



Using the Pythagorean theorem:

$$h^2 = (-4)^2 + (-2)^2$$

$$h^2 = 16 + 4$$

$$h = \sqrt{20}$$

Thus, $\cos x = \dfrac{\text{adjacent side}}{\text{hypotenuse}} = \dfrac{-4}{\sqrt{20}} = \dfrac{-4}{\sqrt{(4)(5)}} = \dfrac{-4}{2\sqrt{5}} = -\dfrac{2}{\sqrt{5}}$.

As you can see, you will arrive at the same answer whether you use a ratio of $\dfrac{-2}{-1}$ or $\dfrac{-4}{-2}$ or any other equivalent ratio.

Method 2: Using Identities

$$\cot^2 x + 1 = \csc^2 x$$

$$2^2 + 1 = \csc^2 x$$

$$\pm\sqrt{5} = \csc x$$

Since $\sin x < 0$, it follows that $\csc x < 0$. Thus, $\csc x = -\sqrt{5}$ and $\sin x = -\dfrac{1}{\sqrt{5}}$.

Now you can find $\cos x$ as required.

$$\cos^2 x + \sin^2 x = 1, \text{ so } \cos^2 x + \left(-\frac{1}{\sqrt{5}}\right)^2 = 1 \text{ or } \cos^2 x = \frac{4}{5}$$

Since $x \in$ Quadrant III, it follows that cosine is negative and $\cos x = -\dfrac{2}{\sqrt{5}}$.

Make sure you complete the following learning activity. This learning activity has multiple questions that extend upon the concepts you have been introduced to in this lesson.

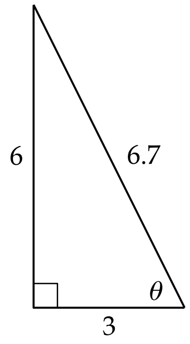
Learning Activity 6.2

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

Use the triangle below to answer Questions 1 to 3.



1. Determine the tangent ratio.
2. Determine the cotangent ratio.
3. Determine the secant ratio.
4. What is the period of the function $f(0) = \sin\theta$?
5. Evaluate: ${}_5C_5$
6. Convert $\frac{\pi}{2}$ to degrees.
7. What is the domain of the function, $f(x) = \frac{1}{x+6}$?
8. Simplify: $\sqrt[4]{16x^4y^8}$

continued

Learning Activity 6.2 (continued)

Part B: Elementary Identities

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. In which quadrant does θ terminate if
   a) $\sin\theta > 0$ and $\cos\theta < 0$?
   b) $\sec\theta > 0$ and $\tan\theta < 0$?
   c) $\cos\theta < 0$ and $\cot\theta > 0$?
   d) $\csc\theta < 0$ and $\tan\theta < 0$?

2. Use the method of triangles to find the exact values of the other remaining circular functions.
   a) $\sin\theta = \frac{5}{13}$ and $\tan\theta < 0$
   b) $\cos x = -\frac{3}{5}$ and $\csc x < 0$
   c) $\tan\theta = 5$ and $\cos < 0$
   d) $\csc\theta = 2$
   e) $\sec x = \frac{13}{5}$ and $\tan x > 0$
   f) $\cot x = -\frac{8}{15}$

3. Use the method of Pythagorean identities to find the exact values of the other remaining circular functions.
   a) $\sin\theta = \frac{5}{13}$ and $\tan\theta < 0$
   b) $\cos x = -\frac{3}{5}$ and $\csc x < 0$
   c) $\tan\theta = 5$ and $\cos\theta < 0$
   d) $\csc\theta = 2$ and $\cos\theta > 0$
   e) $\sec x = \frac{13}{5}$ and $\tan x > 0$
   f) $\cot x = -\frac{8}{15}$ and $\sin x > 0$

continued

Learning Activity 6.2 (continued)

4. Prove that each of the following is an identity.

   a) $\cos x \sec x = 1$

   b) $\csc x \sin x = 1$

   c) $\tan x \cot x = 1$

   d) $\cot x \sin x = \cos x$

   e) $\tan x \cos x = \sin x$

   f) $\dfrac{\sec x}{\csc x} = \tan x$

   g) $\cos^2 x - \sin^2 x = 1 - 2\sin^2 x$

   h) $\cos^4 x - \sin^4 x = 1 - 2\sin^2 x$

   i) $\dfrac{1}{1-\sin\theta} + \dfrac{1}{1+\sin\theta} = 2\sec^2\theta$

   j) $\sec^2\theta - \csc^2\theta = \tan^2\theta - \cot^2\theta$

   k) $\dfrac{\csc^2\theta - 1}{\csc^2\theta} = \cos^2\theta$

   l) $\dfrac{1-\sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1+\sin\theta}$

   m) $\sin x \cos x = \dfrac{1}{\tan x + \cot x}$

   n) $\dfrac{1+\cos^2 x}{\sin^2 x} = 2\csc^2 x - 1$

5. Determine the non-permissible values of each expression.

   a) $\dfrac{\sin x + \cos^2 x \csc x}{\csc x}$

   b) $\dfrac{\cot x}{\cos x \csc x}$

   c) $\sin x \cos x \csc x$

   d) $\dfrac{\cos^2 x}{\sin x} + \sin x$

   e) $\sin^2 x \cos x + \cos^3 x$

   f) $\dfrac{\cos x}{\sin x}$

   g) $\dfrac{1}{\tan x}$

6. Explain why verifying that two sides of a trigonometric identity are equal for a given value is insufficient to conclude the identity is valid. Give an example of a trigonometric equation that works for at least one value of θ but does not work for every value of θ.

continued

Learning Activity 6.2 (continued)

7. Using technology, determine if the following equations may be trigonometric identities. Explain your reasoning. If you believe an equation is a trigonometric identity, verify that the two sides of the identity are equivalent.
   a) $\cos\theta \cot\theta + \sin\theta = \csc\theta$
   b) $\dfrac{\cos x}{1-\sin x} - \tan x = \csc x$

8. Consider the equation $\cot x + 1 = \csc x(\sin x + \cos x)$.
   a) Verify this equation holds for $x = \dfrac{3\pi}{4}$.
   b) Verify this equation is a trigonometric identity and thus holds for all values of x.

Lesson Summary

In this lesson, you were introduced to trigonometric identities. You learned various techniques to prove that trigonometric identities were valid. You also learned what techniques were not valid for proving a trigonometric identity, such as substituting in a specific angle value. In the next lesson, you will continue learning about various trigonometric identities.

LESSON 3: USING ELEMENTARY IDENTITIES

Lesson Focus

In this lesson, you will

☐ learn how to use elementary identities to simplify and solve a trigonometric equation

Lesson Introduction

This lesson is a continuation of the previous lesson, providing more practice using elementary identities. In this lesson, your focus is going to be on using elementary identities to help you solve trigonometric equations.

Using Elementary Identities

In the previous lesson, you learned about eight trigonometric identities. These identities can be used to simplify a trigonometric equation, thus enabling you to solve the equation.

Example 1

Solve tan $x = 2 \sin x$, where $0 \le x \le 2\pi$.

Solution

Steps	**Explanation**
$\tan x = 2 \sin x$	Copy the question.
$\frac{\sin x}{\cos x} = 2 \sin x$	Substitute $\frac{\sin x}{\cos x}$ for tan x.
$(\cos x)\left(\frac{\sin x}{\cos x}\right) = 2 \sin x \cos x$	Multiply both sides by cos x to get rid of the denominator.
$\sin x = 2 \sin x \cos x$	Algebra.
$\sin x - 2 \sin x \cos x = 0$	More algebra.
$\sin x(1 - 2 \cos x) = 0$	Factor.

Steps		**Explanation**
$\sin x = 0$	$1 - 2\cos x = 0$	Zero Product Property.
$x = 0, \pi, 2\pi$	$\cos x = \frac{1}{2}$	
	$x = \frac{\pi}{3}, \frac{5\pi}{3}$	

$\therefore$ The solution is $x = \left\{0, \pi, 2\pi, \frac{\pi}{3}, \frac{5\pi}{3}\right\}$.

Example 2

Solve $\sin^2 x + \cos x = 1$, where $0° \le x < 360°$.

Note: An identity can be used to change the quadratic term, $\sin^2 x$, into an expression involving cosine. However, there is no identity that can be used on the linear term, $\cos x$, to change it to an expression involving sine.

Solution

$(1 - \cos^2 x) + \cos x = 1$ since $\sin^2 x = 1 - \cos^2 x$

$$1 - \cos^2 x + \cos x = 1$$

$$0 = \cos^2 x - \cos x + 1 - 1$$

$$0 = \cos^2 x - \cos x$$

$$0 = \cos x\,(1 - \cos x)$$

$$\cos x = 0 \text{ or } \cos x = 1$$

$$x = \{90°, 270°, 0°\}$$

This next equation is somewhat tricky. Some equations that involve sums and differences of linear expressions of sine and cosine functions are solved by squaring both sides. However, since both sides have been squared, extraneous solutions may have been introduced. Therefore, the solutions must be checked.

Example 3

Solve $\cos x + 1 = \sqrt{3}\sin x$, where $0 \le x < 2\pi$.

Solution

Note: To get rid of the radical sign over the 3, you need to square both sides of the equation. Once you have a $\sin^2 x$, you can convert to $\cos^2 x$ terms using the identity.

$$(\cos x + 1)^2 = \left(\sqrt{3}\sin x\right)^2$$

$$\cos^2 x + 2\cos x + 1 = 3\sin^2 x$$

$$\cos^2 x + 2\cos x + 1 = 3\left(1 - \cos^2 x\right)$$

$$\cos^2 x + 2\cos x + 1 = 3 - 3\cos^2 x$$

$$\cos^2 x + 3\cos^2 x + 2\cos x + 1 - 3 = 0$$

$$4\cos^2 x + 2\cos x - 2 = 0$$

$$2\left(2\cos^2 x + \cos x - 1\right) = 0$$

$$2(2\cos x - 1)(\cos x + 1) = 0$$

$$\cos x = \frac{1}{2} \text{ or } \cos x = -1$$

Possible solutions are: $x = \left\{\frac{\pi}{3}, \frac{5\pi}{3}, \pi\right\}$.

Now you need to check for extraneous roots.

Check: $\frac{\pi}{3}$ in the original equation $\cos x + 1 = \sqrt{3}\sin x$.

LHS: $\cos\frac{\pi}{3} + 1 = \frac{1}{2} + 1 = \frac{3}{2}$

RHS: $\sqrt{3}\sin\frac{\pi}{3} = \sqrt{3}\left(\frac{\sqrt{3}}{2}\right) = \frac{3}{2}$

LHS = RHS; therefore, $\frac{\pi}{3}$ is a root.

Check: $\frac{5\pi}{3}$

LHS: $\cos\frac{5\pi}{3} + 1 = \frac{1}{2} + 1 = \frac{3}{2}$

RHS: $\sqrt{3}\sin\frac{5\pi}{3} = \sqrt{3}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{3}{2}$

LHS ≠ RHS; therefore, $\frac{5\pi}{3}$ is an extraneous root.

Check: π

LHS: $\cos \pi + 1 = -1 + 1 = 0$
RHS: $\sqrt{3}\sin \pi = \sqrt{3}(0) = 0$

LHS = RHS; therefore, π is a root.

The solution is $x = \frac{\pi}{3}$ and $x = \pi$.

Make sure you complete the following learning activity, as you will be introduced to more concepts related to solving trigonometric equations using elementary trigonometric identities throughout the learning activity. Also, make sure you check your answers in the Learning Activity Answer Key. The answer key sometimes provides additional explanations about the problem and solution.

Learning Activity 6.3

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. What is the exact value of cos 120°?
2. Find the positive coterminal angle for $-14°$ in the interval $0° \leq \theta \leq 360°$.
3. What is the last term in the expansion of $(2x^2 - 3y)^6$?
4. Solve $\sin \theta = 2$ in the interval $[0, 2\pi]$.
5. What is the area of a triangle with a height of 16 m and a base of 5 m?
6. Is $x = 5$ a solution to the inequality $x^2 - 4x + 3 \leq 0$?
7. Which is the better deal, a package of 4 pens for \$1.26 or a package of 3 pens for \$0.99?
8. Evaluate: $\frac{121}{7} \div \frac{44}{49}$

continued

Learning Activity 6.3 (continued)

Part B: Solving Trigonometric Equations by Using Trigonometric Identities

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Solve each equation for x where $0° \le x < 360°$. Use exact values where possible; otherwise, round to one decimal place.
   a) $\tan^2 x + \sec^2 x = 3$
   b) $\sin x = \cos x$
   c) $\tan x = \sin x \tan x$
   d) $2 \cos x = 3 \tan x$
   e) $\tan^2 x = \sec x + 1$
   f) $3 \tan^2 x = \sec x \tan x$

2. Solve each equation for x where $0 \le x < 2\pi$. Use exact values where possible; otherwise, round to two decimal places.
   a) $2 \cos^2 x - 1 = -\sin x$
   b) $4 \cos^2 x = 3$
   c) $2 \sec^2 x + \tan x = 2$
   d) $2 \cos^2 x + 3 \sin x = 0$

3. Solve each equation for x where $0 \le x < 2\pi$.
   a) $\sin x + \cos x = 1$
   b) $\sin x + \cos x = 0$
   c) $\sin x - 2 = \cos x$
   d) $\sin x - \cos x = 1$
   e) $\tan x - 1 = \sec x$

4. Test the following identity by substituting the stated values for α and β.
   Difference identity: $\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$

   a) $\alpha = 90°, \beta = 30°$
   b) $\alpha = 150°, \beta = 30°$
   c) $\alpha = 90°, \beta = 45°$
   d) $\alpha = \pi, \beta = \dfrac{\pi}{4}$
   e) $\alpha = \pi, \beta = \dfrac{2\pi}{3}$
   f) $\alpha = \dfrac{\pi}{2}, \beta = \dfrac{\pi}{3}$

Lesson Summary

In this lesson, you learned how to use elementary trigonometric identities to solve trigonometric equations. In the next lesson, you will be looking at more trigonometric identities that you were introduced to in the previous learning activity. These identities are called *sum and difference identities*.

Assignment 6.2

Using Elementary Identities

Total: 44 marks

You need to show all your steps as you work out your solutions, and make sure your answers are clear (include units, where appropriate). Check the Introduction for the Marking Guidelines that will be used for all assignments and examinations.

1. Use the method of Pythagorean identities to find the exact values of the other remaining circular functions, given that $\sec\theta = \frac{13}{7}$ and $\cot\theta < 0$. *(4 marks)*

continued

Assignment 6.2: Using Elementary Identities (continued)

2. Prove that each of the following is an identity.
   a) $\sin x \tan x \csc x \cot x = 1$ *(2 marks)*

   b) $\sin^2 x \sec^2 x + \sin^2 x \csc^2 x = \sec^2 x$ *(2 marks)*

continued

Assignment 6.2: Using Elementary Identities (continued)

c) $\cot x + \tan x = \sec x \csc x$ *(2 marks)*

d) $\frac{1}{\cos x} - \cos x = \tan x \sin x$ *(2 marks)*

continued

e) $\csc x = \dfrac{1 + \sec x}{\sin x + \tan x}$ *(3 marks)*

f) $\dfrac{1}{\cot x - \csc x} + \dfrac{1}{\cot x + \csc x} = -2 \cot x$ *(4 marks)*

continued

Assignment 6.2: Using Elementary Identities (continued)

3. Simplify each expression as much as possible. Determine the non-permissible values of each expression.

   a) $\dfrac{\csc^2 x - 1}{\csc^2 x}$ *(4 marks)*

   b) $\dfrac{\sin x \sec x}{\tan x + \cot x}$ *(5 marks)*

continued

4. Consider the equation $\dfrac{\cos x}{1 - \sin x} = \dfrac{1 + \sin x}{\cos x}$.

   a) Verify this equation holds for $x = \dfrac{\pi}{6}$. *(3 marks)*

   b) Verify this equation is a trigonometric identity, and thus holds for all values of x. *(3 marks)*

continued

Assignment 6.2: Using Elementary Identities (continued)

5. Solve each equation over the indicated interval. Use exact values where possible; otherwise, round to two decimal places.
   a) $\csc^2 \theta = 2 \cot \theta$, $0° \le \theta < 360°$ *(4 marks)*

continued

Assignment 6.2: Using Elementary Identities (continued)

b) $1 - \tan\theta = \sqrt{2}\sec\theta$, $0 \le \theta < 2\pi$ (Make sure you check your solutions for extraneous roots). *(6 marks)*

Lesson 4: Sum and Difference Identities

Lesson Focus

In this lesson, you will

- ☐ learn how to verify the sum and difference identities for particular values
- ☐ learn how to simplify expressions using these identities
- ☐ learn how to evaluate expressions using these identities
- ☐ learn how to prove identities using sum and difference identities

Lesson Introduction

In the learning activity in Lesson 3, you were introduced to one difference identity involving the cosine function. In this lesson, you will be introduced to other identities involving the addition or subtraction of angles. Using all of the identities you have just learned, you will then be able to evaluate and simplify more complicated trigonometric expressions, and you will be able to find the exact value of trigonometric ratios on the unit circle beyond those derived from special triangles.

Sum and Difference Identities

In this lesson, you are going to learn about six different sum and difference identities involving the sine, cosine, and tangent functions. Sum and difference identities are useful, as they allow you to take a problem to which you do not know the answer and break it down into a problem that you can solve.

Difference Identities

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Sum Identities

$$\sin(\alpha + \beta) = \sin\alpha\ \cos\beta + \cos\alpha\ \sin\beta$$

$$\cos(\alpha + \beta) = \cos\alpha\ \cos\beta - \sin\alpha\ \sin\beta$$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\ \tan\beta}$$

If you are interested in seeing a proof of these identities, search the Internet. You will not be required to reproduce the proof on any assignment or examination for this course.

The above six identities can be used to find the exact trigonometric values of more angles by using the exact values of angles you already mastered on the unit circle.

Include the identities above on your resource sheet.

Example 1

Find the exact value of sin 75°.

Solution

Look at the angles you already know on the unit circle (30°, 45°, 60°, 90°, 120°, . . .) and find a sum or difference that gives you 75°. One is 45° + 30°. Next, select the appropriate identity. The question was sin 75°, so you require $\sin(\alpha + \beta)$.

Now write the formula down:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Since $\alpha = 45°$ and $\beta = 30°$,

$$\sin(45° + 30°) = \sin 45° \cos 30° + \cos 45° \sin 30°.$$

Then, substitute the exact values you know from the unit circle:

$$\sin(45° + 30°) = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$$

$$\sin(45° + 30°) = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

$$\therefore \sin 75° = \frac{\sqrt{6} + \sqrt{2}}{4}$$

Note: You could also do the question with 135° − 60° = 75° and using $\sin(\alpha - \beta)$.

Furthermore, you could use your calculator to confirm that the exact value on the RHS is the same as the decimal approximation given by the value of sin 75° on your calculator.

In summary, you will need to

1. find a combination using unit circle angles
2. select the corresponding sum or difference identity
3. substitute exact trigonometric values
4. simplify

Note: You can also find combinations with angles that are in radians. It might be a good opportunity for you to practise your skills with fractions. You will see that the radian values that you can most often find will involve π and a denominator of 12.

Example 2

Find a sum or difference of two angles equal to $\frac{13\pi}{12}$, using angles whose exact trigonometric ratios you already know from the unit circle.

Solution:

Think of angles you already know from the unit circle and write them with a denominator of 12.

$$\frac{\pi}{6} = \frac{2\pi}{12} \qquad \frac{\pi}{4} = \frac{3\pi}{12} \qquad \frac{\pi}{3} = \frac{4\pi}{12}$$

$$\frac{2\pi}{3} = \frac{8\pi}{12} \qquad \frac{3\pi}{4} = \frac{9\pi}{12} \qquad \frac{5\pi}{6} = \frac{10\pi}{12}$$

Using these angles, you can see that there are two combinations that would equal $\frac{13\pi}{2}$.

$$\left(\frac{9\pi}{12} + \frac{4\pi}{12}\right) = \left(\frac{3\pi}{4} + \frac{\pi}{3}\right) \text{ or } \left(\frac{10\pi}{12} + \frac{3\pi}{12}\right) = \left(\frac{5\pi}{6} + \frac{\pi}{4}\right)$$

Example 3

Use an appropriate identity and the fact that $\frac{\pi}{12} = \frac{4\pi}{12} - \frac{3\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$ to find $\cos\left(\frac{\pi}{12}\right)$.

Solution

Using the cosine difference identity, $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$, you get:

$$\begin{aligned}\cos\frac{\pi}{12} &= \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\\ &= \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}\\ &= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right)\\ &= \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}\\ &= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}\\ &= \frac{\sqrt{2} + \sqrt{6}}{4}\end{aligned}$$

Multiply the numerator and denominator of each fraction by $\sqrt{2}$ if you want to rationalize the denominator.

Note: This answer has a **rationalized denominator**, which is the simplified form for radicals. In other words, no radicals are located in the denominator.

Include a description on your resource sheet of the process to find exact values of trigonometric ratios using sum and difference identities.

Example 4

Find the coordinates of $P\left(\frac{\pi}{12}\right)$.

Solution

The notation $P\left(\frac{\pi}{12}\right)$ refers to a point on the unit circle after an angle of rotation of $\frac{\pi}{12}$ radians or moving along the circle an arc length of $\frac{\pi}{12}$ radians.

The x-coordinate is $\frac{\sqrt{2}+\sqrt{6}}{4}$, as shown in Example 3, since the x-coordinate of the point on the unit circle is $\cos\left(\frac{\pi}{12}\right)$.

To find the y-coordinate, you can use the identity:

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

$$\begin{aligned}
\sin\frac{\pi}{12} &= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\
&= \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4} \\
&= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\
&= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\
&= \frac{\sqrt{6}-\sqrt{2}}{4}
\end{aligned}$$

Recall: $\frac{1}{\sqrt{2}}$ is equivalent to $\frac{\sqrt{2}}{2}$.

$\therefore$ The coordinates of $P\left(\frac{\pi}{12}\right) = \left(\cos\frac{\pi}{12}, \sin\frac{\pi}{12}\right) = \left(\frac{\sqrt{2}+\sqrt{6}}{4}, \frac{\sqrt{6}-\sqrt{2}}{4}\right)$.

Note: You can also use the calculated value of $\cos\left(\frac{\pi}{12}\right)$ and the fact that $\sin^2\left(\frac{\pi}{12}\right) + \cos^2\left(\frac{\pi}{12}\right) = 1$ to find $\sin\frac{\pi}{12}$. You need to think about what quadrant $\frac{\pi}{12}$ is in when determining the sign of your final answer.

Now you are capable of finding the exact value of coordinates of any point on the unit circle, which is a sum or a difference of the special angles of rotation.

Example 5

a) If $\sin \alpha = \frac{3}{5}$ with $\alpha \in$ Quadrant II, and $\cos \beta = \frac{5}{13}$ with $\tan \beta > 0$, find the coordinates of $P(\alpha + \beta)$.

b) In what quadrant does the angle of rotation $(\alpha + \beta)$ terminate?

Solutions

a) To find the coordinates of $P(\alpha + \beta)$, you must find the x-coordinate using $\cos(\alpha + \beta)$ and the y-coordinate using $\sin(\alpha + \beta)$.

First, draw a triangle diagram for α in Quadrant II.

Since $\sin \alpha = \frac{3}{5}$, it follows that:

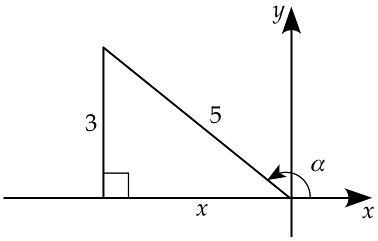
$$5^2 = 3^2 + x^2$$

$$25 = 9 + x^2$$

$$16 = x^2$$

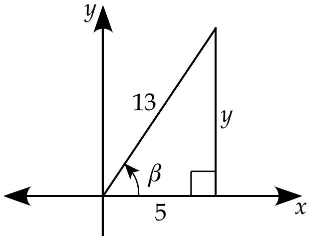
$$\pm\sqrt{16} = x$$

$$x = -4$$



As α lies in Quadrant II, $\cos \alpha = -\frac{4}{5}$.

Now, draw a triangle diagram for β in Quadrant I, since $\tan \beta > 0$ and $\cos \beta > 0$.



Since $\cos\ \beta\ = \dfrac{5}{13}$ and $\tan \beta > 0$, it follows that $\sin \beta > 0$.

$$\begin{aligned} 13^2 &= 5^2 + y^2 \\ 169 &= 25 + y^2 \\ 144 &= y^2 \\ \pm\sqrt{144} &= y^2 \\ y &= 12 \end{aligned}$$

$$\therefore \sin\ \beta\ = \frac{12}{13}$$

Now find $\cos\ (\alpha + \beta)$ and $\sin\ (\alpha + \beta)$ using the sum identities.

$$\begin{aligned} \cos\left(\alpha + \beta\right) &= \cos \alpha\ \cos\ \beta\ - \sin \alpha\ \sin\ \beta \\ &= \left(-\frac{4}{3}\right)\left(\frac{5}{13}\right) - \left(\frac{3}{5}\right)\left(\frac{12}{13}\right) \\ &= -\frac{20}{65} - \frac{36}{65} \\ &= -\frac{56}{65} \end{aligned}$$

$$\begin{aligned} \sin\left(\alpha + \beta\right) &= \sin \alpha\ \cos\ \beta\ + \cos \alpha\ \sin\ \beta \\ &= \left(\frac{3}{5}\right)\left(\frac{5}{13}\right) + \left(-\frac{4}{5}\right)\left(\frac{12}{13}\right) \\ &= \frac{15}{65} - \frac{48}{65} \\ &= -\frac{33}{65} \end{aligned}$$

Therefore, the coordinates of $P(\alpha + \beta)$ are $\left(-\dfrac{56}{65}, -\dfrac{33}{65}\right)$.

b) Since $\cos\ (\alpha + \beta) < 0$ and $\sin\ (\alpha + \beta) < 0$, it follows that the angle of rotation $(\alpha + \beta)$ terminates in the third quadrant.

Example 6

Use the tangent sum and difference identities to answer the following questions.

a) Determine the exact value of tan 165°.

b) Write the expression $\dfrac{\tan\dfrac{\pi}{6}-\tan\dfrac{4\pi}{3}}{1+\tan\dfrac{\pi}{6}\tan\dfrac{4\pi}{3}}$ as a single trigonometric function.

Solutions

a) First, you need to rewrite tan 165° in terms of 2 angles that you already know from the special triangles on the unit circle—either the sum of two angles or the difference of two angles. You need to know the exact tan values of these two angles in order to find the exact value of tan 165°.

There are other combinations you could use but this solution shows tan 165° can be written as tan (135° + 30°). You can determine the exact values of tan 135° and tan 30° from your unit circle.

$$\begin{aligned}
\tan 165^\circ &= \tan\left(135^\circ + 30^\circ\right)\\
&= \frac{\tan 135^\circ + \tan 30^\circ}{1-\tan 135^\circ \tan 30^\circ}\\
&= \frac{-1+\dfrac{\sqrt{3}}{2}}{1-\left(-1\left(\dfrac{\sqrt{3}}{2}\right)\right)}\\
&= \frac{\dfrac{-3+\sqrt{3}}{3}}{\dfrac{3+\sqrt{3}}{3}}\\
&= \frac{-3+\sqrt{3}}{3}\left(\frac{3}{3+\sqrt{3}}\right)\\
&= \frac{-3+\sqrt{3}}{3+\sqrt{3}}
\end{aligned}$$

b) To write this expression as a single trigonometric function, you need to use the tangent difference identity in reverse.

Note: Make sure you identify the correct identity to use. It is easy to get confused between the sum and difference identities.

Consider the tangent difference identity:

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \tan\beta}$$

In the expression $\dfrac{\tan\frac{\pi}{6} - \tan\frac{4\pi}{3}}{1 + \tan\frac{\pi}{6}\tan\frac{4\pi}{3}}$, $\dfrac{\pi}{6} = \alpha$ and $\dfrac{4\pi}{3} = \beta$.

$$\therefore \frac{\tan\frac{\pi}{6} - \tan\frac{4\pi}{3}}{1 + \tan\frac{\pi}{6}\tan\frac{4\pi}{3}} = \tan\left(\frac{\pi}{6} - \frac{4\pi}{3}\right) = \tan\left(\frac{\pi}{6} - \frac{8\pi}{6}\right) = \tan\left(-\frac{7\pi}{6}\right)$$

Make sure you complete the following learning activity, as it will allow you to practice the skills you just learned.

Learning Activity 6.4

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. State the non-permissible values of the function $f(x) = \dfrac{x^2 + x - 20}{x^2 + 5x - 14}$.
2. Find all the values of θ between [0°, 180°] if $\tan\theta = 1$.
3. Convert 1080° to radians.
4. What is the exact value of sin 60°?

continued

Learning Activity 6.4 (continued)

5. If two angles of a triangle are 82° and 34°, what is the measurement of the third angle?

6. In which quadrant is $\csc\theta$ negative and $\sec\theta$ positive?

7. Estimate the amount of tax at 12% on a bill of \$49.95.

8. Determine the inverse function if $f(x) = \frac{3}{2}x$.

Part B: Using Sum and Difference Identities

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Without using a calculator, evaluate the following.

   a) $\sin\frac{\pi}{9}\cos\frac{5\pi}{36} + \cos\frac{\pi}{9}\sin\frac{5\pi}{36}$

   b) $\sin\frac{2\pi}{5}\cos\frac{3\pi}{5} + \cos\frac{2\pi}{5}\sin\frac{3\pi}{5}$

   c) $\cos\frac{7\pi}{12}\cos\frac{\pi}{4} + \sin\frac{7\pi}{12}\sin\frac{\pi}{4}$

   d) $\cos\frac{5\pi}{9}\cos\frac{\pi}{9} - \sin\frac{5\pi}{9}\sin\frac{\pi}{9}$

   e) $\dfrac{\tan\frac{4\pi}{9} + \tan\frac{25\pi}{18}}{1 - \tan\frac{4\pi}{9}\tan\frac{25\pi}{18}}$

   f) $\dfrac{\tan 3\pi - \tan\frac{5\pi}{3}}{1 + \tan 3\pi \tan\frac{5\pi}{3}}$

2. Determine the exact value of $\sin\frac{7\pi}{12}$ by setting $\alpha = \frac{\pi}{4}$ and $\beta = \frac{\pi}{3}$ in the relevant identity.

3. Find the coordinates of $P\left(\frac{7\pi}{12}\right)$.

4. Find the exact value of $\tan\frac{7\pi}{12}$.

continued

Learning Activity 6.4 (continued)

5. Use identities to find the exact value of each of the following. Use a sum or difference of two special angles to get 195°.

   a) $\cos 195°$

   b) $\sin 195°$

   c) $\tan 195°$

6. Given that $\sin \alpha = \frac{7}{25}$ and $\cos \beta = \frac{9}{41}$ and neither $P(\alpha)$ nor $P(\beta)$ are in the first quadrant, find:

   a) $\sin (\alpha + \beta)$

   b) $\cos (\alpha + \beta)$

   c) $\tan (\alpha + \beta)$

   d) $\sec (\alpha + \beta)$

   e) The quadrant in which $(\alpha + \beta)$ terminates

7. Express $\cos\left(\frac{\pi}{3} + \theta\right)$ as a function of θ only.

8. Express $\tan\left(\theta - \frac{\pi}{6}\right)$ as a function of θ only.

9. Prove the following identity.

   $\sin (\alpha + \beta) + \sin (\alpha - \beta) = 2 \sin \alpha \cos \beta$

10. Use identities to simplify the following expressions to a single trigonometric function.

    a) $\sin 2x \cos 3x + \cos 2x \sin 3x$

    b) $\dfrac{\tan x + \tan 3x}{1 - \tan x \tan 3x}$

    c) $\sin \alpha \cos 6\alpha + \cos \alpha \sin 6\alpha$

    d) $\cos^2 3x - \sin^2 3x$

    e) $\cos 2\alpha \cos 3 + \sin 2\alpha \sin 3\alpha$

continued

Learning Activity 6.4 (continued)

11. Prove the following *very important* identities. These double angle identities are discussed further in Lesson 5.
    a) $\sin 2\alpha = 2 \sin \alpha \cos \alpha$
    b) $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$
    c) $\cos 2\alpha = 2 \cos^2 \alpha - 1$
    d) $\cos 2\alpha = 1 - 2 \sin^2 \alpha$
    e) $\tan 2\alpha = \dfrac{2 \tan \alpha}{1 - \tan^2 \alpha}$

12. Use identities to simplify the following to a function involving only x.
    a) $\sin \left(\dfrac{\pi}{2} + x\right)$
    b) $\cos \left(\dfrac{\pi}{2} + x\right)$

13. Use identities to write the general solution of the following equations.
    a) $\cos \left(x - \dfrac{\pi}{2}\right) = 1$
    b) $\sin \left(\dfrac{\pi}{2} - x\right) = 1$

Lesson Summary

In this lesson, you learned about six different identities called sum and difference identities. In the next lesson, you will learn about three more identities called *double angle identities*. You were briefly introduced to these identities in Question 11 of Learning Activity 6.4.

Lesson 5: Double Angle Identities

Lesson Focus

In this lesson, you will learn how to

- ☐ derive the double angle identities
- ☐ simplify expressions using these identities
- ☐ evaluate expressions using these identities
- ☐ prove identities using double angle identities
- ☐ solve equations using these identities

Lesson Introduction

In Learning Activity 6.4, you derived a set of identities called double angle identities. As you may have guessed, these identities involve relationships amongst circular functions of one arc and another arc twice as long, or of one angle and an angle twice as large. These identities can be used to further expand the list of angles for which you can find exact trigonometric ratios.

Double Angle Identities

The double angle identities are listed below. Check the solutions to proofs that were a part of Learning Activity 6.4, question 11. Be sure to include these identities on your resource sheet.

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\cos 2\alpha = \begin{cases} \cos^2 \alpha - \sin^2 \alpha \\ 2 \cos^2 \alpha - 1 \\ 1 - 2 \sin^2 \alpha \end{cases}$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

Example 1

Use the double angle identities to change the following into functions of angles that are half the given angles.

a) $\sin 2A$

b) $\sin 4A$

c) $\sin 10x$

d) $\tan 4A$

e) $\cos 4\alpha$

Solutions

a) $\sin 2A = 2\sin A \cos A$

b) $\sin (2(2A)) = 2 \sin 2A \cos 2A$

Notice that the coefficient of 2 is not changed. To use this identity, all you need is one angle to be twice as large as another—that's all!

c) $\sin 10x = 2 \sin 5x \cos 5x$

Again, notice that $10x = 2(5x)$.

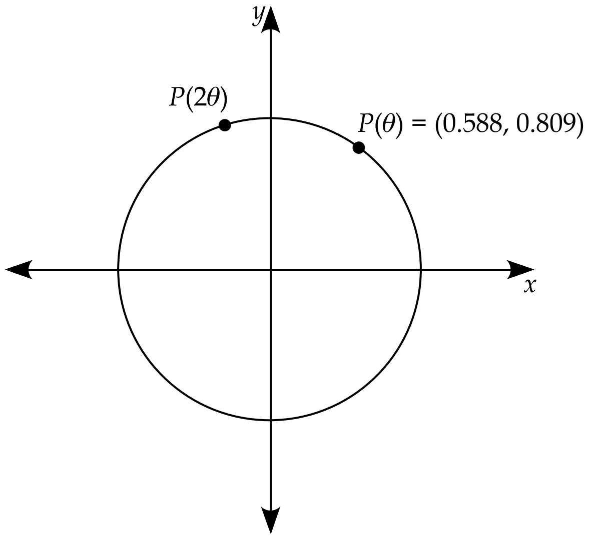
d) $\tan 4A = \dfrac{2 \tan 2A}{1 - \tan^2 2A}$

e) Any one of the three following choices is correct.

$$\cos 4\alpha = \begin{cases} \cos^2 2\alpha - \sin^2 2\alpha \\ 2 \cos^2 2\alpha - 1 \\ 1 - 2 \sin^2 2\alpha \end{cases}$$

Example 2

On the unit circle, the coordinates of the terminal point $P(\theta)$ are (0.588, 0.809). Find the coordinates of $P(2\theta)$.



Solution

The coordinates of $P(2\theta)$ will be $(\cos 2\theta, \sin 2\theta)$.

$$\cos 2\theta = 2\cos^2\theta - 1 = 2(0.588)^2 - 1 = -0.309$$

$$\sin 2\theta = 2\sin\theta\cos\theta = 2(0.809)(0.588) = 0.951$$

Therefore, the coordinates of $P(2\theta) = (-0.309, 0.951)$.

Example 3

Solve the equation $\sin 2\theta = \sin\theta$, where $0 \le \theta < 2\pi$.

Solution

$$\sin 2\theta = \sin\theta$$

$$2\sin\theta\cos\theta = \sin\theta$$

$$2\sin\theta\cos\theta - \sin\theta = 0$$

$$\sin\theta\,(2\cos\theta - 1) = 0$$

$$\sin\theta = 0 \text{ or } \cos\theta = \frac{1}{2}$$

$$\theta = \left\{0, \pi, \frac{\pi}{3}, \frac{5\pi}{3}\right\}$$

The identity enabled you to change the equation to one where all the arguments were θ rather than 2θ.

Note: An argument can also be thought of as the i*ndependent variable* of a function.

Example 4

Find the exact values of the following trigonometric ratios using two different methods.

a) $\sin\left(\frac{10\pi}{3}\right)$

b) $\cos\left(\frac{22\pi}{6}\right)$

c) $\tan\left(\frac{10\pi}{4}\right)$

Solutions

a) Rewrite this expression as a double angle because $\frac{10\pi}{3} = 2\left(\frac{5\pi}{3}\right) \cdot \frac{5\pi}{3}$ is a special angle on the unit circle, and therefore you can determine the exact value of any trigonometric ratio corresponding to this angle.

$\sin\left(\frac{10\pi}{3}\right) = \sin 2\left(\frac{5\pi}{3}\right)$ Rewrite $\frac{10\pi}{3}$ as a double angle.

$= 2\sin\frac{5\pi}{3}\cos\frac{5\pi}{3}$ Use the double angle identity for sine.

$= 2\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)$ Use the unit circle to determine special values.

$= -\frac{\sqrt{3}}{2}$ Simplify.

An alternative method for finding the exact value of $\sin\left(\frac{10\pi}{3}\right)$ is to recognize that the angle $\theta = \frac{10\pi}{3}$ is coterminal with $\theta = \frac{4\pi}{3}$.

$\therefore \sin\frac{10\pi}{3} = \sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$.

b) $\frac{22\pi}{6}$ can be expressed as a double angle: $\frac{22\pi}{6} = 2\left(\frac{11\pi}{6}\right)$.

$$\begin{aligned}
\therefore \cos\left(\frac{22\pi}{6}\right) &= \cos\left(2\left(\frac{11\pi}{6}\right)\right) \\
&= \cos^2\left(\frac{11\pi}{6}\right) - \sin^2\left(\frac{11\pi}{6}\right) \\
&= \left(\frac{\sqrt{3}}{2}\right)^2 - \left(-\frac{1}{2}\right)^2 \\
&= \frac{3}{4} - \frac{1}{4} \\
&= \frac{2}{4} \\
&= \frac{1}{2}
\end{aligned}$$

Another way of evaluating this expression is to notice that $\theta = \frac{22\pi}{6}$ is coterminal with $\theta = \frac{10\pi}{6} = \frac{5\pi}{3}$.

$\therefore \cos\left(\frac{22\pi}{6}\right) = \cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}$.

c) $\frac{10\pi}{4}$ can be written as $2\left(\frac{5\pi}{4}\right)$.

$$\begin{aligned}
\therefore \tan\left(\frac{10\pi}{4}\right) &= \tan\left(2\left(\frac{5\pi}{4}\right)\right) \\
&= \frac{2\tan\frac{5\pi}{4}}{1 - \tan^2\frac{5\pi}{4}} \\
&= \frac{2(1)}{1 - 1^2} \\
&= \frac{2}{0} \\
&= \text{undefined}
\end{aligned}$$

To solve this problem using an alternative method, recognize that $\frac{10\pi}{4} = \frac{5\pi}{2}$ and $\theta = \frac{5\pi}{2}$ is a coterminal angle with $\theta = \frac{\pi}{2}$.

$\therefore \tan \frac{10\pi}{4} = \tan \frac{\pi}{2}$, which is undefined.

From the above example, you can see that these double angle identities provide an alternative method for finding trigonometric ratios in certain circumstances. However, these double angle identities are even more useful for simplifying trigonometric expressions and proving trigonometric identities. Consider the following example.

Example 5

Prove the following identities.

a) $\tan x\left(1 + \cos 2x\right) = \sin 2x$

b) $\dfrac{\sin 2x}{1 - \cos 2x} = \cot x$

Solutions

Recall, it is usually easiest to start on the more complicated side of the equation to see if you can manipulate that expression to match the expression on the other side of the equation. In both (a) and (b), the left-hand side is more complicated.

a) $\text{LHS} = \tan x\left(1 + \cos 2x\right)$

You want to convert this expression to one involving only sine and cosine ratios. You also want to get rid of the double angle. However, you have a choice between 3 equivalent expressions for cos $2x$. As you know, there will be a "cos x" in the denominator once you rewrite tan x. You also know there is no cos x in the denominator of the final expression. Therefore, you want to rewrite cos $2x$ as an expression only involving cosine to see if you can cancel out the cosine in the denominator.

$$= \frac{\sin x}{\cos x}\left(1 + \left(2\cos^2 x - 1\right)\right)$$

$$= \frac{\sin x}{\cos x}\left(2\cos^2 x\right)$$

$$= 2\sin x\cos x$$

$$= \sin 2x$$

$$= \text{RHS}$$

b) $\text{LHS} = \dfrac{\sin 2x}{1 - \cos 2x}$

First, you need to rewrite the expressions involving double angles. Again, you have a choice on how to rewrite cos $2x$. One way of thinking about this choice is to consider how to get rid of the "1" in the denominator. This might lead to a situation where you can cancel factors out of the numerator and the denominator to arrive at cot x. To get rid of the "1," you need to subtract a positive 1. This situation would occur if you used the identity $\cos 2x = 1 - 2\sin^2 x$.

$$= \frac{2\sin x\cos x}{1 - \left(1 - 2\sin^2 x\right)}$$

$$= \frac{2\sin x\cos x}{2\sin^2 x}$$

$$= \frac{\cos x}{\sin x}$$

$$= \cot x$$

$$= \text{RHS}$$

Note: The explanations given in the previous example of how to prove each identity are just suggested ways of thinking. There are multiple ways of proving identities; some may be longer and some may be shorter.

Learning Activity 6.5

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Convert $\frac{\pi}{3}$ to degrees.
2. What is the exact value of tan 45°?
3. Will the function $y = x^2 - 2$ have an inverse that is a function?
4. State the non-permissible values of the function $f(x) = \frac{x-2}{x^2+5x+6}$.
5. Simplify: $49^{-\frac{1}{2}}$
6. You put a chicken in the oven at 2:48 pm. At what time will the chicken be done if it takes 85 minutes to cook?
7. What is the length of the hypotenuse of a right-angled triangle if the two legs of the triangle measure 6 m and 8 m respectively?
8. Factor: $3x^2 + 7x + 2$

Part B: Using Double Angle Identities

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Write sin 4A in terms of functions of A. Do not simplify.

continued

Learning Activity 6.5 (continued)

2. Write each of the following in terms of only one circular function.
   a) $\cos^2 4x - \sin^2 4x$
   b) $2 \sin 2x \cos 2x$
   c) $4 \sin x \cos x$
   d) $1 - 2 \sin^2 5\alpha$
   e) $2 \cos^2 10\alpha - 1$
   f) $\sin \alpha \cos \alpha$
   g) $6 \sin 5x \cos 5x$

3. Use a double angle identity to find $\cos \dfrac{\pi}{5}$, if $\cos \dfrac{\pi}{10} = 0.95$.

4. If $0 < \theta < \dfrac{\pi}{2}$ and $\sin \theta = \dfrac{3}{5}$, find the exact values of $\sin 2\theta$ and $\cos 2\theta$.

5. Given that $\sin \alpha = \dfrac{4}{5}$ and $\cos \beta = \dfrac{5}{13}$, and neither $P(\alpha)$ not $P(\beta)$ are in the first quadrant, find
   a) $\sin 2\alpha$
   b) $\cos (\alpha - \beta)$
   c) $\cos 2\beta$
   d) $\sin (\alpha + \beta)$
   e) $\tan 2\alpha$
   f) $\tan (\alpha - \beta)$

6. Write each of the following in terms of only one circular function or a constant. Include any non-permissible values for each expression.
   a) $(\sin x + \cos x)^2$
   b) $\dfrac{\sin 2x}{\cos x}$

7. Prove the following identities.
   a) $\tan x + \cot x = 2 \csc 2x$

      Note: It is useful if you change all the functions to functions of a single argument, usually x, rather than a double value, $2x$.
   b) $\cos^4 x - \sin^4 x = \cos 2x$
   c) $\dfrac{2 \sin x}{\sin 2x} = \sec x$
   d) $\dfrac{\sec^2 x}{2 - \sec^2 x} = \sec 2x$
   e) $\sin 3x = 3 \sin x - 4 \sin^3 x$

      Hint: Write $3x$ as $(2x + x)$ to use the sum identity for sin. Continue to change everything to functions of x.

continued

Learning Activity 6.5 (continued)

8. Solve for x, where $0 < x < 2\pi$.

   a) $\sin 2x = \sqrt{2} \sin x$

   b) $\sin 2x = \sqrt{2} \cos x$

   c) $\sin 2x = \tan x$

   d) $\cos 2x = \sin x$

   e) $\tan 2x = \tan x$

   f) $\cos 2x = 2 - 5 \cos x$

9. Using a graphing calculator, graph each side of the equation in Question 8 as a separate function. Check for the points of intersection in the interval $(0, 2\pi)$ of the two functions to verify your answers to Question 8.

10. Given $\sin x = \frac{5}{7}$, $0 \le x < \frac{\pi}{2}$, find values for $\sin 2x$, $\cos 2x$, $\tan 2x$, $\csc 2x$, $\sec 2x$, and $\cot 2x$.

11. Consider the equation $\frac{1 - \cos 2x}{2} = \sin^2 x$.

    a) Verify the equation holds for $x = \frac{\pi}{6}$.

    b) Using technology, graph $y = \frac{(1 - \cos 2x)}{2}$ and $y = \sin^2 x$ on the same coordinate grid.

    c) Using the graph you created in (b), do you believe this is a possible identity?

    d) Verify algebraically that the equation is or is not an identity.

Lesson Summary

In this lesson, you learned about an additional form of trigonometric identities called double angle identities. This is the last form of trigonometric identities you are going to learn about. Using all the trigonometric identities you have learned so far, you should be able to solve many trigonometric equations, prove numerous trigonometric identities, and simplify various trigonometric expressions.

Assignment 6.3

Sum and Difference and Double Angle Identities

Total: 39 marks

You need to show all your steps as you work out your solutions, and make sure your answers are clear (include units, where appropriate). Check the Introduction for the Marking Guidelines that will be used for all assignments and examinations.

1. Without using a calculator, evaluate the following. *(2 × 2 marks each = 4 marks)*

   a) $\cos\dfrac{27\pi}{10}\cos\dfrac{19\pi}{20}+\sin\dfrac{27\pi}{10}\sin\dfrac{19\pi}{20}$

   b) $\dfrac{\tan\dfrac{\pi}{18}+\tan\dfrac{\pi}{9}}{1-\tan\dfrac{\pi}{18}\tan\dfrac{\pi}{9}}$

continued

Assignment 6.3: Sum and Difference and Double Angle Identities (continued)

2. Find the exact values of the coordinates of $P\left(\frac{5\pi}{12}\right)$. *(4 marks)*

3. Use a double angle identity to find $\cos\frac{\pi}{9}$, if $\sin\frac{\pi}{18} = 0.17$. *(2 marks)*

continued

Assignment 6.3: Sum and Difference and Double Angle Identities (continued)

4. Prove the following identities.

   a) $\sin(x + y) - \sin(x - y) = 2\cos x \sin y$ *(2 marks)*

   b) $\tan x = \dfrac{\sin 2x}{1 + \cos 2x}$ *(3 marks)*

continued

Assignment 6.3: Sum and Difference and Double Angle Identities (continued)

c) $\cos 3x = 4\cos^3 x - 3\cos x$ *(3 marks)*

Hint: Write $3x$ as $(2x + x)$ and change everything to functions of x.

continued

Assignment 6.3: Sum and Difference and Double Angle Identities (continued)

5. Solve $\sin 2x + \cos x = 0$ for x, where $0 < x < 2\pi$. *(4 marks)*

6. Given $\sin \alpha = -\frac{4}{5}$ and $\pi < \alpha < \frac{3\pi}{2}$ and $\sin \beta = \frac{12}{13}$ and $\frac{\pi}{2} < \beta < \pi$.

   Find the exact value of

   a) $\sin(\alpha - \beta)$ *(3 marks)*

continued

Assignment 6.3: Sum and Difference and Double Angle Identities (continued)

b) $\cos 2\beta$ *(2 marks)*

c) $\tan 2\beta$ *(4 marks)*

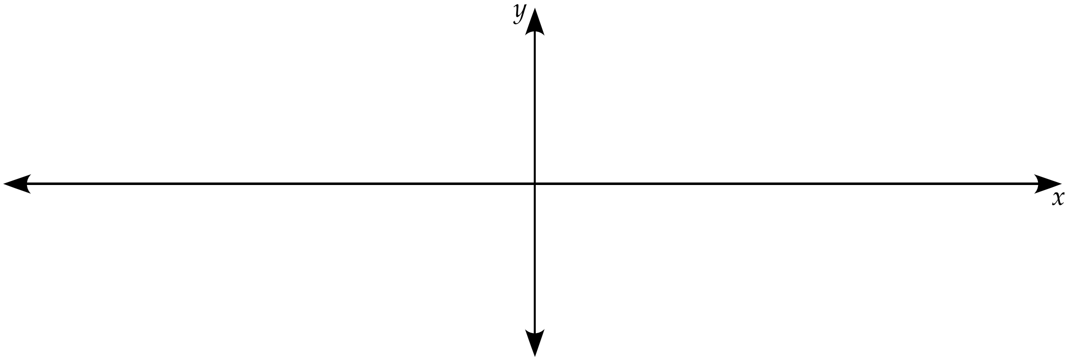
continued

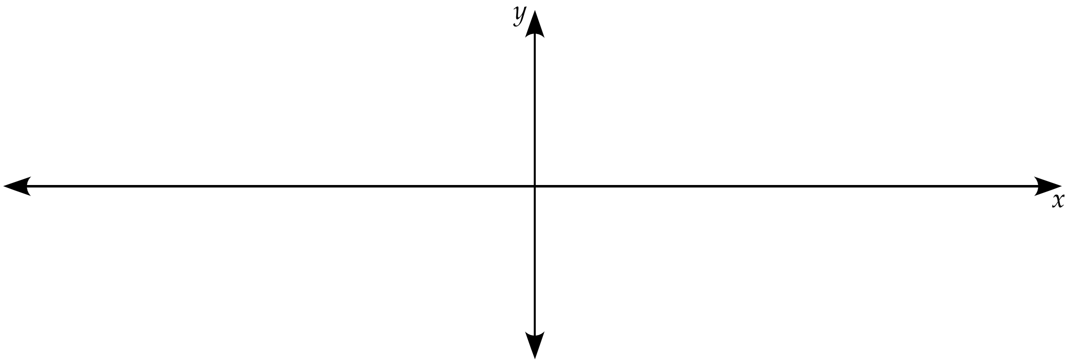
Assignment 6.3: Sum and Difference and Double Angle Identities (continued)

7. Consider the equation $\sin\left(\frac{3\pi}{2} + x\right) = -\cos x$.

   a) Verify the equation holds for $x = \frac{\pi}{3}$. *(1 mark)*

   b) Graph $y = \sin\left(\frac{3\pi}{2} + x\right)$ and $y = -\cos x$ on the coordinate grids below. *(4 marks)*





continued

Assignment 6.3: Sum and Difference and Double Angle Identities (continued)

c) Using the graph you created in (b), do you believe this is a possible identity? *(1 mark)*

d) Verify algebraically that the equation is or is not an identity. *(2 marks)*

MODULE 6 SUMMARY

In this module, you used the concept of the unit circle, which you learned about in Module 5, to help you solve trigonometric equations and develop trigonometric identities. You had previously learned how to solve linear trigonometric equations, and throughout this module you extended that knowledge to learn how to solve quadratic trigonometric equations.

You then learned about various identities built from the unit circle including Pythagorean identities, sum and difference identities, and double angle identities. Using these identities, you were able to simplify various trigonometric expressions, solve more complex trigonometric equations, and prove other trigonometric identities.

In the next module, you will be learning about exponential and logarithmic functions.

Submitting Your Assignments

It is now time for you to submit Assignments 6.1 to 6.3 to the Distance Learning Unit so that you can receive some feedback on how you are doing in this course. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 6 assignments and organize your material in the following order:

☐ Module 6 Cover Sheet (found at the end of the course Introduction)
☐ Assignment 6.1: Solving Trigonometric Equations
☐ Assignment 6.2: Using Elementary Identities
☐ Assignment 6.3: Sum and Difference and Double Angle identities

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Notes

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Module 6
Trigonometric Equations and Identities

Learning Activity Answer Keys

MODULE 6:
TRIGONOMETRIC EQUATIONS AND IDENTITIES

Learning Activity 6.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. What is the equation of the vertical asymptote of the function $y = \frac{1}{3x+5}$?
2. What is the amplitude of the function $y = 3\sin x - 4$?
3. Simplify: $\frac{6!}{3!2!}$
4. State an angle that is coterminal to 122°.
5. In which quadrant is $\theta = \frac{7\pi}{5}$ located?
6. State the non-permissible values of the function $f(x) = \frac{x}{2x-8}$.
7. Rationalize the denominator: $\frac{6}{\sqrt{7}}$
8. Simplify: $\frac{12a^5d^3e^2}{8a^6d^4e}$

Answers:

1. $x = -\frac{5}{3}$
2. 3
3. 60 $\left(\frac{6!}{3!2!} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3!2!} = \frac{6 \cdot 5 \cdot 4}{2} = \frac{120}{2} = 60\right)$
4. 482° (122° + 360° = 482°; other answers are possible)
5. Quadrant III $\left(\frac{7\pi}{5} > \pi \text{ and } \frac{7\pi}{5} < 3\frac{\pi}{2}\right)$
6. $x \neq 4$

7. $\dfrac{6\sqrt{7}}{7}$

8. $\dfrac{3e}{2ad}$

Part B: Solving Trigonometric Equations

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Solve the following equations over the indicated intervals. Provide exact answers using your unit circle values wherever possible. Round to 3 decimal places when necessary.

   a) $2\tan\theta + \dfrac{2}{\sqrt{3}} = 0,\ 0 \le \theta < 2\pi$

   Answer:

$$2\tan\theta = -\frac{2}{\sqrt{3}}$$

$$\tan\theta = -\frac{1}{\sqrt{3}}$$

$$\theta = \frac{5\pi}{6} \text{ and } \frac{11\pi}{6}$$

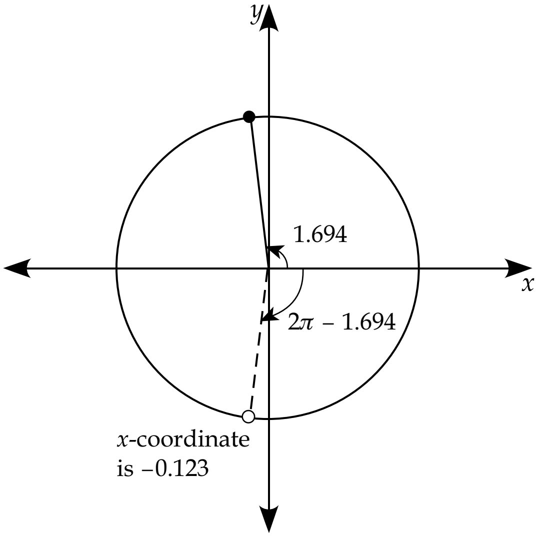
   b) $\cos x = -0.123,\ 0 \le x < 2\pi$

   Answer:

   Method 1

   Use your calculator to find $\cos^{-1}(-0.123) = 1.694$.

   Since 1.694 radians is more than $\dfrac{\pi}{2}$ and less than π, this angle is located in Quadrant II where cosine is negative. Cosine is also negative in Quadrant III. To find the measure of this angle, use the symmetry of the unit circle diagram.



The angle from 0 to 1.694 is the same measure as the angle from 2π back to the terminal arm on Quadrant III.

Quadrant III: $2\pi - 1.694 = 4.590$

Therefore, the two solutions are $x = 1.694$ and $x = 4.590$.

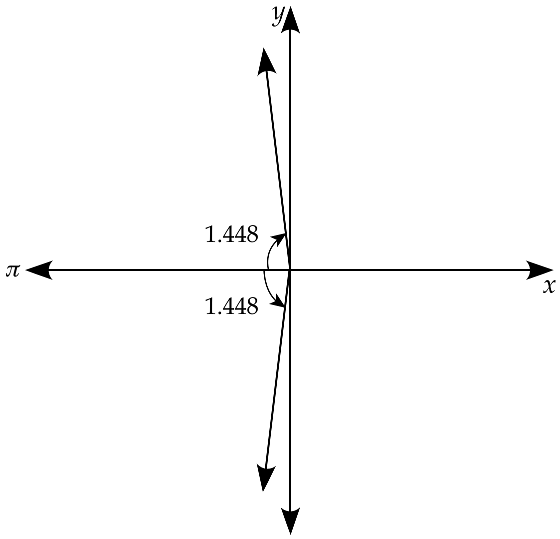
Method 2: Using Reference Angles

Find the reference angle by ignoring the negative.

$x_r = \cos^{-1}(0.123)$

$x_r = 1.448$

Now, using the reference angle, draw it in Quadrants II and III, where cosine has negative values.



$x = \pi - 1.448$
$x = \pi + 1.448$
$\therefore x = 1.694$ and $x = 4.590$

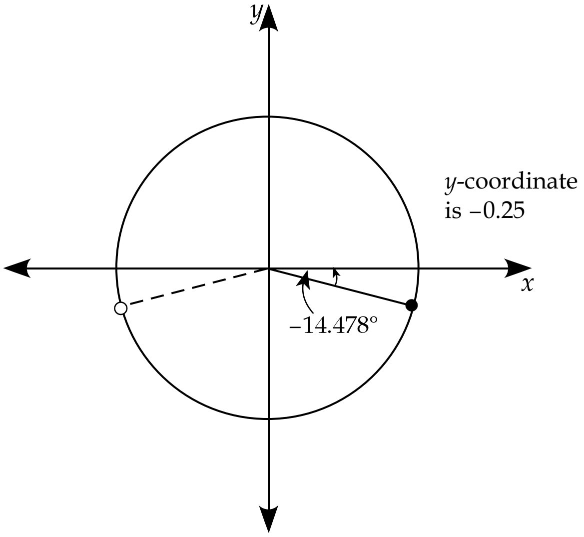
c) $\sin x = -0.25$, $0° \le x < 360°$

Answer:

$x = \sin^{-1}(-0.25) = -14.478°$

This angle is located in Quadrant IV, where sine is negative. However, it is not in the correct domain. To find an angle in the required domain, you can get the coterminal angle by adding 360°. The reference angle for this angle is 14.478°.

Sine is negative in Quadrants III and IV.



Quadrant III:

$x = 180° + 14.478° = 194.478°$

Quadrant IV:

$x = 360° - 14.478° = 345.522°$

The above calculation is based on the coterminal angle. $\theta = -14.478°$ is the calculated angle. $\theta = -14.478° + 360°$ is the coterminal angle in the required domain.

The two solutions are $x = 194.478°$ and $x = 345.522°$.

d) $2 \csc x = 4,\ -2\pi \leq x < 0$

Answer:

$$\csc x = 2$$

$$\frac{1}{\sin x} = 2$$

$$\sin x = \frac{1}{2}$$

$$x = \left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$$

To determine the solutions in the correct domain, subtract 2π from each of the above angles.

$$x = \frac{\pi}{6} - 2\pi = \frac{\pi}{6} - \frac{12\pi}{6} = -\frac{11\pi}{6}$$

$$x = \frac{5\pi}{6} - \frac{12\pi}{6} = -\frac{7\pi}{6}$$

The two solutions are $x = -\frac{11\pi}{6}$ and $x = -\frac{7\pi}{6}$.

2. Solve the following equations over the interval $0 \le \theta < 2\pi$. Provide exact answers.

   a) $\csc\theta - \sin\theta = 0$

   Answer:

$$\begin{aligned} \frac{1}{\sin\theta} - \sin\theta &= 0 \\ \frac{1-\sin^2\theta}{\sin\theta} &= 0 \\ 1-\sin^2\theta &= 0 \\ (1-\sin\theta)(1+\sin\theta) &= 0 \\ \sin\theta &= \pm 1 \\ \theta &= \frac{\pi}{2}, \frac{3\pi}{2} \end{aligned}$$

   Note: Because $y = \csc\theta$ has a restricted domain, check your solutions in the original question. Since $\csc\frac{\pi}{2}$ and $\csc\frac{3\pi}{2}$ are defined, they are both solutions to the question. Remember, when $\sin\theta = 0$, $\csc\theta$ is undefined.

   b) $\cot\theta - \cot\theta\sin\theta = 0$

   Answer:

$$\cot\theta\,(1-\sin\theta) = 0$$

$$\cot\theta = 0 \text{ or } \sin\theta = 1$$

$$\theta = \frac{\pi}{2}, \frac{2\pi}{2} \text{ or } \theta = \frac{\pi}{2}$$

   Thus, $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

3. Find the general solution to each of the following equations.

a) $\sin\theta = -\frac{1}{2}$, and $\tan\theta > 0$

Answer:

These conditions are satisfied in Quadrant III.

$$\theta = \frac{7\pi}{6}$$

The general solution is $\theta = \frac{7\pi}{6} + 2\pi n,\ n \in I$.

b) $2\csc\theta = -4$

Answer:

$$\csc\theta = -2$$

$$\frac{1}{\sin\theta} = -2$$

$$\sin\theta = -\frac{1}{2}$$

$$\theta = \frac{7\pi}{6} \text{ and } \frac{11\pi}{6}$$

The general solution is $\theta = \frac{7\pi}{6} + 2\pi n,\ n \in I$ and $\theta = \frac{11\pi}{6} + 2\pi n,\ n \in I$.

c) $2\cos^2\theta + \cos\theta = 1$

Answer:

Let $w = \cos\theta$

$$2w^2 + w - 1 = 0$$

$$(2w - 1)(w + 1) = 0$$

$$w = \frac{1}{2} \text{ and } w = -1$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3} \text{ and } \frac{5\pi}{3}$$

$$\cos\theta = -1$$

$$\theta = \pi$$

The general solution is $\theta = \frac{\pi}{3} + 2\pi n,\ n \in I, \theta = \frac{5\pi}{3} + 2\pi n,\ n \in I,$ and $\theta = 2\pi n,\ n \in I.$

d) $\tan^2 \theta - \tan \theta = 0$

Answer:

$\tan \theta (\tan \theta - 1) = 0$

$\tan \theta = 0$ and $\tan \theta = 1$

$\tan \theta = 0$

$\theta = 0, \pi, 2\pi$

$\tan \theta = 1$

$$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$$

The general solution is $\theta = \pi n,\ n \in I$ and $\theta = \frac{\pi}{4} + \pi n,\ n \in I.$

4. Solve the following equations over the indicated intervals. Provide exact answers.

a) $\cos^2 x - 2 \cos x = 0, 0 \le \theta < 2\pi$

Answer:

$\cos x(\cos x - 2) = 0$

$\cos x = 0$ or $\cos x = 2$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad$ $x =$ no answer, since range of $\cos x$ is $[-1, 1]$

$\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}$

b) $2 \cos^2 \theta + \cos \theta = 0, 0 \le \theta < 2\pi$

Answer:

$\cos \theta \ (2 \cos \theta + 1) = 0$

$\cos \theta = 0, \cos \theta = -\frac{1}{2}$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}$$

c) $2\sin^2\theta = \sin\theta,\ -\pi \le \theta < \pi$

Answer:

$$2\sin^2\theta - \sin\theta = 0$$

$$\sin\theta\,(2\sin\theta - 1) = 0$$

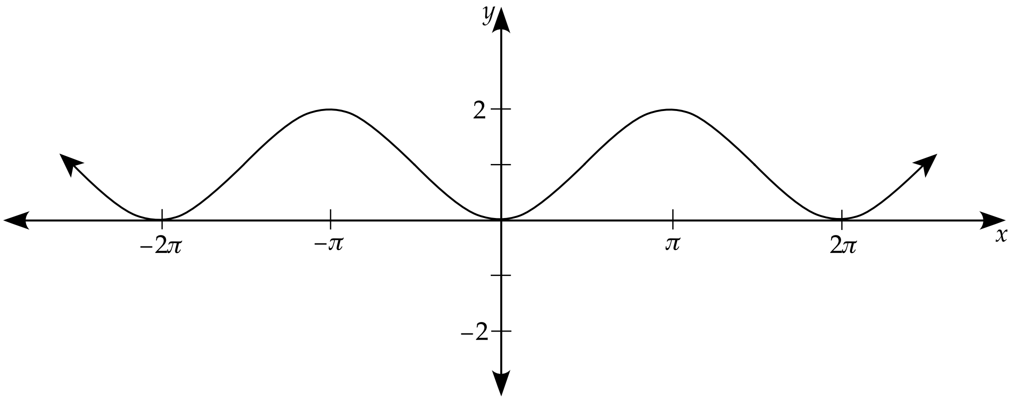
$$\sin\theta = 0 \qquad\qquad \sin\theta = \frac{1}{2}$$

$$\theta = -\pi \qquad\qquad \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

Note: π would be a solution of $\sin\theta = 0$, but π is not in the domain.

5. a) Graph the function $y = \sin\left(\theta - \frac{\pi}{2}\right) + 1$.

Answer:



b) Determine the equation of the zeros of the function $y = \sin\left(\theta - \frac{\pi}{2}\right) + 1$.

Answer:

The equation of the zeros is $\theta = 2\pi n,\ n \in I$.

c) Determine the solution of the equation $\sin\left(\theta - \frac{\pi}{2}\right) = -1$ based on the equation of the zeros of the function $y = \sin\left(\theta - \frac{\pi}{2}\right) + 1$.

Answer:

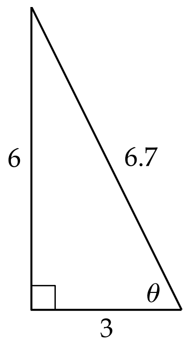
The solution of the equation $\sin\left(\theta - \frac{\pi}{2}\right) = -1$ is identical to the equation of the zeros of the function $y = \sin\left(\theta - \frac{\pi}{2}\right) + 1$ and is thus $\theta = 2\pi n, n \in I$.

Learning Activity 6.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

Use the triangle below to answer Questions 1 to 3.



1. Determine the tangent ratio.
2. Determine the cotangent ratio.
3. Determine the secant ratio.
4. What is the period of the function $f(\theta) = \sin\theta$?
5. Evaluate: ${}_5C_5$
6. Convert $\frac{\pi}{2}$ to degrees.
7. What is the domain of the function, $f(x) = \dfrac{1}{x+6}$?
8. Simplify: $\sqrt[4]{16x^4y^8}$

Answers:

1. $\tan\theta = 2 \left(\tan\theta = \dfrac{6}{3} = 2\right)$
2. $\cot\theta = \dfrac{1}{2} \left(\cot\theta = \dfrac{3}{6} = \dfrac{1}{2}\right)$
3. $\sec\theta = \dfrac{6.7}{3}$
4. 2π
5. 1

6. 90°
7. $\{x \mid x \in \Re, x \neq -6\}$
8. $2xy^2 \left(\sqrt[4]{16x^4y^8} = \sqrt[4]{2^4}\,\sqrt[4]{x^4}\,\sqrt[4]{\left(y^2\right)^4} = 2(x)\left(y^2\right) \right)$

Part B: Elementary Identities

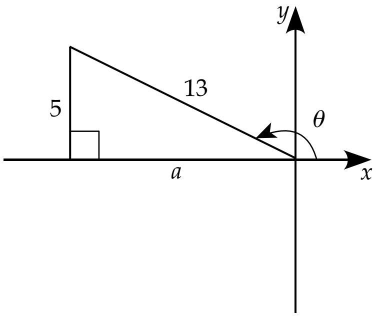
Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. In which quadrant does θ terminate if
   a) $\sin\theta > 0$ and $\cos\theta < 0$? *Answer:* Quadrant II
   b) $\sec\theta > 0$ and $\tan\theta < 0$? *Answer:* Quadrant IV
   c) $\cos\theta < 0$ and $\cot\theta > 0$? *Answer:* Quadrant III
   d) $\csc\theta < 0$ and $\tan\theta < 0$? *Answer:* Quadrant IV

2. Use the method of triangles to find the exact values of the other remaining circular functions.

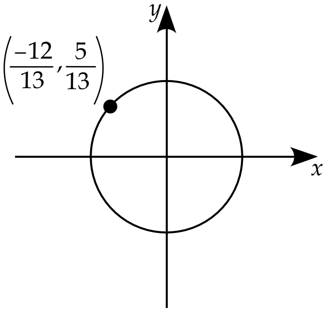
   a) $\sin\theta = \dfrac{5}{13}$ and $\tan\theta < 0$

   Answer:



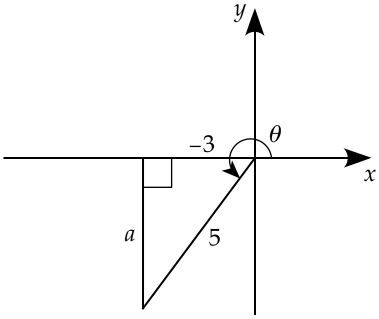
The third side is $a^2 + 5^2 = 13^2$ or $a = \pm 12$. Furthermore, since $\sin\theta > 0$ and $\tan\theta < 0$, it follows that $\theta \in$ Quadrant II. Therefore $a = -12$ and

$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$-\dfrac{12}{13}$	$-\dfrac{5}{12}$	$\dfrac{13}{5}$	$-\dfrac{13}{12}$	$-\dfrac{12}{5}$



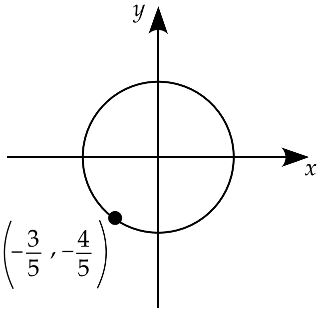
b) $\cos x = -\frac{3}{5}$ and $\csc x < 0$

Answer:



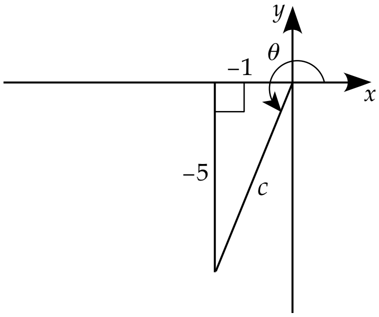
The third side is $a^2 + 3^2 = 5^2$ or $a = \pm 4$. Furthermore, since $\sin x < 0$ and $\cos x < 0$, it follows that $\theta \in$ Quadrant III. Therefore $a = -4$ and

sin *x*	tan *x*	csc *x*	sec *x*	cot *x*
$-\frac{4}{5}$	$\frac{4}{3}$	$-\frac{5}{4}$	$-\frac{5}{3}$	$\frac{3}{4}$



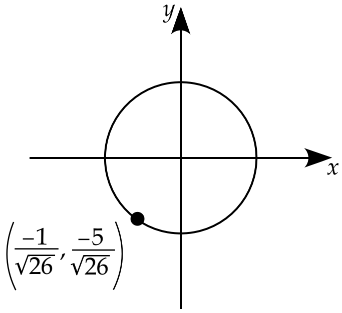
c) $\tan\theta = 5$ and $\cos\theta < 0$

Answer:



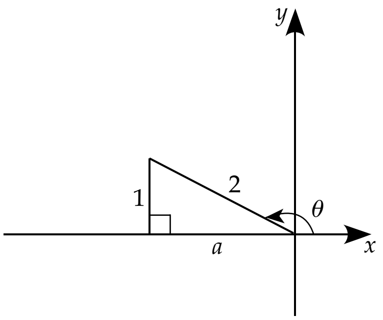
The third side is $5^2 + 1^2 = c^2$ or $c = \pm\sqrt{26}$. We use only $\sqrt{26}$, not $-\sqrt{26}$ because c is the hypotenuse. Furthermore, since $\tan\theta > 0$ and $\cos\theta < 0$, it follows that $\theta \in$ Quadrant III. Therefore,

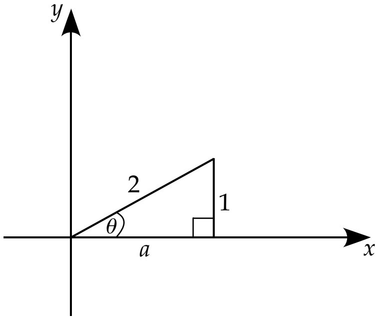
$\sin\theta$	$\cos\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$-\frac{5}{\sqrt{26}}$	$-\frac{1}{\sqrt{26}}$	$-\frac{\sqrt{26}}{5}$	$-\sqrt{26}$	$\frac{1}{5}$



d) csc $\theta = 2$

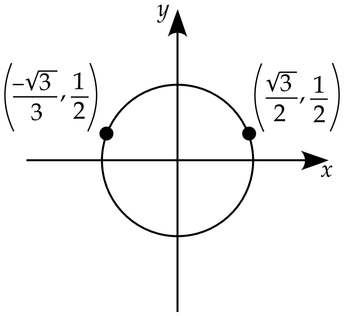
Answer:





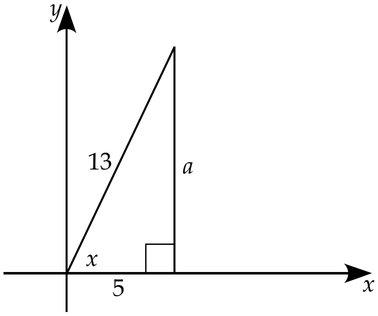
Since csc $\theta = 2$, it follows that sin $\theta = \dfrac{1}{2}$. The third side is $a^2 + 1^2 = 2^2$ or $a = \pm\sqrt{3}$. Furthermore, since sin $\theta > 0$, it follows that $\theta \in$ Quadrant I or Quadrant II. Thus, a can be either $\sqrt{3}$ or $-\sqrt{3}$ and

sin θ	cos θ	tan θ	sec θ	cot θ
$\dfrac{1}{2}$	$\pm\dfrac{\sqrt{3}}{2}$	$\pm\dfrac{1}{\sqrt{3}}$	$\pm\dfrac{2}{\sqrt{3}}$	$\pm\sqrt{3}$



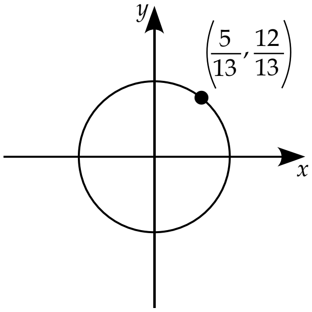
e) $\sec x = \frac{13}{5}$ and $\tan x > 0$

Answer:



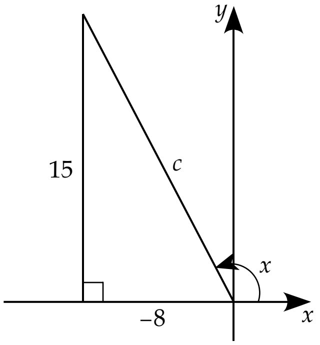
Since $\sec x = \frac{13}{5}$, it follows that $\cos x = \frac{5}{13}$. The third side is $a^2 + 5^2 = 13^2$ or $a = \pm 12$. Furthermore, since $\cos x > 0$ and $\tan x > 0$, then $x \in$ Quadrant I and

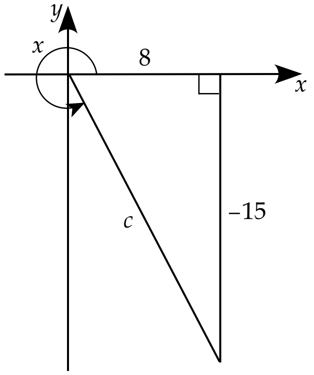
$\sin x$	$\cos x$	$\tan x$	$\csc x$	$\cot x$
$\frac{12}{13}$	$\frac{5}{13}$	$\frac{12}{5}$	$\frac{13}{12}$	$\frac{5}{12}$



f) $\cot x = -\dfrac{8}{15}$

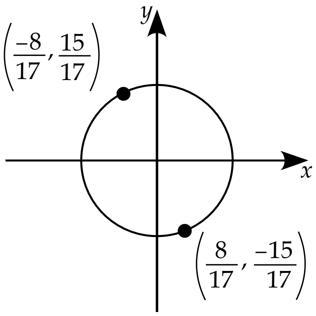
Answer:





Since $\cot x = -\dfrac{8}{15}$, it follows that $\tan x = -\dfrac{15}{8}$. The third side is $15^2 + 8^2 = c^2$ or $c = \pm 17$. Since c is the hypotenuse, $c = 17$. Furthermore, since $\tan x < 0$, it follows that $x \in$ Quadrant II or Quadrant IV, and

$\sin x$	$\cos x$	$\tan x$	$\csc x$	$\sec x$
$\pm\dfrac{15}{17}$	$\mp\dfrac{8}{17}$	$-\dfrac{15}{8}$	$\mp\dfrac{17}{15}$	$\pm\dfrac{17}{8}$



3. Use the method of Pythagorean identities to find the exact values of the other remaining circular functions.

   a) $\sin\theta = \frac{5}{13}$ and $\tan\theta < 0$

   Answer:

   This angle is located in Quadrant II where sine is positive and tangent is negative.

   Use the $\sin^2\theta + \cos^2\theta = 1$ identity to determine the cosine ratio. From the sine and cosine ratio, you can determine every other ratio.

$$\begin{aligned}\sin^2\theta + \cos^2\theta &= 1\\ \left(\frac{5}{13}\right)^2 + \cos^2\theta &= 1\\ \cos^2\theta &= 1 - \frac{25}{169}\\ \cos^2\theta &= \frac{144}{169}\\ \cos\theta &= -\frac{12}{13}\end{aligned}$$

$$\begin{aligned}\csc\theta &= \frac{13}{5}\\ \sec\theta &= -\frac{13}{12}\\ \tan\theta &= \frac{\frac{5}{13}}{-\frac{12}{13}} = -\frac{5}{12}\\ \cot\theta &= -\frac{12}{5}\end{aligned}$$

   b) $\cos x = -\frac{3}{5}$ and $\csc x < 0$

   Answer:

   This angle is located in Quadrant III where cosine (and therefore secant) and sine (and therefore cosecant) are negative.

$$\begin{aligned}\sin^2 x + \cos^2 x &= 1\\ \sin^2 x + \left(-\frac{3}{5}\right)^2 &= 1\\ \sin^2 x &= 1 - \frac{9}{25}\\ \sin^2 x &= \frac{16}{25}\\ \sin x &= -\frac{4}{5}\end{aligned}$$

$$\begin{aligned}\csc x &= -\frac{5}{4}\\ \sec x &= -\frac{5}{3}\\ \tan x &= \frac{-\frac{4}{5}}{-\frac{3}{5}} = \frac{4}{3}\\ \cot x &= \frac{3}{4}\end{aligned}$$

c) $\tan\theta = 5$ and $\cos\theta < 0$

Answer:

This angle is located in Quadrant IV where tangent is positive, cosine is negative, and sine is negative.

$$\cot\theta = \frac{1}{5}$$

Use the identity $\tan^2\theta + 1 = \sec^2\theta$ to determine the secant ratio.

$$\begin{aligned} \tan^2\theta + 1 &= \sec^2\theta \\ (5)^2 + 1 &= \sec^2\theta \\ 26 &= \sec^2\theta \\ -\sqrt{26} &= \sec\theta \end{aligned}$$

$$\therefore \cos\theta = -\frac{1}{\sqrt{26}}$$

Use the identity $\sin^2\theta + \cos^2\theta = 1$ to determine the sine ratio.

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \sin^2\theta + \left(-\frac{1}{\sqrt{26}}\right) &= 1 \\ \sin^2\theta &= 1 - \frac{1}{\sqrt{26}} \\ \sin^2\theta &= \frac{25}{26} \\ \sin\theta &= -\frac{5}{\sqrt{26}} \end{aligned}$$

$$\therefore \csc\theta = -\frac{\sqrt{26}}{5}$$

d) csc $\theta = 2$ and cos $\theta > 0$

Answer:

This angle is located in Quadrant I where sine (and therefore cosecant) and cosine are positive.

$$\sin\theta = \frac{1}{2}$$

Use the $\sin^2\theta + \cos^2\theta = 1$ identity to determine the cosine ratio. Then you can determine the other three ratios.

$$\begin{aligned}\sin^2\theta + \cos^2\theta &= 1\\ \left(\frac{1}{2}\right)^2 + \cos^2\theta &= 1\\ \cos^2\theta &= 1 - \frac{1}{4}\\ \cos^2\theta &= \frac{3}{4}\\ \cos\theta &= \frac{\sqrt{3}}{2}\end{aligned}$$

$$\begin{aligned}\therefore \sec\theta &= \frac{2}{\sqrt{3}}\\ \tan\theta &= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}\\ \cot\theta &= \sqrt{3}\end{aligned}$$

Note: These are special values on the unit circle. You may have found all the ratios by using the unit circle and your knowledge of the relationships among the six functions.

e) $\sec x = \frac{13}{5}$ and $\tan x > 0$

Answer:

This angle is located in Quadrant I where cosine (and therefore secant) and tangent are positive.

$$\cos x = \frac{5}{13}$$

Use the $\sin^2 x + \cos^2 x = 1$ identity to determine the sine ratio and then determine the other three ratios.

$$\sin^2 x + \cos^2 x = 1$$

$$\sin^2 x + \left(\frac{5}{13}\right)^2 = 1$$

$$\sin^2 x = 1 - \frac{25}{169}$$

$$\sin^2 x = \frac{144}{169}$$

$$\sin x = \frac{12}{13}$$

$$\therefore \csc x = \frac{13}{12}$$

$$\tan x = \frac{\frac{12}{13}}{\frac{5}{13}} = \frac{12}{5}$$

$$\cot x = \frac{5}{12}$$

f) $\cot x = -\frac{8}{15}$ and $\sin x > 0$

Answer:

This angle is located in Quadrant II where tangent (and therefore cotangent) is negative and sine is positive.

$$\tan x = -\frac{15}{8}$$

Use the $\tan^2 x + 1 = \sec^2 x$ identity to determine the secant and cosine ratios.

$$\begin{aligned} \tan^2 x + 1 &= \sec^2 x \\ \left(-\frac{15}{8}\right)^2 + 1 &= \sec^2 x \\ \frac{225}{64} + 1 &= \sec^2 x \\ \frac{289}{64} &= \sec^2 x \\ -\frac{17}{8} &= \sec x \end{aligned}$$

Note: Cosine and, therefore, secant are negative in Quadrant II.

$$\therefore \cos x = -\frac{8}{17}$$

Use the $\sin^2 x + \cos^2 x = 1$ identity to determine the sine and cosecant ratios.

$$\begin{aligned} \sin^2 x + \cos^2 x &= 1 \\ \sin^2 x + \left(-\frac{8}{17}\right)^2 &= 1 \\ \sin^2 x &= 1 - \frac{64}{289} \\ \sin^2 x &= \frac{225}{289} \\ \sin x &= \frac{15}{17} \end{aligned}$$

$$\therefore \csc x = \frac{17}{15}$$

4. Prove that each of the following is an identity.

a) $\cos x \sec x = 1$

Answer:

$$\text{LHS} = \cos x\left(\frac{1}{\cos x}\right) = 1 = \text{RHS}$$

b) $\csc x \sin x = 1$

Answer:

$$\text{LHS} = \left(\frac{1}{\sin x}\right)\sin x = 1 = \text{RHS}$$

c) $\tan x \cot x = 1$

Answer:

$$\text{LHS} = \tan x\left(\frac{1}{\tan x}\right) = 1 = \text{RHS}$$

d) $\cot x \sin x = \cos x$

Answer:

$$\text{LHS} = \frac{\cos x}{\sin x}\sin x = \cos x = \text{RHS}$$

e) $\tan x \cos x = \sin x$

Answer:

$$\text{LHS} = \frac{\sin x}{\cos x}\cos x = \sin x = \text{RHS}$$

f) $\dfrac{\sec x}{\csc x} = \tan x$

Answer:

$$\begin{aligned}\text{LHS} &= \frac{\sec x}{\csc x}\\ &= \frac{\dfrac{1}{\cos x}}{\dfrac{1}{\sin x}}\\ &= \frac{1}{\cos x}\cdot\frac{\sin x}{1}\\ &= \frac{\sin x}{\cos x}\\ &= \tan x\\ &= \text{RHS}\end{aligned}$$

g) $\cos^2 x - \sin^2 x = 1 - 2\sin^2 x$

Answer:

$$\begin{aligned}\text{LHS} &= \cos^2 x - \sin^2 x \\ &= (1 - \sin^2 x) - \sin^2 x \\ &= 1 - 2\sin^2 x \\ &= \text{RHS}\end{aligned}$$

h) $\cos^4 x - \sin^4 x = 1 - 2\sin^2 x$

Answer:

$$\begin{aligned}\text{LHS} &= \cos^4 x - \sin^4 x \\ &= \left(\cos^2 x + \sin^2 x\right)\left(\cos^2 x - \sin^2 x\right) \\ &= (1)\left(\cos^2 x - \sin^2 x\right) \\ &= \cos^2 x - \sin^2 x \\ &= \left(1 - \sin^2 x\right) - \sin^2 x \\ &= 1 - 2\sin^2 x \\ &= \text{RHS}\end{aligned}$$

i) $\dfrac{1}{1 - \sin\theta} + \dfrac{1}{1 + \sin\theta} = 2\sec^2\theta$

Answer:

$$\begin{aligned}\text{LHS} &= \frac{1}{1 - \sin\theta} + \frac{1}{1 + \sin\theta} \\ &= \frac{(1 + \sin\theta) + (1 - \sin\theta)}{(1 + \sin\theta)(1 - \sin\theta)} \\ &= \frac{2}{1 - \sin^2\theta} \\ &= \frac{2}{\cos^2\theta} \\ &= 2\sec^2\theta \\ &= \text{RHS}\end{aligned}$$

j) $\sec^2\theta - \csc^2\theta = \tan^2\theta - \cot^2\theta$

Answer:

$$\begin{aligned}\text{LHS} &= \sec^2\theta - \csc^2\theta \\ &= \left(\tan^2\theta + 1\right) - \left(\cot^2\theta + 1\right) \\ &= \tan^2\theta + 1 - \cot^2\theta - 1 \\ &= \tan^2\theta - \cot^2\theta \\ &= \text{RHS}\end{aligned}$$

k) $\dfrac{\csc^2\theta - 1}{\csc^2\theta} = \cos^2\theta$

Answer:

$$\begin{aligned}
\text{LHS} &= \frac{\csc^2\theta - 1}{\csc^2\theta} \\
&= \frac{\dfrac{1}{\sin^2\theta} - 1}{\dfrac{1}{\sin^2\theta}} \\
&= \frac{\dfrac{1-\sin^2\theta}{\sin^2\theta}}{\dfrac{1}{\sin^2\theta}} \\
&= \frac{1-\sin^2\theta}{\sin^2\theta}\cdot\frac{\sin^2\theta}{1} \\
&= \frac{1-\sin^2\theta}{1} \\
&= \cos^2\theta \\
&= \text{RHS}
\end{aligned}$$

l) $\dfrac{1-\sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1+\sin\theta}$

Answer:

Looking at the left-hand side, both $\sin\theta$ and $\cos\theta$ are first degree. The only identities you have are for $\sin^2\theta$ and $\cos^2\theta$. It is necessary to create either $\sin^2\theta$ or $\cos^2\theta$ and then try to simplify. Multiplying $\dfrac{1-\sin\theta}{\cos\theta}$ by 1 doesn't change the expression, just the form of the expression.

Method 1: Multiply the LHS by $\dfrac{1+\sin\theta}{1+\sin\theta}$.

$$\begin{aligned}
\text{LHS} &= \left(\frac{1-\sin\theta}{\cos\theta}\right)\left(\frac{1+\sin\theta}{1+\sin\theta}\right)\\
&= \frac{1-\sin^2\theta}{\cos\theta\,(1+\sin\theta\,)}\\
&= \frac{\cos^2\theta}{\cos\theta\,(1+\sin\theta\,)}\\
&= \frac{\cos\theta}{1+\sin\theta}\\
&= \text{RHS}
\end{aligned}$$

Method 2: Multiply the LHS by $\dfrac{\cos\theta}{\cos\theta}$.

$$\begin{aligned}
\text{LHS} &= \left(\frac{1-\sin\theta}{\cos\theta}\right)\left(\frac{\cos\theta}{\cos\theta}\right)\\
&= \frac{\cos\theta\,(1-\sin\theta\,)}{\cos^2\theta}\\
&= \frac{\cos\theta\,(1-\sin\theta\,)}{1-\sin^2\theta}\\
&= \frac{\cos\theta\,(1-\sin\theta\,)}{(1+\sin\theta\,)(1-\sin\theta\,)}\\
&= \frac{\cos\theta}{1+\sin\theta}\\
&= \text{RHS}
\end{aligned}$$

Note: You could also have multiplied the RHS by $\dfrac{\cos\theta}{\cos\theta}$ or by $\dfrac{1-\sin\theta}{1-\sin\theta}$.

Try both of these to practise the algebraic manipulations for the proof.

m) $\sin x \cos x = \dfrac{1}{\tan x + \cot x}$

Answer:

$$\begin{aligned}
\text{RHS} &= \frac{1}{\tan x + \cot x} \\
&= \frac{1}{\dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}} \\
&= \frac{1}{\dfrac{\sin^2 x + \cos^2 x}{\cos x \sin x}} \\
&= (1)\frac{\cos x \sin x}{\sin^2 x + \cos^2 x} \\
&= \frac{\cos x \sin x}{1} \\
&= \cos x \sin x \\
&= \text{LHS}
\end{aligned}$$

n) $\dfrac{1 + \cos^2 x}{\sin^2 x} = 2\csc^2 x - 1$

Answer:

$$\begin{aligned}
\text{RHS} &= 2\csc^2 x - 1 \\
&= 2\left(\frac{1}{\sin x}\right)^2 - 1 \\
&= \frac{2}{\sin^2 x} - 1 \\
&= \frac{2 - \sin^2 x}{\sin^2 x} \\
&= \frac{2 - \left(1 - \cos^2 x\right)}{\sin^2 x} \\
&= \frac{1 + \cos^2 x}{\sin^2 x} \\
&= \text{LHS}
\end{aligned}$$

5. Determine the non-permissible values of each expression.

   a) $\dfrac{\sin x + \cos^2 x \csc x}{\csc x}$

   Answer:

   Since $\csc x = \dfrac{1}{\sin x}$, $\csc x$ will never be zero. The non-permissible values of this expression occur when $\sin x = 0$. The non-permissible values are $x = k\pi, k \in I$.

b) $\dfrac{\cot x}{\cos x \csc x}$

Answer:

The non-permissible values of this expression occur when $\sin x = 0$ and $\cos x = 0$ and are thus $x = k\pi$, $k \in I$, and $x = \dfrac{(2k+1)\pi}{2}$, $k \in I$. (Recall this expression is the equivalent of saying "all the odd multiples of $\dfrac{\pi}{2}$," which can also be written as $x = \dfrac{\pi}{2} + 2k\pi$ and $x = \dfrac{3\pi}{2} + 2k\pi$, $k \in I$.)

c) $\sin x \cos x \csc x$

Answer:

The non-permissible values of this expression occur when $\sin x = 0$ since $\csc x = \dfrac{1}{\sin x}$, and are thus $x = k\pi$, $k \in I$.

d) $\dfrac{\cos^2 x}{\sin x} + \sin x$

Answer:

The non-permissible values of this expression occur when $\sin x = 0$ and are thus $x = k\pi$, $k \in I$.

e) $\sin^2 x \cos x + \cos^3 x$

Answer:

There are no non-permissible values for this expression.

f) $\dfrac{\cos x}{\sin x}$

Answer:

The non-permissible values of this expression occur when $\sin x = 0$ and are thus $x = k\pi$, $k \in I$.

g) $\dfrac{1}{\tan x}$

Answer:

The non-permissible vales of this expression occur when tan $x = 0$ and, since $\tan x = \dfrac{\sin x}{\cos x}$, the non-permissible values are also where $\cos x = 0$.

$$\tan x = 0 \text{ at } x = k\pi,\ k \in I$$

$$\cos x = 0 \text{ at } x = \frac{\pi}{2} + 2k\pi,\ \frac{3\pi}{2} + 2k\pi,\ k \in I$$

Note that both expressions in (f) and (g) above are identities for cot x. Yet, since their domains are different, their graphs are not exactly the same. The only difference between the graphs of $y = \dfrac{1}{\tan x}$ and $y = \dfrac{\cos x}{\sin x}$ is that there will be more points represented by open circles (indicating holes) in the graph of $y = \dfrac{1}{\tan x}$.

6. Explain why verifying that two sides of a trigonometric identity are equal for a given value is insufficient to conclude the identity is valid. Give an example of a trigonometric equation that works for at least one value of θ but does not work for every value of θ.

   Answer:

   It is insufficient to prove an identity works for all angle values by proving the identity works for one or a few angle values. For example, the equation $\sin^2 x = \cos^2 x$ works when x is an odd multiple of $\dfrac{\pi}{4}$. However, this equation does not work for all angle values. If you let $x = 0$, then $\sin^2 x = 0$ and $\cos^2 x = 1$, which are not equivalent. Therefore, in order to prove a trigonometric identity is valid, you need to manipulate the expression without substituting in a particular angle value.

7. Using technology, determine if the following equations may be trigonometric identities. Explain your reasoning. If you believe an equation is a trigonometric identity, verify that the two sides of the identity are equivalent.

   a) $\cos\theta\cot\theta + \sin\theta = \csc\theta$

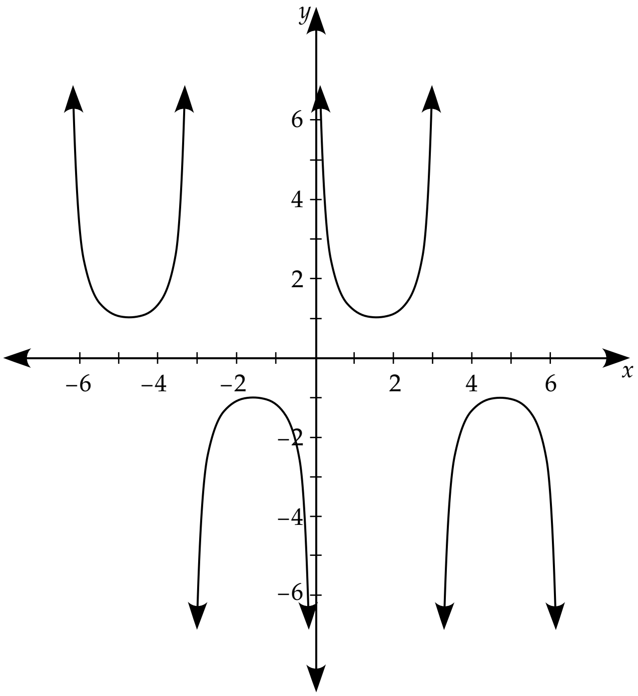
      Answer:

      Graph both the functions $y = \cos\theta\cot\theta + \sin\theta$ and $y = \csc\theta$, using technology, on the same coordinate grid.

      Note: You may need to write each function in terms of sine, cosine, and tangent, as some forms of technology will not accept the reciprocal trigonometric functions.

      Therefore, $y = \cos\theta\cot\theta + \sin\theta$ is the same as

      $y = \cos\theta\left(\dfrac{1}{\tan\theta}\right) + \sin\theta$ and $y = \csc\theta$ is the same as $y = \dfrac{1}{\sin\theta}$.



      As you can see, if you graph both the function $y = \cos\theta\cot\theta + \sin\theta$ and $y = \csc\theta$ on the same coordinate grid, these functions completely overlap. Each of the function values, corresponding to a specific angle for each function, is identical. Therefore, it is likely that this equation represents a trigonometric identity.

Verify the equation is an identity:

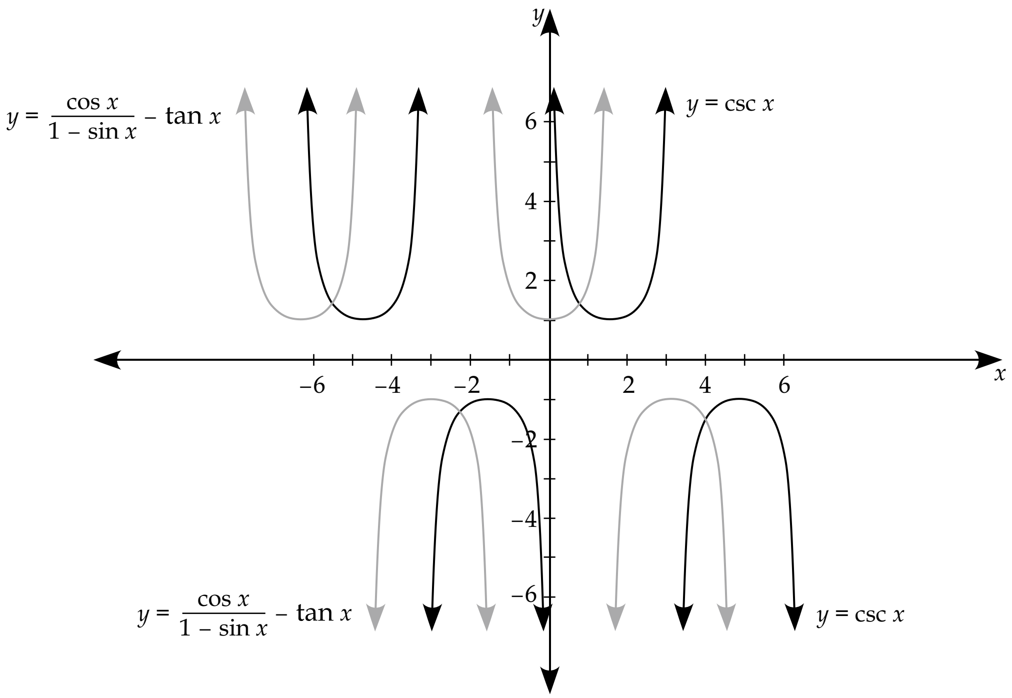
$$\begin{aligned}\text{LHS} &= \cos\theta\left(\frac{\cos\theta}{\sin\theta}\right)+\sin\theta\\ &= \frac{\cos^2\theta}{\sin\theta}+\sin\theta\\ &= \frac{\left(\cos^2\theta+\sin^2\theta\right)}{\sin\theta}\\ &= \frac{1}{\sin\theta}\\ &= \csc\theta\\ &= \text{RHS}\end{aligned}$$

As both sides of the equation are equivalent, this equation is a trigonometric identity.

b) $\dfrac{\cos x}{1-\sin x}-\tan x=\csc x$

Answer:

Graph both the functions $y=\dfrac{\cos x}{1-\sin x}-\tan x$ and $y=\csc x=\dfrac{1}{\sin x}$ on the same coordinate grid.



As you can see, these functions do not overlap completely. They have different function values for most angle values. Therefore, this equation is not a trigonometric identity. If this equation were a trigonometric identity, the function values for each and every angle value would be identical.

8. Consider the equation $\cot x + 1 = \csc x(\sin x + \cos x)$.

   a) Verify this equation holds for $x = \frac{3\pi}{4}$.

   Answer:

   Let $x = \frac{3\pi}{4}$.

$$\begin{aligned} \text{LHS} &= \cot\left(\frac{3\pi}{4}\right) + 1 \\ &= \frac{1}{\tan\left(\frac{3\pi}{4}\right)} + 1 \\ &= \frac{1}{-1} + 1 \\ &= -1 + 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{RHS} &= \csc\left(\frac{3\pi}{4}\right)\left(\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4}\right) \\ &= \frac{1}{\sin\left(\frac{3\pi}{4}\right)}\left(\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4}\right) \\ &= \frac{1}{\frac{\sqrt{2}}{2}}\left(\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{\sqrt{2}}{2}\right)\right) \\ &= \frac{2}{\sqrt{2}}(0) \\ &= 0 \end{aligned}$$

   LHS = RHS

   Therefore, this equation holds for $x = \frac{3\pi}{4}$.

   b) Verify this equation is a trigonometric identity and thus holds for all values of x.

   Answer:

$$\begin{aligned} \text{RHS} &= \csc x \sin x + \csc x \cos x \\ &= \frac{1}{\sin x}\sin x + \frac{1}{\sin x}\cos x \\ &= \frac{\sin x}{\sin x} + \frac{\cos x}{\sin x} \\ &= 1 + \cot x \\ &= \text{LHS} \end{aligned}$$

   Therefore, this trigonometric identity holds for all values of x.

Learning Activity 6.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. What is the exact value of cos 120°?
2. Find the positive coterminal angle for $-14°$ in the interval $0° \leq \theta \leq 360°$.
3. What is the last term in the expansion of $(2x^2 - 3y)^6$?
4. Solve $\sin \theta = 2$ in the interval $[0, 2\pi]$.
5. What is the area of a triangle with a height of 16 m and a base of 5 m?
6. Is $x = 5$ a solution to the inequality $x^2 - 4x + 3 \leq 0$?
7. Which is the better deal, a package of 4 pens for \$1.26 or a package of 3 pens for \$0.99?
8. Evaluate: $\frac{121}{7} \div \frac{44}{49}$

Answers:

1. $-\frac{1}{2}$
2. 346° $(-14° + 360° = 346°)$
3. $729y^6$ $((-3y)^6 = 729y^6)$
4. No solution. ($\sin \theta$ is never greater than 1)
5. $40m^2$ $\left(\frac{16(5)}{2} = 8(5) = 40\right)$
6. No $\left(5^2 - 4(5) + 3 = 25 - 20 + 3 = 8 \not\leq 0\right)$
7. 4 pens for \$1.26 (3 packages of 4 pens would give you 12 pens for \$3.78;
   4 packages of 3 pens would give you 12 pens for \$3.96)
8. $\frac{77}{4}$ $\left(\frac{121}{7} \div \frac{44}{49} = \frac{121}{7} \cdot \frac{49}{44} = \frac{11}{1} \cdot \frac{7}{4} = \frac{77}{4}\right)$

Part B: Solving Trigonometric Equations by Using Trigonometric Identities

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Solve each equation for x where $0° \le x < 360°$. Use exact values where possible; otherwise, round to one decimal place.

   a) $\tan^2 x + \sec^2 x = 3$

   Answer:

$$\begin{aligned} \tan^2 x + \sec^2 x &= 3 \\ \tan^2 x + \left(1 + \tan^2 x\right) &= 3 \\ 2\tan^2 x &= 2 \\ \tan^2 x &= 1 \\ \tan x &= \pm 1 \\ x &= 45°,\ 135°,\ 225°,\ 315° \end{aligned}$$

   b) $\sin x = \cos x$

   Answer:

$$\begin{aligned} \sin x &= \cos x \\ \frac{\sin x}{\cos x} &= 1 \qquad \text{provided } \cos x \neq 0 \\ \tan x &= 1 \\ x &= 45°,\ 225° \end{aligned}$$

   c) $\tan x = \sin x \tan x$

   Answer:

$$\begin{aligned} \tan x &= \sin x \tan x \\ \tan x - \sin x \tan x &= 0 \\ \tan x\left(1 - \sin x\right) &= 0 \\ \tan x = 0 &\text{ or } \sin x = 1 \\ x &= 0°,\ 180°,\ 90° \end{aligned}$$

d) $2 \cos x = 3 \tan x$

Answer:

$$2 \cos x = 3 \tan x$$

$$2 \cos x = 3 \cdot \frac{\sin x}{\cos x}$$

$$2 \cos^2 x = 3 \sin x \quad \text{provided } \cos x \neq 0$$

$$2\left(1 - \sin^2 x\right) = 3 \sin x$$

$$2 - 2 \sin^2 x = 3 \sin x$$

$$0 = 2 \sin^2 x + 3 \sin x - 2$$

$$0 = \left(2 \sin x - 1\right)\left(\sin x + 2\right)$$

$$\sin x = \frac{1}{2} \text{ or } \sin x = \cancel{-2}$$

$$x = 30°, 150° \ \left(\text{No solution for } \sin x = -2.\right)$$

e) $\tan^2 x = \sec x + 1$

Answer:

$$\tan^2 x = \sec x + 1$$

$$\sec^2 x - 1 = \sec x + 1$$

$$\sec^2 x - \sec x - 2 = 0$$

$$\left(\sec x + 1\right)\left(\sec x - 2\right) = 0$$

$$\sec x = -1 \text{ or } \sec x = 2$$

$$\cos x = -1 \text{ or } \cos x = \frac{1}{2}$$

$$x = 180°,\ 60°,\ 300°$$

f) $3 \tan^2 x = \sec x \tan x$

Answer:

$$3 \tan^2 x = \sec x \tan x$$

$$3 \tan^2 x - \sec x \ \tan x = 0$$

$$\tan x(3 \tan x - \sec x) = 0$$

$$\tan x\left(3 \cdot \frac{\sin x}{\cos x} - \frac{1}{\cos x}\right) = 0$$

$$\tan x\left(\frac{3 \sin x - 1}{\cos x}\right) = 0$$

$$\frac{(\tan x)(3 \sin x - 1)}{\cos x} = 0$$

$$(\tan x)(3 \sin x - 1) = 0$$

$$\tan x = 0 \text{ or } \sin x = \frac{1}{3}$$

$$x = 0°,\ 180°,\ 19.5°,\ 160.5°$$

2. Solve each equation for x where $0 \leq x < 2\pi$. Use exact values where possible; otherwise, round to two decimal places.

a) $2 \cos^2 x - 1 = -\sin x$

Answer:

$$2 \cos^2 x - 1 = -\sin x$$

$$2\left(1 - \sin^2 x\right) - 1 = -\sin x$$

$$2 - 2 \sin^2 x - 1 = -\sin x$$

$$0 = 2 \sin^2 x - \sin x - 1$$

$$0 = (2 \sin x + 1)(\sin x - 1)$$

$$\sin x = -\frac{1}{2} \ \text{ or } \ \sin x = 1$$

$$x = \frac{7\pi}{6},\ \frac{11\pi}{6},\ \frac{\pi}{2}$$

b) $4\cos^2 x = 3$

Answer:

$$\begin{aligned} 4\cos^2 x &= 3 \\ \cos^2 x &= \frac{3}{4} \\ \cos x &= \pm\frac{\sqrt{3}}{2} \\ x &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned}$$

c) $2\sec^2 x + \tan x = 2$

Answer:

$$\begin{aligned} 2\sec^2 x + \tan x &= 2 \\ 2\left(\tan^2 x + 1\right) + \tan x &= 2 \\ 2\tan^2 x + 2 + \tan x &= 2 \\ 2\tan^2 x + \tan x &= 0 \\ \tan x\left(2\tan x + 1\right) &= 0 \end{aligned}$$

$$\tan x = 0 \text{ or } \tan x = -\frac{1}{2}$$

$$x = 0, \pi, 2.68, 5.82$$

d) $2\cos^2 x + 3\sin x = 0$

Answer:

$$2\cos^2 x + 3\sin x = 0$$

$$2\left(1 - \sin^2 x\right) + 3\sin x = 0$$

$$2 - 2\sin^2 x + 3\sin x = 0$$

$$0 = 2\sin^2 x - 3\sin x - 2$$

$$0 = \left(2\sin x + 1\right)\left(\sin x - 2\right)$$

$$\sin x = -\frac{1}{2} \text{ or } \sin x = 2$$

$$x = \frac{7\pi}{6}, \frac{11\pi}{6}$$

3. Solve each equation for x where $0 \le x < 2\pi$.

a) $\sin x + \cos x = 1$

Answer:

$$\sin x + \cos x = 1$$

$$\sin x = 1 - \cos x$$

$$\left(\sin x\right)^2 = \left(1 - \cos x\right)^2$$

Square both sides. Using this step means you need to check for extraneous roots.

$$\sin^2 x = 1 - 2\cos x + \cos^2 x$$

$$(1 - \cos^2 x) = 1 - 2\cos x + \cos^2 x$$

$$0 = 2\cos^2 x - 2\cos x$$

$$0 = 2\cos x(\cos x - 1)$$

$$\cos x = 0 \text{ or } \cos x = 1$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}, 0$$

Possible values of $x = 0, \frac{\pi}{2}, \frac{3\pi}{2}$.

Check for extraneous roots:

for $x = 0$, $\sin 0 = 0$ and $1 - \cos 0 = 1 - 1 = 0$ $\quad \therefore$ LHS = RHS

for $x = \frac{\pi}{2}$, $\sin \frac{\pi}{2} = 1$ and $1 - \cos \frac{\pi}{2} = 1 - 0 = 1$ $\quad \therefore$ LHS = RHS

for $x = \frac{3\pi}{2}$, $\sin \frac{3\pi}{2} = -1$ and $1 - \cos \frac{3\pi}{2} = 1 - 0 = 1$ $\quad \therefore$ LHS $\neq$ RHS

Therefore, $\frac{3\pi}{2}$ is not a root.

Only solutions are: $x = 0, \frac{\pi}{2}$

b) $\sin x + \cos x = 0$

Answer:

Since the RHS is 0, this equation can be done quite easily. We don't need to square both sides.

$$\sin x = -\cos x$$

$$\frac{\sin x}{\cos x} = -1$$

$$\tan x = -1$$

Therefore, $x = \frac{3\pi}{4}, \frac{7\pi}{4}$.

c) $\sin x - 2 = \cos x$

Answer:

$$\sin x - 2 = \cos x$$

$$(\sin x - 2)^2 = (\cos x)^2 \qquad \text{Square both sides.}$$

$$\sin^2 x - 4 \sin x + 4 = \cos^2 x$$

$$\sin^2 x - 4 \sin x + 4 = (1 - \sin^2 x)$$

$$2 \sin^2 x - 4 \sin x + 3 = 0$$

The discriminant ($b^2 - 4ac$) of the quadratic is:

$$(-4)^2 - 4(2)(3)$$

$$16 - 24$$

$$-8 < 0$$

Hence, no solution! Observe the original equation is $\sin x - 2 = \cos x$. If you rearrange this equation to get $\sin x - \cos x = 2$, you should be able to reason why there is no solution to this equation. You know the range of both the sine and cosine functions is $[-1, 1]$. Using those values, the only way to get a solution of 2, is for $\sin x = 1$ and $\cos x = -1$ *at the same time*. This never occurs, and therefore there is no solution to the above equation.

d) $\sin x - \cos x = 1$

Answer:

$$\begin{aligned}
\sin x - \cos x &= 1 \\
\sin x &= 1 + \cos x \\
(\sin x)^2 &= (1 + \cos x)^2 \\
\sin^2 x &= 1 + 2\cos x + \cos^2 x \\
(1 - \cos^2 x) &= 1 + 2\cos x + \cos^2 x \\
0 &= 2\cos^2 x + 2\cos x \\
0 &= 2\cos x(\cos x + 1) \\
\cos x = 0 \quad &\text{or} \quad \cos x = -1 \\
x &= \frac{\pi}{2}, \frac{3\pi}{2}, \pi
\end{aligned}$$

Possible values of $x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

Check for extraneous roots:

for $x = \frac{\pi}{2}$, $\sin\frac{\pi}{2} - \cos\frac{\pi}{2} = 1 - 0 = 1$ $\quad\therefore$ LHS = RHS

for $x = \pi$, $\sin\pi - \cos\pi = 0 - (-1) = 1$ $\quad\therefore$ LHS = RHS

for $x = \frac{3\pi}{2}$, $\sin\frac{3\pi}{2} - \cos\frac{3\pi}{2} = -1 - 0 = -1$ $\quad\therefore$ LHS $\neq$ RHS

Therefore, $\frac{3\pi}{2}$ is not a root.

The only solutions are $x = \frac{\pi}{2}$ and π.

e) $\tan x - 1 = \sec x$

Answer:

$$\begin{aligned} \tan x - 1 &= \sec x \\ (\tan x - 1)^2 &= (\sec x)^2 \\ \tan^2 x - 2\tan x + 1 &= \sec^2 x \\ \tan^2 x - 2\tan x + 1 &= \tan^2 x + 1 \\ -2\tan x &= 0 \\ \tan x &= 0 \\ x &= 0,\ \pi \end{aligned}$$

Possible values of $x = 0, \pi$.

Check for extraneous roots:

for $x = 0$, $\tan 0 - 1 = 0 - 1 = -1$ and $\sec 0 = 1$ $\quad\therefore$ LHS ≠ RHS

Therefore, 0 is not a root.

for $x = \pi$, $\tan \pi - 1 = 0 - 1 = -1$ and $\sec \pi = -1$ $\quad\therefore$ LHS = RHS

$x = \pi$ is a root.

$\therefore$ The only solution is $x = \pi$.

4. Test the following identity by substituting the stated values for α and β.
   Difference identity: $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

   a) $\alpha = 90°, \beta = 30°$

   Answer:

$$\cos(90° - 30°) = \cos 90° \cos 30° + \sin 90° \sin 30°$$

$$\text{RHS} = 0\left(\frac{\sqrt{3}}{2}\right) + 1\left(\frac{1}{2}\right) = \frac{1}{2}$$

$$\text{LHS} = \cos(90° - 30°) = \cos 60° = \frac{1}{2}$$

Thus, LHS = RHS.

b) $\alpha = 150°, \beta = 30°$

Answer:

$$\cos(150° - 30°) = \cos 150° \cos 30° + \sin 150° \sin 30°$$

$$\text{RHS} = -\frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right) = -\frac{3}{4} + \frac{1}{4} = -\frac{2}{4} = -\frac{1}{2}$$

$$\text{LHS} = \cos(150° - 30°) = \cos 120° = -\frac{1}{2}$$

Thus, LHS = RHS.

c) $\alpha = 90°, \beta = 45°$

Answer:

$$\cos(90° - 45°) = \cos 90° \cos 45° + \sin 90° \sin 45°$$

$$\text{RHS} = 0\left(\frac{1}{\sqrt{2}}\right) + 1\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$$

$$\text{LHS} = \cos(90° - 45°) = \cos 45° = \frac{1}{\sqrt{2}}$$

Thus, LHS = RHS.

d) $\alpha = \pi, \ \beta = \frac{\pi}{4}$

Answer:

$$\cos\left(\pi - \frac{\pi}{4}\right) = \cos \pi \ \cos \frac{\pi}{4} + \sin \pi \ \sin \frac{\pi}{4}$$

$$\text{RHS} = -1\left(\frac{1}{\sqrt{2}}\right) + 0\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}$$

$$\text{LHS} = \cos\left(\pi - \frac{\pi}{4}\right) = \cos \frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$$

Thus, LHS = RHS.

e) $\alpha = \pi,\ \beta = \dfrac{2\pi}{3}$

Answer:

$$\cos\left(\pi - \frac{2\pi}{3}\right) = \cos \pi \ \cos\frac{2\pi}{3} + \sin \pi \ \sin \frac{2\pi}{3}$$

$$\text{RHS} = -1\left(-\frac{1}{2}\right) + 0\left(\frac{\sqrt{3}}{2}\right) = \frac{1}{2}$$

$$\text{LHS} = \cos\left(\pi - \frac{2\pi}{3}\right) = \cos \frac{\pi}{3} = \frac{1}{2}$$

Thus, LHS = RHS.

f) $\alpha = \dfrac{\pi}{2},\ \beta = \dfrac{\pi}{3}$

Answer:

$$\cos\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \cos \frac{\pi}{2} \ \cos \frac{\pi}{3} + \sin \frac{\pi}{2} \ \sin \frac{\pi}{3}$$

$$\text{RHS} = 0\left(\frac{1}{2}\right) + 1\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}$$

$$\text{LHS} = \cos\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

Thus, LHS = RHS.

The identity seems to be true, since it was valid in all the above cases. However, a proof is required to make sure it is valid for all values of α and β. This will be explored in Lesson 4.

Learning Activity 6.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. State the non-permissible values of the function $f(x) = \dfrac{x^2 + x - 20}{x^2 + 5x - 14}$.
2. Find all the values of θ between [0°, 180°] if $\tan \theta = 1$.
3. Convert 1080° to radians.
4. What is the exact value of sin 60°?
5. If two angles of a triangle are 82° and 34°, what is the measurement of the third angle?
6. In which quadrant is $\csc \theta$ negative and $\sec \theta$ positive?
7. Estimate the amount of tax at 13% on a bill of $49.95.
8. Determine the inverse function if $f(x) = \dfrac{3}{2}x$.

Answers:

1. $x \neq -7$ and $x \neq 2$ $(x^2 + 5x - 14 = (x + 7)(x - 2))$
2. 45°
3. 6π radians $(1080° = 360° \times 3$, so $1080° = 2\pi(3)$ or $6\pi)$
4. $\dfrac{\sqrt{3}}{2}$
5. 64° (180° − 82° − 34° = 64°)
6. Quadrant IV $\left(\csc \theta = \dfrac{1}{\sin \theta} \text{ and } \sec \theta = \dfrac{1}{\cos \theta}\right)$
7. ≈$6.50 (10% of $50.00 is $5.00, 1% of $50.00 is $0.50, 13% of $50.00 is $5.00 + $0.50 + $0.50 + $0.50 ≈ $6.50)
8. $y = \dfrac{2}{3}x$ $\left(y = \dfrac{3}{2}x \rightarrow x = \dfrac{3}{2}y;\ \dfrac{2}{3}x = y\right)$

Part B: Using Sum and Difference Identities

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Without using a calculator, evaluate the following.

   a) $\sin \frac{\pi}{9} \cos \frac{5\pi}{36} + \cos \frac{\pi}{9} \sin \frac{5\pi}{36}$

   Answer:

   If $\alpha = \frac{\pi}{9}$ and $\beta = \frac{5\pi}{36}$, this expression is $\sin \alpha \cos \beta + \cos \alpha \sin \beta$.

   Looking at the identities, this is the same as $\sin (\alpha + \beta)$.

   $$\sin\left(\frac{\pi}{9} + \frac{5\pi}{36}\right) = \sin\left(\frac{4\pi}{36} + \frac{5\pi}{36}\right) = \sin\frac{9\pi}{36} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

   b) $\sin \frac{2\pi}{5} \cos \frac{3\pi}{5} + \cos \frac{2\pi}{5} \sin \frac{3\pi}{5}$

   Answer:

   If $\alpha = \frac{2\pi}{5}$ and $\beta = \frac{3\pi}{5}$, this expression is $\sin \alpha \cos \beta + \cos \alpha \sin \beta = \sin (\alpha + \beta)$.

   $$\sin\left(\frac{2\pi}{5} + \frac{3\pi}{5}\right) = \sin\frac{5\pi}{5} = \sin\pi = 0$$

   c) $\cos \frac{7\pi}{12} \cos \frac{\pi}{4} + \sin \frac{7\pi}{12} \sin \frac{\pi}{4}$

   Answer:

   If $\alpha = \frac{7\pi}{12}$ and $\beta = \frac{\pi}{4}$, this expression is $\cos \alpha \cos \beta + \sin \alpha \sin \beta = \cos (\alpha - \beta)$.

   $$\cos\left(\frac{7\pi}{12} - \frac{\pi}{4}\right) = \cos\left(\frac{7\pi}{12} - \frac{3\pi}{12}\right) = \cos\frac{4\pi}{12} = \cos\frac{\pi}{3} = \frac{1}{2}$$

d) $\cos\frac{5\pi}{9}\cos\frac{\pi}{9}-\sin\frac{5\pi}{9}\sin\frac{\pi}{9}$

Answer:

If $\alpha=\frac{5\pi}{9}$ and $\beta=\frac{\pi}{4}$, this expression is $\cos\alpha\cos\beta-\sin\alpha\sin\beta=$ $\cos(\alpha+\beta)$.

$$\cos\left(\frac{5\pi}{9}+\frac{\pi}{9}\right)=\cos\left(\frac{6\pi}{9}\right)=\cos\frac{2\pi}{3}=-\frac{1}{2}$$

e) $\dfrac{\tan\frac{4\pi}{9}+\tan\frac{25\pi}{18}}{1-\tan\frac{4\pi}{9}\tan\frac{25\pi}{18}}$

Answer:

$$\tan\left(\frac{4\pi}{9}+\frac{25\pi}{18}\right)=\tan\left(\frac{8\pi}{18}+\frac{25\pi}{18}\right)=\tan\left(\frac{33\pi}{18}\right)=\tan\frac{11\pi}{6}=-\frac{\sqrt{3}}{3}$$

f) $\dfrac{\tan 3\pi-\tan\frac{5\pi}{3}}{1+\tan 3\pi\tan\frac{5\pi}{3}}$

Answer:

$$\tan\left(3\pi-\frac{5\pi}{3}\right)=\tan\left(\frac{9\pi}{3}-\frac{5\pi}{3}\right)=\tan\left(\frac{4\pi}{3}\right)=\sqrt{3}$$

2. Determine the exact value of $\sin \frac{7\pi}{12}$ by setting $\alpha = \frac{\pi}{4}$ and $\beta = \frac{\pi}{3}$ in the relevant identity.

Answer:

Since $\frac{\pi}{4} + \frac{\pi}{3} = \frac{3\pi}{12} + \frac{4\pi}{12} = \frac{7\pi}{12}$, then:

$$\begin{aligned}
\sin\left(\frac{7\pi}{12}\right) &= \sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right) \\
&= \sin \frac{\pi}{4} \cos \frac{\pi}{3} + \cos \frac{\pi}{4} \sin \frac{\pi}{3} \\
&= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \\
&= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} \\
&= \frac{\sqrt{2} + \sqrt{6}}{4}
\end{aligned}$$

3. Find the coordinates of $P\left(\frac{7\pi}{12}\right)$.

Answer:

The coordinates of $P\left(\frac{7\pi}{12}\right)$ are $\left(\cos \frac{7\pi}{12}, \sin \frac{7\pi}{12}\right)$. You already know $\sin \frac{7\pi}{12} = \frac{\sqrt{2} + \sqrt{6}}{4}$ from Question 2. Therefore, you only need to find $\cos \frac{7\pi}{12}$.

$$\begin{aligned}\cos\left(\frac{7\pi}{12}\right) &= \cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right)\\ &= \cos\frac{\pi}{4}\cos\frac{\pi}{3}-\sin\frac{\pi}{4}\sin\frac{\pi}{3}\\ &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)-\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\\ &= \frac{\sqrt{2}}{4}-\frac{\sqrt{6}}{4}\\ &= \frac{\sqrt{2}-\sqrt{6}}{4}\end{aligned}$$

Therefore, the coordinates of $P\left(\frac{7\pi}{12}\right)=\left(\frac{\sqrt{2}-\sqrt{6}}{4}, \frac{\sqrt{2}+\sqrt{6}}{4}\right)$.

4. Find the exact value of $\tan\frac{7\pi}{12}$.

Answer:

$$\begin{aligned}\tan\left(\frac{7\pi}{12}\right) &= \tan\left(\frac{\pi}{4}+\frac{\pi}{3}\right)\\ &= \frac{\tan\frac{\pi}{4}+\tan\frac{\pi}{3}}{1-\tan\frac{\pi}{4}\tan\frac{\pi}{3}}\\ &= \frac{\sqrt{3}+1}{1-\sqrt{3}(1)}\\ &= \frac{\sqrt{3}+1}{1-\sqrt{3}}\end{aligned}$$

Or, since you already know the values of $\sin \frac{7\pi}{12}$ and $\cos \frac{7\pi}{12}$ from Questions 3 and 4, another method is:

$$\tan\left(\frac{7\pi}{12}\right) = \frac{\sin\left(\frac{7\pi}{12}\right)}{\cos\left(\frac{7\pi}{12}\right)}$$

$$= \frac{\frac{\sqrt{2}+\sqrt{6}}{4}}{\frac{\sqrt{2}-\sqrt{6}}{4}}$$

$$= \frac{\sqrt{2}+\sqrt{6}}{4}\left(\frac{4}{\sqrt{2}-\sqrt{6}}\right)$$

$$= \frac{\sqrt{2}+\sqrt{6}}{\sqrt{2}-\sqrt{6}}$$

It is an interesting task to show that the two answers are the same number by rationalizing each of the two answers. Or, you could use a calculator.

Remember, to rationalize a denominator containing a binomial, multiply by the conjugate. (If the denominator is $a + b$, the conjugate is $a - b$). For example, if you rationalize

$\frac{\sqrt{2}+\sqrt{6}}{\sqrt{2}-\sqrt{6}}$, you get

$$\frac{\sqrt{2}+\sqrt{6}}{\sqrt{2}-\sqrt{6}} \cdot \frac{\sqrt{2}+\sqrt{6}}{\sqrt{2}+\sqrt{6}} = \frac{\sqrt{4}+\sqrt{12}+\sqrt{12}+\sqrt{36}}{\sqrt{4}-\sqrt{36}} = \frac{2+2\sqrt{12}+6}{2-6}$$

$$= \frac{8+2\left(2\sqrt{3}\right)}{-4} = \frac{8+4\sqrt{3}}{-4} = \frac{8}{-4} + \frac{4\sqrt{3}}{-4} = -2-\sqrt{3}$$

If you rationalize $\frac{\sqrt{3}+1}{1-\sqrt{3}}$, you get:

$$\begin{aligned}
&\frac{\sqrt{3}+1}{1-\sqrt{3}} \cdot \frac{1+\sqrt{3}}{1+\sqrt{3}} \\
&= \frac{\sqrt{3}+3+1+\sqrt{3}}{1+\sqrt{3}-\sqrt{3}-3} \\
&= \frac{4+2\sqrt{3}}{-2} \\
&= -2-\sqrt{3}
\end{aligned}$$

As expected, both methods result in the same answer.

5. Use identities to find the exact value of each of the following. Use a sum or difference of two special angles to get 195°.

   a) cos 195°

   Answer:

$$\begin{aligned}
\cos 195^\circ &= \cos\left(150^\circ + 45^\circ\right) \\
&= \cos 150^\circ \cos 45^\circ - \sin 150^\circ \sin 45^\circ \\
&= \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\
&= -\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\
&= \frac{-\sqrt{6}-\sqrt{2}}{4}
\end{aligned}$$

b) sin 195°

Answer:

$$\begin{aligned}\sin 195° &= \sin(150° + 45°)\\ &= \sin 150° \cos 45° + \cos 150° \sin 45°\\ &= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\\ &= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}\\ &= \frac{\sqrt{2} - \sqrt{6}}{4}\end{aligned}$$

c) tan 195°

Answer:

$$\begin{aligned}\tan 195° &= \tan(150° + 45°)\\ &= \frac{\tan 150° + \tan 45°}{1 - \tan 150° \tan 45°}\\ &= \frac{-\frac{\sqrt{3}}{3} + 1}{1 - \frac{\sqrt{3}}{3}(1)}\\ &= \frac{-\frac{\sqrt{3}}{3} + \frac{3}{3}}{\frac{3}{3} + \frac{\sqrt{3}}{3}}\\ &= \frac{\frac{-\sqrt{3} + 3}{3}}{\frac{3 + \sqrt{3}}{3}}\\ &= \frac{-\sqrt{3} + 3}{3}\left(\frac{3}{3 + \sqrt{3}}\right)\\ &= \frac{-\sqrt{3} + 3}{3 + \sqrt{3}}\end{aligned}$$

6. Given that $\sin\alpha = \frac{7}{25}$ and $\cos\beta = \frac{9}{41}$ and neither $P(\alpha)$ nor $P(\beta)$ are in the first quadrant, find:

   a) $\sin(\alpha + \beta)$
   b) $\cos(\alpha + \beta)$
   c) $\tan(\alpha + \beta)$
   d) $\sec(\alpha + \beta)$
   e) The quadrant in which $(\alpha + \beta)$ terminates

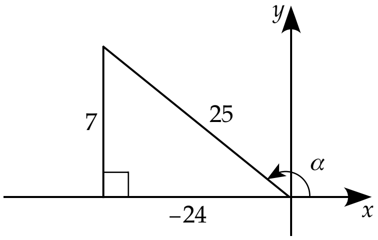
   Answers:

   Since $\sin\alpha = \frac{7}{25}$ and $P(\alpha) \notin$ Quadrant I,

   it follows that $P(\alpha) \in$ Quadrant II, so $\cos\alpha < 0$.

   Since $\sin^2\alpha + \cos^2\alpha = 1$, $\left(\frac{7}{25}\right)^2 + \cos^2\alpha = 1$.

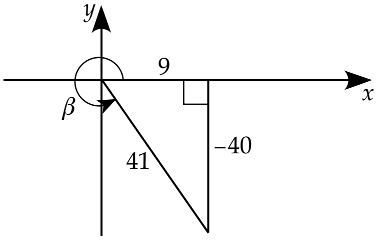
   Therefore, $\cos\alpha = -\frac{24}{25}$.



   Similarly, since $\cos\beta = \frac{9}{41}$ and $P(\beta) \notin$ Quadrant I, it follows that $P(\beta) \in$ Quadrant IV, so $\sin\beta < 0$.

   Since $\sin^2\beta + \cos^2\beta = 1$, $\sin^2\beta + \left(\frac{9}{41}\right)^2 = 1$.

   Therefore, $\sin\beta = -\frac{40}{41}$.



   a) $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \frac{7}{25}\left(\frac{9}{41}\right) + \left(-\frac{24}{25}\right)\left(-\frac{40}{41}\right) = \frac{63 + 960}{1025} = \frac{1023}{1025}$$

   b) $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= -\frac{24}{25}\left(\frac{9}{41}\right) - \left(\frac{7}{25}\right)\left(-\frac{40}{41}\right) = \frac{-216 + 280}{1025} = \frac{64}{1025}$$

   c) $\tan(\alpha + \beta) = \dfrac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \dfrac{\frac{1023}{1025}}{\frac{64}{1025}} = \dfrac{1023}{64}$

d) $\sec(\alpha + \beta) = \dfrac{1}{\cos(\alpha + \beta)} = \dfrac{1}{\frac{64}{1025}} = \dfrac{1025}{64}$

e) Since both sin $(\alpha + \beta) > 0$ and cos $(\alpha + \beta) > 0$, then $(\alpha + \beta)$ must terminate in Quadrant I.

7. Express $\cos\left(\dfrac{\pi}{3} + \theta\right)$ as a function of θ only.

Answer:

Use the cos $(\alpha + \beta)$ identity $\cos\alpha\cos\beta - \sin\alpha\sin\beta$.

$$\cos\left(\frac{\pi}{3} + \theta\right) = \cos\frac{\pi}{3}\cos\theta - \sin\frac{\pi}{3}\sin\theta = \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta$$

$$= \frac{\cos\theta - \sqrt{3}\sin\theta}{2}$$

8. Express $\tan\left(\theta - \dfrac{\pi}{6}\right)$ as a function of θ only.

Answer:

Use the tan $(\alpha - \beta)$ identity.

$$\tan\left(\theta - \frac{\pi}{6}\right) = \frac{\tan\theta - \tan\frac{\pi}{6}}{1 + \tan\theta\tan\frac{\pi}{6}} = \frac{\tan\theta - \frac{1}{\sqrt{3}}}{1 + \tan\theta\left(\frac{1}{\sqrt{3}}\right)}$$

$$= \frac{\sqrt{3}\left(\tan\theta - \frac{1}{\sqrt{3}}\right)}{\sqrt{3}\left(1 + \tan\theta\left(\frac{1}{\sqrt{3}}\right)\right)}$$

$$= \frac{\sqrt{3}\tan\theta - 1}{\sqrt{3} + \tan\theta}$$

9. Prove the following identity.

   $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin\alpha \cos\beta$

   Answer:

   $$\begin{aligned} \text{LHS} &= \sin(\alpha + \beta) + \sin(\alpha - \beta) \\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta + \sin\alpha\cos\beta - \cos\alpha\sin\beta \\ &= 2\sin\alpha\cos\beta \\ &= \text{RHS} \end{aligned}$$

10. Use identities to simplify the following expressions to a single trigonometric function.

    a) $\sin 2x \cos 3x + \cos 2x \sin 3x$

    Answer:

    Use the identity $\sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin(\alpha + \beta)$

    $\sin(2x + 3x) = \sin 5x$

    b) $\dfrac{\tan x + \tan 3x}{1 - \tan x \tan 3x}$

    Answer:

    Use the identity $\dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha + \tan\beta} = \tan(\alpha + \beta)$.

    $\tan(x + 3x) = \tan 4x$

    c) $\sin\alpha \cos 6\alpha + \cos\alpha \sin 6\alpha$

    Answer:

    Use the identity $\sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin(\alpha + \beta)$.

    $\sin(\alpha + 6\alpha) = \sin 7\alpha$

    d) $\cos^2 3x - \sin^2 3x$

    Answer:

    Use the identity $\cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha + \beta)$.

    $\cos 3x \cos 3x - \sin 3x \sin 3x = \cos(3x + 3x) = \cos 6x$

e) $\cos 2\alpha \cos 3\alpha + \sin 2\alpha \sin 3\alpha$

Answer:

Use the identity $\cos \alpha \cos \beta + \sin \alpha \sin \beta = \cos (\alpha - \beta)$.

$\cos (2\alpha - 3\alpha) = \cos (-\alpha)$

Note: Considering the symmetry of the cosine function, $\cos (-\alpha) = \cos \alpha$. Either answer is correct.

11. Prove the following *very important* identities. These double angle identities are discussed further in Lesson 5.

a) $\sin 2\alpha = 2 \sin \alpha \cos \alpha$

Answer:

$\text{LHS} = \sin 2\alpha = \sin (\alpha + \alpha) = \sin \alpha \cos \alpha + \cos \alpha \sin \alpha$

$= 2 \sin \alpha \cos \alpha$

$= \text{RHS}$

b) $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$

Answer:

$\text{LHS} = \cos 2\alpha = \cos (\alpha + \alpha) = \cos \alpha \cos \alpha - \sin a \sin \alpha$

$= \cos^2 \alpha - \sin^2 \alpha$

$= \text{RHS}$

c) $\cos 2\alpha = 2 \cos^2 \alpha - 1$

Answer:

From part (b):

$\text{LHS} = \cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$

$= \cos^2 \alpha - (1 - \cos^2 \alpha)$

$= \cos^2 \alpha - 1 + \cos^2 \alpha$

$= 2 \cos^2 \alpha - 1$

$= \text{RHS}$

d) $\cos 2\alpha = 1 - 2\sin^2 \alpha$

Answer:

From part (b):

$$\begin{aligned} \text{LHS} &= \cos 2\alpha \\ &= \cos^2 \alpha - \sin^2 \alpha \\ &= (1 - \sin^2 \alpha) - \sin^2 \alpha \\ &= 1 - 2\sin^2 \alpha \\ &= \text{RHS} \end{aligned}$$

e) $\tan 2\alpha = \dfrac{2\tan \alpha}{1 - \tan^2 \alpha}$

Answer:

$$\begin{aligned} \text{LHS} &= \tan 2\alpha \\ &= \tan(\alpha + \alpha) \\ &= \frac{\tan \alpha + \tan \alpha}{1 - \tan \alpha \tan \alpha} \\ &= \frac{2\tan \alpha}{1 - \tan^2 \alpha} \\ &= \text{RHS} \end{aligned}$$

12. Use identities to simplify the following to a function involving only x.

a) $\sin\left(\dfrac{\pi}{2} + x\right)$

Answer:

$$\begin{aligned} \sin\left(\frac{\pi}{2} + x\right) &= \sin\left(\frac{\pi}{2}\right)\cos x + \cos\left(\frac{\pi}{2}\right)\sin x \\ &= 1(\cos x) + 0(\sin x) \\ &= \cos x \end{aligned}$$

b) $\cos\left(\frac{\pi}{2}+x\right)$

Answer:

$$\begin{aligned}\cos\left(\frac{\pi}{2}+x\right) &= \cos\left(\frac{\pi}{2}\right)\cos x - \sin\left(\frac{\pi}{2}\right)\sin x\\ &= 0\,(\cos x) - 1\,(\sin x)\\ &= -\sin x\end{aligned}$$

13. Use identities to write the general solution of the following equations.

a) $\cos\left(x-\frac{\pi}{2}\right)=1$

Answer:

$$\begin{aligned}\text{LHS: } \cos\left(x-\frac{\pi}{2}\right) &= \cos x\cos\left(\frac{\pi}{2}\right)+\sin x\sin\left(\frac{\pi}{2}\right)\\ &= \cos x(0)+\sin x\,(1)\\ &= \sin x\end{aligned}$$

$$\sin x = 1$$

$$x=\frac{\pi}{2}$$

Thus, all solutions are $\left\{x \mid x=\frac{\pi}{2}+2k\pi,\ k\in I\right\}$

b) $\sin\left(\frac{\pi}{2}-x\right)=1$

Answer:

$$\begin{aligned}\sin\left(\frac{\pi}{2}-x\right) &= \sin\left(\frac{\pi}{2}\right)\cos x - \cos\left(\frac{\pi}{2}\right)\sin x\\ &= 1\,(\cos x) - 0\,(\sin x)\\ &= \cos x\end{aligned}$$

$$\cos x = 1$$

$$x = 0$$

Thus, all solutions are $\{x \mid x = 0 + 2k\pi,\ k \in I\}$ or $\{x \mid x = 2k\pi,\ k \in I\}$.

Learning Activity 6.5

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Convert $\frac{\pi}{3}$ to degrees.
2. What is the exact value of tan 45°?
3. Will the function $y = x^2 - 2$ have an inverse that is a function?
4. State the non-permissible values of the function $f(x) = \frac{x-2}{x^2+5x+6}$.
5. Simplify: $49^{-\frac{1}{2}}$
6. You put a chicken in the oven at 2:48 pm. At what time will the chicken be done if it takes 85 minutes to cook?
7. What is the length of the hypotenuse of a right-angled triangle if the two legs of the triangle measure 6 m and 8 m respectively?
8. Factor: $3x^2 + 7x + 2$

Answers:

1. 60°
2. 1
3. No (A parabola is not a one-to-one function.)
4. $x \neq -2$ and $x \neq -3$ $(x^2 + 5x + 6 = (x + 2)(x + 3))$
5. $\frac{1}{7}\left(49^{-\left(\frac{1}{2}\right)} = \frac{1}{49^{\frac{1}{2}}} = \frac{1}{\sqrt{49}} = \frac{1}{7}\right)$
6. 4:13 (2:48 + 12 minutes is 3:00 pm; another 60 minutes is 4:00 pm;
   60 + 12 = 72, so there are 13 minutes left for the 85 minutes)
7. 10 m (Let x = length of hypotenuse: $6^2 + 8^2 = x^2$; $36 + 64 = x^2$; $100 = x^2$, $10 = x$)
8. $(3x + 1)(x + 2)$

Part B: Using Double Angle Identities

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Write $\sin 4A$ in terms of functions of A. Do not simplify.

   Answer:

   $\sin 4A = 2 \sin 2A \cos 2A$

   $= 2(2 \sin A \cos A)(\cos^2 A - \sin^2 A)$

   This is just one possibility.

2. Write each of the following in terms of only one circular function.

   a) $\cos^2 4x - \sin^2 4x$

      Answer:

      $\cos 8x$

      (Using the identity $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$.)

   b) $2 \sin 2x \cos 2x$

      Answer:

      $\sin 4x$

      (Using the identity $\sin 2\alpha = 2 \sin \alpha \cos \alpha$.)

   c) $4 \sin x \cos x$

      Answer:

      $2 \sin 2x$

   d) $1 - 2 \sin^2 5\alpha$

      Answer:

      $\cos 10\alpha$

   e) $2 \cos^2 10\alpha - 1$

      Answer:

      $\cos 20\alpha$

f) $\sin \alpha \cos \alpha$

Answer:

$$\sin \alpha \ \cos \alpha = \frac{2 \sin \alpha \ \cos \alpha}{2} = \frac{\sin 2\alpha}{2}$$

g) $6 \sin 5x \cos 5x$

Answer:

$6 \sin 5x \cos 5x = 3(2 \sin 5x \cos 5x) = 3(\sin 10x)$

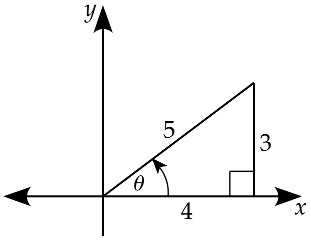
3. Use a double angle identity to find $\cos \frac{\pi}{5}$, if $\cos \frac{\pi}{10} = 0.95$.

Answer:

$$\cos \frac{\pi}{5} = 2 \cos^2 \frac{\pi}{10} - 1 = 2(0.95)^2 - 1 = 0.805$$

4. If $0 < \theta < \frac{\pi}{2}$ and $\sin \theta = \frac{3}{5}$, find the exact values of $\sin 2\theta$ and $\cos 2\theta$.

Answer:



Since $0 < \theta < \frac{\pi}{2}$ and $\sin \theta = \frac{3}{5}$, it follows that $\cos^2 \theta + \left(\frac{3}{5}\right)^2 = 1$, so $\cos \theta = \frac{4}{5}$ with $\theta \in$ Quadrant I.

$\sin 2\theta = 2 \sin \theta \ \cos \theta = 2\left(\frac{3}{5}\right)\frac{4}{5} = \frac{24}{25}$ and

$\cos 2\theta = 1 - 2 \sin^2 \theta$

$= 1 - 2\left(\frac{3}{5}\right)^2$

$= 1 - 2\left(\frac{9}{25}\right)$

$= 1 - \frac{18}{25} = \frac{25}{25} - \frac{18}{25} = \frac{7}{25}$

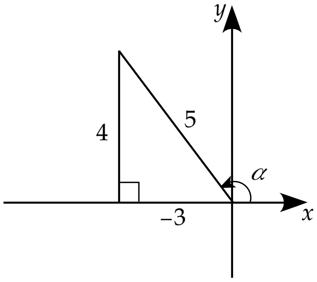
5. Given that $\sin \alpha = \frac{4}{5}$ and $\cos \beta = \frac{5}{13}$, and neither $P(\alpha)$ not $P(\beta)$ are in the first quadrant, find

   a) $\sin 2\alpha$
   b) $\cos (\alpha - \beta)$
   c) $\cos 2\beta$
   d) $\sin (\alpha + \beta)$
   e) $\tan 2\alpha$
   f) $\tan (\alpha - \beta)$

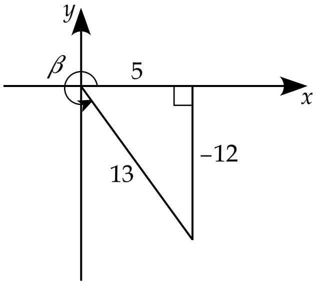
   Answers:

   Since $\sin \alpha > 0$ and not in Quadrant I, it follows that $P(\alpha)$ is in Quadrant II. Similarly, since $\cos \beta > 0$ and not in Quadrant I, it follows that $P(\beta)$ is in Quadrant IV.

   $\cos^2 \alpha + \left(\frac{4}{5}\right)^2 = 1$. Hence, in Quadrant II $\cos \alpha = -\frac{3}{5}$



   $\sin^2 \beta + \left(\frac{5}{13}\right)^2 = 1$. Hence, in Quadrant IV $\sin \beta = -\frac{12}{13}$



a) $\sin 2\alpha = 2\sin\alpha\cos\alpha = 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) = -\frac{24}{25}$

b) $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \left(-\frac{3}{5}\right)\frac{5}{13} + \frac{4}{5}\left(-\frac{12}{13}\right)$

$$= \frac{-15 - 48}{65} = -\frac{63}{65}$$

c) $\cos 2\beta = \cos^2\beta - \sin^2\beta = \left(\frac{5}{13}\right)^2 - \left(-\frac{12}{13}\right)^2 = \frac{25 - 144}{169} = -\frac{119}{169}$

d) $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \frac{4}{5}\left(\frac{5}{13}\right) + \left(-\frac{3}{5}\right)\left(-\frac{12}{13}\right) = \frac{20 + 36}{65} = \frac{56}{65}$$

e) $\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{4}{5}}{-\frac{3}{5}} = -\frac{4}{3}$

Hence, $\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha} = \frac{2\left(-\frac{4}{3}\right)}{1 - \left(-\frac{4}{3}\right)^2}$

$$= \frac{-\frac{8}{3}}{1 - \frac{16}{9}} = \frac{-\frac{8}{3}}{-\frac{7}{9}} = \frac{24}{7}.$$

f) $\tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{-\frac{12}{13}}{\frac{5}{13}} = -\frac{12}{5}$

Hence, $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \frac{-\frac{4}{3} - \left(-\frac{12}{5}\right)}{1 + \left(-\frac{4}{3}\right)\left(-\frac{12}{5}\right)}.$

Therefore, $\tan(\alpha - \beta) = \frac{\frac{16}{15}}{1 + \frac{48}{15}} = \frac{\frac{16}{15}}{\frac{63}{15}} = \frac{16}{63}.$

6. Write each of the following in terms of only one circular function or a constant. Include any non-permissible values for each expression.

   a) $(\sin x + \cos x)^2$

      Answer:

      There are no non-permissible values for this expression. The domain of $\sin x$ and $\cos x$ is $x \in \Re$.

      $$\begin{aligned}(\sin x + \cos x)^2 &= \sin^2 x + 2 \sin x \cos x + \cos^2 x \\ &= 1 + 2 \sin x \cos x \\ &= 1 + \sin 2x\end{aligned}$$

   b) $\dfrac{\sin 2x}{\cos x}$

      Answer:

      The non-permissible values for this expression occur when $\cos x = 0$ and are thus $\left\{x \mid x = \dfrac{(2k+1)\pi}{2},\ k \in I\right\}$.

      $$\begin{aligned}\frac{\sin 2x}{\cos x} &= \frac{2 \sin x \cos x}{\cos x} \\ &= 2 \sin x\end{aligned}$$

7. Prove the following identities.

a) $\tan x + \cot x = 2 \csc 2x$

Answer:

$$\begin{aligned} \text{LHS} &= \tan x + \cot x \\ &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \\ &= \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} \\ &= \frac{1}{\sin x \cos x} \\ &= \frac{2}{2 \sin x \cos x} \end{aligned}$$

$$\begin{aligned} \text{RHS} &= 2 \csc 2x \\ &= 2\left(\frac{1}{\sin 2x}\right) \\ &= 2\left(\frac{1}{2 \sin x \cos x}\right) \\ &= \frac{2}{2 \sin x \cos x} \end{aligned}$$

So LHS = RHS.

b) $\cos^4 x - \sin^4 x = \cos 2x$

Answer:

$$\begin{aligned} \text{LHS} &= \cos^4 x - \sin^4 x \\ &= (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x) \\ &= 1(\cos 2x) \\ &= \cos 2x \\ &= \text{RHS} \end{aligned}$$

c) $\dfrac{2 \sin x}{\sin 2x} = \sec x$

Answer:

$$\begin{aligned}
\text{LHS} &= \frac{2 \sin x}{\sin 2x} \\
&= \frac{2 \sin x}{2 \sin x \cos x} \\
&= \frac{1}{\cos x} \\
&= \sec x \\
&= \text{RHS}
\end{aligned}$$

d) $\dfrac{\sec^2 x}{2 - \sec^2 x} = \sec 2x$

Answer:

$$\begin{aligned}
\text{LHS} &= \frac{\sec^2 x}{2 - \sec^2 x} \\
&= \frac{\dfrac{1}{\cos^2 x}}{2 - \left(\dfrac{1}{\cos^2 x}\right)} \\
&= \frac{\dfrac{1}{\cos^2 x}}{\dfrac{2\cos^2 x - 1}{\cos^2 x}} \\
&= \frac{1}{\cos^2 x}\left(\frac{\cos^2 x}{2 \cos^2 x - 1}\right) \\
&= \frac{1}{2 \cos^2 x - 1} \\
&= \frac{1}{\cos 2x} \\
&= \sec 2x \\
&= \text{RHS}
\end{aligned}$$

e) $\sin 3x = 3\sin x - 4\sin^3 x$

Hint: Write $3x$ as $(2x + x)$ and change everything to functions of x.

Answer:

$$
\begin{aligned}
\text{LHS} &= \sin 3x \\
&= \sin(2x + x) \qquad \text{Use } \sin(\alpha + \beta) \text{ identity.} \\
&= \sin 2x \cos x + \cos 2x \sin x \\
&= (2\sin x \cos x)\cos x + (1 - 2\sin^2 x)\sin x \\
&= 2\sin x \cos^2 x + \sin x - 2\sin^3 x \\
&= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x \\
&= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x \\
&= 3\sin x - 4\sin^3 x \\
&= \text{RHS}
\end{aligned}
$$

8. Solve for x, where $0 < x < 2\pi$.

a) $\sin 2x = \sqrt{2}\sin x$

Answer:

$$
\begin{aligned}
\sin 2x &= \sqrt{2}\sin x \\
2\sin x \cos x &= \sqrt{2}\sin x \\
2\sin x \cos x - \sqrt{2}\sin x &= 0 \\
\sin x\left(2\cos x - \sqrt{2}\right) &= 0 \\
\sin x = 0 \text{ or } \cos x &= \frac{\sqrt{2}}{2} \\
x = \pi \text{ or } x &= \frac{\pi}{4}, \frac{7\pi}{4}
\end{aligned}
$$

b) $\sin 2x = \sqrt{2} \cos x$

Answer:

$$\sin 2x = \sqrt{2} \cos x$$

$$2 \sin x \cos x = \sqrt{2} \cos x$$

$$2 \sin x \cos x - \sqrt{2} \cos x = 0$$

$$\cos x\left(2\sin x - \sqrt{2}\right) = 0$$

$$\cos x = 0 \text{ or } \sin x = \frac{\sqrt{2}}{2}$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2} \text{ or } x = \frac{\pi}{4}, \frac{3\pi}{4}$$

c) $\sin 2x = \tan x$

Answer:

$$\sin 2x = \tan x$$

$$2 \sin x \cos x = \frac{\sin x}{\cos x}$$

$$2 \sin x \cos^2 x = \sin x$$

$$2 \sin x \cos^2 x - \sin x = 0$$

$$\sin x\left(2\cos^2 x - 1\right) = 0$$

$$\sin x = 0 \text{ or } \cos x = \pm\frac{1}{\sqrt{2}}$$

$$x = \pi \text{ or } x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

d) $\cos 2x = \sin x$

Answer:

$$\cos 2x = \sin x$$

$$1 - 2\sin^2 x = \sin x$$

$$0 = 2\sin^2 x + \sin x - 1$$

$$0 = (2\sin x - 1)(\sin x + 1)$$

$$\sin x = \frac{1}{2} \text{ or } \sin x = -1$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6} \text{ or } x = \frac{3\pi}{2}$$

e) $\tan 2x = \tan x$

Answer:

$$\tan 2x = \tan x$$

$$\frac{2\tan x}{1 - \tan^2 x} = \tan x$$

$$2\tan x = \tan x\left(1 - \tan^2 x\right)$$

$$2\tan x = \tan x - \tan^3 x$$

$$\tan^3 x + \tan x = 0$$

$$\tan x\left(\tan^2 x + 1\right) = 0$$

$$\tan x = 0 \text{ or } \tan x = \pm\sqrt{-1}$$

Note: There is no solution to $\tan x = \pm\sqrt{-1}$.

$\therefore x = \pi$

f) $\cos 2x = 2 - 5 \cos x$

Answer:

$$\begin{aligned} \cos 2x &= 2 - 5\cos x \\ 2\cos^2 x - 1 &= 2 - 5\cos x \\ 2\cos^2 x + 5\cos x - 3 &= 0 \\ (2\cos x - 1)(\cos x + 3) &= 0 \\ \cos x &= \frac{1}{2} \text{ or } \cos x = -3 \end{aligned}$$

Note: $\cos x \neq -3$

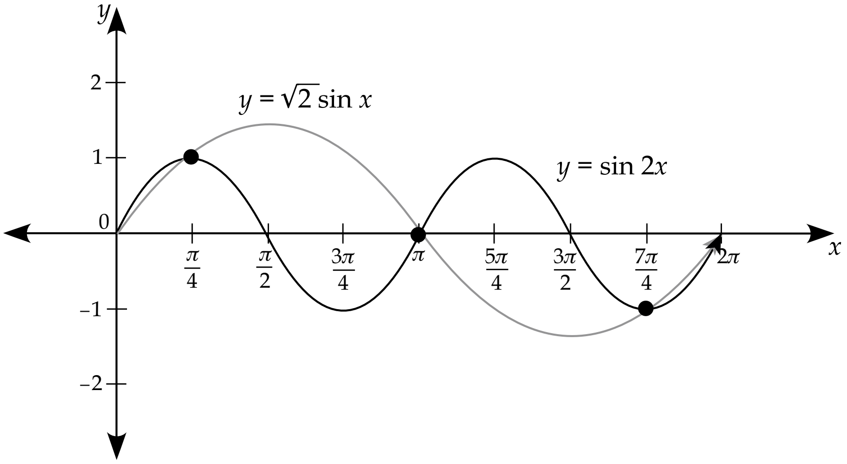
$\therefore x = \frac{\pi}{3}, \frac{5\pi}{3}$

9. Using a graphing calculator or graphing technology, graph each side of the equation in Question 8 as a separate function. Check for the points of intersection in the interval $(0, 2\pi)$ of the two functions to verify your answers to Question 8.

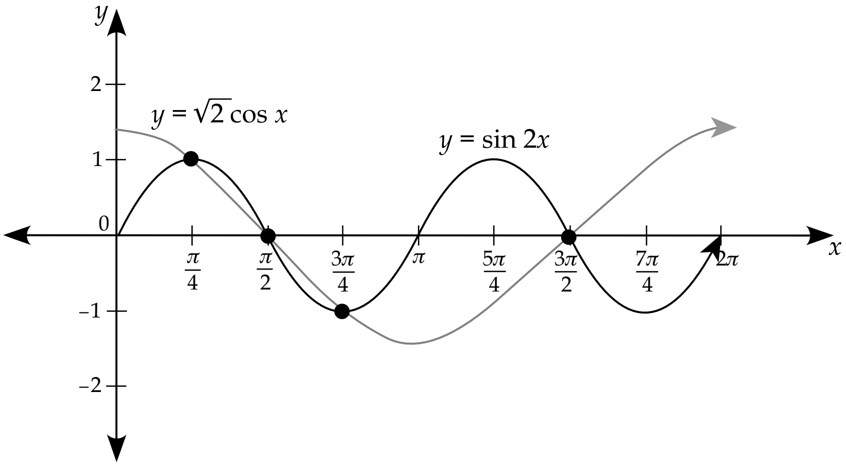
Answer:

Set the `MODE` to `Radians` and the `WINDOW XMIN = 0` and `XMAX = 2`π. Then graph `LHS` and `RHS` of the equation on the same graph and solve for the points of intersection. You may have to change the y-scale to see the graphs properly. The points of intersection are shown as black dots and their coordinates can be read from the sketches.

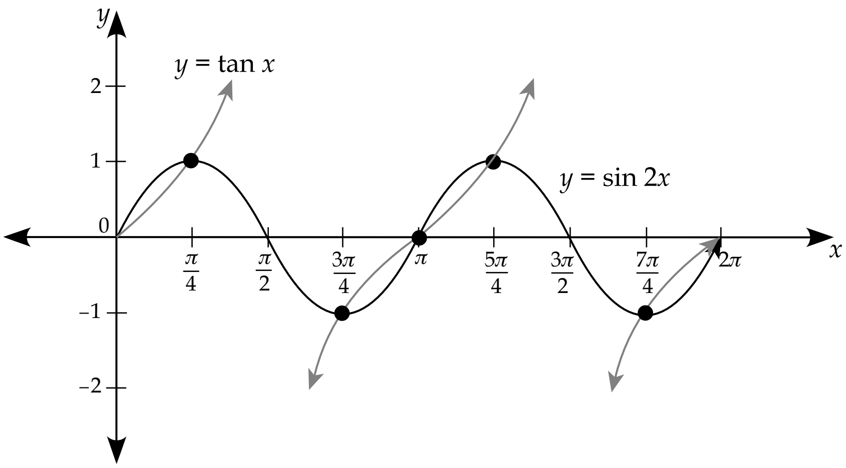
a) $y_1 = \sin 2x$ and $y_2 = \sqrt{2} \sin x$



b) $y_1 = \sin 2x$ and $y_2 = \sqrt{2}\cos x$

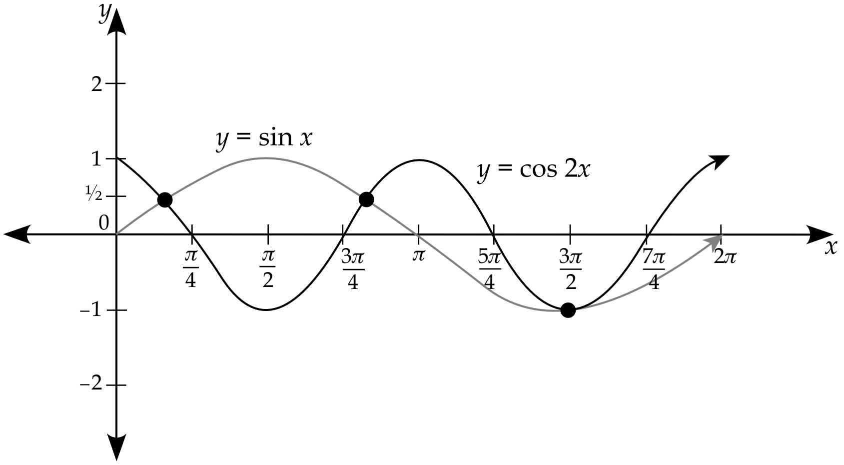


c) $y_1 = \sin 2x$ and $y_2 = \tan x$



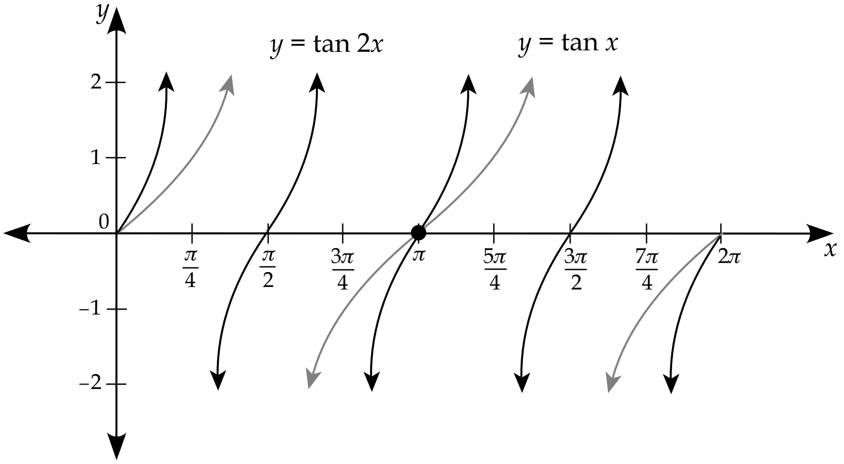
d) $y_1 = \cos 2x$ and $y_2 = \sin x$

Notice that the y-coordinate of the points of intersection are either 0.5 or −1. The corresponding x-values can be found by substituting into the original equations, or by using `2nd Calc` intersection on your graphing calculator.



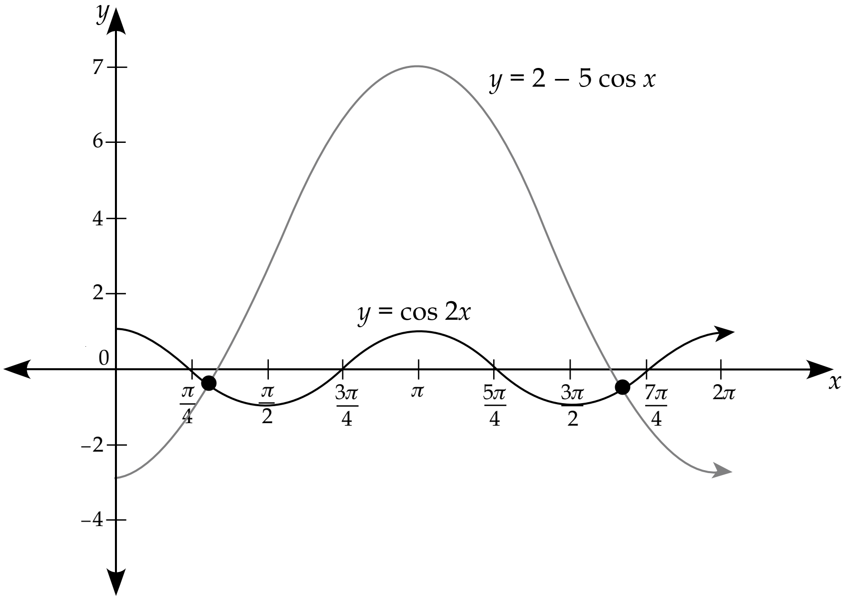
e) $y_1 = \tan 2x$ and $y_2 = \tan x$

Notice that the curves also intersect at 0 and 2π, but these values are outside of the given restrictions.



f) $y_1 = \cos 2x$ and $y_2 = 2 - 5 \cos x$

In this case, the coordinates of the points of intersection are impossible to read from the graph. You would need to solve for them using `2nd Calc` intersection on your graphing calculator.



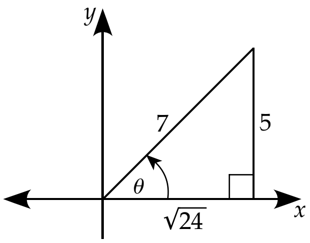
10. Given $\sin x = \frac{5}{7}, 0 \le x < \frac{\pi}{2}$, find values for $\sin 2x$, $\cos 2x$, $\tan 2x$, $\csc 2x$, $\sec 2x$, and $\cot 2x$.

Answer:

This angle is located in Quadrant I where every trigonometric ratio is positive.

As $\sin x = \frac{5}{7}$, then $\cos^2 x + \left(\frac{5}{7}\right)^2 = 1$.

$\therefore \cos x = \frac{\sqrt{24}}{7}$ and $\tan x = \frac{5}{\sqrt{24}}$.



$$\sin 2x = 2 \sin x \cos x = 2\left(\frac{5}{7}\right)\left(\frac{\sqrt{24}}{7}\right) = \frac{10\sqrt{24}}{49}$$

$$\cos 2x = 1 - 2 \sin^2 x = 1 - 2\left(\frac{5}{7}\right)^2 = 1 - 2\left(\frac{25}{49}\right) = 1 - \frac{50}{49} = -\frac{1}{49}$$

$$\tan 2x = \frac{\sin 2x}{\cos 2x} = \frac{\frac{10\sqrt{24}}{49}}{-\frac{1}{49}} = -10\sqrt{24}$$

$$\csc 2x = \frac{1}{\sin 2x} = \frac{49}{10\sqrt{24}}$$

$$\sec 2x = \frac{1}{\cos 2x} = -49$$

$$\cot 2x = \frac{1}{\tan 2x} = -\frac{1}{10\sqrt{24}}$$

11. Consider the equation $\dfrac{1 - \cos 2x}{2} = \sin^2 x$.

    a) Verify the equation holds for $x = \dfrac{\pi}{6}$.

    Answer:

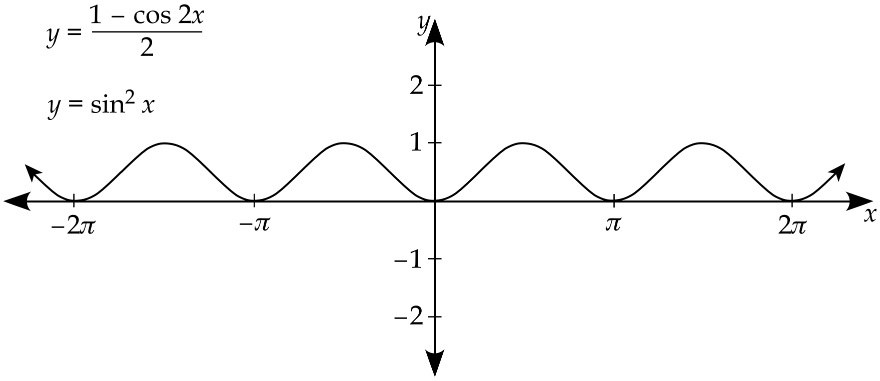
    $$\text{LHS} = \frac{\left(1 - \cos\left(2\left(\frac{\pi}{6}\right)\right)\right)}{2} = \frac{\left(1 - \cos\frac{\pi}{3}\right)}{2} = \frac{1 - \frac{1}{2}}{2} = \frac{\frac{1}{2}}{2} = \frac{1}{4}$$

    $$\text{RHS} = \sin^2 \frac{\pi}{6} = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

    $\therefore$ LHS = RHS

b) Using technology, graph $y = \frac{(1 - \cos 2x)}{2}$ and $y = \sin^2 x$ on the same coordinate grid.

Answer:



c) Using the graph you created in (b), do you believe this is a possible identity?

Answer:

This is a possible identity because the graphs of each function are identical on the interval graphed.

d) Verify algebraically that the equation is or is not an identity.

Answer:

$$\begin{aligned}\text{LHS} &= \frac{1 - \cos 2x}{2}\\ &= \frac{1 - (1 - 2\sin^2 x)}{2}\\ &= \frac{1 - 1 + 2\sin^2 x}{2}\\ &= \frac{2\sin^2 x}{2}\\ &= \sin^2 x\\ &= \text{RHS}\end{aligned}$$

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Module 7
Exponents and Logarithms

MODULE 7:
EXPONENTS AND LOGARITHMS

Introduction

This module introduces you to two very important functions: exponential and logarithmic functions. As the name suggests, these functions are based on the properties of exponents.

These types of functions occur when modelling financial situations and natural processes such as population growth, the decay of a radioactive substance, sound, light, the intensity of an earthquake, the cooling of a dead organism, and even the spread of epidemics. In the last lesson of this module, you will be presented with many application problems that will allow you to see the types of situations in which exponential and logarithmic functions are used as models.

Assignments in Module 7

When you have completed the assignments for Module 7, submit your completed assignments to the Distance Learning Unit either by mail or electronically through the learning management system (LMS). The staff will forward your work to your tutor/marker.

Lesson	Assignment Number	Assignment Title
2	Assignment 7.1	Exponential Functions and Logarithms
4	Assignment 7.2	Dealing with Logarithms
7	Assignment 7.3	Solving Exponential and Logarithmic Equations

Resource Sheet

When you write your final examination, you are encouraged to take a Final Examination Resource Sheet with you into the examination. This sheet will be one letter-sized page, 8½ ″ by 11″, with both sides in your handwriting or typewritten. You will submit it with your examination, but you do not receive any marks for it.

Many students have found that preparing a resource sheet is an excellent way to review. It provides you with a summary of the important facts of each module. You should complete a resource sheet for each module to help with your studying and reviewing. Lesson summaries and module summaries are included for you to use as a guide.

You may use the list of instructions provided below to help you with preparing your resource sheet for the material in Module 7. On this sheet, you should record math terms and definitions, formulas, sample questions, or a list of places where you often make mistakes. You should also identify special areas that require extra attention or review by writing the page numbers.

After you have completed each module's resource sheet, you may summarize the sheets from Modules 1 to 8 to prepare your Final Examination Resource Sheet. The final examination for this course is based on Modules 1 to 8.

Resource Sheet for Module 7

As you go through the lessons of this module, you may want to consider the following suggestions regarding the creation of a resource sheet.

1. List all the important math terms, and define them if necessary.
2. List all the formulas and perhaps a sample problem that shows how the formula is used.
3. If necessary, write the solutions to some problems, showing in detail how you did the calculations.
4. Copy any questions that represent the key points of the lesson, and perhaps include the solutions as well.
5. Identify the problems you found most difficult, and copy the page numbers onto the resource sheet so that you can review them before writing the examination. You may also copy the problems and the solutions onto your resource sheet, and later write them onto your Final Examination Resource Sheet.
6. Write any comments, ideas, shortcuts, or other reminders that may be helpful during an examination.

LESSON 1: EXPONENTIAL FUNCTIONS

Lesson Focus

In this lesson, you will

☐ learn the definition of an exponential function

☐ learn how to sketch exponential functions

☐ learn how to state the properties of these exponential functions

Lesson Introduction

In Grade 11 Pre-Calculus Mathematics, you studied geometric sequences where you were asked a question similar to the following:

> You have been hired for one month as a consultant for a top technology company. Would you rather accept a salary of $1,000,000 for that month or accept a salary beginning at $0.01 the first day, and doubling every consecutive day?

The largest salary is surprisingly the one that begins at only a penny a day. This salary grows extremely fast so that by day 30 you are making $5,368,709.12. After the 30 days, you will have made a total of $10,737,418.23. This is known as **exponential growth**.

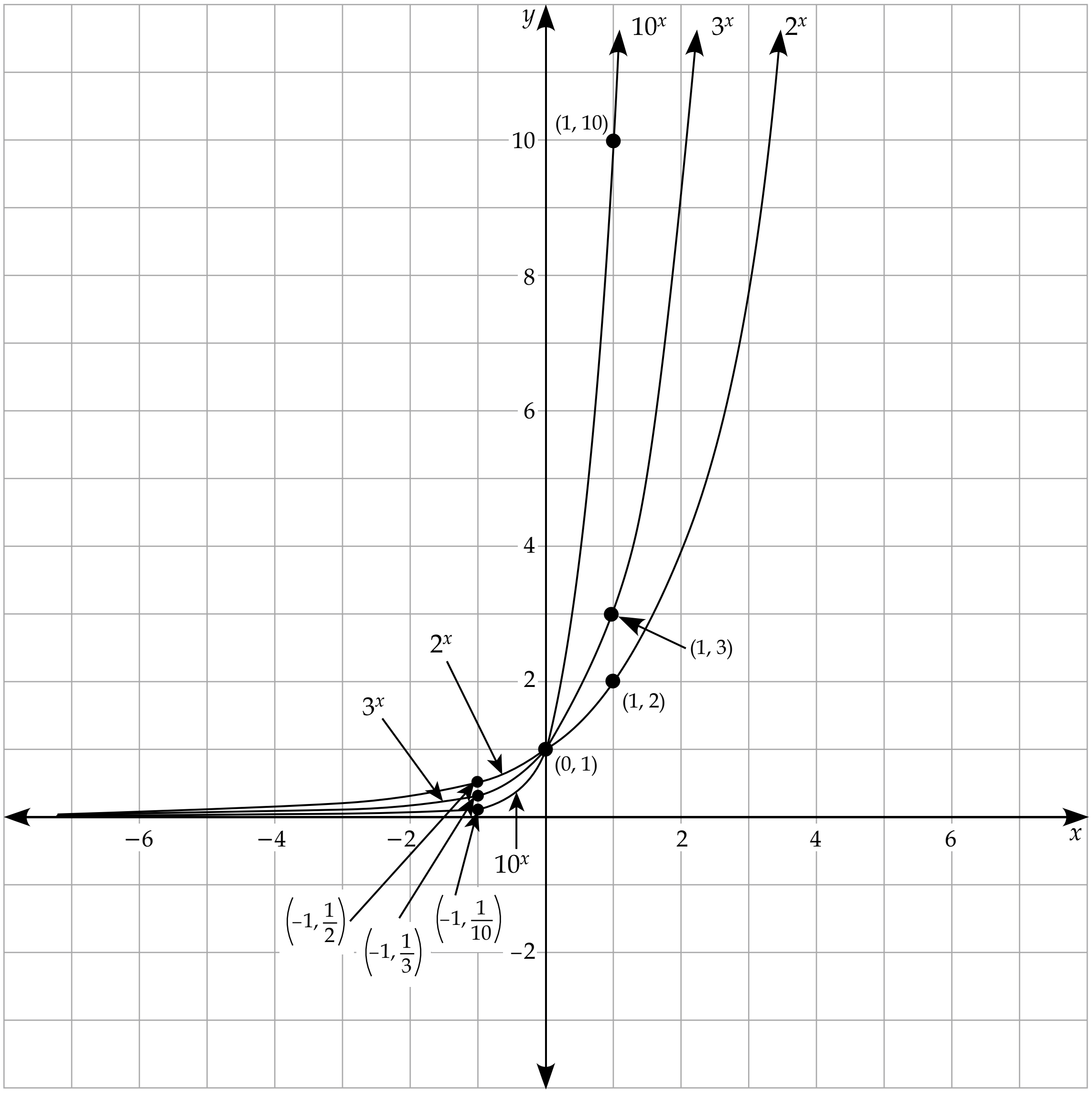
So far, throughout your study of mathematics, you have considered functions that contain variables in the base of an expression. For example, you studied quadratic functions, $f(x) = ax^2 + bx + c$, where the base was x, and the largest exponent was 2. Exponential functions are different, as they contain the variable in the exponent instead of the base. For example, $f(x) = 2^x$ is an exponential function.

Exponential Functions

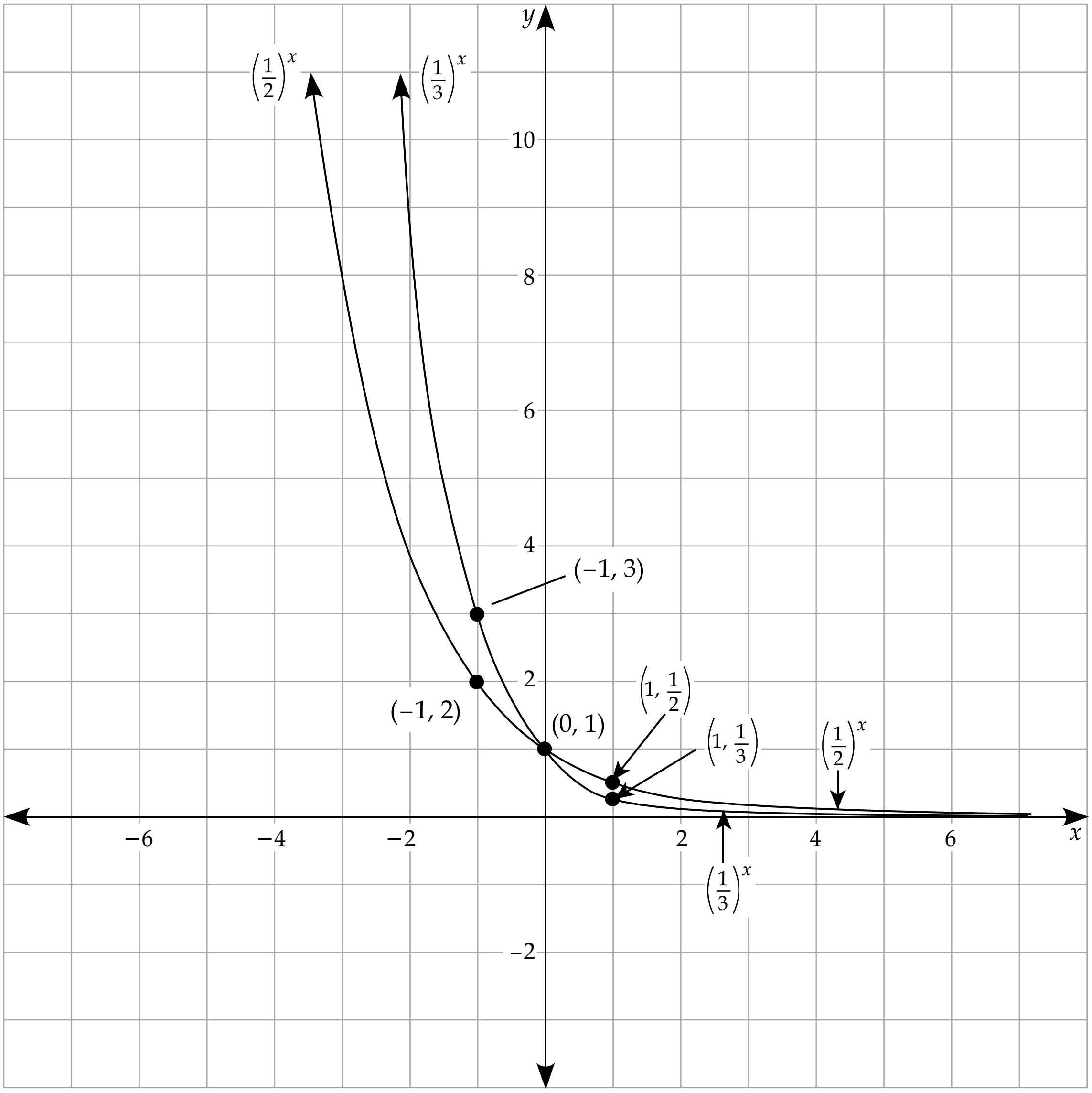
To begin your understanding of exponential growth, start by sketching various graphs of exponential functions. These graphs will give you a good understanding of how quickly exponential functions can increase.

Use the following table of values to sketch exponential functions and make some observations.

x	3	2	1	0	–1	–2	–3
$y = 2^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
$y = 3^x$	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$
$y = 10^x$	1000	100	10	1	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$



x	3	2	1	0	−1	−2	−3
$y = \left(\frac{1}{2}\right)^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$y = \left(\frac{1}{3}\right)^x$	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27

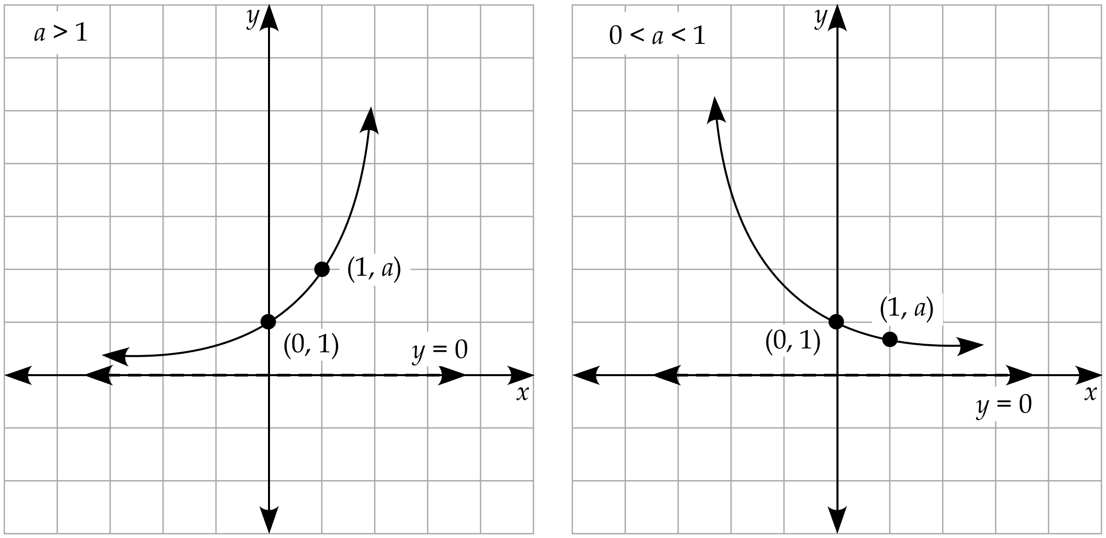


All graphs of the exponential functions have the following properties.

1. They pass through the point with coordinates (0, 1), since $a^0 = 1$.
2. Also, they pass through the point with coordinates (1, *base*) where a = *base*.
3. Their domain is $\Re$.
4. Their range is $(0, \infty)$.
5. They have $y = 0$ as a horizontal asymptote.

An **exponential function** is a function of the form $y = a^x$, where $a > 0$, $a \neq 1$, and $x \in \Re$. The **independent variable** is in the exponent for all exponential functions. In general, all functions of the form $y = a^x$, $a > 0$ have the above four properties.

Furthermore, if $a > 1$, the function is always increasing, and if $0 < a < 1$, the function is decreasing. Consider the two graphs below.



You may find it helpful to include a summary of the above information on your resource sheet.

Note: In the learning activity at the end of this lesson, you will analyze why an exponential function of the form $y = a^x$ has an asymptote located at $y = 0$.

Example 1

Use your knowledge of transformations to sketch each of the following exponential functions and state its domain, range, y-intercept, and equation of asymptote.

a) $y = 2^{x-1}$

b) $y = 2^{x} - 1$

c) $y = -2^{x}$

d) $y = -3\left(2^{x-1}\right)$

e) $y = \left|2^{x} - 1\right|$

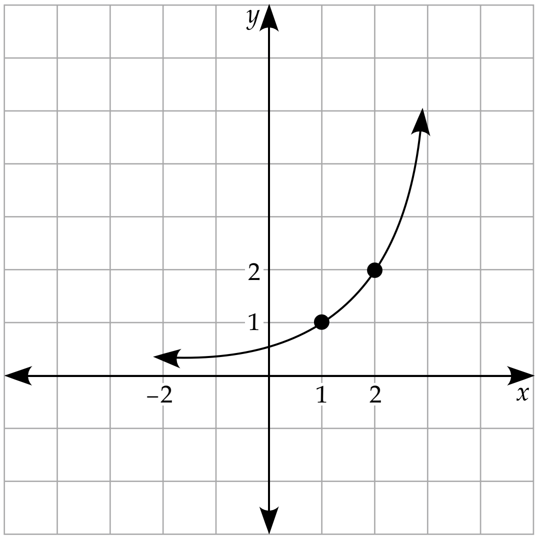
f) $y = \left(\frac{1}{2}\right)^{x}$

Solutions

You should be familiar with transformations from previous modules.

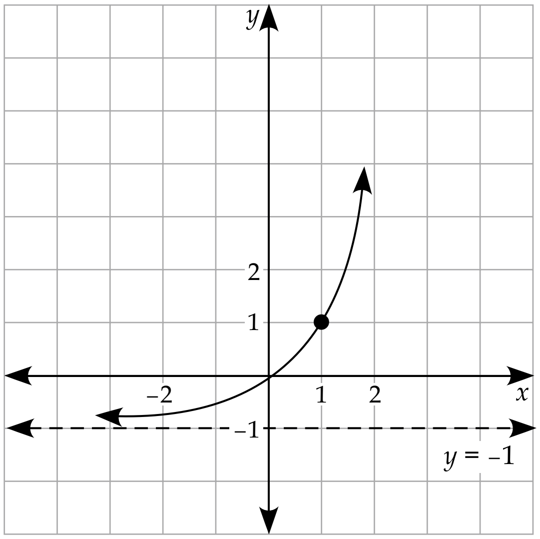
a) The graph of $y = 2^{x-1}$ is the graph of $y = 2^{x}$ shifted 1 unit to the right.

This can also be done algebraically. Use the points of $y = 2^{x}$ as (x, y) and apply the algebraic transformation of $(x, y) \rightarrow (x + 1, y)$.



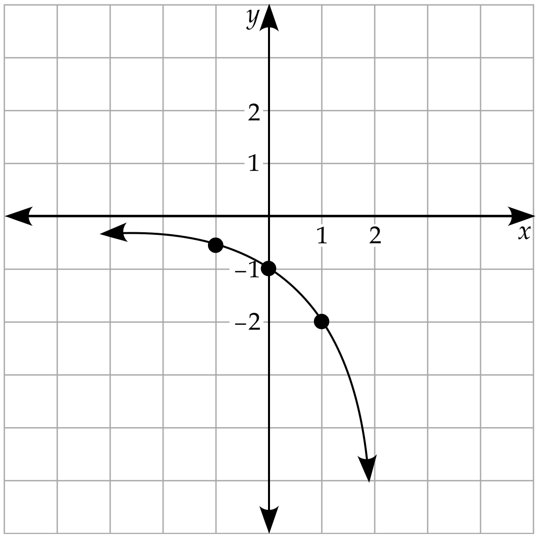
b) The graph of $y = 2^x - 1$ is the graph of $y = 2^x$ shifted 1 unit down.

Algebraically: $(x, y) \rightarrow (x, y - 1)$



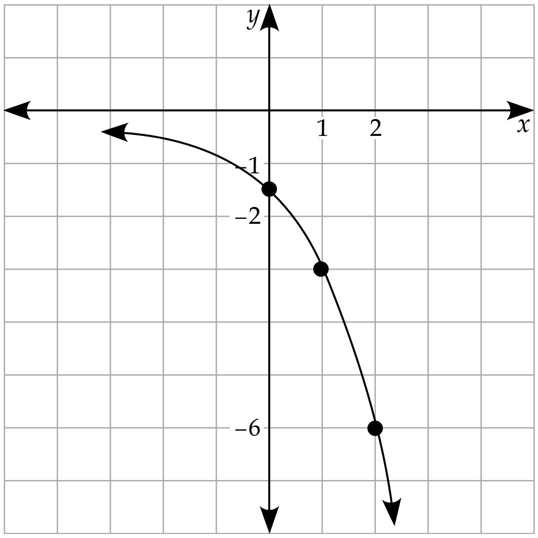
c) The graph of $y = -2^x$ is the graph of $y = 2^x$ reflected over the x-axis.

Algebraically: $(x, y) \rightarrow (x, -y)$

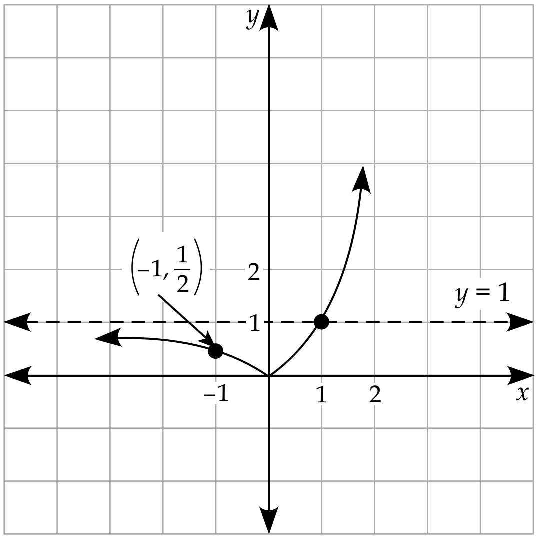


d) The graph of $y = -3(2^{x-1})$ is the graph of $y = 2^x$ shifted 1 unit to the right, reflected over the x-axis, and vertically stretched by a factor of 3.

Algebraically: $(x, y) \rightarrow (x + 1, -3y)$

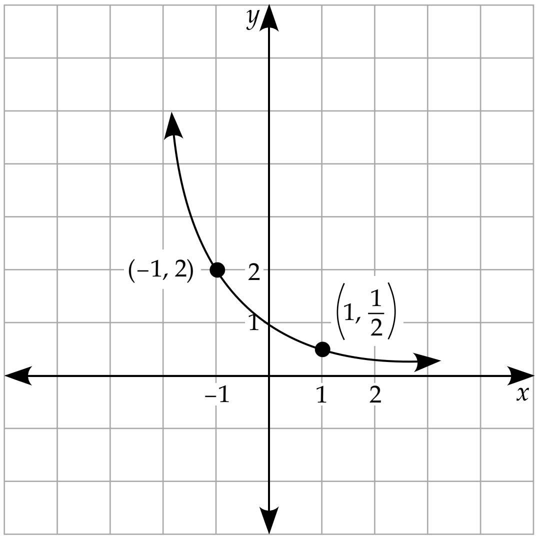


e) The graph of $y = | 2^x - 1 |$ is the graph shown in (b) with the part below the x-axis reflected above the x-axis.



f) **Note:** $y = \left(\frac{1}{2}\right)^x = \frac{1}{2^x} = 2^{-x}$

The graph of $y = \left(\frac{1}{2}\right)^x$ is the same as the graph of $y = 2^x$ reflected in the y-axis.



The table below is easily filled in from reading each graph. If you are not sure of the value of the y-intercept, substitute 0 for x and find the corresponding value of y.

Question	Domain	Range	y-intercept	Equation of Asymptote
(a)	$\Re$	$(0, \infty)$	$\frac{1}{2}$	$y = 0$
(b)	$\Re$	$(-1, \infty)$	0	$y = -1$
(c)	$\Re$	$(-\infty, 0)$	-1	$y = 0$
(d)	$\Re$	$(-\infty, 0)$	$-1\frac{1}{2}$	$y = 0$
(e)	$\Re$	$[0, \infty)$	0	$y = 1$
(f)	$\Re$	$(0, \infty)$	1	$y = 0$

Make sure you complete the following Learning Activity, as it will allow you to practice graphing exponential functions using transformations.

Learning Activity 7.1

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. State the non-permissible values of the function $f(x) = \dfrac{x^2 + 4x + 3}{x^2 + 5x + 6}$.
2. Express $\cot\theta$ in terms of $\csc\theta$ and $\sec\theta$.
3. Evaluate: ${}_5C_1$
4. Convert $\dfrac{\pi}{2}$ to degrees.
5. If $f(x) = \dfrac{x + 2}{x^2 - 4}$, evaluate $f(x)$ at $x = 1$.

continued

Learning Activity 7.1 (continued)

6. Determine the vertex of the function $f(x) = -3(x - 4)^2 + 4$.
7. Multiply: $\left(3\sqrt{4} + \sqrt{2}\right)\left(3\sqrt{4} - \sqrt{2}\right)$
8. Write an expression that represents 11 less than the reciprocal of x.

Part B: Graphing Exponential Functions

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Sketch the curve of $y = 1^x$.
2. Sketch the graph of $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$ on the same coordinate system.

   How are the graphs related? Write $y = \left(\frac{1}{2}\right)^x$ in a different form.
3. Write the function representing the reflection of $y = 5^x$, in the y-axis, in two different ways.
4. Sketch each of the following exponential functions and state the domain, range, y-intercept, and equation of asymptote for each.
   - a) $y = 3^x + 1$
   - b) $y = 3^x - 2$
   - c) $y = -3^x + 2$
   - d) $y = 2(3^x)$
   - e) $y = -2(3^{(x-1)})$
   - f) $y = 3^{-x}$
   - g) $y = 3^{(1-x)}$
   - h) $y = -3^{(2-x)}$
   - i) $y = -3^{(x-1)} + 2$
   - j) $y = |\,3^x - 1\,|$
   - k) $y = \left(\frac{1}{3}\right)^x - 2$
5. The definition of an exponential function, $f(x) = a^x$, states that $a \neq 1$ and $a > 0$.
   - a) Why does a need to be greater than zero?
   - b) Why can't $a = 1$?
   - c) Explain why every exponential function, $f(x) = a^x$, contains a horizontal asymptote at $y = 0$.

continued

Learning Activity 7.1 (continued)

6. When dealing with exponential functions, a special constant appears so frequently that it has been given the symbol e. This constant is equal to approximately 2.718. Sketch the graph of e^x using technology.

   Note: Your calculator or graphing applet will most likely contain the constant e, just as it contains the constant π.

7. If $f(x) = a^x$, $a \neq 1$, $a > 0$, then $f(x)$ passes the horizontal line test and is a one-to-one function. Its inverse will also be a function. Sketch $f^{-1}(x)$ for the following functions by reflecting $f(x)$ over the line $y = x$.
   a) $f(x) = 2^x$
   b) $f(x) = 10^x$
   c) $f(x) = e^x$

8. List the common properties each of the inverses in Question 7 seem to have.

Lesson Summary

In this lesson, you were introduced to exponential functions. You learned how exponential functions contain variables in the exponent. You then learned how to graph multiple variations of exponential functions by using transformations. In the next lesson, you are going to be introduced to logarithms.

LESSON 2: LOGARITHMS AND EXPONENTS

Lesson Focus

In this lesson, you will

- ☐ learn about the relationship between logarithms and exponents
- ☐ learn how to convert between logarithmic and exponential forms
- ☐ learn how to evaluate a logarithmic expression
- ☐ learn how to estimate the value of a logarithm

Lesson Introduction

In Question 7 of Learning Activity 7.1, you were asked to sketch the inverse of various exponential functions. The inverse of an exponential function is such an important function that it has a name. The name for the function curves that you drew in Question 7 is **logarithm**.

Logarithms

Imagine you have $15 000 and you invest it at a rate of 5% each year. How long do you have to wait until you can put a down payment of $20,000 on a house?

Each year, your investment is worth 105% of what it was worth the previous year. Therefore, to get the value of your investment for the following year, multiply the value of the investment each year by 1.05.

Year	Investment Growth	Investment Worth
0	$15 000	$15 000
1	$15 000 (1.05)	$15 750
2	[$15 000(1.05)](1.05)	$15 537.50
3	[$15 000(1.05)](1.05)](1.05)	$17 364.375
4	[$15 000(1.05)](1.05)(1.05)](1.05)	$18 232.59375
5	[$15 000(1.05)](1.05)(1.05)(1.05)](1.05)	$19 144.22344
6	[$15 000(1.05)](1.05)(1.05)(1.05)(1.05)](1.05)	$20 101.43461
n	$\$15\ 000(1.05)^n$	$\$15\ 000(1.05)^n$

Somewhere between the 5th and 6th year, you will have enough money to put a down payment on a house. But is there a way to know this exact date?

You need to solve the equation $20\,000 = \$15\,000(1.05)^n$. This is where logarithms come into play. **Logarithms** are defined to be the inverse operation of exponents, and thus they can *undo* exponents.

To write algebraically the inverse of $y = 2^x$, you switch the x-value and y-value in the statement $y = 2^x$, forming $x = 2^y$. This is the inverse of $y = 2^x$. Logarithms were developed because it would be impossible to write this equation in terms of y, or $f^{-1}(x)$ without introducing this new function type.

The y-form of the function $x = 2^y$ is re-written as $y = \log_2 x$, where y is the exponent to which the base 2 must be raised to yield x. The log form means exactly the same thing as the exponential form. Logarithms were developed in order to be able to solve for y when looking at an equation such as $x = 2^y$.

For $a > 0$, $x = a^y$ is equivalent to $y = \log_a x$.

The statement $y = \log_2 x$ is read as "y = the logarithm of x to the base 2." $x = a^y$ is called the **exponential form**, while $y = \log_a x$ is called the **logarithmic form**.

For example, $5^2 = 25$ can be re-written as $\log_5 (25) = 2$.

You can also think of logarithms as a question.

For example, "evaluate $\log_4 64$" is identical to asking "What is the exponent when the base is 4 and the value is 64?" or to solving $4^x = 64$.

$\therefore \log_4 64 = 3$

Note: Before you continue on with the remainder of this lesson, you should recall the following two facts related to exponents.

$$x^{\frac{1}{2}} = \sqrt[2]{x} = \sqrt{x}$$

$$x^{\frac{1}{3}} = \sqrt[3]{x}$$

Exponential Form: $C = B^A$

Logarithmic Form: $\log_B C = A$

The base "B" is the same for both the exponential form and the logarithmic form. Remember that the exponential function and the logarithmic function are inverses. OR

You know $10^2 = 100$. If you can remember that $\log_{10} 100$ is equal to 2, you can use that to figure out where all the values or variables go when you switch from one form to the other. Record a way that will help you remember on your resource sheet.

Now would be an excellent time to update your resource sheet with a summary of logarithms.

Evaluating Logarithms

It is very important that you know how to convert between exponential form and logarithmic form. Logarithmic form may look confusing to you because it's new. So, the best way to deal with the confusion is to convert to something you know, such as exponential form.

Example 1

Write each logarithmic statement in exponential form and solve for the value of y.

a) $y = \log_2 8$

b) $y = \log_3 9$

c) $y = \log_5\left(\frac{1}{25}\right)$

Solutions

a) $y = \log_2 8$

$2^y = 8$

$2^y = 2^3$

$y = 3$

b) $y = \log_3 9$

$3^y = 9$

$3^y = 3^2$

$y = 2$

c) $y = \log_5\left(\frac{1}{25}\right)$

$5^y = \frac{1}{25}$

$5^y = \frac{1}{5^2}$

$5^y = 5^{-2}$

$y = -2$

In the above example, you were given different logarithmic functions and you found the function values, the y-coordinates, for given values of x. To reverse the process (that is, find the x-coordinate given a y-coordinate), you can use the same method of converting from logarithmic form to exponential form. Some students find working with exponents easier because they are more familiar with exponents than logarithms, even though they are different representations of the same concept! Therefore, you should feel comfortable working with both types of notations.

Example 2

Write each logarithmic statement in exponential form and solve for the value x.

a) $3 = \log_2 x$

b) $1 = \log_{10} x$

c) $-1 = \log_5 x$

d) $\frac{1}{2} = \log_9 x$

e) $2 = \log_e x$

Solutions

a) $2^3 = x$

$8 = x$

b) $10^1 = x$

$10 = x$

c) $5^{-1} = x$

$\frac{1}{5} = x$

d) $9^{\frac{1}{2}} = x$

$\sqrt{9} = 3 = x$

e) $e^2 = x$

$x \cong 7.389$

Therefore, if you are given one of the coordinates of a point on a logarithmic curve, you are able to find the other coordinate by changing the logarithmic form to exponential form.

You can also use the form change to solve for an unknown base. Consider the following example.

Note: Since the base of an exponential function is always positive and logarithms are just the inverse function, it follows that the base of a logarithm is always positive.

Example 3

Write each logarithmic statement in exponential form and solve for the unknown base value b.

a) $3 = \log_b 8$

b) $\dfrac{1}{2} = \log_b 4$

c) $-\dfrac{2}{3} = \log_b 9$

d) $2 = \log_b \dfrac{2}{3}$

Solutions

Step 1: Change to exponential form.

a) $b^3 = 8$

b) $b^{\frac{1}{2}} = 4$

c) $b^{-\frac{2}{3}} = 9$

d) $b^2 = \dfrac{2}{3}$

Step 2: Solve for b.

You must make the exponent of b equal to 1. Therefore, use inverse operations to solve.

a) $\left(b^3\right)^{\frac{1}{3}} = 8^{\frac{1}{3}}$

$$b = 8^{\frac{1}{3}} = \sqrt[3]{8} = 2$$

b) $\left(b^{\frac{1}{2}}\right)^2 = 4^2$

$$b = 4^2 = 16$$

c) $\left(b^{-\frac{2}{3}}\right)^{\left(-\frac{3}{2}\right)} = 9^{-\frac{3}{2}}$

$$b = 9^{\left(\frac{1}{2}\right)(-3)} = \left(\sqrt{9}\right)^{-3} = 3^{-3} = \frac{1}{3^3} = \frac{1}{27}$$

d) $\left(b^2\right)^{\frac{1}{2}} = \left(\dfrac{2}{3}\right)^{\frac{1}{2}}$

$$b = \sqrt{\frac{2}{3}}$$

Note: Normally, for part (d), you would write $b = \pm\sqrt{\dfrac{2}{3}}$ but remember for logarithms that the base, b, is always greater than zero.

Question: What do you think log 100 might represent?

If you see a logarithm written without a base, by convention this is understood to be a logarithm with base 10. This is because our number system is base 10 and logarithms were originally used to multiply very large and very small base 10 numbers.

Therefore, log 100 is the same as $\log_{10} 100$, which is asking $10^x = 100$. The answer is 2 as $10^2 = 100$.

Include this convention on your resource sheet.

Additionally, the base used for many applications is base e, so a shorthand notation is used: ln 5 means $\log_e 5$. The letter “l” stands for logarithm and the letter “n” stands for natural. Logarithms with a base of e are called natural logarithms and are written as $\log_e x$ or as $\ln x$. If you use your calculator for logarithms, either base 10 or base e are usually the only 2 bases provided.

Example 4

Solve for the variable.

a) $\log 1000 = x$

b) $\log x = 5$

Solutions

a) log 1000 is the same as $\log_{10} 1000$.

Rewrite in exponential form.

$10^x = 1000$

Solve for x.

$10^x = 10^3$

$x = 3$

b) $\log x = 5$ is the same as $\log_{10} x = 5$.

Rewrite in exponential form and then solve for x.

$10^5 = x$

$100\ 000 = x$

You may wish to include a few examples of switching between logarithmic and exponential form and of determining the value of a logarithm on your resource sheet.

Estimating Logarithms Using Benchmarks

How would you go about estimating the value of $\log_2 7$?

First, you could write this expression in exponential form as $2^x = 7$.

You can also conclude that the answer is not a whole number, as you know that $2^2 = 4$ and $2^3 = 8$. Therefore, $2 < x < 3$, or x must be between 2 and 3.

Using these values as benchmarks, you should be able to notice that x should be closer to 3 than to 2, as 2^3 is closer to 7 than 2^2.

Therefore, an appropriate estimate would be $x = 2.9$.

Check your estimate:

$2^{2.9} = 7.464$

$\therefore \log_2 7 \approx 2.9$

This is a good estimate.

Example 5

Estimate the value of each of the following logarithms.

a) $\log_3 11$
b) $\log_4 27$
c) $\log_2 19$
d) $\log 59$
e) $\ln 6$

Solutions

a) $\log_3 11$

First, rewrite in exponential form: $3^x = 11$

Then, determine values using benchmarks: $3^2 = 9$ and $3^3 = 27$
$\therefore 2 < x < 3$

Now, use the closest value to estimate x: 11 is closer to 9 than 27, so x is closer to 2.

Estimate is 2.2.

$\therefore \log_3 11 \approx 2.2$

Check: $3^{2.2} = 11.212$

b) $\log_4 27$

Abbreviate the process:

Exponential form: $4^x = 27$

Benchmarks: $4^2 = 16$ and $4^3 = 64$

Estimate: 27 is closer to 16, so x is closer to 2
$\therefore x = 2.3$

Check: $4^{2.3} = 24.25$ (estimate is a little low)

$\therefore \log_4 27 \approx 2.3$

c) $\log_2 19$

Exponential form: $2^x = 19$

Benchmarks: $2^4 = 16$ and $2^5 = 32$

Estimate: 19 is closer to 16, so x is 4.2

Check: $2^{4.2} = 18.379$

$\therefore \log_2 19 \approx 4.2$

d) $\log 59$

Exponential form: $10^x = 59$

Benchmarks: $10^1 = 10$ and $10^2 = 100$

Estimate: 59 is closer to 100, so x is closer to 2
x is 1.8

Check: $10^{1.8} = 63.096$ (estimate is a little high)

$\therefore \log 59 \approx 1.8$

e) $\ln 6$

(Remember: $\ln 6$ is $\log_e 6$)

Exponential form: $e^x = 6$

Benchmarks (recall e is 2.718 . . .): $e^1 = 2.718$ and $e^2 = 7.389$

Estimate: 6 is closer to 7.389, so x is 1.8

Check: $e^{1.8} = 6.05$ (this is a good estimate)

$\therefore \ln 6 \approx 1.8$

Solving Exponential Equations

To determine the value of the variable in a logarithmic equation, you sometimes need to be able to solve exponential equations.

In order to solve exponential equations, you need to write each expression as a power with the same base. Once the bases are equivalent, you can then equate the exponents on either side of the equal sign. Example 1 of this lesson used this technique as well.

Note: You will learn more about solving exponential equations in Lesson 5. However, it is useful to know how to solve the simpler forms of exponential equations when you are finding logarithms. Exponential equations can sometimes be solved by converting both sides to the same base.

Example 6

Solve for x.

a) $5^x = 125$

b) $3^{x-1} = 27$

c) $4^{x-2} = 2$

d) $81^{2x} = 27^{x+1}$

e) $25^{2x+3} = \left(\frac{1}{125}\right)^{x+4}$

Solutions

Solve for x:

a) $5^x = 125$

First, write both sides with the same base: $5^x = 5^3$

Simplify, if necessary, using exponent laws, and equate the exponents: $x = 3$

Solve the equation (already solved): $x = 3$

b) $3^{x-1} = 27$

Same base both sides: $3^{x-1} = 3^3$

Equate exponents: $x - 1 = 3$

Solve: $x = 4$

c) $4^{x-2} = 2$

Same base: $(2^2)^{x-2} = 2^1$

Equate exponents: $2x - 4 = 1$

(Remember $(2^2)^{x-2} = 2^{2x-4}$ because you multiply when there is an exponent raised to another exponent.)

Solve:

$$2x - 4 = 1$$
$$2x = 5$$
$$x = \frac{5}{2}$$

d) $81^{2x} = 27^{x+1}$

Same base of 3: $(3^4)^{2x} = (3^3)^{x+1}$

Exponent laws: $3^{8x} = 3^{3x+3}$

Equate exponents: $8x = 3x + 3$

Solve equation: $5x = 3$

$$x = \frac{3}{5}$$

e) $25^{2x+3} = \left(\frac{1}{125}\right)^{x+4}$

Same base of 5: $(5^2)^{2x+3} = (5^{-3})^{x+4}$

Exponent laws: $5^{4x+6} = 5^{-3x-12}$

Equate exponents: $4x + 6 = -3x - 12$

Solve: $7x = -18$

$$x = -\frac{18}{7}$$

Make sure you complete the following Learning Activity, as it will give you practice converting between logarithmic form and exponential form. This is an essential skill you need to have before continuing on to complete the rest of the module.

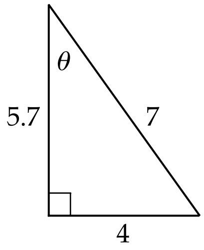
Learning Activity 7.2

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

Use the triangle below to answer Questions 1 to 3.



1. Determine the cosine ratio.
2. Determine the cotangent ratio.
3. Determine the secant ratio.
4. How many ways can three students be awarded first, second, and third place in a contest?
5. State the non-permissible values of the function $f(x) = \dfrac{4}{x^2 + 4x}$.
6. Convert 4π to degrees.
7. Find all the values of θ between $[0, 2\pi]$, if $\sin \theta = -1$.
8. Multiply: $(-3x^{-2})(4x^3)$

continued

Learning Activity 7.2 (continued)

Part B: Logarithms

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Evaluate each of the following.

   a) $16^{\frac{1}{2}}$

   b) $27^{\frac{2}{3}}$

   c) $16^{-\frac{1}{2}}$

   d) $27^{-\frac{2}{3}}$

   e) $\left(\frac{1}{4}\right)^{-\frac{1}{2}}$

   f) $81^{\frac{1}{4}}$

   g) $-4^{-\frac{1}{2}}$

   h) $(-8)^{-\frac{1}{3}}$

   i) $100^{-\frac{3}{2}}$

   j) $4\left(\frac{1}{9}\right)^{\frac{1}{2}}$

2. Express each of the following in exponential form.

   a) $\log_2 16 = 4$

   b) $\log_4 64 = 3$

   c) $\log_{10} 0.01 = -2$

   d) $\log_5 \frac{1}{5} = -1$

3. Write $\log_A B = C$ in exponential form.

4. Express each of the following in logarithmic form.

   a) $3^4 = 81$

   b) $32 = 2^5$

   c) $\left(\frac{1}{4}\right)^2 = \frac{1}{16}$

   d) $2^{-3} = \frac{1}{8}$

continued

Learning Activity 7.2 (continued)

5. Write $A^B = C$ in logarithmic form.

6. Solve for x (without using a calculator).

   a) $2^x = 32$ b) $2^{3x-5} = 16$

   c) $5^{4x-7} = 125$ d) $\dfrac{1}{3^{x-1}} = 81$

   e) $2\left(5^{2x-9}\right) = 250$ f) $32^{3x-2} = 16$

   g) $3^{8x} = \dfrac{1}{81}$ h) $\dfrac{1}{4^{x-2}} = 64$

7. Solve for the variable.

   a) $y = \log_5 125$ b) $\log_5 x = 2$

   c) $\log_x 16 = -\dfrac{4}{3}$ d) $y = \log_{\frac{1}{2}} \left(\dfrac{1}{2}\right)^4$

   e) $\log_7 1 = y$ f) $y = \log_2 \dfrac{1}{4}$

   g) $\log_3 x = -4$ h) $\log_8 x = \dfrac{4}{3}$

   i) $\log_4 x = -\dfrac{5}{2}$ j) $\log_b 125 = \dfrac{3}{4}$

   k) $\log_b \sqrt{5} = \dfrac{1}{4}$ l) $\log_{\sqrt{3}} 9 = y$

8. For all values of a, $a > 0$, what is the value of $\log_a a$?

9. For all values of a, $a > 0$, what is the value of $\log_a 1$?

10. Evaluate the following logarithms without using a calculator.

    a) $\log_2 2$ b) $\log_7 \left(7^6\right)$

    c) $\log_5 625$ d) $\log_2 256$

    e) $\log_3 1$ f) $\log_5 \dfrac{1}{25}$

    g) $\log_6 \dfrac{1}{6}$

continued

Learning Activity 7.2 (continued)

11. Estimate the following logarithms to one decimal place without using a calculator.
    a) $\log_{10} 800$
    b) $\log_2 47$
    c) $\log_3 76$
    d) $\log_4 15$

Lesson Summary

In this lesson, you were introduced to logarithms. You learned how logarithms are exponents and how exponential equations and logarithmic equations are related. You also learned how to determine the exact values of logarithms by relating them back to exponents. When the logarithms did not relate back to whole number exponents, you learned techniques to estimate the value of a logarithm. In the next lesson, you will learn about the properties of logarithms, which are related to the properties of exponents.

Assignment 7.1

Exponential Functions and Logarithms

Total: 48 marks

You need to show all your steps as you work out your solutions, and make sure your answers are clear (include units, where appropriate). Check the Introduction for the Marking Guidelines that will be used for all assignments and examinations.

1. a) Sketch the graph of $y = 3^x$ and $y = \left(\frac{1}{3}\right)^x$ on the same coordinate system. *(2 marks)*

b) Use transformation language to describe how the graphs are related. *(1 mark)*

c) Write $y = \left(\frac{1}{3}\right)^x$ in an alternate form. *(1 mark)*

continued

Assignment 7.1: Exponential Functions and Logarithms (continued)

2. Sketch each of the following exponential functions and state the domain, range, y-intercept, and equation of asymptote for each. *(5 × 5 marks each = 25 marks)*

   a) $y = 2^{x-3} + 4$

continued

Assignment 7.1: Exponential Functions and Logarithms (continued)

b) $y = \frac{1}{2}\left(3^{x+2}\right)$

continued

Assignment 7.1: Exponential Functions and Logarithms (continued)

c) $y = -2^{x-1} + 3$

continued

Assignment 7.1: Exponential Functions and Logarithms (continued)

d) $y = 2(3^{-x-1})$

continued

Assignment 7.1: Exponential Functions and Logarithms (continued)

e) $y = -e^{x+2}$

continued

Assignment 7.1: Exponential Functions and Logarithms (continued)

3. Express each of the following in exponential form. *(4 × 0.5 mark each = 2 marks)*

   a) $\log_7 49 = 2$

   b) $\log \dfrac{1}{1000} = -3$

   c) $\log_{\frac{1}{4}} \dfrac{1}{64} = 3$

   d) $\ln 2 = x$

continued

Assignment 7.1: Exponential Functions and Logarithms (continued)

4. Express each of the following in logarithmic form. *(4 × 0..5 mark each = 2 marks)*

   a) $6^3 = 216$

   b) $\left(\frac{1}{3}\right)^{-5} = 243$

   c) $512 = 8^3$

   d) $e^5 = x$

continued

Assignment 7.1: Exponential Functions and Logarithms (continued)

5. Solve for the variable.

   a) $\log_4 x = 2$ *(1 mark)*

   b) $\log_x 36 = -2$ *(2 marks)*

   c) $\log_9 \sqrt{3} = x$ *(2 marks)*

continued

Assignment 7.1: Exponential Functions and Logarithms (continued)

d) $\log_8 x = -\frac{2}{3}$ *(2 marks)*

e) $\log x = 4$ *(1 mark)*

f) $\log_3 \frac{1}{81} = x$ *(2 marks)*

g) $\ln x = 3$ *(1 mark)*

continued

Assignment 7.1: Exponential Functions and Logarithms (continued)

6. Estimate the following logarithms to one decimal place without using a calculator. Show your work. *(2 × 2 marks each = 4 marks)*

   a) $\log_4 71$

   b) $\log_3 35$

Notes

LESSON 3: THE LOGARITHMIC THEOREMS

Lesson Focus

In this lesson, you will

- ☐ learn how to prove the logarithmic theorems
- ☐ learn about the relationship between the logarithmic theorems and the exponent laws
- ☐ learn how to use the logarithmic theorems to simplify logarithmic expressions

Lesson Introduction

In the last lesson, you were introduced to the concept of logarithms. You learned how logarithms are related to exponents. In previous courses, you dealt with exponent laws that allowed you to simplify expressions that contain different powers with the same base that are being multiplied, divided, or raised to a power. In this lesson, you will see how these three exponent laws relate to the three logarithmic theorems you will need to know for this course.

The Logarithmic Theorems

Right now, if you were asked to find the zero of a logarithmic function such as $f(x) = \log_3 x - 2$, you would probably answer this question as follows:

$f(x) = \log_3 x - 2$	First, you would set $f(x)$ equal to zero.
$0 = \log_3 x - 2$ $2 = \log_3 x$	Next, you would rearrange the equation in order to be able to rewrite the equation in exponential form.
$3^2 = x$	Then, you would rewrite the equation in exponential form.
$x = 9$	Finally, you would solve for x.

Therefore, the zero of this function is 9. However, if the function was $h(x) = \log_2 x + \log_2 (x - 1) - 1$, you would have a difficult time trying to find these x-intercepts.

In this lesson, you will prove the three logarithmic theorems, which are actually different representations of the exponent laws that you already know. These logarithmic theorems will help you to solve more complex logarithmic equations and determine the zeros of more complex logarithmic functions.

Recall: A **logarithm** of a number, N, to a base, b, is the **exponent**, y, to which the base must be raised to yield the number N. It is easier than it sounds, $\log_b N = y$ means $b^y = N$.

Before you are introduced to the logarithm rules, you need to be comfortable with the exponent laws. The following chart provides a review.

Exponent Laws		
Product Law	When you multiply powers with the same base, add the exponents.	$b^n b^m = b^{n+m}$
Quotient Law	When you divide powers with the same base, subtract the exponents.	$\dfrac{b^n}{b^m} = b^{n-m}$
Power Law	When a power is raised to an exponent, multiply the exponents.	$\left(b^m\right)^n = b^{mn}$

The Product Rule for Logarithms

To determine the product rule for logarithms, first consider what pattern seems to hold for the expressions in the following example.

Example 1

Evaluate these expressions using your knowledge of logarithms.

a) $\log_2 8 + \log_2 16$

b) $\log_3 3 + \log_3 27$

c) $\log_4 16 + \log_4 64$

Solutions

a) $\log_2 8 = 3$ as $2^3 = 8$ and $\log_2 16 = 4$ as $2^4 = 16$

$\therefore \log_2 8 ++ \log_2 16 = 3 + 4 = 7$

b) $\log_3 3 = 1$ as $3^1 = 3$ and $\log_3 27 = 3$ as $3^3 = 27$

$\therefore \log_3 3 + \log_3 27 = 1 + 3 = 4$

c) $\log_4 16 = 2$ as $4^2 = 16$ and $\log_4 64 = 3$ as $4^3 = 64$

$\therefore \log_4 16 + \log_4 64 = 2 + 3 = 5$

Example 2

Solve the following equations. Relate the solutions of these equations to the logarithmic expressions in Example 1.

a) $7 = \log_2 x$

b) $4 = \log_3 x$

c) $5 = \log_4 x$

Solutions

a) $2^7 = x$

$128 = x$

$x = 128$ is equivalent to the product of 8 and 16.

That is, $2^7 = 8 \cdot 16$ and in log form, $\log_2 (8 \cdot 16) = 7$.

Relating this to Example 1 (a), $\log_2 8 + \log_2 16 = 7$.

b) $3^4 = x$

$81 = x$

$x = 81$ is equivalent to the product of 3 and 27.

That is, $3^4 = 3 \cdot 27$ and in log form, $\log_3 (3 \cdot 27) = 4$.

Relating this to Example 1 (b), $\log_3 3 + \log_3 27 = 4$.

c) $4^5 = x$

$1024 = x$

$x = 1024$ is equivalent to the product of 16 and 24.

That is, $4^5 = 16 \cdot 64$ and in log form, $\log_4 (16 \cdot 64) = 5$.

Relating this to Example 1 (c), $\log_4 16 + \log_4 64 = 5$.

What rule seems to hold for logarithmic expressions of the form $\log_b \boldsymbol{P} + \log_b \boldsymbol{Q}$?

From the above examples, you can make the conjecture that $\log_b \boldsymbol{P} + \log_b \boldsymbol{Q}$ is equivalent to $\log_b (\boldsymbol{PQ})$. Consider the following proof.

Product Rule for Logarithms Proof

$\log_b (\boldsymbol{PQ}) = \log_b \boldsymbol{P} + \log_b \boldsymbol{Q}$

Translation: The logarithm of a product is the sum of the logarithms of each factor.

Proof:

Let $x = \log_b P$ and $y = \log_b Q$

$b^x = P$ and $b^y = Q$	Change to exponential form.
$PQ = b^x b^y$	Perform multiplication.
$PQ = b^{x+y}$	Add the exponents (use the exponent laws).
$\log_b (PQ) = x + y$	Change back to logarithmic form.
$\log_b (PQ) = \log_b P + \log_b Q$	Substitute for x and y.

The Quotient Rule for Logarithms

To determine the quotient rule for logarithms, first consider what pattern seems to hold for the expressions in the following example.

Example 3

Evaluate these expressions using your knowledge of logarithms.

a) $\log_2 16 - \log_2 4$

b) $\log_3 81 - \log_3 3$

c) $\log_4 256 - \log_4 16$

Solutions

a) $\log_2 16 = 4$ as $2^4 = 16$ and $\log_2 4 = 2$ as $2^2 = 4$

$\therefore \log_2 16 - \log_2 4 = 4 - 2 = 2$

b) $\log_3 81 = 4$ as $3^4 = 81$ and $\log_3 3 = 1$ as $3^1 = 3$

$\therefore \log_3 81 - \log_3 3 = 4 - 1 = 3$

c) $\log_4 256 = 4$ as $4^4 = 256$ and $\log_4 16 = 2$ as $4^2 = 16$

$\therefore \log_4 256 - \log_4 16 = 4 - 2 = 2$

Example 4

Solve the following equations. Relate the solutions of these equations to the logarithmic expressions in Example 3.

a) $2 = \log_2 x$

b) $3 = \log_3 x$

c) $2 = \log_4 x$

Solutions

a) $2^2 = x$

$4 = x$

$x = 4$ is equivalent to the quotient of 16 and 4.

That is, $2^2 = 16 \div 4$ and in log form, $\log_2\left(\frac{16}{4}\right) = 2$.

Relating this to Example 3 (a), $\log_2 16 - \log_2 4 = 2$.

b) $3^3 = x$

$27 = x$

$x = 27$ is equivalent to the quotient of 81 and 3.

That is, $3^3 = 81 \div 3$ and in log form, $\log_3\left(\frac{81}{3}\right) = 3$.

Relating this to Example 3 (b), $\log_3 81 - \log_3 3 = 3$.

c) $4^2 = x$

$16 = x$

$x = 16$ is equivalent to the quotient of 256 and 16.

That is, $4^2 = 256 \div 16$ and in log form, $\log_4\left(\frac{256}{16}\right) = 2$.

Relating this to Example 3 (c), $\log_4 256 - \log_4 16 = 2$.

What rule seems to hold for logarithmic expressions of the form $\log_b \boldsymbol{P} - \log_b \boldsymbol{Q}$?

From the above examples, you can make the conjecture that $\log_b \boldsymbol{P} - \log_b \boldsymbol{Q}$ is equivalent to $\log_b\left(\frac{P}{Q}\right)$. Consider the following proof.

Quotient Rule for Logarithms Proof

$$\log_b \left(\frac{P}{Q}\right) = \log_b P - \log_b Q$$

Translation: The logarithm of a quotient is the difference of the logarithms of numerator and denominator.

Proof:

Let $x = \log_b P$ and $y = \log_b Q$

$b^x = P$ and $b^y = Q$ Change to exponential form.

$\frac{P}{Q} = \frac{b^x}{b^y}$ Perform the division.

$\frac{P}{Q} = b^{x-y}$ Subtract the exponents (use the exponent laws).

$\log_b \left(\frac{P}{Q}\right) = x - y$ Change back to logarithmic form.

$\log_b \left(\frac{P}{Q}\right) = \log_b P - \log_b Q$ Substitute for x and y.

The Power Rule for Logarithms

To determine the power rule for logarithms, first consider what pattern seems to hold for the expressions in the following example.

Example 5

Evaluate these expressions using your knowledge of logarithms.

a) $\log_2 4^2$

b) $\log_3 3^7$

c) $\log_4 16^3$

Solutions

a) $\log_2 4^2$ can be written with a variable as $x = \log_2 4^2$, so write $2^x = 4^2$

$2^x = 16$

$x = 4$

$\therefore \log_2 4^2 = 4$

b) $\log_3 3^7$ can be written with a variable as $x = \log_3 3^7$, so write $3^x = 3^7$

$x = 7$

$\therefore \log_3 3^7 = 7$

c) $\log_4 16^3$ can be written with a variable as $x = \log_4 16^3$, so write $4^x = 16^3$

$4^x = 4096$

$x = 6$

$\therefore \log_4 16^3 = 6$

Example 6

Solve the following equations. Relate the solutions of these equations to the logarithmic expressions in Example 5.

a) $4 = x \log_2 4$

b) $7 = x \log_3 3$

c) $6 = x \log_4 16$

Solutions

a) $\log_2 4 = 2$ since $2^2 = 4$

$4 = x \log_2 4$

$4 = x(2)$

$2 = x$

That is, $4 = 2 \log_2 4$

Relating this to Example 5 (a), $\log_2 4^2 = 4$.

b) $\log_3 3 = 1$ since $3^1 = 3$

$7 = x \log_3 3$

$7 = x(1)$

That is, $7 = 7 \log_3 3$

Relating this to Example 5 (b), $\log_3 3^7 = 7$.

c) $\log_4 16 = 2$ since $4^2 = 16$

$6 = x \log_4 16$

$6 = x(2)$

$3 = x$

That is, $6 = 3 \log_4 16$

Relating this to Example 5 (c), $\log_4 16^3 = 6$.

What rule seems to hold for logarithmic expressions of the form $\log_b P^n$?

From the above examples, you can make the conjecture that $\log_b P^n$ is equivalent to $n(\log_b P)$. Consider the following proof.

Power Rule for Logarithms Proof

$\log_b P^n = n(\log_b P)$

Translation: The logarithm of a power is the product of the exponent and the logarithm of the base of the power.

Proof:

Let $x = \log_b P$

$b^x = P$	Change to exponential form.
$P^n = (b^x)^n$	Raise both sides to the power "*n*."
$P^n = b^{nx}$	Multiply the exponents (use the exponent laws).
$\log_b P^n = nx$	Change back to logarithmic form.
$\log_b P^n = n(\log_b P)$	Substitute for *x*.

The following table summarizes the three logarithmic theorems that you have just learned. It would be a good idea to copy this table onto your resource sheet.

Table of Logarithmic Theorems and Their Equivalent Exponent Properties

If $P > 0$, and $Q > 0$, then

Logarithmic Theorem	Corresponding Exponent Law	Name
$\log_b (PQ) = \log_b P + \log_b Q$	$b^n b^m = b^{n+m}$	Product Rule
$\log_b \left(\frac{P}{Q}\right) = \log_b P - \log_b Q$	$\frac{b^n}{b^m} = b^{n-m}$	Quotient Rule
$\log_b P^n = n(\log_b P)$	$(b^m)^n = b^{mn}$	Power Rule

These theorems enable you to combine, or expand, logarithmic expressions.

In the beginning of this lesson, one of the more difficult problems involved finding the zeros of the function $h(x) = \log_2 x + \log_2 (x - 1) - 1$. You are now able to solve this problem using the logarithmic Product Rule.

$h(x) = \log_2 x + \log_2 (x - 1) - 1$

$0 = \log_2 x + \log_2 (x - 1) - 1$ Let $h(x) = 0$ to find the zeros.

$1 = \log_2 x + \log_2 (x - 1)$ Rearrange.

$1 = \log_2 [x(x - 1)]$ Use the Product Rule for logarithms.

$2^1 = x^2 - x$ Change to exponential form.

$0 = x^2 - x - 2$ Rearrange.

$0 = (x - 2)(x + 1)$ Factor.

$x = 2$ or $x = -1$ Use the Zero Product Property.

However, $\log_2 (-1)$ does not exist since the argument must be greater than zero. Therefore, $x = 2$ is the solution.

Reminder of what the argument is: In the expression $\log_4 (x + 1) = 2$, $(x + 1)$ would be the argument. In the expression $\log_5 m = -3$, m would be the argument and the argument must be greater than zero, which you could confirm by converting to exponential form.

Before you begin solving logarithmic equations, it is important that you first practise using these theorems in simplifying expressions. Notice that you need to always check your answers for extraneous solutions when solving equations.

Example 7

Express the following as a single logarithm.

a) $\log_a 5 + \log_a 8 - \log_a 4$

b) $\log_2 A - 2 \log_2 B + \frac{1}{2} \log_2 C$

Solutions

a) $\log_a 5 + \log_a 8 - \log_a 4$

$= \log_a \left(\frac{5(8)}{4}\right)$

$= \log_a 10$

b) $\log_2 A - 2\log_2 B + \frac{1}{2}\log_2 C$

$= \log_2 A - \log_2 B^2 + \log_2 C^{\frac{1}{2}}$

$= \log_2 \left(\frac{A\sqrt{C}}{B^2}\right)$

Notice how addition and subtraction of logarithmic terms becomes the logarithm of a product or quotient of numbers. A constant multiplying of a logarithmic term becomes the logarithm of a power.

Notice that this example provides a process for changing an expression with many logarithms to an expression with only one logarithm. Similar to working with the exponent rules, the logarithmic rules can only be done when all the bases are the same.

Example 8

Expand as a sum and/or difference of logarithmic terms.

a) $\log\left(\dfrac{A^2B^3}{\sqrt{C}}\right)$

b) $\log\left(\dfrac{10\sqrt{x+y}}{z^3}\right)$

Solutions

a) $\log\left(\dfrac{A^2B^3}{\sqrt{C}}\right)$

$= \log A^2 + \log B^3 - \log \sqrt{C}$ Remember, $\sqrt{C} = C^{\frac{1}{2}}$.

$= 2 \log A + 3 \log B - \dfrac{1}{2} \log C$

b) $\log\left(\dfrac{10\sqrt{x+y}}{z^3}\right)$

$= \log 10 + \log\left(\sqrt{x+y}\right) - \log z^3$

$= \log 10 + \dfrac{1}{2}\log(x+y) - 3\log z$

$= 1 + \dfrac{1}{2}\log(x+y) - 3\log z$

To simplify log 10, think, “To what exponent does the base 10 need to be raised to result in 10?”

Note: It is a common error for students to think that log $(x + y)$ can be expanded using logarithmic rules; however, there is no rule to deal with the log of a sum.

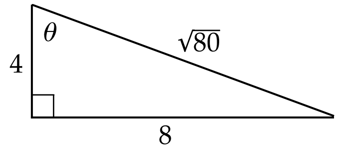
Learning Activity 7.3

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

Use the triangle below to answer Questions 1 to 3.



1. Determine the cosine ratio.
2. Determine the tangent ratio.
3. Determine the cosecant ratio.
4. Factor: $x^2 + 2x - 24$
5. What is $\frac{3}{7}$ of $\frac{5}{8}$?
6. Simplify: $\sqrt{20} + \sqrt{45}$
7. If the graph of $y = -3x^2 + 2$ is translated 4 units right, what is an equation of the translated parabola?
8. In which quadrant is $\theta = \frac{15\pi}{8}$ located?

continued

Learning Activity 7.3 (continued)

Part B: Using the Logarithmic Theorems

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Expand as a sum and/or difference of individual logarithmic expressions, and simplify if possible.

   a) $\log_b \left(\frac{a^2}{\sqrt{r}}\right)$ b) $\log_7 \left(49\sqrt[3]{A}\right)$

   c) $\log \left(\frac{A^2}{B^3C}\right)$ d) $\log \left(\frac{2xy}{\sqrt{z}}\right)$

   e) $\log\left(\frac{ABC}{PQ^2}\right)$ f) $\log \left(\frac{3b\sqrt{c+1}}{4d^2}\right)$

   g) $\log_3 \left(x\sqrt{y}\right)$ h) $\log \left(\frac{8a^3}{b^4c^5}\right)$

2. Write as a single logarithm.

   a) $\log_3 A + 3\log_3 B - \log_3 C$ b) $\frac{1}{3}\log A - \log B + \log C$

   c) $\log (a+b) - \log(a-b)$ d) $\frac{1}{2}\left(\log_a x - 3\log_a y\right)$

   e) $2\left(\log (x+y) - \log z\right)$ f) $\frac{1}{2}\log A + 2\log B - \frac{1}{5}\log C$

3. Use the logarithmic theorems to write log y as an expression of logarithms (i.e., take log y of each side of the equation and simplify).

   a) $y = \frac{(x-1)(x+3)^2}{\sqrt{x^2+2}}$

   b) $y = \sqrt{x^2(x+1)}$

continued

Learning Activity 7.3 (continued)

4. Express each of the following as a single logarithm. Simplify, if possible.
   a) $\log_2 5 + \log_2 7 + \log_2 6$
   b) $\log 2 + \log 5$
   c) $\frac{1}{2}\log_5 4 + \frac{1}{3}\log_5 27$
   d) $2\log_3 7 - \left(\log_3 14 + \log_3 35\right)$

5. Given:

   $\log_b 2 = A$
   $\log_b 3 = B$
   $\log_b 5 = C$

   Use the three equations given above to find expressions for the following in terms of *A*, *B*, and *C*.

   a) $\log_b 6$
   b) $\log_b 10$
   c) $\log_b \left(\frac{15}{2}\right)$
   d) $\log_b 8$
   e) $\log_b 25$
   f) $\log_b \sqrt{25}$
   g) $\log_b \sqrt{\frac{6}{5}}$

6. Given $\log_3 5 = D$, express $\log_3 45$ in terms of D.

Lesson Summary

In this lesson, you learned about three logarithmic theorems and how these theorems are related to the exponent laws. You then practiced using these theorems to expand and simplify logarithmic expressions in various circumstances. In the next lesson, you will be looking at the logarithmic function in more depth, including various graphs of logarithmic functions.

Notes

LESSON 4: THE LOGARITHMIC FUNCTION

Lesson Focus

In this lesson, you will

☐ learn how to sketch logarithmic functions and state their properties

☐ learn about the relationship between exponential and logarithmic functions

Lesson Introduction

In Lesson 2, you were introduced to logarithmic notation and the definition of a logarithm. In this lesson, you will be looking at logarithmic functions and their properties. Just as any other function can be transformed, you will use the concepts of transformations that you developed previously to help you graph various logarithmic functions.

The Logarithmic Function

Recall: $x = a^y$ is equivalent to $y = \log_a x$, where $a > 0$.

Sketching Logarithmic Functions

When you are asked to sketch a logarithmic function, you can sketch its inverse, the exponential function, and then reflect this inverse in the line $y = x$ to draw the desired curve. However, you may consider this strategy to be time consuming. You may find that instead you prefer to sketch the logarithmic curves directly by remembering the sketch of a basic log curve.

Switching the role of x and y from the exponential function, the key features of $y = \log_b x$ are:

- $y = \log_b x$ passes through the points $(1, 0)$ and $(b, 1)$
  - These key points exist due to the meaning of a logarithm: for any base b, $\log_b 1 = 0$, and $\log_b b = 1$.
- There will be a vertical asymptote at $x = 0$.

Example 1

Sketch each of the following logarithmic functions and state the domain, range, x-intercept, and equation of asymptote for each.

a) $y = \log_2 x$

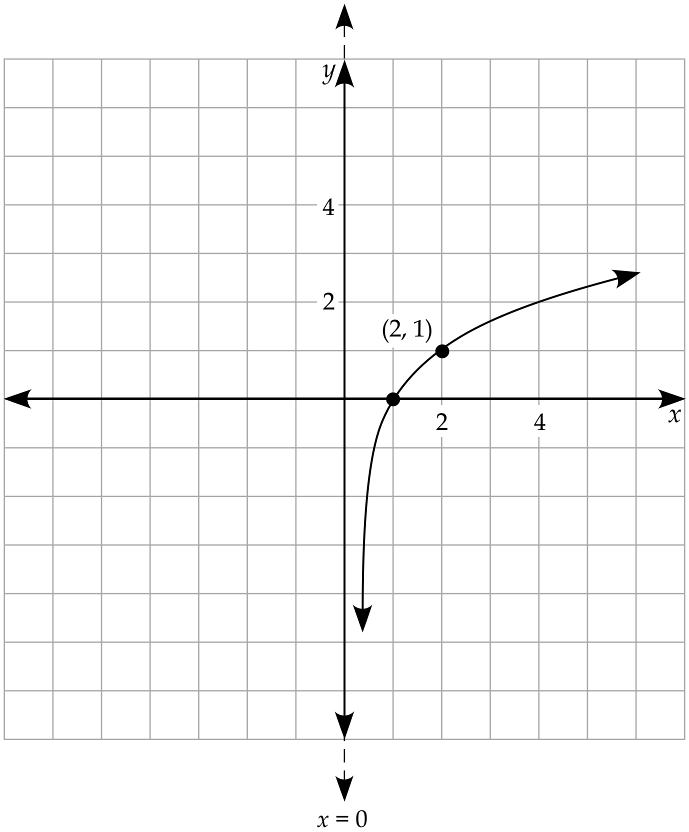
b) $y = \log x$

c) $y = \ln x$

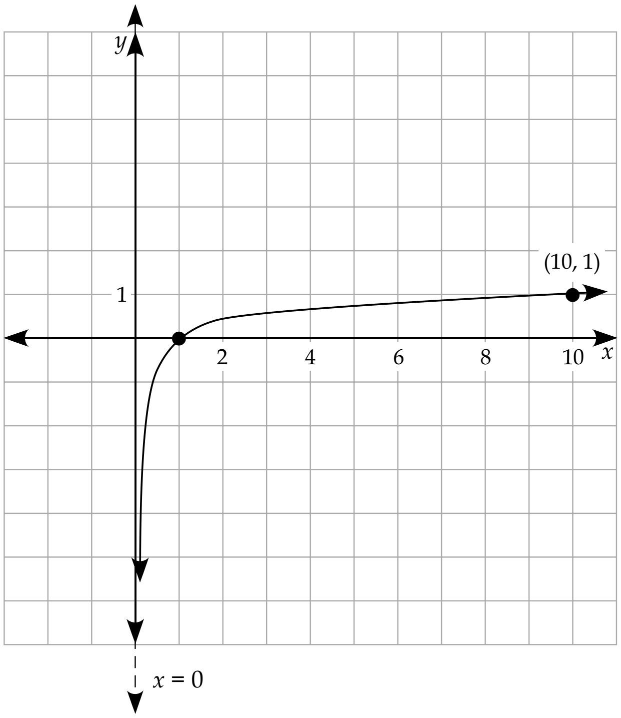
Solutions

These are exactly the same sketches as in Question 7 of Learning Activity 7.1, except these functions are written in the y-form (logarithmic form).

a) $y = \log_2 x$ is equivalent to $2^y = x$, which is the inverse of $2^x = y$. This function is also the same as the reflection of $y = 2^x$ in the line $y = x$.

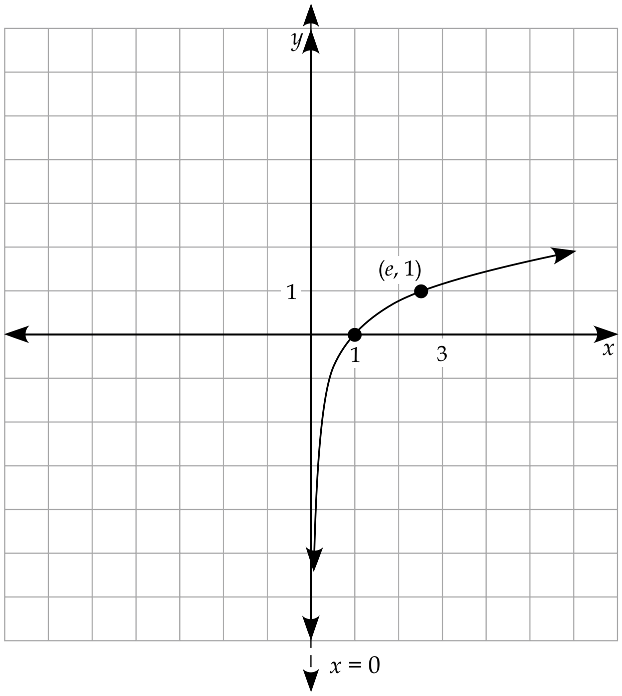


b) $y = \log x$ is equivalent to $10^y = x$, which is the inverse of $10^x = y$. This function is also the same as the reflection of $y = 10^x$ in the line $y = x$.



This function passes through the point (10, 1) as $\log 10 = 1$ because $10^1 = 10$.

c) $y = \ln x$ is equivalent to $e^y = x$, which is the inverse of $e^x = y$. This function is also the same as the reflection of $y = e^x$ in the line $y = x$. Remember, $e \approx 2.718$.



This function passes through the point $(e, 1)$ as $\ln e = 1$ because $e^1 = e$.

Question	Domain	Range	x-intercept	Equation of Asymptote
(a)	$(0, \infty)$	$\Re$	1	$x = 0$
(b)	$(0, \infty)$	$\Re$	1	$x = 0$
(c)	$(0, \infty)$	$\Re$	1	$x = 0$

As you listed in Question 8 of Learning Activity 7.1, the basic logarithmic curve has the following properties.

a) The domain is $(0, \infty)$.

b) The range is $\Re$.

c) The x-intercept is 1.

d) The y-axis is a vertical asymptote (equation of this asymptote is $x = 0$).

Recall: The functions $y = \log_{10} x$ and $y = \log x$ are equivalent; $y = \log x$ is the shorthand notation for writing $y = \log_{10} x$.

Also, the functions $y = \log_e x$ and $y = \ln x$ are equivalent. Traditionally, $\log x$ is called a **common logarithm** and $\ln x$ is called a **natural logarithm**.

	Written as
common logarithm	$y = \log_{10} x$ or $y = \log x$
natural logarithm	$y = \log_e x$ or $y = \ln x$

The term "ln" is sometimes pronounced *lawn*, *ell n*, or *log base e*. This function is the inverse function of $y = e^x$. As the function $y = e^x$ appears so frequently in nature, so does its inverse $y = \ln x$. The base e is the base that is used most often in calculus and its applications.

Example 2

Sketch each of the following transformed logarithmic functions and state the domain, range, and equation of asymptote for each.

Note: Transformations can be performed on logarithms just like any other function.

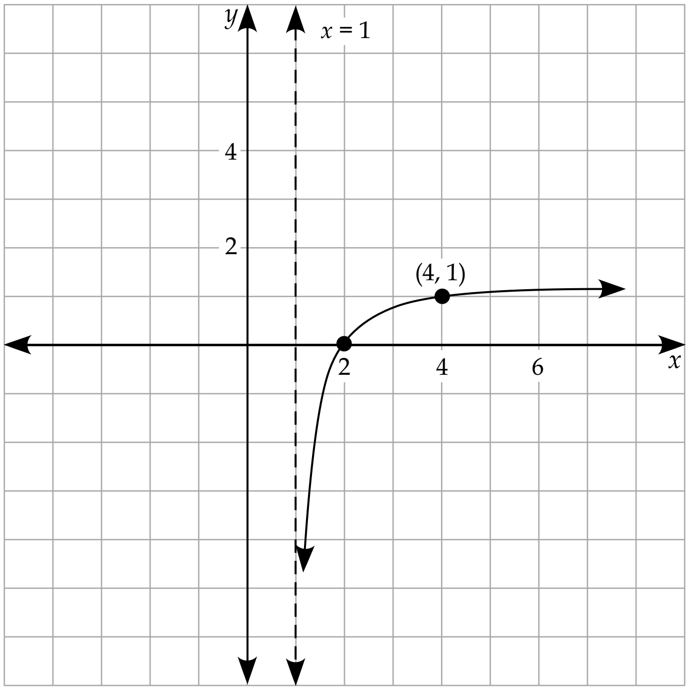
a) $y = \log_3 (x - 1)$

b) $y = \log x + 2$

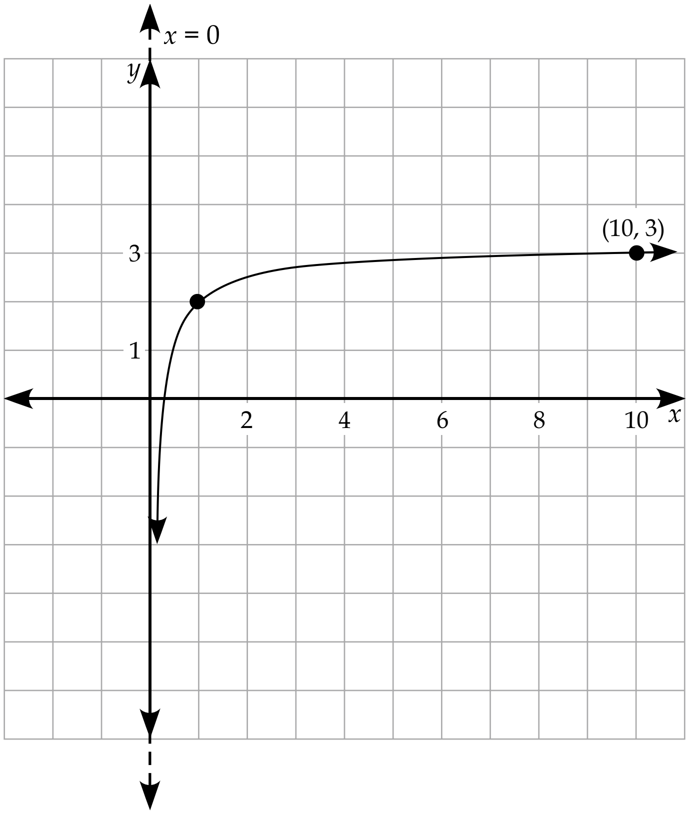
c) $y = -2 \ln x$

Solutions

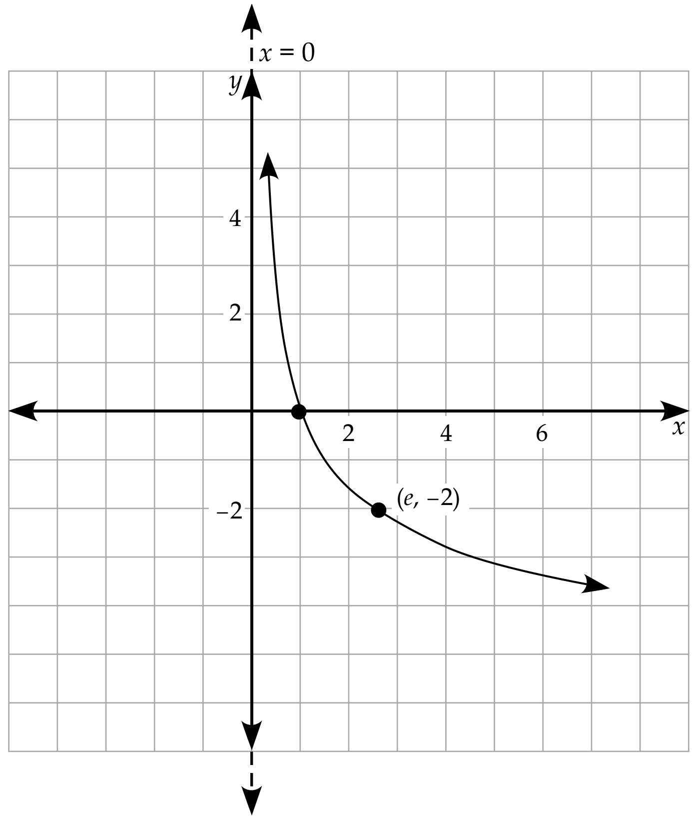
a) This is the graph of $y = \log_3 x$ shifted 1 unit to the right.



b) This is the graph of $y = \log x$ shifted up 2 units.



c) This is the graph of ln x stretched vertically by a factor of 2 and then reflected over the x-axis.



Question	Domain	Range	Equation of Asymptote
(a)	$(1, \infty)$	$\Re$	$x = 1$
(b)	$(0, \infty)$	$\Re$	$x = 0$
(c)	$(0, \infty)$	$\Re$	$x = 0$

Example 3

Graph the following functions. Explain how the different bases affect the graph of each logarithmic function.

a) $y = \log_4 x$

b) $y = \log_5 x$

c) $y = \log_6 x$

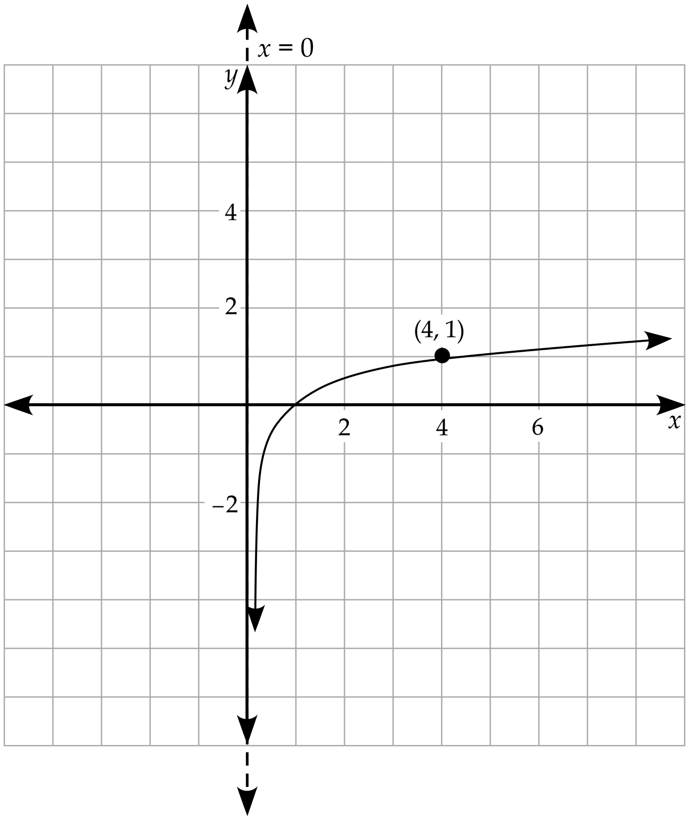
Solutions

Note: For your reference, the point (base, 1) is included on each of the graphs where the base is 4, 5, and 6 respectively.

a) To sketch the graph of $y = \log_4 x$, determine the two key points, (1, 0) and (b, 1), and the location of the vertical asymptote.

Two points on the graph of $y = \log_4 x$ are (1, 0) and (4, 1).

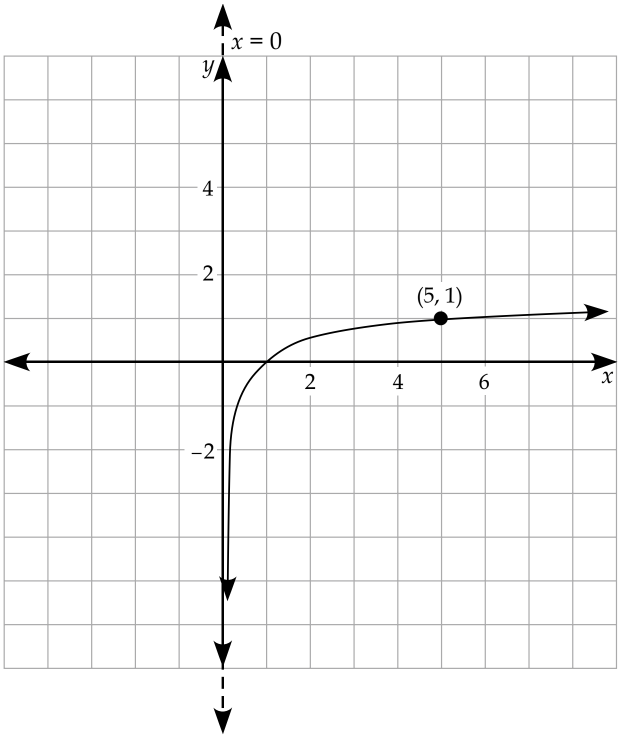
The vertical asymptote is located at $x = 0$.



b) To sketch the graph of $y = \log_5 x$, determine the two key points ((1, 0) and (b, 1)) and the location of the vertical asymptote.

Two points on the graph of $y = \log_5 x$ are (1, 0) and (5, 1).

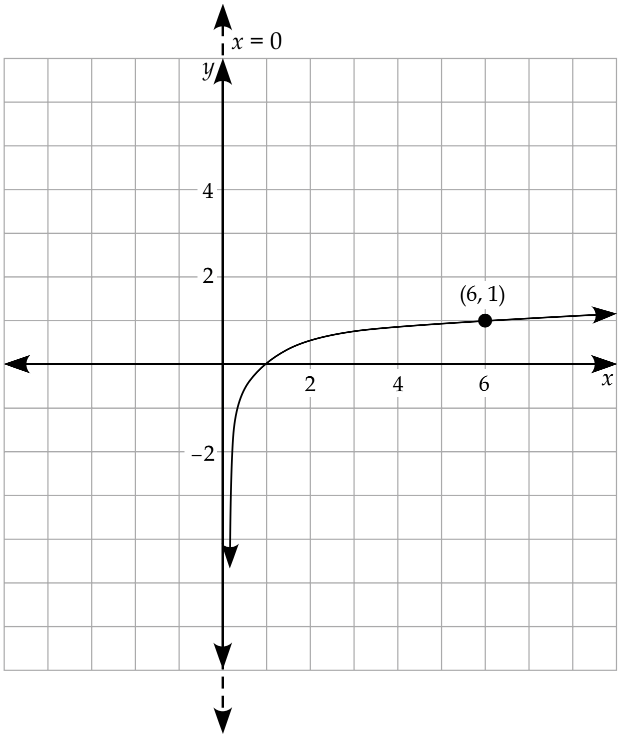
The vertical asymptote is located at $x = 0$.



c) To sketch the graph of $y = \log_6 x$, determine the two key points ((1, 0) and (b, 1)) and the location of the vertical asymptote.

Two points on the graph of $y = \log_6 x$ are (1, 0) and (6, 1).

The vertical asymptote is located at $x = 0$.



As the base of each logarithmic function gets larger, the function increases slowly. This is the *opposite* of what occurs in exponential functions. As the base of an exponential function gets larger, the function increases faster.

Therefore, the larger the base of a logarithmic function, the slower the function will increase. You can use this idea to help you sketch logarithmic functions with more ease.

Learning Activity 7.4

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Simplify: $\dfrac{\left(\sin^2 x + \cos^2 x\right) + 1}{\cos x}$
2. What is the exact value of $\sin\left(\dfrac{\pi}{6} + \dfrac{\pi}{3}\right)$?
3. Simplify: $\dfrac{(x+3)!}{(x+1)!}$
4. State an angle that is coterminal to 52°.
5. State the non-permissible values of the function $f(x) = \dfrac{x}{x^2 - 4}$.
6. Express $\log_3 8 - \log_3 2$ as one logarithm.
7. Simplify: $\log 10$
8. Factor: $6x^2 + 7x - 3$

continued

Part B: Graphing Logarithmic Functions

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Sketch each of the following logarithmic functions and state the domain, range, and equation of asymptote for each.
   a) $y = \log_8 x$
   b) $y = -\log x$
   c) $y = \left|\log x\right|$
   d) $y = \ln x - 2$
   e) $y = \log_2 (x + 2)$
   f) $y = \log (x + 1) - 2$
   g) $y = -\ln (x + 2) + 1$
   h) $y = \left|\log_3 x - 2\right|$

2. Sketch each of the following logarithmic functions and state the domain, range, and equation of asymptote for each.
   a) $f(x) = \log (-x)$
   b) $g(x) = \dfrac{1}{\log x}$

   Hint: Graph $f(x) = \log (x)$. Then use that graph to sketch the reciprocal, $y = \dfrac{1}{f(x)}$.

3. Sketch $f^{-1}(x)$ if $f(x) = \log_4 x$. State the domain, range, and equations of asymptotes of both $f(x)$ and $f^{-1}(x)$.

4. Sketch $f(x) = \log_3 (x - 2)$ and $g(x) = 3^x + 2$ on the same coordinate system. How are these two functions related?

5. Sketch $y = \log x$ and $10^y = x$ on the same coordinate system. How are these two graphs related?

6. Compare and contrast the graphs of $y = 2 \log_2 x$ and $y = \log_2 (x^2)$. Is there any consistency with or contradiction of the log Power Theorem? Explain your answer.

Note: By now your graphing skills and your knowledge of transformations of functions should be well honed. The ability to quickly recognize the shape and position of the graph of a function is important. Now might be the time to update your resource sheet with the basic functions and to review their possible transformations.

Lesson Summary

In this lesson, you learned how to graph logarithmic functions by first graphing the inverse exponential function and then reflecting that function through the line $y = x$. In order to graph logarithmic functions more easily, you learned the basic shape of a logarithmic function. You then learned how various transformations, including changing the base of the function, alter the graph of a logarithmic function.

In the next lesson, you will be learning how to solve exponential and logarithmic equations.

Notes

Assignment 7.2

Dealing with Logarithms

Total: 46 marks

You need to show all your steps as you work out your solutions, and make sure your answers are clear (include units, where appropriate). Check the Introduction for the Marking Guidelines that will be used for all assignments and examinations.

1. Write as a single logarithm. *(2 × 3 marks each = 6 marks)*

   a) $\log_3 A - 2\log_3 B + \frac{1}{2}\log_3 C$

   b) $5\log m - 4\log n - \frac{1}{2}\log p$

2. Express the following expression as a single logarithm. Simplify, if possible. *(2 marks)*

   $\log_6 4 + \log_6 9 + \log_6 6$

continued

Assignment 7.2: Dealing with Logarithms (continued)

3. Expand as a sum and/or difference of individual logarithmic expressions, and simplify if possible.

   a) $\log_2\left(\dfrac{\sqrt{x}\,(x-5)}{y^3}\right)$ *(3 marks)*

   b) $\log\left(\dfrac{\sqrt{A}}{B^2\sqrt[3]{C}}\right)$ *(3 marks)*

   c) $\log_3\left(\left(27^3\right)\sqrt[3]{\dfrac{x^2}{y}}\right)$ *(5 marks)*

continued

Assignment 7.2: Dealing with Logarithms (continued)

4. Given:

   $\log_b 3 = A$

   $\log_b 4 = B$

   $\log_b 7 = C$

   Use the three equations given above to write an expression for the following.

   a) $\log_b 12$ *(2 marks)*

   b) $\log_b \dfrac{21}{4}$ *(3 marks)*

continued

Assignment 7.2: Dealing with Logarithms (continued)

5. Sketch each of the following logarithmic functions and state the domain, range, and equation of asymptote for each. *(3 × 4 marks each = 12 marks)*
   a) $y = \log_5 x - 1$

   b) $y = -\log(x + 4)$

continued

Assignment 7.2: Dealing with Logarithms (continued)

c) $y = \log_2 (x - 3) + 4$

continued

Assignment 7.2: Dealing with Logarithms (continued)

6. Sketch $f(x) = 7^x$ and $f^{-1}(x)$ on the coordinate grids given below. State the domain, range, and equations of asymptotes of both $f(x)$ and $f^{-1}(x)$. *(5 marks)*

continued

Assignment 7.2: Dealing with Logarithms (continued)

7. Sketch $f(x) = \log_4 (x + 3)$ and $g(x) = 4^x - 3$ on the coordinate grids given below. Give two ways in which these functions are related. *(5 marks)*

Notes

LESSON 5: EXPONENTIAL EQUATIONS

Lesson Focus

In this lesson, you will

- ☐ learn how to find common and natural logarithms of numbers using a calculator
- ☐ learn how to evaluate logarithms of bases other than 10 and e
- ☐ learn how to solve exponential equations

Lesson Introduction

You have been briefly introduced to solving exponential equations throughout this module. Recall, exponential equations are equations that have variables in the exponent. Until now, you have solved equations having bases that are powers of a common base. In this lesson, you will use your knowledge of logarithms to help you to solve exponential equations having bases that are not powers of a common base.

Solving Exponential Equations

Before you learn how to solve exponential equations having bases that are *not* powers of a common base, it will be helpful to review how to solve exponential equations having bases that are powers of a common base.

Example 1

a) Solve for x.

$4^x = 8^{1-x}$

b) Solve for x.

$27(3^x) = 81^{2x+1}$

Solutions

a) $4^x = 8^{1-x}$

$\left(2^2\right)^x = \left(2^3\right)^{1-x}$ Express each power with the same base.

$2^{2x} = 2^{3-3x}$ Use the exponent rules to simplify.

$2x = 3 - 3x$ Equal bases → Equal exponents

$5x = 3$

$x = \frac{3}{5}$

b) $27\left(3^x\right) = 81^{2x+1}$

$3^3\left(3^x\right) = \left(3^4\right)^{2x+1}$

$3^{3+x} = 3^{8x+4}$

$3 + x = 8x + 4$

$-7x = 1$

$x = -\frac{1}{7}$

However, if the terms of the equation do not share a common base, the solution is not so simple! For example, to find a value for x for the equation $2^x = 7$, you can only approximate your answer. You know $2^2 = 4$ and $2^3 = 8$, so $2^x = 7$ means x is a little less than 3. Without a calculator, the approximation stops at this stage. In this lesson, you will learn to use your calculator to find much closer approximations.

Finding Logarithms Using the Calculator

In order to solve more complex exponential equations, you first need to learn how to find the exact values of logarithms. This involves the use of your calculator.

Solving equations that do not share a common base requires you to select an arbitrary base and find the exponents by using logarithms. For example, solve the following equation:

$2^x = 3$

Since your calculator has a button for base 10 logarithms, select base 10 and change the equation to equivalent values using base 10.

$$(10^{0.30103})^x = 10^{0.47712}$$

$$10^{0.30103x} = 10^{0.47712}$$

$$0.30103x = 0.47712$$

$$x = 1.585 \text{ (Correct to 3 digits after the decimal.)}$$

How do you find those exponents with a base of 10? Use your calculator! Use your calculator and find the following:

log 2 = 0.30103: This means that $10^{0.30103} \approx 2$.

log 3 = 0.47712: This means that $10^{0.47712} \approx 3$.

As stated throughout this module, the logarithm function gives you an exponent. Therefore, to find the exponent of a given base that will produce a certain value, you need to use the logarithm. Furthermore, since the number is not a rational power of the base, the logarithm is an approximation (although it is a very good approximation!).

Most calculators are limited to finding logarithms of either base 10 or base e since those two bases are the ones most commonly used.

Example 2

Use your calculator to find the following logarithms.

a) $\log 100$	e) $\ln 100$
b) $\log \frac{1}{10}$	f) $\ln e^3$
c) $\log 5$	g) $\ln 2$
d) $\log 2.73$	h) $\ln 1$

Solutions

The LOG button on your calculator calculates logarithms with a base of 10. Therefore, use the LOG button for calculating log x.

a) log 100

log 100 = 2 (using the calculator). You could also use the method of changing to exponential form:

Let $y = \log 100$, then:

$10^y = 100$

$10^y = 10^2$

$y = 2$ The same answer is obtained in both ways.

b) $\log \dfrac{1}{10}$

$-1\left(\text{since } 10^{-1} = 0.1 = \dfrac{1}{10}\right)$

c) $\log 5$

0.69897 (means $10^{0.69897} = 5$). You need to use your calculator for this logarithm.

d) $\log 2.73$

0.43515 (means $10^{0.43515} = 2.73$)

e) $\ln 100$

Use the LN button on your calculator to calculate a logarithm base e, or $\ln x$.

4.60517 (means $e^{4.60517} = 100$)

f) $\ln e^3$

This question can be answered easily without the calculator.

Recall: $\ln x = \log_e x$

Let $x = \log_e e^3$. Then:

$$e^x = e^3$$
$$\therefore\ x = 3$$

You could also write:

$$\begin{aligned}\ln e^3 &= 3 \ln e \\ &= 3(1) \\ &= 3\end{aligned}$$

g) $\ln 2$

0.69315 (means $e^{0.69315} = 2$)

h) $\ln 1$

0 (since $e^0 = 1$)

The calculator has the two special buttons LOG and LN to find the common and natural logarithms. However, what would you press to find $\log_2 5$? There is no $\log_2 x$ button! You can still find $\log_2 5$ by using your knowledge of exponents.

Example 3

Find each of the following logarithms.

a) $\log_2 10$

b) $\log_7 11$

c) $\log_3 (0.123)$

d) $\log_5 125$

Solutions

a) $\log_2 10$

Let $\log_2 10 = x$

$$
\begin{aligned}
2^x &= 10 && \text{Change to exponential form.}\\
\log 2^x &= \log 10 && \text{Take common log of both sides.}\\
x \log 2 &= \log 10 && \text{Power rule for logarithms.}\\
x &= \frac{\log 10}{\log 2} && \text{Divide both sides by log 2.}
\end{aligned}
$$

Now, you can use your calculator to find log 10 (which you could have found to be 1 without your calculator) and log 2.

$$x = \frac{1}{0.30103} = 3.32193$$

Why did you take the common logarithm of both sides? Because the LOG button is on the calculator, but so is the LN button. Try this with LN:

Let $\log_2 10 = x$

$$
\begin{aligned}
2^x &= 10 && \text{Change to exponential form.}\\
\ln 2^x &= \ln 10 && \text{Take natural log of both sides.}\\
x \ln 2 &= \ln 10 && \text{Power rule for logarithms.}\\
x &= \frac{\ln 10}{\ln 2} && \text{Divide both sides by ln 2.}\\
x &= \frac{2.30259}{0.69315} = 3.32193 && \text{Use LN button on your calculator.}
\end{aligned}
$$

Note: You can use one of the two logarithm buttons provided by the calculator. The base of the logarithms are different, but the ratios produce the same answer.

Note that $\dfrac{\log 10}{\log 2} \neq \log\left(\dfrac{10}{2}\right)$ or $\log 5$. There is no log theorem that deals with the quotient of two logs. Dividing the two logs is not the same as dividing two numbers and then finding the log of the result. Check your logarithm theorems to be sure you understand the difference.

b) $\log_7 11$

$$\log_7 11 = x$$

$$7^x = 11$$

$$\log 7^x = \log 11$$ Use either log or ln here.

$$x \log 7 = \log 11$$

$$x = \frac{\log 11}{\log 7} = 1.23227$$

c) $\log_3 (0.123)$

$$\log_3 (0.123) = x$$

$$3^x = 0.123$$

$$\ln 3^x = \ln (0.123)$$ Use either log or ln here.

$$x \ln 3 = \ln (0.123)$$

$$x = \frac{\ln (0.123)}{\ln 3} = 1.90747$$

d) $\log_5 125$

$$\log_5 125 = x$$

$$5^x = 125$$

$$5^x = 5^3$$

$$x = 3$$

Note: You don't need to use your calculator to solve part (d).

When finding logarithms, it is common practice to state the logarithm to four or five decimal places because small changes in an exponent can produce fairly large changes in the powers.

Change of Base Theorem

Change of Base Theorem

$$\log_b A = \frac{\log_n A}{\log_n b}$$

Note: The base has been changed from base b to base n. It is most convenient for n to be either 10 or e and then you can use the LOG or LN buttons on your calculator.

Proof

$$\text{Let } x = \log_b A$$

$$b^x = A$$

Exponential form.

$$\log_n b^x = \log_n A$$

Take the logarithm of base n (the new base) of each side of the equation.

$$x(\log_n b) = \log_n A$$

$$x = \frac{\log_n A}{\log_n b}$$

$$\log_b A = \frac{\log_n A}{\log_n b}$$

Substituting in for x.

You can use this Change of Base theorem to find the logarithm of any positive number to any positive base using logs base 10 or base e:

$$\log_b A = \frac{\log A}{\log b} \quad \text{or} \quad \log_b A = \frac{\ln A}{\ln B}$$

Solving Exponential Equations

You have just learned how to solve exponential equations that did not have a common base by taking the logarithm of both sides of the equation. This process can be simplified by using the Change of Base formula, which uses the logarithmic theorems.

To solve **exponential equations** in which the bases cannot be expressed as powers of a common base, take l*ogarithms of both sides* and then use the logarithmic theorems to get the variable out of the exponent. Be sure you do not try to simplify logarithms of a sum or a difference. The theorem you have for simplifying logarithms only involves products, quotients, and powers.

Example 4

Solve the following exponential equations. Round to two decimal places.

a) $5^x = 10$

b) $3^{x+1} = 17$

c) $2(3)^x = 5$

d) $e^x = 3^{x+2}$

Solutions

a)

$5^x = 10$	
$\log 5^x = \log 10$	Take log of both sides.
$x(\log 5) = \log 10$	Use Power Rule to move variable out of the exponent.
$x = \dfrac{\log 10}{\log 5}$	
$x = 1.43$	

b)

$3^{x+1} = 17$	
$\log 3^{x+1} = \log 17$	
$(x+1)\log 3 = \log 17$	Use Power Rule.
$x \log 3 + \log 3 = \log 17$	Distribute log 3 to clear brackets.
$x \log 3 = \log 17 - \log 3$	Isolate x-terms on one side.
$x = \dfrac{\log 17 - \log 3}{\log 3}$	Use brackets for the numerator when typing this expression into the calculator, so that the correct order of operations is preserved.
$x = 1.58$	

c) $2(3)^x = 5$

$\log 2(3)^x = \log 5$

$\log 2 + \log 3^x = \log 5$

Use the Product Rule here to separate factors.

$\log 2 + x \log 3 = \log 5$

Use the Power Rule to move variable out of the exponent.

$x \log 3 = \log 5 - \log 2$

$$x = \frac{\log 5 - \log 2}{\log 3}$$

$x = 0.83$

d) $e^x = 3^{x+2}$

$\ln e^x = \ln 3^{x+2}$

It is more convenient to use ln rather than log since a base of *e* is in the question.

$x \ln e = (x + 2) \ln 3$

$x(1) = x \ln 3 + 2 \ln 3$

$x - x \ln 3 = 2 \ln 3$

Isolate all x-terms on one side.

$x(1 - \ln 3) = 2 \ln 3$

Factor out the common factor of x.

$$x = \frac{2 \ln 3}{1 - \ln 3}$$

Use brackets for the denominator here to preserve order of operations.

$x = -22.28$

Learning Activity 7.5

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Determine the inverse function if $f(x) = 2x + 5$.
2. Simplify: $\cos\theta \sec\theta$
3. Convert 180° to radians.
4. State the non-permissible values of the function $f\left(x\right) = \dfrac{x}{x^2 - 3x - 4}$.
5. Express log 7 + log 8 as one logarithm.
6. Which is the better deal, three dress shirts for $65 or two dress shirts for $41?
7. What is the length of the remaining leg of a right-angled triangle if the hypotenuse measures 15 m and one leg measures 9 m?
8. Simplify: $\sqrt{72x^3y^7}$

continued

Learning Activity 7.5 (continued)

Part B: Solving Exponential Equations

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Solve the following exponential equations without using a calculator.

   a) $8\left(4^x\right) = 32$

   b) $3^{4x-1} = 27^{2x}$

   c) $25\left(125^{3-x}\right) = 5^{x+2}$

   d) $4^{x+1} = 2^x\sqrt{2}$

   e) $(-8)^{\frac{5}{3}} = 2\left(4^{\frac{x}{2}}\right)$

   f) $4^{6x} = \dfrac{1}{64}$

2. Find these logarithms without using a calculator.

   a) $\log_5 25$

   b) $\log_2 1024$

   c) $\log_3 \dfrac{1}{27}$

   d) $\log_7 49$

   e) $\log_4 \dfrac{1}{64}$

   f) $\log_2 8$

   g) $\log_3 \sqrt[3]{9}$

   h) $\log_{\sqrt{7}} 49$

3. Use your calculator to find each logarithm.

   a) $\log 15$

   b) $\log 1000$

   c) $\ln 7$

   d) $\ln 4$

   e) $\log (2.589)$

   f) $\log (-4)$

   g) $\ln (0.527)$

   h) $\ln 250$

   i) $\log \left(\dfrac{13}{15}\right)$

   j) $\log (0.5)$

   k) $\ln e$

   l) $\ln 8$

continued

Learning Activity 7.5 (continued)

4. Use your calculator to find each logarithm.
   - a) $\log^5 10$
   - b) $\log_3 20$
   - c) $\log_{100} 1000$
   - d) $\log_7 e$
   - e) $\log_{100} 1000$
   - f) $\log_9 81$
   - g) $\log_2 0.40$
   - h) $\log_3 15$
5. Solve the following exponential equations. State your answer to two decimal places.
   - a) $2(3^x) = 75$
   - b) $2(3^x) = 5^{x-1}$
   - c) $e^x = 2^{x+1}$
   - d) $19^{x-5} = 3^{x+2}$
   - e) $e^{2x-5} = 25$
   - f) $6^{3x} = 2^{2x-3}$
   - g) $5e^{x-1} = 6^x$
   - h) $(1.05)^x = 3$
   - i) $5(15)^x = 3^{x+1}$
6. Find the x-intercepts of each of the following exponential functions.
   - a) $y = 3^x - 4$
   - b) $y = 2^x - 1$
   - c) $y = 4^x + 2$
   - d) $y = -2^x + 3$

Lesson Summary

In this lesson, you learned how to solve exponential equations, including exponential equations where the bases were not powers of a common base. To solve these more complex exponential equations, you used logarithms and the logarithmic theorems you learned previously. In the next lesson, you will be solving logarithmic equations.

LESSON 6: LOGARITHMIC EQUATIONS

Lesson Focus

In this lesson, you will

☐ learn how to solve logarithmic equations

Lesson Introduction

In the previous lesson, you learned how to solve exponential equations by using your knowledge of logarithms. In this lesson, you will reverse this process and use your knowledge of exponential equations to help you solve logarithmic equations.

Solving Logarithmic Equations

A logarithmic equation is an equation with one or more terms that use a logarithm function and a variable argument. To solve **logarithmic equations**, you use the logarithmic theorems to transform all the terms having a logarithm to the same base into a single logarithm and then change the equation into exponential form. Since logarithms have domains restricted to positive numbers, be sure to check for extraneous roots.

There are two kinds of logarithmic equation problems.

1. If the terms on both sides of the equations are logarithmic terms, you can rewrite the equation, setting the arguments equal to each other. You will notice that the bases of the logs are the same.

   For example,

$$\begin{aligned} &\log_2 (x + 1) = \log_2 3 \\ &\therefore x + 1 = 3 \\ &\qquad\quad x = 2 \end{aligned}$$

2. If the terms on one side are logarithmic equations and the terms on the other side are numbers, then you can solve the equation by converting to exponential form.

   For example,

   $$\begin{aligned} \log_2 (x+1) &= 4 \\ \therefore x + 1 &= 2^4 \\ x + 1 &= 16 \\ x &= 15 \end{aligned}$$

Example 1

Solve the following logarithmic equations. Notice that the terms in each logarithmic equation use the same base. Check your solutions.

a) $\log_5 (3x + 1) + \log_5 (x - 3) = 3$

b) $\log_2 3x = \log_2 (2x + 7)$

c) $\log (x + 1) + \log (x - 1) = \log 8$

d) $\ln (x - 2) + \ln (2x - 3) = 2 \ln x$

e) $\log_2 (2 - x) + \log_2 (3 - x) = 1$

Solutions

a) $\log_5 (3x + 1) + \log_5 (x - 3) = 3$

Notice that there are two terms containing a logarithm to base 5. Thus, you can combine the two terms by using the Product Theorem.

$$\log_5 \left[(3x+1)(x-3)\right] = 3$$

Change to exponential form.

$$\begin{aligned} \log_5 \left[(3x+1)(x-3)\right] &= 3 \\ (3x+1)(x-3) &= 5^3 \\ 3x^2 - 8x - 3 &= 125 \\ 3x^2 - 8x - 128 &= 0 \end{aligned}$$

At this point, you can either factor or use the Quadratic Formula.

Factoring:

$$(3x+16)(x-8) = 0$$

$$x = -\frac{16}{3} \text{ or } x = 8$$

Quadratic Formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{8 \pm \sqrt{64 - 4(3)(-128)}}{2(3)}$$

$$x = \frac{8 \pm \sqrt{1600}}{6}$$

$$x = \frac{8 \pm 40}{6}$$

$$x = \frac{8 + 40}{6} \text{ or } x = \frac{8 - 40}{6}$$

$$x = 8 \text{ or } x = -\frac{16}{3}$$

When you check your answers after solving a logarithmic equation, you need to check if the arguments of the logarithms are positive numbers.

In this example, you need to determine if $3x + 1$ and $x - 3$ are positive for each possible value of x.

When $x = 8$, $3x + 1$ and $x - 3$ are both positive. However, when $x = -\frac{16}{3}$, both expressions are negative. Therefore, $-\frac{16}{3}$ must be discarded because you cannot take the logarithm of a negative number.

$\therefore$ The solution is $x = 8$.

b) $\log_2 3x = \log_2 (2x + 7)$

Since both sides are using the logarithm function of base 2, then the two arguments must be equal.

$$\log_2 (3x) = \log_2 (2x + 7)$$

$$\therefore 3x = 2x + 7$$

$$x = 7$$

Note: This is the same as taking the inverse logarithm of both sides.

Check:

$3(7) >^{?} 0$

Yes

$2(7) + 7 >^{?} 0$

Yes

$\therefore$ The solution to this equation is $x = 7$.

c) $\log(x + 1) + \log(x - 1) = \log 8$

There are two possible ways you could solve this equation.

Method 1:

$$\log(x+1) + \log(x-1) = \log 8$$

$$\log\left[(x+1)(x-1)\right] = \log 8$$ Use the Product Rule.

$$\therefore (x+1)(x-1) = 8$$ Equate the arguments.

$$x^2 - 1 = 8$$

$$x^2 - 9 = 0$$

$$(x+3)(x-3) = 0$$

$$x = -3 \text{ or } x = 3$$

$x = -3$ is an extraneous root as it will make the arguments negative.

$\therefore x = 3$ is the solution.

Method 2:

$$\log(x+1) + \log(x-1) = \log 8$$

$$\log(x+1) + \log(x-1) - \log 8 = 0$$

$$\log\frac{(x+1)(x-1)}{8} = 0$$

$$10^0 = \frac{(x+1)(x-1)}{8}$$

$$1(8) = x^2 - 1$$

$$0 = x^2 - 9$$

$$0 = (x-3)(x+3)$$

$$x + 3 = 0 \text{ or } x - 3 = 0$$

$$x = -3 \text{ or } x = 3$$

$x = -3$ is an extraneous root, as it will make the arguments negative.

$\therefore x = 3$ is the solution.

d) $\ln(x-2)+\ln(2x-3)=2\ln x$

$$\ln\left[(x-2)(2x-3)\right]=\ln\left(x^2\right)$$

$$\therefore (x-2)(2x-3)=x^2$$

$$2x^2-3x-4x+6-x^2=0$$

$$x^2-7x+6=0$$

$$(x-6)(x-1)=0$$

$$x=6 \text{ or } x=1$$

Check $x = 6$:

$x - 2 > 0$	$2x - 3 > 0$
$6 - 2 \overset{?}{>} 0$	$2(6) - 3 \overset{?}{>} 0$
Yes	Yes

Check $x = 1$:

$x - 2 > 0$	$2x - 3 > 0$
$1 - 2 \overset{?}{>} 0$	$2(1) - 3 \overset{?}{>} 0$
No	No

The only solution to the above equation is $x = 6$.

e) $\log_2(2-x)+\log_2(3-x)=1$

$$\log_2\left[(2-x)(3-x)\right]=1$$

$$2^1=(2-x)(3-x)$$

$$2=6-2x-3x+x^2$$

$$x^2-5x+4=0$$

$$(x-4)(x-1)=0$$

$$\therefore x=4 \text{ or } x=1$$

Check: $x = 4$:

$2 - x > 0$	$3 - x > 0$
$2 - 4 \overset{?}{>} 0$	$3 - 4 \overset{?}{>} 0$
No	No

Check: $x = 1$:

$2 - x > 0$	$3 - x > 0$
$2 - 1 >^{?} 0$	$3 - 1 >^{?} 0$
Yes	Yes

Therefore, the only solution to this equation is $x = 1$.

Make sure you complete the following learning activity, as it will allow you to practise solving logarithmic equations. You will need to be comfortable solving both exponential and logarithmic equations before moving on to the next lesson.

Learning Activity 7.6

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. What is the exact value of sin 180°?
2. Find the positive coterminal angle for 427° in the interval $0° \le \theta \le 360°$.
3. Express $\frac{1}{2}\log 4$ as one logarithm.
4. What is the area of a circular rug that has a diameter of 15 feet?
5. Rationalize the denominator: $\frac{2}{5\sqrt{3}}$
6. Simplify: $\frac{8x^{-3}y^{-2}}{2x^{2}y^{-3}}$
7. Simplify: $\sqrt{80} + \sqrt{45}$
8. Factor: $8a^2b^5c^3 + 2a^2bc^3$

continued

Learning Activity 7.6 (continued)

Part B: Solving Logarithmic Equations

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Solve the following logarithmic equations. Remember to check for extraneous roots!
   a) $\log_2 (x-4) + \log_2 (x-3) = 1$
   b) $\log_2 x + \log_2 (x-2) = \log_2 (9-2x)$
   c) $\log_4 (x+2) + \log_4 (2x-3) = 1$
   d) $2 \log_2 x - \log_2 (x-1) = 2$
   e) $\log_{\frac{1}{7}} x + \log_{\frac{1}{7}} (5x-28) = -2$
   f) $\log_2 (2-x) + \log_2 (-x) = \log_2 3$
   g) $\log_2 (x-1) + \log_2 (x+2) = 2$
   h) $\log_3 x = 3 + \log_3 (x+6)$

2. Solve for x. Round to 3 decimal places, if necessary.
   a) $e^{-0.01x} = 27$
   b) $e^{\ln(1-x)} = 2x$
   c) $\ln\left(e^{\sqrt{x+1}}\right) = 3$
   d) $e^{2x-1} = 5$

3. Solve for x. Round to 2 decimal places, if necessary.
   a) $\log_3 (x+2) - 1 = \log_3 x$
   b) $2^{x+3} = 7\left(3^{x-2}\right)$
   c) $\log (x+3) - \log x = 1$
   d) $4^{2x} = \left(2^x\right)^{x-2}$

continued

Learning Activity 7.6 (continued)

Questions 4 to 11 may prove to be tricky, but they are not difficult. Good luck!

4. Simplify. (Inverse Operations Rules)
   a) $e^{\ln x}$
   b) $\ln\left(e^{x}\right)$
   c) $\log\left(10^{x}\right)$
   d) $10^{\log x}$

5. Simplify.
   a) $3^{\log 3\ 4+\log 3\ 5}$
   b) $e^{\ln 3-\ln 2}$

6. Solve for x.
   a) $\log_2\left(\log_3 x\right) = 2$
   b) $\log_3\left(\log_2 (x-2)\right) = 1$

7. Solve for x.
   a) $5^{\sin x} = \frac{1}{5}$
   b) $\left(\ln x\right)^{-1} = \ln x$

8. Use the Change of Base formula to solve the following.
   a) $\log_2 x + \log_4 x = 3$
   b) $\log_3 (x+1) + \log_{27} (x+1) = 4$

9. Sketch the graph of $f(x) = \log(\sin x)$.

   Using a graphing calculator is helpful but not necessary. Think about where $\sin x > 0$, and then think about the values of the function $\log x$, if x takes on the values between 0 and 1, and then between 1 and 0.

10. Compare and contrast the graphs of $f(x) = e^{\ln x}$ and $g(x) = \ln(e^{x})$.

11. Solve for x: $\log x^2 + (\log x)^2 = 3$.

Lesson Summary

In this lesson, you learned how to solve logarithmic equations by using the logarithmic theorems and then converting the resulting logarithmic equation into exponential form. You will use this knowledge, as well as the knowledge of how to solve exponential equations, in the next lesson. In the next lesson, you will look at applications of exponential and logarithmic functions.

Notes

LESSON 7: EXPONENTIAL AND LOGARITHMIC APPLICATIONS

Lesson Focus

In this lesson, you will
- ☐ learn how to solve compound interest problems
- ☐ learn how to solve continuous compounding problems
- ☐ learn how to solve growth and decay problems
- ☐ learn how to solve problems involving logarithmic scales

Lesson Introduction

One of the most common applications of exponential functions is the calculation of compound interest. In Lesson 2, you saw a compound interest problem that led to the need for an inverse function to exponents, called the logarithm function. In this lesson, you will examine application problems, such as compound interest problems, in more depth. As you now know how to solve logarithmic and exponential equations, you will be able to solve these more complex equations.

Exponential and Logarithmic Applications

Exponential and logarithmic functions have many applications including compound interest problems (such as those involving loans, mortgages, and investments), exponential growth and decay (such as the growth of a population of a species or the breakdown of natural carbon compounds), and logarithmic scales (including the Richter scale that measures the energy contained in an earthquake). You will look at each of these three types of problems in more depth in this lesson.

Compound Interest

You were introduced to compound interest in Lesson 2. In this lesson, you will consider three different ways of compounding interest and the effect that different types of compounding have on the amount of interest you earn.

Interest Compounded Annually

Suppose you had invested $1000 in a financial institution that pays 10% compound interest per year. If you do not make any further deposits or withdrawals from this account, you can calculate the amount of money in this account after a given number of years. Compound interest means you receive interest on the new balance in each time period (i.e., interest on the interest). The table on the following page illustrates this situation.

Amount (P) at the Beginning of the Year	**Time in Years**	**Amount (A) at the End of the Year Principal + Interest ($P + I$)**
$1000	1	$1000 + I$ $= 1000 + 1000 \cdot 10\%$ $= 1000 + 1000 \cdot 0.10$ $= 1000\ (1 + 0.10)$ common factor $= 1000(1.10)$ $= 1100$
$1100	2	$1100 + I$ $= 1100 + 1100 \cdot 0.10$ $= 1100\ (1 + 0.10)$ $= 1100(1.10)$ $= 1220$ Or substitute 1000(1.10) for 1100 from Year 1. $= [1000(1.10)](1.10)$ $= 1000(1.10)(1.10)$ $= 1000(1.10)^2$ $= 1220$
$1210	3	$1210 + I$ $= 1210 + 1210 \cdot 0.10$ $= 1210\ (1 + 0.10)$ $= 1210(1.10)$ $= 1331$ Or substitute $1000(1.10)^2$ for 1210 from Year 2. $= [1000(1.10)^2](1.10)$ $= 1000(1.10)^3$ $= 1331$
$1331	4	$1331 + I$ $= 1331 + 1331 \cdot 0.10$ $= 1331\ (1 + 0.10)$ $= 1331(1.10)$ $= 1464.10$ Or substitute $1000(1.10)^3$ for 1331 from Year 3. $= [1000(1.10)^3](1.10)$ $= 1000(1.10)^4$ $= 1464.10$

A pattern has developed.

After five years the amount would be $1000(1 + 0.10)^5 = \$1610.51$.

After 10 years the amount would be $1000(1 + 0.10)^{10} = \$2593.74$.

After 25 years the amount would be $1000(1 + 0.10)^{25} = \$10,834.71$.

After t years the amount would be $1000(1 + 0.10)^t$.

The general compound interest formula is $A = P(1 + r)^t$.

where P is the principal, the original amount, in dollars
r is the rate for each compounding period, written as a decimal
t is the number of years
A is the final amount, in dollars

Include this formula on your resource sheet.

Example 1

Find the amount you must pay for a 10-year loan for $10,000 at 8%.

Solution

$A = P(1 + r)^t$

$A = 10\,000(1 + 0.08)^{10} = 21\,589.25$

You must pay $21,589.25.

(Use the x^y, y^x, or ^ button on your calculator to find $(1.08)^{10}$.)

Example 2

What amount of money must be invested today, at 6%, if you need $5000, seven years from today?

Solution

$$A = P(1 + r)^t$$

$$5000 = P(1 + 0.06)^7$$

$$P = \frac{5000}{(1.06)^7} = 3325.29$$

You must invest $3325.29 today.

Notice that the formula contains four variables—namely, A, P, r, and t. The first two examples solved for A and P. The remaining types require you to find r or t.

Example 3

At what rate must you invest your money to increase a \$1000 investment into \$2500 in eight years?

Solution

$$A = P(1+r)^t$$

$$2500 = 1000(1+r)^8$$

$$\frac{2500}{1000} = (1+r)^8$$

$$2.5 = (1+r)^8$$

$$\sqrt[8]{2.5} = 1 + r \qquad \text{Note: } \sqrt[8]{2.5} = 2.5^{\frac{1}{8}}$$

$$1.1214 = 1 + r$$

$$1.1214 - 1 = r$$

$$0.1214 = r$$

You would require a rate of approximately 12.14%, compounded annually.

Example 4

If you can invest money at 6%, how long will it take your investment to double itself?

Solution

For money to double itself, $\$x$ will become $\$2x$. Hence,

$$A = P(1+r)^t$$

$$2x = x(1+0.06)^t$$

$$\frac{2x}{x} = (1.06)^t$$

$$2 = (1.06)^t \qquad \text{To solve an exponential equation, use logarithms.}$$

$$\log 2 = t \log(1.06)$$

$$t = \frac{\log 2}{\log(1.06)}$$

$$t \approx 11.9$$

It would take about 11.9 years.

Interest Compounded More Than Once a Year

Often, interest is calculated more than once a year. When this happens, the number of calculations per year increases, and the interest earned per compounding period decreases.

When compounding more than once per year, the formula is modified to account for the number of compounding periods per year, n.

You may see the formula rewritten as

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where n is the number of payments per year and t is the number of years.

You may wish to add this formula to your resource sheet as well.

Example 5

Find the final amount on deposit, if \$5000 is invested for eight years at 12% compounded (a) semi-annually, (b) quarterly, and (c) monthly.

Solution

Note: Unless stated otherwise, the percent statement, 12%, is always the yearly rate. Therefore, to change it to the period rate, you must divide this yearly rate by the number of compounding periods in a year.

Change Period From Annually To:	Divided the Yearly Interest Rate By:	Multiply the Number of Compounding Periods Per Year By:
semi-annually	2	2
quarterly	4	4
monthly	12	12
weekly	52	52
daily	365	365

a) $A = P\left(1 + \frac{r}{n}\right)^{nt}$

$$A = 5000\left(1 + \frac{0.12}{2}\right)^{2(8)}$$

$$A = 5000(1 + 0.06)^{16}$$

$$A = \$12{,}701.76$$

b) $A = P\left(1 + \frac{r}{n}\right)^{nt}$

$$A = 5000\left(1 + \frac{0.12}{4}\right)^{4(8)}$$

$$A = 5000(1 + 0.03)^{32}$$

$$A = \$12{,}875.41$$

c) $A = P\left(1 + \frac{r}{n}\right)^{nt}$

$$A = 5000\left(1 + \frac{0.12}{12}\right)^{12(8)}$$

$$A = 5000(1 + 0.01)^{96}$$

$$A = \$12{,}996.36$$

Note: As the number of periods increases, the amount your deposit earns also increases. This is because your interest earned begins to earn interest at an earlier stage of the investment.

Continuous Compounding

Suppose you decide to deposit \$100 in five different accounts that each earn 12%. The only difference is how often they are compounded. The investments are compounded:

a) annually
b) monthly
c) weekly
d) daily
e) hourly

Of course, the more it is compounded, the more you earn. The question is, "How much difference does it make?"

The answer may surprise you!

Value of *A*	**Increase Over Previous Method**
a) $100(1+0.12)^1 = \$112$	Base amount
b) $100\left(1+\frac{0.12}{12}\right)^{1(2)}$ $= 100(1+0.01)^{12}$ $= \$112.68$	112.68 − 112.00 = 0.68
c) $100\left(1+\frac{0.12}{52}\right)^{52(1)}$ $= 100(1+0.0023077)^{52}$ $= \$112.73$	112.73 − 112.68 = 0.05
d) $100\left(1+\frac{0.12}{365}\right)^{365(1)}$ $= 100(1+0.0003288)^{365}$ $= \$112.75$	112.75 − 112.73 = 0.02
e) $100\left(1+\frac{0.12}{(365)(24)}\right)^{(365)(24)(1)}$ $= 100(1+0.000013699)^{8760}$ $= \$112.75$	112.75 − 112.75 = 0.00

As the number of compounding periods increases, the amount of your return also increases but by a decreasing amount. The question is, "Is there a ceiling?" In other words, is there a point at which it doesn't matter how many compounding periods there are, as the amount of your return will no longer increase? As suggested by the change from daily to hourly compounding, such a point, or ceiling, may exist.

Example 6

In the formula, $A = P\left(1+\frac{r}{n}\right)^{nt}$, where n is the number of payments per year,

an equivalent formula is $A = P\left(1+\frac{1}{{}^{n}\!/_{r}}\right)^{\frac{n}{r}\cdot rt}$. For a constant rate of increase,

the part affected by increasing the number of payments, n, is $\left(1+\frac{1}{{}^{n}\!/_{r}}\right)^{\frac{n}{r}}$.

You can choose any rate of interest (and for simplicity we will choose

$r = 0.1$). Now, evaluate $\left(1+\frac{1}{{}^{n}\!/_{0.1}}\right)^{\frac{n}{0.1}}$ for the following values of n representing

the number of payments per year.

$n = 10$
$n = 100$
$n = 1000$
$n = 10\,000$
$n = 100\,000$
$n = 1\,000\,000$

What do you notice?

Solution

The values are:

n	$\left(1+\frac{1}{{}^{n}\!/_{0.1}}\right)^{\frac{n}{0.1}}$
10	2.7048138
100	2.7169239
1000	2.7181459
10 000	2.7182682
100 000	2.7182805
1 000 000	2.7182817

You may recognize the number as the value of e. As n increases to infinity, this represents continuous growth (moment by moment). As n approaches infinity, the expression $\left(1+\frac{1}{n/r}\right)^{\frac{n}{r}}$ approaches the number e, the base of the natural logarithms, a number with which you are already familiar.

Thus, the formula for continuous compounding growth can be simplified to $A = Pe^{rt}$.

Going back to the original question, if you had \$100 invested at 12% for one year and compounding was happening continuously, the final amount was \$112.75. Now calculate the following expression:

$$A = 100e^{0.12}$$

Did you also get \$112.75?

This is the best you can get for \$100 invested at 12%. No matter how frequently your interest is calculated, this is the maximum amount of money your investment will be worth after one year.

The compound interest formula $A = P(1 + r)^n$ becomes $A = Pe^{rt}$ if the compounding is continuous. Continuous compound growth has applications to population growth where increases don't happen annually or semi-annually. You may wish to include this formula and a brief description of continuous compounding on your resource sheet.

Example 7

What is the value of a \$22 000 investment after five years if the rate is 10% compounded continuously?

Solution

Use the formula $A = Pe^{rt}$ (affectionately known as the "Pert" formula).

$$A = 22\ 000e^{0.10(5)} = 22\ 000e^{0.5} = 22\ 000(1.6487213) = \$36{,}217.87$$

Law of Natural Growth and Decay

The continuous compounding formula is used in situations of population or biological growth where compounding takes place "all the time." Under these circumstances, the same formula is often written using different variables such as:

$$y_t = y_0 e^{kt}$$

where y_0 is the original amount

y_t is the amount after time t

t is the number of time periods elapsed

k is the rate of growth for the time period

This formula is called the **Law of Natural Growth**. If a substance is decaying, the final amount, y_t, will be less than the original amount, y_0, and the law may be called the **Law of Natural Decay**. In this case, the rate of growth, k, will be negative.

Add this formula, as well as a brief description including why and how it is used, to your resource sheet.

Example 8

At the present time, there are 1000 Type A bacteria. If the rate of increase per hour is 0.025, how many bacteria can you expect in 24 hours?

Solution

Use the formula $y_t = y_0 e^{kt}$.

$$y_{24} = 1000e^{0.025(24)} = 1000e^{0.6} = 1000(1.8221188) = 1822 \text{ bacteria}$$

Example 9

A radioactive substance decays at a daily rate of 0.13. How long does it take for this substance to decompose to half its size?

Solution

If you represent the initial amount of this substance by $2x$, then the final amount of this substance must be x, or half of the original amount.

Method 1

As a decay question, change the rate to a negative.

$$y_t = y_0 e^{-kt}$$

$$x = (2x)e^{-0.13t}$$

$$\frac{x}{2x} = e^{-0.13t}$$

$$0.5 = e^{-0.13t}$$

$$\ln (0.5) = \ln \left(e^{-0.13t}\right)$$

$$\ln (0.5) = -0.13t(\ln e)$$

$$t = \frac{\ln (0.5)}{-0.13}$$

$$t = 5.332$$

Method 2

Positive rate;
interpret the answer

$$y_t = y_0 e^{kt}$$

$$x = 2xe^{0.13t}$$

$$0.5 = e^{0.13t}$$

$$\ln (0.5) = \ln\left(e^{0.13t}\right)$$

$$\ln (0.5) = 0.13t(\ln e)$$

$$t = \frac{\ln (0.5)}{0.13}$$

$$t = -5.332 \text{ (interpret the negative sign)}$$

∴ The substance decays to half its amount after 5.332 days.

Note: Make sure you answer the question with the same units given in the question. In this example, the substance decays at a *daily* rate; therefore, the amount of time it will take to decay will also be measured in *days*.

Whichever method is used, the conclusion is the same. The radioactive substance takes about five and one-third days to decompose to half its amount. This is called the **half-life of the substance**.

Definition: The **half-life** of a substance is the length of time required for the substance to decay to half its current amount.

You can use any method to solve natural decay problems. It is a good idea to become comfortable with one of these methods and stick with it.

Example 10

The beetle population multiplies so that the population t days from today is given by the formula $b_t = 2000e^{0.01t}$.

a) How many beetles are there today?

b) How many beetles will there be tomorrow? In one week?

c) How long will it take for the beetle population to double itself?

Solutions

a) Let $t = 0$ and solve for b_t.

$b_t = 2000e^{0.01t}$

$b_0 = 2000e^{0.01(0)}$

$= 2000e^0$

$= 2000(1)$

$= 2000$ beetles

Notice that this is the value of y_0, as expected!

b) Since t is measured in days, let $t = 1$.

$b_1 = 2000e^{0.01(1)}$

$= 2000e^{0.01}$

$= 2020$ beetles

In one week, $t = 7$.

$b_7 = 2000e^{0.01(7)}$

$= 2000e^{0.07}$

$= 2145$ beetles

c) From part (a) or from the equation, you know the original number of beetles is $b_0 = 2000$. The doubled population will be 4000 beetles.

$$4000 = 2000e^{0.01t}$$

$$\frac{4000}{2000} = e^{0.01t}$$

$$2 = e^{0.01t}$$

$$\ln 2 = 0.01t(\ln e)$$

$$\ln 2 = 0.01t(1)$$

$$t = \frac{\ln 2}{0.01}$$

$$t = 69.3$$

It will take 69.3 days for the beetle population to double itself.

These exponential functions have four variables, y_t, y_0, k, and t. The formula is often given and if three of these variables are known, your task is to find the fourth variable. This process constitutes the majority of the problems you are expected to solve involving growth or decay.

Learning Activity 7.7

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. What is the exact value of $\sin 30^\circ$?
2. In which quadrant is $\sin \theta$ positive and $\cos \theta$ negative?
3. Estimate the taxes, 13%, on a $327 item.
4. Evaluate: $\sqrt[4]{625}$
5. Simplify: $|-20(4 - 5)|$
6. Solve for x: $(x - 2)^2 = 9$
7. Simplify: $\frac{3}{5} + \frac{3}{7}$
8. Factor: $64x^4 - 81y^8$

Part B: Exponential Growth Lab

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

In this learning activity you will be completing an Exponential Growth Lab. To complete this lab, you will need a pencil, and approximately 100 coins.

Modelling Exponential Growth

In this lab, you are going to model the spread of bacteria. You will conduct 10 trials and record how many new bacteria appear after each trial.

1. Place 2 coins in a cup (representing 2 bacteria initially).
2. Shake the cup and dump the coins out of the cup onto a flat surface.

continued

Learning Activity 7.7 (continued)

3. For each coin that lands with the “heads” side facing up, add another coin, representing some of the bacteria replicating.
4. Repeat steps 2 and 3 until you have completed 10 trials.
5. Fill in the *Difference in Amount of Coins* column by subtracting the amount of coins of the previous trial from the current trial. For example, in trial 2, the difference will be:

   # of Coins in Trial 2 − # of Coins in Trial 1

6. Fill in the *Percent Change* column by using the formula:

   $$\frac{\text{new amount of coins} - \text{old amount of coins}}{\text{old amount of coins}}$$

   Note: This value should be close to $\frac{1}{2}$. When you drop a handful of coins, theoretically, half of them should land on heads and half should land on tails. Therefore, the percent change for each trial should be close to $\frac{1}{2}$ theoretically. However, your results will vary. Convert the fraction to a decimal.

7. Calculate the average percent change by totalling the percent change column and then dividing by 9 (the number of percent change values).

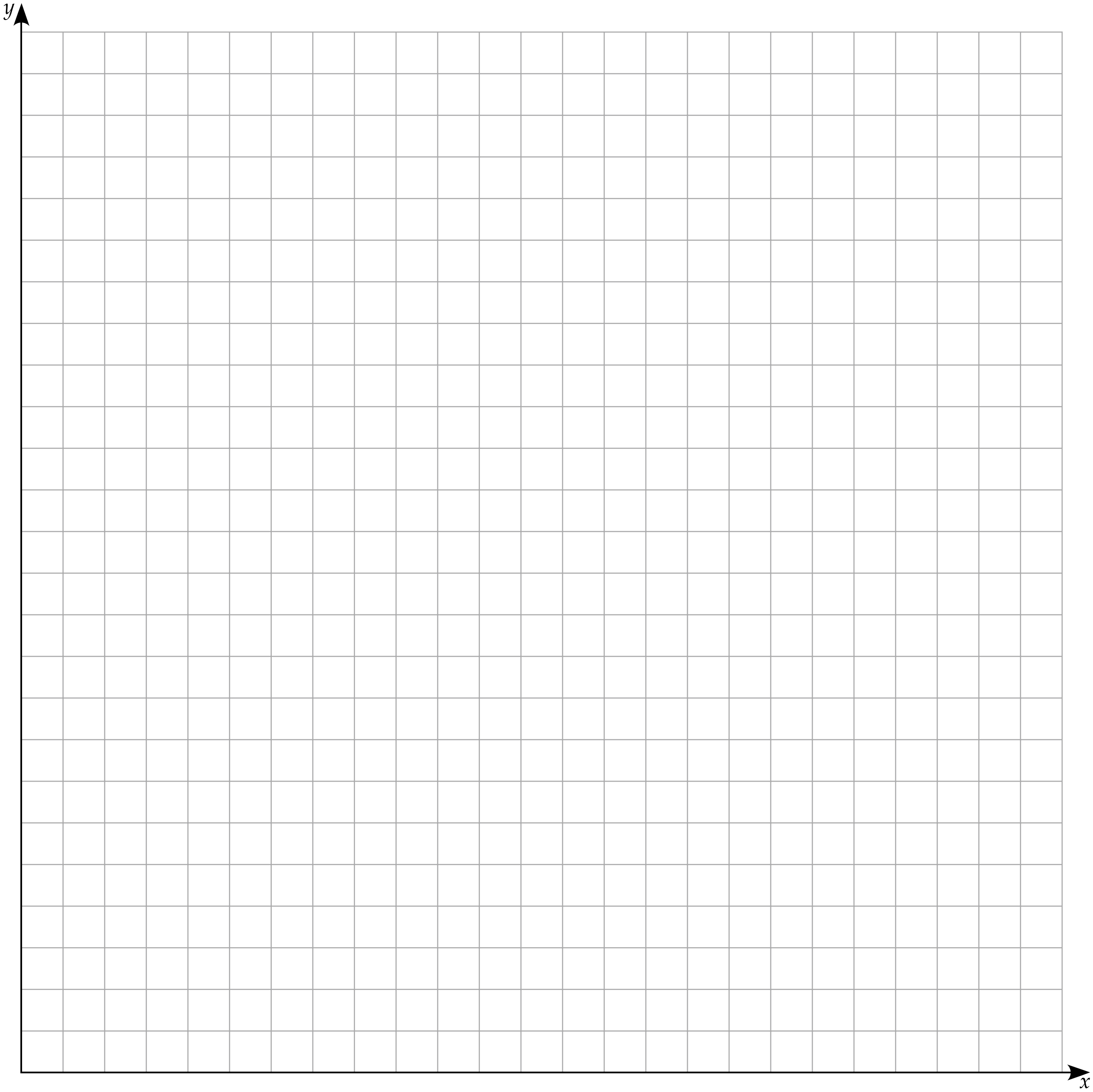
Note: Your first trial begins with you having 2 coins.

Trial Number	# of Coins	Difference in Amount of Coins	Percent Change
0	2		
1			
2			
3			
4			
5			
6			
7			
8			
9			
		Average Percent Change	

continued

Learning Activity 7.7 (continued)

8. Graph your data below. Label the y-axis with the title *Number of Coins (Bacteria)*.



Trial Number

continued

Learning Activity 7.7 (continued)

9. Write an exponential growth function in the form $y = P(1 + r)^t$ that models your data.

   P = The initial amount of coins (bacteria)

   r = The rate of growth (use the average percent change)

   t = Time (**Note:** Time is the independent variable in this equation represented by the trial number.)

   The exponential growth function that models my data is ____________ .

10. Use your exponential growth function model to predict how many bacteria there will be after

    a) 20 trials

    b) 50 trials

11. If you started this trial with more coins, would you expect the pattern to stay the same? Why or why not?

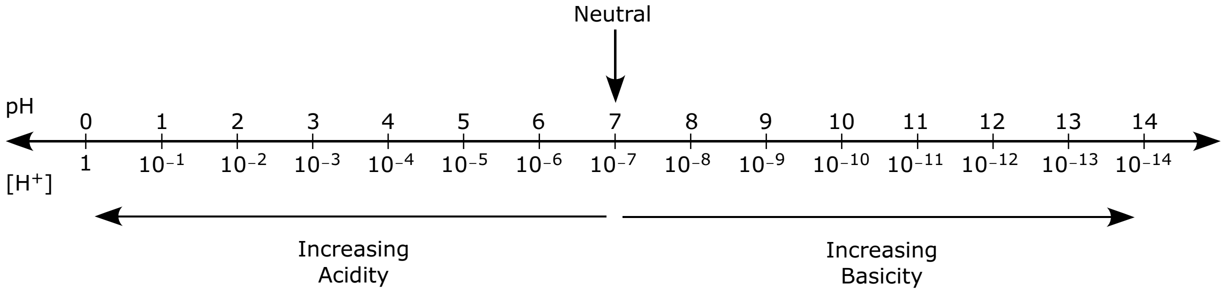
Logarithmic Scales

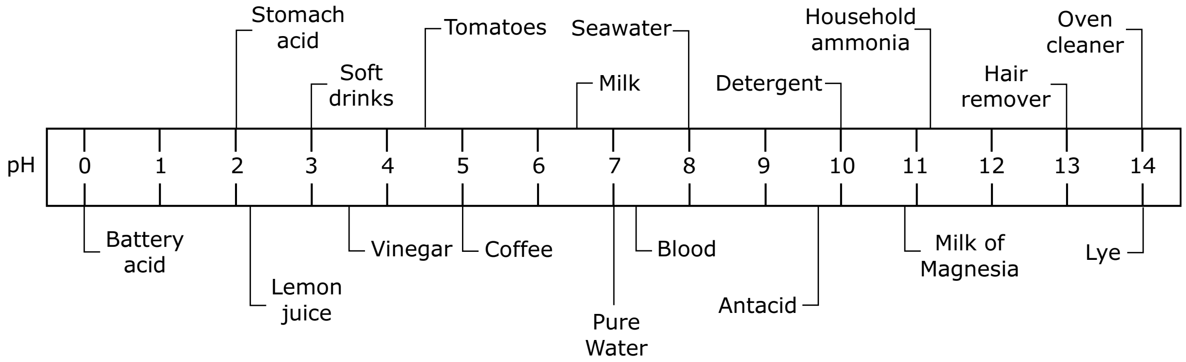
Logarithmic scales are used to present data that represents the difference in magnitude for extreme values, such as between 1 and 1 billion. However, they can be difficult to read if you don't understand the difference between the levels on the scale. Larger by one order of magnitude is ten times larger.

The Richter scale is a base ten logarithmic scale. The Richter scale is used to measure the energy contained within an earthquake. If an earthquake measures in at 3 on the Richter scale (this would be an earthquake that you can feel, but that does not cause any damage), this earthquake would contain 10 times as much energy as an earthquake that measures in at 2 on the Richter scale. If an earthquake measures in at 4 on the Richter scale (this would be an earthquake that rattles the dishes in your cupboard), this earthquake would contain 10 times 10, or 100 times as much energy as an earthquake that measures in at 2 on the Richter scale.

As this is a logarithmic scale of base ten, each energy level is ten times greater than the previous energy level. Since it is a logarithmic relationship, the Richter scale number represents an exponent on a base of 10. Therefore, every increase of 1 in the exponent increases the energy by a factor of 10.

Another base ten logarithmic scale is the pH scale used to measure the acidity of various liquids. Consider the two diagrams below.





The pH scale measures the concentration of hydrogen ions ([H+]) using a logarithmic scale. As you can see, each scale value is 10 times smaller, and 10 times more basic, than the previous scale value. For example, pH 5 is 10 times more basic than pH 4, and 100 times more basic than pH 3.

Example 11

Use your knowledge of logarithms to answer the following questions.

a) How many times more energy is contained within an earthquake that is rated a 7 on the Richter scale than an earthquake that is rated a 4 on the Richter scale?

b) How many times less energy is contained within an earthquake that is rated a 3 on the Richter scale than an earthquake that is rated a 7 on the Richter scale?

c) If coffee has a pH level of 5, how many times more basic is detergent that has a pH level of 10?

d) If soft drinks have a pH level of 3, how many times more acidic is battery acid that has a pH level of 0?

Solutions

a) An earthquake with a rating of 7 on the Richter scale and an earthquake with a rating of 4 on the Richter scale are 3 points away. Therefore, the earthquake with a rating of 7 contains 10^3 or $10 \cdot 10 \cdot 10$ or 1000 times more energy than an earthquake with a rating of 4.

b) An earthquake with a rating of 3 on the Richter scale and an earthquake with a rating of 7 on the Richter scale are 4 points away. Therefore, the earthquake with a rating of 3 contains 10^4 or or 10 000 times less energy than an earthquake with a rating of 7.

c) A pH level of 5 and a pH level of 10 are 5 points away on the pH scale. Therefore, detergent is 10^5 or 100 000 times more basic than coffee.

d) A pH level of 3 and a pH level of 0 are 3 points away on the pH scale. Therefore, battery acid is 10^3 or 1000 times more acidic than a soft drink.

Example 12

The pH of a substance is defined by the equation pH = $-\log\,[\mathrm{H}^+]$, where $[\mathrm{H}^+]$ is the hydrogen ion concentration measured in moles/L. If the pH of milk is 6.5, find its hydrogen ion concentration.

Solution

$\mathrm{pH} = -\log\left[\mathrm{H}^+\right]$	
$6.5 = -\log\left[\mathrm{H}^+\right]$	Substitute in the pH value of milk.
$-6.5 = \log\left[\mathrm{H}^+\right]$	Divide each side of the equation by negative one.
$10^{-6.5} = \left[\mathrm{H}^+\right]$	Change to exponential form (the base of the log is 10).
$\left[\mathrm{H}^+\right] = 3.2 \times 10^{-7}$ moles per litre	Evaluate and use the correct units of moles/L.

The following learning activity will provide you with lots of practice dealing with applications of exponential and logarithmic equations. Make sure you complete this Learning Activity before you complete the final Assignment for this module.

Learning Activity 7.8

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Convert 45° to radians.
2. The square root of 200 is between which two whole numbers?
3. Write as an entire radical: $3x^2\sqrt{x}$
4. Evaluate: $50 - 1 \cdot 6$
5. Solve for x: $5 + 3x = 41$
6. Which two numbers have a product of -30 and a sum of 13?
7. Evaluate: $\frac{3}{8} + \frac{1}{3} - \frac{7}{24}$
8. Simplify: $\dfrac{\frac{5}{x}}{7}$

Part B: Exponential and Logarithmic Applications

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. A \$5000 investment earns interest at the rate of 8.4% compounded monthly.
   a) What is the investment worth after one year?
   b) What is the investment worth after 10 years?
   c) How much interest was earned in 10 years?

continued

Learning Activity 7.8 (continued)

2. What sum of money must be invested now so that $20,000 is available in five years, if the rate is 8.4% compounded monthly?
3. What interest rate must be charged so that money doubles itself in 10 years if the rate is compounded
   a) annually?
   b) semi-annually?
   c) monthly?
   d) weekly?
4. With which plan would an investor earn more?
   Plan A: 8% compounded annually.
   Plan B: 7.5% compounded daily.
5. The population of gophers in a particular field is modelled by the equation $P = 100(1.1)^t$, where t is measured in years.
   a) How many gophers will there be after 20 years?
   b) How long will it take for the gopher population to triple?
6. A radioactive substance is decaying according to the formula $y = y_0e^{kx}$, where x is measured in years. The initial amount is 10 grams, and eight grams remain after five years.
   a) Find the value of k. Round to 5 decimal places.
   b) Estimate the amount remaining after 10 years.
   c) Find the half-life to the nearest tenth of a year.
7. When the population growth of a city was first measured, the population was 22 000. It was found that the population, P, grew by the formula $P = 22\,000(10^{0.0163t})$. If t is measured in years, how long will it take for the city to double its population?
8. The pH of a substance is defined by $\text{pH} = -\log\,[\text{H}^+]$, where $[\text{H}^+]$ is the hydrogen ion concentration in moles/L. If the pH of lemon juice is 2.3, find its ion concentration.

continued

Learning Activity 7.8 (continued)

9. Use your knowledge of logarithms to answer the following questions.
   a) How many times more energy is contained within an earthquake that is rated an 8 on the Richter scale than an earthquake that is rated a 3 on the Richter scale?
   b) If detergent has a pH level of 10, how many times more acidic is stomach acid that has a pH level of 2?

10. If the lights are left on when a car is parked, the battery discharges and the voltage, V, of the battery is given by $V = V_0e^{-kt}$, where t is the time in minutes. If the original voltage, V_9, was 12 volts and $k = 0.01$, find how long it takes to reduce the voltage to nine volts.

11. The atmospheric pressure, P, at height h kilometres above sea level is given by $P = P_0e^{-kh}$. The pressure at sea level, P_0, is 101.3 kPa. (**Note:** kPa is the abbreviation for kiloPascals, a unit of measure for pressure, and does not refer to the k in the formula.)
    a) Find the value of k if $P = 89$ kPa when $h = 1$ km. Round to 5 decimal places.
    b) Use your answer in part (a) to find the pressure at a height of 2 km.

12. The temperature in a room is 20°C. If a container of boiling water at 100°C is brought into this room, the water in the container cools according to the formula $T = 20 + 80e^{-0.03t}$, where T is the temperature of the water t minutes after being placed in this room. How long will it take for the water to cool to 40°C in this room?

13. The number of bacteria in a certain culture, t hours from now, grows according to the formula $A = 800(3)^t$.
    a) What will be the bacteria count 3.12 hours from now?
    b) When will the bacteria count reach 100 000?

14. For a period of its life, a tree grows according to the formula $D = D_0e^{kt}$, where D is the diameter of the tree, in centimetres, t years after the beginning of the period. After two years, the diameter of the tree is 12.724 cm. After five years, the diameter is 15.62 cm. Find the value of D_0 and the value of k.

15. James attempts to increase the value of his investment by increasing the number of compounding periods. If the best rate James can get is 6%, is it possible for James to obtain enough compounding periods to achieve a rate equivalent to 7%? Explain.

continued

Learning Activity 7.8 (continued)

16. The population of a small northern Manitoba town increased by 15% two years ago, and then decreased by 15% last year. The current population of the town is 5000 people. What was the population of the town before the 15% increase two years ago? **Hint:** You can answer this question using logical reasoning without the use of any population formula.

Lesson Summary

In this lesson, you used everything you have learned about exponential and logarithmic equations to solve exponential and logarithmic application problems. These problems ranged from those dealing with finances to determining radioactive decay and population growth. This is the last lesson in this module.

Assignment 7.3

Solving Exponential and Logarithmic Equations

Total: 37 marks

You need to show all your steps as you work out your solutions, and make sure your answers are clear (include units, where appropriate). Check the Introduction for the Marking Guidelines that will be used for all assignments and examinations.

1. Solve the following exponential equations without using a calculator.
   (3 × 1 mark each = 3 marks)

   a) $2^{\frac{x}{3}} = 32$

   b) $2^{3x+5} = \frac{1}{16}$

   c) $4^{2x-6} = 1$

continued

Assignment 7.3: Solving Exponential and Logarithmic Equations (continued)

2. Solve the following exponential equations. State your answer to two decimal places. *(3 × 4 marks each = 12 marks)*

   a) $2\left(5^x\right) = 751$

   b) $3e^{2x-5} = 25$

continued

Assignment 7.3: Solving Exponential and Logarithmic Equations (continued)

c) $4\left(3^x\right) = 91$

3. Find these logarithms by using a calculator. State your answer to four decimal places. *(3 × 1 mark each = 3 marks)*

a) $\log 6$

b) $\ln 3$

c) $\log(-0.123)$

continued

Assignment 7.3: Solving Exponential and Logarithmic Equations (continued)

4. Solve the following logarithmic equations. Remember to check for extraneous roots!
   (3 × 3 marks each = 9 marks)

   a) $\log_3(x) + \log_3(x - 2) = 1$

   b) $\log_2(x - 4) - \log_2(x - 3) = 1$

continued

Assignment 7.3: Solving Exponential and Logarithmic Equations (continued)

c) $\log_6 \left(x^2 - 16\right) - \log_6 x = \log_6 6$

continued

Assignment 7.3: Solving Exponential and Logarithmic Equations (continued)

5. A radioactive substance is decaying according to the formula $y = Ae^{-0.2t}$, where y is the amount of material remaining after t years.

   a) If the initial amount is $A = 80$ grams, how much remains after three years? *(1 mark)*

   b) Find the half-life of this substance. *(3 marks)*

continued

Assignment 7.3: Solving Exponential and Logarithmic Equations (continued)

6. Julie invests $100 at 10%.
   a) How much is the investment worth after one year? *(1 mark)*

   b) How long will it take for the money to double itself? *(3 marks)*

continued

Assignment 7.3: Solving Exponential and Logarithmic Equations (continued)

7. Use your knowledge of logarithms to answer the following questions.
   (2 × 1 mark each = 2 marks)

   a) How many times more energy is contained within an earthquake that is rated a 7 on the Richter scale than an earthquake that is rated a 1 on the Richter scale?

   b) If a certain brand of dish soap has a pH level of 8, how many times more acidic is lime juice that has a pH level of 3.5?

MODULE 7 SUMMARY

In this module, you learned about exponential and logarithmic functions. First, you built on your knowledge of exponents to graph exponential functions. You then learned about the inverse functions of exponential functions called logarithmic functions. You were able to graph these functions using two methods—the method of first graphing the inverse exponential function and then reflecting that function through the line $y = x$, and the method of using transformations of the standard logarithmic function.

Just as there are rules for simplifying exponents, there are also rules for simplifying logarithmic expressions. You learned about the logarithmic theorems that helped you to solve both logarithmic and exponential equations. Once you were able to solve logarithmic and exponential equations, you could solve many application problems. As you saw throughout Lesson 7, exponential and logarithmic functions have many applications in nature and financial situations.

In the next module, you are going to be studying rational and radical functions.

Submitting Your Assignments

It is now time for you to submit Assignments 7.1 to 7.3 to the Distance Learning Unit so that you can receive some feedback on how you are doing in this course. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 7 assignments and organize your material in the following order:

☐ Module 7 Cover Sheet (found at the end of the course Introduction)
☐ Assignment 7.1: Exponential Functions and Logarithms
☐ Assignment 7.2: Dealing with Logarithms
☐ Assignment 7.3: Solving Exponential and Logarithmic Equations

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Notes

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Module 7
Exponents and Logarithms

Learning Activity Answer Keys

MODULE 7:
EXPONENTS AND LOGARITHMS

Learning Activity 7.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. State the non-permissible values of the function $f(x) = \dfrac{x^2 + 4x + 3}{x^2 + 5x + 6}$.
2. Express cot θ in terms of csc θ and sec θ.
3. Evaluate: ${}_5C_1$
4. Convert $\dfrac{\pi}{2}$ to degrees.
5. If $f(x) = \dfrac{x+2}{x^2 - 4}$, evaluate $f(x)$ at $x = 1$.
6. Determine the vertex of the function $f(x) = -3(x - 4)^2 + 4$.
7. Multiply: $\left(3\sqrt{4} + \sqrt{2}\right)\left(3\sqrt{4} - \sqrt{2}\right)$
8. Write an expression that represents 11 less than the reciprocal of x.

Answers:

1. $x \neq -2, -3 \quad \left(\dfrac{x^2 + 4x + 3}{x^2 + 5x + 6} = \dfrac{(x+1)(x+3)}{(x+2)(x+3)}\right)$
2. $\dfrac{\csc\theta}{\sec\theta} \quad \left(\cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{\dfrac{1}{\sec\theta}}{\dfrac{1}{\csc\theta}} = \dfrac{\csc\theta}{\sec\theta}\right)$
3. 5
4. 90°
5. $f(1) = -1 \quad \left(f(1) = \dfrac{1+2}{1^2 - 4} = \dfrac{3}{1-4} = \dfrac{3}{-3} = -1\right)$

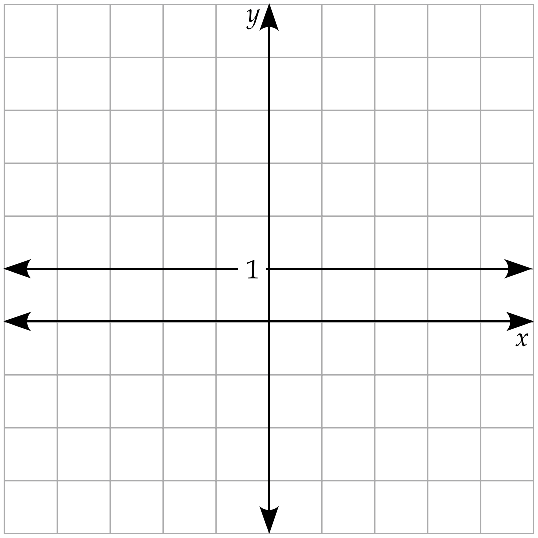
6. (4, 4)
7. 34 ($9 \cdot 4 - 2 = 34$)
8. $\frac{1}{x} - 11$

Part B: Graphing Exponential Functions

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Sketch the curve of $y = 1^x$.

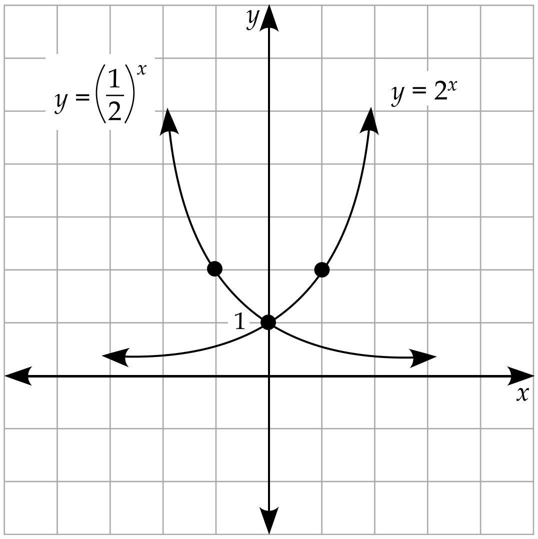
   Answer:



2. Sketch the graph of $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$ on the same coordinate system.

   How are the graphs related? Write $y = \left(\frac{1}{2}\right)^x$ in a different form.

   Answer:



   $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$ are horizontal reflections of each other.

   $$y = \left(\frac{1}{2}\right)^x = \frac{1^x}{2^x} = \frac{1}{2^x} = 2^{-x}$$

   So $y = 2^{-x}$ and $y = \left(\frac{1}{2}\right)^x$ are the same function written in different forms.

3. Write the function representing the reflection of $y = 5^x$, in the y-axis, in two different ways.

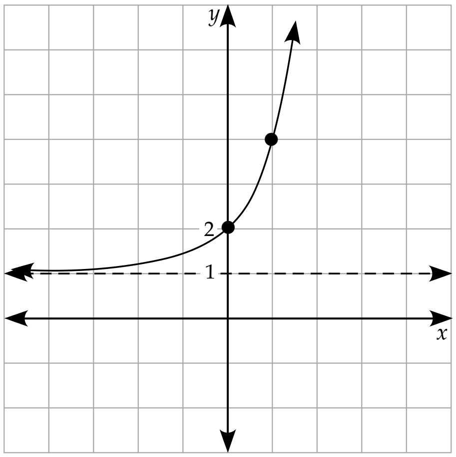
   Answer:

   $y = 5^{-x}$ or $y = \left(\frac{1}{5}\right)^x$

4. Sketch each of the following exponential functions and state the domain, range, y-intercept, and equation of asymptote for each.

a) $y = 3^x + 1$

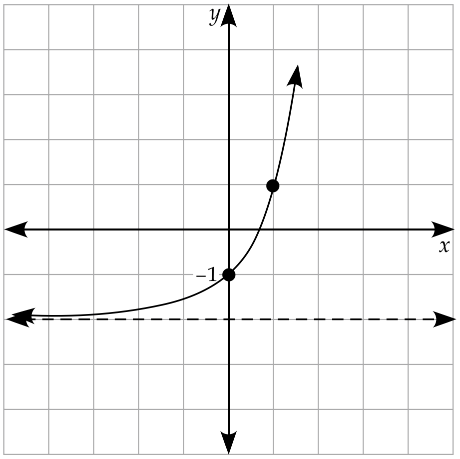
Answer:



This is the graph of $y = 3^x$ shifted 1 unit up.

b) $y = 3^x - 2$

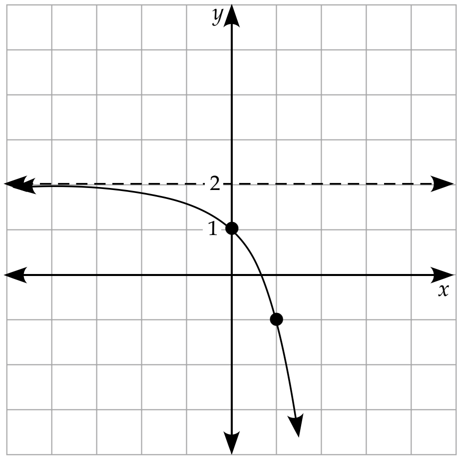
Answer:



This is the graph of $y = 3^x$ shifted 2 units down.

c) $y = -3^x + 2$

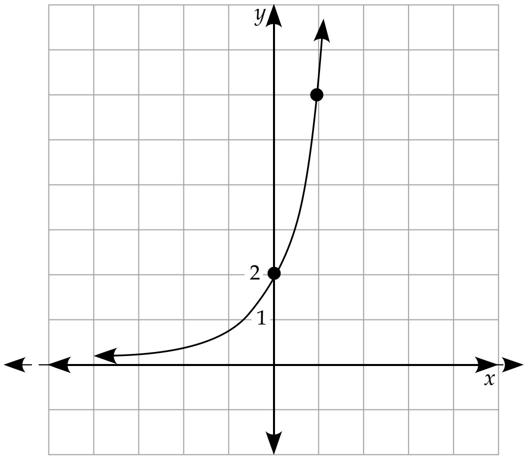
Answer:



This is the graph of $y = 3^x$ reflected through the x-axis and then shifted 2 units upward.

d) $y = 2(3^x)$

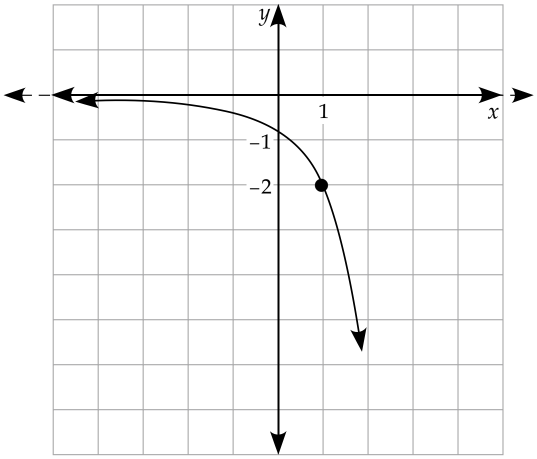
Answer:



This is the graph of $y = 3^x$ stretched vertically by a factor of 2.

e) $y = -2(3^{(x-1)})$

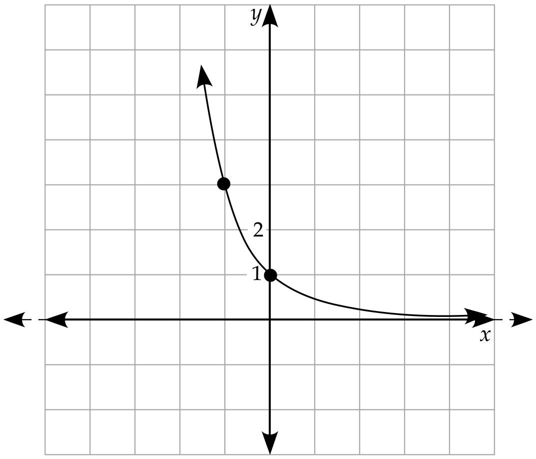
Answer:



This is the graph of $y = 3^x$ stretched vertically by a factor of 2, reflected through the x-axis, and then moved 1 unit to the right.

f) $y = 3^{-x}$

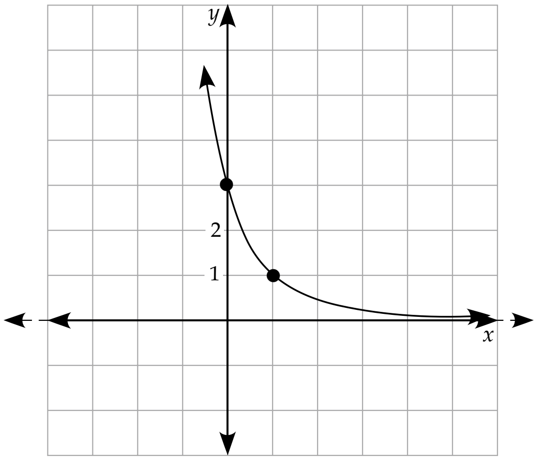
Answer:



This is the graph of $y = 3^x$ reflected through the y-axis.

g) $y = 3^{(1-x)}$

Answer:

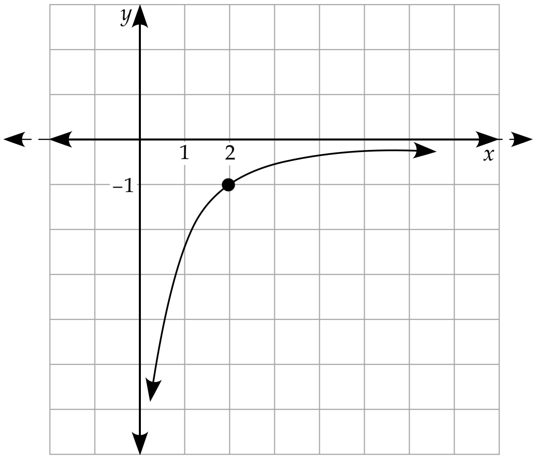


$y = 3^{1-x} = 3^{-1(x-1)}$

This is the graph of $y = 3^x$ reflected through the y-axis and then moved 1 unit to the right.

h) $y = -3^{(2-x)}$

Answer:

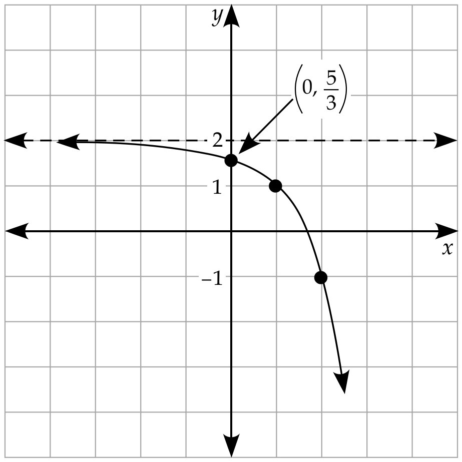


$y = -3^{-(x-2)}$

This is the graph of $y = 3^x$ reflected through the y-axis, reflected through the x-axis, and then moved 2 units to the right.

i) $y = -3^{(x-1)} + 2$

Answer:

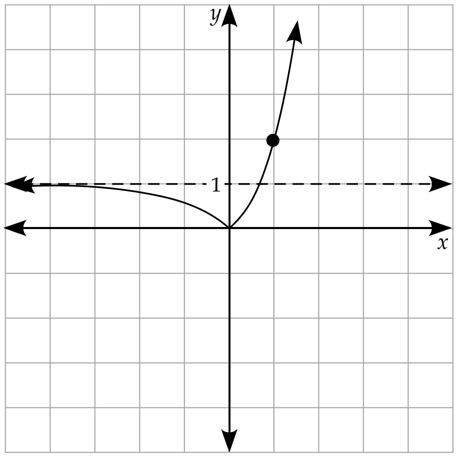


$y = -3^{x-1} + 2$

This is the graph of $y = 3^x$ reflected through the x-axis, moved 1 unit to the right and 2 units upward.

j) $y = |\ 3^x - 1\ |$

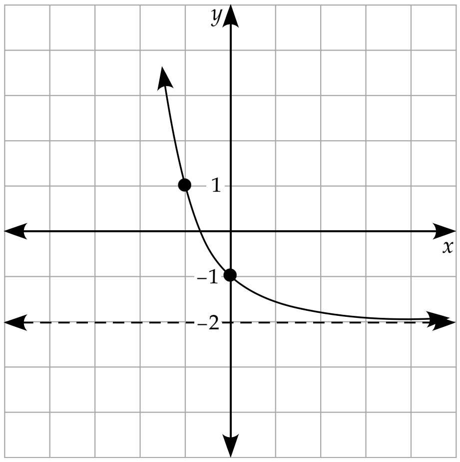
Answer:



This is the graph of $y = 3^x$ moved 1 unit down. Every part of this graph that is below the x-axis is then reflected over the x-axis.

k) $y = \left(\frac{1}{3}\right)^x - 2$

Answer:



$$y = \left(\frac{1}{3}\right)^x - 2 = 3^{-x} - 2$$

This is the graph of $y = 3^x$ reflected through the y-axis and then moved 2 units down.

Question	Domain	Range	y-intercept	Asymptote
(a)	$\Re$	$(1, \infty)$	2	$y = 1$
(b)	$\Re$	$(-2, \infty)$	-1	$y = -2$
(c)	$\Re$	$(-\infty, 2)$	1	$y = 2$
(d)	$\Re$	$(0, \infty)$	2	$y = 0$
(e)	$\Re$	$(-\infty, 0)$	$-\frac{2}{3}$	$y = 0$
(f)	$\Re$	$(0, \infty)$	1	$y = 0$
(g)	$\Re$	$(0, \infty)$	3	$y = 0$
(h)	$\Re$	$(-\infty, 0)$	-9	$y = 0$
(i)	$\Re$	$(-\infty, 2)$	$\frac{5}{3}$	$y = 2$
(j)	$\Re$	$[0, \infty)$	0	$y = 1$
(k)	$\Re$	$(-2, \infty)$	-1	$y = -2$

5. The definition of an exponential function, $f(x) = a^x$, states that $a \neq 1$ and $a > 0$.

   a) Why does a need to be greater than zero?

   Answer:

   If $a = 0$, then the entire function would be trivial, since it would always equal zero.

   If $a < 0$, this function would alternate between positive and negative values for whole number exponents. This is because a negative number to an *even* exponent will be *positive*. However, a negative number to an *odd* exponent will be *negative*.

   For fractional exponents, this function is not continuous.

   Consider $a = -2$ and $f\left(\frac{1}{2}\right)$. Then, $f\left(\frac{1}{2}\right) = (-2)^{\frac{1}{2}} = \sqrt{-2}$. This value is not defined. However, consider $a = -2$ and $f\left(\frac{1}{3}\right)$. Then,

   $$f\left(\frac{1}{3}\right) = (-2)^{\frac{1}{3}} = \sqrt[3]{-2} = -1.2599$$

   Therefore, this function is not continuous and presents problems when graphing. To ensure the exponential function is continuous and has consistent properties, $a > 0$.

b) Why can't $a = 1$?

Answer:

As you saw in Question 1, if $a = 1$, the entire function would always equal 1 as 1^x always equals 1 for any value of x. This degenerates to a linear function rather than an exponential function.

c) Explain why every exponential function, $f(x) = a^x$, contains a horizontal asymptote at $y = 0$.

Answer:

Let's consider the cases for $f(x) = a^x$.

- If $0 < a < 1$, then as x gets bigger, a^x will get smaller. Even though a^x can become arbitrarily small, it will never become zero.

  For example, $\left(\frac{1}{2}\right)^x$ can be written as $\frac{1}{2^x}$. As x gets larger, the fraction value gets smaller but the numerator will never be zero.

- If $a > 1$, then as x decreases and approaches negative infinity, a^x will get smaller. Even though x is negative, a^x will be positive and will become arbitrarily small but it will never become zero.

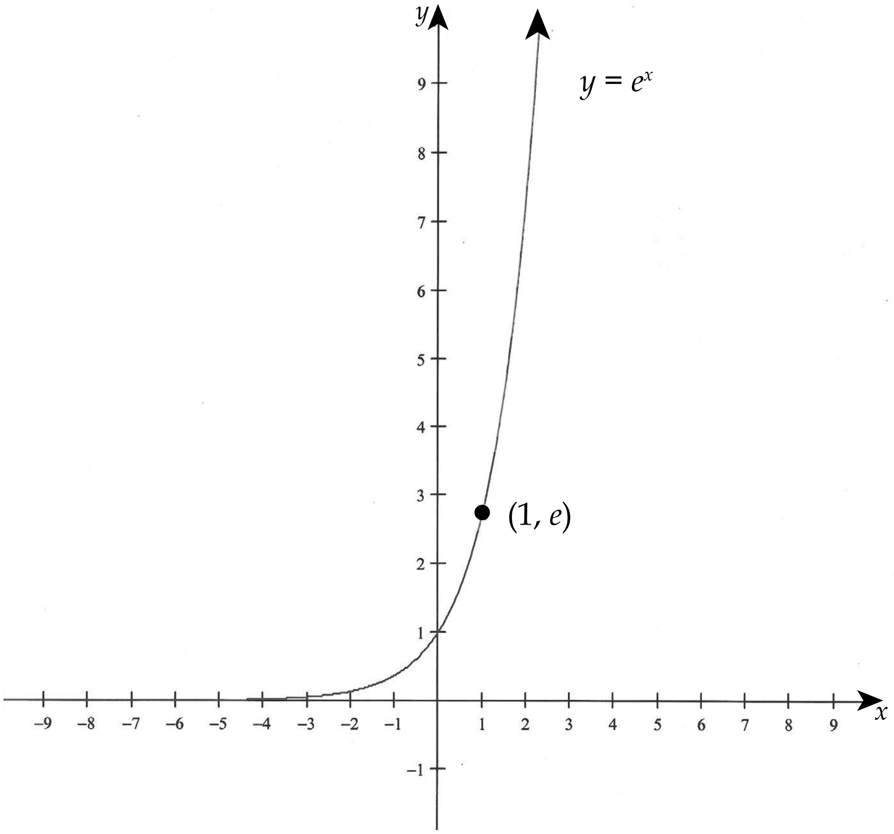
  For example, $2^{-1} = \frac{1}{2}$; $2^{-5} = \frac{1}{2^5} = \frac{1}{32}$; $2^{-10} = \frac{1}{2^{10}} = \frac{1}{1024}$.

  As the exponent becomes a "bigger" negative value, the fraction value gets smaller but the numerator will never be zero.

6. When dealing with exponential functions, a special constant appears so frequently that it has been given the symbol e. This constant is an irrational number, so it has decimal values that continue indefinitely without repeating. Its value is approximately 2.718. Sketch the graph of e^x using technology.

   Note: Your calculator or graphing applet will most likely contain the constant e, just as it contains the constant π.

   Answer:

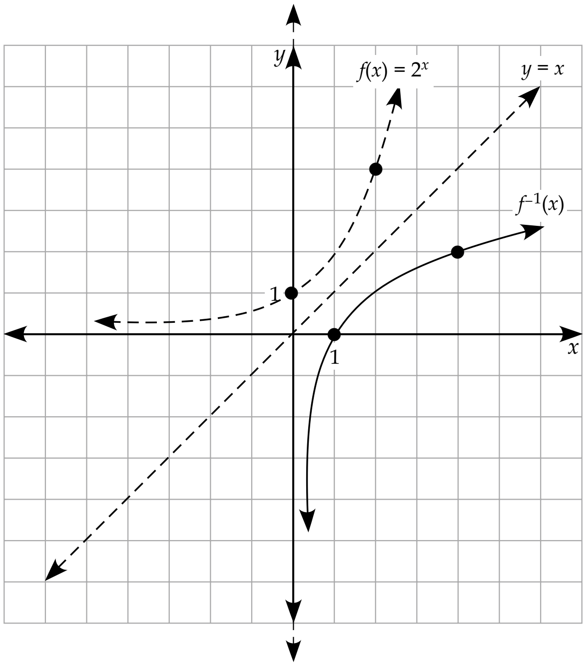


Note: If you don't have access to graphing technology, make a table of values for $y = 2.718^x$ to get an approximate graph of $y = e^x$. You will not be required to have technology for the hand-in assignment and you will not be allowed to have graphing technology for the final examination.

7. If $f(x) = a^x$, $a \neq 1$, $a > 0$, then $f(x)$ passes the horizontal line test and is a one-to-one function. Its inverse will also be a function. Sketch $f^{-1}(x)$ for the following functions by reflecting $f(x)$ over the line $y = x$.

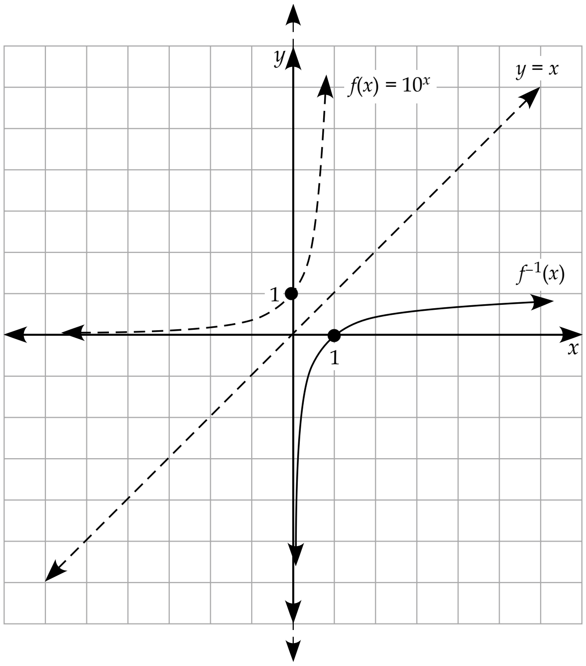
   a) $f(x) = 2^x$

   Answer:



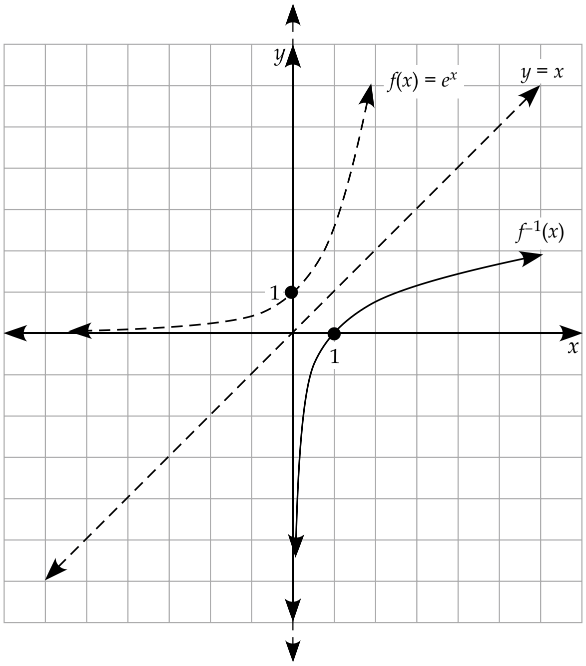
   b) $f(x) = 10^x$

   Answer:



c) $f(x) = e^x$

Answer:



8. List the common properties each of the inverses in Question 7 seem to have.

Answer:

All three graphs possess the following properties:

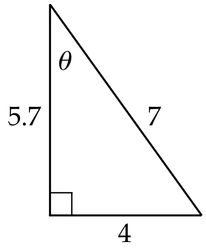
- The domains are $(0, \infty)$.
- The ranges are $\Re$.
- The x-intercepts are 1.
- They all have the y-axis as a vertical asymptote.
- They are all increasing on the domain.

Learning Activity 7.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

Use the triangle below to answer Questions 1 to 3.



1. Determine the cosine ratio.
2. Determine the cotangent ratio.
3. Determine the secant ratio.
4. How many ways can three students be awarded first, second, and third place in a contest?
5. State the non-permissible values of the function $f(x) = \dfrac{4}{x^2 + 4x}$.
6. Convert 4π to degrees.
7. Find all the values of θ between $[0, 2\pi]$, if $\sin\theta = -1$.
8. Multiply: $(-3x^{-2})(4x^3)$

Answers:

1. $\cos\theta = \dfrac{5.7}{7}$
2. $\cot\theta = \dfrac{5.7}{4}$
3. $\sec\theta = \dfrac{7}{5.7}$
4. 6 (3! = 6)
5. $x \neq 0$ and $x \neq -4$ $(x^2 + 4x = x(x + 4))$
6. 720°

7. $\frac{3\pi}{2}$

8. $-12x$

Part B: Logarithms

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Evaluate each of the following.

a) $16^{\frac{1}{2}}$

Answer:

$$16^{\frac{1}{2}} = \sqrt{16} = 4$$

b) $27^{\frac{2}{3}}$

Answer:

$$27^{\frac{2}{3}} = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9$$

c) $16^{-\frac{1}{2}}$

Answer:

$$16^{-\frac{1}{2}} = \frac{1}{16^{\frac{1}{2}}} = \frac{1}{\sqrt{16}} = \frac{1}{4}$$

d) $27^{-\frac{2}{3}}$

Answer:

$$27^{-\frac{2}{3}} = \frac{1}{27^{\frac{2}{3}}} = \frac{1}{9}$$

e) $\left(\frac{1}{4}\right)^{-\frac{1}{2}}$

Answer:

$$\left(\frac{1}{4}\right)^{-\frac{1}{2}} = 4^{\frac{1}{2}} = \sqrt{4} = 2$$

f) $81^{\frac{1}{4}}$

Answer:

$$81^{\frac{1}{4}} = \sqrt[4]{81} = 3$$

g) $-4^{-\frac{1}{2}}$

Answer:

$$-4^{-\frac{1}{2}} = -\frac{1}{4^{\frac{1}{2}}} = -\frac{1}{\sqrt{4}} = -\frac{1}{2}$$

h) $(-8)^{-\frac{1}{3}}$

Answer:

$$(-8)^{-\frac{1}{3}} = \frac{1}{(-8)^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{-8}} = \frac{1}{-2} = -\frac{1}{2}$$

i) $100^{-\frac{3}{2}}$

Answer:

$$100^{-\frac{3}{2}} = \frac{1}{100^{\frac{3}{2}}} = \frac{1}{\left(\sqrt{100}\right)^3}$$

$$= \frac{1}{10^3} = \frac{1}{1000}$$

j) $4\left(\frac{1}{9}\right)^{\frac{1}{2}}$

Answer:

$$4\left(\frac{1}{9}\right)^{\frac{1}{2}} = 4\left(\sqrt{\frac{1}{9}}\right) = 4\left(\frac{1}{3}\right) = \frac{4}{3}$$

2. Express each of the following in exponential form.

a) $\log_2 16 = 4$

Answer:

$2^4 = 16$

b) $\log_4 64 = 3$

Answer:

$4^3 = 64$

c) $\log_{10} 0.01 = -2$

Answer:

$10^{-2} = 0.01$

d) $\log_5 \frac{1}{5} = -1$

Answer:

$5^{-1} = \frac{1}{5}$

3. Write $\log_A B = C$ in exponential form.

Answer:

$A^C = B$

4. Express each of the following in logarithmic form.

a) $3^4 = 81$

Answer:

$\log_3 81 = 4$

b) $32 = 2^5$

Answer:

$\log_2 32 = 5$

c) $\left(\frac{1}{4}\right)^2 = \frac{1}{16}$

Answer:

$\log_{\frac{1}{4}} \frac{1}{16} = 2$

d) $2^{-3} = \frac{1}{8}$

Answer:

$\log_2 \frac{1}{8} = -3$

5. Write $A^B = C$ in logarithmic form.

   Answer:

   $\log_A C = B$

   Recall: The logarithm is the value of the exponent, B.

6. Solve for x (without using a calculator).

   a) $2^x = 32$

      Answer:

      Write each expression as a power with a base of 2.

      $2^x = 32$

      $2^x = 2^5$

      As the bases are now equivalent, you can equate the exponents.

      $x = 5$

   b) $2^{3x-5} = 16$

      Answer:

      Write each expression as a power with a base of 2.

      $2^{3x-5} = 16$

      $2^{3x-5} = 2^4$

      You can equate the exponents on either side of the equals sign, as the bases are equivalent.

      $3x - 5 = 4$

      Solve for x.

      $3x = 9$

      $x = 3$

   c) $5^{4x-7} = 125$

      Answer:

      Write each expression as a power with a base of 5.

      $5^{4x-7} = 125$

      $5^{4x-7} = 5^3$

      Equate the exponents on either side of the equals sign.

      $4x - 7 = 3$

      $4x = 10$

      $$x = \frac{10}{4} = \frac{5}{2}$$

d) $\dfrac{1}{3^{x-1}} = 81$

Answer:

Write each expression as a power with a base of 3.

$$\frac{1}{3^{x-1}} = 81$$

$$3^{-(x-1)} = 3^4$$

$$\therefore -(x-1) = 4$$

$$-x + 1 = 4$$

$$-x = 3$$

$$x = -3$$

e) $2(5^{2x-9}) = 250$

Answer:

First, divide each side of the equation by 2. Then, rewrite each expression as a power with a base of 5.

$$2(5^{2x-9}) = 250$$

$$5^{2x-9} = 125$$

$$5^{2x-9} = 5^3$$

$$\therefore 2x - 9 = 3$$

$$2x = 12$$

$$x = 6$$

f) $32^{3x-2} = 16$

Answer:

To solve this equation, you need to recognize that both of these expressions can be written as a power with a base of 2.

$$32^{3x-2} = 16$$

$$2^{5(3x-2)} = 2^4$$

Now you need to recall the rules of exponents, specifically the power of a power law—when a power is raised to a power, you need to multiply the exponents.

$$\begin{aligned} \therefore 5(3x-2) &= 4 \\ 15x - 10 &= 4 \\ 15x &= 14 \\ x &= \frac{14}{15} \end{aligned}$$

g) $3^{8x} = \dfrac{1}{81}$

Answer:

$$3^{8x} = \frac{1}{3^4}$$

$$3^{8x} = 3^{-4}$$

$$\therefore 8x = -4$$

$$x = -\frac{1}{2}$$

h) $\dfrac{1}{4^{x-2}} = 64$

Answer:

$$\frac{1}{4^{x-2}} = 4^3$$

$$4^{-(x-2)} = 4^3$$

$$\begin{aligned} \therefore -(x-2) &= 3 \\ -x + 2 &= 3 \\ -x &= 1 \\ x &= -1 \end{aligned}$$

7. Solve for the variable.

a) $y = \log_5 125$

Answer:

$5^y = 125$

$5^y = 5^3$

$y = 3$

b) $\log_5 x = 2$

Answer:

$5^2 = x$

$25 = x$

c) $\log_x 16 = -\frac{4}{3}$

Answer:

$$x^{-\frac{4}{3}} = 16$$

$$x^{\left(-\frac{4}{3}\right)\left(-\frac{3}{4}\right)} = 16^{-\frac{3}{4}}$$

$$x = \frac{1}{16^{\frac{3}{4}}}$$

$$x = \frac{1}{\left(\sqrt[4]{16}\right)^3}$$

$$x = \frac{1}{2^3}$$

$$x = \frac{1}{8}$$

d) $y = \log_{\frac{1}{2}} \left(\frac{1}{2}\right)^4$

Answer:

$$\left(\frac{1}{2}\right)^y = \left(\frac{1}{2}\right)^4$$

$y = 4$

e) $\log_7 1 = y$

Answer:

$7^y = 1$

$y = 0$

f) $y = \log_2 \frac{1}{4}$

Answer:

$$2^y = \frac{1}{4}$$

$$2^y = \frac{1}{2^2}$$

$2^y = 2^{-2}$

$y = -2$

g) $\log_3 x = -4$

Answer:

$$3^{-4} = x$$

$$\frac{1}{3^4} = x$$

$$\frac{1}{81} = x$$

h) $\log_8 x = \frac{4}{3}$

Answer:

$$8^{\frac{4}{3}} = x$$

$$\left(\sqrt[3]{8}\right)^4 = x$$

$$2^4 = x$$

$$16 = x$$

i) $\log_4 x = -\frac{5}{2}$

Answer:

$$4^{-\frac{5}{2}} = x$$

$$\frac{1}{4^{\frac{5}{2}}} = x$$

$$\frac{1}{\left(\sqrt{4}\right)^5} = x$$

$$\frac{1}{2^5} = x$$

$$\frac{1}{32} = x$$

j) $\log_b 125 = \frac{3}{4}$

Answer:

$$b^{\frac{3}{4}} = 125$$

$$b^{\left(\frac{3}{4}\right)\left(\frac{4}{3}\right)} = 125^{\frac{4}{3}}$$

$$b = \left(\sqrt[3]{125}\right)^4$$

$$b = 5^4$$

$$b = 625$$

k) $\log_b \sqrt{5} = \frac{1}{4}$

Answer:

$$b^{\frac{1}{4}} = \sqrt{5}$$

$$b^{\frac{1}{4}} = 5^{\frac{1}{2}}$$

$$b^{\left(\frac{1}{4}\right)(4)} = 5^{\left(\frac{1}{2}\right)(4)}$$

$$b = 5^{\frac{4}{2}}$$

$$b = 5^2$$

$$b = 25$$

l) $\log_{\sqrt{3}} 9 = y$

Answer:

$$\sqrt{3}^{y} = 9$$

$$\left(3^{\frac{1}{2}}\right)^{y} = 3^2$$

$$3^{\frac{y}{2}} = 3^2$$

$$\frac{y}{2} = 2$$

$$y = 4$$

8. For all values of a, $a > 0$, what is the value of $\log_a a$?

Answer:

This question is asking you to solve for x in the following equation:

$a^x = a, a > 0$

$\therefore x = 1$, since $a^1 = a$

9. For all values of a, $a > 0$, what is the value of $\log_a 1$?

Answer:

This question is asking you to solve for x in the following equation:

$a^x = 1, a > 0$

$\therefore x = 0$, since $a^0 = 1$

10. Evaluate the following logarithms without using a calculator.

a) $\log_2 2$

Answer:

$2^x = 2$

$x = 1$

$\therefore \log_2 2 = 1$

b) $\log_7 \left(7^6\right)$

Answer:

$7^x = 7^6$

$x = 6$

$\therefore \log_7 \left(7^6\right) = 6$

c) $\log_5 625$

Answer:

$5^x = 625$

$x = 4$

$\therefore \log_5 625 = 4$

d) $\log_2 256$

Answer:

$2^x = 256$

$x = 8$

$\therefore \log_2 256 = 8$

e) $\log_3 1$

Answer:

$3^x = 1$

$x = 0$

$\therefore \log_3 1 = 0$

f) $\log_5 \dfrac{1}{25}$

Answer:

$5^x = \dfrac{1}{25}$

$5^x = \dfrac{1}{5^2}$

$5^x = 5^{-2}$

$x = -2$

$\therefore \log_5 \dfrac{1}{25} = -2$

g) $\log_6 \frac{1}{6}$

Answer:

$$6^x = \frac{1}{6}$$

$$6^x = 6^{-1}$$

$$x = -1$$

$$\therefore \log_6 \frac{1}{6} = -1$$

11. Estimate the following logarithms to one decimal place without using a calculator.

a) $\log_{10} 800$

Answer:

First, write the expression in exponential form:

$10^x = 800$

Then, find benchmarks:

$10^2 = 100$ and $10^3 = 1000$

$\therefore 2 < x < 3$

Finally, estimate the solution and check to see if your estimation is reasonable:

As 800 is closer to 1000 than 100, x is closer to 3 than to 2.

Estimate: $x \approx 2.9$

Check: $10^{29} = 794.328$

This is a good estimation.

$\therefore \log_{10} 800 \approx 2.9$

b) $\log_2 47$

Answer:

Write in exponential form:

$2^x = 47$

Find benchmarks:

$2^5 = 32$ and $2^6 = 64$

$\therefore 5 < x < 6$

Estimate and check:

As 47 is approximately right in the middle of 32 and 64, x should be around 5.5.

Estimate: $x \approx 5.5$

Check: $2^{5.5} = 42.255$

This is a good estimation.

$\therefore \log_2 47 \approx 5.5$

c) $\log_3 76$

Answer:

Write in exponential form:

$3^x = 76$

Find benchmarks:

$3^3 = 27$ and $3^4 = 81$

$\therefore 3 < x < 4$

Estimate and check:

As 76 is closer to 81 than 27, x should be closer to 4.

Estimate: $x \approx 39$

Check: $3^{3.9} = 72.573$

This is as close an estimate as you are going to get.

$\therefore \log_3 76 \approx 3.9$

d) $\log_4 15$

Answer:

Write in exponential form:

$4^x = 15$

Find benchmarks:

$4^1 = 4$ and $4^2 = 16$

$\therefore 1 < x < 2$

Estimate and check:

As 15 is closer to 16 than 4, x should be closer to 2.

Estimate: $x \approx 1.9$

Check: $4^{1.9} = 13.929$

This is as close an estimate as you are going to get.

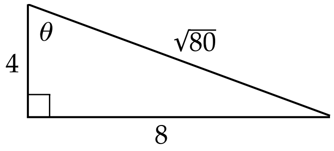
$\therefore \log_4 15 \approx 1.9$

Learning Activity 7.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

Use the triangle below to answer Questions 1 to 3.



1. Determine the cosine ratio.
2. Determine the tangent ratio.
3. Determine the cosecant ratio.
4. Factor: $x^2 + 2x - 24$
5. What is $\frac{3}{7}$ of $\frac{5}{8}$?
6. Simplify: $\sqrt{20} + \sqrt{45}$
7. If the graph of $y = -3x^2 + 2$ is translated 4 units right, what is an equation of the translated parabola?
8. In which quadrant is $\theta = \frac{15\pi}{8}$ located?

Answers:

1. $\cos\theta = \frac{4}{\sqrt{80}}$
2. $\tan\theta = \frac{8}{4} = 2$
3. $\csc\theta = \frac{\sqrt{80}}{8}$
4. $(x + 6)(x - 4)$
5. $\frac{15}{56}\ \left(\frac{3}{7} \cdot \frac{5}{8} = \frac{15}{56}\right)$
6. $5\sqrt{5}\ \left(2\sqrt{5} + 3\sqrt{5} = 5\sqrt{5}\right)$
7. $y = -3(x - 4)^2 + 2$

8. Quadrant IV $\left(\frac{15\pi}{8} > \frac{3\pi}{2} \text{ or } \frac{12\pi}{8} \text{ and } \frac{15\pi}{8} < 2\pi \text{ or } \frac{16\pi}{8}\right)$

Part B: Using the Logarithmic Theorems

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Expand as a sum and/or difference of individual logarithmic expressions, and simplify if possible.

   a) $\log_b\left(\frac{a^2}{\sqrt{r}}\right)$

   Answer:

   $$\log_b\left(\frac{a^2}{\sqrt{r}}\right) = \log_b a^2 - \log_b \sqrt{r}$$

   $$= 2\log_b a - \log_b r^{\frac{1}{2}}$$

   $$= 2\log_b a - \frac{1}{2}\log_b r$$

   b) $\log_7\left(49\sqrt[3]{A}\right)$

   Answer:

   $$\log_7\left(49\sqrt[3]{A}\right) = \log_7 49 + \log_7 \sqrt[3]{A}$$

   $$= 2 + \log_7 A^{\frac{1}{3}}$$

   $$= 2 + \frac{1}{3}\log_7 A$$

   c) $\log\left(\frac{A^2}{B^3C}\right)$

   Answer:

   $$\log\left(\frac{A^2}{B^3C}\right) = \log A^2 - \log B^3 - \log C$$

   $$= 2\log A - 3\log B - \log C$$

d) $\log\left(\dfrac{2xy}{\sqrt{z}}\right)$

Answer:

$$\log\left(\frac{2xy}{\sqrt{z}}\right) = \log 2 + \log x + \log y - \log \sqrt{z}$$

$$= \log 2 + \log x + \log y - \frac{1}{2}\log z$$

e) $\log\left(\dfrac{ABC}{PQ^2}\right)$

Answer:

$$\log\left(\frac{ABC}{PQ^2}\right) = \log A + \log B + \log C - \log P - 2\log Q$$

f) $\log\left(\dfrac{3b\sqrt{c+1}}{4d^2}\right)$

Answer:

$$\log\left(\frac{3b\sqrt{c+1}}{4d^2}\right) = \log 3 + \log b + \frac{1}{2}\log(c+1) - \log 4 - 2\log d$$

g) $\log_3\left(x\sqrt{y}\right)$

Answer:

$$\log_3\left(x\sqrt{y}\right) = \log_3 x + \frac{1}{2}\log_3 y$$

h) $\log\left(\dfrac{8a^3}{b^4c^5}\right)$

Answer:

$$\log\left(\frac{8a^3}{b^4c^5}\right) = \log 8 + 3\log a - 4\log b - 5\log c$$

2. Write as a single logarithm.

   a) $\log_3 A + 3\log_3 B - \log_3 C$

   Answer:

   $$\log_3 \frac{AB^3}{C}$$

   b) $\frac{1}{3}\log A - \log B + \log C$

   Answer:

   $$\log \frac{\sqrt[3]{A}C}{B}$$

   c) $\log(a+b) - \log(a-b)$

   Answer:

   $$\log\left(\frac{a+b}{a-b}\right)$$

   d) $\frac{1}{2}(\log_a x - 3\log_a y)$

   Answer:

   $$\log_a \sqrt{\frac{x}{y^3}}$$

   e) $2\left(\log(x+y) - \log z\right)$

   Answer:

   $$\log\left(\frac{x+y}{z}\right)^2$$

   f) $\frac{1}{2}\log A + 2\log B - \frac{1}{5}\log C$

   Answer:

   $$\log\left(\frac{\sqrt{A}B^2}{\sqrt[5]{C}}\right)$$

3. Use the logarithmic theorems to write log y as an expression of logarithms (i.e., take log y of each side of the equation and simplify).

   a) $y = \dfrac{(x-1)(x+3)^2}{\sqrt{x^2+2}}$

   Answer:

   $$\begin{aligned}\log y &= \log \frac{(x-1)(x+3)^2}{\sqrt{x^2+2}} \\ &= \log(x-1) + 2\log(x+3) - \frac{1}{2}\log\left(x^2+2\right)\end{aligned}$$

b) $y = \sqrt{x^2(x+1)}$

Answer:

$$\begin{aligned}\log y &= \log\left(\sqrt{x^2(x+1)}\right)\\ &= \frac{1}{2}(\log x^2 + \log(x+1)\\ &= \frac{1}{2}\left(2\log x + \log(x+1)\right)\end{aligned}$$

4. Express each of the following as a single logarithm. Simplify, if possible.

a) $\log_2 5 + \log_2 7 + \log_2 6$

Answer:

$$\begin{aligned}\log_2 5 + \log_2 7 + \log_2 6 &= \log_2 (5)(7)(6)\\ &= \log_2 210\end{aligned}$$

b) $\log 2 + \log 5$

Answer:

$$\begin{aligned}\log 2 + \log 5 &= \log (2)(5)\\ &= \log 10\\ &= 1\end{aligned}$$

c) $\frac{1}{2}\log_5 4 + \frac{1}{3}\log_5 27$

Answer:

$$\begin{aligned}\frac{1}{2}\log_5 4 + \frac{1}{3}\log_5 27 &= \log_5 \sqrt{4} + \log_5 \sqrt[3]{27}\\ &= \log_5 2 + \log_5 3\\ &= \log_5 2(3)\\ &= \log_5 6\end{aligned}$$

d) $2 \log_3 7 - (\log_3 14 + \log_3 35)$

Answer:

$$\begin{aligned} 2 \log_3 7 - (\log_3 14 + \log_3 35) &= \log_3 7^2 - (\log_3 (14)(35)) \\ &= \log_3 49 - \log_3 490 \\ &= \log_3 \left(\frac{49}{490}\right) \\ &= \log_3 \left(\frac{1}{10}\right) \\ &= \log_3 1 - \log_3 10 \\ &= 0 - \log_3 10 \\ &= -\log_3 10 \end{aligned}$$

5. Given:

$\log_b 2 = A$
$\log_b 3 = B$
$\log_b 5 = C$

Use the three equations given above to find expressions for the following in terms of A, B, and C.

a) $\log_b 6$

Answer:

$$\begin{aligned} \log_b 6 &= \log_b 2(3) \\ &= \log_b 2 + \log_b 3 \\ &= A + B \end{aligned}$$

b) $\log_b 10$

Answer:

$$\begin{aligned} \log_b 10 &= \log_b 2(5) \\ &= \log_b 2 + \log_b 5 \\ &= A + C \end{aligned}$$

c) $\log_b \left(\frac{15}{2}\right)$

Answer:

$$\begin{aligned} \log_b \left(\frac{15}{2}\right) &= \log_b \frac{(3)(5)}{2} \\ &= \log_b 3 + \log_b 5 - \log_b 2 \\ &= B + C - A \end{aligned}$$

d) $\log_b 8$

Answer:

$$\begin{aligned} \log_b 8 &= \log_b 2^3 \\ &= 3 \log_b 2 \\ &= 3A \end{aligned}$$

e) $\log_b 25$

Answer:

$$\begin{aligned}\log_b 25 &= \log_b 5^2\\ &= 2\log_b 5\\ &= 2C\end{aligned}$$

f) $\log_b \sqrt{30}$

Answer:

$$\begin{aligned}\log_b \sqrt{30} &= \log_b 30^{\frac{1}{2}}\\ &= \frac{1}{2}\log_b 30\\ &= \frac{1}{2}\left(\log_b (5)(3)(2)\right)\\ &= \frac{1}{2}\left(\log_b 5 + \log_b 3 + \log_b 2\right)\\ &= \frac{1}{2}(C + B + A)\end{aligned}$$

g) $\log_b \sqrt{\frac{6}{5}}$

Answer:

$$\begin{aligned}\log_b \sqrt{\frac{6}{5}} &= \log_b \left(\frac{6}{5}\right)^{\frac{1}{2}}\\ &= \frac{1}{2}\log_b \left(\frac{6}{5}\right)\\ &= \frac{1}{2}\log_b \frac{(2)(3)}{5}\\ &= \frac{1}{2}\left(\log_b 2 + \log_b 3 - \log_b 5\right)\\ &= \frac{1}{2}(A + B - C)\end{aligned}$$

6. Given $\log_3 5 = D$, express $\log_3 45$ in terms of D.

Answer:

$$\begin{aligned}\log_3 45 &= \log_3 (3 \cdot 3 \cdot 5)\\ &= \log_3\left(3^2\right) + \log_3 5\\ &= 2\log_3 3 + \log_3 5\\ &= 2 + D\end{aligned}$$

Learning Activity 7.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Simplify: $\dfrac{\left(\sin^2 x + \cos^2 x\right) + 1}{\cos x}$
2. What is the exact value of $\sin\left(\dfrac{\pi}{6} + \dfrac{\pi}{3}\right)$?
3. Simplify: $\dfrac{(x+3)!}{(x+1)!}$
4. State an angle that is coterminal to 52°.
5. State the non-permissible values of the function $f(x) = \dfrac{x}{x^2 - 4}$.
6. Express $\log_3 8 - \log_3 2$ as one logarithm.
7. Simplify: $\log 10$
8. Factor: $6x^2 + 7x - 3$

Answers:

1. $\dfrac{2}{\cos x}$ or $2 \sec x$ $\left(\dfrac{\left(\sin^2 x + \cos^2 x\right) + 1}{\cos x} = \dfrac{1+1}{\cos x} = \dfrac{2}{\cos x} \text{ or } 2 \sec x\right)$
2. 1 $\left(\sin\left(\dfrac{\pi}{6} + \dfrac{\pi}{3}\right) = \sin\left(\dfrac{\pi}{6} + \dfrac{2\pi}{6}\right) = \sin\left(\dfrac{3\pi}{6}\right) = \sin\left(\dfrac{\pi}{2}\right) = 1\right)$
3. $x^2 + 5x + 6$ $\left(\dfrac{(x+3)!}{(x+1)!} = \dfrac{(x+3)(x+2)(x+1)!}{(x+1)!} = (x+3)(x+2) = x^2 + 5x + 6\right)$
4. 412° (52° + 360° = 412°; other answers are possible—add or subtract multiples of 360°)
5. $x \neq 2$ and $x \neq -2$ ($x^2 - 4 = (x + 2)(x - 2)$)
6. $\log_3 4$ $\left(\log_3 8 - \log_3 2 = \log_3\left(\dfrac{8}{2}\right) = \log_3 4\right)$
7. 1 ($\log 10 = x$, $10^x = 10$, $\therefore x = 1$)
8. $(3x - 1)(2x + 3)$

Part B: Graphing Logarithmic Functions

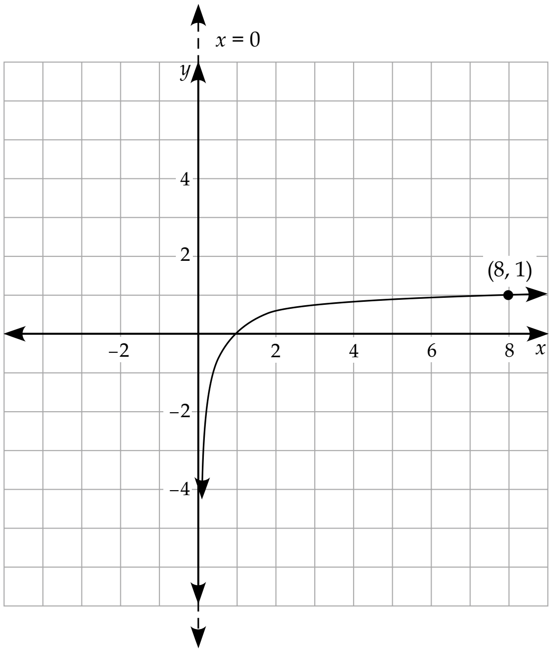
Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Sketch each of the following logarithmic functions and state the domain, range, and equation of asymptote for each.

   Note: These are sketches of the basic curve $f(x) = \log_a x$ with the following transformations:

   a) $y = \log_8 x$

   Answer:



As the base of this function is 8, this function will increase slower than the logarithmic function of base 6 that you graphed in Example 3 (c).

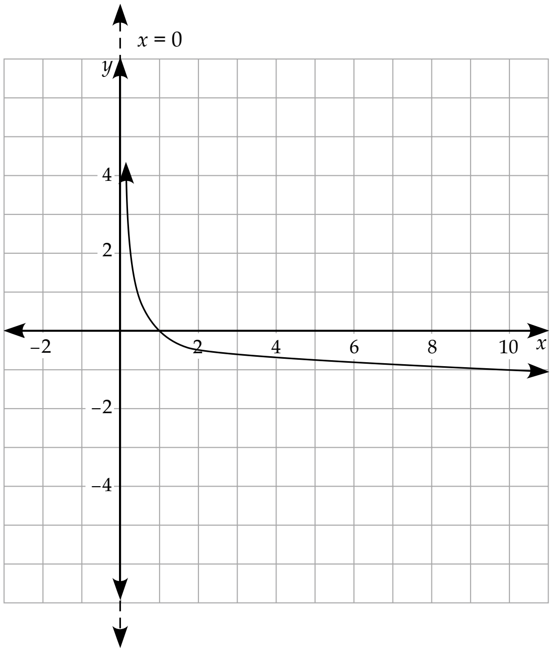
Domain: $(0, \infty)$

Range: $(-\infty, \infty)$ or $\Re$

Asymptote: $x = 0$

b) $y = -\log x$

Answer:



Base 10 logarithm reflected in the x-axis.

Domain: $(0, \infty)$

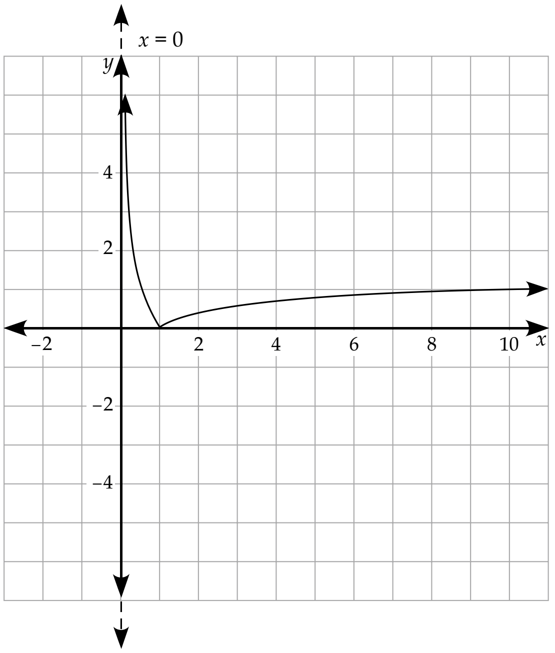
Range: $(-\infty, \infty)$

Asymptote: $x = 0$

Note: The ranges that are given as $(-\infty, \infty)$ can also be written as $y \in \Re$.

c) $y = |\log x|$

Answer:



Base 10 logarithm where all points below the x-axis are reflected in the x-axis.

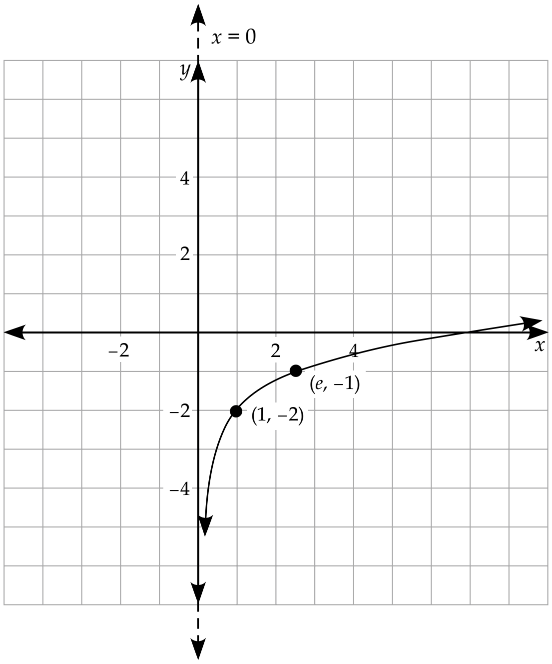
Domain: $(0, \infty)$

Range: $[0, \infty)$

Asymptote: $x = 0$

d) $y = \ln x - 2$

Answer:



Base *e* logarithm translated two units downward.

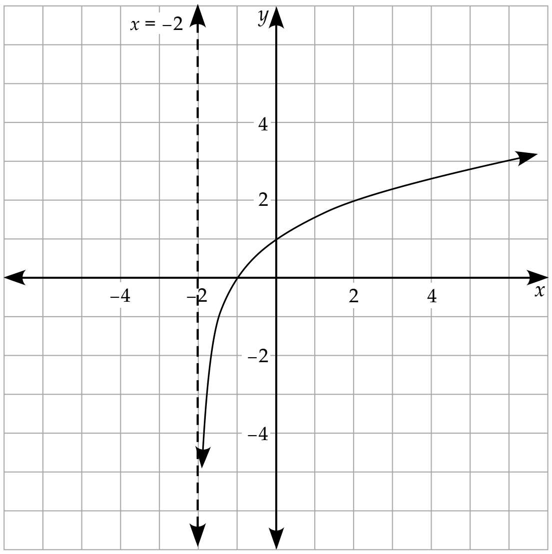
Domain: $(0, \infty)$

Range: $(-\infty, \infty)$ or $\Re$

Asymptote: $x = 0$

e) $y = \log_2 (x + 2)$

Answer:



Base 2 logarithm, which is translated 2 units to the left.

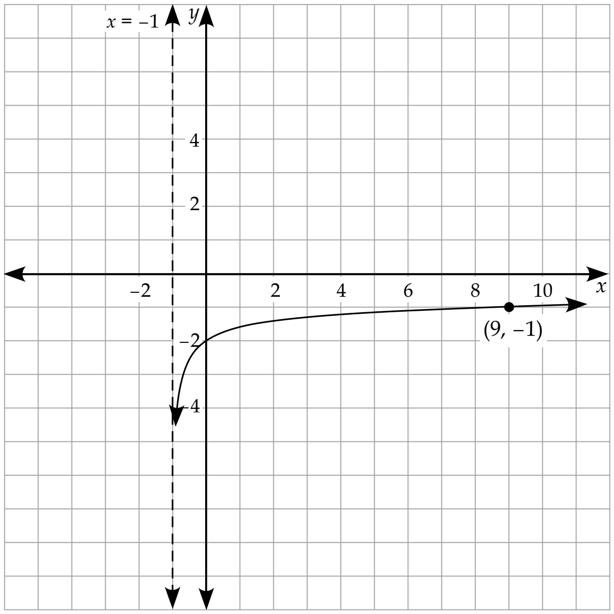
Domain: $(-2, \infty)$

Range: $(-\infty, \infty)$ or $\Re$

Asymptote: $x = -2$

f) $y = \log(x + 1) - 2$

Answer:



Base 10 logarithm translated 1 unit to the left and 2 units downward.

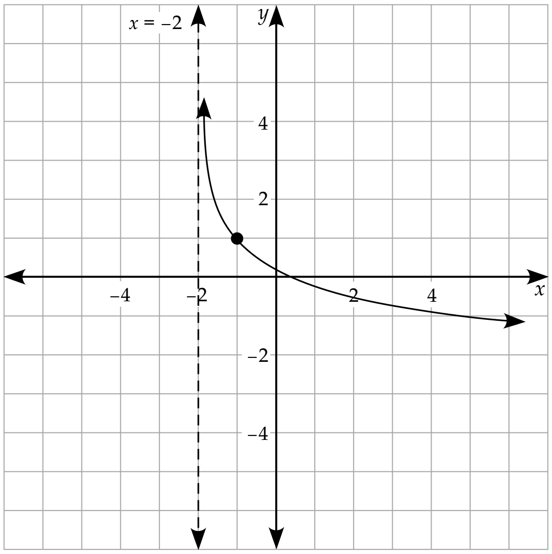
Domain: $(-1, \infty)$

Range: $(-\infty, \infty)$ or $\Re$

Asymptote: $x = -1$

g) $y = -\ln(x + 2) + 1$

Answer:



Base *e* logarithm reflected in the *x*-axis, translated 2 units to the left, and 1 unit upward.

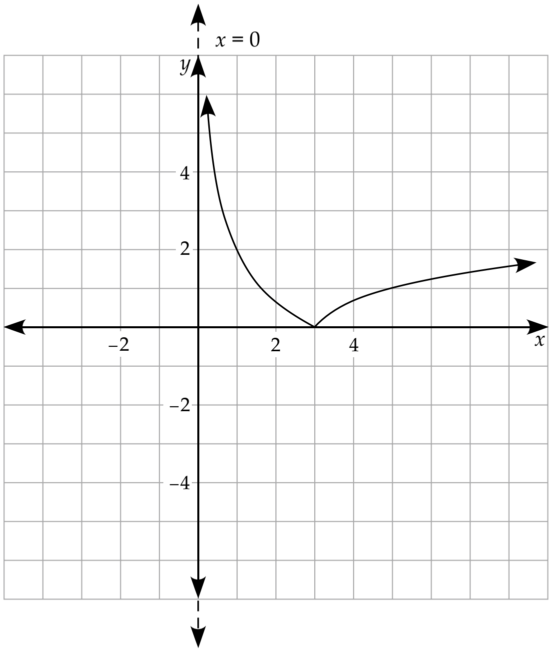
Domain: $(-2, \infty)$

Range: $(-\infty, \infty)$ or $\Re$

Asymptote: $x = -2$

h) $y = | \log_3 x - 2 |$

Answer:



Base 3 logarithm translated 2 units down, followed by reflecting all points below the x-axis in the x-axis.

Domain: $(0, \infty)$

Range: $[0, \infty)$

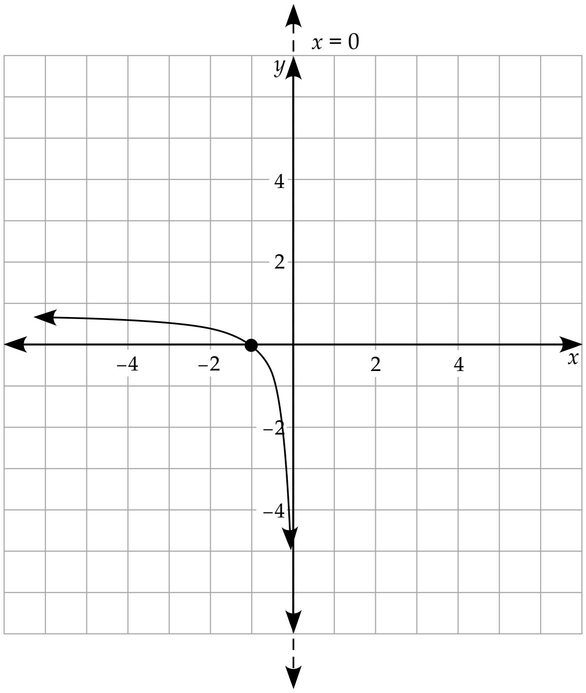
Asymptote: $x = 0$

2. Sketch each of the following logarithmic functions and state the domain, range, and equation of asymptote for each.

   a) $f(x) = \log(-x)$

   Answer:

   This is a base 10 logarithm reflected in the y-axis.



Domain: $(-\infty, 0)$

Range: $\Re$

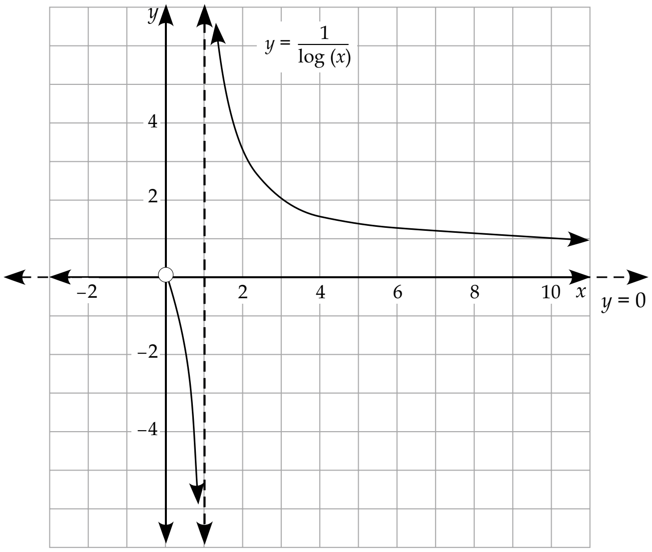
Asymptote: $x = 0$

b) $g(x) = \dfrac{1}{\log x}$

Hint: Graph $f(x) = \log(x)$. Then use that graph to sketch the reciprocal, $y = \dfrac{1}{f(x)}$.

Answer:

Draw $y = \log x$ and find the reciprocal of all the y-values.



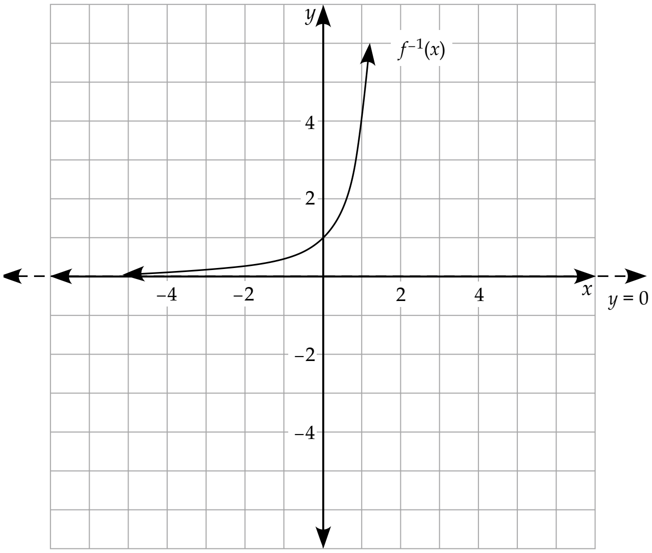
Domain: $(0, 1) \cup (1, \infty)$

Range: $\{y \mid y \neq 0\}$

Asymptote: $x = 1$ and $y = 0$

3. Sketch $f^{-1}(x)$ if $f(x) = \log_4 x$. State the domain, range, and equations of asymptotes of both $f(x)$ and $f^{-1}(x)$.

Answer:



$f(x)$:

Domain: $(0, \infty)$

Range: $y \in \Re$

Asymptote: $x = 0$

$f^{-1}(x)$:

Domain: $x \in \Re$

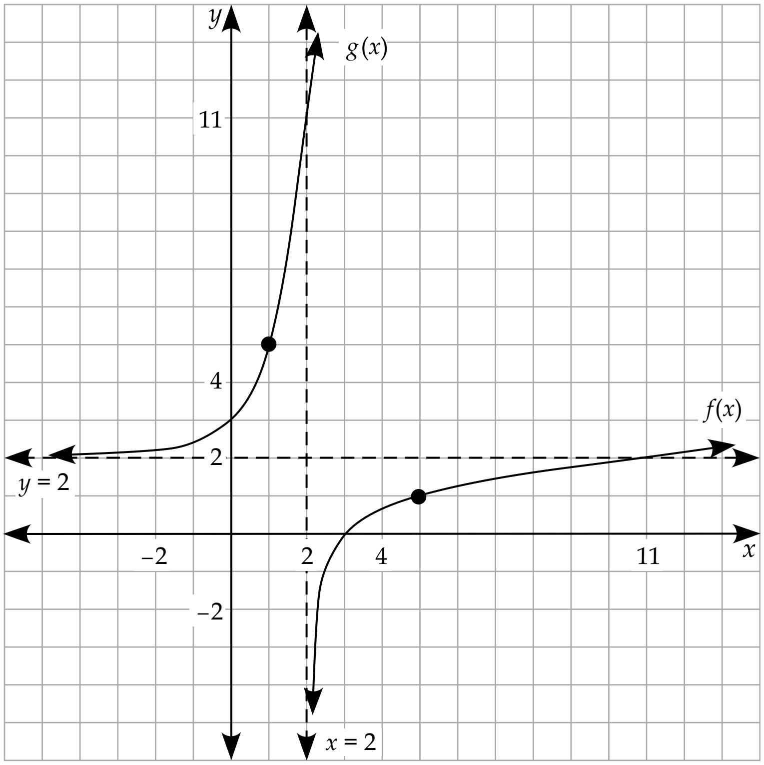
Range: $(0, \infty)$

Asymptote: $y = 0$

Note: Reflect in the line with equation $y = x$. The graph is the graph of $y = 4^x$, since the exponential must be the inverse of the logarithmic function. Also, the roles of x and y change for f and f^{-1}.

4. Sketch $f(x) = \log_3 (x - 2)$ and $g(x) = 3^x + 2$ on the same coordinate system. How are these two functions related?

Answer:

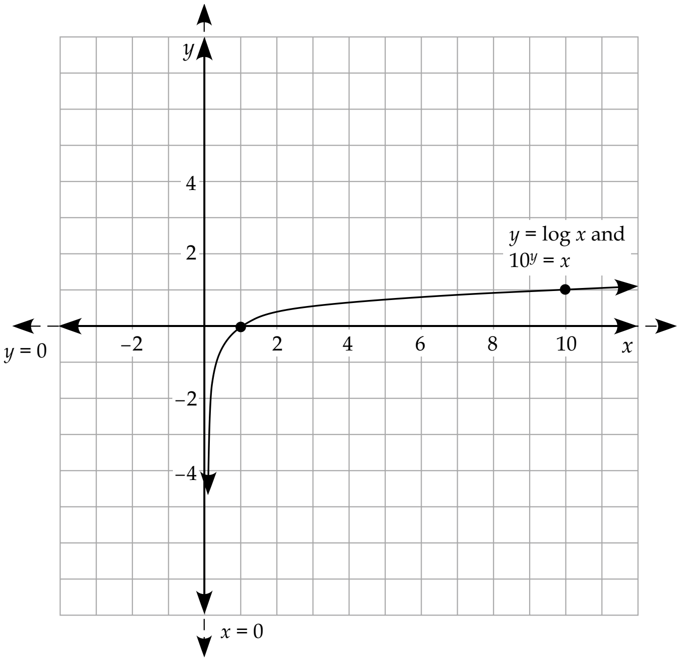


The curves are a reflection of each other, in the line with equation $y = x$, which means they are inverses of each other. The inverse of $y = \log_3 (x - 2)$ is

$$x = \log_3 (y - 2)$$
$$3^x = y - 2$$
$$3^x + 2 = y$$

5. Sketch $y = \log x$ and $10^y = x$ on the same coordinate system. How are these two graphs related?

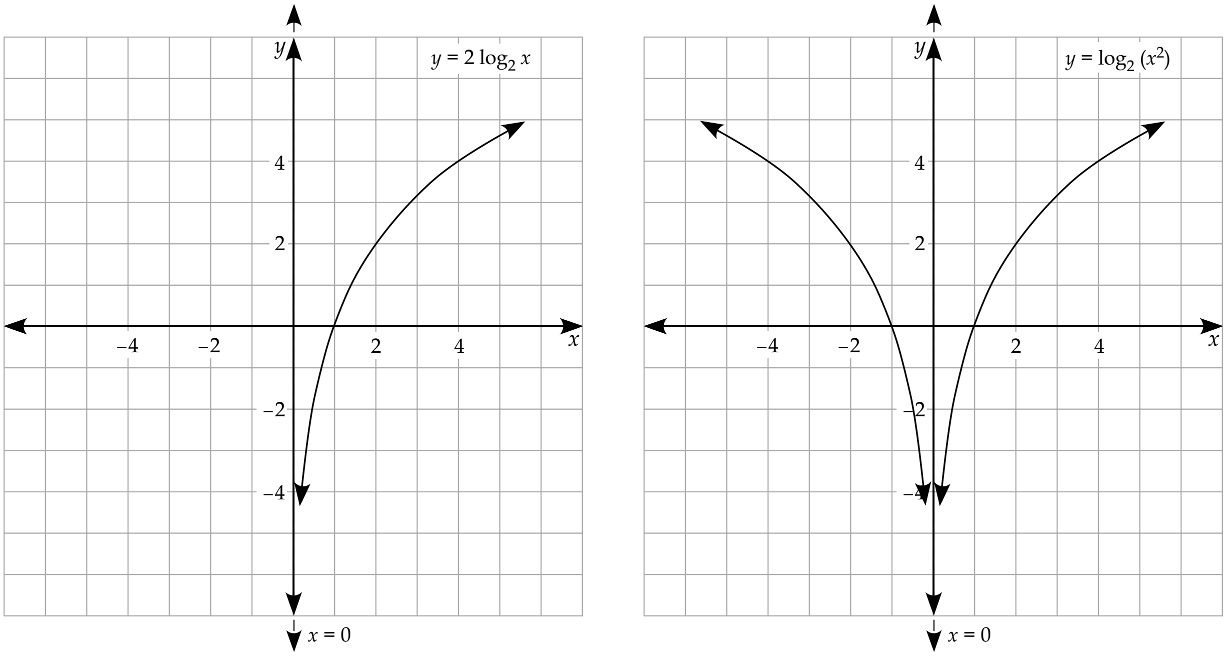
Answer:



They are exactly the same curve because $y = \log x$ and $10^y = x$ are the same statement, except the former is written in log form while the latter is written in exponential form.

6. Compare and contrast the graphs of $y = 2 \log_2 x$ and $y = \log_2 (x^2)$. Is there any consistency with or contradiction of the log Power Theorem? Explain your answer.

   Answer:



By carefully defining a function, a mathematician is able to indicate the desired domain.

Consider $y = 2 \log_2 x$. Its domain is $(0, \infty)$. Its graph is created by doubling the y-values of $y = \log_2 x$ and, as such, is only in Quadrants I and IV, as shown in the diagram above.

Now consider $y = \log_2 (x^2)$. Its domain is $\{x \mid x \neq 0\}$. Its graph can be drawn where $x < 0$. It is an even function. The part of its graph in Quadrants II and III is a reflection in the y-axis of the part of the graph in Quadrants I and IV. Its equation could be written as $y = 2 \log_2 |x|$. Thus, for $x > 0$, the graphs are the same. However, for $x < 0$, only $y = \log_2 (x^2)$ has a graph.

This is not a contradiction of the Power Theorem, since the theorem only holds on the domain for which a function is defined. In conclusion, the two functions are not the same because of the way they are defined: $y = \log_2 (x^2)$ becomes $y = 2 \log_2 |x|$ and is defined as long as $x \neq 0$, whereas $y = 2 \log_2 x$ is defined only for $x > 0$.

Learning Activity 7.5

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Determine the inverse function if $f(x) = 2x + 5$.
2. Simplify: $\cos\theta \sec\theta$
3. Convert 180° to radians.
4. State the non-permissible values of the function $f(x) = \dfrac{x}{x^2 - 3x - 4}$.
5. Express log 7 + log 8 as one logarithm.
6. Which is the better deal, three dress shirts for $65 or two dress shirts for $41?
7. What is the length of the remaining leg of a right-angled triangle if the hypotenuse measures 15 m and one leg measures 9 m?
8. Simplify: $\sqrt{72x^3y^7}$

Answers:

1. $y = \dfrac{x-5}{2}$ $\left(x = 2y + 5;\ x - 5 = 2y,\ y = \dfrac{x-5}{2}\right)$
2. 1 $\left(\cos\theta\ \sec\theta\ = \cos\theta\left(\dfrac{1}{\cos\theta}\right) = 1\right)$
3. π
4. $x \neq 4$ and $x \neq -1$ $(x^2 - 3x - 4 = (x - 4)(x + 1))$
5. log 56 (log 7 + log 8 = log (7 · 8) = log 56)
6. Two dress shirts for $41 (3 shirts for $65 ⇒ one shirt is slightly more than $21; 2 shirts for $41 ⇒ one shirt is less than $21)
7. 12 m $(15^2 = 9^2 + x^2;\ 225 = 81 + x^2;\ 144 = x^2;\ 12 = x)$
8. $6xy^3\sqrt{2xy}$ $\left(\sqrt{72x^3y^7} = \sqrt{36}\sqrt{2}\sqrt{x^2}\sqrt{x}\sqrt{y^6}\sqrt{y} = 6xy^3\sqrt{2xy}\right)$

Part B: Solving Exponential Equations

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Solve the following exponential equations without using a calculator.

a) $8\left(4^x\right) = 32$

Answer:

$$2^3\left(2^2\right)^x = 2^5$$
$$2^{3+2x} = 2^5$$
$$3 + 2x = 5$$
$$2x = 2$$
$$x = 1$$

b) $3^{4x-1} = 27^{2x}$

Answer:

$$3^{4x-1} = \left(3^3\right)^{2x}$$
$$3^{4x-1} = 3^{6x}$$
$$4x - 1 = 6x$$
$$-2x = 1$$
$$x = -\frac{1}{2}$$

c) $25\left(125^{3-x}\right) = 5^{x+2}$

Answer:

$$5^2\left(5^3\right)^{3-x} = 5^{x+2}$$
$$5^2\left(5^{9-3x}\right) = 5^{x+2}$$
$$5^{11-3x} = 5^{x+2}$$
$$11 - 3x = x + 2$$
$$-4x = -9$$
$$x = \frac{9}{4}$$

d) $4^{x+1} = 2^x\sqrt{2}$

Answer:

$$\left(2^2\right)^{x+1} = 2^x 2^{\frac{1}{2}}$$
$$2^{2x+2} = 2^{x+\frac{1}{2}}$$
$$2x + 2 = x + \frac{1}{2}$$
$$x = \frac{1}{2} - 2$$
$$x = -\frac{3}{2}$$

e) $(-8)^{\frac{5}{3}} = 2\left(4^{\frac{x}{2}}\right)$

Answer:

$$\left((-2)^3\right)^{\frac{5}{3}} = 2\left(2^2\right)^{\frac{x}{2}}$$

$$(-2)^5 = 2\left(2^x\right)$$

$$(-2)^5 = 2^{x+1}$$

Impossible, since a negative cannot equal a positive.

f) $4^{6x} = \frac{1}{64}$

Answer:

$$4^{6x} = 4^{-3}$$

$$6x = -3$$

$$x = -\frac{1}{2}$$

2. Find these logarithms without using a calculator.

a) $\log_5 25$

Answer:

$$x = \log_5 25$$

$$5^x = 25$$

$$5^x = 5^2$$

$$x = 2$$

b) $\log_2 1024$

Answer:

$$x = \log_2 1024$$

$$2^x = 1024$$

$$2^x = 2^{10}$$

$$x = 10$$

c) $\log_3 \frac{1}{27}$

Answer:

$$x = \log_3 \frac{1}{27}$$

$$3^x = \frac{1}{27}$$

$$3^x = 27^{-1}$$

$$3^x = \left(3^3\right)^{-1}$$

$$3^x = 3^{-3}$$

$$x = -3$$

d) $\log_7 49$

Answer:

$$x = \log_7 49$$

$$7^x = 49$$

$$7^x = 7^2$$

$$x = 2$$

e) $\log_4 \dfrac{1}{64}$

Answer:

$$x = \log_4 \frac{1}{64}$$

$$4^x = 64^{-1}$$

$$4^x = \left(4^3\right)^{-1}$$

$$4^x = 4^{-3}$$

$$x = -3$$

f) $\log_2 8$

Answer:

$$x = \log_2 8$$

$$2^x = 8$$

$$2^x = 2^3$$

$$x = 3$$

g) $\log_3 \sqrt[3]{9}$

Answer:

$$x = \log_3 \sqrt[3]{9}$$

$$3^x = 9^{\frac{1}{3}}$$

$$3^x = (3^2)^{\frac{1}{3}}$$

$$3^x = 3^{\frac{2}{3}}$$

$$x = \frac{2}{3}$$

h) $\log_{\sqrt{7}} 49$

Answer:

$$x = \log_{\sqrt{7}} 49$$

$$\left(\sqrt{7}\right)^x = 49$$

$$\left(7^{\frac{1}{2}}\right)^x = 7^2$$

$$7^{\frac{1}{2}x} = 7^2$$

$$\frac{1}{2}x = 2$$

$$x = 4$$

Note: You may not need to show all these steps when evaluating logarithms, but they are provided so you can follow the reasoning.

3. Use your calculator to find each logarithm.

a) log 15
b) log 1000
c) ln 7
d) ln 4
e) log (2.589)
f) log (−4)
g) ln (0.527)
h) ln 250
i) $\log\left(\dfrac{13}{15}\right)$
j) log (0.5)
k) ln *e*
l) ln 8

Answers:

a) 1.17609
b) 3
c) 1.94591
d) 1.38629
e) 0.41313
f) Impossible, domain is $x > 0$; the argument of a log must be positive.
g) −0.64055
h) 5.52146
i) −0.06215
j) −0.30103
k) 1 (since $e^1 = e$)
l) 2.07944

4. Use your calculator to find each logarithm.

a) $\log_5 10$

Answer:

$$\begin{aligned} \log_5 10 &= x \\ 5^x &= 10 \\ \log 5^x &= \log 10 \\ x \log 5 &= \log 10 \\ x &= \frac{\log 10}{\log 5} \\ &= 1.43068 \end{aligned}$$

b) $\log_3 20$

Answer:

You can use Change of Base Theorem to shorten the work needed.

$$\begin{aligned} x &= \frac{\log 20}{\log 3} \\ &= 2.72683 \end{aligned}$$

c) $\log_2 e$

Answer:

$$\log_2 e = x$$
$$2^x = e$$
$$\ln 2^x = \ln e$$
$$x \ln 2 = \ln e$$
$$x = \frac{\ln e}{\ln 2}$$
$$x = 1.44270$$

d) $\log_7 e$

Answer:

You can use Change of Base Theorem with log base e to shorten the work.

$$x = \frac{\ln e}{\ln 7}$$
$$x = \frac{1}{1.9459}$$
$$x = 0.51390$$

e) $\log_{100} 1000$

Answer:

$$\log_{100} 1000 = x$$
$$100^x = 1000$$
$$\left(10^2\right)^x = 10^3$$
$$2x = 3$$
$$x = \frac{3}{2}$$

f) $\log_9 81$

Answer:

Let $x = \log_9 81$

$$9^x = 81$$
$$9^x = 9^2$$
$$x = 2$$

g) $\log_2 0.40$

Answer:

$$\log_2 0.40 = x$$
$$2^x = 0.40$$
$$\log 2^x = \log 0.40$$
$$x \log 2 = \log 0.40$$
$$x = \frac{\log 0.40}{\log 2}$$
$$x = -1.32193$$

h) $\log_3 15$

Answer:

Again, you can use Change of Base Theorem as an alternative method.

$$x = \frac{\log 15}{\log 3}$$
$$x = 2.46497$$

5. Solve the following exponential equations. State your answer to two decimal places.

a) $2\left(3^x\right) = 75$

Answer:

$$\log 2\left(3^x\right) = \log 75$$

$$\log 2 + \log 3^x = \log 75$$

Use Product Rule first.

$$x \log 3 = \log 75 - \log 2$$

Use Power Rule and isolate *x*.

$$x = \frac{\log 75 - \log 2}{\log 3}$$

$$x = 3.30$$

b) $2\left(3^x\right) = 5^{x-1}$

Answer:

$$\log 2\left(3^x\right) = \log 5^{x-1}$$

$$\log 2 + x\log 3 = (x-1)\log 5$$

$$\log 2 + x \log 3 = x \log 5 - \log 5$$

Use Product Rule and then Power Rule.

$$x\log 3 - x\log 5 = -\log 5 - \log 2$$

Isolate *x*-terms to one side.

$$x\left(\log 3 - \log 5\right) = -\log 5 - \log 2$$

Factor out common *x*.

$$x = \frac{-\log 5 - \log 2}{\log 3 - \log 5}$$

$$x = 4.51$$

c) $e^x = 2^{x+1}$

Answer:

$$x \ln e = (x+1)\ln 2$$

$$x(1) = x \ln 2 + \ln 2$$

$$x - x \ln 2 = \ln 2$$

$$x\left(1 - \ln 2\right) = \ln 2$$

$$x = \frac{\ln 2}{1 - \ln 2}$$

$$x = 2.26$$

Note: Whenever *e* is included in the exponential equation, it is usually better to use ln *x* rather than log *x* because you can simplify knowing ln *e* = 1.

d) $19^{x-5} = 3^{x+2}$

Answer:

$$(x-5)\log 19 = (x+2)\log 3$$

Take log of both sides, then use Power Rule.

$$x\log 19 - 5\log 19 = x\log 3 + 2\log 3$$

$$x\log 19 - x\log 3 = 2\log 3 + 5\log 19$$

Isolate x-terms to one side.

$$x(\log 19 - \log 3) = 2\log 3 + 5\log 19$$

Factor out common x.

$$x = \frac{2\log 3 + 5\log 19}{\log 19 - \log 3}$$

$$x = 9.17$$

e) $e^{2x-5} = 25$

Answer:

$$(2x-5)\ln e = \ln 25$$

Take log of both sides, then use Power Rule.

$$(2x-5)(1) = \ln 25$$

$$2x = \ln 25 + 5$$

$$x = \frac{\ln 25 + 5}{2}$$

$$x = 4.11$$

f) $6^{3x} = 2^{2x-3}$

Answer:

$$3x\log 6 = (2x-3)\log 2$$

$$3x\log 6 = 2x\log 2 - 3\log 2$$

$$3x\log 6 - 2x\log 2 = -3\log 2$$

$$x(3\log 6 - 2\log 2) = -3\log 2$$

$$x = \frac{-3\log 2}{3\log 6 - 2\log 2}$$

$$x = -0.52$$

g) $5e^{x-1} = 6^x$

Answer:

$$\ln 5 + \ln e^{x-1} = \ln 6^x$$

Take log of both sides, then use Product Rule.

$$\ln 5 + (x-1)\ln e = x \ln 6$$

Use Power Rule.

$$\ln 5 + (x-1)(1) = x \ln 6$$

$$\ln 5 + x - 1 = x \ln 6$$

$$x - x \ln 6 = 1 - \ln 5$$

$$x(1 - \ln 6) = 1 - \ln 5$$

$$x = \frac{1 - \ln 5}{1 - \ln 6}$$

$$x = 0.77$$

h) $(1.05)^x = 3$

Answer:

$$x \log 1.05 = \log 3$$

$$x = \frac{\log 3}{\log 1.05}$$

$$x = 22.52$$

i) $5(15)^x = 3^{x+1}$

Answer:

$$\log 5 + \log 15^x = \log 3^{x+1}$$

Take log of both sides, then use Product Rule.

$$\log 5 + x \log 15 = (x+1)\log 3$$

Use Power Rule.

$$\log 5 + x \log 15 = x \log 3 + \log 3$$

$$x \log 15 - x \log 3 = \log 3 - \log 5$$

$$x(\log 15 - \log 3) = \log 3 - \log 5$$

$$x = \frac{\log 3 - \log 5}{\log 15 - \log 3}$$

$$x = -0.32$$

6. Find the x-intercepts of each of the following exponential functions.

Note: To find the x-intercepts of each exponential function, set $y = 0$ and solve the corresponding exponential equation.

a) $y = 3^x - 4$

Answer:

$$y = 3^x - 4$$

$$0 = 3^x - 4$$

$$4 = 3^x$$

$$\log 4 = \log \left(3^x\right)$$

$$\log 4 = x \log 3$$

$$x = \frac{\log 4}{\log 3}$$

$$x = 1.262$$

b) $y = 2^x - 1$

Answer:

$$y = 2^x - 1$$

$$0 = 2^x - 1$$

$$1 = 2^x$$

$$x = 0$$

c) $y = 4^x + 2$

Answer:

$$y = 4^x + 2$$

$$0 = 4^x + 2$$

$$-2 = 4^x$$

There is no solution to this equation. This exponential function is entirely above the x-axis, so there are no x-intercepts.

d) $y = -2^x + 3$

Answer:

$$y = -2^x + 3$$

$$0 = -2^x + 3$$

$$-3 = -2^x$$

$$3 = 2^x$$

$$\log 3 = \log \left(2^x\right)$$

$$\log 3 = x \log 2$$

$$x = \frac{\log 3}{\log 2}$$

$$x = 1.585$$

Learning Activity 7.6

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. What is the exact value of sin 180°?
2. Find the positive coterminal angle for 427° in the interval $0° \leq \theta \leq 360°$.
3. Express $\frac{1}{2}\log 4$ as one logarithm.
4. What is the area of a circular rug that has a diameter of 15 feet?
5. Rationalize the denominator: $\frac{2}{5\sqrt{3}}$
6. Simplify: $\frac{8x^{-3}y^{-2}}{2x^2y^{-3}}$
7. Simplify: $\sqrt{80} + \sqrt{45}$
8. Factor: $8a^2b^5c^3 + 2a^2bc^3$

Answers:

1. 0
2. 67° ($427° - 360° = 67°$)
3. $\log 2 \left(\frac{1}{2}\log 4 = \log 4^{\frac{1}{2}} = \log 2\right)$
4. $A = \frac{225}{4}\pi$ feet $\left(A = \pi r^2 = \pi\left(\frac{15}{2}\right)^2 = \frac{225}{4}\pi\right)$
5. $\frac{2\sqrt{3}}{15}$ $\left(\frac{2}{5\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{15}\right)$
6. $\frac{4y}{x^5}$
7. $7\sqrt{5}$ $\left(4\sqrt{5} + 3\sqrt{5} = 7\sqrt{5}\right)$
8. $2a^2bc^3(4b^4 + 1)$

Part B: Solving Logarithmic Equations

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Solve the following logarithmic equations. Remember to check for extraneous roots!

   a) $\log_2(x-4)+\log_2(x-3)=1$

   Answer:

   $\log_2[(x-4)(x-3)]=1$

   $(x-4)(x-3)=2^1$ Convert to exponential form.

   $x^2-3x-4x+12=2$

   $x^2-7x+10=0$

   $(x-2)(x-5)=0$

   $x=2$ or $x=5$

   Check: $x = 2$

   Substitute into original equation.

   $\log_2(2-4)+\log_2(2-3)=1$

   $\log_2(-1)+\log_2(-1)=1$

   ∴ Negative arguments.

   ∴ Discard $x = 2$.

   Check: $x = 5$

   Substitute into original equation.

   $\log_2(5-4)+\log_2(5-3)=1$

   $\log_2(1)+\log_2(2)=1$

   ∴ Arguments are positive.

   ∴ Keep $x = 5$.

   ∴ Solution is $x = 5$.

b) $\log_2 x + \log_2 (x - 2) = \log_2 (9 - 2x)$

Answer:

$$\log_2 [x(x - 2)] = \log_2 (9 - 2x)$$

$$x(x - 2) = (9 - 2x)$$ Equate arguments.

$$x^2 - 2x = 9 - 2x$$

$$x^2 = 9, \text{ so}$$

$$x = \pm 3$$

Discard −3 since $\log_2 (-3)$ is not defined; keep 3.

∴ Solution is $x = 3$.

c) $\log_4 (x + 2) + \log_4 (2x - 3) = 1$

Answer:

$$\log_4 [(x + 2)(2x - 3)] = 1$$

$$(x + 2)(2x - 3) = 4^1$$ Convert to exponential form.

$$2x^2 - 3x + 4x - 6 = 4$$

$$2x^2 + x - 10 = 0$$

$$(2x + 5)(x - 2) = 0$$

$$x = -\frac{5}{2} \text{ or } x = 2$$

Discard $-\frac{5}{2}$.

∴ Solution is $x = 2$.

d) $2\log_2 x - \log_2 (x-1) = 2$

Answer:

$$\log_2 x^2 - \log_2 (x-1) = 2$$

Power Rule.

$$\log_2 \frac{x^2}{x-1} = 2$$

Quotient Rule.

$$\frac{x^2}{x-1} = 2^2$$

Convert to exponential form.

$$\frac{x^2}{x-1} = 4$$

$$x^2 = 4(x-1)$$

$$x^2 = 4x - 4$$

$$x^2 - 4x + 4 = 0$$

$$(x-2)(x-2) = 0$$

$$(x-2)^2 = 0$$

$$x = 2$$

Upon checking, $x = 2$, we have the solution, $x = 2$.

e) $\log_{\frac{1}{7}} x + \log_{\frac{1}{7}} (5x-28) = -2$

Answer:

$$\log_{\frac{1}{7}} \left[x(5x-28)\right] = -2$$

Product Rule.

$$5x^2 - 28x = \left(\frac{1}{7}\right)^{-2}$$

Convert to exponential form.

$$5x^2 - 28x = 49$$

$$5x^2 - 28x - 49 = 0$$

$$(5x+7)(x-7) = 0$$

$$x = -\frac{7}{5} \text{ or } x = 7$$

Discard $-\frac{7}{5}$ since $\log_{\frac{1}{7}}\left(-\frac{7}{5}\right)$ is not defined.

$\therefore$ Solution is $x = 7$.

f) $\log_2 (2 - x) + \log_2 (-x) = \log_2 3$

Answer:

$\log_2 [(2 - x)(-x)] = \log_2 3$ Product Rule.

$(2 - x)(-x) = 3$ Equate arguments.

$x^2 - 2x = 3$

$x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$

$x = 3$ or $x = -1$

Discard 3 since log (2 − 3) = log (−1) is not defined.

∴ Solution is $x = -1$.

g) $\log_2 (x - 1) + \log_2 (x + 2) = 2$

Answer:

$\log_2 [(x - 1)(x + 2)] = 2$

$x^2 + x - 2 = 2^2$

$x^2 + x - 6 = 0$

$(x + 3)(x - 2) = 0$

$x = -3$ or $x = 2$

Discard −3.

∴ Solution is $x = 2$.

h) $\log_3 x = 3 + \log_3 (x + 6)$

Answer:

$$\begin{aligned}
\log_3 x - \log_3 (x + 6) &= 3 \\
\log_3 \frac{x}{x+6} &= 3 \\
\frac{x}{x+6} &= 3^3 \\
\frac{x}{x+6} &= 27 \\
x &= 27(x + 6) \\
x &= 27x + 162 \\
-26x &= 162 \\
x &= -\frac{162}{26} = -\frac{81}{13}
\end{aligned}$$

Discard $-\frac{81}{13}$.

There is no solution.

2. Solve for x. Round to 3 decimal places, if necessary.

a) $e^{-0.01x} = 27$

Answer:

$$\begin{aligned}
-0.01x \ln e &= \ln 27 \\
-0.01x(1) &= 3.29584 \\
-0.01x &= 3.29584 \\
x &= \frac{3.29584}{-0.01} \\
x &= -329.584
\end{aligned}$$

b) $e^{\ln(1-x)} = 2x$

Answer:

$$\begin{aligned}
\ln\left(e^{\ln(1-x)}\right) &= \ln 2x \\
\ln(1-x)(\ln e) &= \ln(2x) \\
\ln(1-x)(1) &= \ln(2x) \\
\ln(1-x) &= \ln(2x) \\
1-x &= 2x \\
-3x &= -1 \\
x &= \frac{1}{3}
\end{aligned}$$

Note: We started with the question $e^{\ln(1-x)} = 2x$, and on the fifth line of the solution, we wrote $1 - x = 2x$. Note that these expressions are part of the original question. We could save steps by realizing that e^x and $\ln x$ are inverse operations. In other words,

$$e^{\ln A} = A$$

in all situations where $\ln A$ is defined.

A simpler solution for 3(b) becomes

$$\begin{aligned}
e^{\ln(1-x)} &= 2x \\
1-x &= 2x \\
-3x &= -1 \\
x &= \frac{1}{3}
\end{aligned}$$

c) $\ln\left(e^{\sqrt{x+1}}\right) = 3$

Answer:

$e^{\sqrt{x+1}} = e^3$ Change to exponential form.

$\sqrt{x+1} = 3$ Square both sides.

$x + 1 = 9$

$x = 8,$ and is a solution.

We need to check for an extraneous root because we have a radical equation.

Note: This time we started with $e^{\sqrt{x+1}} = 3$ and eventually wrote in the solution $\sqrt{x+1} = 3$. Again, because of $\ln x$ and e^x being inverse operations, we could save steps by realizing that the following rule applies:

$$\ln e^A = A$$

A simpler solution for 3(c) becomes:

$$\ln\left(e^{\sqrt{x+1}}\right) = 3$$

$$\sqrt{x+1} = 3$$

$$x + 1 = 9$$

$$x = 8$$

Check:

$$\ln e^{\sqrt{8+1}} = 3$$

$$\sqrt{8+1} = 3$$

$$\sqrt{9} = 3$$

$$3 = 3$$

$\therefore$ The solution is $x = 8$.

d) $e^{2x-1} = 5$

Answer:

$$(2x-1)\ln e = \ln 5$$

$$(2x-1)(1) = \ln 5$$

$$2x = \ln 5 + 1$$

$$x = \frac{\ln 5 + 1}{2}$$

$$x = 1.305$$

3. Solve for x. Round to 2 decimal places, if necessary.

   a) $\log_3 (x + 2) - 1 = \log_3 x$

   Answer:

$$\log_3 (x + 2) - \log_3 x = 1$$

$$\log_3 \frac{x + 2}{x} = 1$$

$$\frac{x + 2}{x} = 3^1$$

$$x + 2 = 3x$$

$$-2x = -2$$

$$x = 1, \text{ which is a solution}$$

   b) $2^{x+3} = 7\left(3^{x-2}\right)$

   Answer:

$$\log 2^{x+3} = \log \left[7\left(3^{x-2}\right)\right]$$

$$\log 2^{x+3} = \log 7 + \log 3^{x-2}$$

$$(x + 3)\log 2 = \log 7 + (x - 2)\log 3$$

$$x \log 2 + 3 \log 2 = \log 7 + x \log 3 - 2 \log 3$$

$$x \log 2 - x \log 3 = \log 7 - 2 \log 3 - 3 \log 2$$

$$x(\log 2 - \log 3) = \log 7 - 2\log 3 - 3\log 2$$

$$x = \frac{\log 7 - 2\log 3 - 3\log 2}{\log 2 - \log 3}$$

$$x = 5.75, \text{ which is a valid solution}$$

c) $\log(x+3) - \log x = 1$

Answer:

$$\log \frac{x+3}{x} = 1$$

$$\frac{x+3}{x} = 10^1$$

$$x + 3 = 10x$$

$$3 = 9x$$

$$x = \frac{3}{9} = \frac{1}{3}, \text{ which is a valid solution}$$

d) $4^{2x} = \left(2^x\right)^{x-2}$

Answer:

$$\left(2^2\right)^{2x} = 2^{x^2-2x}$$

$$2^{4x} = 2^{x^2-2x}$$

$$4x = x^2 - 2x$$

$$x^2 - 6x = 0$$

$$x(x-6) = 0$$

$$x = 0 \text{ or } x = 6$$

Questions 4 to 11 may prove to be tricky, but they are not difficult. Good luck!

4. Simplify. (Inverse Operations Rules)

   a) $e^{\ln x}$

   Answer:

   Let $y = e^{\ln x}$ and solve for y.

   $$\begin{aligned} \ln\left(y\right) &= \ln\left(e^{\ln x}\right) \\ \ln y &= \ln\left(e^{\ln x}\right) \\ &= \ln x \ln e \\ &= \ln x(1) \\ &= \ln x \end{aligned}$$

   $\therefore y = x$ and $e^{\ln x} = x$

   b) $\ln\left(e^{x}\right)$

   Answer:

   $$\begin{aligned} \ln\left(e^{x}\right) &= x \ln e \\ &= x(1) \\ &= x \end{aligned}$$

   c) $\log\left(10^{x}\right)$

   Answer:

   $$\begin{aligned} \log\left(10^{x}\right) &= x \log 10 \\ &= x(1) \\ &= x \end{aligned}$$

   d) $10^{\log x}$

   Answer :

   Let $y = 10^{\log x}$ and solve for y.

   $$\begin{aligned} \log y &= \log\left(10^{\log x}\right) \\ &= \log x \log 10 \\ &= \log x(1) \\ &= \log x \end{aligned}$$

   $\therefore y = x$ and $10^{\log x} = x$.

   Note: In each part, the answer is x. What you have proven in this exercise is that logs and exponents are inverse operations. You might add these four rules to your resource sheet. For example, the rule corresponding to question 5(a) is $e^{\ln x} = x$.

5. Simplify.

   a) $3^{\log_3 4+\log_3 5}$

   Answer:

   Method 1:

   $3^{\log_3 [4(5)]} = 3^{\log_3 20} = 20$, since 3^x and $\log_3 x$ are inverse functions, or

   Method 2:

   let $y = 3^{\log_3 20}$

   $\log y = (\log_3 20)(\log 3)$ Log both sides, then use Power Rule.

   $\log y = \dfrac{\log 20}{\log 3} \cdot \log 3$ Change of Base Theorem.

   $\log y = \log 20$ Cancel the log 3s.

   $y = 20$ The arguments must be equal if the logs are equal.

   b) $e^{\ln 3 - \ln 2}$

   Answer:

   $e^{\ln \frac{3}{2}} = \dfrac{3}{2}$

   Same reasoning as in part (a).

6. Solve for x.

   a) $\log_2 (\log_3 x) = 2$

   Answer:

   Convert to exponential form twice: from the outside, in.

   $\log_3 x = 2^2$

   $x = 3^4$

   $x = 81$

   b) $\log_3 (\log_2 (x-2)) = 1$

   Answer:

   Convert to exponential form twice: from the outside, in.

   $\log_2 (x-2) = 3^1$

   $x - 2 = 2^3$

   $x - 2 = 8$

   $x = 10$

7. Solve for x.

a) $5^{\sin x} = \frac{1}{5}$

Answer:

$$5^{\sin x} = 5^{-1}$$

$$\sin x = -1$$

$$\left\{x \,\middle|\, x = \frac{3\pi}{2} + 2k\pi,\ k \in \mathrm{I}\right\} \text{ or}$$

$$\left\{x \,\middle|\, x = 270^{\circ} + k360^{\circ},\ k \in \mathrm{I}\right\}$$

b) $(\ln x)^{-1} = \ln x$

Answer:

$$\frac{1}{\ln x} = \ln x$$

$$1 = (\ln x)^2$$

$$\pm 1 = \ln x$$

$$1 = \ln x \text{ or } -1 = \ln x$$

$$e^1 = x \text{ or } e^{-1} = x$$

$$x = e \text{ or } x = e^{-1} = \frac{1}{e}$$

Check: $x = e$ and $x = \frac{1}{e}$ in the original equation. Both are acceptable.

∴ The solutions are $x = e$ and $\frac{1}{e}$.

8. Use the Change of Base formula to solve the following.

a) $\log_2 x + \log_4 x = 3$

Answer:

Change the second term using base 2.

$$\log_2 x + \frac{\log_2 x}{\log_2 4} = 3$$

$$\log_2 x + \frac{\log_2 x}{2} = 3$$

$$2\log_2 x + \log_2 x = 6$$

$$3\log_2 x = 6$$

$$\frac{3\log_2 x}{3} = \frac{6}{3}$$

$$\log_2 x = 2$$

$$x = 2^2 = 4$$

b) $\log_3 (x+1) + \log_{27} (x+1) = 4$

Answer:

Change the second term using base 3.

$$\log_3(x+1) + \frac{\log_3(x+1)}{\log_3 27} = 4$$

$$\log_3(x+1) + \frac{\log_3(x+1)}{3} = 4$$

$$3\log_3(x+1) + \log_3(x+1) = 12$$

$$4\log_3(x+1) = 12$$

$$\log_3(x+1) = 3$$

$$x+1 = 3^3$$

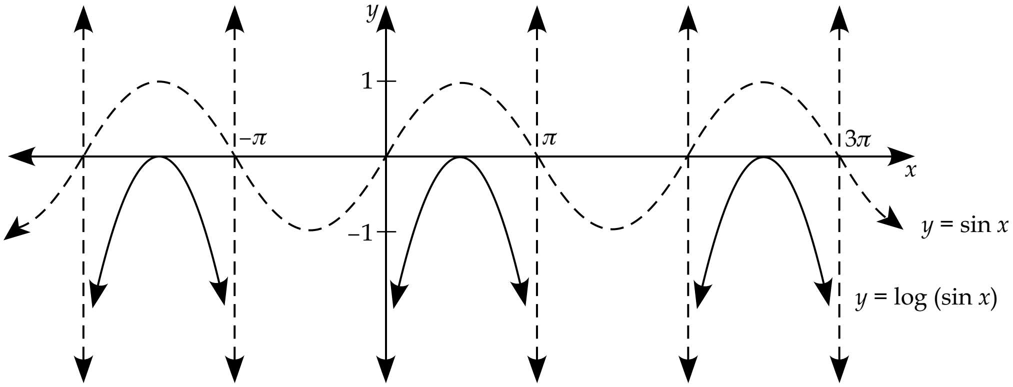
$$x+1 = 27$$

$$x = 26$$

9. Sketch the graph of $f(x) = \log(\sin x)$.

Using a graphing calculator is helpful but not necessary. Think about where $\sin x > 0$, and then think about the values of the function $\log x$, if x takes on the values between 0 and 1, and then between 1 and 0.

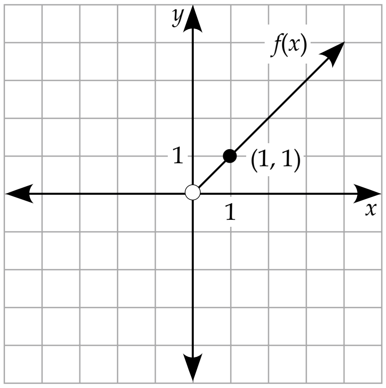
Answer:

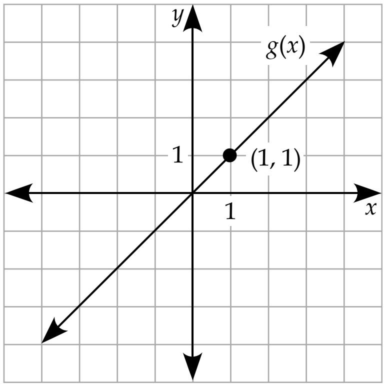


Note that $f(x)$ is undefined when $\sin x$ is negative.

10. Compare and contrast the graphs of $f(x) = e^{\ln x}$ and $g(x) = \ln(e^x)$.

Answer:





Both graphs have the same formula, $y = x$, except the domains are different. Since you can take logarithms of only positive numbers, it follows that the domain of f is $x > 0$, while the domain of g is $\mathfrak{R}$.

11. Solve for x: $\log(x^2) + (\log x)^2 = 3$.

Answer:

Be careful—the exponents play different roles! You are squaring $\log x$, not its argument.

$$2\log x + (\log x)^2 = 3$$

This is a quadratic form in $\log x$.

$$(\log x)^2 + 2\log x - 3 = 0$$

Think of $a^2 + 2a - 3 = 0$, where $a = \log x$.

$$(\log x - 1)(\log x + 3) = 0$$

$$\log x = 1 \text{ or } \log x = -3$$

Therefore, $x = 10^1$ or $x = 10^{-3}$

$$\therefore x = 10, \frac{1}{1000}$$

Learning Activity 7.7

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. What is the exact value of sin 30°?
2. In which quadrant is $\sin\theta$ positive and $\cos\theta$ negative?
3. Estimate the taxes, 13%, on a $327 item.
4. Evaluate: $\sqrt[4]{625}$
5. Simplify: $|-20(4-5)|$
6. Solve for x: $(x-2)^2 = 9$
7. Simplify: $\frac{3}{5}+\frac{3}{7}$
8. Factor: $64x^4 - 81y^8$

Answers:

1. $\frac{1}{2}$
2. Quadrant II
3. ≈ $42.00 (10% of 327 is 32.7; 1% is 3.27; 13% is 32.7 + 3.27 + 3.27 + 3.27 ≈ $42.00)
4. 5 (since $5^4 = 625$)
5. 20 ($|-20(4-5)| = |-20(-1)| = |20| = 20$)
6. $x = -1, 5$ ($(x-2)^2 = 9$; $x - 2 = \pm 3$; $x = -1$ or $x = 5$)
7. $1\frac{1}{35}$ $\left(\frac{3}{5}+\frac{3}{7}=\frac{21}{35}+\frac{15}{35}=\frac{36}{35}=1\frac{1}{35}\right)$
8. $(8x^2 - 9y^4)(8x^2 + 9y^4)$ [difference of squares pattern]

Part B: Exponential Growth Lab

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

In this learning activity you will be completing an Exponential Growth Lab. To complete this lab, you will need a pencil, and approximately 100 coins.

Modelling Exponential Growth

In this lab, you are going to model the spread of bacteria. You will conduct 10 trials and record how many new bacteria appear after each trial.

1. Place 2 coins in a cup (representing 2 bacteria initially).
2. Shake the cup and dump the coins out of the cup onto a flat surface.
3. For each coin that lands with the "heads" side facing up, add another coin, representing some of the bacteria replicating.
4. Repeat steps 2 and 3 until you have completed 10 trials.
5. Fill in the *Difference in Amount of Coins* column by subtracting the amount of coins of the previous trial from the current trial. For example, in trial 2, the difference will be:

   # of Coins in Trial 2 − # of Coins in Trial 1
6. Fill in the *Percent Change* column by using the formula:

$$\frac{\text{new amount of coins} - \text{old amount of coins}}{\text{old amount of coins}}$$

Note: This value should be close to $\frac{1}{2}$. When you drop a handful of coins, theoretically, half of them should land on heads and half should land on tails. Therefore, the percent change for each trial should be close to $\frac{1}{2}$ theoretically. However, your results will vary. Convert the fraction to a decimal.

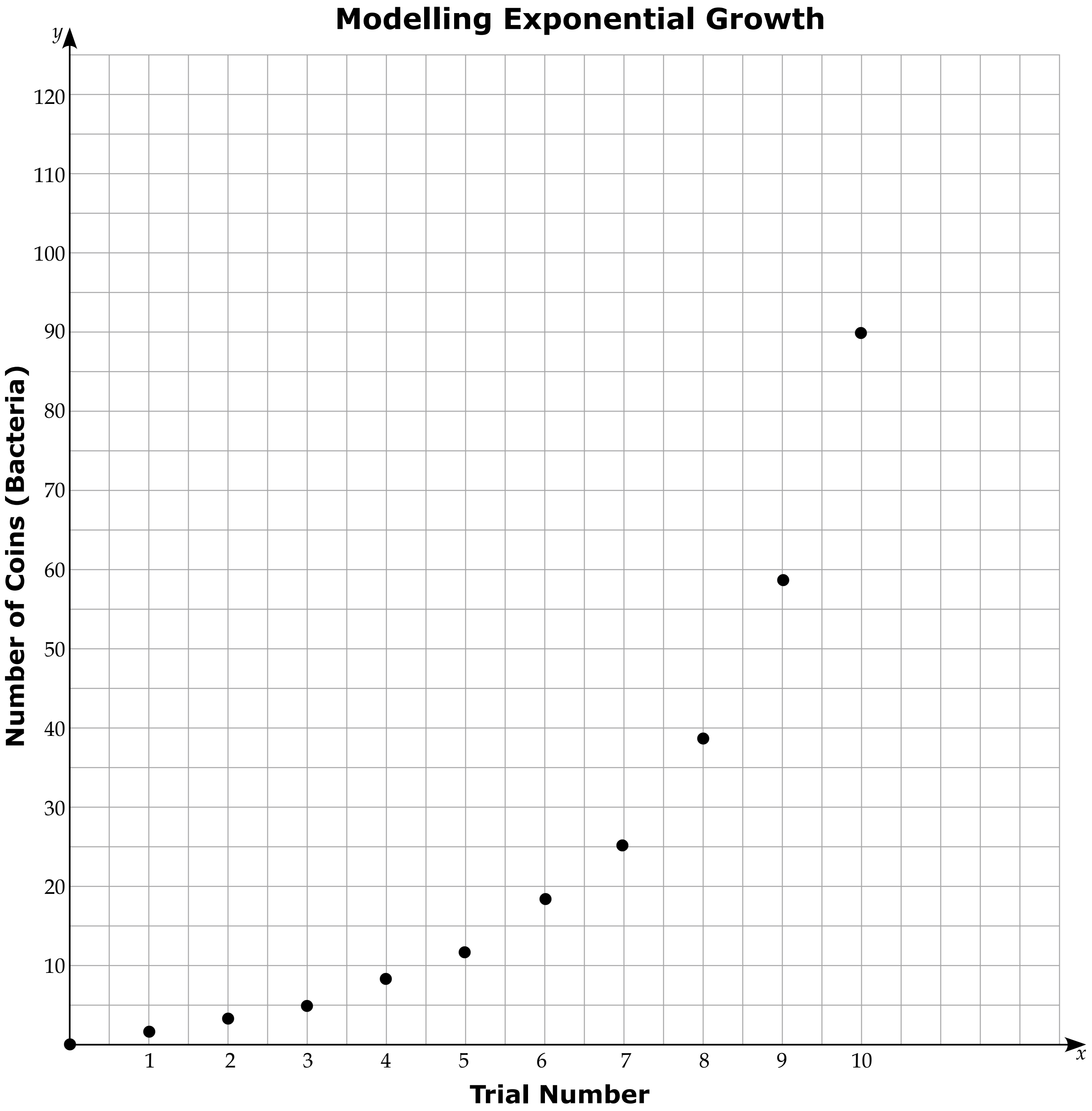
7. Calculate the average percent change by totalling the percent change column and then dividing by 9 (the number of percent change values).

Note: Your first trial begins with you having 2 coins.

Trial Number	# of Coins (Bacteria)	Difference in Amount of Coins	Percent Change
0	2		
1	3	1	$\frac{1}{2} = 0.5$
2	5	2	$\frac{2}{3} = 0.66667$
3	7	2	$\frac{2}{5} = 0.4$
4	11	3	$\frac{3}{7} = 0.42857$
5	17	6	$\frac{6}{11} = 0.54545$
6	25	8	$\frac{8}{17} = 0.47059$
7	39	14	$\frac{14}{25} = 0.56$
8	58	19	$\frac{19}{39} = 0.48718$
9	90	32	$\frac{32}{58} = 0.55172$
		Average Percent Change	0.512

Note: Your data will be different. This data will be used to complete the rest of the solutions. Use your data. Your answers will be different but you should get similar information.

8. Graph your data below. Label the y-axis with the title *Number of Coins (Bacteria)*.



9. Write an exponential growth function in the form $y = P(1 + r)^t$ that models your data.

   P = The initial amount of coins (bacteria)

   r = The rate of growth (use the average percent change)

   t = Time (**Note:** Time is the independent variable in this equation represented by the trial number.)

   The exponential growth function that models my data is ____________ .

   Answer:

   (This answer is for the data given. Your answer may be slightly different based on your data.)

   $y = 2(1 + 0.512)^t$ or $y = 2(1.512)^t$.

10. Use your exponential growth function model to predict how many bacteria there will be after

    a) 20 trials

       Answer:

       $y = 2(1.512)^{20}$

       $y = 7799$

    b) 50 trials

       Answer:

       $y = 2(1.512)^{50}$

       $y = 1899413647$

11. If you started this trial with more coins, would you expect the pattern to stay the same? Why or why not?

    Answer:

    Yes. The pattern should stay the same. The growth will occur slowly at the beginning and then progress to grow faster and faster.

Learning Activity 7.8

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Convert 45° to radians.
2. The square root of 200 is between which two whole numbers?
3. Write as an entire radical: $3x^2\sqrt{x}$
4. Evaluate: $50 - 1 \cdot 6$
5. Solve for x: $5 + 3x = 41$
6. Which two numbers have a product of -30 and a sum of 13?
7. Evaluate: $\frac{3}{8} + \frac{1}{3} - \frac{7}{24}$
8. Simplify: $\dfrac{\frac{5}{x}}{7}$

Answers:

1. $\frac{\pi}{4}$
2. 14 and 15 ($14^2 = 196$ and $15^2 = 225$)
3. $\sqrt{9x^5}$
4. 44 ($50 - 6 = 44$)
5. $x = 12$ ($5 + 3x = 41$; $3x = 36$; $x = 12$)
6. 15 and -2
7. $\frac{5}{12}$ $\left(\frac{3}{8} + \frac{1}{3} - \frac{7}{24} = \frac{9}{24} + \frac{8}{24} - \frac{7}{24} = \frac{10}{24} = \frac{5}{12}\right)$
8. $\frac{5}{7x}$ $\left(\dfrac{\frac{5}{x}}{7} = \frac{5}{x} \div 7 = \frac{5}{x} \cdot \frac{1}{7} = \frac{5}{7x}\right)$

Part B: Exponential and Logarithmic Applications

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. A $5000 investment earns interest at the rate of 8.4% compounded monthly.
   a) What is the investment worth after one year?

      Answer:

      $$A = 5000\left(1 + \frac{0.084}{12}\right)^{12}$$

      $$A = \$5436.55$$

   b) What is the investment worth after 10 years?

      Answer:

      $$A = 5000\left(1 + \frac{0.084}{12}\right)^{12(10)}$$

      $$A = \$11{,}547.99$$

   c) How much interest was earned in 10 years?

      Answer:

      The interest earned is $11,547.99 – $5,000.00 = $6547.99.

2. What sum of money must be invested now so that $20,000 is available in five years, if the rate is 8.4% compounded monthly?

   Answer:

   Use the following formula:

   $$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

   $$20\ 000 = P\left(1 + \frac{0.084}{12}\right)^{12(5)}$$

   $$P = \frac{20\ 000}{(1.007)^{60}} = \$13{,}160.18$$

3. What interest rate must be charged so that money doubles itself in 10 years if the rate is compounded

   a) annually?

   Answer:

$$2x = x(1+r)^{10}$$

$$2 = (1+r)^{10}$$

$$\sqrt[10]{2} = 1 + r$$

$$r = 0.07177$$

$$r \approx 7.18\%$$

   b) semi-annually?

   Answer:

$$2x = x\left(1 + \frac{r}{2}\right)^{20}$$

$$2 = \left(1 + \frac{r}{2}\right)^{20}$$

$$\sqrt[20]{2} = 1 + \frac{r}{2}$$

$$2.07053 = 2 + r$$

$$r = 0.07053 \approx 7\%$$

   c) monthly?

   Answer:

$$2x = x\left(1 + \frac{r}{12}\right)^{120}$$

$$2 = \left(1 + \frac{r}{12}\right)^{120}$$

$$\sqrt[120]{2} = 1 + \frac{r}{12}$$

$$r = 0.0695 \approx 6.95\%$$

   d) weekly?

   Answer:

$$2x = x\left(1 + \frac{r}{52}\right)^{520}$$

$$2 = \left(1 + \frac{r}{52}\right)^{520}$$

$$\sqrt[520]{2} = 1 + \frac{r}{52}$$

$$r = 0.06936 \approx 6.94\%$$

4. With which plan would an investor earn more?

   Plan A: 8% compounded annually.

   Plan B: 7.5% compounded daily.

   Answer:

$$A = 1(1 + 0.08)^1$$

$$A = \$1.08$$

$$B = 1\left(1 + \frac{0.075}{365}\right)^{365}$$

$$B = \$1.0779$$

   They are almost equal, but Plan A is a little better.

   Note: An initial principal amount could be any number but 1 was used for simplicity. Also, the time frame could be any number but 1 year was used for simplicity.

5. The population of gophers in a particular field is modelled by the equation $P = 100(1.1)^t$, where t is measured in years.

   a) How many gophers will there be after 20 years?

      Answer:

      After 20 years there will be $100(1.1)^{20} \approx 673$ gophers.

   b) How long will it take for the gopher population to triple?

      Answer:

$$
\begin{aligned}
300 &= 100(1.1)^n \\
3 &= (1.1)^n \\
\log 3 &= n \log 1.1 \\
n &= \frac{\log 3}{\log 1.1} = 11.5 \text{ years}
\end{aligned}
$$

6. A radioactive substance is decaying according to the formula $y = y_0e^{kx}$, where x is measured in years. The initial amount is 10 grams, and eight grams remain after five years.

   a) Find the value of k. Round to 5 decimal places.

      Answer:

$$
\begin{aligned}
8 &= 10e^{k(5)} \\
\frac{8}{10} &= \frac{10e^{5k}}{10} \\
0.8 &= e^{5k} \\
\ln 0.8 &= 5k \ln e \\
k &= \frac{\ln 0.8}{5} = -0.04463
\end{aligned}
$$

   b) Estimate the amount remaining after 10 years.

      Answer:

      This is called an estimate because the k-value is approximate and other factors make it hard to predict the actual value in 10 years.

$$y = 10e^{-0.04463(10)}$$

$$y = 10(0.63999) \qquad [\text{use the } e^x \text{ button}]$$

$$y = 6.4 \text{ grams}$$

c) Find the half-life to the nearest tenth of a year.

Answer:

$$\frac{1}{2}a = ae^{-0.04463x} \qquad \left[\text{since } a \text{ must decrease to } \frac{1}{2}a\right]$$

$$\frac{1}{2} = e^{-0.04463x}$$

$$\ln(0.5) = -0.04463x(\ln e) \qquad [\text{but } \ln e = 1]$$

$$x = \frac{\ln(0.5)}{-0.04463} = 15.53 \text{ years}$$

The half-life is approximately 15.5 years.

7. When the population growth of a city was first measured, the population was 22 000. It was found that the population, P, grew by the formula $P = 22\,000(10^{0.0163t})$. If t is measured in years, how long will it take for the city to double its population?

Answer:

$$44\,000 = 22\,000\left(10^{0.0163t}\right)$$

$$2 = 10^{0.0163t}$$

$$\log 2 = 0.0163t(\log 10) \qquad [\text{but } \log 10 = 1]$$

$$t = \frac{\log 2}{0.0163} = 18.468 \text{ years}$$

8. The pH of a substance is defined by pH = −log [H+], where [H+] is the hydrogen ion concentration in moles/L. If the pH of lemon juice is 2.3, find its ion concentration.

Answer:

$$\text{pH} = -\log[\text{H}^+]$$

$$2.3 = -\log[\text{H}^+]$$

$$-2.3 = \log[\text{H}^+]$$

$$10^{-2.3} = [\text{H}^+]$$

$$[\text{H}^+] = 0.00501 \text{ moles/L}$$

9. Use your knowledge of logarithms to answer the following questions.

   a) How many times more energy is contained within an earthquake that is rated an 8 on the Richter scale than an earthquake that is rated a 3 on the Richter scale?

      Answer:

      An earthquake with a rating of 8 on the Richter scale and an earthquake with a rating of 3 on the Richter scale are 5 points away. Therefore, the earthquake with a rating of 8 contains 10^5 or 100 000 times more energy than an earthquake with a rating of 3.

   b) If detergent has a pH level of 10, how many times more acidic is stomach acid that has a pH level of 2?

      Answer:

      A pH level of 10 and a pH level of 2 are 8 points away on the pH scale. Therefore, stomach acid is $10^8 = 100\,000\,000$ times more acidic than detergent.

10. If the lights are left on when a car is parked, the battery discharges and the voltage, V, of the battery is given by $V = V_0e^{-kt}$, where t is the time in minutes. If the original voltage, V_0, was 12 volts and $k = 0.01$, find how long it takes to reduce the voltage to nine volts.

    Answer:

$$9 = 12e^{-(0.01)t}$$

$$\frac{9}{12} = \frac{12e^{-(0.01)t}}{12}$$

$$0.75 = e^{-(0.01)t}$$

$$\ln 0.75 = -0.01t(\ln e) \qquad [\text{but } \ln e = 1]$$

$$t = \frac{\ln 0.75}{-0.01} = 28.77 \text{ minutes}$$

11. The atmospheric pressure, P, at height h kilometres above sea level is given by $P = P_0e^{-kh}$. The pressure at sea level, P_0, is 101.3 kPa. (**Note:** kPa is the abbreviation for kiloPascals (a unit of measure for pressure), and does not refer to the k in the formula.)

   a) Find the value of k if $P = 89$ kPa when $h = 1$ km. Round to 5 decimal places.

   Answer:

$$
\begin{aligned}
89 &= 101.3e^{-k(1)} \\
\frac{89}{101.3} &= \frac{101.3e^{k(1)}}{101.3} \\
0.87858 &= e^{-k(1)} \\
\ln 0.87858 &= -k(\ln e) \qquad\qquad [\text{but } \ln e = 1] \\
k &= -\ln 0.87858 = 0.12945
\end{aligned}
$$

   b) Use your answer in part (a) to find the pressure at a height of 2 km.

   Answer:

$$
\begin{aligned}
P &= 101.3e^{-0.12945(2)} \\
P &= 101.3(0.7719) \\
P &= 78.2 \text{ kPa}
\end{aligned}
$$

12. The temperature in a room is 20°C. If a container of boiling water at 100°C is brought into this room, the water in the container cools according to the formula $T = 20 + 80e^{-0.03t}$, where T is the temperature of the water t minutes after being placed in this room. How long will it take for the water to cool to 40°C in this room?

    Answer:

$$
\begin{aligned}
40 &= 20 + 80e^{-0.03t} \\
20 &= 80e^{-0.03t} \\
0.25 &= e^{-0.03t} \\
\ln 0.25 &= -0.03t(\ln e) \qquad\qquad [\text{but } \ln e = 1] \\
t &= \frac{\ln 0.25}{-0.03} \\
t &= 46.2 \text{ minutes}
\end{aligned}
$$

13. The number of bacteria in a certain culture, t hours from now, grows according to the formula $A = 800(3)^t$.

    a) What will be the bacteria count 3.12 hours from now?

    Answer:

    The bacteria count will be $800(3)^{3.12} = 24\ 643.8 \approx 24\ 644$ bacteria.

    b) When will the bacteria count reach 100 000?

    Answer:

$$100\ 000 = 800(3)^t$$

$$\frac{100\ 000}{800} = \frac{800(3)^t}{800}$$

$$125 = 3^t$$

$$\log 125 = t \log 3$$

$$t = \frac{\log 125}{\log 3}$$

$$t = 4.4 \text{ hours}$$

14. For a period of its life, a tree grows according to the formula $D = D_0e^{kt}$, where D is the diameter of the tree, in centimetres, t years after the beginning of the period. After two years, the diameter of the tree is 12.724 cm. After five years, the diameter is 15.62 cm. Find the value of D_0 and the value of k.

    Answer:

    $15.62 = D_0e^{k(5)}$

    $12.724 = D_0e^{k(2)}$

    Solve by division method:

$$\frac{15.62}{12.724} = \frac{D_0e^{5k}}{D_0e^{2k}}$$

$$1.2276 = e^{5k-2k}$$

$$1.2276 = e^{3k}$$

$$\ln 1.2276 = 3k \ln e$$

$$k = \frac{\ln 1.2276}{3}$$

$$k = 0.068354$$

To find the value of D_0, substitute $k = 0.068354$ into one of the original equations.

$$15.62 = D_0 e^{(0.068354)(5)}$$

$$\frac{15.62}{e^{(0.068354)(5)}} = D_0$$

$$D_0 = 11.098 \text{ cm}$$

15. James attempts to increase the value of his investment by increasing the number of compounding periods. If the best rate James can get is 6%, is it possible for James to obtain enough compounding periods to achieve a rate equivalent to 7%? Explain.

Answer:

The most compounding periods Janes can get is continuous compounding. Using the formula A = Pert, where

$P = \$1.00$

$r = 0.06$

$t = 1$ year

then, $A = 1e^{0.06(1)}$

$A = \$1.0618$

However, at 7%, his return on \$1.00 would be \$1.07. Since \$1.07 > 1.0618, James will not be able to "compound" his 6% rate to a 7% rate.

16. The population of a small northern Manitoba town increased by 15% two years ago, and then decreased by 15% last year. The current population of the town is 5000 people. What was the population of the town before the 15% increase two years ago? **Hint:** You can answer this question using logical reasoning without the use of any population formula.

Answer:

To answer this question, start from the current population of the town and work backwards.

The current population of the town is 5000 people. Last year, the population of the town decreased by 15%. Therefore, the population of the town this year is 85% of the population of the town last year. This can be represented by the following equation:

Let t_{-1} be the population of the town last year. Let t_0 be the population of the town this year.

$$\begin{aligned} t_{-1}(85\%) &= t_0 \\ t_{-1}(0.85) &= 5000 \\ t_{-1} &= \frac{5000}{0.85} \\ t_{-1} &\approx 5882 \text{ people} \end{aligned}$$

Therefore, the population of the town last year was approximately 5882 people.

Follow the same line of reasoning to determine t_{-2}, the population of the town two years ago.

The population of the town last year was 5882 people. The year before that, the population increased by 15%. Therefore, the population of the town last year is 115% larger than the population two years ago. This can be represented by the following equation:

$$\begin{aligned} t_{-2}(115\%) &= t_{-1} \\ t_{-2}(1.15) &= 5882 \\ t_{-2} &= \frac{5882}{1.15} \\ t_{-2} &\approx 5115 \text{ people} \end{aligned}$$

Therefore, the population of the town before the 15% increase two years ago was 5115 people.

From this question, you can see how an equal percentage increase and an equal percentage decrease do not “undo” each other.

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Module 8
Radical and Rational Functions

MODULE 8:
RADICAL AND RATIONAL FUNCTIONS

Introduction

This module builds on concepts you learned throughout Grade 11 Pre-Calculus Mathematics and throughout the first seven modules of this course. In Grade 11 Pre-Calculus Mathematics you learned how to solve radical and rational equations algebraically. You will be combining this skill with the function transformation skills you have learned throughout this course in order to be able to solve radical and rational equations graphically. Before you do this, you first need to learn how to graph radical and rational functions using transformations and other properties of functions.

Are you planning on a career in either the medical field or in law enforcement? If you are, you may encounter these types of functions. Radical functions can be used in the reconstruction of traffic accidents, as they can be used to model the stopping distance of a vehicle. Rational functions can be used to model the amount of a drug found in a person's bloodstream over time. There are many more applications of these functions in different fields, including astronomy and business analysis.

Assignments in Module 8

When you have completed the assignments for Module 8, submit your completed assignments to the Distance Learning Unit either by mail or electronically through the learning management system (LMS). The staff will forward your work to your tutor/marker.

Lesson	Assignment Number	Assignment Title
2	Assignment 8.1	Radical Functions
3	Assignment 8.2	Solving Radical Equations
6	Assignment 8.3	Rational Functions

Resource Sheet

When you write your final examination, you are encouraged to take a Final Examination Resource Sheet with you into the examination. This sheet will be one letter-sized page, 8½ ʺ by 11ʺ, with both sides in your handwriting or typewritten. You will submit it with your examination, but you do not receive any marks for it.

Many students have found that preparing a resource sheet is an excellent way to review. It provides you with a summary of the important facts of each module. You should complete a resource sheet for each module to help with your studying and reviewing. Lesson summaries and module summaries are included for you to use as a guide.

You may use the list of instructions provided below to help you with preparing your resource sheet for the material in Module 8. On this sheet, you should record math terms and definitions, formulas, sample questions, or a list of places where you often make mistakes. You should also identify special areas that require extra attention or review by writing the page numbers.

After you have completed each module's resource sheet, you may summarize the sheets from Modules 1 to 8 to prepare your Final Examination Resource Sheet. The final examination for this course is based on Modules 1 to 8.

Resource Sheet for Module 8

As you go through the lessons of this module, you may want to consider the following suggestions regarding the creation of a resource sheet.

1. List all the important math terms, and define them if necessary.
2. List all the formulas and perhaps a sample problem that shows how the formula is used.
3. If necessary, write the solutions to some problems, showing in detail how you did the calculations.
4. Copy any questions that represent the key points of the lesson, and perhaps include the solutions as well.
5. Identify the problems you found most difficult, and copy the page numbers onto the resource sheet so that you can review them before writing the examination. You may also copy the problems and the solutions onto your resource sheet, and later write them onto your Final Examination Resource Sheet.
6. Write any comments, ideas, shortcuts, or other reminders that may be helpful during an examination.

Writing Your Final Examination

You will write the final examination when you have completed Module 8 of this course. The final examination is based on Modules 1 to 8, and is worth 25 percent of your final mark in the course. To do well on the final examination, you should review all the work you complete in Modules 1 to 8, including all the learning activities and assignments. You will write the final examination under supervision.

Notes

LESSON 1: TRANSFORMATIONS OF RADICAL FUNCTIONS

Lesson Focus

In this lesson, you will

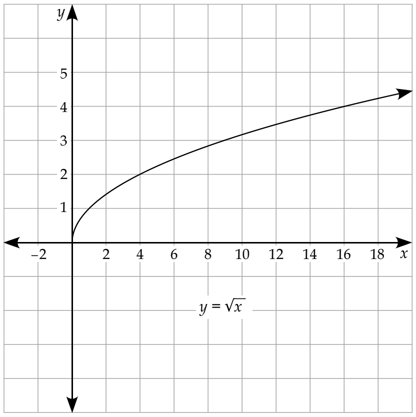
- ☐ learn how to sketch the graph of the function $y = a\sqrt{b(x-h)} + k$ by using transformations
- ☐ learn how the domain of $y = \sqrt{x}$ is related to the domain of $y = a\sqrt{b(x-h)} + k$
- ☐ learn how the range of $y = \sqrt{x}$ is related to the range of $y = a\sqrt{b(x-h)} + k$

Lesson Introduction

In previous modules, you had experience graphing the square root function, using what you knew about transformations. In this lesson, you are going to look in more detail at the square root function, which is one type of radical function. You are also going to analyze the domain and range of radical functions to determine how they are affected when a radical function is transformed.

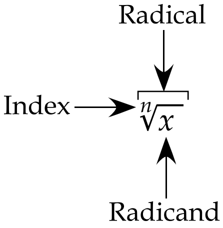
Radical Functions

Consider the standard graph of the square root function and its corresponding table of values.



$y = \sqrt{x}$	
x	y
–1	undefined
0	0
1	1
4	2
9	3
16	4

Radical functions contain a variable inside a radical (as part of a radicand).



For example, the following functions are radical functions.

$$y = \sqrt{x} + 4$$

$$y = \sqrt{x^2 - 3x + 2}$$

The following function is *not* a radical function.

$$y = \sqrt{3} + x$$

The square root function is not defined for negative values. This is because the square root of a negative number is not defined in the real number system.

Just as other functions can be transformed (such as the quadratic function, the cubic function, and the absolute value function), so can the radical function. You have even completed previous examples and questions using transformations on the radical function.

The **standard form of the square root function** is $y = a\sqrt{b(x - h)} + k$. The effects each of the variables *a*, *b*, *h*, and *k* have on a function are shown below.

Variable	Name	Transformation	Effect on (x, y)
a	vertical stretch/compression	$f(x) \rightarrow af(x)$	(x, ay)
b	horizontal stretch/compression	$f(x) \rightarrow f(bx)$	$\left(\frac{1}{b}x, y\right)$
h	horizontal translation	$f(x) \rightarrow \mathrm{f}(x - h)$	$(x + h, y)$
k	vertical translation	$f(x) \rightarrow f(x) + k$	$(x, y + k)$

It may be helpful for you to include the standard form of the square root function and a definition of the radical function on your resource sheet.

Using your knowledge of transformations and radical functions, you should be able to graph any square root function in standard form. Remember, vertical stretches and compressions are completed before vertical translations (similarly with horizontal transformations).

Graphing a Radical Function

Example 1

Graph the following functions using transformations.

a) $y = 3\sqrt{x+1}$

b) $y = \sqrt{\frac{1}{4}x} + 3$

c) $y = 2\sqrt{2x}$

d) $y = 3\sqrt{\left(\frac{1}{2}\right)(x+3)} - 6$

Solutions

a) $y = 3\sqrt{x+1}$

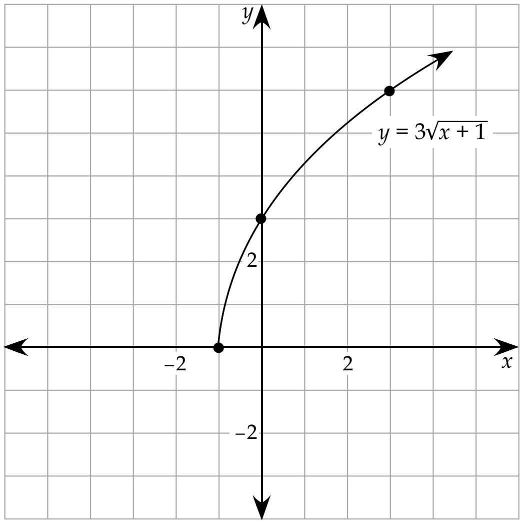
Compare this function with the standard square root function $y = a\sqrt{b(x-h)} + k$. This function is being vertically stretched by a factor of 3 and being moved horizontally 1 unit to the left.

Therefore, each of the x-values are being reduced by 1, while each of the y-values are being multiplied by 3.

Algebraically: $(x, y) \rightarrow (x - 1, 3y)$

Creating a table of values, such as the following, may be helpful.

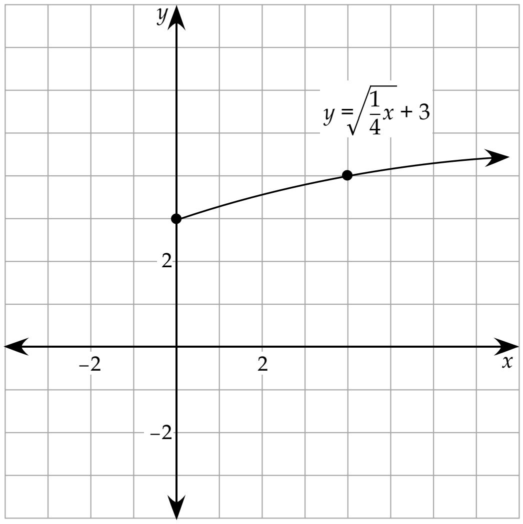
$y = \sqrt{x}$		$y = 3\sqrt{x+1}$	
x	y	$x - 1$	$3y$
0	0	−1	0
1	1	0	3
4	2	3	6
9	3	8	9



b) $y = \sqrt{\frac{1}{4}x} + 3$

This function is being horizontally compressed by a factor of $\frac{1}{4}$ (or stretched by a factor of 4). Recall: values inside the function next to the x affect the function in the *opposite* way that you may expect them to. The function is also translated vertically 3 units up.

Therefore, each of the x-values are being divided by $\frac{1}{4}$ (or multiplied by 4) and each of the y-values are being increased by 3.

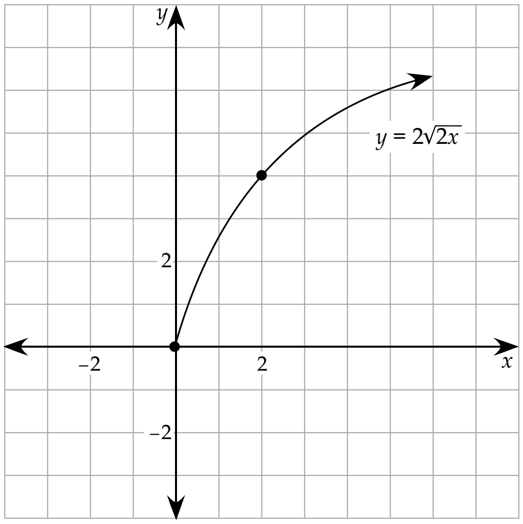


Algebraically:
$(x, y) \rightarrow (4x, y + 3)$

c) $y = 2\sqrt{2x}$

This function is being vertically stretched by a factor of 2 and horizontally compressed by a factor of 2. Therefore, each of the y-values is being multiplied by 2 and each of the x-values is being divided by 2.

Algebraically: $(x, y) \rightarrow \left(\frac{x}{2}, 2y\right)$

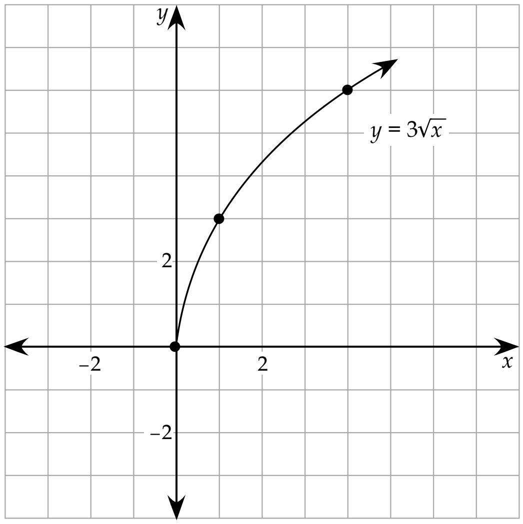


d) $y = 3\sqrt{\left(\frac{1}{2}\right)(x + 3)} - 6$

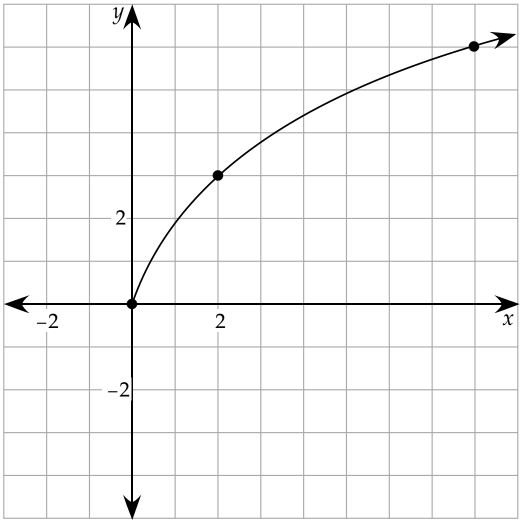
This function is being vertically stretched by a factor of 3, horizontally compressed by a factor of $\frac{1}{2}$ (or stretched by a factor of 2), moved 3 units to the left, and moved 6 units down.

One of the easier ways to graph functions with multiple transformations is to graph them in steps.

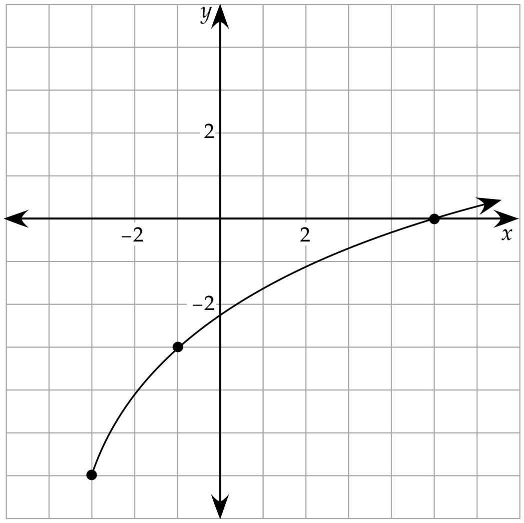
First, graph the vertical stretch.



Then graph the horizontal stretch.



Now, it is possible to graph both the horizontal and vertical translations at once.



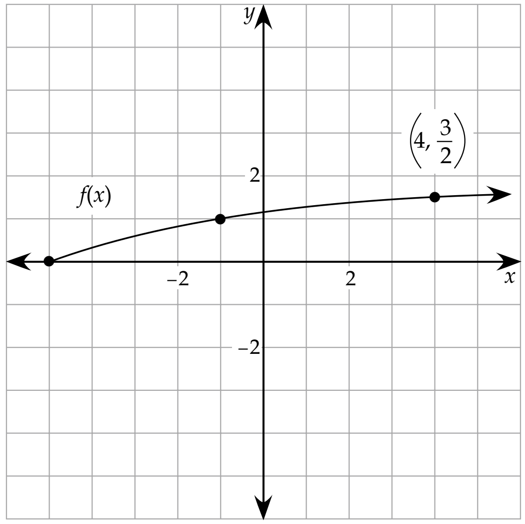
Algebraically:
$(x, y) \rightarrow (2x - 3, 3y - 6)$

Determining a Radical Function Equation From a Given Graph

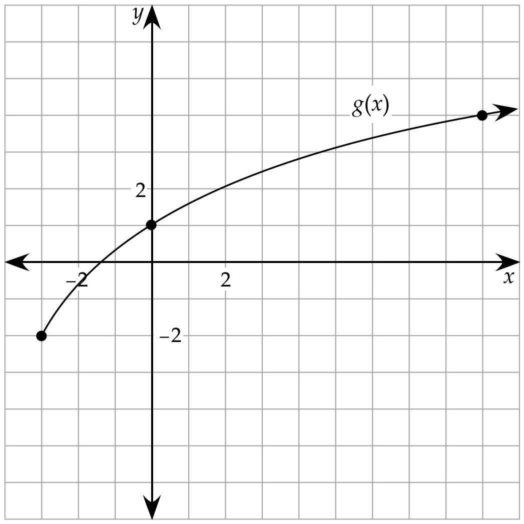
Example 2

Determine an equation of the function shown on each of the graphs below.

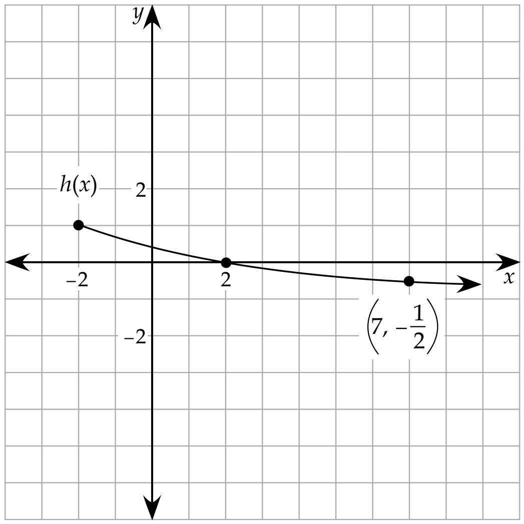
a)



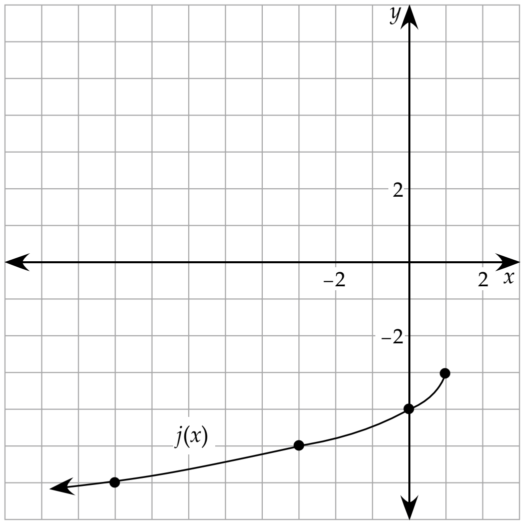
b)



c)

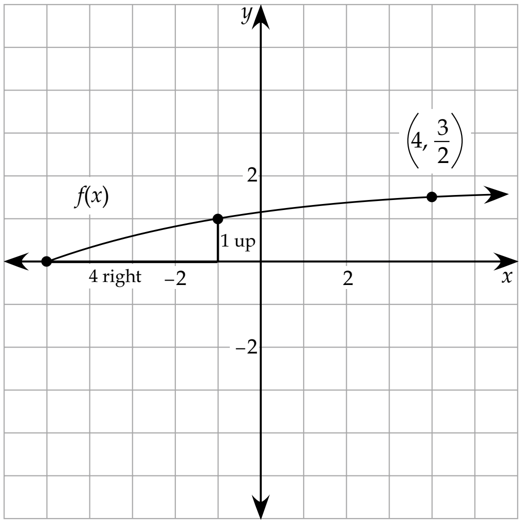


d)



Solutions

a)



In order to determine the equation of the function, consider the translation. This graph has been shifted 5 units to the left. Therefore, $y = a\sqrt{b(x+5)}$.

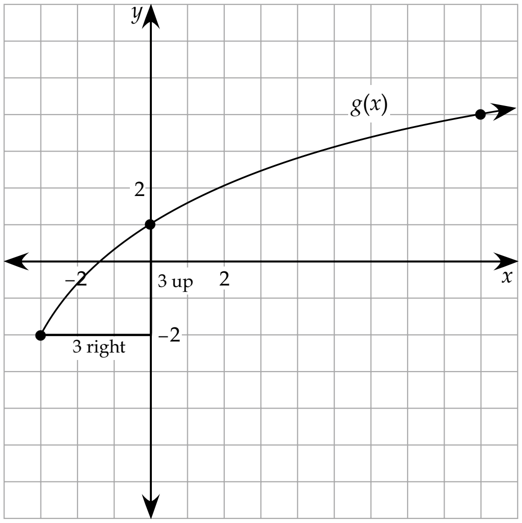
The standard function $y = \sqrt{x}$ has the following points:

x	y
0	0
1	1
4	2
9	3

So in a standard radical function, you would move 4 units right and 2 units up from the starting point. In the given graph, the second point (1, −1) is 4 units right but only 1 unit up (as shown in the graph above). Therefore, a vertical compression has occurred, so $a = \frac{1}{2}$.

Final solution: $y = \frac{1}{2}\sqrt{x+5}$

b)

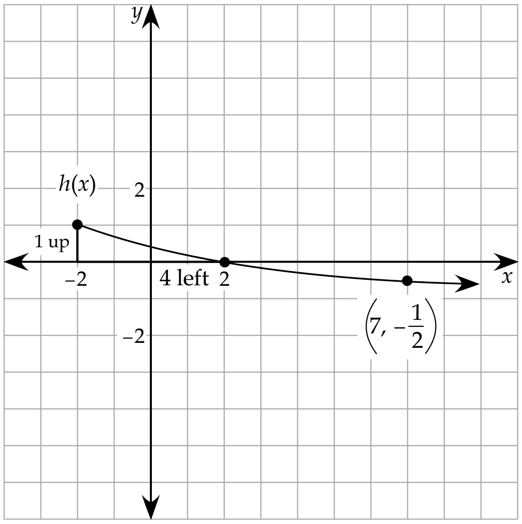


This graph has been shifted 3 units left and 2 units down. Therefore, $y = a\sqrt{b(x+3)} - 2$.

From the starting point, the second point is 3 units right and 3 units up (as shown in the graph above). In a standard radical function, you would move 9 units right and 3 units up. Therefore, a horizontal compression has occurred and $b = 3$.

Final solution: $y = \sqrt{3(x+3)} - 2$

c)



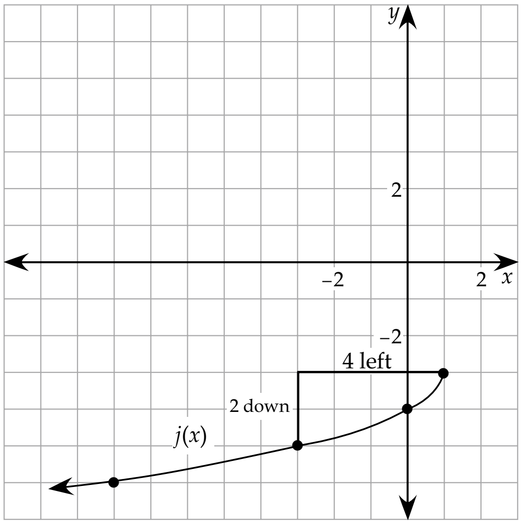
This graph has been translated 2 units left and 1 unit up. Therefore, $y = a\sqrt{b(x+2)} + 1$.

It is pointing down to the right, so it has been reflected over the x-axis. This means a is negative.

Again, considering a second point and comparing it to the starting point, you can determine the horizontal or vertical stretch/compression. In this graph, a second point is 4 units right and the vertical move is only 1 unit (as shown in the graph above). In a standard function $y = \sqrt{x}$, it would move 2 units vertically, so a vertical compression has occurred and $a = \frac{1}{2}$.

Final solution: $y = \frac{1}{2}\sqrt{(x+2)} + 1$

d)



This function has undergone two reflections, both through the *x*-axis and through the *y*-axis. This function has also been moved 1 unit to the right and 3 units down. Therefore, the corresponding *a*, *b*, *h*, and *k* values in the standard radical equation are −1, −1, 1, and −3 respectively.

The *a* value is −1 because this function has been reflected through the *x*-axis.

The *b* value is −1 because this function has been reflected through the *y*-axis.

The *h* value is 1 because this function has been translated horizontally 1 unit to the right.

The *k* value is −3 because this function has been translated vertically 3 units down.

Final solution: $j(x) = -\sqrt{-(x-1)} - 3$.

Finding the Domain and Range of a Radical Function

Example 3

Graph the function $y = 2\sqrt{x - 4} + 5$, and state its domain and range.

Compare the domain and range of $y = 2\sqrt{x - 4} + 5$ with the domain and range of $y = \sqrt{x}$. What do you notice?

Solution

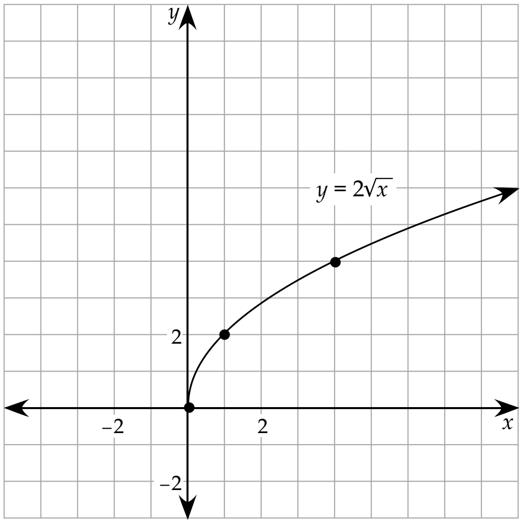
To graph the function $y = 2\sqrt{x - 4} + 5$, you can use transformations.

The graph of $y = \sqrt{x}$ has been shifted 4 units to the right, 5 units up, and stretched by a factor of 2.

To graph these multiple transformations, you can either use multiple steps to graph one transformation at a time, or you can use a table of values.

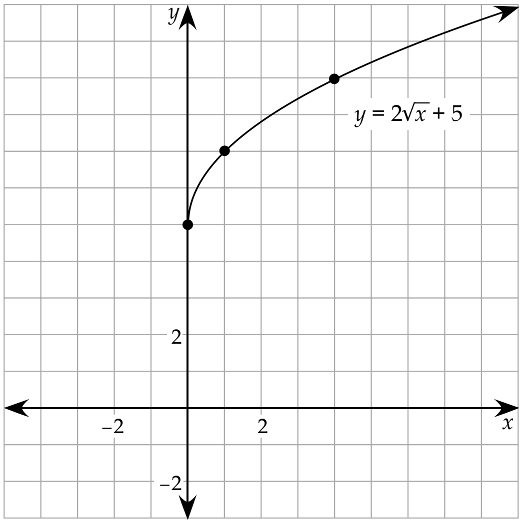
Method 1: Multiple Graphs

To use this method, first graph the vertical stretch by a factor of 2.

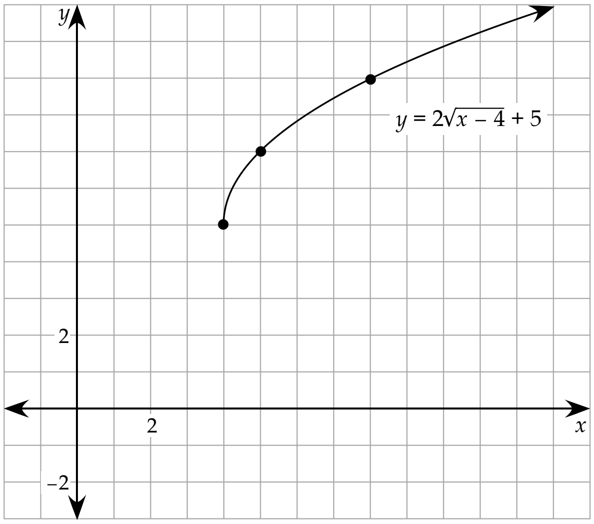


Now, you can either graph the vertical translation or the horizontal translation. It does not matter in what order these transformations are completed.

In this example, the vertical translation of 5 units up is graphed first.



Finally, graph the horizontal translation of 4 units to the right.



Method 2: Table of Values

To create a table of values, use coordinate transformation notation.

This function is being vertically stretched by a factor of 2. Therefore, each y-coordinate is being multiplied by 2.

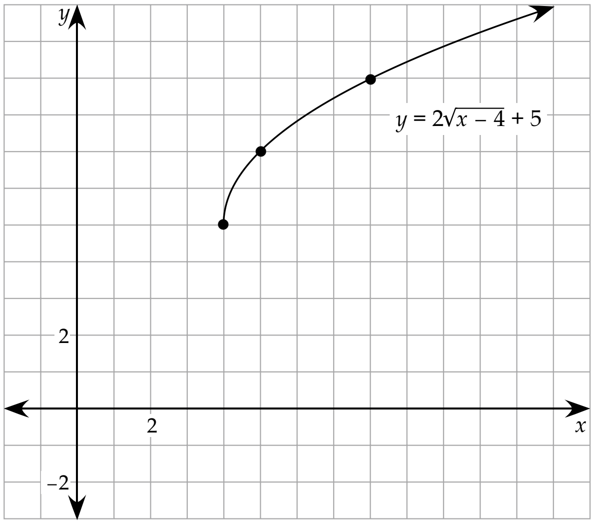
This function is also being vertically translated 5 units up. Therefore, each y-coordinate is also increased by 5.

Because of the horizontal translation of 4 units to the right, each x-coordinate is being increased by 4.

In coordinate transformation notation, this looks like $(x, y) \rightarrow (x + 4, 2y + 5)$.

Thus, each original point on the graph of $y = \sqrt{x}$ is transformed according to the notation above.

$(0, 0) \rightarrow (4, 5)$
$(1, 1) \rightarrow (5, 7)$
$(4, 2) \rightarrow (8, 9)$
$(9, 3) \rightarrow (13, 11)$



The domain and range of $y = 2\sqrt{x - 4} + 5$ are:

Domain: $\{x \mid x \geq 4\}$
Range: $\{y \mid y \geq 5\}$

The domain and range of $y = \sqrt{x}$ are:

Domain: $\{x \mid x \geq 0\}$
Range: $\{y \mid y \geq 0\}$

From this example, you may notice that domain and range are related to the h- and k-values in the standard form of the square root function, $y = a\sqrt{b(x - h)} + k$.

The domain of a square root function consists of all the x-values such that the radicand is non-negative. This is because the square root is only defined when the radicand is greater than or equal to zero. If you analyze the radicand when it is greater than or equal to zero (such as below), the b-value will have no effect on the domain of the function. This is because zero divided by any real number is still zero. However, the value that will have an effect on the domain is the h-value.

$$
\begin{aligned}
x - h &\geq \frac{0}{b}, b \in \Re \\
x - h &\geq 0 \\
x &\geq h
\end{aligned}
$$

The range of a radical function consists of all the y-values that the function achieves when the radicand is non-negative. The smallest value the radicand can achieve is zero. The square root of zero is zero. Also, anything multiplied by zero, such as the variable a, is still zero. Therefore, the value that will have an effect on the range is the k-value.

$$
\begin{aligned}
y &= a\sqrt{b(x - h)} + k \\
y &= a\sqrt{0} + k \\
y &= 0 + k \\
y &= k
\end{aligned}
$$

What happens, however, when you reflect a radical function through the x-axis or the y-axis? Does this affect the domain and/or the range? In the following learning activity, you will determine the answers to these questions. Make sure you complete this learning activity, as it is designed to help you do well on the following assignment and your examinations.

Learning Activity 8.1

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. State the equations of the non-permissible values of the function $f(x) = \dfrac{2}{x^2 - 9}$.
2. Simplify: $\sec \theta \sin \theta$
3. Express $\log_5 125 = 3$ in exponential form.
4. Simplify: $\ln e$.
5. Convert 30° to radians.
6. Find all the values of θ between $[0, 2\pi]$ if $\cos \theta = 1$.
7. In which quadrant is $\theta = \dfrac{11\pi}{7}$ located?
8. Evaluate: $\left| -\dfrac{4}{7} - \dfrac{9}{49} \right|$

Part B: Transformations of the Radical Function

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Graph the following functions using transformations. State the domain and range of each function.

   a) $y = \dfrac{1}{2}\sqrt{x + 7}$

   b) $y = \sqrt{3x} - 2$

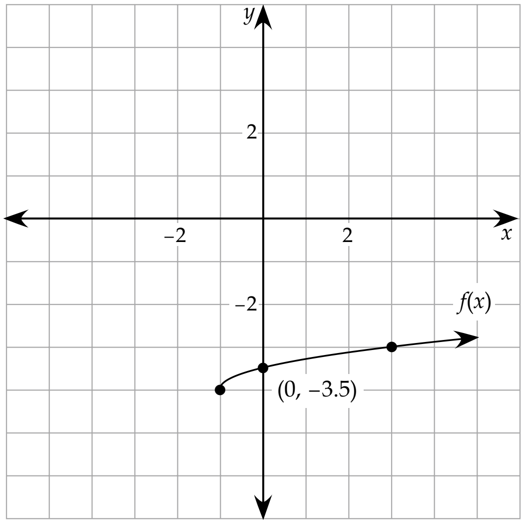
   c) $y = \dfrac{1}{3}\sqrt{-\dfrac{1}{4}x - 1}$

   d) $y = -2\sqrt{\left(\dfrac{1}{3}\right)(x + 5)} - 3$

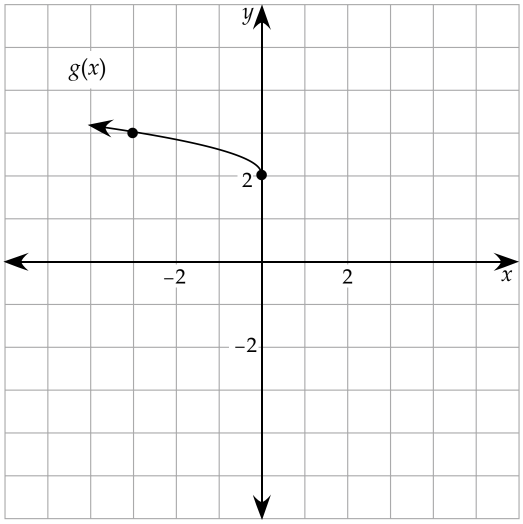
continued

2. Write the equation of each of the following functions, given the steps taken to transform the function.
   a) The square root function is stretched horizontally by a factor of 3 and translated 2 units to the left and 3 units down.
   b) The square root function is compressed vertically by a factor of 2, reflected through the *x*-axis, and translated 6 units to the right.
   c) The square root function is compressed horizontally by a factor of 7, reflected through the *y*-axis, and translated 3 units up.
   d) The square root function is reflected through the *x*-axis, reflected through the *y*-axis, stretched vertically by a factor of 2, and moved 1 unit to the left.
3. Determine the radical functions that correspond to the following graphs.

a)



b)



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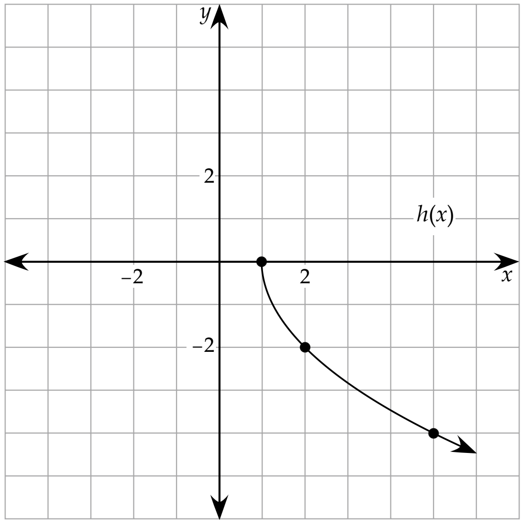
Learning Activity 8.1 (continued)

c)



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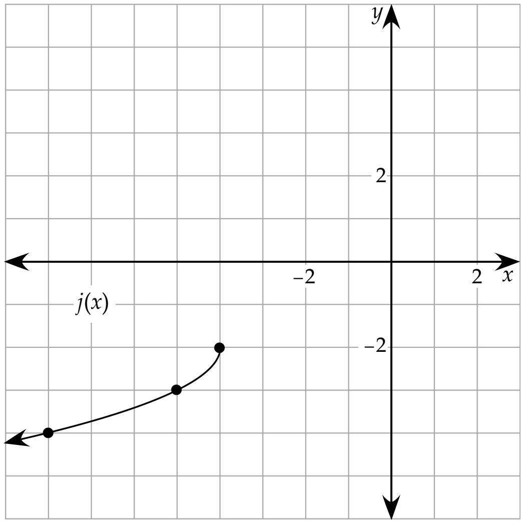


4. Graph the following functions:

$$f(x) = \sqrt{x+2} - 3 \text{ and } g(x) = -\sqrt{x+2} - 3$$

State the domain and range of each function. What do you notice about the domain and range of each function? How are they related? Generalize the effect a reflection through the x-axis has on the domain and range of a function.

5. Graph the following functions:

$$f(x) = \sqrt{x-1} - 4 \text{ and } g(x) = \sqrt{-(x-1)} - 4$$

State the domain and range of each function. What do you notice about the domain and range of each function? How are they related? Generalize the effect a reflection through the y-axis has on the domain and range of a function.

continued

Learning Activity 8.1 (continued)

6. Match each function to its corresponding graph.

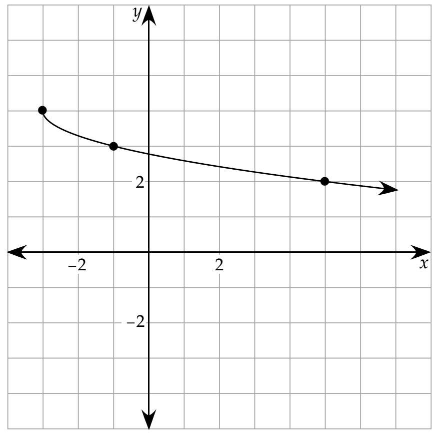
   a) $y = -\sqrt{\left(\frac{1}{2}\right)(x+3)} + 4$

   b) $y = \sqrt{\left(\frac{1}{2}\right)(x+3)} + 4$

   c) $y = -\sqrt{2(x+3)} + 4$

   d) $y = \sqrt{2(x+3)} + 4$

i)



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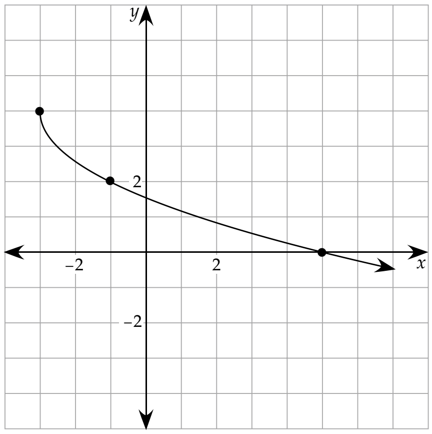

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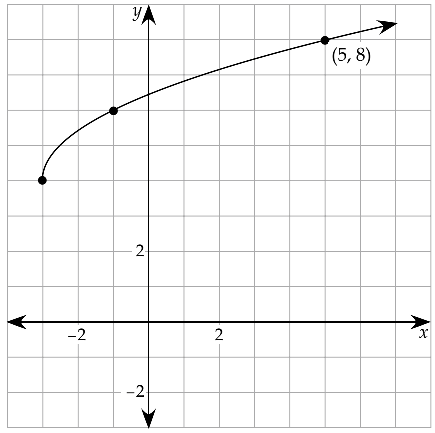

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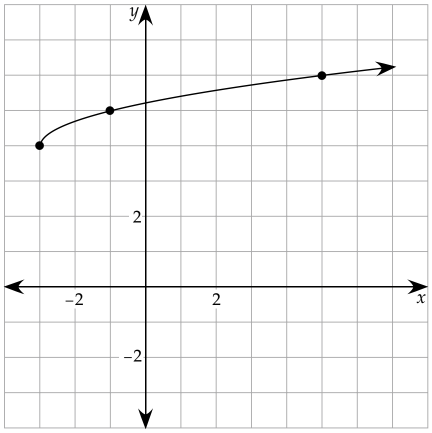


7. The approximate distance, *d*, in miles, that a person can see to the horizon from a height, *h*, in feet, is given by the equation $d = \sqrt{\frac{3}{2}h}$. If a person who is 6 feet tall is standing on a 72-foot cliff, can that person see a sailboat that is 11 miles away?

Lesson Summary

In this lesson, you learned how to graph radical functions using transformations. These skills are similar to those you learned in Module 2. You also saw how the domain and range of a function are affected by transformations, especially translations and reflections.

In the next lesson, you will be analyzing the connections between a function $y = f(x)$ and $y = \sqrt{f(x)}$.

Lesson 2: The Square Root of a Function

Lesson Focus

In this lesson, you will

- ☐ learn how to sketch the graph of the function $y = \sqrt{f(x)}$ when you are given the function $y = f(x)$ by using transformations
- ☐ learn how sketch the graph of the function $y = \sqrt{f(x)}$ when you are given the graph of the function $y = f(x)$
- ☐ learn about the relationship between the domain and range of $y = f(x)$ and $y = \sqrt{f(x)}$

Lesson Introduction

In the last lesson, you looked at graphing radical functions by using transformations. In this lesson, you will be looking at graphing the square root of different types of functions, including linear and quadratic functions.

The square roots of functions are sometimes graphed in order to find constants of proportionality. These constants are useful in many physics applications, including pendulums and even the length of skid marks found at automobile accidents. You will discover more about this type of application in the learning activity at the end of this lesson.

The Square Root of a Function

In order to determine how a function is affected when the square root of the function is taken, consider the standard square root function $y = \sqrt{f(x)}$ when $f(x) = x$.

When $f(x)$ is less than zero, the square root does not exist. Therefore, you only need to consider $f(x)$ values that are greater than or equal to zero.

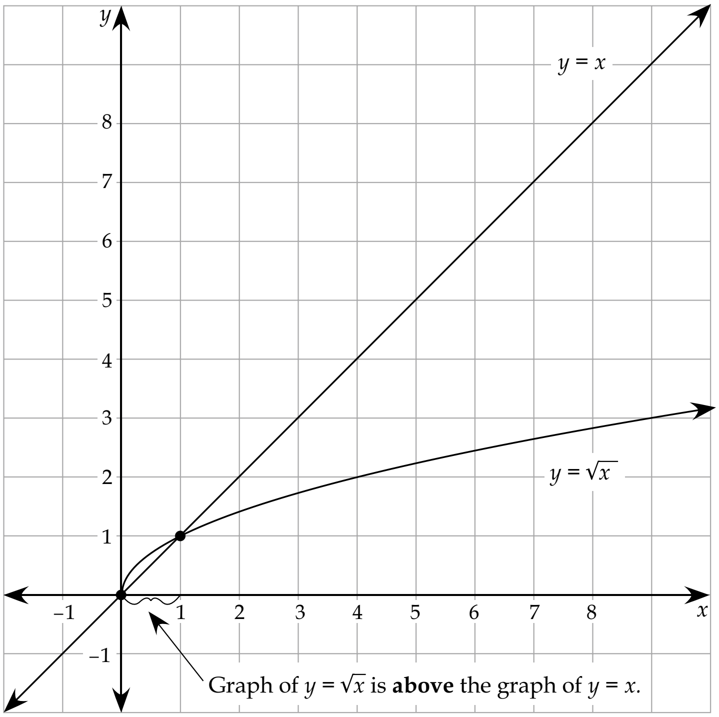
Consider the following table of values for points between zero and one.

$0 \le x \le 1$	
$f(x) = x$	$y = \sqrt{f(x)}$
0	0
0.1	0.316
0.2	0.447
0.3	0.548
0.4	0.632
0.5	0.707
0.6	0.775
0.7	0.837
0.8	0.894
0.9	0.949
1	1

What pattern do you notice between each x-value and its corresponding square root?

In all function values between zero and one, the square root of each function value is *greater* than the original function value itself.

In terms of graphing, this means, for function values between zero and one, that the graph of $y = \sqrt{f(x)}$ will be *above* the graph of $f(x)$ for function values between zero and one.



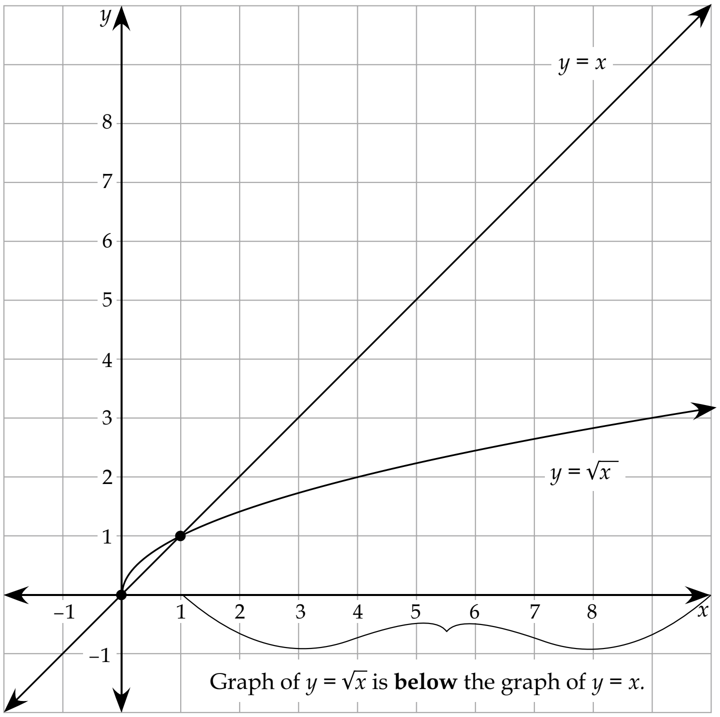
Consider the following table of values for function values greater than one.

$x \geq 1$	
x	$\sqrt{x}$
1	1
4	2
9	3
36	6
64	8
81	9
100	10
500	22.361
1,000	31.623
500,000	707.107
1,000,000	1000

What pattern do you notice between each x-value and its corresponding square root?

For all function values greater than one, the square root of each function value is less than the original function value.

In terms of graphing, this means that the graph of $y = \sqrt{f(x)}$ will be *below* the graph of $f(x)$ when the function value is greater than one.

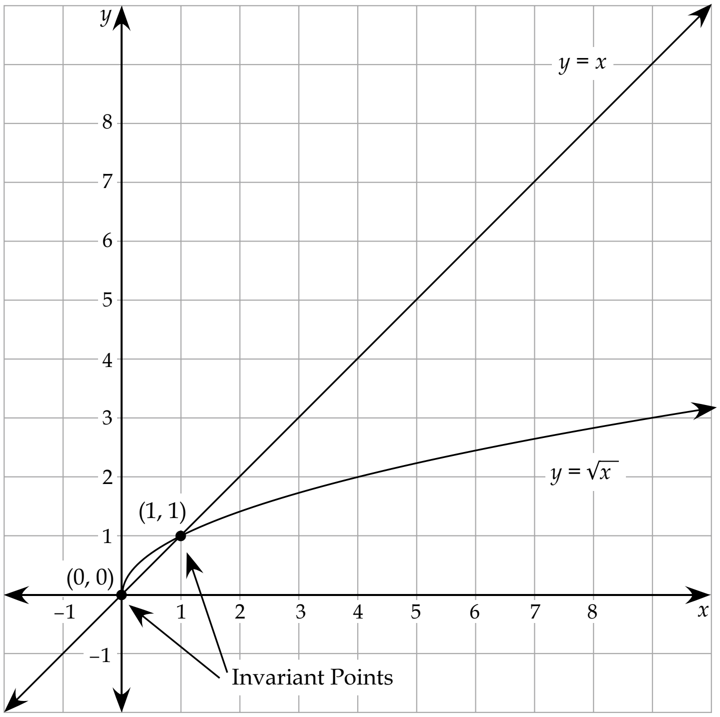


There are two points at which the graphs of $y = f(x)$ and $y = \sqrt{f(x)}$ are the same. This occurs because $f(x) = \sqrt{f(x)}$ at these points.

When $f(x) = 0$ and when $f(x) = 1$, $f(x) = \sqrt{f(x)}$ because $\sqrt{0} = 0$ and $\sqrt{1} = 1$.

Since the positions of the points do not change as a result of the radical transformation, these points are invariant points. Recall: invariant points are points that stay the same on the graph of $f(x)$ and the transformed graph of $f(x)$.

Note that when you take the square root of a function, you are taking the square root of the y-values, not the square root of the x-values. Therefore, when the function takes on a y-value of 0 or 1, these points stay the same and are invariant points.



You may wish to include a summary of the above information on your resource sheet. This will be helpful to you when you are graphing the square root of a function.

Graphing the Square Root of a Function

It is now possible, using what you know about square root functions, to sketch the graph of the square root of a function when you are given the graph of the function.

The following table summarizes the properties of square root graphs.

Value of $f(x)$	$f(x) < 0$	$f(x) = 0$	$0 < f(x) < 1$	$f(x) = 1$	$f(x) > 1$
Relative location of graph of $y = \sqrt{f(x)}$	The graph of $y = \sqrt{f(x)}$ is undefined.	The graphs of $y = \sqrt{f(x)}$ and $y = f(x)$ intersect on the x-axis.	The graph of $y = \sqrt{f(x)}$ is **above** the graph of $y = f(x)$.	The graph of $y = \sqrt{f(x)}$ intersects the graph of $y = f(x)$.	The graph of $y = \sqrt{f(x)}$ is **below** the graph of $y = f(x)$.

It may be helpful for you to include a copy of this chart, or the following steps for graphing the square root of a function, in your own words, on your resource sheet.

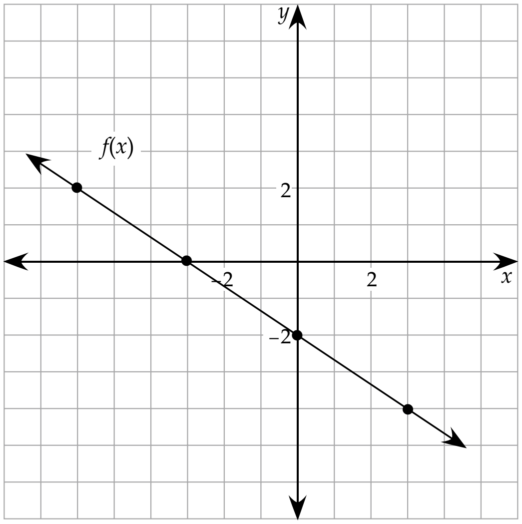
Steps for Graphing the Square Root of a Function:

1. Determine where the square root function is not defined. This will occur whenever the function is below the x-axis, as $y = \sqrt{f(x)}$ is not defined for negative values of $f(x)$.
2. Determine the invariant points, or where $f(x) = 0$ and $f(x) = 1$.
3. Determine where the graph of $y = \sqrt{f(x)}$ is *above* the graph of $y = f(x)$.
4. Determine where the graph of $y = \sqrt{f(x)}$ is *below* the graph of $y = f(x)$.
5. Use all of the above to graph $y = \sqrt{f(x)}$.

Example 1

Graph the square root of each of the functions below.

a)



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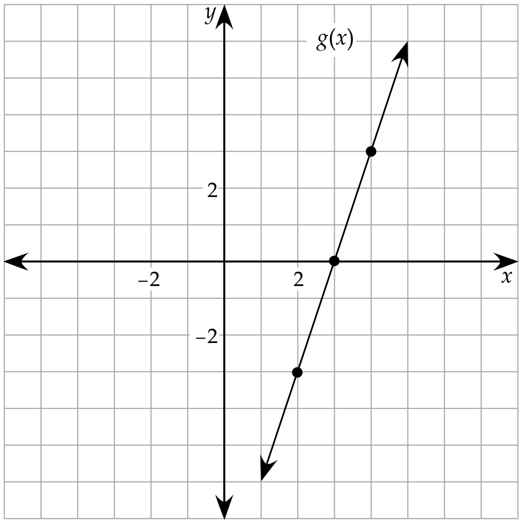
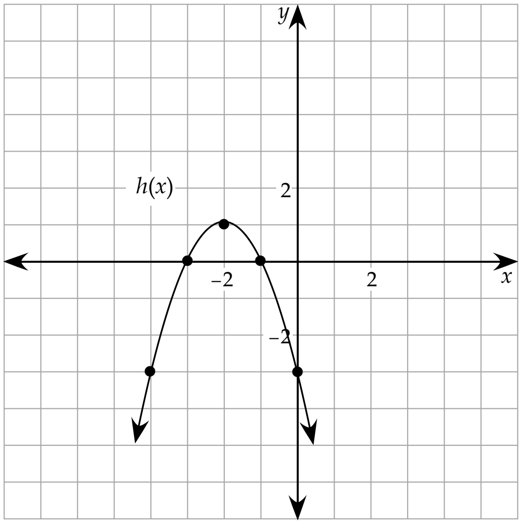
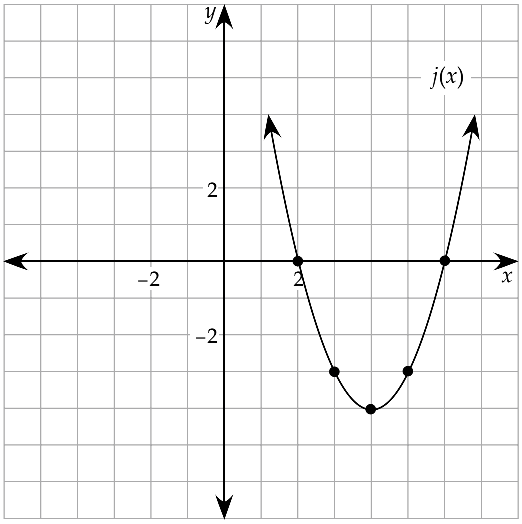


c)



d)

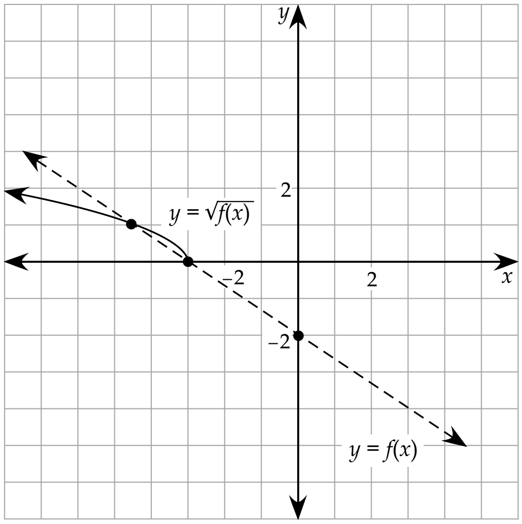


Solutions

a) To graph the square root of this function, you can use the above chart as a reference or use the steps for graphing the square root of a function.

The square root function is not defined for values less than zero. The graph of the square root function, therefore, will only exist when $x \leq -3$.

The invariant points exist when $f(x) = 0$ and 1. The point at which $f(x) = 0$ is easy to find, (−3, 0). Therefore, $f(x) = 1$ is somewhere between −4 and −5. Call this point A. Thus, $\sqrt{f(x)}$ will be greater than, or above, $f(x)$ between −3 and point A. Also, $\sqrt{f(x)}$ will be less than, or below, $f(x)$ between point A and infinity. Using this information, you can graph $y = \sqrt{f(x)}$.



Note: It is not necessary to find the x-value when the y-value is 1 because it is an invariant point. In this case, you could find that Point A is $\left(-\frac{9}{2}, 1\right)$, using the original graph.

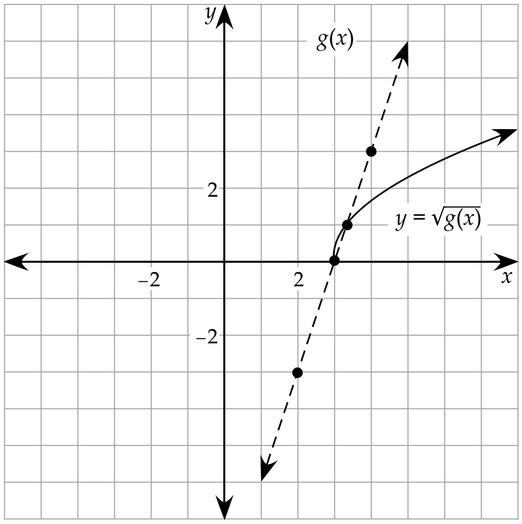
b) As the square root function is not defined for values less than zero, $y = \sqrt{g(x)}$ will not be defined when $x < 3$.

The point at which $g(x) = 0$ is (3, 0).

The point at which $g(x) = 1$ is somewhere between 3 and 4, but close to 4. Call this Point B.

$\sqrt{g(x)}$ will be above $g(x)$ between 3 and B.

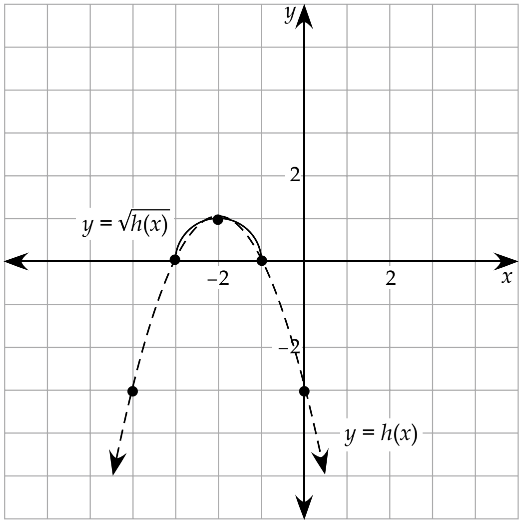
$\sqrt{g(x)}$ will be below $g(x)$ between B and infinity.



Note: Again, you could find Point B to have coordinates $\left(\frac{10}{3}, 1\right)$, but it is not necessary for graphing the radical function from a graph.

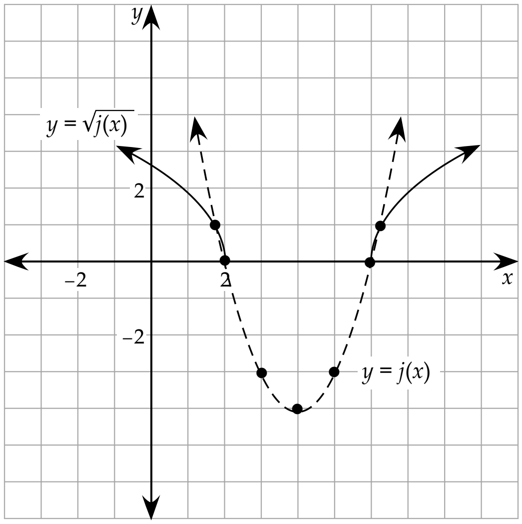
c) $\sqrt{h(x)}$ is not defined when $x < -3$ and when $x > -1$.

The points at which $h(x) = 0$ are $(-3, 0)$ and $(-1, 0)$.
The point at which $h(x) = 1$ is $(-2, 1)$.
$\sqrt{h(x)}$ will be above $h(x)$, between -3 and -2, and also between -2 and -1.



d) $\sqrt{j(x)}$ is not defined when $2 < x < 6$.

The points at which $j(x) = 0$ are $(2, 0)$ and $(6, 0)$.
The points which $j(x) = 1$ are located just before 2 and just after 6.
Using the graph of $j(x)$ as a guide, you can now graph $y = \sqrt{j(x)}$, keeping in mind the invariant points, and when $\sqrt{j(x)}$ should be above or below $j(x)$.



Graphing the Square Root of Linear Functions

Graphing the square root of linear functions is similar to graphing radical functions using transformations. You just need to rearrange the equation into standard form.

Example 2

Graph the following functions as well as their square roots. State the domain and range of the original function as well as the square root of that function.

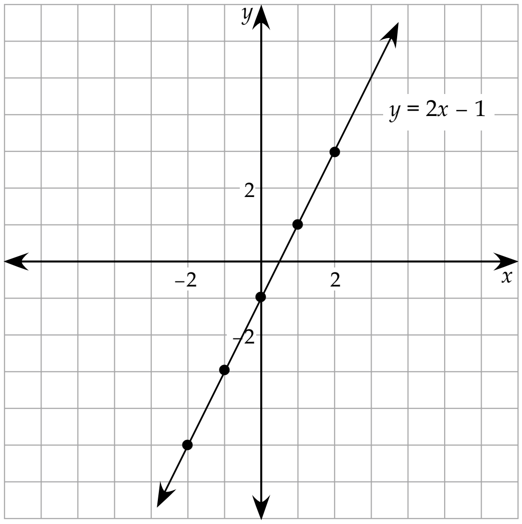
a) $y = 2x - 1$

b) $y = \frac{1}{2}x + 4$

Solutions

a) **Method 1:**

First, graph the function $y = 2x - 1$.

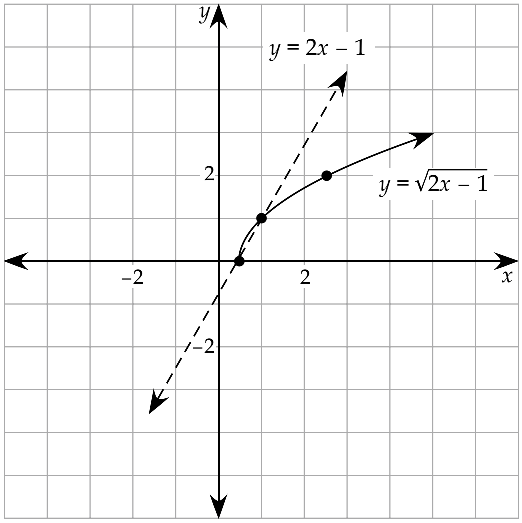


Domain: $\{x \mid x \in \Re\}$

Range: $\{y \mid y \in \Re\}$

To graph the function $y = \sqrt{2x - 1}$, think of $y = \sqrt{f(x)}$, where $f(x) = 2x - 1$.

Plot key points of $f(x)$. Key points include the invariant points where $f(x) = 0$ and $f(x) = 1$. You could also choose integer square-root values such as $\sqrt{f(x)}$ when $f(x) = 4$. As well, take note of the domain of $y = \sqrt{f(x)}$. In this case, $x \geq \frac{1}{2}$, since the function values to the left of $x = \frac{1}{2}$ are all negative.



x	$f(x)$	$y = \sqrt{f(x)}$
$\frac{1}{2}$	0	0
1	1	1
$2\frac{1}{2}$	4	2

Notice the function values at $y = 0$ and $y = 1$ are invariant points. The square root of the function values between 0 and 1 are above $f(x) = 2x - 1$ and the square root of the function values greater than 1 are below the $f(x) = 2x - 1$ line. Finally, notice the domain of the square root function is only where $f(x) = 2x - 1$ is on or above the x-axis.

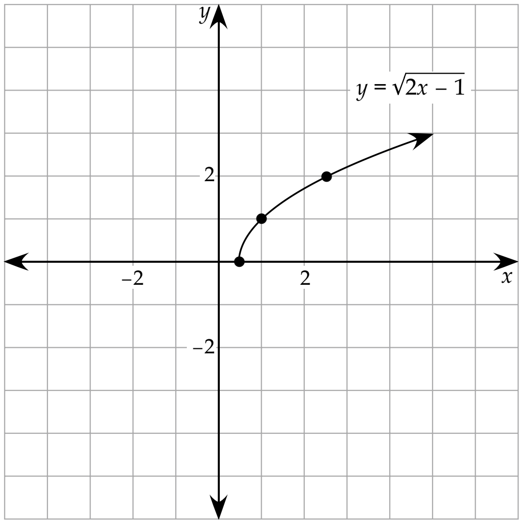
Method 2:

Graph the square root of the function, or $y = \sqrt{2x - 1}$. In order to graph this function using transformations of the standard radical $y = \sqrt{x}$, you need to first factor the radicand.

$$y = \sqrt{2\left(x - \frac{1}{2}\right)}$$

This function is the standard radical function $y = \sqrt{x}$, which has been moved $\frac{1}{2}$ unit to the right and compressed horizontally by a factor of 2.

$(x, y) \rightarrow \left(\frac{1}{2}x + \frac{1}{2}, y\right)$	
(0, 0)	$\left(\frac{1}{2}, 0\right)$
(1, 1)	(1, 1)
(4, 2)	$\left(\frac{5}{2}, 2\right)$
(9, 3)	(5, 3)

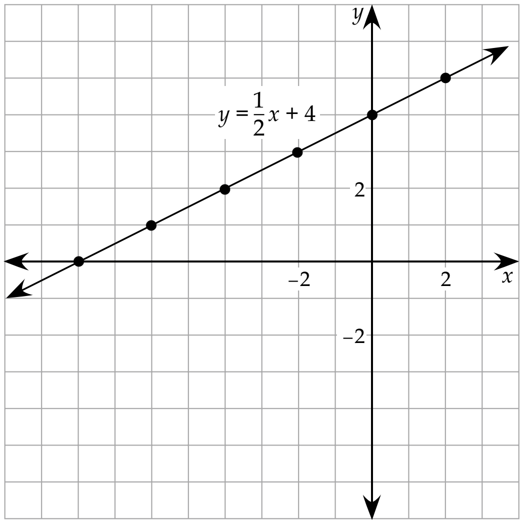


Domain: $\left\{x \mid x \geq \frac{1}{2}\right\}$

Range: $\{y \mid y \geq 0\}$

b) **Method 1: Analyzing Invariant Points**

Graph the function $y = \dfrac{1}{2}x + 4$.

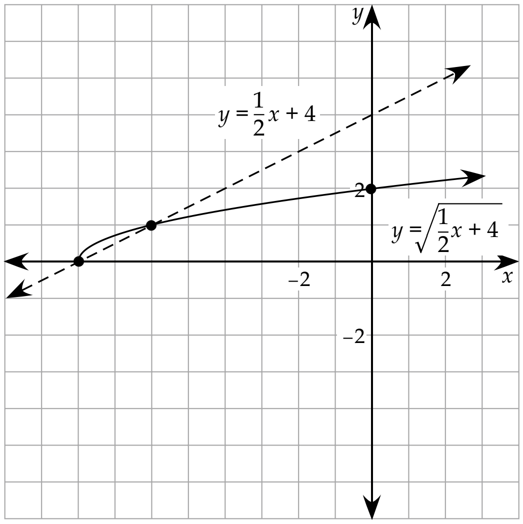


Domain: $\{x \mid x \in \Re\}$
Range: $\{y \mid y \in \Re\}$

To graph the function $y = \sqrt{\dfrac{1}{2}x + 4}$, you can think of $y = \sqrt{f(x)}$ where

$f(x) = \dfrac{1}{2}x + 4$.

Plot key points of $f(x)$. Also note the domain of $y = \sqrt{f(x)}$. In this case, $x \geq -8$ since the function values to the left of $x = -8$ are all negative.



x	$f(x)$	$y = \sqrt{f(x)}$
-8	0	0
-6	1	1
0	4	2

Notice the function values at $y = 0$ and $y = 1$ are invariant points. The square root of the function values between 0 and 1 are above $f(x) = \frac{1}{2}x + 4$ and the square root of the function values greater than 1 are below the $f(x) = \frac{1}{2}x + 4$ line. Finally, notice the domain of the square root function is only where $f(x) = \frac{1}{2}x + 4$ is on or above the x-axis.

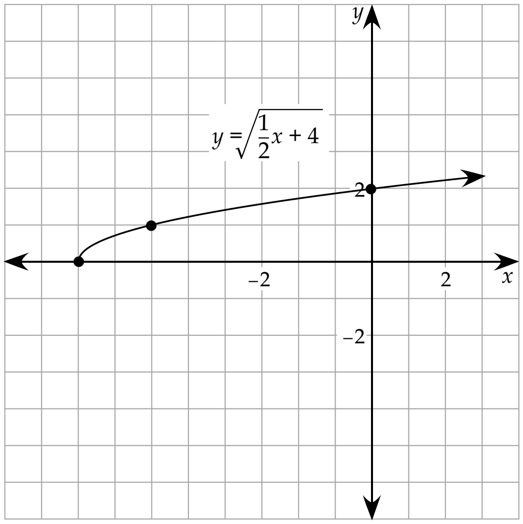
Method 2: By Transformations

Graph the square root of the function $y = \frac{1}{2}x + 4$.

$$y = \sqrt{\frac{1}{2}x + 4} = \sqrt{\frac{1}{2}(x + 8)}$$

This function is a transformation of the standard radical function $y = \sqrt{x}$, which has been moved 8 units to the left and stretched horizontally by a factor of 2.

(x, y)	$(2x - 8, y)$
(0, 0)	$(-8, 0)$
(1, 1)	$(-6, 1)$
(4, 2)	(0, 2)
(9, 3)	(10, 3)



Domain: $\{x \mid x \geq -8\}$

Range: $\{y \mid y \geq 0\}$

Learning Activity 8.2

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Express $3^{(-4)} = \frac{1}{81}$ in log form.
2. Express $\log_2 32 = 5$ in exponential form.
3. Simplify: $\tan^2 \theta \cot \theta$
4. Solve for x: $\ln e^{2x-1} = 1$.
5. Convert $\frac{\pi}{2}$ to degrees.
6. What is the exact value of sin 30°?
7. State an angle that is coterminal to 319°.
8. If $f(x) = x^2 - 3x$, find $f(-3)$.

Part B: The Square Root of a Function

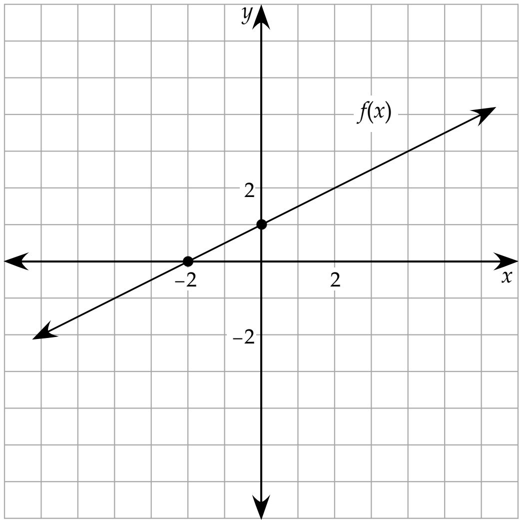
Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Graph the following functions, as well as their square roots. State the domain and range of the original function, as well as the square root of that function.
   a) $y = 3x - 7$
   b) $y = -\frac{1}{4}x + 2$
   c) $y = -x - 2$

continued

2. Graph the square root of each of the functions below.

a)



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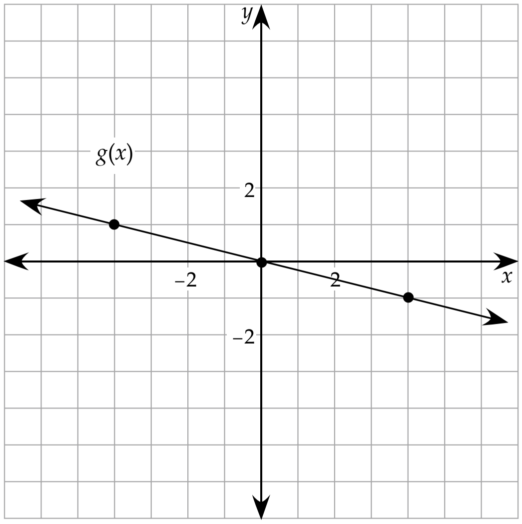

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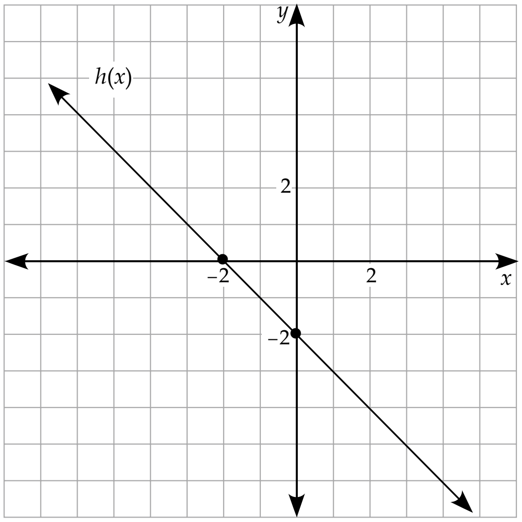


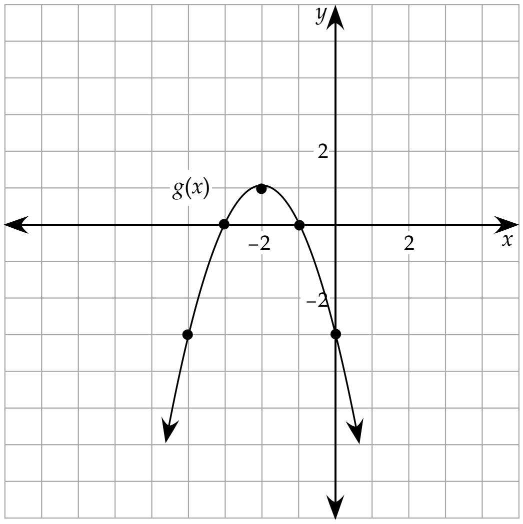
3. Compare the domains and ranges of the linear functions above with their corresponding square root functions. What do you notice?

continued

Learning Activity 8.2 (continued)

4. Match the following graphs with their corresponding square root graphs.

a)



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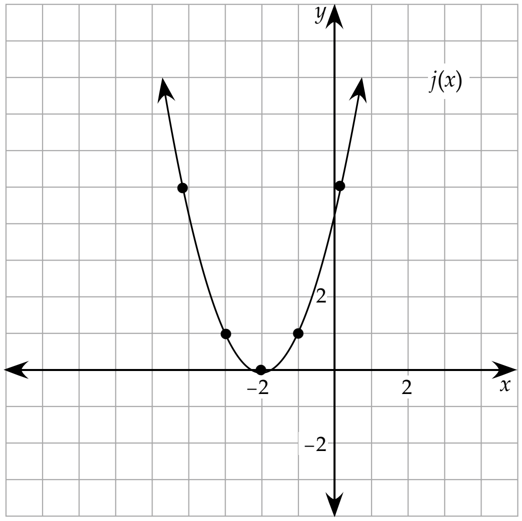
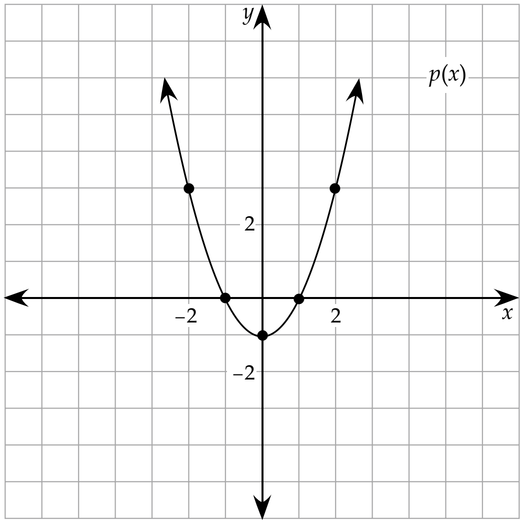
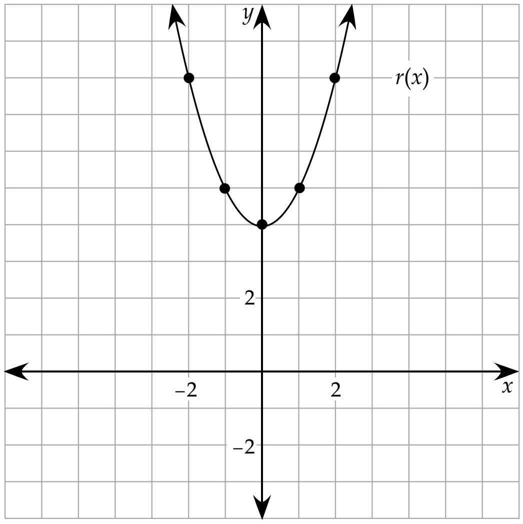


c)



d)



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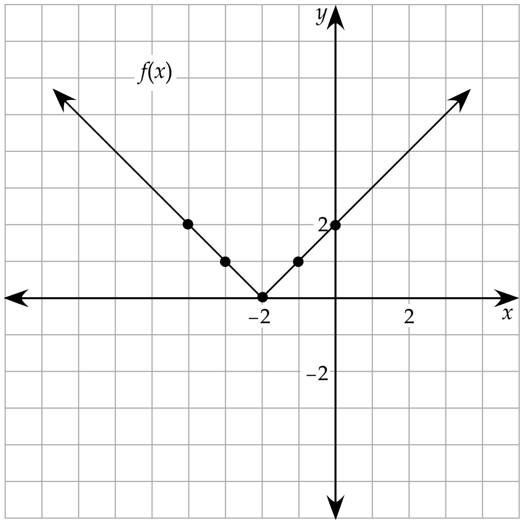
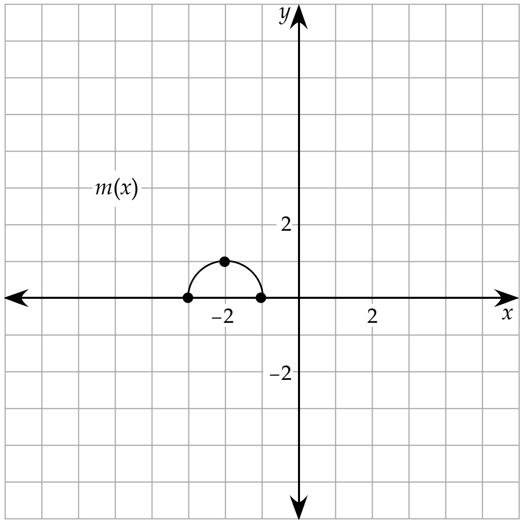
Learning Activity 8.2 (continued)

i)



ii)



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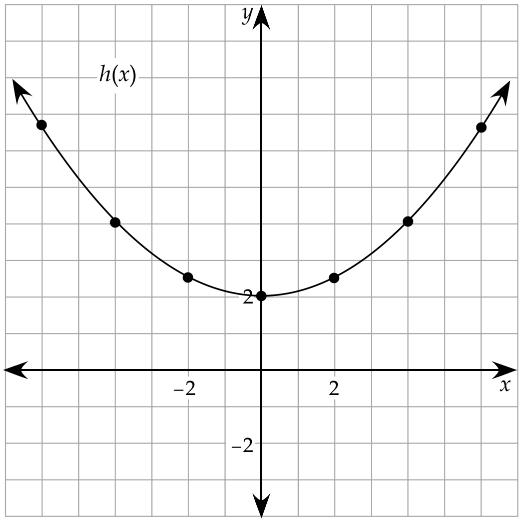

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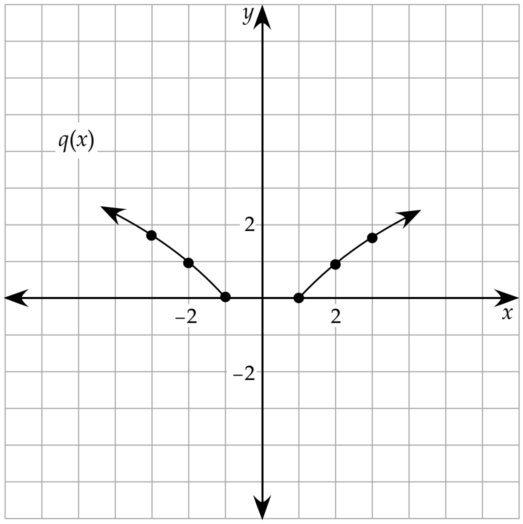


5. The speed a car is travelling is proportional to the square root of the length of the skid mark it creates during an automobile accident. This can be expressed as $V = k\sqrt{D}$, where V represents speed, D represents distance, and k is the constant of proportionality.

   In order to determine how fast a vehicle was travelling before an accident, police are able to measure the length of the skid mark. However, in order to use the above formula, they need to find the constant of proportionality. To do this, they have acquired the following data about the speed of a vehicle and the corresponding length of its skid mark when stopped suddenly.

continued

Learning Activity 8.2 (continued)

Speed (miles)	Length (feet)
0	0
20	19.8
25	30.9
30	44.4
40	79.1
50	123.5
60	177.8

To determine how fast a vehicle was travelling in order to create a skid mark 130 feet long, you need to complete the following steps.

a) Create a graph of speed versus length.

b) What type of function is created from the data (for example, linear, quadratic, radical)?

c) What type of function would be created from the square root of the graph you just created (for example, linear, quadratic, radical)?

d) Take the square root of this graph using the transformation properties you have learned about throughout this lesson.

e) What is the slope of this graph?

f) What is the relationship between the slope of the graph and the constant, *k*, in the equation $V = k\sqrt{D}$?

g) What is the value of *k* in the equation $V = k\sqrt{D}$?

h) Using the value you found in (g), determine how fast the vehicle must have been travelling in order to create a 130-foot skid mark.

Lesson Summary

In this lesson, you compared the graphs of linear and quadratic functions with the graphs of the square root of linear and quadratic functions. As well, you located invariant points on a linear or quadratic graph in order to graph the corresponding square root function.

In the next lesson, you will be looking at radical equations.

Assignment 8.1

Radical Functions

Total: 34 marks

You need to show all your steps as you work out your solutions, and make sure your answers are clear (include units, where appropriate). Check the Introduction for the Marking Guidelines that will be used for all assignments and examinations.

1. Graph the following functions using transformations. State the domain and range of each function. *(4 × 3 marks each = 12 marks)*

   a) $y = 2\sqrt{x + 2} - 5$

continued

Assignment 8.1: Radical Functions (continued)

b) $y = \sqrt{-\frac{1}{2}x} + 3$

c) $y = -2\sqrt{-(x+1)}$

continued

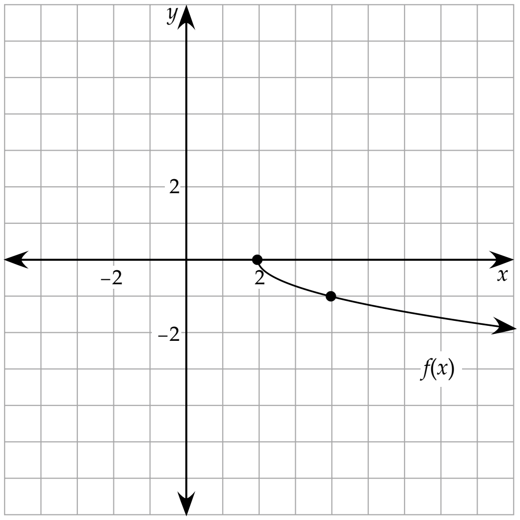
Assignment 8.1: Radical Functions (continued)

d) $y = -\sqrt{\left(\frac{1}{3}\right)(x-3)} - 5$

continued

2. Determine the equations of the functions that correspond to the following graphs.
(2 × 2 marks each = 4 marks)

a)



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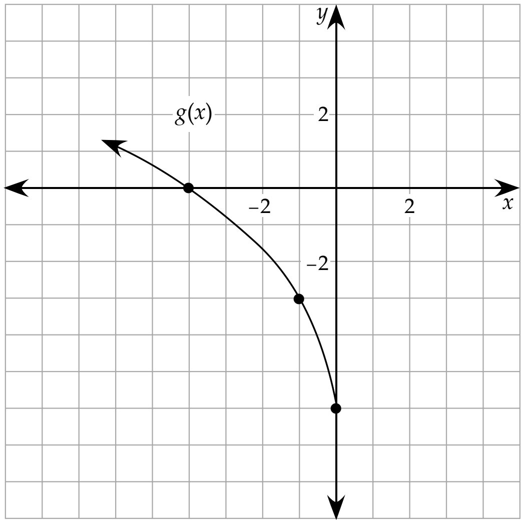
b)



continued

Assignment 8.1: Radical Functions (continued)

3. Express the following function, $f(x)$, algebraically as a mapping from $y = \sqrt{x}$. Sketch the function $f(x)$, and state the domain and range. *(5 marks)*

$$f(x) = -\frac{1}{2}\sqrt{-(x+2)} - 3$$

Graphically: **Algebraically:**

continued

Assignment 8.1: Radical Functions (continued)

4. Graph the square root of the following function. State the domain and range of the square root of the function. *(3 marks)*

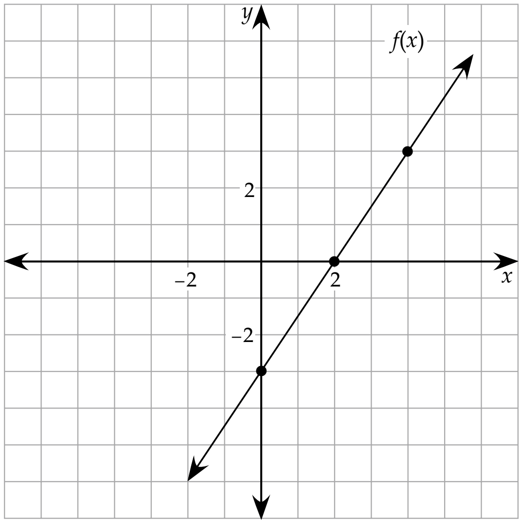
   $y = -2x + 4$

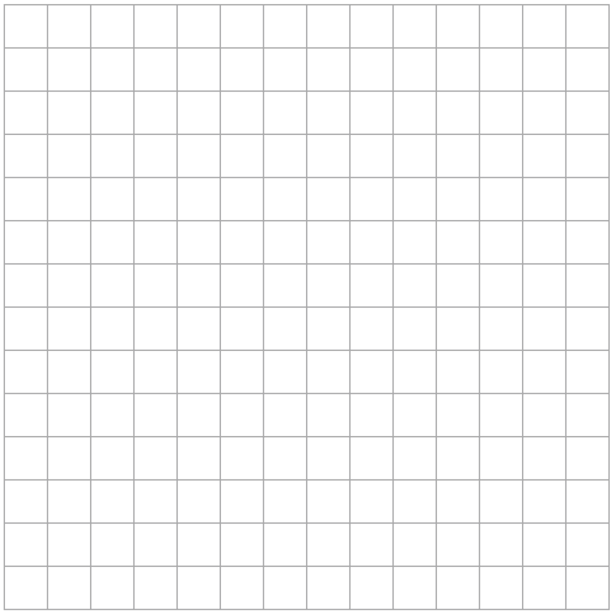
continued

Assignment 8.1: Radical Functions (continued)

5. Graph the square root of each of the functions below. *(3 × 2 marks each = 6 marks)*

a)





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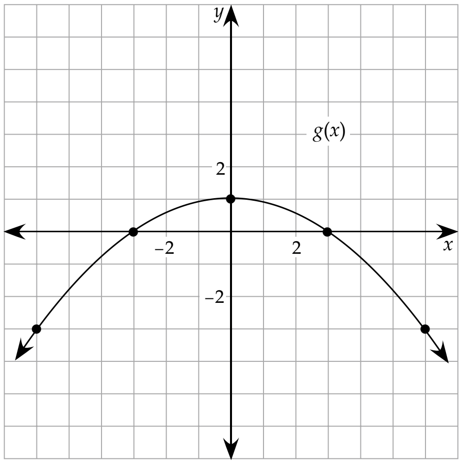
Assignment 8.1: Radical Functions (continued)

b)



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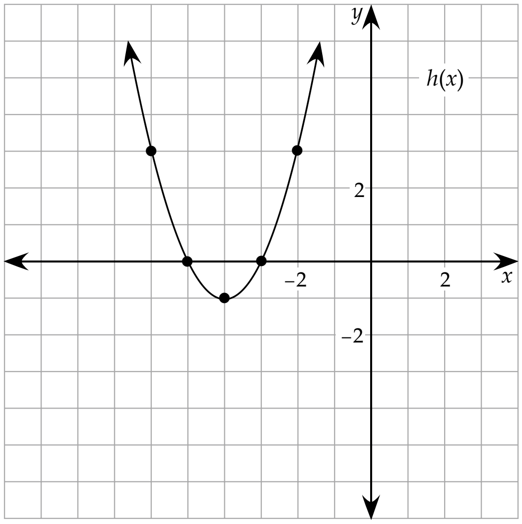
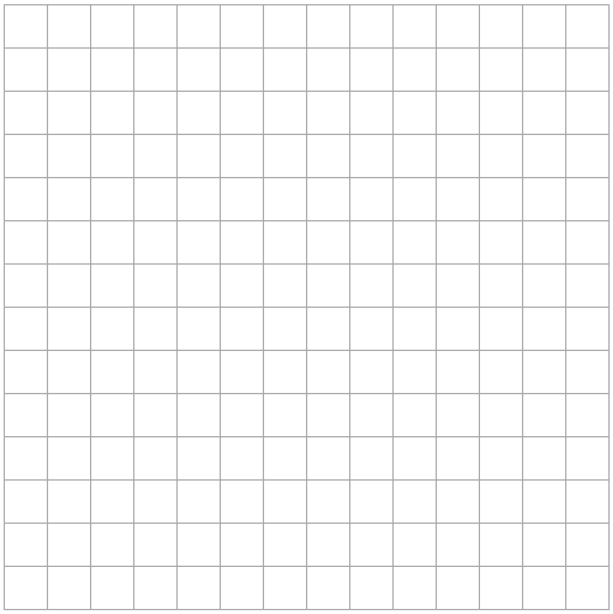
Assignment 8.1: Radical Functions (continued)

c)





continued

Assignment 8.1: Radical Functions (continued)

6. Compare the domain and range of $f(x) = 2x - 6$ with the domain of $y = \sqrt{f(x)}$. Explain why the domains and ranges of $y = f(x)$ and $y = \sqrt{f(x)}$ are different. *(4 marks)*

Notes

LESSON 3: RADICAL EQUATIONS

Lesson Focus

In this lesson, you will

- ☐ review how to solve radical equations algebraically
- ☐ learn how to solve radical equations graphically
- ☐ learn about the relationship between the roots of a radical equation and the *x*-intercepts of the corresponding radical function

Lesson Introduction

In Grade 11 Pre-Calculus Mathematics, you learned how to solve radical equations algebraically. In this lesson, you are going to take a graphical approach to solving radical equations now that you have the skills to graph multiple types of radical equations. Once you graph radical equations, you will be able to see how the roots of a radical equation relate to the *x*-intercepts of the corresponding radical function. This is similar to a concept you learned in Grade 11 Pre-Calculus Mathematics where you related the roots of a quadratic equation to the *x*-intercepts of the corresponding quadratic function.

Radical Equations

A **radical equation** is an equation containing at least one radical expression with a variable as part of the radicand.

Examples of radical equations:	Examples that *are not* radical equations:
$\sqrt{x^2+1} = 7$	$\sqrt{7} + x = 1$
$\sqrt{x-1} + 5 = \sqrt{x}$	$\sqrt{\pi} = x^2 - x$

Remember that π is not a variable; it is a constant.

Include this definition on your resource sheet.

Solving Radical Equations Algebraically

In Grade 11 Pre-Calculus Mathematics, you learned that in order to solve a radical equation, it is helpful to square both sides of the equation. In other words, you can raise each side of the equation to the power of 2. However, this can sometimes lead to extraneous roots. Therefore, you need to check your answers to ensure they work in the original equation. The following example will allow you to practise and review how to solve radical equations algebraically.

Example 1

Solve the following radical equations for x algebraically. Check your solutions for extraneous roots.

a) $\sqrt{3x-1}+1=2$

b) $\sqrt{2x+3}=\sqrt{4x-1}$

Solutions

a) Step one is always to isolate the radical.

$$\sqrt{3x-1}=1$$

Now, you can raise each side to the power of 2.

$$\left(\sqrt{3x-1}\right)^2=1^2$$

Solve the resulting equation for x.

$$3x-1=1$$
$$3x=2$$
$$x=\frac{2}{3}$$

Check your solution for extraneous roots.

LHS	RHS
$\sqrt{3\left(\frac{2}{3}\right)-1}+1$	2
$\sqrt{1}+1$	2
$1+1$	2

As the left-hand side (LHS) equals the right-hand side (RHS), the solution is $x=\frac{2}{3}$.

b) There is no need to isolate the radical in this equation, as both radicals are already on separate sides of the equation.

Square both sides:

$$\left(\sqrt{2x+3}\right)^2 = \left(\sqrt{4x-1}\right)^2$$

Isolate x:

$$\begin{aligned} 2x + 3 &= 4x - 1 \\ 2x - 2x + 3 + 1 &= 4x - 2x - 1 + 1 \\ 4 &= 2x \\ \frac{4}{2} &= \frac{2x}{2} \\ x &= 2 \end{aligned}$$

Check for extraneous roots:

LHS	RHS
$\sqrt{2(2)+3}$	$\sqrt{4(2)-1}$
$\sqrt{4+3}$	$\sqrt{8-1}$
$\sqrt{7}$	$\sqrt{7}$

As the left-hand side equals the right-hand side, the solution is $x = 2$.

Example 2

a) Determine the root of $\sqrt{2x+4}+3=7$ algebraically.

b) Graph $y=\sqrt{2x+4}-4$ and state the x-intercept.

c) What is the relationship between the root of $\sqrt{2x+4}+3=7$ and the x-intercept of the function $y=\sqrt{2x+4}-4$? Explain.

Solutions

a) Step one is to isolate the radical.

$$\begin{aligned} \sqrt{2x+4}+3 &= 7 \\ \sqrt{2x+4} &= 4 \end{aligned}$$

Now, square both sides of the equation.

$$\left(\sqrt{2x+4}\right)^2 = 4^2$$

$$2x + 4 = 16$$

Isolate for x.

$$2x = 12$$

$$x = 6$$

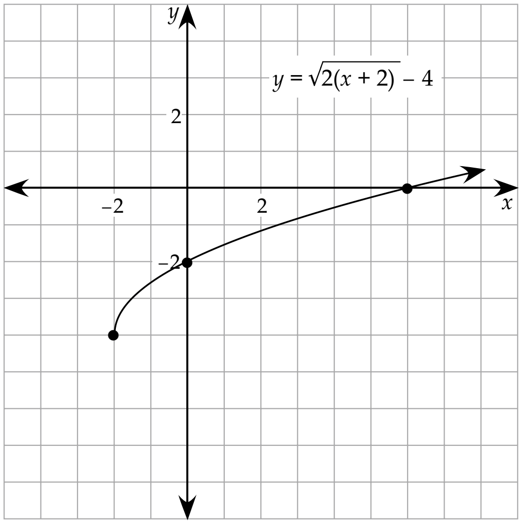
Check for extraneous roots.

LHS	RHS
$\sqrt{2(6)+4}+3$	7
$\sqrt{12+4}+3$	7
$\sqrt{16}+3$	7
$4+3$	7

As the LHS equals the RHS, $x = 6$ is the root of the above radical equation.

b) Graph $y = \sqrt{2x+4} - 4$ using transformations.

This function is the standard square root function, which has been compressed horizontally by a factor of 2, moved 2 units to the left, and translated 4 units down.



The x-intercept is located at (6, 0).

c) The root of $\sqrt{2x+4}+3=7$ is 6 while the x-intercept of $y=\sqrt{2x+4}-4$ is 6 as well. The reason for this is that they are two different representations of the same concept.

If you rearrange the equation $\sqrt{2x+4}+3=7,$ you get $\sqrt{2x+4}-4=0.$ Solving this equation algebraically is identical to solving $y=\sqrt{2x+4}-4$ for the x-intercept. The reason for this is that the x-intercepts of any function occur when $y = 0$. In this particular question, the x-intercept occurs when $\sqrt{2x+4}-4$ equals 0.

Solving Radical Equations Graphically

In order to solve radical equations graphically, you can graph each side of the equation to determine where the two functions intersect. This intersection point is the solution to the radical equations.

Example 3

Graph $y=\sqrt{x}$, $y = -2$, $y = -1$, $y = 0$, $y = 1$, and $y = 2$ on the same coordinate grid. Use this coordinate grid to solve the following radical equations.

a) $\sqrt{x}=-2$

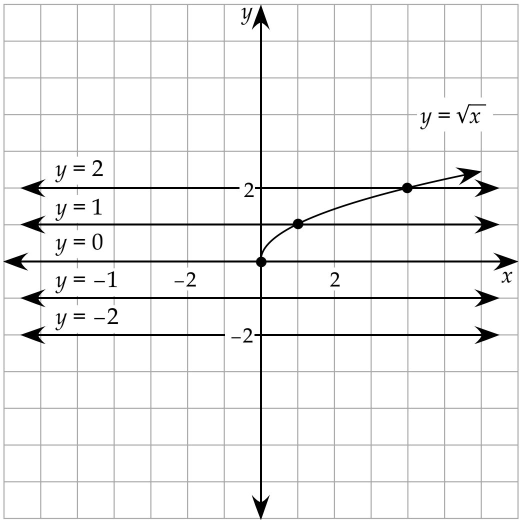
b) $\sqrt{x}=-1$

c) $\sqrt{x}=0$

d) $\sqrt{x}=1$

e) $\sqrt{x}=2$

Solutions



a) In order to solve this equation, you need to determine if and where the two functions, $y = \sqrt{x}$ and $y = -2$, intersect.

   As $\sqrt{x}$ never goes below the x-axis, these two functions will never intersect. Therefore, there is no solution.

b) The functions $y = \sqrt{x}$ and $y = -1$ never intersect. Therefore, there is no solution.

c) The functions $y = \sqrt{x}$ and $y = 0$ intersect at (0, 0). The solution to this radical equation is thus $x = 0$.

d) The functions $y = \sqrt{x}$ and $y = 1$ intersect at the point (1, 1). Therefore, the solution is $x = 1$.

e) The functions $y = \sqrt{x}$ and $y = 2$ intersect at the point (4, 2). The solution to this equation is the x-coordinate of this point, or $x = 4$.

It is also possible to solve radical equations graphically by moving all terms to one side of the equation and then graphing the related function after replacing "0 =" with "y =." The x-intercepts of the related function are solutions to the radical equation.

Example 4

Graph $y = \sqrt{x} + 2$, $y = \sqrt{x} + 1$, $y = \sqrt{x}$, $y = \sqrt{x} - 1$, and $y = \sqrt{x} - 2$ on the same coordinate grid. Use this coordinate grid to solve the following radical equations.

a) $\sqrt{x} + 2 = 0$

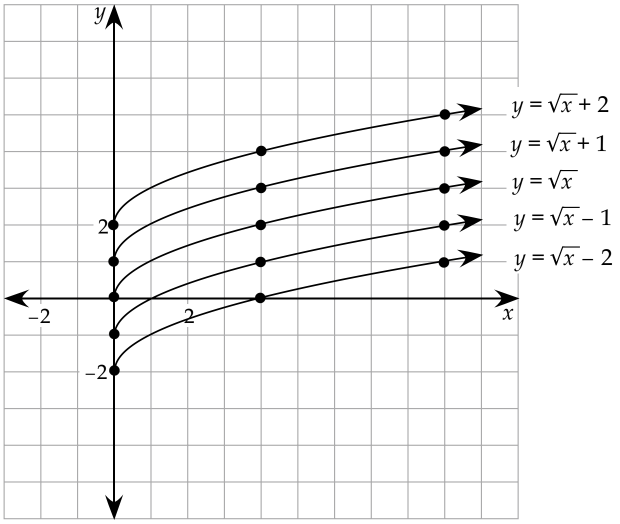
b) $\sqrt{x} + 1 = 0$

c) $\sqrt{x} = 0$

d) $\sqrt{x} - 1 = 0$

e) $\sqrt{x} - 2 = 0$

Solutions



a) Consider the corresponding function $y = \sqrt{x} + 2$. When this function touches the x-axis, the corresponding equation will be $\sqrt{x} + 2 = 0$. Therefore, there is no solution to this radical equation, as this function never touches the x-axis. It is always above the x-axis.

b) The corresponding function is $y = \sqrt{x} + 1$. There is no solution to the corresponding equation $\sqrt{x} + 1 = 0$ because the function $y = \sqrt{x} + 1$ never touches the x-axis. It is always above the x-axis.

c) The corresponding function is $y = \sqrt{x}$. The solution to the equation $\sqrt{x} = 0$ is 0, as $y = \sqrt{x}$ touches the x-axis at (0, 0).

d) The corresponding function is $y = \sqrt{x} + 1$. This function crosses the x-axis at $x = 1$. Therefore, the solution to the equation $\sqrt{x} - 1 = 0$ is $x = 1$.

e) The corresponding function is $y = \sqrt{x} + 2$. This function crosses the x-axis at $x = 4$. Therefore, the solution to the equation $\sqrt{x} - 2 = 0$ is $x = 4$.

The equations you solved in Examples 3 and 4 are identical. You just solved the equations in two different ways. In Example 3, you graphed both sides of the equation to determine where the functions intersected. In Example 4, you first moved all terms to one side of the equation, graphed the related function, and then determined where that function crossed the x-axis. These two methods can be used interchangeably to solve radical equations graphically.

Example 5

Solve the following radical equations graphically using both methods.

$\sqrt{2x+11}=5$

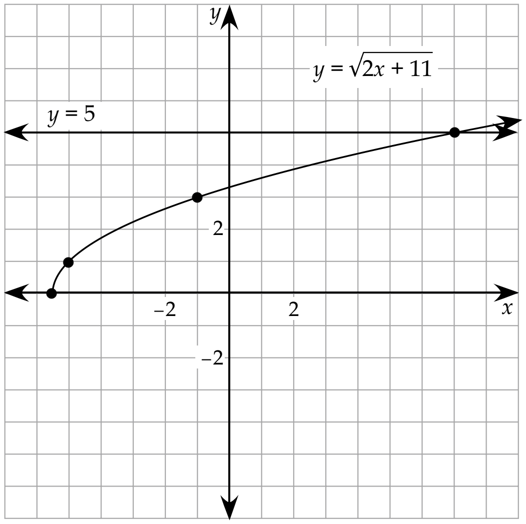
Solution

Method 1: Graph each side of the equation.

To solve this equation using Method 1, graph $y=\sqrt{2x+11}$ and $y = 5$.

To graph $y=\sqrt{2x+11}$, first factor the radicand and then graph the resulting function using transformations.

$$y=\sqrt{2x+11}=\sqrt{2\left(x+\frac{11}{2}\right)}$$



These functions intersect at the point (7, 5). The solution to this equation is $x = 7$.

Method 2: Graph one function.

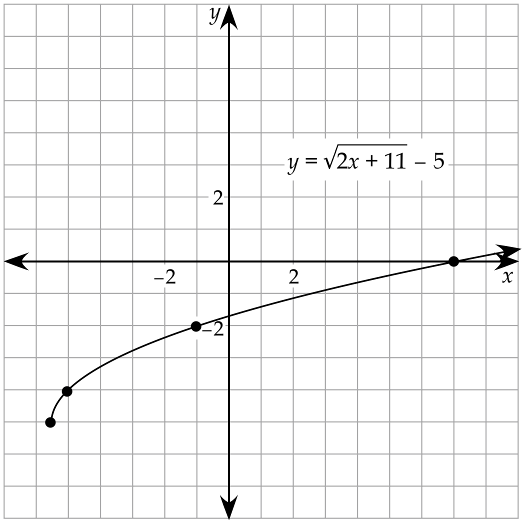
To use this method, first move everything to one side of the equation and then graph the corresponding function.

$$\sqrt{2x+11}-5=0$$

The corresponding function is $y=\sqrt{2x+11}-5$.

To graph this function, use transformations.

$y=\sqrt{2\left(x+\frac{11}{2}\right)}-5$ is the standard radical graph compressed horizontally by a factor of 2, moved $\frac{11}{2}$ units to the left, and moved 5 units down.



This function crosses the x-axis at $x = 7$. Notice the x-intercept of the function $y=\sqrt{2x+11}-5$ is the same as the root of the corresponding radical equation $\sqrt{2x+11}=5$.

Before you complete the following learning activity, include a brief summary of the two different methods for solving a radical equation graphically on your Resource Sheet. This will help you in the following learning activity, assignment, and final examination when you are asked to answer questions regarding this concept.

Learning Activity 8.3

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. If $f(x) = -x^2 + 5$, find $f(5)$.
2. What are the zeros of the function $f(x) = x^2 + 7x + 12$?
3. Express $2^7 = 128$ in log form.
4. Express $\log_6 36 = 2$ in exponential form.
5. Convert 60° to radians.
6. State the non-permissible values of the function $f(x) = \dfrac{x-2}{x^2-2x}$.
7. Simplify: $\sqrt[3]{54x^4y^2z^3}$
8. Determine the inverse function $f(x) = \dfrac{3}{x-2}$.

continued

Learning Activity 8.3 (continued)

Part B: Radical Equations

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Consider the equation $\sqrt{x+3} = x + 3$.
   a) Determine the roots of the above equation algebraically.
   b) Sketch $y = \sqrt{x+3}$ and $y = x + 3$ to determine the solution(s) of $\sqrt{x+3} = x + 3$.
2. Solve the following radical equation graphically.
   $\sqrt{x+7} = 3$
3. The root of the radical equation $2\sqrt{x-3} = 4$ is $x = 7$. Write the related function that will have an x-intercept of 7.
4. Estimate the solutions to the following radical equations graphically.
   $\sqrt{3x-3} - 2 = 6$

Lesson Summary

In this lesson, you learned how to solve radical equations. In order to solve radical equations, you had to be confident in your ability to graph radical functions, as you were mainly asked to solve radical equations graphically. You learned two methods of solving radical equations graphically—graphing one function to find the x-intercept(s) and graphing two functions to determine where they intersect.

In the next lesson, you will switch your focus to begin learning about rational functions.

Notes

Assignment 8.2

Solving Radical Equations

Total: 13 marks

You need to show all your steps as you work out your solutions, and make sure your answers are clear (include units, where appropriate). Check the Introduction for the Marking Guidelines that will be used for all assignments and examinations.

1. Consider the equation $2\sqrt{x + 3} = 6$.

   a) Determine the roots of the above equation algebraically. *(2 marks)*

continued

b) Determine the x-intercept of the corresponding function $y = 2\sqrt{x+3} - 6$. *(2 marks)*

c) Explain the relationship between the roots of $2\sqrt{x+3} = 6$ and the x-intercepts of $y = 2\sqrt{x+3} - 6$. *(1 mark)*

continued

Assignment 8.2: Solving Radical Equations (continued)

2. Estimate the solution to the following radical equation graphically.

   $\sqrt{3x-6} = 2x - 7$ *(4 marks)*

continued

Assignment 8.2: Solving Radical Equations (continued)

3. Estimate the solution to the following radical equation graphically.

   $\sqrt{4x-8} = \frac{1}{5}x$ *(4 marks)*

LESSON 4: RATIONAL FUNCTIONS

Lesson Focus

In this lesson, you will

- ☐ learn how to graph a rational function using transformations
- ☐ learn how to graph a rational function using graphical properties
- ☐ learn how to identify common characteristics of rational functions

Lesson Introduction

In Module 2, Lesson 5, you briefly looked at reciprocal functions. Reciprocal functions are rational functions where the numerator is equal to 1. This relates to fractions where the reciprocal of 2 is $\frac{1}{2}$. In this lesson, you will learn about different variations of rational functions, including functions that have a trinomial in the numerator and the denominator.

Rational functions can be used to model how much of a drug will be in a patient's bloodstream after a certain period of time. Rational functions are also used in businesses as cost-benefit models. For example, is it worth it for the Government of Manitoba to provide free influenza shots for the entire population if it decreases the amount in potential health care costs by a certain value? How much do the potential health-care costs need to be decreased in order for the free influenza shots to be deemed worthwhile?

Rational Functions

A **rational function** is a function of the form $f(x) = \frac{p(x)}{q(x)}$, where $q(x) \neq 0$ and where $p(x)$ and $q(x)$ are polynomials in x.

Examples of rational functions are:

$$f(x) = \frac{1}{2x+1} \text{ and } g(x) = \frac{x^2+1}{(x-1)(x+5)}$$

Examples of functions that are not rational are:

$$f(x) = x^2 + 2x + 1 \text{ and } g(x) = \frac{2}{\sqrt{3x}}$$

When you are graphing rational functions, you need to be aware that some x-values may not be in the domain. Since you can't divide by zero, the x-values that make the denominator of a rational function equal to zero are not allowed. Therefore, you should give special attention to the shape of the graph near x-values that are not in the domain.

Graphing Rational Functions using Transformations

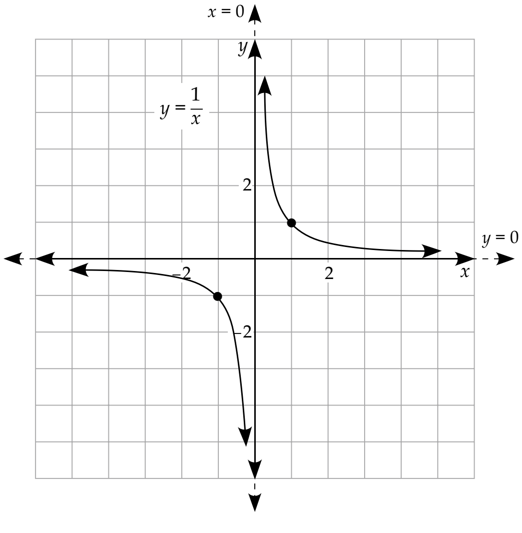
In previous modules, you learned to graph many functions using transformations. It is also possible to graph rational functions using transformations. The standard form of a rational function with a constant in the numerator and a binomial in the denominator is $y = \frac{a}{x-h} + k$. To understand the effect each of the variables a, h, and k has on the function $y = \frac{1}{x}$, consider the following examples.

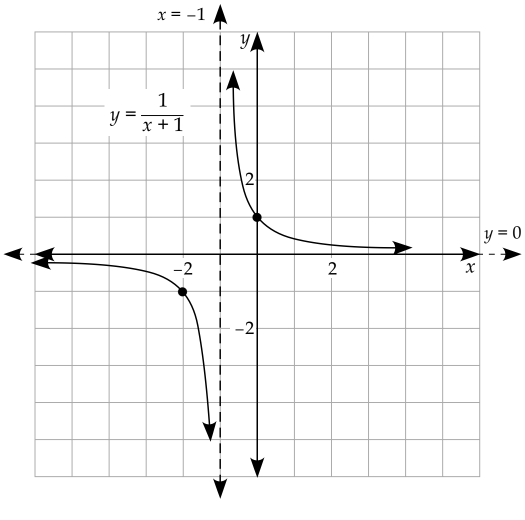
Example 1

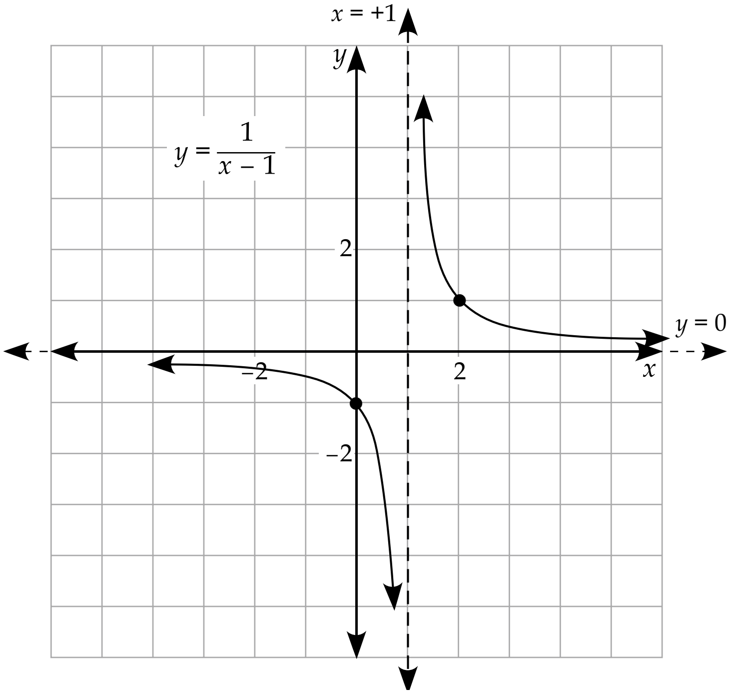
Graph $y = \frac{1}{x}$, $y = \frac{1}{x+1}$, and $y = \frac{1}{x-1}$ on different coordinate grids.

Determine the effect the variable h has on the function $y = \frac{1}{x-h}$.

Solution







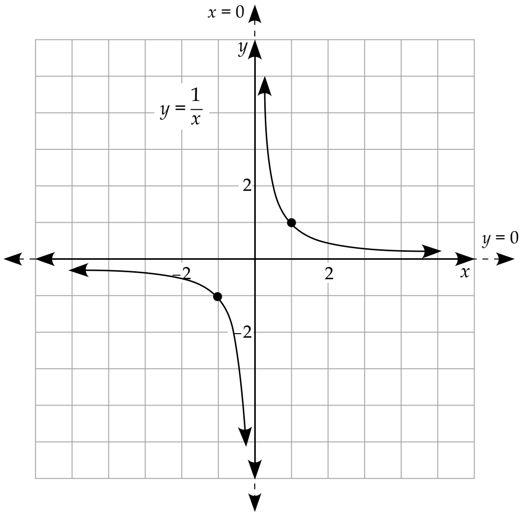
From the above graphs, you will notice that the variable h in $y = \dfrac{1}{x-h}$ shifts the graph horizontally h units to the right if h is positive and h units to the left if h is negative. Therefore, the function $y = \dfrac{1}{x-h}$ is a horizontal shift of the function $y = \dfrac{1}{x}$. The vertical asymptote also shifts and must be drawn in the new location.

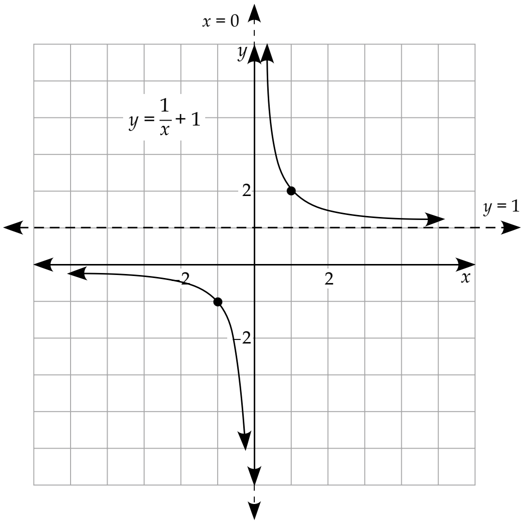
Example 2

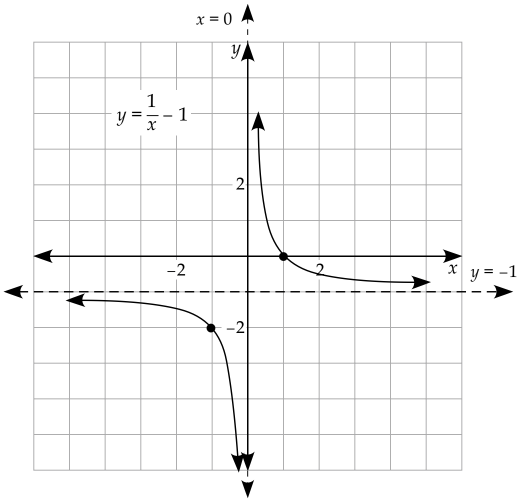
Graph $y = \dfrac{1}{x}$, $y = \dfrac{1}{x} + 1$, and $y = \dfrac{1}{x} - 1$ on different coordinate grids.

Determine the effect the variable k has on the function $y = \dfrac{1}{x} + k$.

Solution







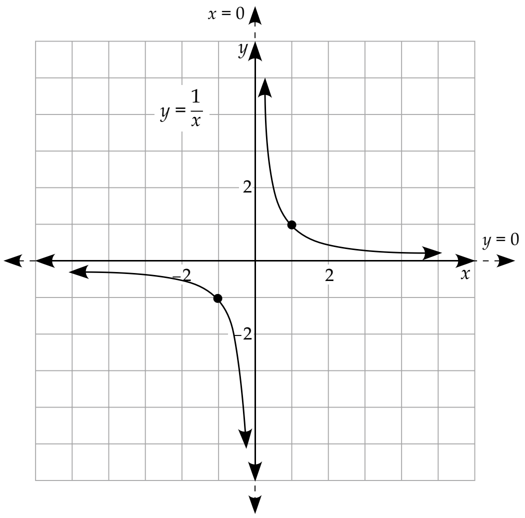
From the above graphs, you will notice that the variable *k* in $y = \frac{1}{x} + k$ shifts the graph vertically *k* units up if *k* is positive and *k* units down if *k* is negative. Therefore, the function $y = \frac{1}{x} + k$ is a vertical shift of the function $y = \frac{1}{x}$.

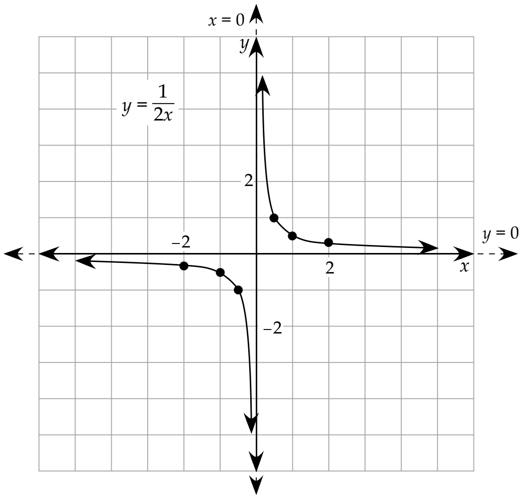
The horizontal asymptote also shifts and must be drawn in the new location.

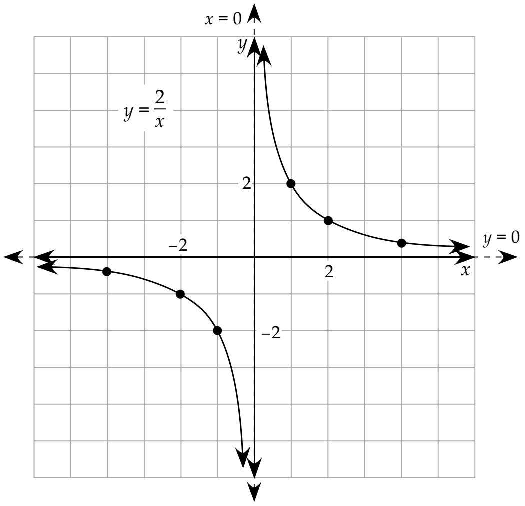
Example 3

Graph $y = \frac{1}{x}$, $y = \frac{1}{2x}$, and $y = \frac{2}{x}$ on different coordinate grids. Determine the effect the variable a has on the function $y = \frac{a}{x}$.

Solution







From the above graphs, you will notice that the variable a in $y = \frac{a}{x}$ stretches the graph by a factor of a if $a > 1$ and compresses the graph by a factor of $\frac{1}{a}$, if $0 < a < 1$. Therefore, the function $y = \frac{a}{x}$ can be thought of as a stretch or a compression of the function $y = \frac{1}{x}$.

The following chart summarizes the transformations you just discovered. It may be helpful for you to include a similar chart on your resource sheet.

Transformations of the function $y = \frac{1}{x}$: $y = \frac{a}{x - h} + k$		
Variable	Effect on graph of $y = \frac{1}{x}$	Effect on (x, y)
h	Horizontal Translation	$(x, y) \rightarrow (x + h, y)$
k	Vertical Translation	$(x, y) \rightarrow (x, y + k)$
a	Vertical Stretch/Compression	$(x, y) \rightarrow (x, ay)$

Example 4

Graph the following rational functions using transformations.

a) $y = \dfrac{3}{x + 2} - 1$

b) $y = \dfrac{1}{2(x - 4)} + 3$

Solutions

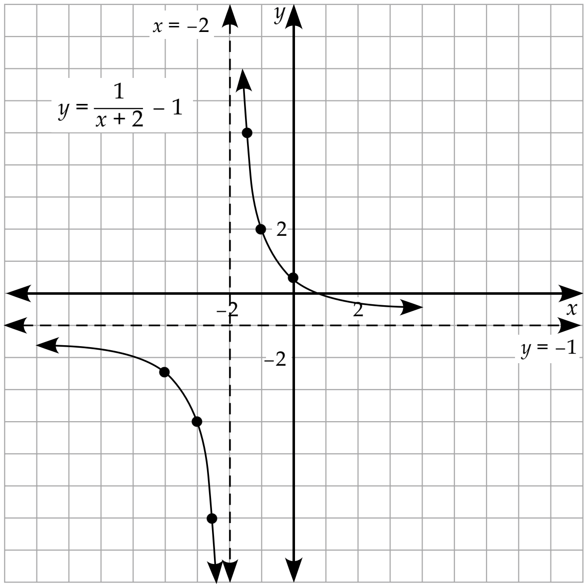
a) Remember, stretches and compressions are always performed before translations. Therefore, this function is a vertical stretch by a factor of 3 of the function $y = \frac{1}{x}$, which is then translated 2 units to the left and 1 unit down.

You also need to transform the vertical asymptote located at $x = 0$ and the horizontal asymptote located at $y = 0$ of the base function, $y = \frac{1}{x}$.

Asymptotes will not be affected by stretches or compressions, but they will be affected by translations.

Create a table of values to help you graph this function:

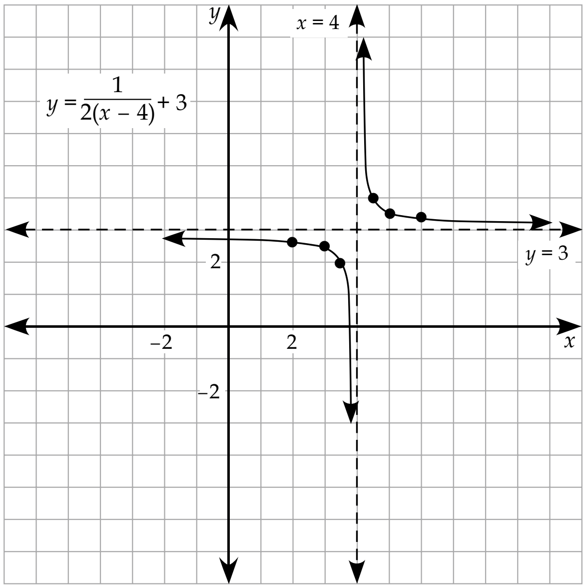
x	$y = \frac{1}{x}$	$(x - 2)$	$(3y - 1)$
−2	−0.5	−4	−2.5
−1	−1	−3	−4
−0.5	−2	−2.5	−7
Vertical Asymptote at $x = 0$		Vertical Asymptote at $x = -2$	
Horizontal Asymptote at $y = 0$		Horizontal Asymptote at $y = -1$	
0.5	2	−1.5	5
1	1	−1	2
2	0.5	0	0.5



b) This function is a vertical compression by a factor of 2 of the graph of $y = \frac{1}{x}$, which is then translated 4 units to the right and 3 units up.

x	$y = \frac{1}{x}$	$(x + 4)$	$\left(\frac{1}{2}y + 3\right)$
−2	−0.5	2	2.75
−1	−1	3	2.5
−0.5	−2	3.5	2
Vertical Asymptote at $x = 0$		Vertical Asymptote at $x = 4$	
Horizontal Asymptote at $y = 0$		Horizontal Asymptote at $y = 3$	
0.5	2	4.5	4
1	1	5	3.5

2	0.5	6	3.25



Graphing Rational Functions Using Graphical Properties

There are many different variations of rational functions that can become more and more complex. In this course, you will only study rational functions that have monomials, binomials, or trinomials in the numerator and denominator.

Recall the following definitions:

- A **monomial** is a polynomial with one term having a whole number exponent that is greater than or equal to 0.
- A **binomial** is a polynomial with two terms (for example, $6x + 3$ is a binomial).
- A **trinomial** is a polynomial with three terms.

In order to graph these rational functions of various forms, their graphical properties are used. These graphical properties include asymptotes, intercepts, and points of discontinuity. You will learn about points of discontinuity in the next lesson.

Holes (or Points of Discontinuity)

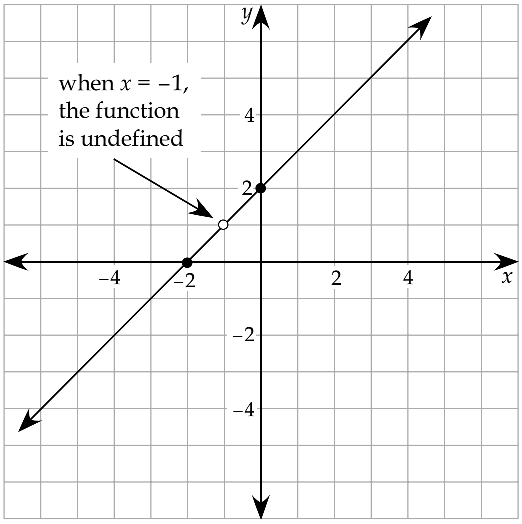
Some rational functions that have a common factor in the numerator and the denominator result in a curve with a "hole" in it. Check the function $y = \frac{x^2 + 3x + 2}{x + 1}$ when $x = -1$. The calculation works out to $\frac{0}{0}$, which is undefined.

Sketch:

$$y = \frac{x^2 + 3x + 2}{x + 1}$$

$$y = \frac{(x + 1)(x + 2)}{x + 1}$$

$$y = x + 2,\ x \neq -1$$

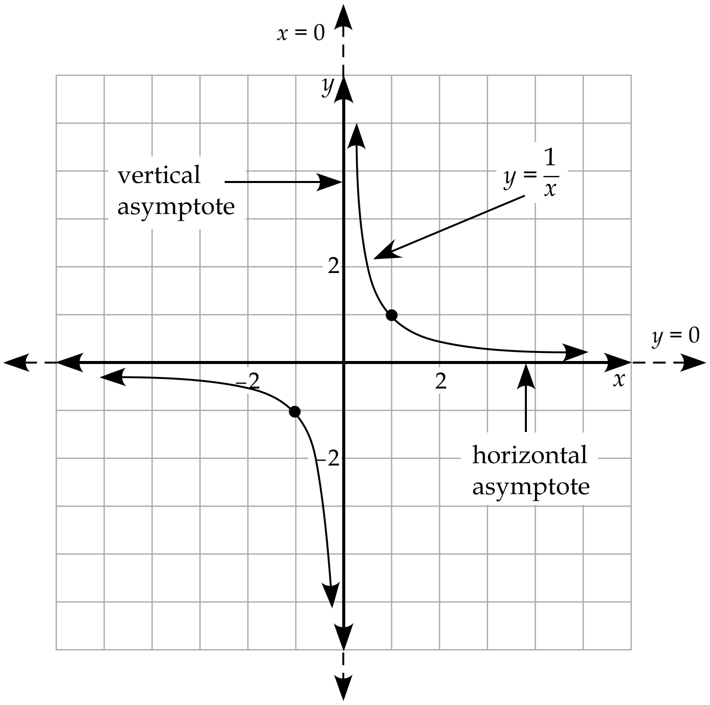


The graph of the function $y = \frac{x^2 + 3x + 2}{x + 1}$ looks just like $y = x + 2$ with a hole where $x = -1$ (that is, $(-1, 1)$).

The Asymptotes

In order to graph rational functions, you need to be familiar with asymptotes and when they occur. You have worked with asymptotes throughout this course.

An asymptote describes a value that the function approaches but does not reach as it goes off to infinity, either horizontally or vertically.



Vertical Asymptotes

Vertical Asymptote: The line $x = a$ is a vertical asymptote of the graph of f if $f(x)$ approaches $+\infty$ or $f(x)$ approaches $-\infty$ as x approaches a, either from the right or from the left.

Vertical asymptotes can occur when the denominator of a rational function equals zero. When the denominator of a rational function equals zero, two things may happen. If the numerator and the denominator of a rational function equal zero for a certain value of x, then there is a hole in the function. If only the denominator equals zero for a certain value of x, then there is a vertical asymptote located at that value.

Horizontal Asymptotes

Horizontal Asymptote: The line $y = b$ is a horizontal asymptote of the graph of f if $f(x)$ approaches b as x approaches $+\infty$ or x approaches $-\infty$.

A horizontal asymptote is the value that a function approaches as x goes to negative infinity (left on the x-axis) or to positive infinity (right on the x-axis). **It is possible for a function to cross a horizontal asymptote, since the asymptotic behaviour only applies to what is happening way off to the left and way off to the right.**

The expression, $\frac{1}{x}$, gets smaller as x gets larger $\left(\text{that is, } \frac{1}{2} > \frac{1}{10} > \frac{1}{100} > \ldots\right)$.

The value of $\frac{1}{x}$ approaches zero as x approaches either negative infinity ($-\infty$) or positive infinity ($+\infty$). Similarly, since the denominator increases in value faster than the numerator, the following expressions also approach zero as x approaches a large number to the left or to the right: $\frac{1}{x^2}, \frac{x}{x^2}, \frac{x+1}{x^2+2}$.

Generally speaking, if the power of x in the denominator is larger than the power of x in the numerator, the rational expression will approach zero as x approaches a larger number to the left or to the right.

Examples of functions with horizontal asymptotes at $y = 0$ are:

$$f(x) = \frac{x+1}{x^2 - 5x + 6}$$

and

$$g(x) = \frac{5}{x^2 + 4}$$

If the power of x in the denominator and numerator are the same, the horizontal asymptote will not be at $y = 0$.

You can use long division to find the horizontal asymptote, but once you see the pattern, long division will not be needed to graph a rational function.

Below are four rational functions:

1. $y = \dfrac{x+3}{x-1}$

$$\begin{array}{r} 1 \\ x-1\overline{\smash{)}x+3} \\ \underline{x-1} \\ 4 \end{array}$$

$$y = 1 + \frac{4}{x-1}$$

As x gets very large ($\pm\infty$), the second term, $\dfrac{4}{x-1}$, will approach zero, so the horizontal asymptote is $y = 1$.

2. $y = \dfrac{3x^2 + 5x + 1}{2x^2}$

After long division, you get $y = \dfrac{3}{2} + \dfrac{5x+1}{2x^2}$.

As x gets very large ($\pm\infty$), the second term, $\dfrac{5x+1}{2x^2}$, will approach zero, so the horizontal asymptote is $y = \dfrac{3}{2}$.

After doing the long division, you may have noticed how the coefficients can be used to determine the horizontal asymptote in the last two examples. In general, when the degree of x for the polynomial in the numerator is the same as the degree of x for the polynomial in the denominator, the horizontal asymptote equation can be determined using the quotient of the two leading coefficients.

3. $y = \dfrac{x+1}{x^2 - 4}$

$$\begin{array}{r} 0 \\ x^2-4\overline{\smash{)}x+1} \end{array}$$

$$y = 0 + \frac{x+1}{x^2-4}$$

As x gets very large ($\pm\infty$), the second term, $\dfrac{x+1}{x^2-4}$, will approach zero, so the horizontal asymptote is $y = 0$.

In general, any rational function with a numerator that is a polynomial with a lower degree than the denominator will have a horizontal asymptote of $y = 0$.

4. $y = \dfrac{2x^2 - x - 3}{x + 2}$

$$\begin{array}{r} 2x - 5 \\ x + 2\overline{)\,2x^2 - x - 3} \\ \underline{2x^2 + 4x} \\ -5x - 3 \\ \underline{-5x - 10} \\ 7 \end{array}$$

$$y = 2x - 5 + \frac{7}{x + 2}$$

As x gets very large (±∞), the last term, $\dfrac{7}{x+2}$, will approach zero, so the asymptote will be the function $y = 2x - 5$.

That means that there is no horizontal asymptote. Instead, as the function goes off to the right (or to the left), it gets closer and closer to the oblique line with a slope of 2 and y-intercept of -5. Note that an oblique line is the name given to a line that is neither horizontal nor vertical.

In general, any rational function with a numerator that is a polynomial with a degree higher than the denominator will not have a horizontal asymptote.

The following guidelines pertain to the asymptotes. Let $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ have no common factors. **Note:** You will learn more about the situation where $p(x)$ and $q(x)$ have common factors in the next lesson. You may wish to add these guidelines to your resource sheet.

1. The graph of f has a vertical asymptote at each real zero of $q(x)$, the denominator, after determining the factors common to $p(x)$ that create holes.
2. The graph of f has a horizontal asymptote determined as follows:
   - If the degree of $p(x)$ is less than the degree of $q(x)$, then the line, $y = 0$, is a horizontal asymptote.
   - If the degree of $p(x)$ is equal to the degree of $q(x)$, then the line, $y = \dfrac{a}{b}$, is a horizontal asymptote, where a is the leading coefficient of $p(x)$ and b is the leading coefficient of $q(x)$.
   - If the degree of $p(x)$ is greater than the degree of $q(x)$, then the graph has no horizontal asymptote. It may have an asymptote; the asymptote is just not a horizontal line.
3. Any vertical and horizontal asymptotes that occur (unless they are right on the x- or y-axis) should be shown on the graph as a dotted line.

Example 5

Find the vertical and horizontal asymptotes of:

a) $f(x) = \dfrac{1}{(x+2)^2}$

b) $f(x) = \dfrac{2x^2}{3x^2 + 1}$

c) $f(x) = \dfrac{x^3 + 1}{x^2}$

Solutions

a) The vertical asymptote(s) occurs at the zeros of denominator $q(x)$. Therefore, set the denominator equal to zero to determine the equation of the vertical asymptote.

$x + 2 = 0$

$x = -2$ is the equation of the vertical asymptote.

Since the degree of the numerator is less than the degree of the denominator, the x-axis is the horizontal asymptote and the equation is $y = 0$.

Note: The degree of the numerator is zero as $x^0 = 1$.

b) The vertical asymptote occurs at the zeros of the denominator $q(x)$. Since $3x^2 + 1$ has no real zeros, then the function has no vertical asymptote.

The degree of the numerator is equal to the degree of the denominator. Thus, the horizontal asymptote is given by the ratio of the leading coefficients of the numerator and denominator.

The leading coefficient of the numerator is 2.

The leading coefficient of the denominator is 3.

The horizontal asymptote is located at $y = \dfrac{2}{3}$.

c) The vertical asymptote occurs at the zeros of the denominator.

$x^3 = 0$

$x = 0$

The equation of the vertical asymptote is $x = 0$; this is the y-axis.

Because the degree of the numerator is larger than the degree of the denominator, there is no horizontal asymptote.

However, as $x \to \infty$, $y \to \frac{x^3}{x^2} = x$. Thus, $y = x$ is an oblique asymptote. In this course, you are only required to draw curves with vertical and/or horizontal asymptotes.

The above procedure for finding the asymptotes becomes a part of the following guidelines to graph a rational function.

Rational functions can be difficult to graph by merely plotting points. Identifying asymptotes before you graph can help you find key features. The following guidelines will help you to graph rational functions. You may want to include these guidelines, or a brief summary, on your resource sheet.

Guidelines

Let $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials, in factored form, with no common factors.

1. Find and plot the *y-intercept* (if any) by evaluating $f(0)$. Remember the y-intercept is at a point on the y-axis and so the x-coordinate is zero at that point.
2. Find the zeros of the numerator (if any) by solving the equation $p(x) = 0$. Then plot the corresponding *x-intercepts* for $f(x)$.
3. Determine the location of any *holes* if there are common factors between $p(x)$ and $q(x)$. Simplify the function. **Note:** You will learn more about situations where there are common factors in the next lesson.
4. Sketch the *vertical asymptotes* by solving the equation of the denominator of the simplified function $q(x) = 0$ to find the zeros of the denominator.
5. Find and sketch the *horizontal asymptote* (if any) by long division or by using the rule for finding the horizontal asymptote given earlier.
6. Use *sign analysis* to show where the function is positive and where it is negative.
7. Find one point on either side of each vertical asymptote.
8. Use *smooth curves* to complete the graph between the vertical asymptotes.

The following examples will help you use these guidelines to graph rational functions.

Example 6

Sketch the graph of $f(x) = \dfrac{3}{x-2}$.

Solution

You could graph this function using transformations. However, in this example, this function is graphed using the guidelines outlined above.

There are no common factors to the numerator or denominator.

y-intercept:

Let $x = 0$

$$f(0) = \frac{3}{0-2} = -\frac{3}{2}$$

$\therefore$ The point is at $\left(0, -\dfrac{3}{2}\right)$.

x-intercept:

Because there is no variable in the numerator, there is no x-intercept.

Vertical asymptote:

Find the zero by letting the expression in the denominator equal 0.

$$x - 2 = 0$$
$$x = 2$$

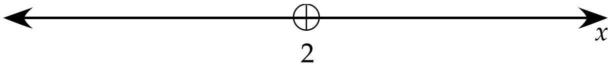
Horizontal asymptote:

Because the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is the x-axis and its equation is $y = 0$.

Sign analysis:

Critical values occur where the sign of a function may change between positive and negative. Critical values can only be at zeros and at vertical asymptotes.

For $y = \dfrac{3}{x-2}$, there is only one critical value, $x = 2$. Thus, on the number line:



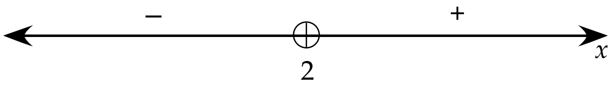
Interval: $(-\infty, 2)$: test $x = 1$

$$f(1) = \frac{3}{1-2} = \frac{3}{-1} = -3 \rightarrow \text{negative}$$

Interval: (2, ∞): test $x = 3$

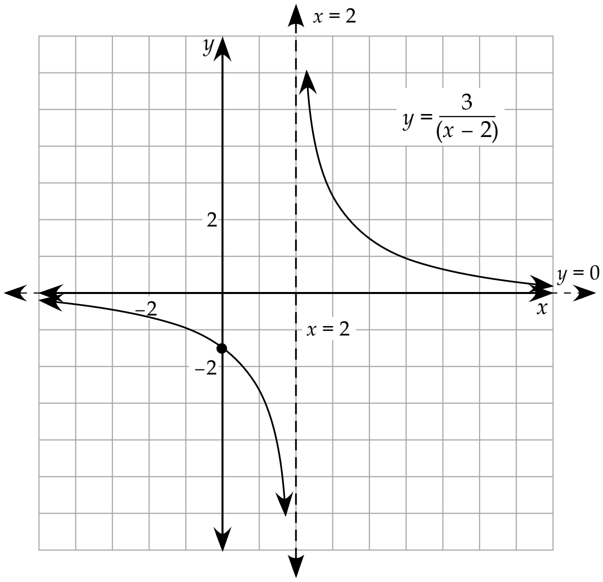
$$f(2) = \frac{3}{3-2} = \frac{3}{1} = 3 \rightarrow \text{positive}$$

Summary diagram:



Draw in the curve:

You may wish to locate additional points to help with the sketch.



Example 7

Sketch the graph of $f(x) = \dfrac{2x^2}{x^2 - 4}$.

Solution

The numerator and denominator have no *common factors*.

y-intercept:

Find $f(0)$.

$$f(0) = \frac{2(0)^2}{0^2 - 4} = 0$$

x-intercept:

Let $2x^2 = 0 \rightarrow x = 0$

Vertical asymptote:

Let the denominator $x^2 - 4 = 0$, find the zeros, and draw the dotted vertical lines through those zeros.

$(x - 2)(x + 2) = 0$

$x = 2$ or -2

Horizontal asymptote:

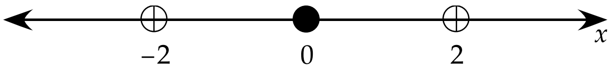
Because the degree of the numerator and denominator are the same, the horizontal asymptote equation is the ratio of the leading coefficient of the numerator and denominator.

Thus, the horizontal asymptote is $y = \frac{2}{1} = 2$.

Sign analysis:

The zero of the function is 0 and the vertical asymptotes are at -2 and 2. Therefore, the critical numbers are 0, -2, and 2.

The number line is:



Test the intervals:

Interval $(-\infty, -2)$: test $x = -3$

$$f(-3) = \frac{2(-3)^2}{(-3)^2 - 4} = \frac{18}{5} = 3.6 \rightarrow \text{positive}$$

The curve approaches the asymptote, $y = 2$, through values greater than 2. Thus, the curve is above the asymptote.

Interval $(-2, 0)$: test $x = -1$

$$f(-1) = \frac{2(-1)^2}{(-1)^2 - 4} = \frac{2}{-3} \rightarrow \text{negative}$$

The curve approaches the asymptote, $x = -2$, on the right, through negative values.

Interval $(0, 2)$: test $x = 1$

$$f(1) = \frac{2(1)^2}{1^2 - 4} = \frac{2}{-3} \rightarrow \text{negative}$$

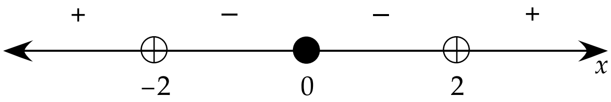
The curve approaches the asymptote, $x = 2$, on the left, through negative values.

Interval (2, ∞): test $x = 3$

$$f(3) = \frac{2(3)^2}{(3)^2 - 4} = \frac{18}{5} = \frac{3}{6} \rightarrow \text{positive}$$

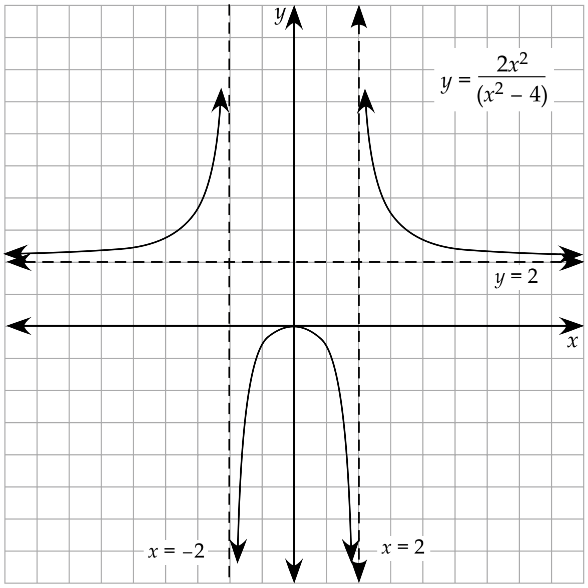
The curve approaches the asymptote, $y = 2$, through values greater than 2. Thus, the curve is above the asymptote.

Summary diagram:



The x- and y-intercepts are at the origin. The graph approaches the horizontal asymptote from above the line where $x > 2$ and $x < -2$.

Note: You can always determine the coordinates of some points in the region to help graph the function.



Example 8

Sketch the graph of $f(x) = \dfrac{8}{x^2 + 4}$.

Solution

There are no *common factors* to the numerator or denominator.

y-intercept:

Find $f(0)$.

$$f(0) = \frac{8}{0^2 + 4} = 2$$

x-intercept:

None, because there is only a constant in the numerator, and no variable.

Vertical asymptote:

None, because $x^2 + 4$ has no real zeros.

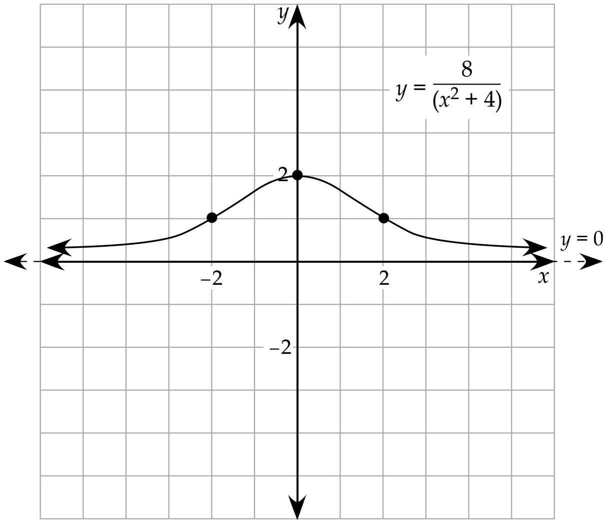
Horizontal asymptote:

Because the degree of the numerator is less than the degree of the denominator, the x-axis is the horizontal asymptote and the equation is $y = 0$.

Sign analysis:

The sign analysis yields a positive region throughout; there are no critical values and a test of $x = 0$ yields a positive y-value.

Determine points that lie on the curve in areas where you are unsure of the shape.



Example 9

Sketch the graph of $f(x) = \dfrac{x + 2}{x + 4}$.

Solution

You could use a transformational approach to graphing this curve, but first of all you would need to rewrite it using long division.

You can also graph this function using graphical properties, as seen below.

There are no *common factors* to the numerator or denominator.

y-intercept:

Find $f(0)$.

$$f(0) = \frac{0 + 2}{0 + 4} = \frac{2}{4} = \frac{1}{2}$$

x-intercept:

Set the numerator equal to zero.

$x + 2 = 0$

$x = -2$

Vertical asymptote:

Set the denominator equal to zero.

$x + 4 = 0$

$x = -4$

Horizontal asymptote:

Because the degree of the numerator is the same as the degree of the denominator, the horizontal asymptote equation is the ratio of the leading coefficient of the numerator and denominator. The horizontal asymptote is $y = \dfrac{1}{1} = 1$.

Sign analysis:

There is a zero at −2 and a vertical asymptote at −4.

There are two critical points, −2 and −4.

Test the intervals:

Interval $(-\infty, -4)$: test $x = -5$

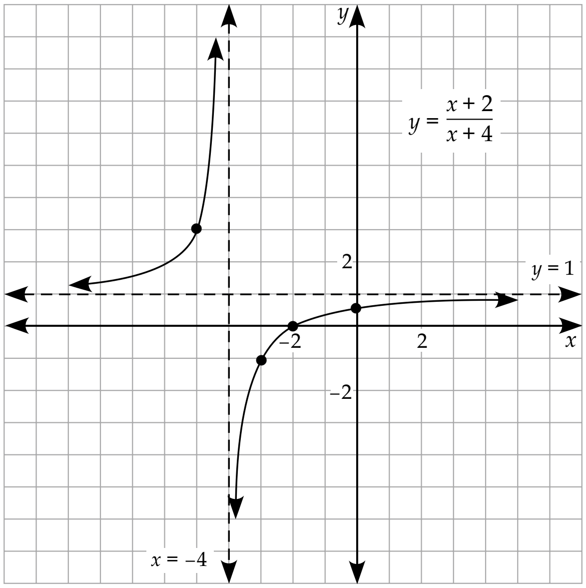
$$f(-5) = \frac{-5 + 2}{-5 + 4} = \frac{-3}{-1} = 3 \rightarrow \text{positive}$$

Interval $(-4, -2)$: test $x = -3$

$$f(-3) = \frac{-3+2}{-3+4} = \frac{-1}{1} = -1 \rightarrow \text{negative}$$

Interval $(-2, \infty)$: test $x = 0$

$$f(0) = \frac{0+2}{0+4} = \frac{2}{4} = \frac{1}{2} \rightarrow \text{positive}$$



From this graph, can you determine the equation of the rational function in the form $y = \frac{a}{x-h} + k$?

It is easy to determine the h- and k-values, as this graph has been shifted 4 units to the left, $h = -4$, and 1 unit up, $k = 1$. This graph has also been reflected through the x-axis and stretched by a factor of 2, which results in a negative a-value of $a = -2$.

Therefore, the function is $y = -\frac{2}{x+4} + 1$.

Make sure you complete the following learning activity, as it will allow you to practice all of the skills you just learned.

Learning Activity 8.4

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Solve for x: $(x + 1)! = 6x!$
2. Simplify: $\sqrt{\dfrac{81x^2}{9y^4}}$
3. Calculate: $\sin 15° \csc 15°$
4. In which quadrant is $\cot \theta$ positive and $\tan \theta$ negative?
5. Express $7^2 = 49$ in logarithmic form.
6. Express $\log_4 256 = 4$ in exponential form.
7. Solve for x: $\log 10^{(x+2)} = 3$
8. Simplify: $\left(5 + \sqrt{x}\right)^2$

continued

Part B: Graphing Rational Functions

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Identify the (i) x-intercepts, (ii) y-intercept, and (iii) the equation of any vertical or horizontal asymptotes for the following.

   a) $y = \dfrac{x-1}{(x-2)(x-3)}$

   b) $y = \dfrac{x}{x^2-9}$

   c) $y = \dfrac{x^2-4}{x^2+4}$

2. Match the function with its graph. Look for asymptotes.

   a) $f(x) = \dfrac{x+3}{x-2}$

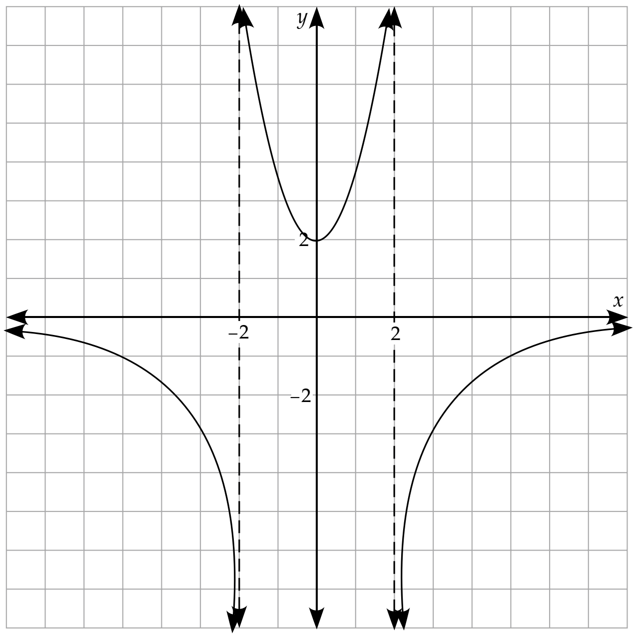
   b) $f(x) = \dfrac{-8}{x^2-4}$

   c) $f(x) = \dfrac{3}{x+2}$

   d) $f(x) = \dfrac{x-3}{2x+4}$

   e) $f(x) = \dfrac{3}{x^2-x-6}$

   i)



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Learning Activity 8.4 (continued)

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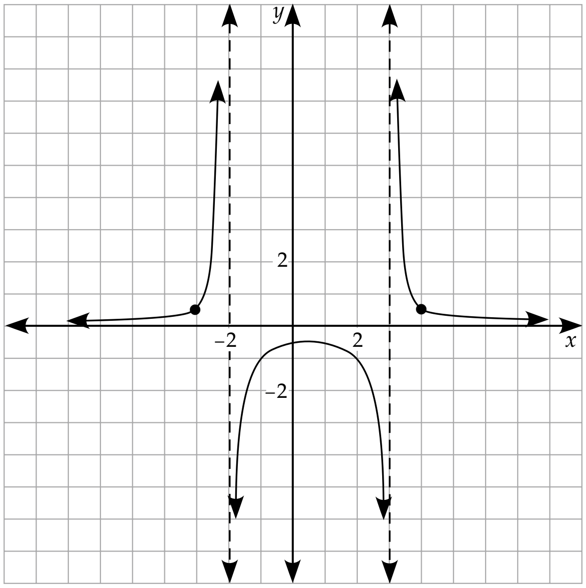

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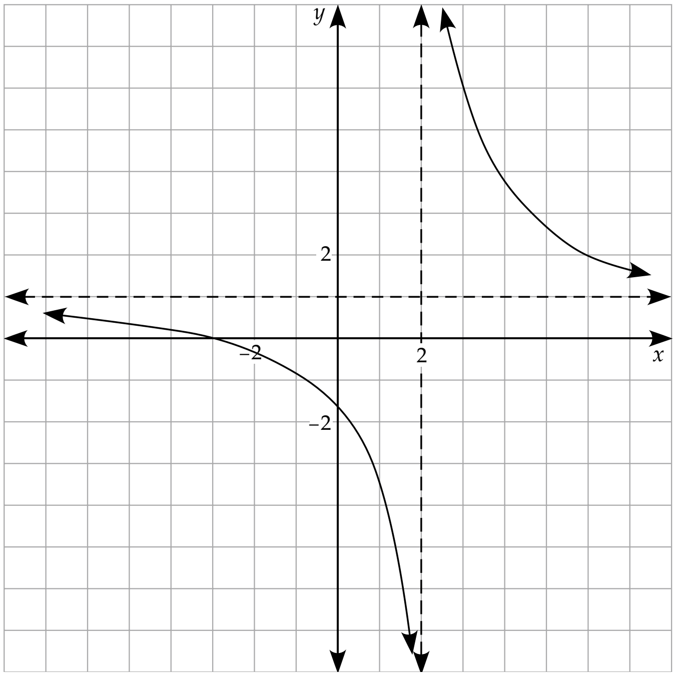

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Learning Activity 8.4 (continued)

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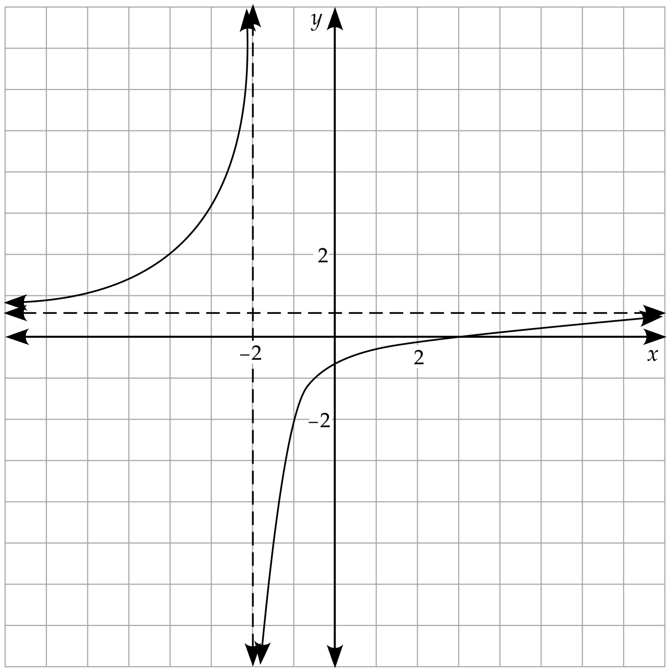

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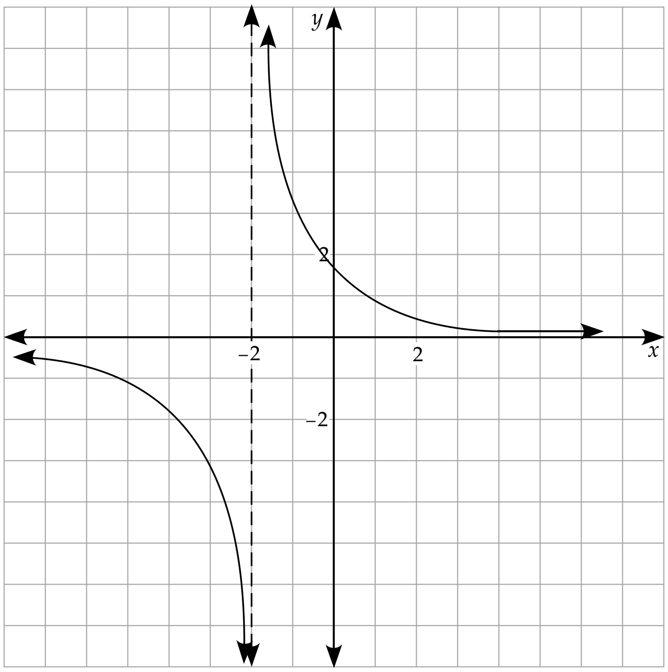

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Learning Activity 8.4 (continued)

3. Sketch the graph of each rational function. Label the x-intercept(s), y-intercept, and state the equation of the horizontal and vertical asymptotes (if they exist), and the domain and range.

   a) $f(x) = \dfrac{-1}{x-4}$

   b) $f(x) = \dfrac{2}{3(x-1)}$

   c) $f(x) = \dfrac{3x}{x+2}$

   d) $f(x) = \dfrac{x+2}{x-3}$

   e) $f(x) = \dfrac{3}{x^2-9}$

   f) $f(x) = \dfrac{x^2-4}{x^2-9}$

   g) $f(x) = \dfrac{x^2}{x^2-1}$

   h) $f(x) = \dfrac{x}{x^2-1}$

   i) $f(x) = \dfrac{3x}{x^2-9}$

   j) $f(x) = \dfrac{2}{x^2+1}$

   k) $f(x) = \dfrac{3}{x^2+2x+1}$

   l) $f(x) = \dfrac{x+3}{x^2+7x-8}$

4. Determine the equation of the function in the form $f(x) = \dfrac{a}{x-h}$ that has a vertical asymptote at $x = 3$ and a y-intercept of (0, 5).

5. The function $c(t) = \dfrac{45t}{t^2+15}$ describes the concentration of a drug in a patient's bloodstream over time. Time, t, is measured in hours while $c(t)$, or the concentration, is measured in milligrams per litre.
   a) Graph this function.
   b) What is the domain in this context?
   c) What is the approximate range in this context?
   d) What happens to the concentration of the drug during the first 4 hours?
   e) What happens to the concentration of the drug over 1 day, or 24 hours?
   f) Is this a reasonable representation of the concentration of a drug in a patient's bloodstream over time?

Lesson Summary

In this lesson, you learned how to graph rational functions using two methods. First, you learned how to graph rational functions using transformations, similar to what you did previously in this module with radical functions. Second, you learned how to graph rational functions using their properties.

In the next lesson, you will be looking at two of these properties in more detail—asymptotes and points of discontinuity.

Lesson 5: Asymptotes Versus Discontinuities

Lesson Focus

In this lesson, you will

- ☐ learn how to identify when a rational function has an asymptote or a discontinuity
- ☐ learn how to identify when a rational function has an asymptote or a point of discontinuity
- ☐ learn about properties of various types of rational functions

Lesson Introduction

In this lesson, you will be introduced to rational functions that contain a common factor in their numerator and denominator. When this occurs, the function is undefined at the x-value, causing the common factor to equal zero. This value becomes a non-permissible value. However, there are different types of non-permissible values. Sometimes, non-permissible values lead to asymptotes and sometimes they lead to points of discontinuity, which are graphed as holes. In this lesson, you will learn to distinguish between when a non-permissible value indicates an asymptote and when it indicates a point of discontinuity (or hole).

Analyzing Functions

In order to determine whether a function has an asymptote or a discontinuity, consider the following two functions, which are identical except at one point.

$$y = \frac{1}{x+4} \text{ and } y = \frac{x+2}{(x+4)(x+2)}$$

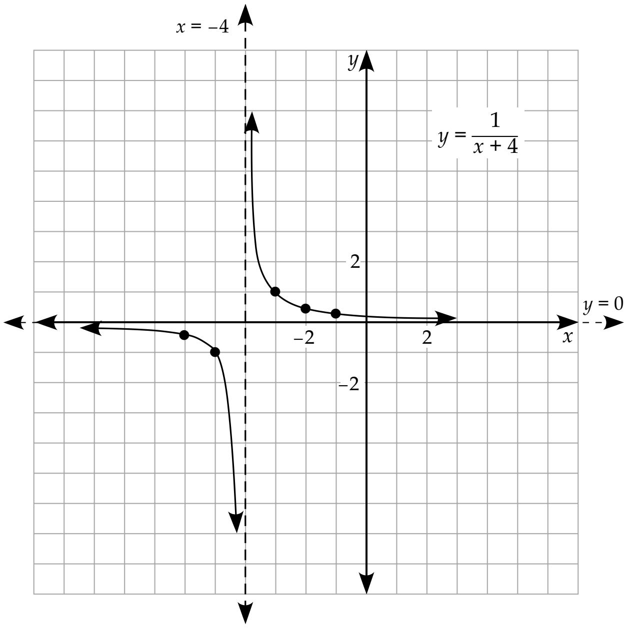
These functions are identical except at the point $x = -2$. Create a table of values for both these functions to determine what happens around the point $x = -2$.

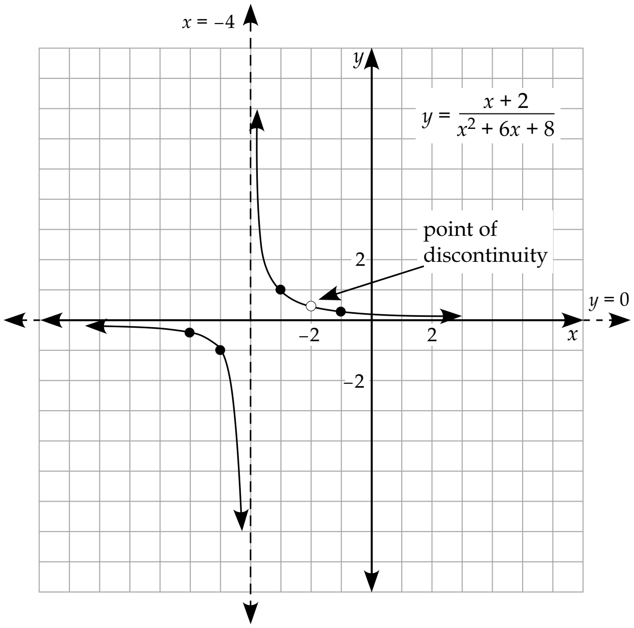
x	$y = \frac{1}{x+4}$	$y = \frac{x+2}{(x+4)(x+2)}$
-3	1	1
-2.5	$\frac{2}{3}$	$\frac{2}{3}$
-2.3	$\frac{10}{17}$	$\frac{10}{17}$
-2.2	$\frac{5}{9}$	$\frac{5}{9}$
-2.1	$\frac{10}{19}$	$\frac{10}{19}$
-2	$\frac{1}{2}$	$\frac{0}{0}$ = undefined
-1.9	$\frac{10}{21}$	$\frac{10}{21}$
-1.8	$\frac{5}{11}$	$\frac{5}{11}$
-1.7	$\frac{10}{23}$	$\frac{10}{23}$
-1.5	$\frac{2}{5}$	$\frac{2}{5}$
-1	$\frac{1}{3}$	$\frac{1}{3}$

In the function $y = \frac{x+2}{x^2 + 6x + 8}$, the y-values exist for x-values approaching -2, but not at -2. Also, the values around -2 do not approach positive or negative infinity as you would expect them to if $x = -2$ was a vertical asymptote. Therefore, there is a point of discontinuity in the function $y = \frac{x+2}{x^2 + 6x + 8}$ at the point $x = -2$. In general, points of discontinuity occur when the numerator and the denominator of a rational function have a common factor. The value of the function evaluates to $\frac{0}{0}$ for a particular value of x.

An expression that evaluates to $\frac{0}{0}$ is undefined and is the subject of further study in calculus courses. It is a conundrum since, as you know, zero over any number is 0. However, any number over itself is always 1. To further complicate it, any number divided by a very small number (close to zero) is infinitely large. As a result, $\frac{0}{0}$ is undefined since it can be thought of in several contradictory ways.

The graphs of both $y = \frac{1}{x + 4}$ and $y = \frac{x + 2}{x^2 + 6x + 8}$ are shown below. Notice how on the graph of $y = \frac{x + 2}{x^2 + 6x + 8}$, which factors as $y = \frac{x + 2}{(x + 4)(x + 2)}$, the point of discontinuity is shown with an empty circle, or a hole.





Example 1

Decide whether each of the following functions will have an asymptote or a point of discontinuity at the indicated value(s) without graphing the function.

a) $y = \dfrac{x^2 - 1}{x - 1}, x = 1$

b) $y = \dfrac{x^2 + 2x + 1}{x^2 + 4x + 4}, x = -2$

c) $y = \dfrac{x^2 - 5x + 6}{x^2 - 2x - 3}, x = 3, x = -1$

d) $y = \dfrac{x^2 + 6x + 5}{x^2 - 2x - 8}, x = -2, x = -1$

Solutions

a) Your first step should always be to factor both the numerator and the denominator.

$$y = \frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1}, x = 1$$

As $x - 1$ is a common factor to both the numerator and the denominator, the y-value will evaluate to $\dfrac{0}{0}$ when $x = 1$, so there will be a point of discontinuity at $x = 1$.

b) $y = \dfrac{x^2 + 2x + 1}{x^2 + 4x + 4} = \dfrac{(x + 1)(x + 1)}{(x + 2)(x + 2)}$

There are no common factors to the numerator and the denominator, and therefore there will be no points of discontinuity. The y-value will evaluate to $\dfrac{1}{0}$ when $x = -2$.

At $x = -2$, there will be a vertical asymptote because, as x gets closer to -2, the y-value gets closer to $\pm\infty$.

c) $y = \dfrac{x^2 - 5x + 6}{x^2 - 2x - 3} = \dfrac{(x-2)(x-3)}{(x-3)(x+1)} = \dfrac{x-2}{x+1}, \; x \neq 3$

At the point $x = 3$, there will be a point of discontinuity, as $x - 3$ is a common factor to the numerator and the denominator of the original function. The y-value will evaluate to $\dfrac{0}{0}$ when $x = 3$.

At the point $x = -1$, there will be a vertical asymptote, as the denominator equals zero at this point. The y-value will evaluate to $\dfrac{-3}{0}$ when $x = -1$.

d) $y = \dfrac{x^2 + 6x + 5}{x^2 - 2x - 8} = \dfrac{(x+5)(x+1)}{(x-4)(x+2)}$

There are no common factors to the numerator and the denominator. Therefore, there will be no points of discontinuity.

The denominator equals zero at the point $x = -2$, and therefore there will be a vertical asymptote at $x = -2$.

The numerator equals zero at the point $x = -1$. When the numerator equals zero, this is the x-intercept. Therefore, there is neither a point of discontinuity nor an asymptote at the point $x = -1$.

Graphing Functions with Points of Discontinuity

Now that you know when rational functions contain a point of discontinuity, you should be able to graph these functions. To graph rational functions, simply follow the guidelines outlined in the previous lesson. Step 3 can be written as follows:

- If the numerator and the denominator have a common factor, plot a hole at the x-value for the common factor in $f(x)$. Solve for the corresponding y-value. This point is the point of discontinuity.

You may wish to add this information to Step 3 of the guidelines for graphing rational functions that you previously included on your resource sheet.

Example 2

Graph the following functions. Pay attention to whether each graph should have a point of discontinuity or a vertical asymptote (or both).

a) $f(x) = \dfrac{x^2 - 4}{x + 2}$

b) $g(x) = \dfrac{x + 3}{x^2 - 4x - 21}$

c) $h(x) = \dfrac{x^2 + 3x - 4}{x^2 + 9x + 20}$

Solutions

a) This function has a factor in common.

$$f(x) = \frac{(x-2)(x+2)}{x+2} = x - 2,\ x \neq -2$$

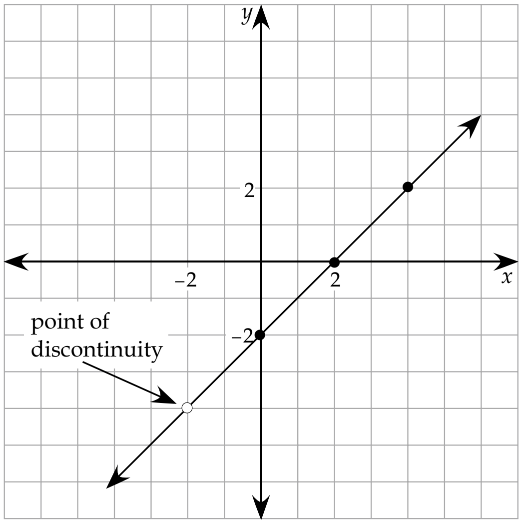
Note that the cancellation of $x + 2$ adds the restriction that $x \neq -2$ and $f(x)$ is not a rational function. Instead, it is a linear function with a hole where $x = -2$. To find the y-value of the hole, substitute $x = -2$ into the simplified function $f(x) = x - 2$. Then the y-value is $y = -2 - 2 = -4$. The point $(-2, -4)$ is called a point of discontinuity and is shown on a graph using an open circle.

y-intercept: -2, evaluate: $f(0) = 0 - 2 = -2$

x-intercept: 2, solve: $x - 2 = 0$, $x = 2$

Vertical asymptote: None

Horizontal asymptote: None



b) This function has a *factor in common*.

$$g(x) = \frac{x+3}{x^2 - 4x - 21} = \frac{x+3}{(x+3)(x-7)} = \frac{1}{x-7},\ x \neq -3$$

y-value of the hole: substitute $x = -3$ into $y = \frac{1}{x-7} = \frac{1}{-3-7} = -\frac{1}{10}$

As $x + 3$ is a common factor to the numerator and the denominator, there will be a point of discontinuity at $\left(-3, -\frac{1}{10}\right)$.

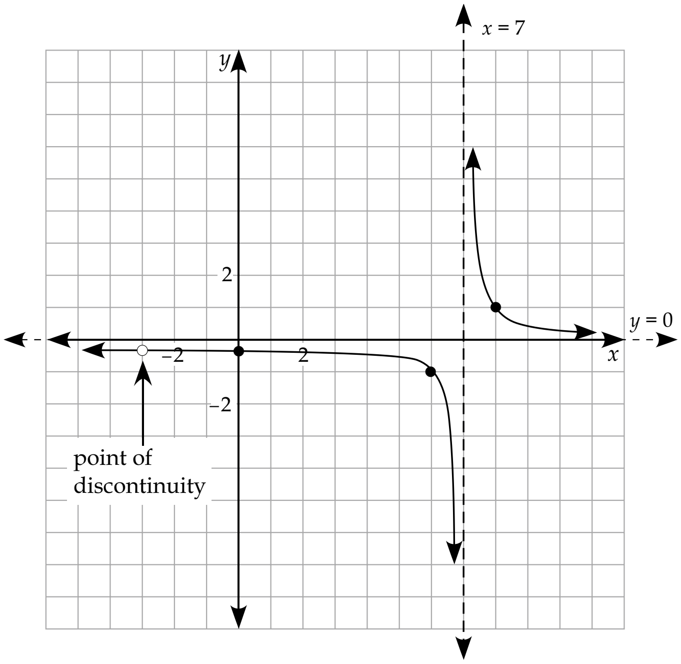
y-intercept: $f(0) = \frac{1}{0-7} = -\frac{1}{7}$

x-intercept: None

Vertical asymptote: $x - 7 = 0$, $x = 7$

Horizontal asymptote: $y = 0$

Point right of asymptote: when $x = 8$, $y = \frac{1}{8-7} = 1$



c) This function has a *factor in common*.

$$h(x) = \frac{x^2 + 3x - 4}{x^2 + 9x + 20} = \frac{(x+4)(x-1)}{(x+5)(x+4)} = \frac{x-1}{x+5},\ x \neq -4$$

y-value of the hole: substitute $x = -4$ into $y = \frac{x-1}{x+5} = y = \frac{-4-1}{-4+5} = -5$

As $x + 4$ is a common factor to the numerator and the denominator, there will be a point of discontinuity at $(-4, -5)$.

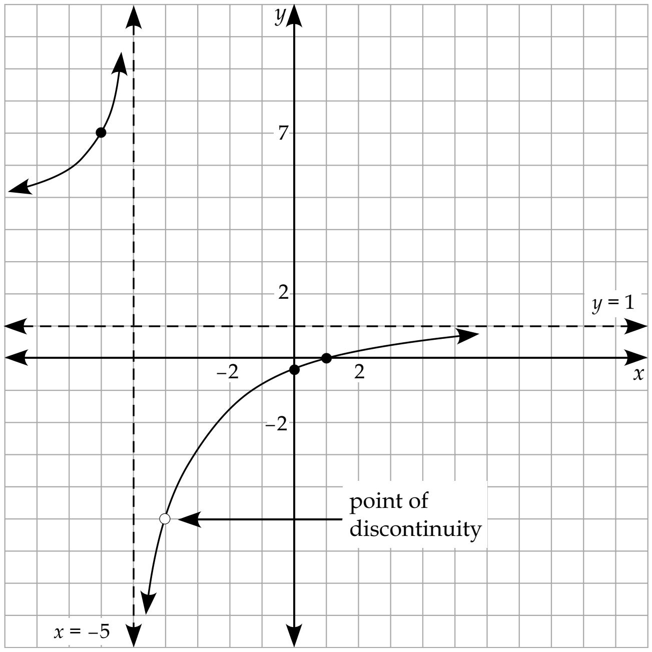
y-intercept: $f(0) = \frac{0-1}{0+5} = -\frac{1}{5}$

x-intercept: $x - 1 = 0,\ x = 1$

Vertical asymptote: $x + 5 = 0,\ x = -5$

Horizontal asymptote: $y = 1$

Point left of asymptote: when $x = -6,\ y = \frac{-6-1}{-6+5} = \frac{-7}{-1} = 7$



Example 3

Compare the functions $y = \dfrac{1}{x - 4}$, $y = \dfrac{x - 4}{x - 4}$, and $y = \dfrac{x - 4}{x^2 - 5x + 4}$ using the following chart. What do the functions have in common? Explain.

Solution

	$y = \dfrac{1}{x - 4}$	$y = \dfrac{x - 4}{x - 4}$	$y = \dfrac{x - 4}{\left(x^2 - 5x + 4\right)} = \dfrac{x - 4}{(x - 4)(x - 1)}$
x-intercept	none	none	none
y-intercept	$-\dfrac{1}{4}$	1	–1
vertical asymptote	$x = 4$	none	$x = 1$
horizontal asymptote	$y = 0$	none	$y = 0$
points of discontinuity	none	$x = 4$	$x = 4$
behaviour at positive infinity	approaches zero through positive y-values	equals 1	approaches zero through positive y-values
behaviour at negative infinity	approaches zero through negative y-values	equals 1	approaches zero through negative y-values

- All three functions have no x-intercepts because the functions are not defined when the numerators equal zero or their numerators cannot equal zero.
- All three functions also have one y-intercept.
- From this chart, it is possible to notice that points of discontinuity, vertical asymptotes, and horizontal asymptotes can exist in multiple combinations.
- The functions $y = \dfrac{1}{x - 4}$ and $\dfrac{x - 4}{x^2 - 5x + 4}$ have similar behaviour near positive and negative infinity, while the function $y = \dfrac{x - 4}{x - 4}$ always equals 1 except for one point of discontinuity.

Make sure you complete the following learning activity before you complete the next assignment, as the learning activity gives you an opportunity to practice determining whether a function will have a point of discontinuity or an asymptote.

Learning Activity 8.5

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. List the non-permissible values of the function: $f(x) = \dfrac{2x+1}{(x+3)(x-6)}$

2. Factor: $2x^2 + 11x - 21$

3. Is $f(x) = \dfrac{3x+5}{2}$ a rational function?

4. Write an expression that represents 2 less than the product of 3 and a number.

5. Evaluate: $|\,2(3 - 6) - 8\,|$

6. If $f(x) = -x^3 + 2x - 7$, find $f(-2)$.

7. Simplify: $\dfrac{2}{7} + \dfrac{5}{3} - \dfrac{1}{21}$

8. What is 15% of 316?

continued

Learning Activity 8.5 (continued)

Part B: Asymptotes Versus Discontinuities

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Decide whether each of the following functions will have an asymptote or a point of discontinuity at the indicated value(s) without graphing the function.

   a) $y = \dfrac{x-2}{x^2+x-6}$, $x = 2$

   b) $y = \dfrac{x^2-4x-5}{x-5}$, $x = 5$

   c) $y = \dfrac{x^2-x-72}{x^2-6x-27}$, $x = 9$, $x = -8$

   d) $y = \dfrac{x^2-7x+6}{x^2-4x+12}$, $x = -2$, $x = 6$

2. Graph the following functions. Pay attention to whether each graph should have a point of discontinuity or a vertical asymptote (or both).

   a) $f(x) = \dfrac{x+4}{x^2+3x-4}$

   b) $g(x) = \dfrac{x^2+3x-40}{x+8}$

   c) $h(x) = \dfrac{x^2+3x+2}{x^2+7x+6}$

continued

Learning Activity 8.5 (continued)

3. Compare the functions $y = \dfrac{1}{x+1}$, $y = \dfrac{x+2}{x^2+3x+2}$, and $y = \dfrac{x-3}{x^2-2x-3}$ using the following chart. What do the functions have in common? Explain.

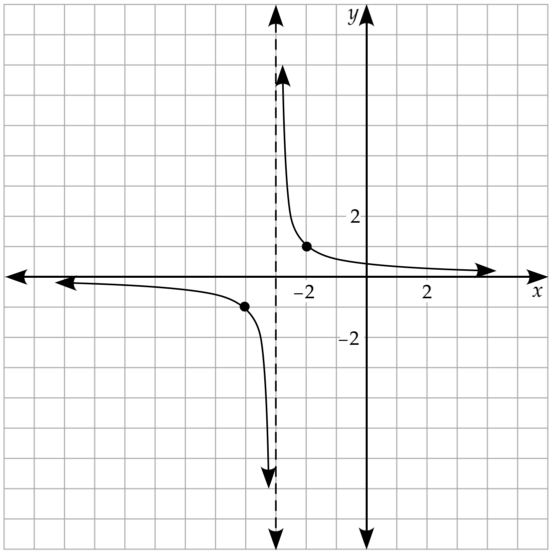
	$y = \dfrac{1}{x+1}$	$y = \dfrac{x+2}{x^2+3x+2} = \dfrac{x+2}{(x+2)(x+1)}$	$y = \dfrac{x-3}{x^2-2x-3} = \dfrac{x-3}{(x-3)(x+1)}$
x-intercept			
y-intercept			
vertical asymptote			
horizontal asymptote			
points of discontinuity			
behaviour at positive infinity			
behaviour at negative infinity			

4. Match each function to its corresponding graph and explain your reasoning.
   a) $y = \dfrac{1}{x+3}$
   b) $y = \dfrac{x-5}{x^2-2x-15}$
   c) $y = \dfrac{x+3}{x^2-2x-15}$
   d) $y = \dfrac{1}{x^2-2x-15}$

continued

Learning Activity 8.5 (continued)

e)



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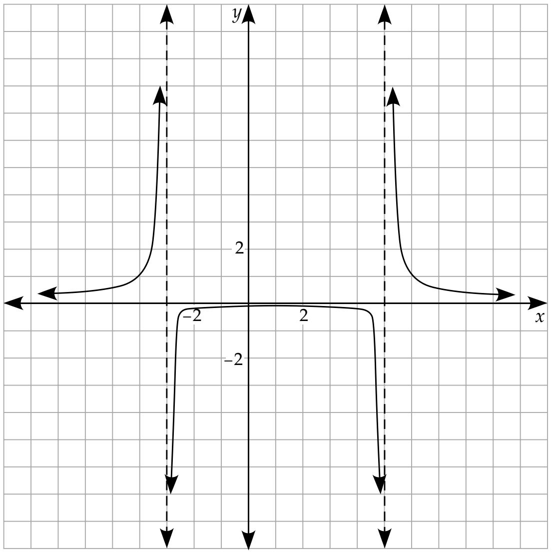

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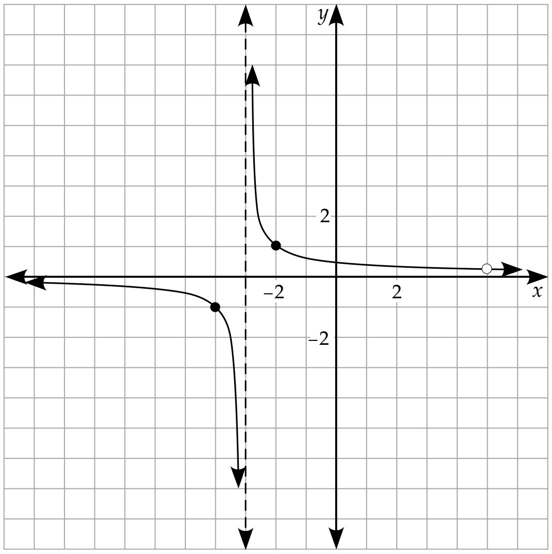

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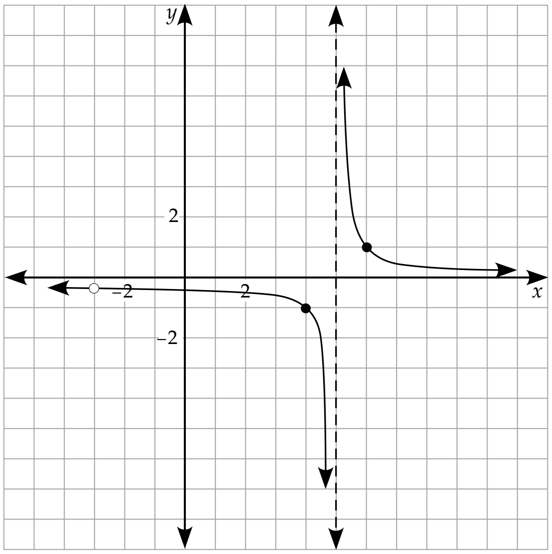


5. Write a possible equation for each rational function described below.
   a) This rational function has a discontinuity at $x = 2$ and a vertical asymptote at $x = 6$.
   b) This rational function has a vertical asymptote at $x = -4$ and a horizontal asymptote at $y = 1$.
   c) This rational function has a discontinuity at $x = -1$, vertical asymptote at $x = 3$, and a horizontal asymptote at $y = 2$.

Lesson Summary

In this lesson, you were introduced to points of discontinuity and you learned to distinguish between when points of discontinuity and vertical asymptotes occur. Points of discontinuity occur at x-values that evaluate to $\frac{0}{0}$ and that happens when the numerator and the denominator of a rational function have a common factor. Vertical asymptotes occur at x-values that evaluate to a number over zero and that happens when a factor common to only the denominator equals zero.

In the next lesson, you will be learning how to solve rational equations graphically.

LESSON 6: SOLVING RATIONAL EQUATIONS

Lesson Focus

In this lesson, you will

- ☐ learn how to solve rational equations graphically
- ☐ learn about the relationship between the roots of a rational equation and the *x*-intercepts of the graph of the corresponding rational function

Lesson Introduction

In the previous two lessons, you learned techniques to graph rational functions. These skills will be useful in this lesson when you are asked to solve rational equations graphically. Just as it was possible to solve radical equations by either graphing one function and finding the *x*-intercepts or graphing two functions and determining where they intersect, it is also possible to solve rational equations this way.

Rational Equations

Just like radical equations and many other equations you have learned about, it is possible to solve rational equations both algebraically and graphically. A **rational equation** is an equation involving rational expressions. In Grade 11 Pre-Calculus Mathematics, you learned how to solve rational equations algebraically. In this course, you learn how to find approximate solutions to rational equations by graphing.

Solving Rational Equations Graphically

Example 1

Determine the roots of $\frac{5}{x-1} = 3$ by graphing. Then, determine the x-intercepts of the corresponding rational function.

Solution

Sketch the graph of $y = \frac{5}{x-1}$ and the graph of $y = 3$.

$y = 3$ is a horizontal line with a y-intercept of 3.

$y = \frac{5}{x-1}$ has: vertical asymptote: $x = 1$

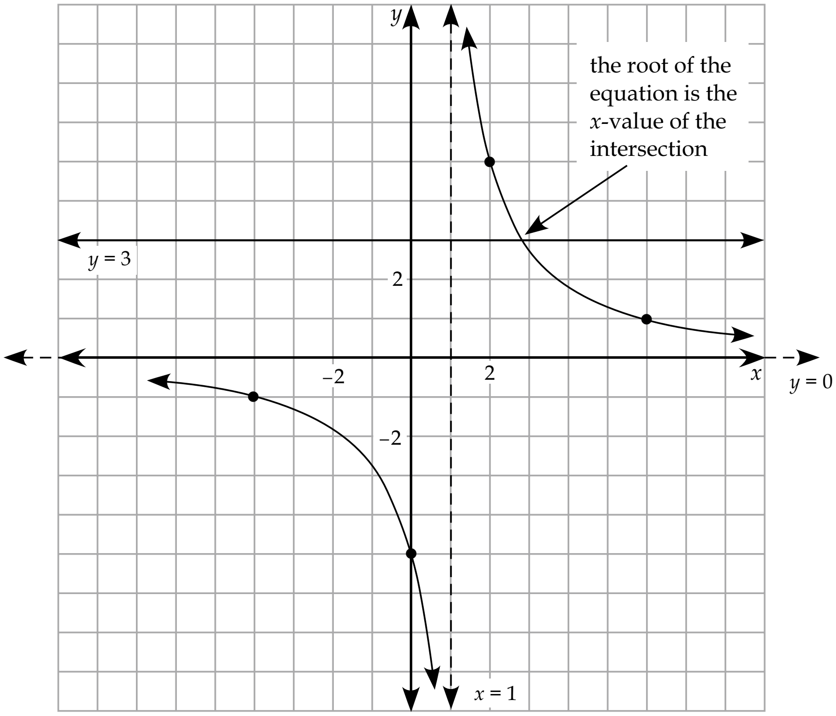
horizontal asymptote: $y = 0$

y-intercept: $y = \frac{5}{0-1} = -5$

x-intercept: none

other points:

x	−4	2	6
y	−1	5	1



Since the x-value of the intersection point is a little less than 3, then the root is somewhere between 2.5 and 3.0.

The corresponding function is related to the equation:

$$\frac{5}{x-1} - 3 = 0$$

The corresponding function is:

$$f(x) = \frac{5}{x-1} - 3$$

The root of the equation is the same as the x-intercept of the corresponding function.

Solve $\frac{5}{x-1} - 3 = 0$ to find the x-intercept.

$$\frac{5}{x-1} = 3$$

$$5 = 3x - 3$$

$$8 = 3x$$

$$\frac{8}{3} = x$$

$$2.6\overline{6} = x$$

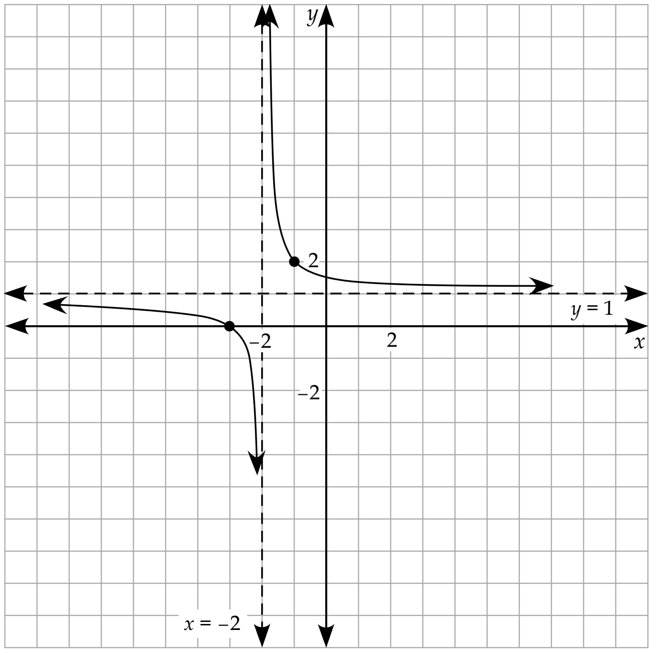
x-intercept is $2.6\overline{6}$

As you can see, the root of an equation is the same as the x-intercept of the corresponding function. You learned about this relationship in Lesson 3 when you solved radical equations and also in Grade 11 Pre-Calculus Mathematics when you solved quadratic equations.

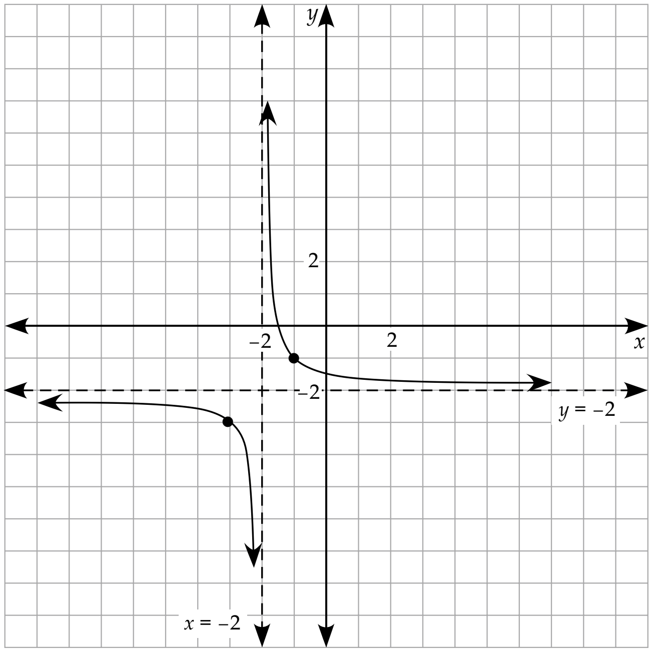
Example 2

Explain how to find the solution of the equation $\frac{x+3}{x+2} = 3$ graphically using:

a) the graph of $f(x) = \frac{x+3}{x+2}$ as shown.

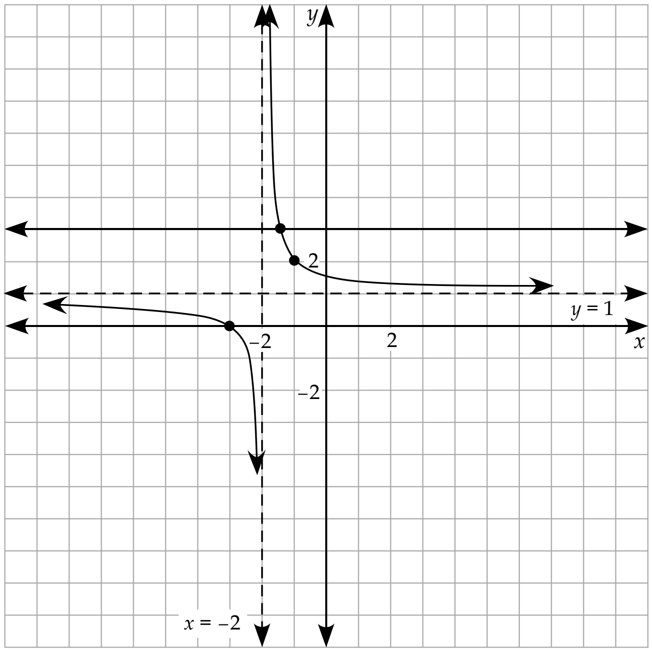


b) The graph of $g(x) = \frac{x+3}{x+2} - 3$ as shown.

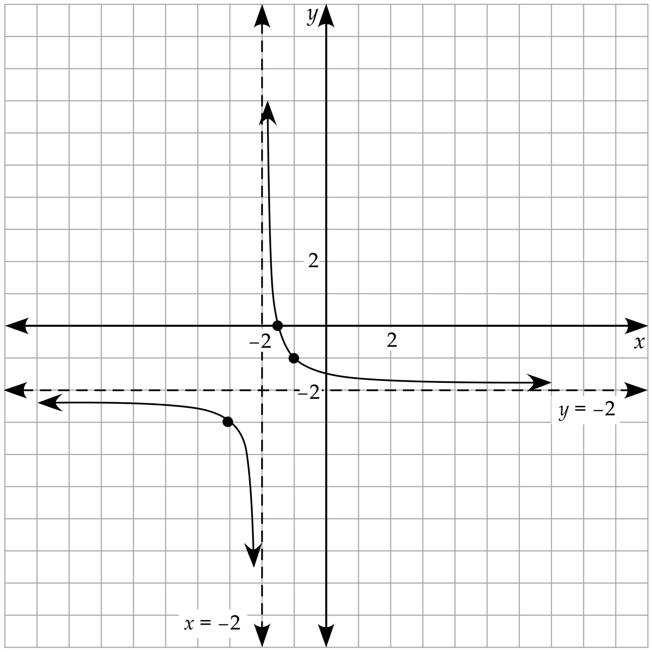


Solutions

a) Draw the horizontal line $y = 3$ and find the point of intersection. That point is $x = -1.5$ (approximately).



b) The x-intercept of this corresponding function is the same as the root of the equation. That point is $x = -1.5$ (approximately).



Make sure you complete the following learning activity and assignment. These are the last two pieces of work you should complete before you start studying for your final examination. If you haven't done so already, now would be a good time for you to update your resource sheet. Include any strategies you learned for solving rational equations that you find helpful.

Learning Activity 8.6

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

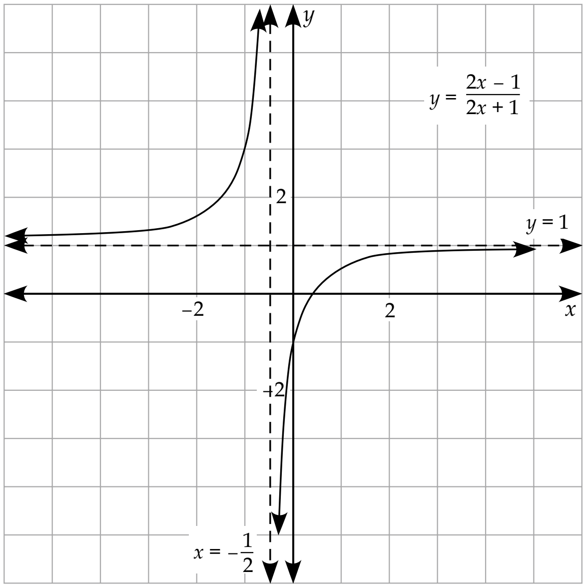
1. Factor: $6x^2 - 17x - 14$
2. Does the function $f(x) = \dfrac{(x-3)(x+7)}{(x-7)(x-3)}$ have a hole or an asymptote at $x = 7$?
3. Simplify: $\sqrt{192x^3y^6z}$
4. Simplify: $\dfrac{9a^4b^8c^2}{27a^3b^2c^8}$
5. Simplify: $\dfrac{\frac{7}{6}}{\frac{14}{8}}$
6. Determine the x-intercept of the function $y = \dfrac{2}{3}x + 7$.
7. Determine the reciprocal of $\dfrac{6x}{7-2x}$.
8. Solve for x: $4x - 5 = 18$

continued

Part B: Solving Rational Equations

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Find the x-intercepts of the rational functions algebraically.
   a) $f(x) = \dfrac{1}{x+3} - 2$
   b) $g(x) = \dfrac{x-3}{x-1}$
2. Find the approximate roots of the following rational equations by graphing.
   a) $\dfrac{1}{x+3} = 2$
   b) $\dfrac{x-3}{x-1} = 2$
3. Approximate the roots to the equations using the related graph of $y = \dfrac{2x-1}{2x+1}$, as shown. Explain how you found your answer.
   a) Solve: $\dfrac{2x-1}{2x+1} = 0$
   b) Solve: $\dfrac{2x-1}{2x+1} = 2$
   c) Solve: $\dfrac{2x-1}{2x+1} = 4$



Lesson Summary

In this lesson, you learned how to solve rational equations graphically using two different methods. In Grade 11 Pre-Calculus Mathematics, you had learned how to solve rational equations algebraically and, therefore, you were able to check whether your answers were correct in multiple ways.

This is the last lesson before you complete your final examination. Make sure you read the information at the end of this module about studying and registering for your final examination.

Assignment 8.3

Rational Functions

Total: 34 marks

You need to show all your steps as you work out your solutions, and make sure your answers are clear (include units, where appropriate). Check the Introduction for the Marking Guidelines that will be used for all assignments and examinations.

1. Graph the following functions. Pay attention to whether each graph should have a point of discontinuity or a vertical asymptote (or both).

   a) $f(x) = \dfrac{2}{x+6} - 5$ *(3 marks)*

continued

Assignment 8.3: Rational Functions (continued)

b) $f(x) = \dfrac{x}{x^2 - 2x - 8}$ *(5 marks)*

continued

Assignment 8.3: Rational Functions (continued)

c) $f(x) = \dfrac{x+5}{x-5}$ *(3 marks)*

continued

Assignment 8.3: Rational Functions (continued)

d) $f(x) = \dfrac{x^2 - 11x + 28}{x - 4}$ *(2 marks)*

continued

Assignment 8.3: Rational Functions (continued)

e) $f(x) = \dfrac{x^2}{x^2 - 2x - 15}$ *(5 marks)*

continued

Assignment 8.3: Rational Functions (continued)

2. Compare the functions $y = \dfrac{1}{x^2 - 2x - 35}$ and $y = \dfrac{x + 6}{2x^2 + 13x + 6}$ using the following chart. What do the functions have in common? Explain. *(6 marks)*

	$y = \dfrac{1}{x^2 - 2x - 35} =$	$y = \dfrac{x + 6}{2x^2 + 13x + 6} =$
x-intercept		
y-intercept		
vertical asymptote		
horizontal asymptote		
points of discontinuity		

continued

Assignment 8.3: Rational Functions (continued)

3. Match each function to its corresponding graph, and explain your reasoning. *(6 × 1 mark each = 6 marks)*

a) $y = \dfrac{5}{x-3}$

b) $y = \dfrac{x-3}{x-3}$

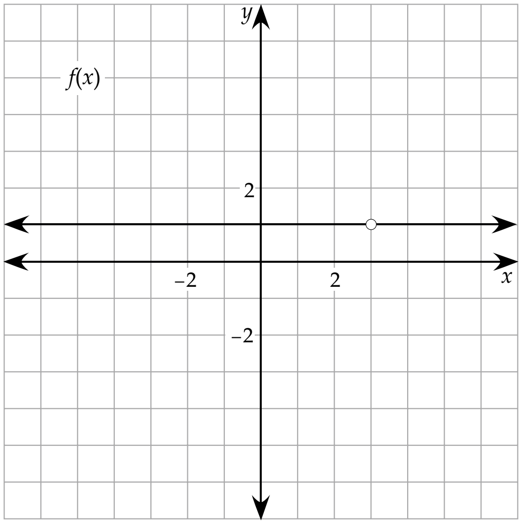
c) $y = \dfrac{x-6}{x^2-9x+18}$

d) $y = \dfrac{1}{x^2-9x+18}$

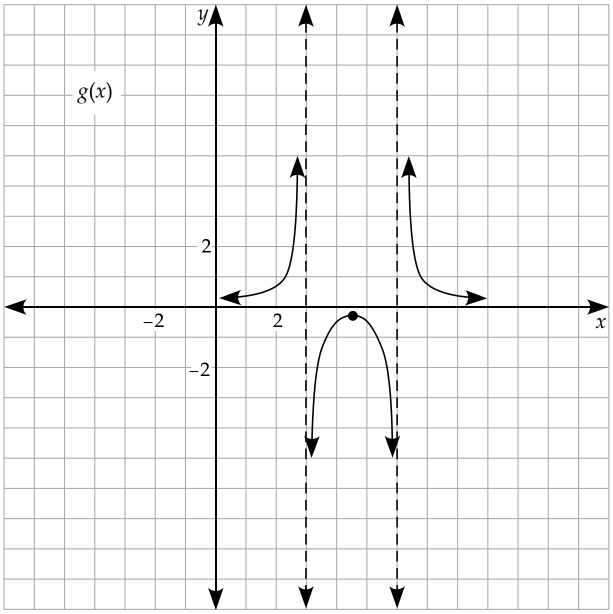
e) $y = \dfrac{x^2-2x-24}{x^2-9x+18}$

f) $y = \dfrac{x-6}{x-3}$

i)



ii)



continued

Assignment 8.3: Rational Functions (continued)

a) $y = \dfrac{5}{x-3}$

b) $y = \dfrac{x-3}{x-3}$

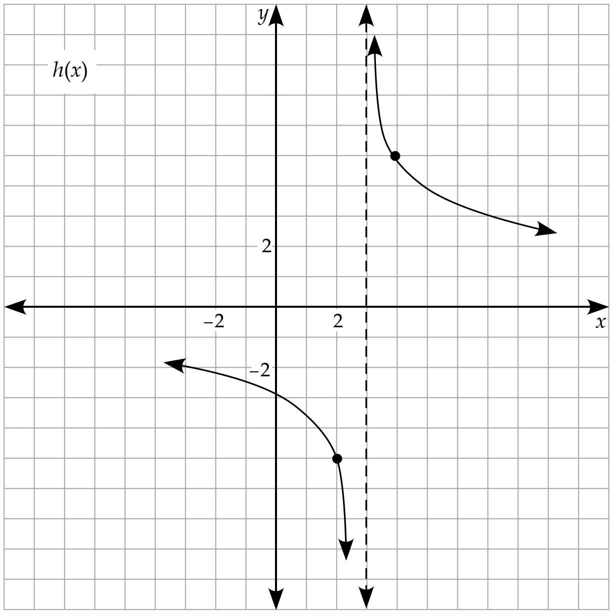
c) $y = \dfrac{x-6}{x^2-9x+18}$

d) $y = \dfrac{1}{x^2-9x+18}$

e) $y = \dfrac{x^2-2x-24}{x^2-9x+18}$

f) $y = \dfrac{x-6}{x-3}$

iii)



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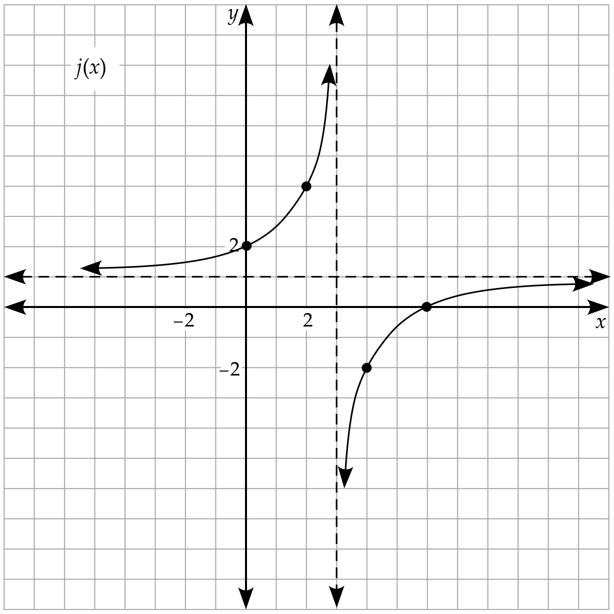


continued

Assignment 8.3: Rational Functions (continued)

a) $y = \dfrac{5}{x - 3}$

b) $y = \dfrac{x - 3}{x - 3}$

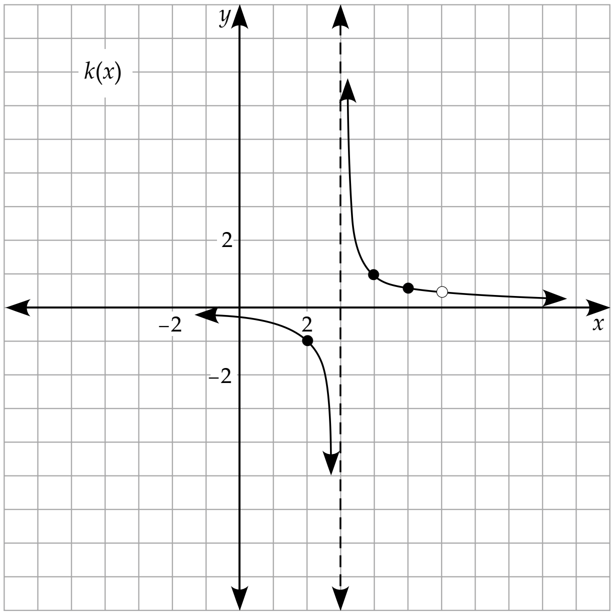
c) $y = \dfrac{x - 6}{x^2 - 9x + 18}$

d) $y = \dfrac{1}{x^2 - 9x + 18}$

e) $y = \dfrac{x^2 - 2x - 24}{x^2 - 9x + 18}$

f) $y = \dfrac{x - 6}{x - 3}$

v)



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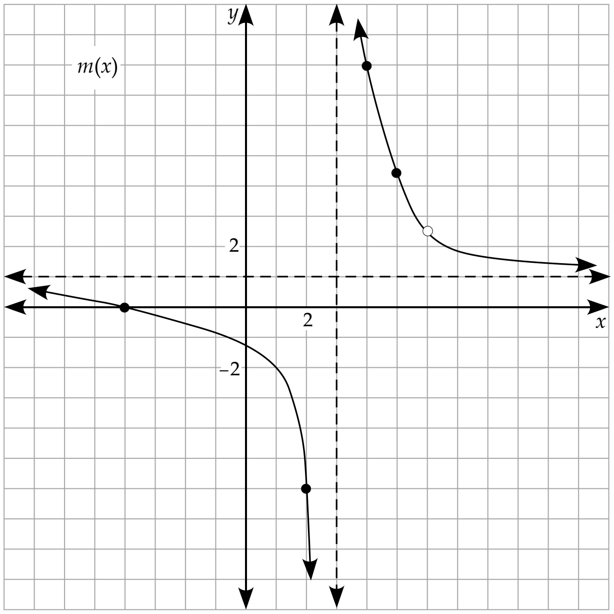


continued

Assignment 8.3: Rational Functions (continued)

4. Find the approximate solution to the following equation graphically.

   $\dfrac{1}{x+6} = -1$ *(4 marks)*

MODULE 8 SUMMARY

Congratulations! You have completed this course!

In this module, you learned about radical and rational functions and how to solve radical and rational equations. Radical functions are functions that contain a radical sign, similar to radical equations. Rational functions are functions that contain polynomials in their numerator and in their denominator, similar to rational equations.

You learned how to graph both radical and rational functions by using transformations. You also learned how to graph rational functions by analyzing the functions to determine their properties, such as asymptotes, intercepts, and points of discontinuity.

To solve radical and rational equations, you built on your knowledge from previous courses in order to solve these equations graphically.

Submitting Your Assignments

It is now time for you to submit Assignments 8.1 to 8.3 to the Distance Learning Unit so that you can receive some feedback on how you are doing in this course. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 8 assignments and organize your material in the following order:

☐ Module 8 Cover Sheet (found at the end of the course Introduction)
☐ Assignment 8.1: Radical Functions
☐ Assignment 8.2: Solving Radical Equations
☐ Assignment 8.3: Rational Functions

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Final Examination

Congratulations, you have finished Module 8 in the course. The final examination is out of 100 marks and worth 25% of your final mark. In order to do well on this examination, you should review all of your learning activities and assignments from Modules 1 to 8.

You will complete this examination while being supervised by a proctor. You should already have made arrangements to have the examination sent to the proctor from the Distance Learning Unit. If you have not yet made arrangements to write it, then do so now. The instructions for doing so are provided in the Introduction to this module.

You will need to bring the following items to the examination: pens/pencils (2 or 3 of each), blank paper, a ruler, a scientific calculator, and your Examination Resource Sheet. A maximum of 3 hours is available to complete your final examination. When you have completed it, the proctor will then forward it for assessment. Good luck!

At this point you will also have to combine your resource sheets from Modules 1 to 8 onto one 8½″ × 11″ paper (you may use both sides). Be sure you have all the formulas, definitions, and strategies that you think you will need. This paper can be brought into the examination with you.

Examination Review

You are now ready to begin preparing for your final examination. Please review the content, learning activities, and assignments from Modules 1 to 8.

The final practice examination is also an excellent study aid for reviewing Modules 1 to 8.

You will learn what types of questions will appear on the examination and what material will be assessed. Remember, your mark on the final examination determines 25% of your final mark in this course and you will have 3 hours to complete the examination.

Final Practice Examination and Answer Key

To help you succeed in your examination, a practice examination can be found in the learning management system (LMS). The final practice examination is very similar to the actual examination that you will be writing. The answer key is also included so that, when you have finished writing the practice examination, you can check your answers. This will give you the confidence that you need to do well on your examination. If you do not have access to the Internet, contact the Distance Learning Unit at 1-800-465-9915 to get a copy of the practice examination and the answer key.

To get the most out of your final practice examination, follow these steps:

1. Study for the final practice examination as if it were an actual examination.
2. Review those learning activities and assignments from Modules 1 to 8 that you found the most challenging. Reread those lessons carefully and learn the concepts.
3. Contact your learning partner and your tutor/marker if you need help.
4. Review your lessons from Modules 1 to 8, including all of your notes, learning activities, and assignments.
5. Use your module resource sheets to make a draft of your Final Examination Resource Sheet. You can use both sides of an 8½” by 11” piece of paper.
6. Bring the following to the final practice examination: pens/pencils (2 or 3 of each), blank paper, a ruler, a scientific calculator, and your Final Examination Resource Sheet.
7. Write your final practice examination as if it were an actual examination. In other words, write the entire examination in one sitting, and don’t check your answers until you have completed the entire examination. Remember that the time allowed for writing the final examination is 3 hours.
8. Once you have completed the entire practice examination, check your answers against the answer key. Review the questions that you got wrong. For each of those questions, you will need to go back into the course and learn the things that you have missed.
9. Go over your resource sheet. Was anything missing or is there anything that you didn’t need to have on it? Make adjustments to your Final Examination Resource Sheet. Once you are happy with it, make a photocopy that you can keep.

Notes

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Module 8
Radical and Rational Functions

Learning Activity Answer Keys

MODULE 8: RADICAL AND RATIONAL FUNCTIONS

Learning Activity 8.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. State the equations of the non-permissible values of the function $f(x) = \dfrac{2}{x^2 - 9}$.
2. Simplify: $\sec\theta \sin\theta$
3. Express $\log_5 125 = 3$ in exponential form.
4. Simplify: $\ln e$.
5. Convert 30° to radians.
6. Find all the values of θ between $[0, 2\pi]$ if $\cos\theta = 1$.
7. In which quadrant is $\theta = \dfrac{11\pi}{7}$ located?
8. Evaluate: $\left| -\dfrac{4}{7} - \dfrac{9}{49} \right|$

Answers:

1. $x \neq 3$ and $x \neq -3$ $(x^2 - 9 = (x - 3)(x + 3))$
2. $\sec\theta \sin\theta = \tan\theta$ $\left(\sec\theta \sin\theta = \dfrac{1}{\cos\theta}\sin\theta = \tan\theta\right)$
3. $5^3 = 125$
4. 1 (let $x = \ln e$; $x = \log_e e$; $e^x = e$; $x = 1$)
5. $\dfrac{\pi}{6}$
6. $0, 2\pi$
7. Quadrant IV $\left[\dfrac{11\pi}{7} = \dfrac{22\pi}{14}, \dfrac{3\pi}{2} = \dfrac{21\pi}{14}, \text{ and } 2\pi = \dfrac{28\pi}{14}, \text{ so } \dfrac{3\pi}{2} < \dfrac{11\pi}{7} < 2\pi\right]$
8. $\left(\left| -\dfrac{4}{7} - \dfrac{9}{49} \right| = \left| -\dfrac{28}{49} - \dfrac{9}{49} \right| = \left| -\dfrac{37}{49} \right| = \dfrac{37}{49}\right)$

Part B: Transformations of the Radical Function

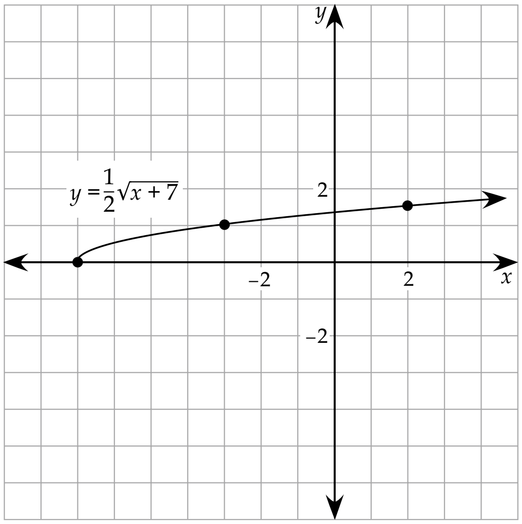
Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Graph the following functions using transformations. State the domain and range of each function.

   a) $y = \frac{1}{2}\sqrt{x+7}$

   Answer:

   This function has been compressed vertically by a factor of 2, and then moved 7 units to the left.



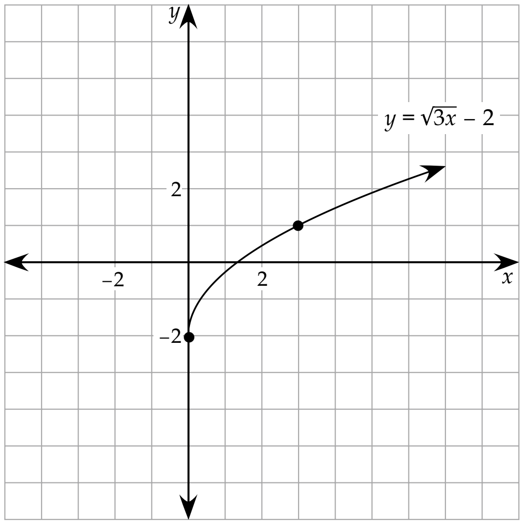
   Domain: $\{x \mid x \geq -7\}$

   Range: $\{y \mid y \geq 0\}$

b) $y = \sqrt{3x} - 2$

Answer:

This function has been compressed horizontally by a factor of 3, and then translated 2 units down.



Domain: $\{x \mid x \geq 0\}$

Range: $\{y \mid y \geq -2\}$

c) $y = \frac{1}{3}\sqrt{-\frac{1}{4}x - 1}$

Answer:

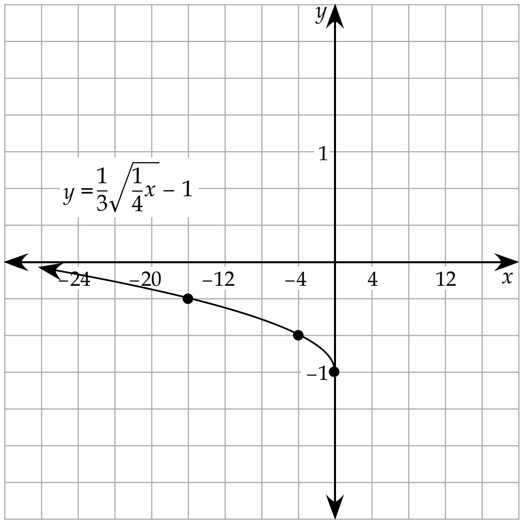
Solution using the coordinates and mapping points.

This function has been compressed vertically by a factor of 3, stretched horizontally by a factor of 4, reflected through the y-axis, and then moved 1 unit down.

To graph these transformations, use coordinate transformation notation.

Each of the x-values is being multiplied by 4, because of the horizontal stretch, and then being multiplied by negative 1, because of the reflection through the y-axis. Each of the y-values is being divided by 3, because of the vertical compression, and then decreased by 1 because of the translation 1 unit down.

$(x, y) \rightarrow \left(-4x, \frac{1}{3}y - 1\right)$
$(0, 0) \rightarrow (0, -1)$
$(1, 1) \rightarrow \left(-4, -\frac{2}{3}\right)$
$(4, 2) \rightarrow \left(-16, -\frac{2}{3}\right)$
$(9, 3) \rightarrow (-36, 0)$



Domain: $\{x \mid x \leq 0\}$

Range: $\{y \mid y \geq -1\}$

d) $y = -2\sqrt{\left(\frac{1}{3}\right)(x + 5)} - 3$

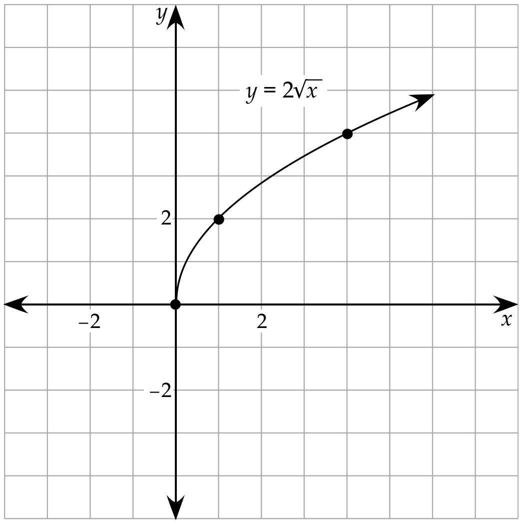
Answer:

Solution using transformations of the standard function $y = \sqrt{x}$.

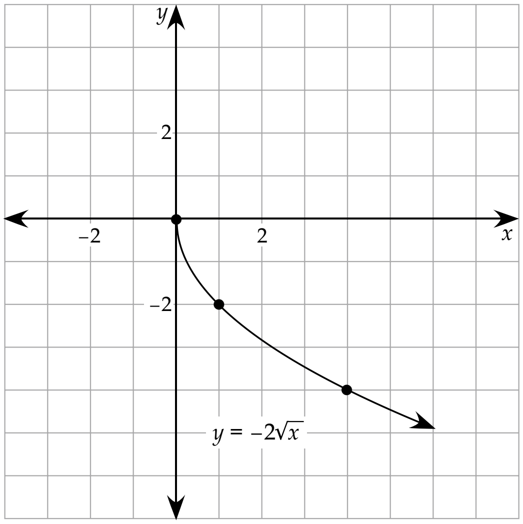
This function has been stretched vertically by a factor of 2, reflected through the x-axis, stretched horizontally by a factor of 3, translated 5 units to the left, and translated 3 units down.

To graph these transformations, graph one transformation at a time.

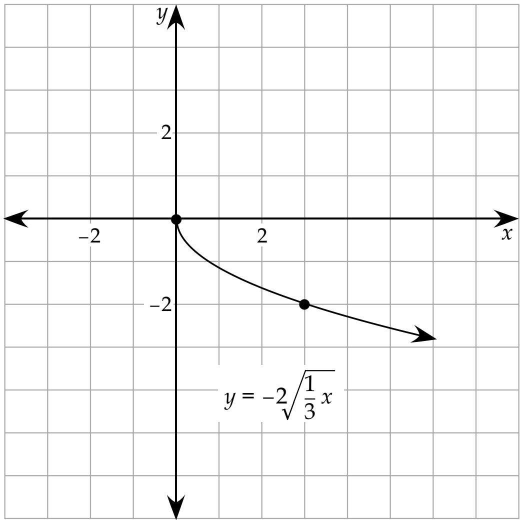
First, graph the vertical stretch by a factor of 2.



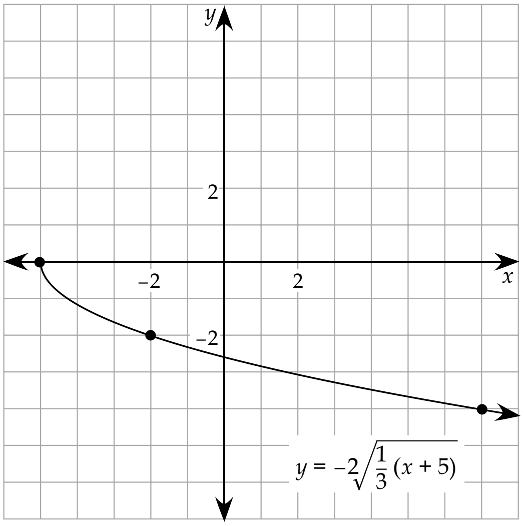
Graph the reflection through the *x*-axis.



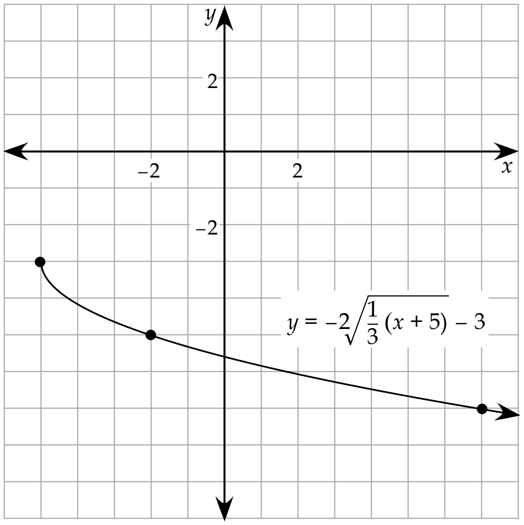
Graph the horizontal stretch by a factor of 3.



Translate the graph 5 units to the left.



Translate the graph 3 units down. This is the graph of the function $y = -2\sqrt{\left(\frac{1}{3}\right)(x+5)} - 3$.



Domain: $\{x \mid x \geq -5\}$

Range: $\{y \mid y \leq -3\}$

2. Write the equation of each of the following functions, given the steps taken to transform the function.

   a) The square root function is stretched horizontally by a factor of 3 and translated 2 units to the left and 3 units down.

      Answer:

      This function is stretched horizontally by a factor of 3, which leads to a b-value of $\frac{1}{3}$. This function is translated 2 units to the left, which leads to an h-value of -2. This function is translated 3 units down, which leads to a k-value of -3.

      $$y = a\sqrt{b(x - h)} + k$$

      $$y = \sqrt{\left(\frac{1}{3}\right)(x + 2)} - 3$$

   b) The square root function is compressed vertically by a factor of 2, reflected through the x-axis, and translated 6 units to the right.

      Answer:

      This function is compressed vertically by a factor of 2, which leads to an a-value of $\frac{1}{2}$. This function is reflected through the x-axis, which leads to a negative a-value. This function is translated 6 units to the right, which leads to an h-value of 6.

      $$y = a\sqrt{b(x - h)} + k$$

      $$y = -\frac{1}{2}\sqrt{x - 6}$$

   c) The square root function is compressed horizontally by a factor of 7, reflected through the y-axis, and translated 3 units up.

      Answer:

      This function is compressed horizontally by a factor of 7, which leads to a b-value of 7. This function is reflected through the y-axis, which leads to a negative radicand. This function is translated 3 units up, which leads to a k-value of 3.

      $$y = a\sqrt{b(x - h)} + k$$

      $$y = \sqrt{-7x} + 3$$

d) The square root function is reflected through the x-axis, reflected through the y-axis, stretched vertically by a factor of 2, and moved 1 unit to the left.

Answer:

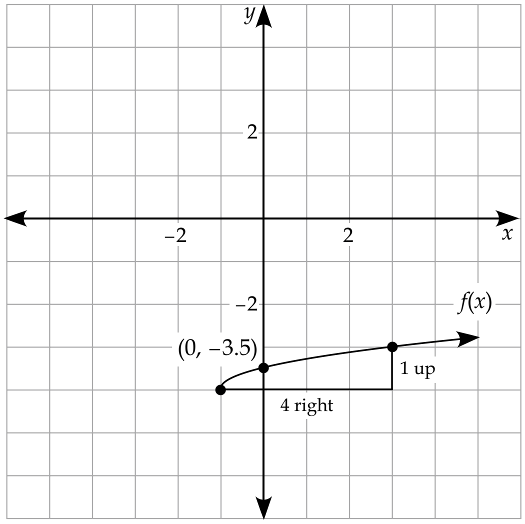
This function is reflected through the x-axis, which leads to a negative a-value. This function is reflected through the y-axis, which leads to a negative radicand. This function is stretched vertically by a factor of 2, which leads to an a-value of 2. This function is translated 1 unit to the left, which leads to an h-value of −1.

$$y = a\sqrt{b(x-h)} + k$$

$$y = -2\sqrt{-(x+1)}$$

3. Determine the radical functions that correspond to the following graphs.

a)



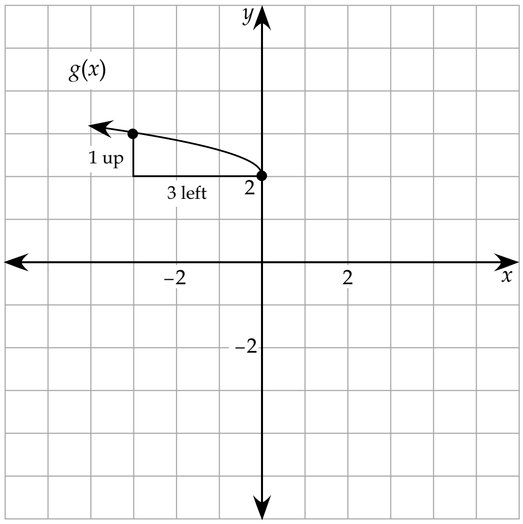
Answer:

Right away you can tell that this graph has been shifted to the left 1 unit. The graph has also been shifted 4 units down. It has also been compressed vertically.

Any stretch or compression will not affect the origin. However, you can now see that the y-coordinates of the original points on the graph of $y = \sqrt{x}$, (1, 1) and (4, 2), have been multiplied by $\frac{1}{2}$. In other words, a vertical compression by a factor of 2 has occurred.

Therefore, the function has undergone a horizontal shift of 1 unit to the left, a vertical shift of 4 units down, and a vertical compression by a factor of 2. The resulting function is $(x) = \frac{1}{2}\sqrt{x+1} - 4$.

b)



Answer:

This function has obviously undergone a reflection through the y-axis, a vertical shift of 2 units up, and some type of compression or stretch.

In this graph, you have moved 3 units horizontally, while moving 1 unit vertically. Again, consider the standard function.

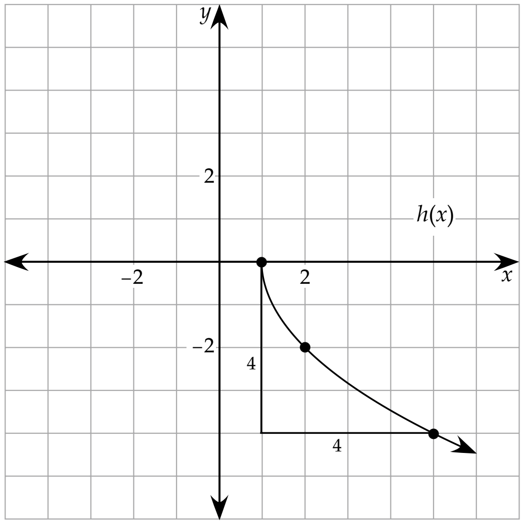
x	y
0	0
1	1
4	2

Normally, a movement of 1 unit vertically is produced by a horizontal movement of 1 unit. In the given graph, it is 3 units.

Now you can determine that a horizontal stretch took place, as it took 3 times as long to reach a y-coordinate of 1. Recall: (1, 1) is a point on the graph of $y = \sqrt{x}$. Therefore, a horizontal stretch by a factor of 3 has occurred.

The resulting function is $y = \sqrt{-\frac{1}{3}x} + 2$.

c)

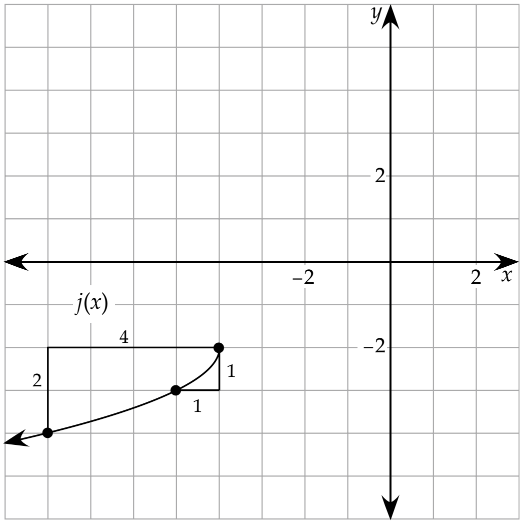


Answer:

This function has been reflected in the x-axis and moved 1 unit to the right. This function has also undergone a stretch or compression.

The original coordinates on the graph of $y = \sqrt{x}$ are (0, 0), (1, 1), and (4, 2). As you can see, each y-coordinate has been multiplied by 2. Therefore, there has been a vertical stretch by a factor of 2. The resulting function is $y = -2\sqrt{x - 1}$.

d)



No stretch/compression

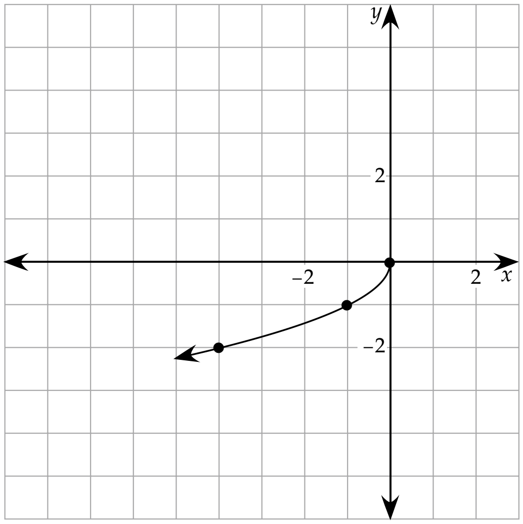
Answer:

This function has been translated 2 units down and 4 units to the left. This function has also been reflected in the x-axis as well as in the y-axis.

Note: Translations are performed after stretches, compressions, and reflections. Therefore, when determining a function when you are given a graph, you need to determine the transformations in opposite order. Determine any translations first, and then follow with reflections, compressions, and stretches. Add this note to your resource sheet.

From this graph, it is possible to see the translations and the reflections all at once. No calculations are needed.

However, to see the reflections more clearly, undo the vertical and horizontal shifts by moving this graph up 2 units and 4 units to the right.



Once you undo both translations, you can see that this graph has been reflected over both axes. The resulting function is $y = -\sqrt{-(x+4)} - 2$.

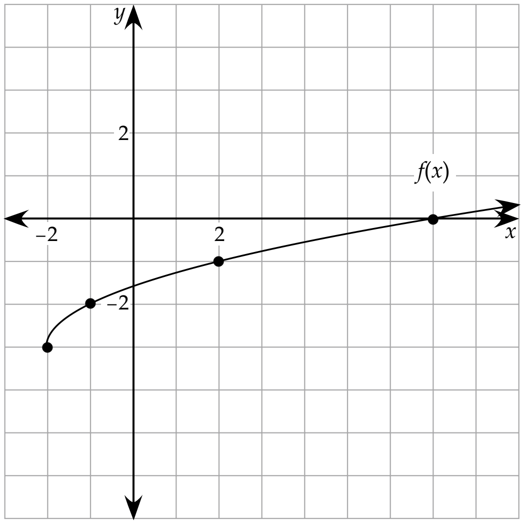
4. Graph the following functions:

$$f(x) = \sqrt{x+2} - 3 \text{ and } g(x) = -\sqrt{x+2} - 3$$

State the domain and range of each function. What do you notice about the domain and range of each function? How are they related? Generalize the effect a reflection through the x-axis has on the domain and range of a function.

Answer:

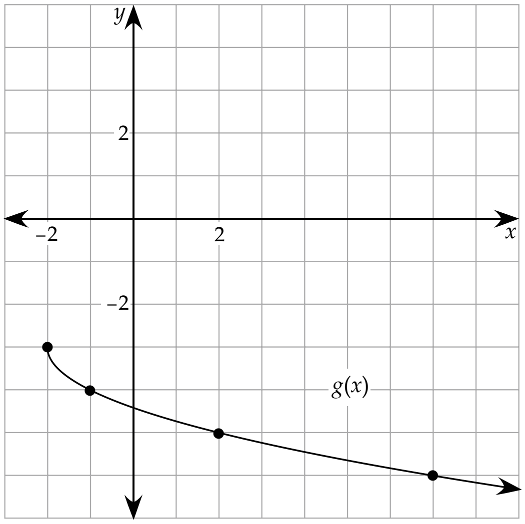
$f(x)$ is the square root function, which has been moved 2 units to the left and 3 units down.



Domain: $\{x \mid x \geq -2\}$

Range: $\{y \mid y \geq -3\}$

$g(x)$ is the square root function, which has been reflected over the x-axis, and then moved 2 units to the left and 3 units down.



Domain: $\{x \mid x \geq -2\}$

Range: $\{y \mid y \leq -3\}$

These two functions have the same domain. However, the range of $g(x)$ is less than or equal to negative three, while the range of $f(x)$ is greater than or equal to negative three.

In general, when a function is reflected through the x-axis, the domain stays the same, but the range changes. The value the range depends on does not change; instead, the inequality sign flips. In other words, if the inequality sign was originally ≤, it becomes ≥. Also, if the inequality sign was originally ≥, it becomes ≤.

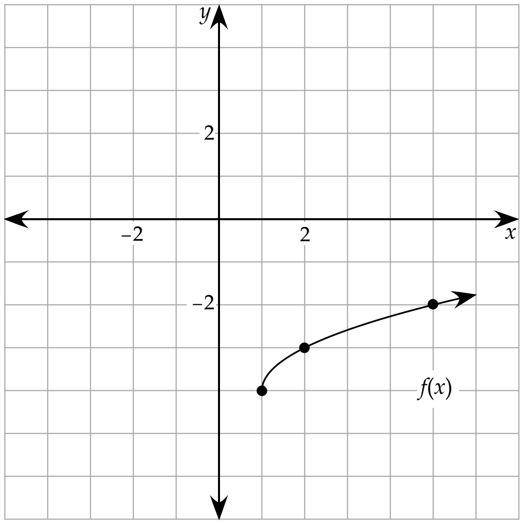
5. Graph the following functions:

$$f(x) = \sqrt{x-1} - 4 \text{ and } g(x) = \sqrt{-(x-1)} - 4$$

State the domain and range of each function. What do you notice about the domain and range of each function? How are they related? Generalize the effect a reflection through the y-axis has on the domain and range of a function.

Answer:

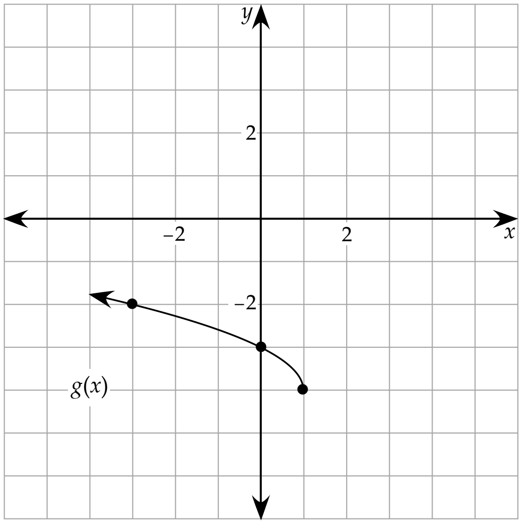
$f(x)$ is the square root function, which has been moved 1 unit to the right and 4 units down.



Domain: $\{x \mid x \geq 1\}$

Range: $\{y \mid y \geq -4\}$

$g(x)$ is the square root function, which has been reflected over the y-axis, and then moved 1 unit to the right and 4 units down.



Domain: $\{x \mid x \leq 1\}$

Range: $\{y \mid y \geq -4\}$

These two functions have the same range. However, the domain of $g(x)$ is less than or equal to negative one, while the domain of $f(x)$ is greater than or equal to negative one.

In general, when a function is reflected through the y-axis, the range stays the same, but the domain changes. If the domain was originally going from h to infinity, the domain changes to going from negative infinity to h and vice versa.

6. Match each function to its corresponding graph.

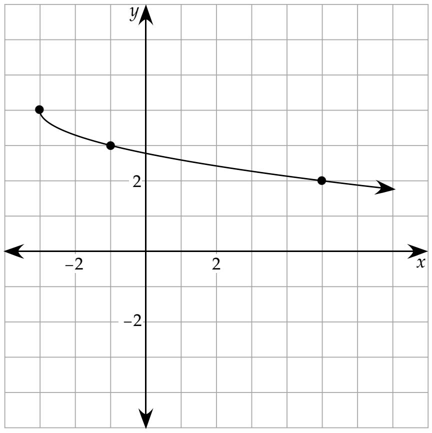
a) $y = -\sqrt{\left(\frac{1}{2}\right)(x+3)} + 4$

b) $y = \sqrt{\left(\frac{1}{2}\right)(x+3)} + 4$

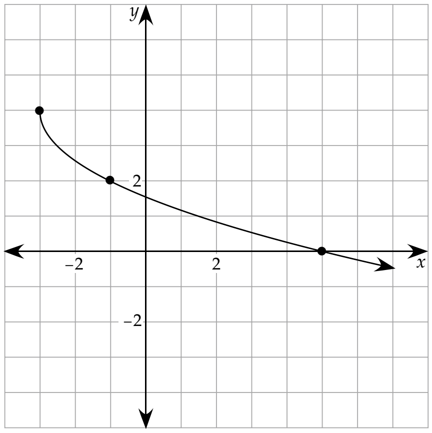
c) $y = -\sqrt{2(x+3)} + 4$

d) $y = \sqrt{2(x+3)} + 4$

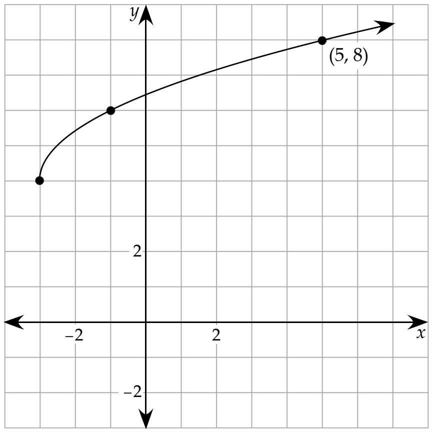
i)



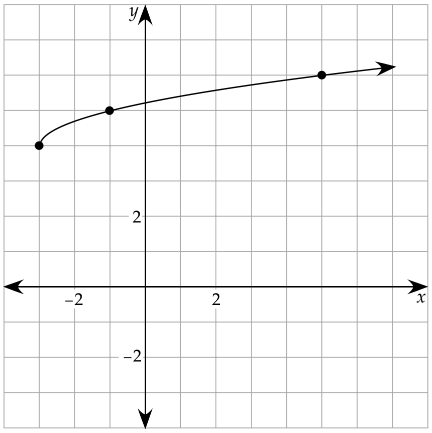
ii)



iii)



iv)



Answer:

The graphs that are reflected through the x-axis, (a) and (c), are going to be decreasing or heading towards negative y-values, while the graphs in (b) and (d) are increasing.

Therefore, (a) and (c) must match with (i) and (ii). Also, (b) and (d) must match with (iii) and (iv).

Also, a b-value of $\frac{1}{2}$ means a horizontal stretch has occurred. Therefore, these graph are going to be skinnier, or increase slower, than the graph of $y = \sqrt{x}$. A b-value of 2 means a horizontal compression has occurred. Therefore, these graphs are going to be fatter, or increase faster, than the graph of $y = \sqrt{x}$.

Thus, (a) and (b) must match with (i) and (iv) while (c) and (d) match with (ii) and (iii). Putting this information together allows you to find the correct matches.

(a) (i)

(b) (iv)

(c) (ii)

(d) (iii)

7. The approximate distance, d, in miles, that a person can see to the horizon from a height, h, in feet, is given by the equation $d = \sqrt{\frac{3}{2}h}$. If a person who is 6 feet tall is standing on a 72-foot cliff, can that person see a sailboat that is 11 miles away?

   Answer:

   The height h from the ground is the combined height of the person and the cliff.

   $6 + 72 = 78$

   $h = 78$

   $$d = \sqrt{\left(\frac{3}{2}\right)(78)} = \sqrt{(3)(39)} = \sqrt{117} \approx 10.82 \text{ miles}$$

   As the sailboat is greater than 10.82 miles away, this person is not able to see the sailboat.

Learning Activity 8.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Express $3^{(-4)} = \frac{1}{81}$ in log form.
2. Express $\log_2 32 = 5$ in exponential form.
3. Simplify: $\tan^2 \theta \cot \theta$
4. Solve for x: $\ln e^{2x-1} = 1$.
5. Convert $\frac{\pi}{2}$ to degrees.
6. What is the exact value of sin 30°?
7. State an angle that is coterminal to 319°.
8. If $f(x) = x^2 - 3x$, find $f(-3)$.

Answers:

1. $\log_3 \frac{1}{81} = -4$
2. $2^5 = 32$
3. $\tan^2 \theta \ \cot \theta = \tan \theta \ \left(\tan^2 \theta \ \cot \theta = \tan^2 \theta \left(\frac{1}{\tan \theta}\right) = \tan \theta\right)$
4. $x = 1$ $(2x - 1 = 1, 2x = 2, x = 1)$
5. 90°
6. $\frac{1}{2}$
7. −41° or 679° (add or subtract 360° to 319°; other answers are possible)
8. $f(-3) = 18$ $(f(-3) = (-3)^2 - 3(-3) = 9 + 9)$

Part B: The Square Root of a Function

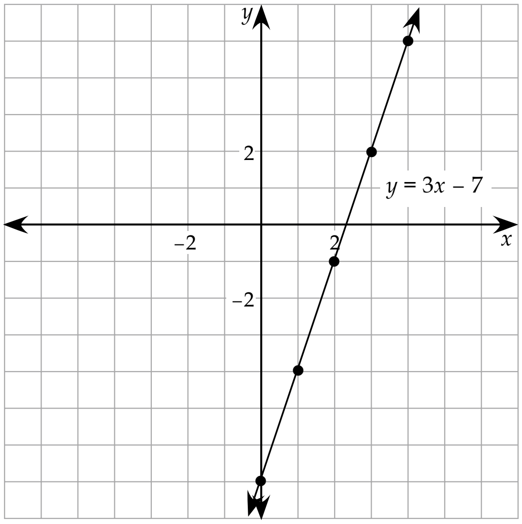
Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Graph the following functions, as well as their square roots. State the domain and range of the original function, as well as the square root of that function.
   a) $y = 3x - 7$

   Answer:

   Solution using the graph of $y = f(x)$ to graph $y = \sqrt{f(x)}$.

   The y-intercept of this function is -7 and it has a slope of 3.



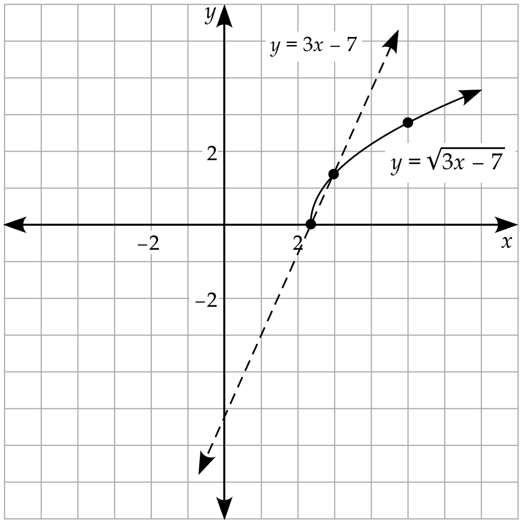
Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

To graph the square root of this function, $y = \sqrt{3x - 7}$, consider $y = \sqrt{f(x)}$ when $f(x) = 3x - 7$.

You could transform the graph of $f(x)$ into $y = \sqrt{f(x)}$ by considering the graph of $y = 3x - 7$.

Take the square root of the $f(x)$ values for each point of $y = \sqrt{f(x)}$.



Domain: $\left[\frac{7}{3}, \infty\right)$

Range: $[0, \infty)$

The x-value can be found by setting $3x - 7 = 0$; therefore, $x = \frac{7}{3}$.

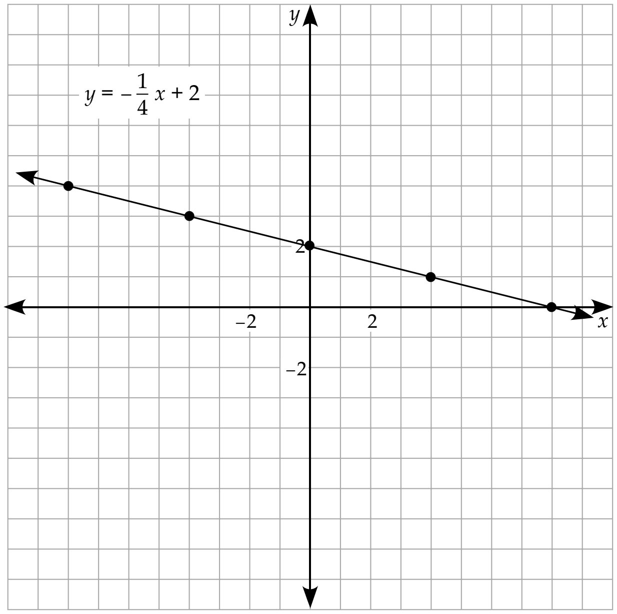
b) $y = -\frac{1}{4}x + 2$

Answer:

Solution using the key invariant points.

The y-intercept of this function is 2 and it has a slope of $-\frac{1}{4}$.

It also includes the points (4, 1) and (8, 0).

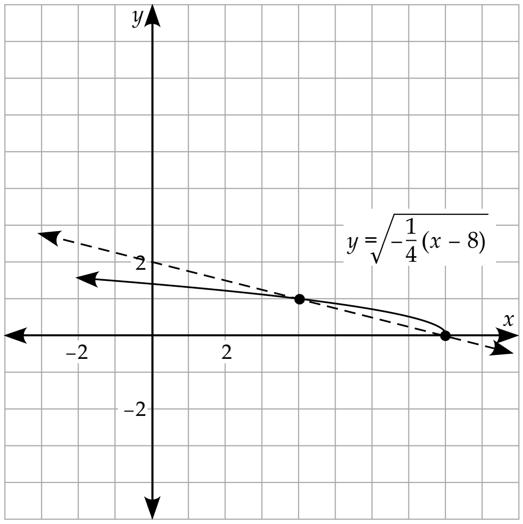


Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

The square root of this function is $y = \sqrt{-\frac{1}{4}x + 2}$. To graph this function, consider $y = \sqrt{f(x)}$ when $f(x) = -\frac{1}{4}x + 2$ and analyze key points.

Notice how the invariant points are (4, 1) and (8, 0).



Domain: $[-\infty, 8]$

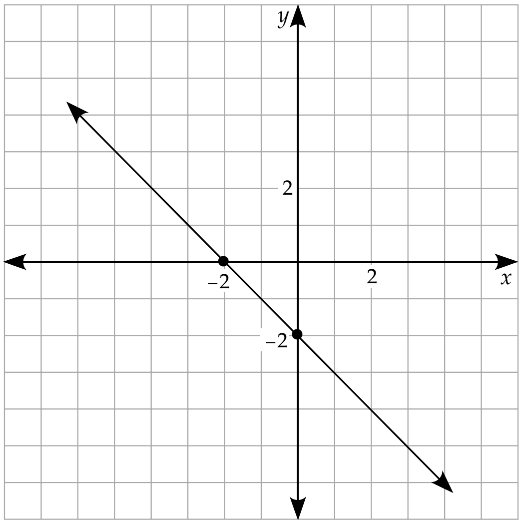
Range: $[0, \infty)$

c) $y = -x - 2$

Answer:

Solution using transformations.

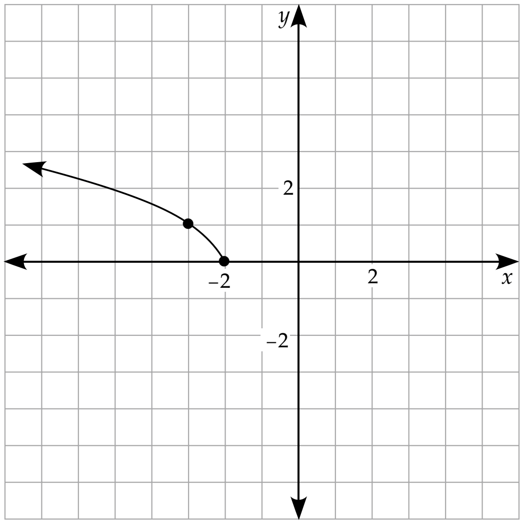
The *y*-intercept of this function is −2 and it has a slope of −1.



Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

The square root of this function is $y = \sqrt{-x - 2}$, which could be written as $y = \sqrt{-(x + 2)}$. Now you could also use transformations to graph this. This is the standard function, $y = \sqrt{x}$, reflected over the y-axis and shifted 2 units left. The invariant points are (−2, 0) and (−3, 1).



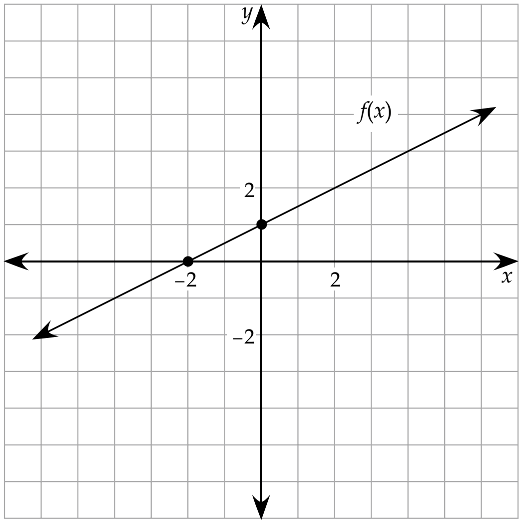
Domain: (−∞, −2]

Range: [0, ∞)

You can see that you can use three different methods to find the graph of $y = \sqrt{f(x)}$ when $f(x)$ is a linear function. Select the method that works the best for you.

2. Graph the square root of each of the functions below.

a)

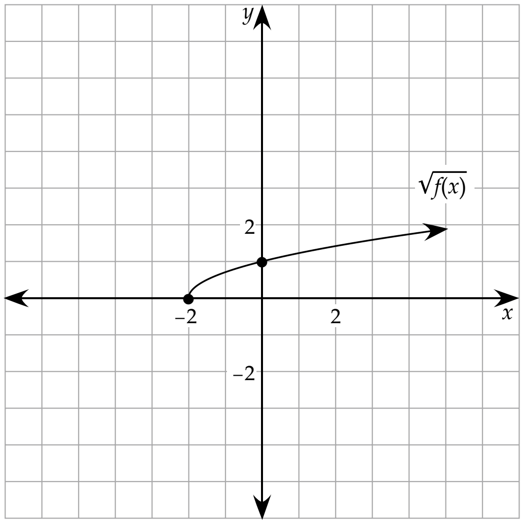


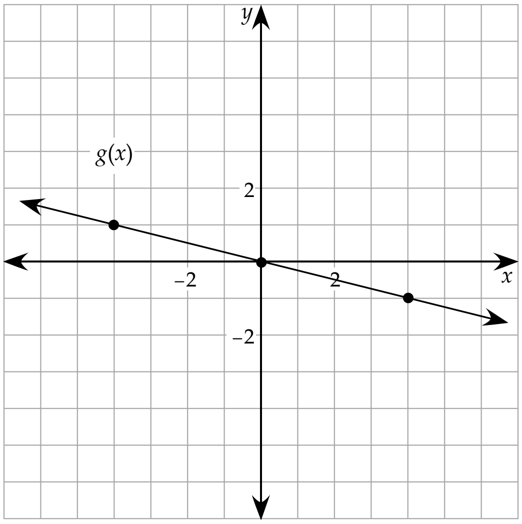
Answer:

The square root of this function is undefined when $x < -2$.

The invariant points exist when $f(x) = 0$ and when $f(x) = 1$, which are $(-2, 0)$ and $(0, 1)$. Between these two points, $y = \sqrt{f(x)}$ is above $y = f(x)$.

To the right of $(0, 1)$, $y = f(x)$ is above $y = \sqrt{f(x)}$.

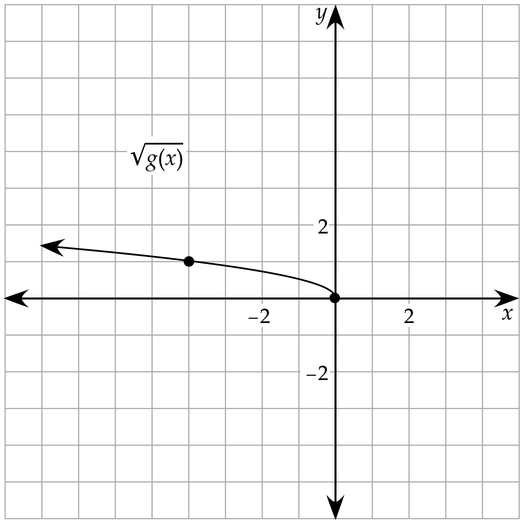


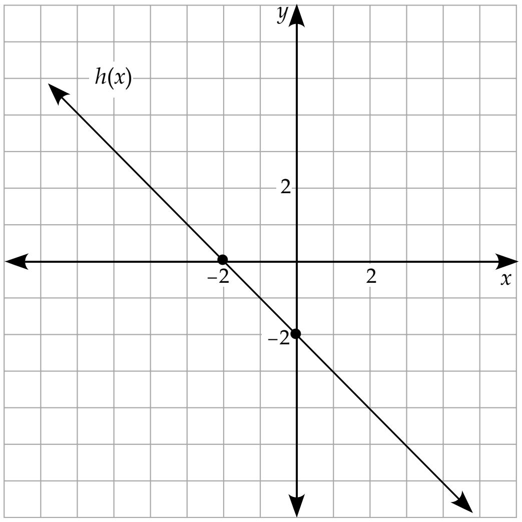
b) 

Answer:

The square root of this function is undefined when $x > 0$.

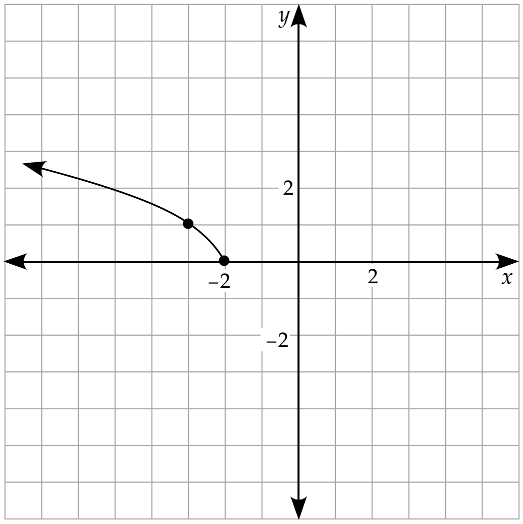
The invariant points are (−4, 1) and (0, 0). Between these points, $y = \sqrt{g(x)}$ is above $y = g(x)$. To the left of (−4, 1), $y = \sqrt{g(x)}$ is below $y = g(x)$.



c) 

Answer:

The square root of this function is undefined when $x > -2$. The invariant points are (−3, 1) and (−2, 0). Between these points, $y = \sqrt{h(x)}$ is above $y = h(x)$. To the left of (−3, 1), $y = \sqrt{h(x)}$ is below $y = h(x)$.



3. Compare the domains and ranges of the linear functions above with their corresponding square root functions. What do you notice?

   Answers:

   a) Domain of $f(x)$: $(-\infty, \infty)$ Domain of $\sqrt{f(x)}$: $[-2, \infty)$

      Range of $f(x)$: $(-\infty, \infty)$ Range of $\sqrt{f(x)}$: $[0, \infty)$

   b) Domain of $g(x)$: $(-\infty, \infty)$ Domain of $\sqrt{g(x)}$: $(-\infty, 0]$

      Range of $g(x)$: $(-\infty, \infty)$ Range of $\sqrt{g(x)}$: $[0, \infty)$

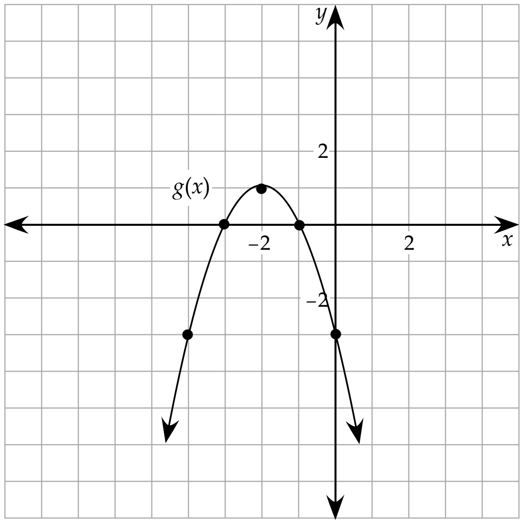
   c) Domain of $h(x)$: $(-\infty, \infty)$ Domain of $\sqrt{h(x)}$: $(-\infty, -2]$

      Range of $h(x)$: $(-\infty, \infty)$ Range of $\sqrt{h(x)}$: $[0, \infty)$

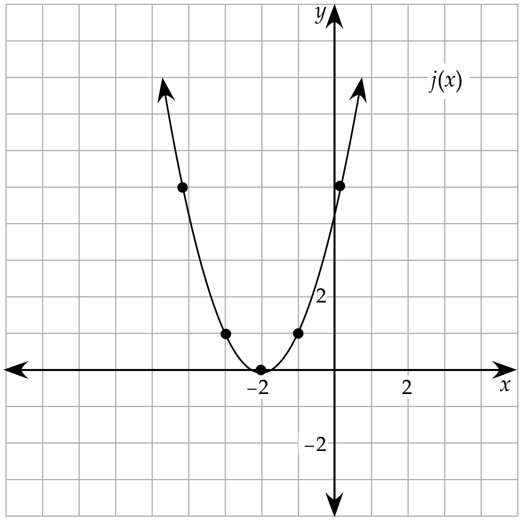
   The domain of the radical function depends on which y-values of the original linear function are greater than or equal to zero. The range is the same for all three radical functions. Only when a y-value is greater than or equal to zero in the original function is the square root of the y-value corresponding to that point included.

4. Match the following graphs with their corresponding square root graphs.

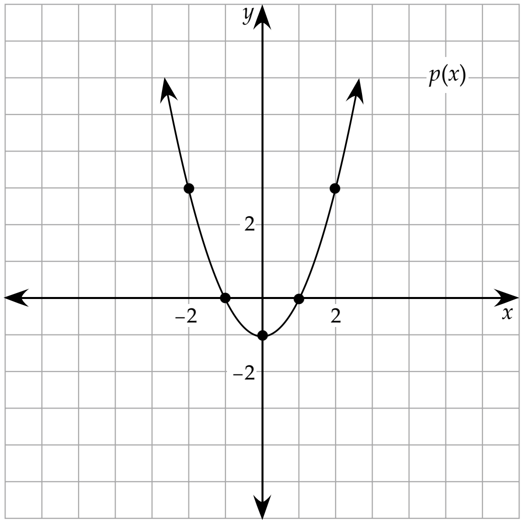
a)



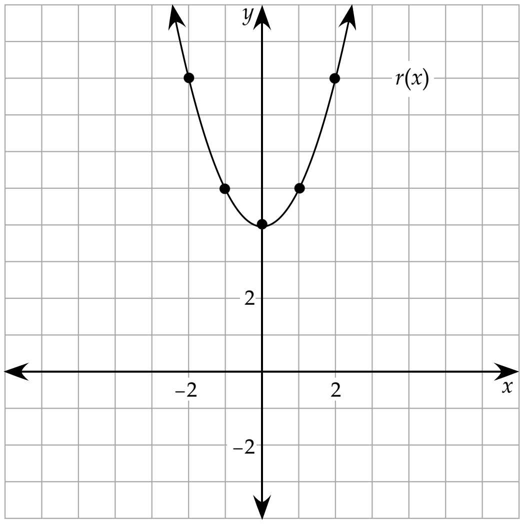
b)



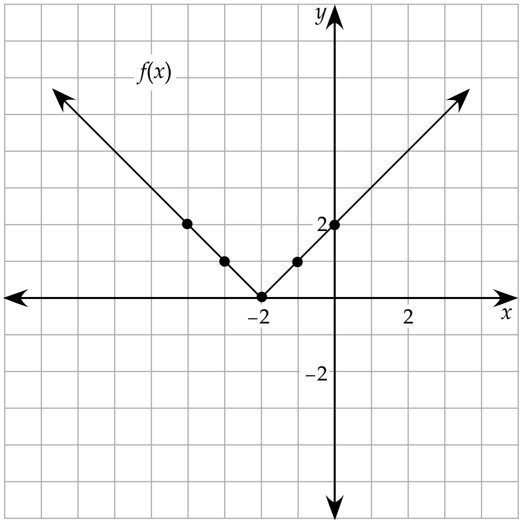
c)



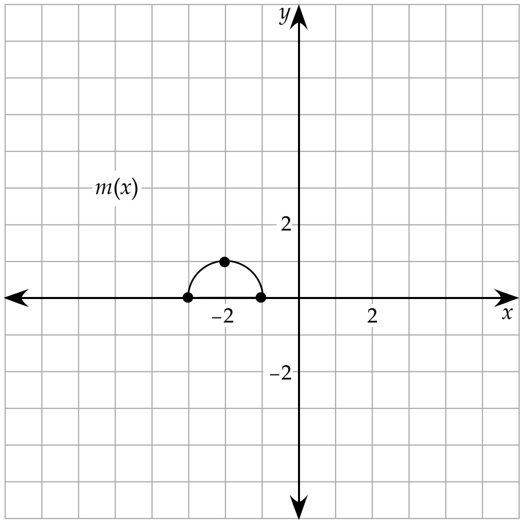
d)



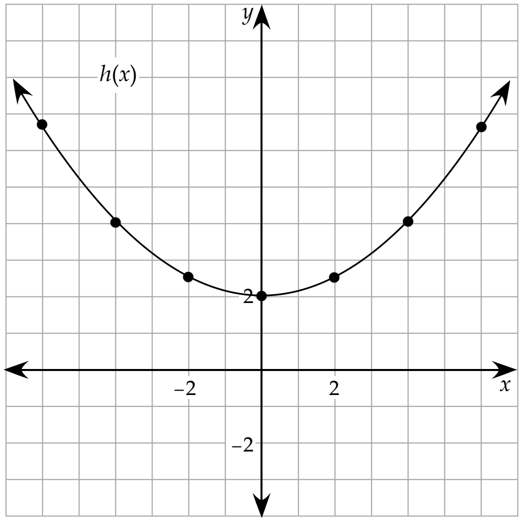
i)



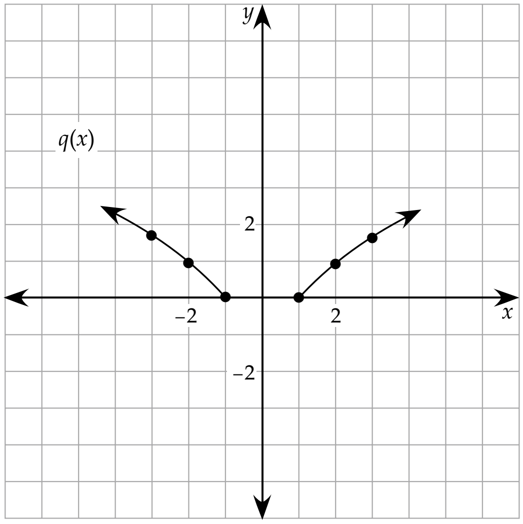
ii)



iii)



iv)



Answer:

The quadratic function $g(x)$ that is only above the x-axis between -3 and -1 belongs with the square root graph that only exists between -3 and -1. This is the graph of $m(x)$. Therefore, $m(x) = \sqrt{g(x)}$.

The absolute value graph $f(x)$ is the graph of the square root function that contains a double root at $x = -2$. This is the function $j(x)$. Therefore, $f(x) = \sqrt{j(x)}$.

The quadratic function $p(x)$, which is below the x-axis between −1 and 1, corresponds to the square root graph $q(x)$, which doesn't exist between −1 and 1. Therefore, $q(x) = \sqrt{p(x)}$.

The function $h(x)$ is the square root of $r(x)$, as they are both completely above the x-axis, and $h(x)$ is wider than $r(x)$, which indicates $h(x) = \sqrt{r(x)}$.

(a) → (ii) $m(x) = \sqrt{g(x)}$

(b) → (i) $f(x) = \sqrt{j(x)}$

(c) → (iv) $q(x) = \sqrt{p(x)}$

(d) → (iii) $h(x) = \sqrt{r(x)}$

5. The speed a car is travelling is proportional to the square root of the length of the skid mark it creates during an automobile accident. This can be expressed as $V = k\sqrt{D}$, where V represents speed, D represents distance, and k is the constant of proportionality.

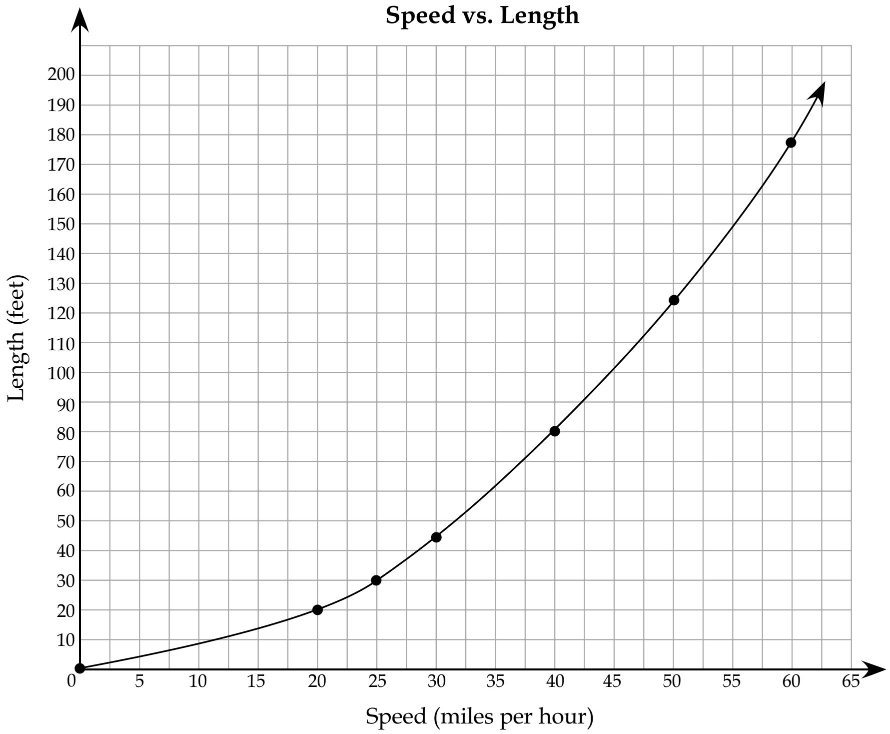
In order to determine how fast a vehicle was travelling before an accident, police are able to measure the length of the skid mark. However, in order to use the above formula, they need to find the constant of proportionality. To do this, they have acquired the following data about the speed of a vehicle and the corresponding length of its skid mark when stopped suddenly.

Speed (miles)	Length (feet)
0	0
20	19.8
25	30.9
30	44.4
40	79.1
50	123.5
60	177.8

To determine how fast a vehicle was travelling in order to create a skid mark 130 feet long, you need to complete the following steps.

a) Create a graph of speed versus length.

Answer:



b) What type of function is created from the data (for example, linear, quadratic, radical)?

Answer:

This is a quadratic function as it resembles a parabola.

c) What type of function would be created from the square root of the graph you just created (for example, linear, quadratic, radical)?

Answer:

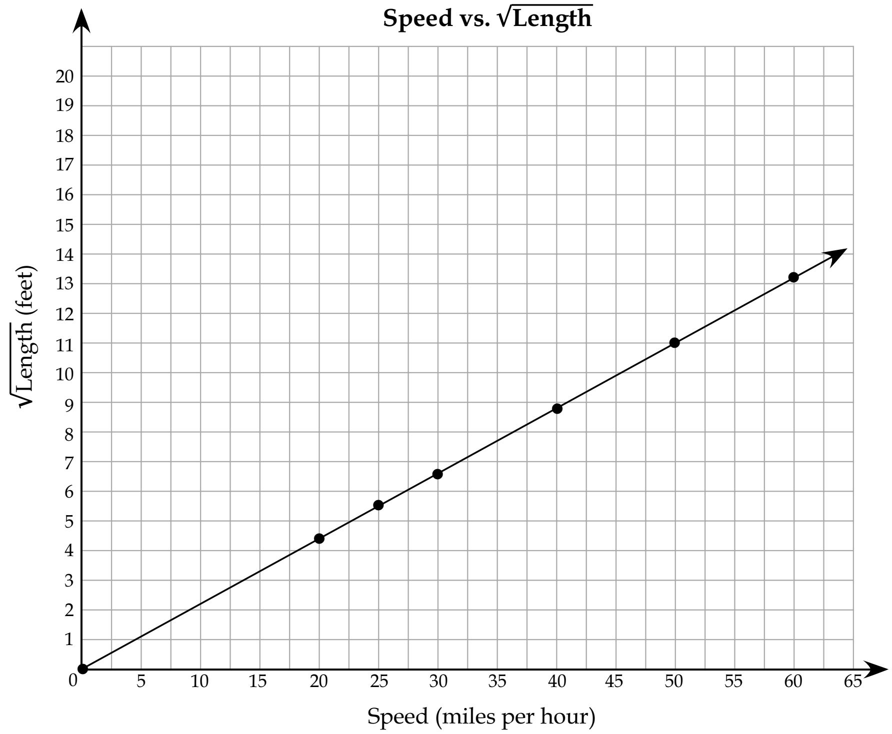
The square root of a quadratic function whose vertex is on the x-axis (as the vertex of this quadratic function seems to be) is an absolute value function. However, note how it is impossible to have a negative speed and thus this graph does not continue for negative values of x. Therefore, only half of the absolute value function would exist. This can also be called a linear function.

d) Take the square root of this graph using the transformation properties you have learned about throughout this lesson.

Answer:

The one easily identifiable invariant point is (0, 0). The majority of the square root of this graph will be somewhere below the Speed vs. Length graph. To get a more accurate picture of what the square root graph looks like, create a table of values.

Speed (miles)	Length (feet)	$\sqrt{\text{Length}}$ (feet)
0	0	0
20	19.8	4.45
25	30.9	5.56
30	44.4	6.66
40	79.1	8.89
50	123.5	11.11
60	177.8	13.33



e) What is the slope of this graph?

Answer:

To determine the slope of this graph, use the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Note: You may want to use this formula twice with two different sets of values in order to get an accurate value for the slope.

Using the points (0, 0) and (20, 4.45):

$$m = \frac{4.45 - 0}{20 - 0} = \frac{4.45}{20} = 0.223$$

Using the points (50, 11.11) and (25, 5.56):

$$m = \frac{5.56 - 11.11}{25 - 50} = \frac{-5.55}{-25} = 0.222$$

An approximate value for the slope of this graph is 0.222.

f) What is the relationship between the slope of the graph and the constant, *k*, in the equation $V = k\sqrt{D}$?

Answer:

The graph represents Speed versus $\sqrt{\text{Length}}$. As speed is graphed on the *x*-axis, let speed or *V* be represented by *x*. As $\sqrt{\text{length}}$ is graphed on the *y*-axis, let $\sqrt{\text{length}}$ or $\sqrt{D}$ be represented by *y*. This will allow you to identify the slope, *m*, when the equation is in the form $y = mx$.

Therefore, $V = k\sqrt{D}$ becomes:

$$x = ky \text{ or } y = \frac{x}{k}$$

The slope of this line is $\frac{1}{k}$.

Thus, $m = \frac{1}{k}$ and $\frac{1}{m} = \frac{1}{0.222} = 4.50$.

g) What is the value of *k* in the equation $V = k\sqrt{D}$?

Answer:

From the above reasoning in (f), $k = 4.50$.

h) Using the value you found in (g), determine how fast the vehicle must have been travelling in order to create a 130-foot skid mark.

Answer:

$V = 4.5\sqrt{D}$

$V = 4.5\sqrt{130}$

$V = 51.31$ miles per hour

Learning Activity 8.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. If $f(x) = -x^2 + 5$, find $f(5)$.
2. What are the zeros of the function $f(x) = x^2 + 7x + 12$?
3. Express $2^7 = 128$ in log form.
4. Express $\log_6 36 = 2$ in exponential form.
5. Convert 60° to radians.
6. State the non-permissible values of the function $f(x) = \dfrac{x-2}{x^2-2x}$.
7. Simplify: $\sqrt[3]{54x^4y^2z^3}$
8. Determine the inverse function $f(x) = \dfrac{3}{x-2}$.

Answers:

1. -20 $(f(5) = -(5)^2 + 5 = -25 + 5)$
2. $x = -3$ and $x = -4$ $(f(x) = (x + 3)(x + 4))$
3. $\log_2 128 = 7$
4. $6^2 = 36$
5. $\dfrac{\pi}{3}$
6. $x \neq 0$ and $x \neq 2$ $(x^2 - 2x = x(x - 2))$
7. $3xz\sqrt[3]{2xy^2}$ $\left(\sqrt[3]{54x^4y^2z^3} = \sqrt[3]{27}\sqrt[3]{x^3}\sqrt[3]{z^3}\sqrt[3]{2xy^2}\right)$
8. $y = \dfrac{3}{x} + 2$ $\left(x = \dfrac{3}{y-2};\ y - 2 = \dfrac{3}{x}\right)$

Part B: Radical Equations

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Consider the equation $\sqrt{x+3} = x + 3$.

   a) Determine the roots of the above equation algebraically.

   Answer:

   Isolate the radical.

   $$\sqrt{x+3} = x + 3$$

   Square both sides of the equation.

   $$\left(\sqrt{x+3}\right)^2 = (x+3)^2$$
   $$x + 3 = x^2 + 6x + 9$$
   $$0 = x^2 + 5x + 6$$

   Solve for x.

   $$x^2 + 5x + 6 = 0$$
   $$(x+2)(x+3) = 0$$
   $$x = -2,\ x = -3$$

   Check for extraneous roots.

   $x = -2$

LHS	RHS
$\sqrt{x+3}$	$x + 3$
$\sqrt{-2+3}$	$-2 + 3$
$\sqrt{1}$	1

   LHS = RHS

   $x = -3$

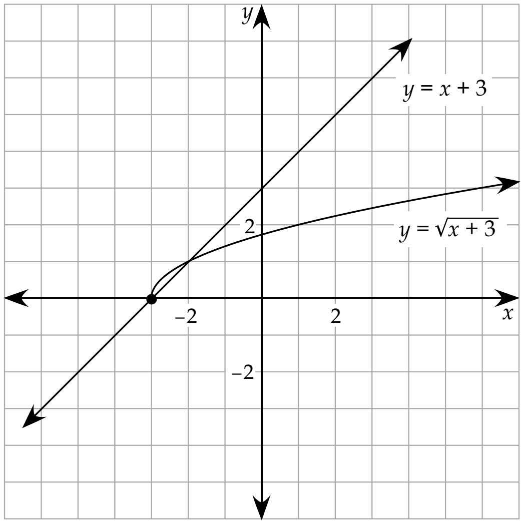
LHS	RHS
$\sqrt{x+3}$	$x + 3$
$\sqrt{-3+3}$	$-3 + 3$
$\sqrt{0}$	0

   LHS = RHS

   As the LHS equals the RHS in both cases, $x = -2$ and $x = -3$ are both roots of the above radical equation.

b) Sketch $y = \sqrt{x + 3}$ and $y = x + 3$ to determine the solution(s) of $\sqrt{x + 3} = x + 3$.

Answer:



The solutions of $\sqrt{x + 3} = x + 3$ occur when the two functions intersect. These points of intersection are located at $x = -3$ and $x = -2$. The solutions are thus $x = -3$ and $x = -2$.

2. Solve the following radical equation graphically.

$\sqrt{x + 7} = 3$

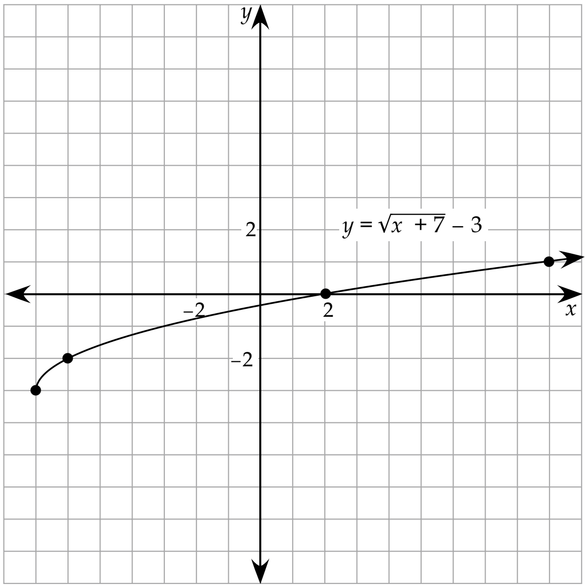
Answer:

Generally, Method 2 is easier to use when the function created from moving everything to one side of the equals sign is easy to graph by transformations. Otherwise, Method 1 is used.

To use Method 2, move everything to one side of the equation.

$\sqrt{x + 7} - 3 = 0$

Graph the corresponding function, $y = \sqrt{x+7} - 3$, using transformations.



This function crosses the x-axis at $x = 2$ and the solution to the related radical equation, $\sqrt{x+7} = 3$ is $x = 2$.

3. The root of the radical equation $2\sqrt{x-3} = 4$ is $x = 7$. Write the related function that will have an x-intercept of 7.

   Answer:

   The function $y = 2\sqrt{x-3} - 4$ has an x-intercept of 7.

4. Estimate the solutions to the following radical equations graphically.

$$\sqrt{3x-3}-2=6$$

Answer:

The restrictions on the variable in this equation are:

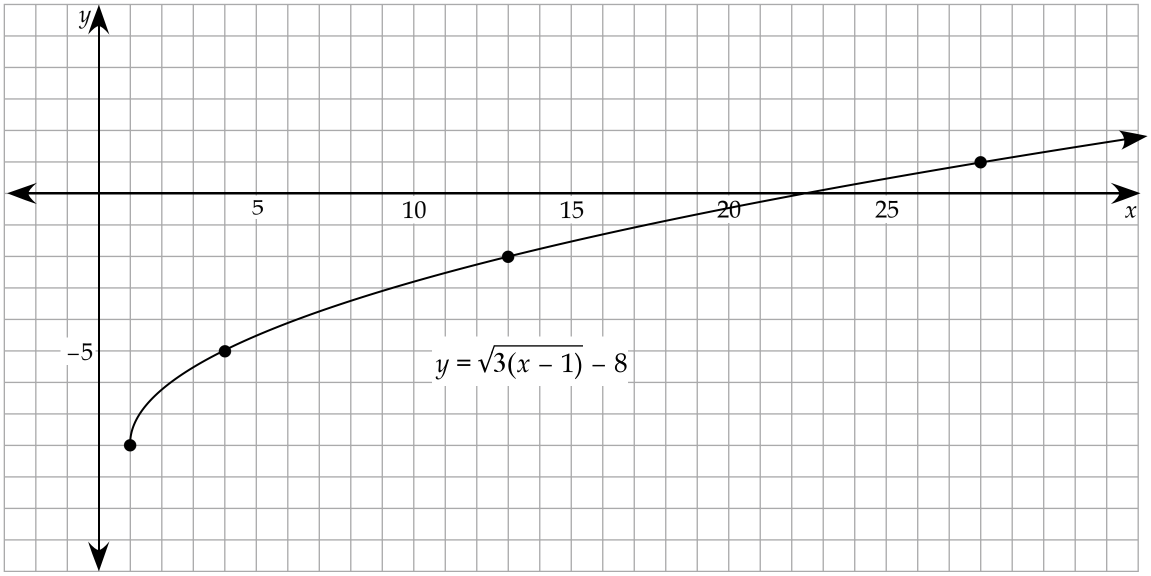
$3x - 3 \geq 0$ or

$x \geq 1$

To graph this equation, first move everything to one side of the equation and then graph the corresponding function.

$$\sqrt{3(x-1)}-8=0$$

$$\sqrt{3(x-1)}-8=y$$



From the graph, you will notice that the x-intercept is located at approximately $x = 22$. This is the approximate solution to the above equation.

Learning Activity 8.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Solve for x: $(x + 1)! = 6x!$
2. Simplify: $\sqrt{\dfrac{81x^2}{9y^4}}$
3. Calculate: $\sin 15° \csc 15°$
4. In which quadrant is $\cot \theta$ positive and $\tan \theta$ negative?
5. Express $7^2 = 49$ in logarithmic form.
6. Express $\log_4 256 = 4$ in exponential form.
7. Solve for x: $\log 10^{(x+2)} = 3$
8. Simplify: $\left(5 + \sqrt{x}\right)^2$

Answers:

1. $x = 5$ $\left[(x+1)! = (x+1)x!\right]$
2. $\dfrac{3x}{y^2}$ $\left(\sqrt{\dfrac{81x^2}{9y^4}} = \dfrac{\sqrt{81}\sqrt{x^2}}{\sqrt{9}\sqrt{y^4}}\right)$
3. 1 $\left[\sin 15° \csc 15° = \sin 15°\left(\dfrac{1}{\sin 15°}\right)\right]$
4. This is impossible because cotangent and tangent are reciprocals and, therefore, always have the same sign.
5. $\log_7 49 = 2$
6. $4^4 = 256$
7. $x = 1$ $(x + 2 = 3, x = 1)$
8. $25 + 10\sqrt{x} + x$

Part B: Graphing Rational Functions

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Identify the (i) x-intercepts, (ii) y-intercept, and (iii) the equation of any vertical or horizontal asymptotes for the following.

   a) $y = \dfrac{x-1}{(x-2)(x-3)}$

   Answer:

   i) x-intercept is the zero of the numerator

   $x - 1 = 0$; $x = 1$

   ii) y-intercept is the value of

   $$f(0) = \frac{0-1}{(0-2)(0-3)} = \frac{-1}{(-2)(-3)} = \frac{-1}{6}$$

   iii) Equations of vertical asymptotes are vertical lines through the zero of the denominator.

   $(x - 2)(x - 3) = 0$

   $x - 2 = 0 \qquad x - 3 = 0$

   $x = 2 \qquad x = 3$

   Equation of horizontal asymptote: Because the degree of the numerator is less than the degree of the denominator, the x-axis is the horizontal asymptote, which has the equation $y = 0$.

   b) $y = \dfrac{x}{x^2 - 9}$

   Answer:

   i) x-intercept is the zero of the numerator; that is, when $x = 0$

   ii) y-intercept is the value of

   $$f(0) = \frac{0}{0-9} = 0$$

iii) Equations of vertical asymptotes are vertical lines drawn through the zero of the denominator.

$x - 3 = 0 \qquad x + 3 = 0$

$x = 3 \qquad x = -3$

Equation of horizontal asymptote: Because the degree of the numerator is less than the degree of the denominator, the x-axis is the horizontal asymptote. Its equation is $y = 0$.

c) $y = \dfrac{x^2 - 4}{x^2 + 4}$

Answer:

i) x-intercept: $x^2 - 4 = 0$, $x = \pm 2$

ii) y-intercept

$$f(0) = \frac{0 - 4}{0 + 4} = -1$$

iii) Vertical asymptotes: Because the denominator $x^2 + 4$ has no real zeros, there are no vertical asymptotes.

Horizontal asymptotes: Because the degree of the numerator is equal to the degree of the denominator, y = the ratio of the leading coefficient of the numerator to that of the denominator. Therefore, $y = 1$.

2. Match the function with its graph. Look for asymptotes.

a) $f(x) = \dfrac{x + 3}{x - 2}$

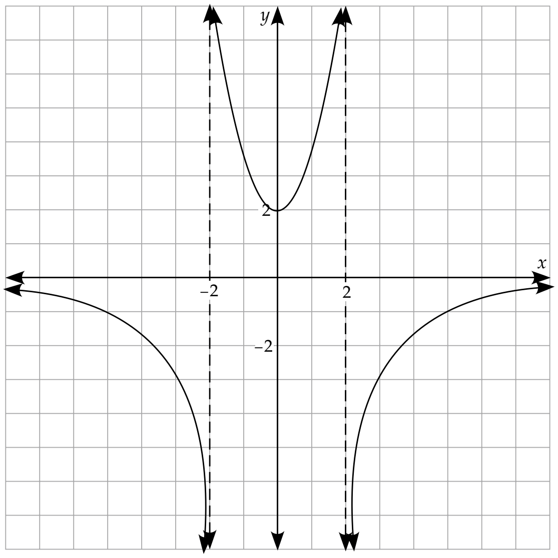
b) $f(x) = \dfrac{-8}{x^2 - 4}$

c) $f(x) = \dfrac{3}{x + 2}$

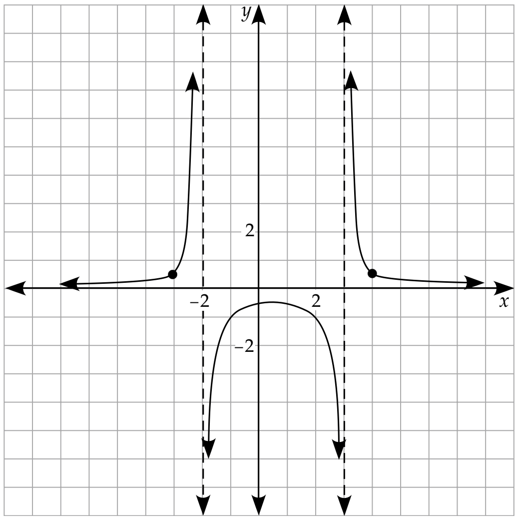
d) $f(x) = \dfrac{x - 3}{2x + 4}$

e) $f(x) = \dfrac{3}{x^2 - x - 6}$

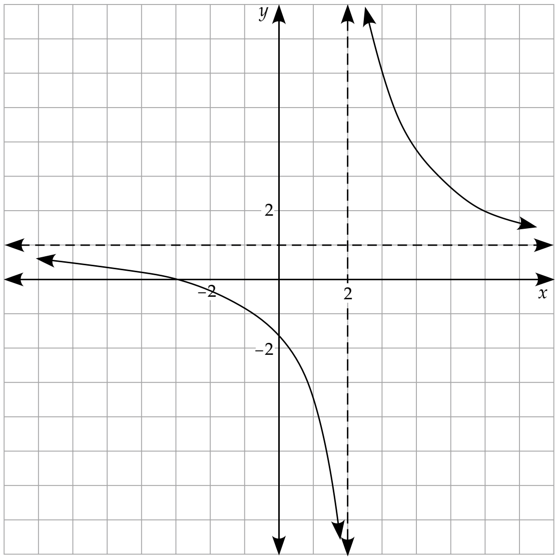
i)



ii)



iii)



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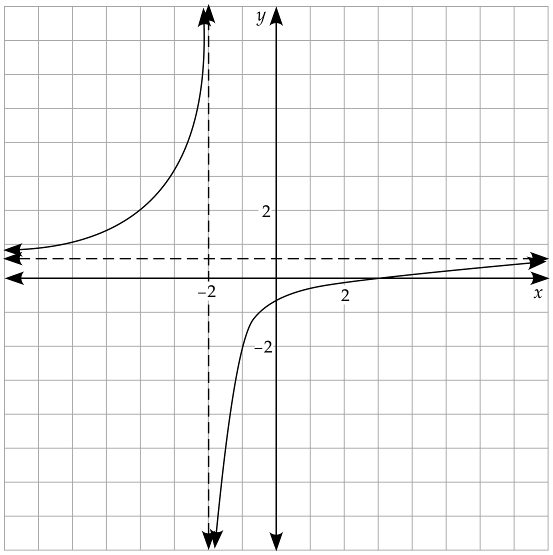
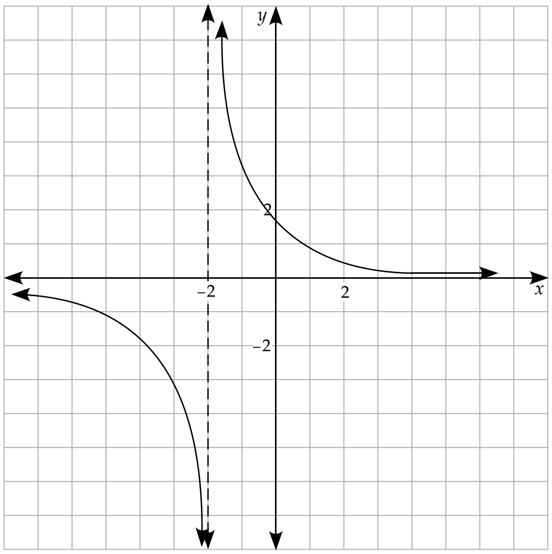


v)



Answer:

(a) (iii) vertical asymptote at $x = 2$

(b) (i) two vertical asymptotes at $x = 2$ and $x = -2$

(c) (v) vertical asymptote at $x = -2$, horizontal asymptote at $y = 0$

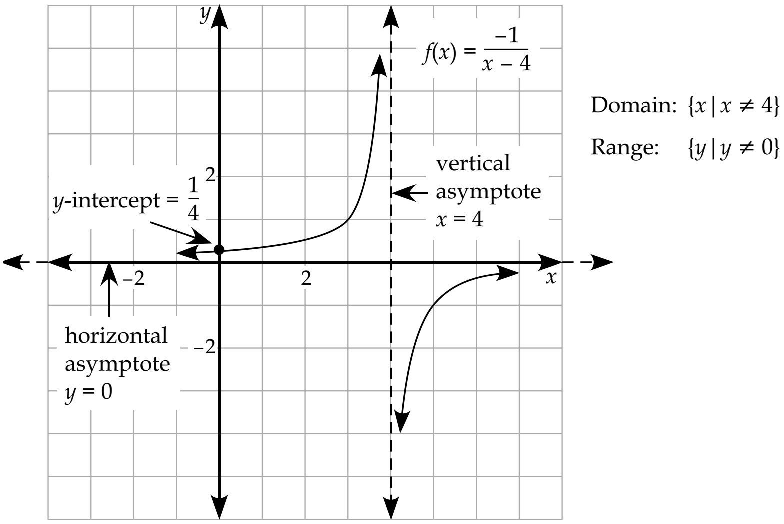
(d) (iv) vertical asymptote at $x = -2$, horizontal asymptote at $y = \frac{1}{2}$

(e) (ii) two vertical asymptotes at $x = -2$ and $x = 3$

3. Sketch the graph of each rational function. Label the x-intercept(s), y-intercept, and state the equation of the horizontal and vertical asymptotes (if they exist), and the domain and range.

a) $f(x) = \dfrac{-1}{x - 4}$

Answer:



There are no points of discontinuity (holes) since there are no factors common to the numerator and the denominator.

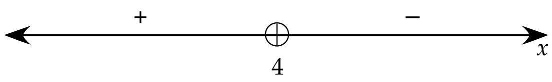
x-intercept: None (no variable in numerator)

y-intercept: $f(0) = \dfrac{-1}{0 - 4} = \dfrac{1}{4}$

Vertical asymptote: $x - 4 = 0$; $x = 4$

Horizontal asymptote: $y = 0$

Sign analysis:



Critical value: $x = 4$

Interval $(-\infty, 4)$: test $x = 3$

$$f(3) = \frac{-1}{(3-4)} = \frac{-1}{-1} = 1 = \text{positive}$$

Interval $(4, \infty)$: test $x = 5$

$$f(5) = \frac{-1}{(5-4)} = \frac{-1}{1} = -1 = \text{negative}$$

Note that you could also use the transformational approach to this graph.

If you start with $y = \frac{1}{x}$, then shift it 4 units to the right to get $y = \frac{1}{x-4}$, and then reflect it over the x-axis to get $f(x) = -\frac{1}{x-4}$, you will have the final graph as shown.

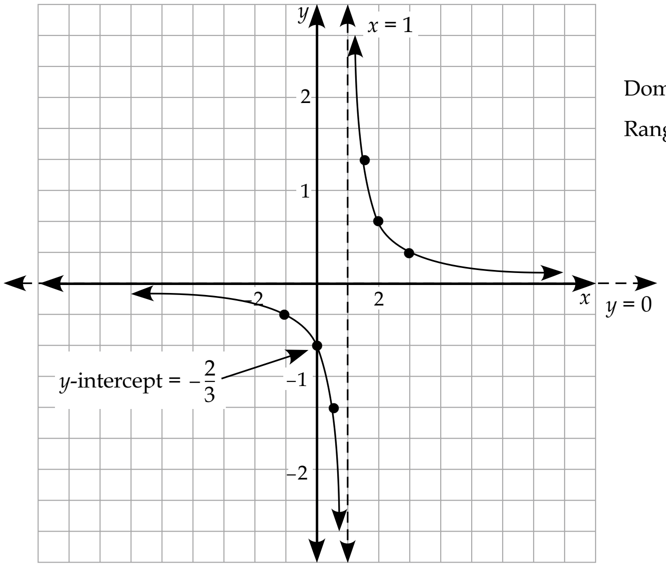
b) $f(x) = \dfrac{2}{3(x-1)}$

Answer:

Graphical Properties could also be used to answer this question.

This is the graph of the radical function $y = \dfrac{1}{x}$, stretched by a factor of $\dfrac{2}{3}$ and then moved 1 unit to the right.

x	$y = \dfrac{1}{x}$	$(x + 1)$	$\left(\dfrac{2}{3}y\right)$
-2	-0.5	-1	$-\dfrac{1}{3}$
-1	-1	0	$-\dfrac{2}{3}$
-0.5	-2	0.5	$-\dfrac{4}{3}$
Vertical Asymptote at $x = 0$		Vertical Asymptote at $x = 1$	
Horizontal Asymptote at $y = 0$		Horizontal Asymptote at $y = 0$	
0.5	2	1.5	$\dfrac{4}{3}$
1	1	2	$\dfrac{2}{3}$
2	0.5	3	$\dfrac{1}{3}$

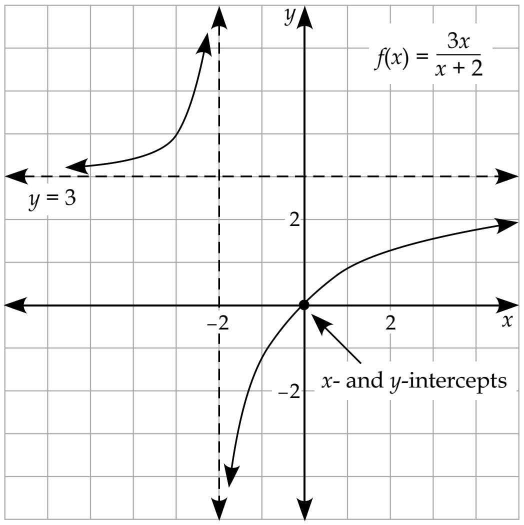


Domain: $\{x \mid x \neq 1\}$

Range: $\{y \mid y \neq 0\}$

c) $f(x) = \dfrac{3x}{x+2}$

Answer:



There are no points of discontinuity (holes) since there are no factors common to the numerator and the denominator.

x-intercept: $3x = 0$, $x = 0$

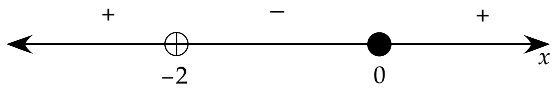
y-intercept: $f(0) = \dfrac{3 \cdot 0}{0+2} = 0$

Vertical asymptote: $x + 2 = 0$, $x = -2$

Horizontal asymptote: Since the degrees are equal, the horizontal asymptote equation is $y = \dfrac{3}{1}$ or $y = 3$.

Sign analysis:

Critical values: $x = 0$ and $x = -2$



Interval $(-\infty, -2)$: test $x = -3$

$$f(-3) = \frac{3(-3)}{(-3+2)} = \frac{-9}{-1} = 9 = \text{positive}$$

Interval $(-2, 0]$: test $x = -1$

$$f(-1) = \frac{3(-1)}{(-1+2)} = \frac{-3}{1} = -3 = \text{negative}$$

Interval [0, ∞): test $x = 1$

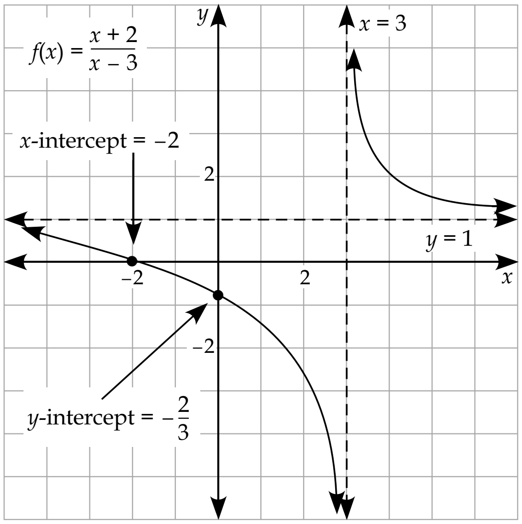
$$f(1) = \frac{3(1)}{(1+2)} = \frac{3}{3} = 1 = \text{positive}$$

Domain: $\{x \mid x \neq -2\}$

Range: $\{y \mid y \neq 3\}$

d) $f(x) = \dfrac{x+2}{x-3}$

Answer:



x-intercept: Let numerator = 0

$x + 2 = 0$, $x = -2$

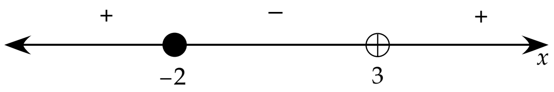
y-intercept: $f(0) = \dfrac{0+2}{0-3} = -\dfrac{2}{3}$

Vertical asymptote: $x - 3$, $x = 3$

Horizontal asymptote: Since degrees of numerator and denominator are equal, $y = \dfrac{1}{1} = 1$.

Sign analysis:

Critical values: $x = -2$ and $x = 3$



Interval $(-\infty, -2]$: test $x = -3$

$$f(-3) = \frac{(-3+2)}{(-3-3)} = \frac{-1}{-6} = \frac{1}{6} = \text{positive}$$

Interval $[-2, 3)$: test $x = 0$

$$f(0) = \frac{(0+2)}{(0-3)} = -\frac{2}{3} = \text{negative}$$

Interval $(3, \infty)$: test $x = 4$

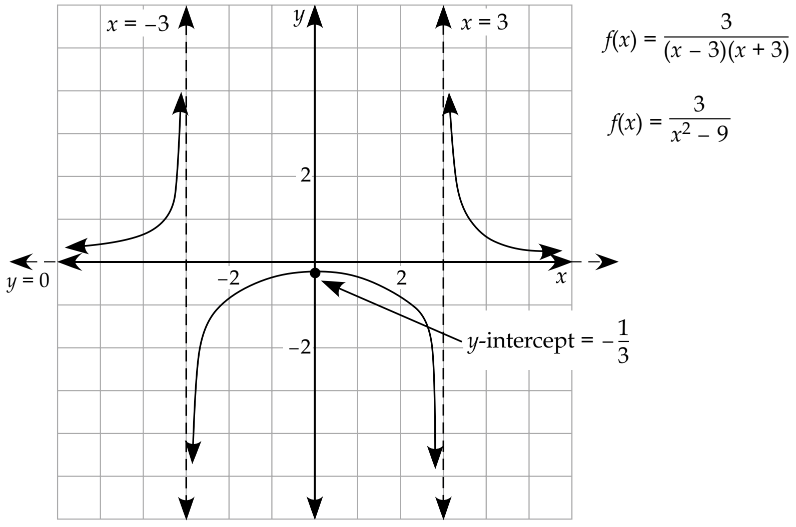
$$f(4) = \frac{(4+2)}{(4-3)} = \frac{6}{1} = 6 = \text{positive}$$

Domain: $\{x \mid x \neq 3\}$

Range: $\{y \mid y \neq 1\}$

e) $f(x) = \dfrac{3}{x^2 - 9}$

Answer:



There are no points of discontinuity (holes) since there are no factors common to the numerator and the denominator.

x-intercept: Since the numerator has no variable, there is no x-intercept.

y-intercept: $f(0) = \dfrac{3}{0-9} = -\dfrac{1}{3}$

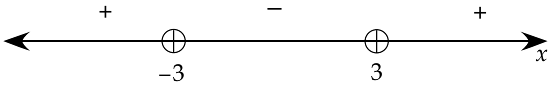
Vertical asymptote: $(x - 3)(x + 3) = 0$

$x = 3$ or $x = -3$

Horizontal asymptote: $y = 0$, because the degree of the numerator is less than the degree of the denominator.

Sign analysis:

Critical values: $x = -3$ and $x = 3$



Interval $(-\infty, -3)$ test $x = -4$

$$f(-4) = \frac{3}{(-4-3)(-4+3)} = \frac{3}{(-7)(-1)} = \frac{3}{7} = \text{positive}$$

Interval $(-3, 3)$: test $x = 0$

$$f(0) = \frac{3}{(0-3)(0+3)} = \frac{3}{-9} = -\frac{1}{3} = \text{negative}$$

Interval $(3, \infty)$: test $x = 4$

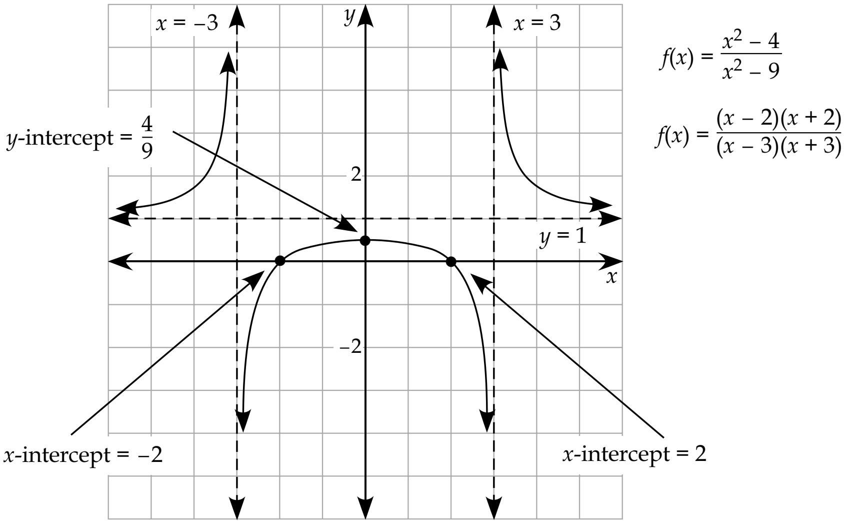
$$f(4) = \frac{3}{(4-3)(4+3)} = \frac{3}{7} = \text{positive}$$

Domain: $\{x \mid x \neq \pm 3\}$

Range: $\left(-\infty, -\dfrac{1}{2}\right) \cup (0, \infty)$

f) $f(x) = \dfrac{x^2 - 4}{x^2 - 9}$

Answer:



There are no points of discontinuity (holes) since there are no factors common to the numerator and the denominator.

x-intercept: Let numerator = 0 to find its zeros.

$$x - 2 = 0 \qquad x + 2 = 0$$
$$x = 2 \qquad x = -2$$

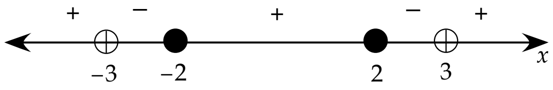
y-intercept: $f(0) = \dfrac{0 - 4}{0 - 9} = \dfrac{4}{9}$

Vertical asymptote: $(x - 3) = 0$, $(x + 3) = 0$
$x = -3$ or $x = 3$

Horizontal asymptote: $y = 1$, because the numerator and denominator are of the same degree with leading coefficients of 1.

Sign analysis:

Critical values: $x = \pm 3$ and $x = \pm 2$



Interval $(-\infty, -3)$: test $x = -4$

$$f(-4) = \frac{(16 - 4)}{(16 - 9)} = \frac{12}{7} = \text{positive}$$

Interval $(-3, -2]$: test $x = -2.5$

$$f(-2.5) = \frac{(6.25 - 4)}{(6.25 - 9)} = \frac{2.25}{-2.75} = -0.82 = \text{negative}$$

Interval $[-2, 2]$: test $x = 0$

$$f(0) = \frac{(0-4)}{(0-9)} = \frac{4}{9} = \text{positive}$$

Interval $[2, 3)$: test $x = 2.5$

$$f(2.5) = \frac{(6.25-4)}{(6.25-9)} = -0.82 = \text{negative}$$

Interval $(3, \infty)$: test $x = 4$

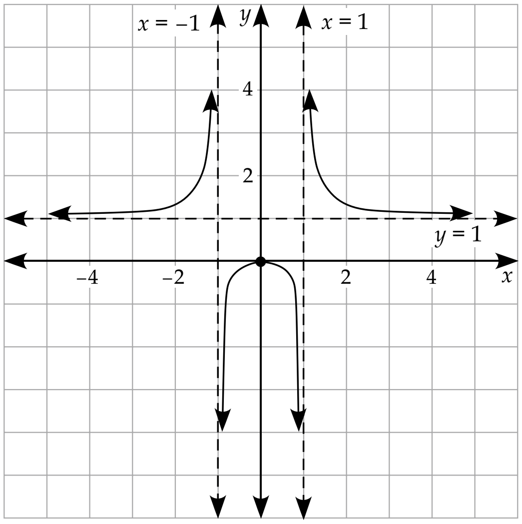
$$f(4) = \frac{(16-4)}{(16-9)} = \frac{12}{7} = \text{positive}$$

Domain: $\{x \mid x \neq \pm 3\}$

Range: $\left(-\infty, \frac{4}{9}\right] \cup (1, \infty)$

g) $f(x) = \dfrac{x^2}{x^2 - 1}$

Answer:



$f(x) = \dfrac{x^2}{x^2 - 1}$

$f(x) = \dfrac{x^2}{(x-1)(x+1)}$

There are no points of discontinuity (holes) since there are no factors common to numerator and denominator.

x-intercept: Let $x^2 = 0$, $\therefore x = 0$

y-intercept: $f(0) = \dfrac{0}{0-1} = 0$

Vertical asymptote: $(x - 1)(x + 1) = 0$

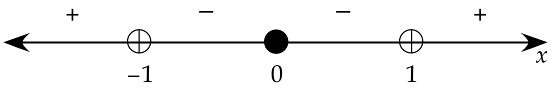
$x - 1 = 0$, $(x + 1) = 0$

$x = 1$ or $x = -1$

Horizontal asymptote: $y = 1$

Sign analysis:

Critical values: $x = \pm 1$ and $x = 0$



Interval $(-\infty, -1)$: test $x = -2$

$$f(-2) = \frac{4}{(4-1)} = \frac{4}{3} = \text{positive}$$

Interval $(-1, 0]$: test $x = 0.5$

$$f(-0.5) = \frac{0.25}{(0.25-1)} = \frac{0.25}{-0.75} = -\frac{1}{3} = \text{negative}$$

Interval $[0, 1)$: test $x = 0.5$

$$f(0.5) = \frac{0.25}{(0.25-1)} = \frac{0.25}{-0.75} = -\frac{1}{3} = \text{negative}$$

Interval $(1, \infty)$: test $x = 2$

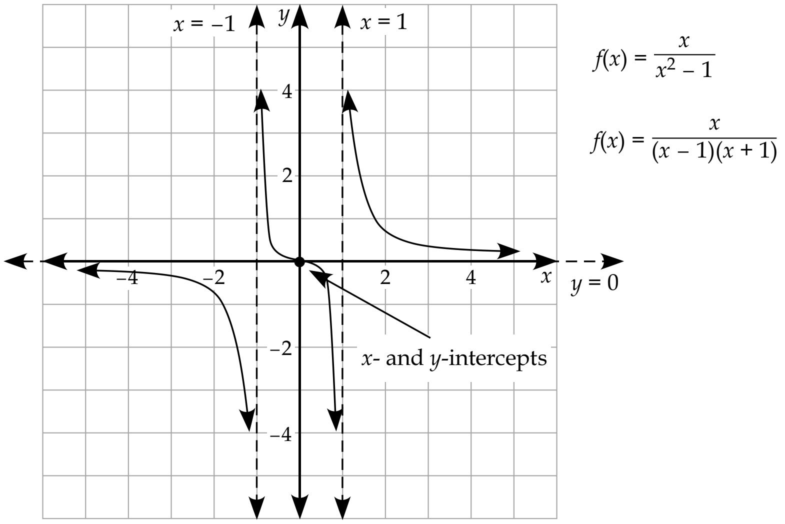
$$f(2) = \frac{4}{(4-1)} = \frac{4}{3} = \text{positive}$$

Domain: $\{x \mid x \neq \pm 1\}$

Range: $(-\infty, 0] \cup (1, \infty)$

h) $f(x) = \dfrac{x}{x^2 - 1}$

Answer:



There are no points of discontinuity (holes) since there are no factors common to the numerator and the denominator.

x-intercept: $x = 0$

y-intercept: $f(0) = 0$

Vertical asymptote: $x^2 - 1 = 0$

$(x - 1)(x + 1) = 0$

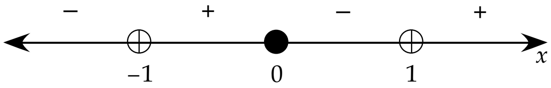
$x - 1 = 0 \qquad x + 1 = 0$

$x = 1 \qquad x = -1$

Horizontal asymptote: $y = 0$

Sign analysis:

Critical values: $x = \pm 1, 0$



Interval $(-\infty, -1)$: test $x = -2$

$$f(-2) = \frac{-2}{(-2-1)(-2+1)} = \frac{-2}{(-3)(-1)} = \frac{-2}{3} = \text{negative}$$

Interval $(-1, 0]$: test $x = -0.5$

$$f(-0.5) = \frac{-0.5}{(-0.5-1)(-0.5+1)} = \frac{-0.5}{(-1.5)(0.5)} = \frac{2}{3} = \text{positive}$$

Interval $[0, 1)$: test $x = 0.5$

$$f(0.5) = \frac{0.5}{(0.5-1)(0.5+1)} = \frac{0.5}{(-0.5)(1.5)} = -\frac{2}{3} = \text{negative}$$

Interval $(1, \infty)$: test $x = 2$

$$f(2) = \frac{2}{(2-1)(2+1)} = \frac{2}{3} = \text{positive}$$

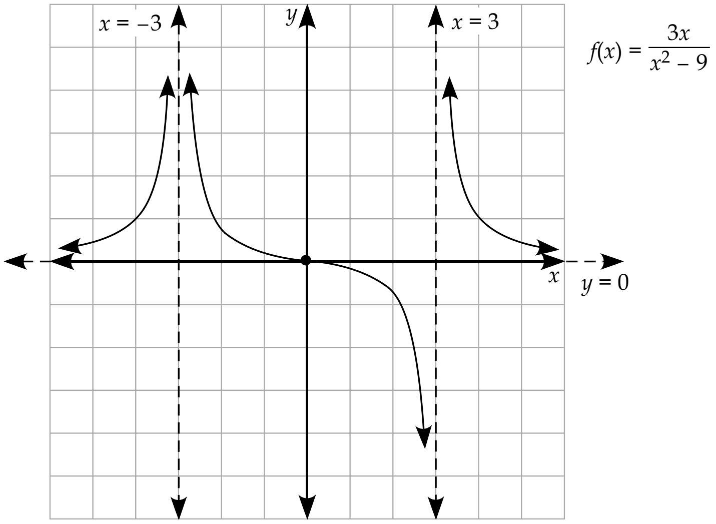
Domain: $\{x \mid x \neq \pm 1\}$

Range: $(-\infty, \infty)$

Notice that this function does cross the horizontal asymptote at (0, 0). The horizontal asymptote only describes, as the graph goes way off to the right to infinity or way off to the left to negative infinity. Although a function may sometimes cross a horizontal asymptote, it will never cross a vertical asymptote since the function is undefined for that value of x.

i) $f(x) = \dfrac{3x}{x^2 - 9}$

Answer:



There are no points of discontinuity (holes) since there are no factors common to the numerator and the denominator.

x-intercept: $3x = 0$, $\therefore x = 0$

y-intercept: $f(0) = 0$

Vertical asymptote: $x^2 - 9 = 0$

$$(x - 3)(x + 3) = 0$$

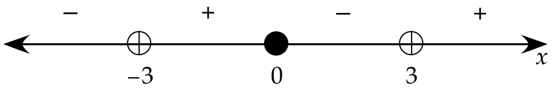
$$x - 3 = 0 \qquad x + 3 = 0$$

$$x = 3 \qquad x = -3$$

Horizontal asymptote: $y = 0$

Sign analysis:

Critical values: $x = \pm 3, 0$



Interval $(-\infty, -3)$: test $x = -4$

$$f(-4) = \frac{-12}{(16 - 9)} = \frac{-12}{7} = \text{negative}$$

Interval $(-3, 0]$: test $x = -1$

$$f(-1) = \frac{-3}{(1 - 9)} = \frac{-3}{-8} = \frac{3}{8} = \text{positive}$$

Interval [0, 3): test $x = 1$

$$f(1) = \frac{3}{(1-9)} = \frac{3}{-8} = -\frac{3}{8} = \text{negative}$$

Interval (3, ∞): test $x = 4$

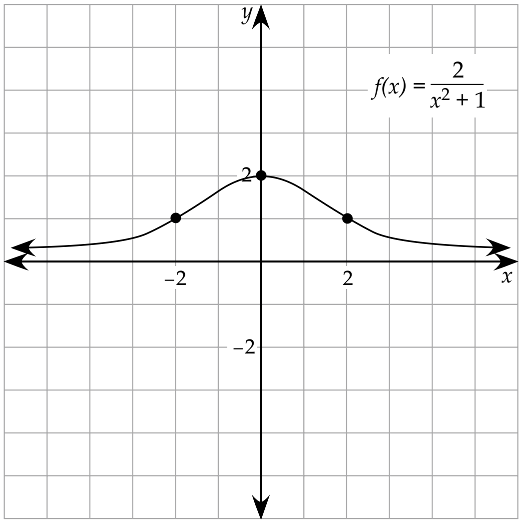
$$f(4) = \frac{12}{(16-9)} = \frac{12}{7} = \text{positive}$$

Domain: $\{x \mid x \neq \pm 3\}$

Range: $(-\infty, \infty)$

j) $f(x) = \dfrac{2}{x^2 + 1}$

Answer:



There are no points of discontinuity (holes) since there are no factors common to the numerator and the denominator.

x-intercept: none

y-intercept: $f(0) = 2$

Vertical asymptote: none

Horizontal asymptote: $y = 0$

The value of $f(x)$ is always positive.

Domain: $(-\infty, \infty)$

Range: $(0, 2]$

k) $f(x) = \dfrac{3}{x^2 + 2x + 1}$

Answer:

There are no points of discontinuity (holes) since there are no factors common to the numerator and the denominator.

There is no x-intercept.

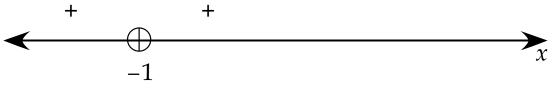
y-intercept: $f(0) = \dfrac{3}{0^2 + 2(0) + 1} = 3$

Vertical asymptote:
$$\begin{aligned} x^2 + 2x + 1 &= 0 \\ (x + 1)^2 &= 0 \\ x + 1 &= 0 \\ x &= -1 \end{aligned}$$

Horizontal asymptote: $y = 0$

Sign analysis:

Critical values: $x = -1$



Interval $(-\infty, -1)$: test $x = -2$

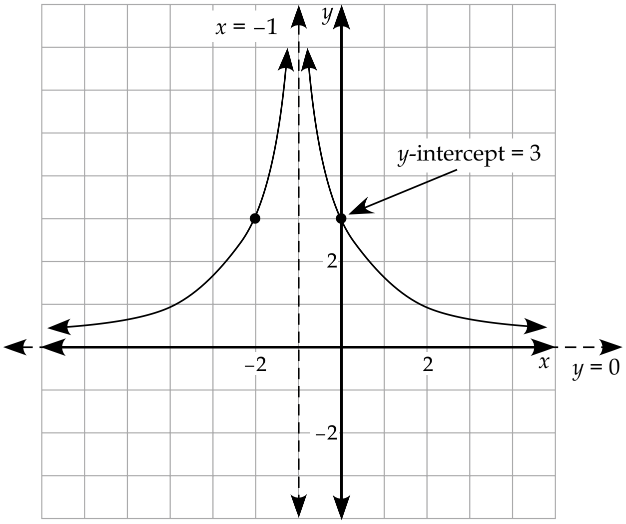
$f(-2) = \dfrac{3}{(-2)^2 + 2(-2) + 1} = 3 = \text{positive}$

Interval $[-1, \infty)$: test $x = 0$

$f(0) = 3 = \text{positive}$

Domain: $\{x \mid x \neq -1\}$

Range: $\{y \mid y > 0\}$



1) $f(x) = \dfrac{x+3}{x^2 + 7x - 8}$

Answer:

There are no points of discontinuity (holes) since there are no factors common to the numerator and the denominator.

x-intercept: $x + 3 = 0$, $x = -3$

y-intercept: $f(0) = \dfrac{0+3}{0^2 + 7(0) - 8} = -\dfrac{3}{8}$

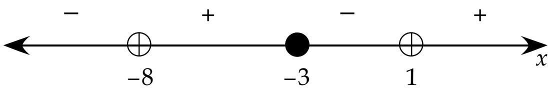
Vertical asymptote:

$$x^2 + 7x - 8 = 0$$
$$(x + 8)(x - 1) = 0$$
$$x + 8 = 0 \qquad x - 1 = 0$$
$$x = -8 \qquad x = 1$$

Horizontal asymptote: $y = 0$

Sign analysis:

Critical values: $x = -8, -3$, and 1



Interval $(-\infty, -8)$: test $x = -9$

$$f(-9) = \frac{-9+3}{(-9)^2 + 7(-9) - 8} = -\frac{6}{10} = -\frac{3}{5} = \text{negative}$$

Interval $[-8, -3)$: test $x = -4$

$$f(-4) = \frac{-4+3}{(-4)^2 + 7(-4) - 8} = \frac{1}{20} = \text{positive}$$

Interval $[-3, 1)$: test $x = 0$

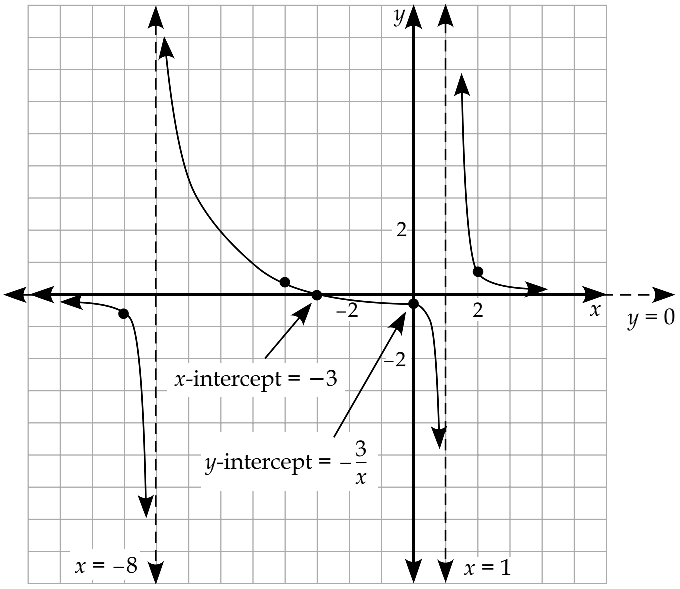
$$f(0) = -\frac{3}{8} = \text{negative}$$

Interval $[1, \infty)$: test $x = 2$

$$f(2) = \frac{2+3}{2^2 + 7(2) - 8} = \frac{5}{10} = \frac{1}{2} = \text{positive}$$

Domain: $\{x \mid x \neq -8, 1\}$

Range: $\{y \mid y \neq 0\}$



4. Determine the equation of the function in the form $f(x) = \frac{a}{x-h}$ that has a vertical asymptote at $x = 3$ and a y-intercept of (0, 5).

Answer:

A rational function has a vertical asymptote when the denominator equals zero.

$x - h = 0$

When the denominator of this function equals zero, $x = 3$.

$$\therefore 3 - h = 0$$
$$3 = h$$

Now, you can substitute the point (0, 5) into the equation $y = \frac{a}{x-3}$ to solve for the missing a-value.

$$y = \frac{a}{x-3}$$

$$5 = \frac{a}{0-3}$$

$$5 = \frac{a}{-3}$$

$$-15 = a$$

The resulting function is $f(x) = -\frac{15}{x-3}$.

5. The function $c(t) = \dfrac{45t}{t^2 + 15}$ describes the concentration of a drug in a patient's bloodstream over time. Time, t, is measured in hours while $c(t)$, or the concentration, is measured in milligrams per litre.

a) Graph this function.

Answer:

The numerator and the denominator have no common factors.

x-intercept: $45t = 0$, $t = 0$

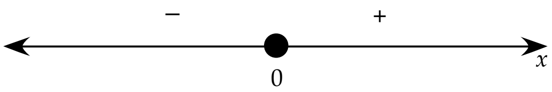
y-intercept: $c(0) = \dfrac{45(0)}{0^2 + 15} = 0$

There are no vertical asymptotes, as the denominator never equals zero.

Horizontal asymptote: $y = 0$

Sign analysis:

Critical value: $x = 0$



Interval $(-\infty, 0)$: test $x = -1$

$$c(-1) = \frac{45(-1)}{(-1)^2 + 15} = \frac{-45}{16} = \text{negative}$$

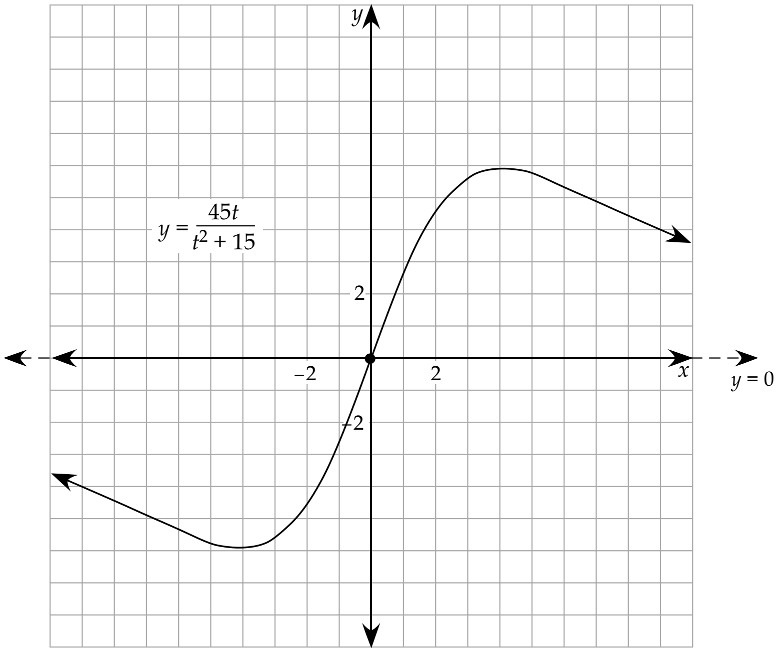
Interval $[0, \infty)$: test $x = 1$

$$c(1) = \frac{45(1)}{1^2 + 15} = \frac{45}{16} = \text{positive}$$

Find additional points to help you graph the function.

For example:

t	$c(t)$
2	$\frac{90}{19} \approx 4.74$
3	$\frac{135}{24} = 5.625$
4	$\frac{180}{31} \approx 5.81$
5	$\frac{225}{40} = 5.625$
6	$\frac{270}{51} \approx 5.29$



b) What is the domain in this context?

Answer:

The domain in this context is [0, ∞), as it is not possible for a drug to be in a patient's bloodstream before they took the drug ($t \geq 0$) and a drug can potentially stay with you forever. However, the effects of the drug will eventually wear off.

c) What is the approximate range in this context?

Answer:

The maximum y-value is just less than 6. The minimum is 0, the amount of the drug in the bloodstream before taking the first pill.

If you used graphing technology, you could find the maximum point to be (3.873, 5.809) and, therefore, your range would be [0, 5.809]. The approximate range would be a good enough answer for an assignment or on the final examination.

d) What happens to the concentration of the drug during the first 4 hours?

Answer:

During the first four hours, the concentration of the drug slowly increases in the patient's bloodstream.

e) What happens to the concentration of the drug over 1 day, or 24 hours?

Answer:

Over 24 hours, the concentration of the drug slowly decreases, after approximately 4 hours of increasing, until at 24 hours the concentration in the patient's bloodstream is:

$$c(24) = \frac{45(24)}{24^2 + 15} = \frac{1080}{591} = \frac{360}{197} \approx 1.83 \text{ milligrams per liter.}$$

f) Is this a reasonable representation of the concentration of a drug in a patient's bloodstream over time?

Answer:

Yes this is a reasonable representation, as some drugs take longer to have an effect (the graph slowly increases between 0 and 4) and then eventually decrease their effectiveness until they are essentially ineffective (the graph slowly decreases toward zero after 4).

Learning Activity 8.5

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. List the non-permissible values of the function: $f(x) = \dfrac{2x+1}{(x+3)(x-6)}$
2. Factor: $2x^2 + 11x - 21$
3. Is $f(x) = \dfrac{3x+5}{2}$ a rational function?
4. Write an expression that represents 2 less than the product of 3 and a number.
5. Evaluate: $|\,2(3 - 6) - 8\,|$
6. If $f(x) = -x^3 + 2x - 7$, find $f(-2)$.
7. Simplify: $\dfrac{2}{7} + \dfrac{5}{3} - \dfrac{1}{21}$
8. What is 15% of 316?

Answers:

1. $x \neq -3$ and $x \neq 6$
2. $(x + 7)(2x - 3)$
3. No (there is no variable in the denominator)
4. $y = 3x - 2$
5. $14\ \left(\left|2(3-6)-8\right| = \left|2(-3)-8\right| = \left|-6-8\right| = \left|-14\right|\right)$
6. $f(-2) = -3$; $(f(-2) = -(-2)^3 + 2(-2) - 7 = -(-8) - 4 - 7 = 8 - 4 - 7)$
7. $\dfrac{40}{21}\ \left(\dfrac{2}{7} + \dfrac{5}{3} - \dfrac{1}{21} = \dfrac{6}{21} + \dfrac{35}{21} - \dfrac{1}{21} = \dfrac{6+35-1}{21}\right)$
8. 47.4 (10% of 316 = 31.6; 5% of 316 = 15.8; 31.6 + 15.8 = 47.4)

Part B: Asymptotes Versus Discontinuities

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Decide whether each of the following functions will have an asymptote or a point of discontinuity at the indicated value(s) without graphing the function.

   a) $y = \dfrac{x-2}{x^2+x-6}$, $x = 2$

   Answer:

   Method 1:

   Evaluate y at $x = 2$; $y = \dfrac{2-2}{2^2+2-6} = \dfrac{0}{0}$.

   Therefore, there is a point of discontinuity.

   Method 2:

   $$y = \frac{x-2}{x^2+x-6} = \frac{x-2}{(x-2)(x+3)}$$

   As $x - 2$ is a factor of both the numerator and the denominator, $x = 2$ is a point of discontinuity of the function $y = \dfrac{x-2}{x^2+x-6}$.

   b) $y = \dfrac{x^2-4x-5}{x-5}$, $x = 5$

   Answer:

   Method 1:

   Evaluate y at $x = 5$; $y = \dfrac{5^2-4(5)-5}{5-5} = \dfrac{0}{0}$.

   Therefore, there is a point of discontinuity.

   Method 2:

   $$y = \frac{x^2-4x-5}{x-5} = \frac{(x-5)(x+1)}{x-5}$$

   As $x - 5$ is a factor of both the numerator and the denominator, $x = 5$ is a point of discontinuity of the function $y = \dfrac{x^2-4x-5}{x-5}$.

c) $y = \dfrac{x^2 - x - 72}{x^2 - 6x - 27}$, $x = 9$, $x = -8$

Answer:

$$y = \frac{x^2 - x - 72}{x^2 - 6x - 27} = \frac{(x-9)(x+8)}{(x-9)(x+3)}$$

As $x - 9$ is a factor of both the numerator and the denominator, $x = 9$ is a point of discontinuity.

As $x + 8$ is a factor of the numerator, $x = -8$ is neither an asymptote nor a point of discontinuity.

d) $y = \dfrac{x^2 - 7x + 6}{x^2 - 4x + 12}$, $x = -2$, $x = 6$

Answer:

$$y = \frac{x^2 - 7x + 6}{x^2 - 4x + 12} = \frac{(x-6)(x-1)}{(x-6)(x+2)}$$

As $x - 6$ is a factor of both the numerator and the denominator, $x = 6$ is a point of discontinuity.

As $x + 2$ is a factor of only the denominator, $x = -2$ is the equation of a vertical asymptote.

2. Graph the following functions. Pay attention to whether each graph should have a point of discontinuity or a vertical asymptote (or both).

   a) $f(x) = \dfrac{x+4}{x^2+3x-4}$

   Answer:

   $$f(x) = \frac{x+4}{x^2+3x-4} = \frac{x+4}{(x+4)(x-1)} = \frac{1}{x-1},\ x \neq -4$$

   y-value of the hole: substitute $x = -4$ into $y = \dfrac{1}{x-1} =$

   $$y = \frac{1}{-4-1} = -\frac{1}{5}$$

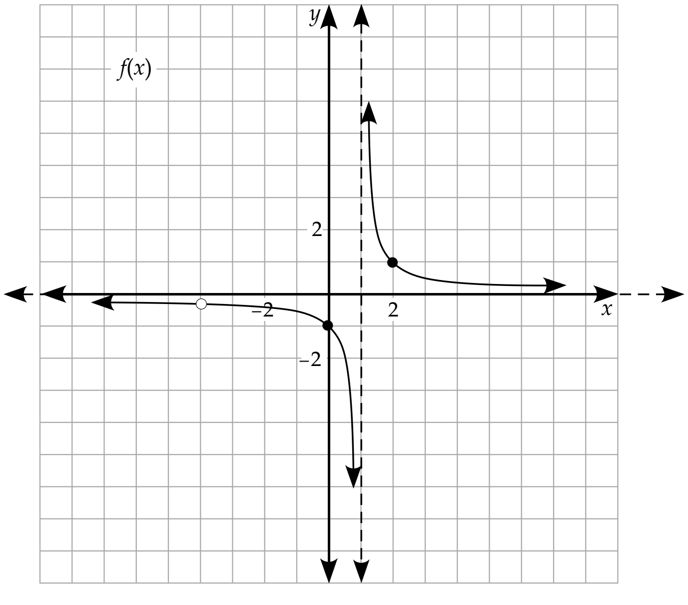
   As $x + 4$ is a common factor to the numerator and the denominator, there will be a *point of discontinuity* at $\left(-4, -\dfrac{1}{5}\right)$.

   y-intercept: $f(0) = \dfrac{1}{0-1} = -1$

   x-intercept: None

   Vertical asymptote: $x - 1 = 0$, $x = 1$

   Horizontal asymptote: $y = 0$



b) $g(x) = \dfrac{x^2 + 3x - 40}{x + 8}$

Answer:

$$g(x) = \frac{x^2 + 3x - 40}{x + 8} = \frac{(x + 8)(x - 5)}{x + 8} = x - 5,\ x \neq -8$$

y-value of the hole: substitute $x = -8$ into $y = x - 5 =$

$$y = -8 - 5 = -13$$

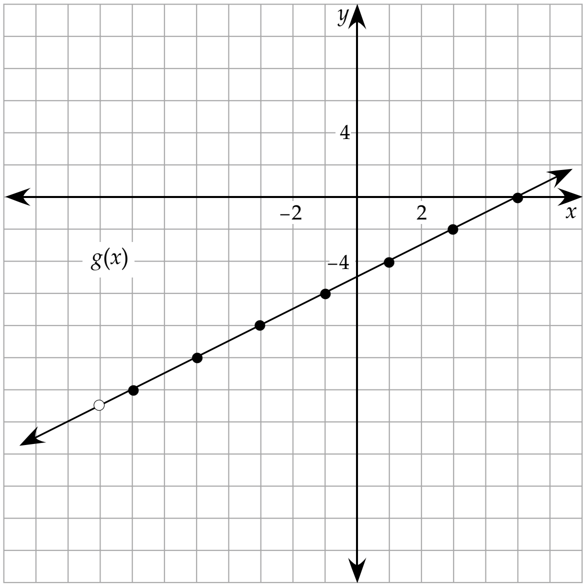
As $x + 8$ is a common factor to the numerator and the denominator, there will be a *point of discontinuity* at $(-8, -13)$.

y-intercept: $g(0) = 0 - 5 = -5$

x-intercept: $x - 5 = 0$, $x = 5$

Vertical asymptote: none

Horizontal asymptote: none



c) $h(x) = \dfrac{x^2 + 3x + 2}{x^2 + 7x + 6}$

Answer:

$$h(x) = \frac{x^2 + 3x + 2}{x^2 + 7x + 6} = \frac{(x+2)(x+1)}{(x+6)(x+1)} = \frac{x+2}{x+6},\ x \neq -1$$

y-value of the hole: substitute $x = -1$ into $y = \dfrac{x+2}{x+6} =$

$$y = \frac{-1+2}{-1+6} = \frac{1}{5}$$

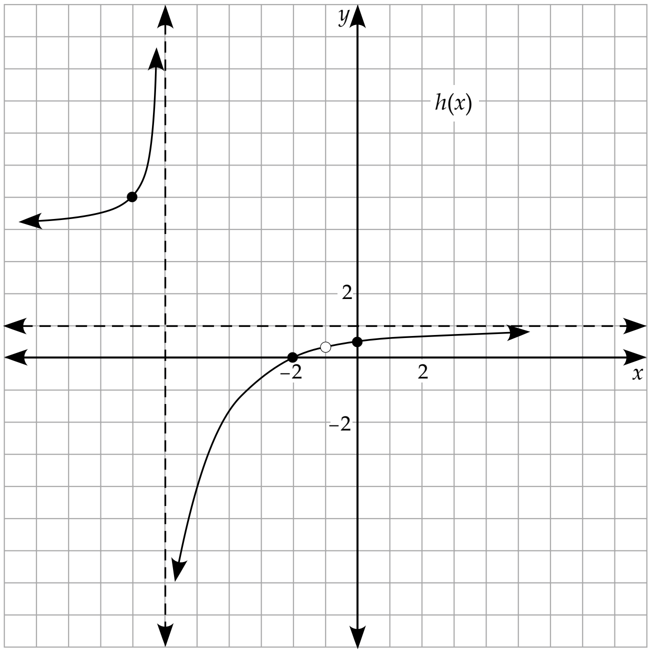
As $x + 1$ is a common factor to the numerator and the denominator, there will be a *point of discontinuity* at $\left(-1, \dfrac{1}{5}\right)$.

y-intercept: $h(0) = \dfrac{0+2}{0+6} = \dfrac{2}{6} = \dfrac{1}{3}$

x-intercept: $x + 2 = 0$, $x = -2$

Vertical asymptote: $x + 6 = 0$, $x = -6$

Horizontal asymptote: $y = 1$



3. Compare the functions $y = \dfrac{1}{x+1}$, $y = \dfrac{x+2}{x^2 + 3x + 2}$, and $y = \dfrac{x-3}{x^2 - 2x - 3}$ using the following chart. What do the functions have in common? Explain.

Answer:

	$y = \dfrac{1}{x+1}$	$y = \dfrac{x+2}{x^2+3x+2} = \dfrac{x+2}{(x+2)(x+1)}$	$y = \dfrac{x-3}{x^2-2x-3} = \dfrac{x-3}{(x-3)(x+1)}$
x-intercept	none	none	none
y-intercept	1	1	1
vertical asymptote	$x = -1$	$x = -1$	$x = -1$
horizontal asymptote	$y = 0$	$y = 0$	$y = 0$
points of discontinuity	none	$x = -2$	$x = 3$
behaviour at positive infinity	approaches zero through positive y-values	approaches zero through positive y-values	approaches zero through positive y-values
behaviour at negative infinity	approaches zero through negative y-values	approaches zero through negative y-values	approaches zero through negative y-values

These functions are identical except for their points of discontinuity. This is because each of the functions is a multiple of $y = \dfrac{1}{x+1}$. When the original function is multiplied by a binomial over itself, a point of discontinuity exists wherever the binomial equals zero. This is why the second and third functions have points of discontinuity at $x = -2$ and $x = 3$ respectively.

4. Match each function to its corresponding graph, and explain your reasoning.

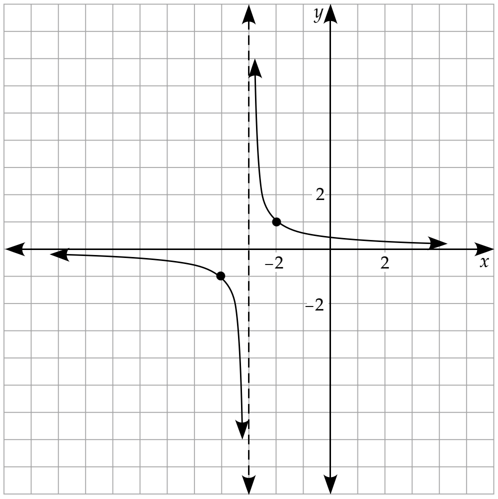
   a) $y = \dfrac{1}{x+3}$

   b) $y = \dfrac{x-5}{x^2 - 2x - 15}$

   c) $y = \dfrac{x+3}{x^2 - 2x - 15}$

   d) $y = \dfrac{1}{x^2 - 2x - 15}$

e)



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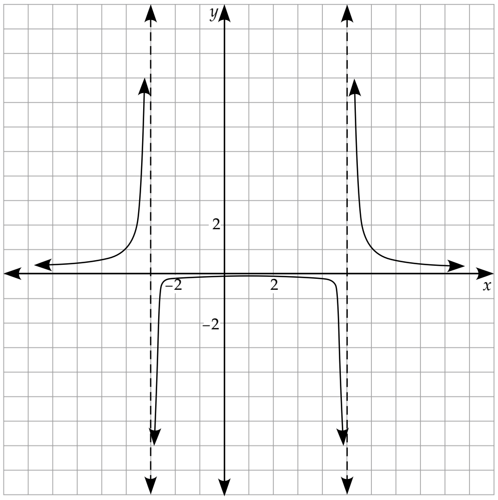
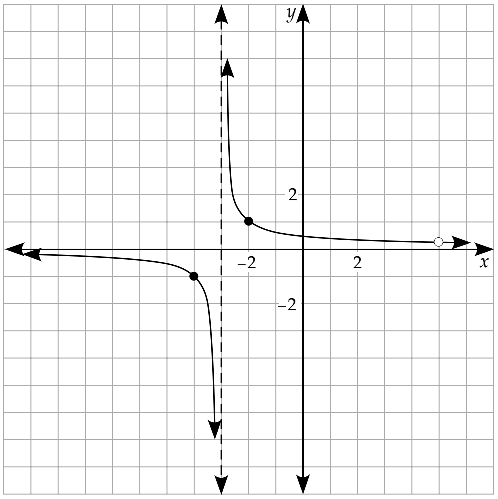
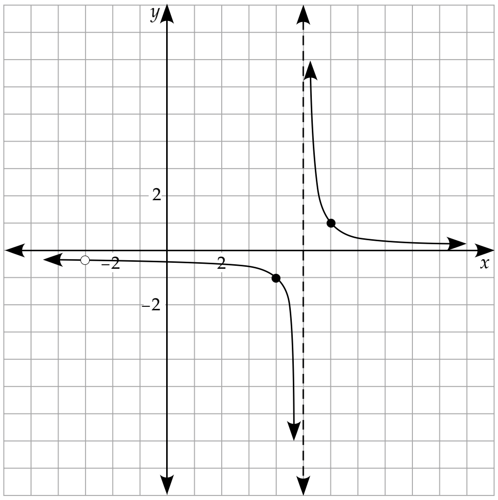


g)



h)



Answer:

(a) → (e) This function has a vertical asymptote at $x = -3$ and no other points of interest. This is because $(x + 3)$ is a factor of only the denominator of this function.

(b) → (g) This function has a vertical asymptote at $x = -3$ and a point of discontinuity at $x = 5$. This is because $(x + 3)$ is a factor of only the denominator of this function while $(x - 5)$ is a factor of both the numerator and the denominator of this function.

(c) → (h) This function has a vertical asymptote at $x = 5$ and a point of discontinuity at $x = -3$. This is because $(x - 5)$ is a factor of only the denominator of this function while $(x + 3)$ is a factor of both the numerator and the denominator of this function.

(d) → (f) This function has two vertical asymptotes at $x = 5$ and $x = -3$ because $(x - 5)$ and $(x + 3)$ are both factors of the denominator of this function but not the numerator.

5. Write a possible equation for each rational function described below.

   a) This rational function has a discontinuity at $x = 2$ and a vertical asymptote at $x = 6$.

   Answer:

   When a function has a discontinuity at a certain value d, $x - d$ must be a factor of both the numerator and the denominator.

   Therefore, $x - 2$ needs to be a factor of the numerator and the denominator of this function.

   When a function has a vertical asymptote at a certain value v, $x - v$ must be a factor of the denominator.

   Therefore, $x - 6$ needs to be a factor of the denominator of this function.

   This function could be:

   $$y = \frac{x-2}{(x-2)(x-6)} \text{ or any multiple of this function.}$$

   b) This rational function has a vertical asymptote at $x = -4$ and a horizontal asymptote at $y = 1$.

   Answer:

   When a function has a vertical asymptote at a certain value v, $x - v$ must be a factor of the denominator.

   Therefore, $x + 4$ needs to be a factor of the denominator of this function.

   When a function has a horizontal asymptote at $y = 1$, both the numerator and the denominator need to be functions of the same degree and have the same leading coefficients.

   Thus, three possible functions are:

   $$y = \frac{x+7}{x+4} \text{ or } y = \frac{(x+2)(x-1)}{(x+4)(x-1)} \text{ or } y = \frac{(x+5)(x+2)}{(x+4)(x-3)}$$

   c) This rational function has a discontinuity at $x = -1$, vertical asymptote at $x = 3$, and a horizontal asymptote at $y = 2$.

   Answer:

   This function needs to contain the factor $x + 1$ in the numerator and the denominator, as $x = -1$ is a point of discontinuity.

   Also, this function needs to contain the factor $x - 3$ in the denominator only, as $x = 3$ is the equation of the vertical asymptote.

In order to achieve a horizontal asymptote at $y = 2$, the numerator and the denominator need to be functions of the same degree. However, the ratio of the leading coefficient of the numerator to the leading coefficient of the denominator needs to be $\frac{2}{1}$.

Possible functions are:

$$y = \frac{(x+1)(2x+1)}{(x+1)(x-3)} = \frac{2x^2 + 2x + 1}{x^2 - 2x - 3} \text{ or}$$

$$y = \frac{(x+1)(2x-3)}{(x+1)(x-3)} = \frac{2x^2 - x - 3}{x^2 - 2x - 3}.$$

Learning Activity 8.6

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Factor: $6x^2 - 17x - 14$
2. Does the function $f(x) = \dfrac{(x-3)(x+7)}{(x-7)(x-3)}$ have a hole or an asymptote at $x = 7$?
3. Simplify: $\sqrt{192x^3y^6z}$
4. Simplify: $\dfrac{9a^4b^8c^2}{27a^3b^2c^8}$
5. Simplify: $\dfrac{\frac{7}{6}}{\frac{14}{8}}$
6. Determine the x-intercept of the function $y = \frac{2}{3}x + 7$.
7. Determine the reciprocal of $\dfrac{6x}{7-2x}$.
8. Solve for x: $4x - 5 = 18$

Answers:

1. $(3x + 2)(2x - 7)$
2. Vertical asymptote
3. $8xy^3\sqrt{3xz}$ $\left(\sqrt{(64x^2y^6)\cdot(3xz)}\right)$
4. $\dfrac{ab^6}{3c^6}$
5. $\dfrac{2}{3}$ $\left(\dfrac{7}{6}\cdot\dfrac{8}{14} = \dfrac{7}{3\cdot 2}\cdot\dfrac{2\cdot 2\cdot 2}{2\cdot 7}\right)$
6. $-\dfrac{21}{2}$ $\left(0 = \dfrac{2}{3}x + 7,\ -7 = \dfrac{2}{3}x - 21 = 2x,\ -\dfrac{21}{2} = x\right)$

7. $\dfrac{7-2x}{6x}$

8. $x = \dfrac{23}{4}\ \left(4x = 23 \rightarrow x = \dfrac{23}{4}\right)$

Part B: Solving Rational Equations

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Find the x-intercepts of the rational functions algebraically.

   a) $f(x) = \dfrac{1}{x+3} - 2$

   Answer:

   To find the x-intercept, let $f(x) = 0$.

$$0 = \frac{1}{x+3} - 2$$

$$2 = \frac{1}{x+3}$$

$$\frac{2(x+3)}{x+3} = \frac{1}{x+3}$$

$$\therefore\ 2x + 6 = 1$$

$$2x = -5$$

$$x = -\frac{5}{2} = -2.5$$

   b) $g(x) = \dfrac{x-3}{x-1}$

   Answer:

   To find the x-intercept, let $g(x) = 0$.

$$0 = \frac{x-3}{x-1}$$

$$0 = x - 3$$

$$x = 3$$

2. Find the approximate roots of the following rational equations by graphing.

   a) $\dfrac{1}{x+3} = 2$

   Answer:

   Sketch $y = \dfrac{1}{x+3}$.

   Vertical asymptote at $x = -3$.

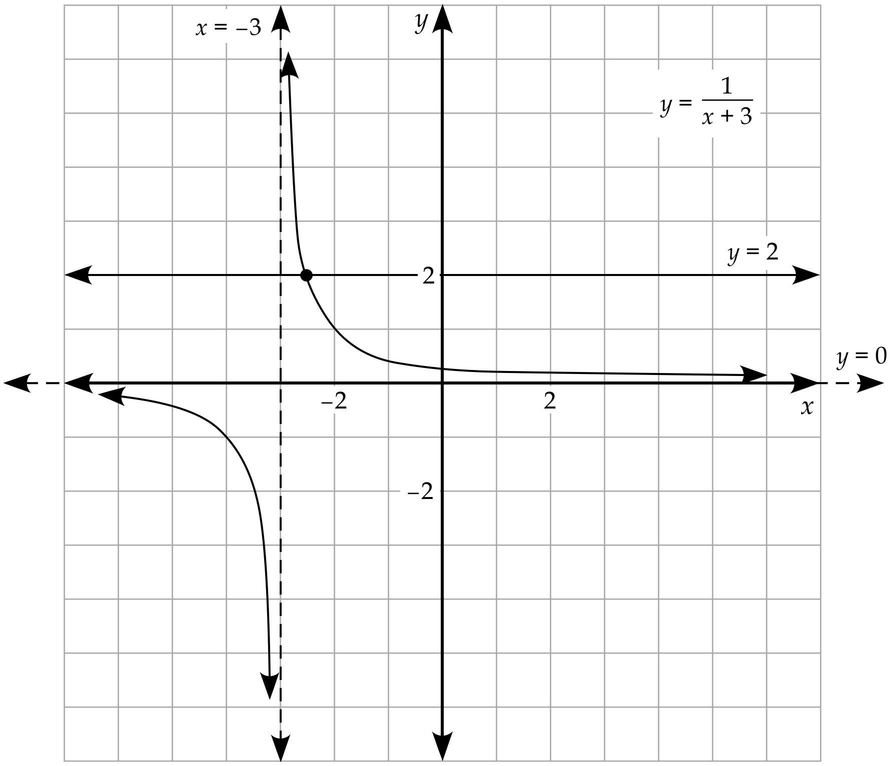
   Horizontal asymptote at $y = 0$.

   y-intercept at $\dfrac{1}{3}$.

   Points at $(-2, 1)$ and $(-4, -1)$.

   Also, sketch $y = 2$, a horizontal line.

   Find the x-value at the intersection point: $x = -2.5$ (approximately).



b) $\dfrac{x-3}{x-1} = 2$

Answer:

Sketch $y = \dfrac{x-3}{x-1}$.

Vertical asymptote at $x = 1$.

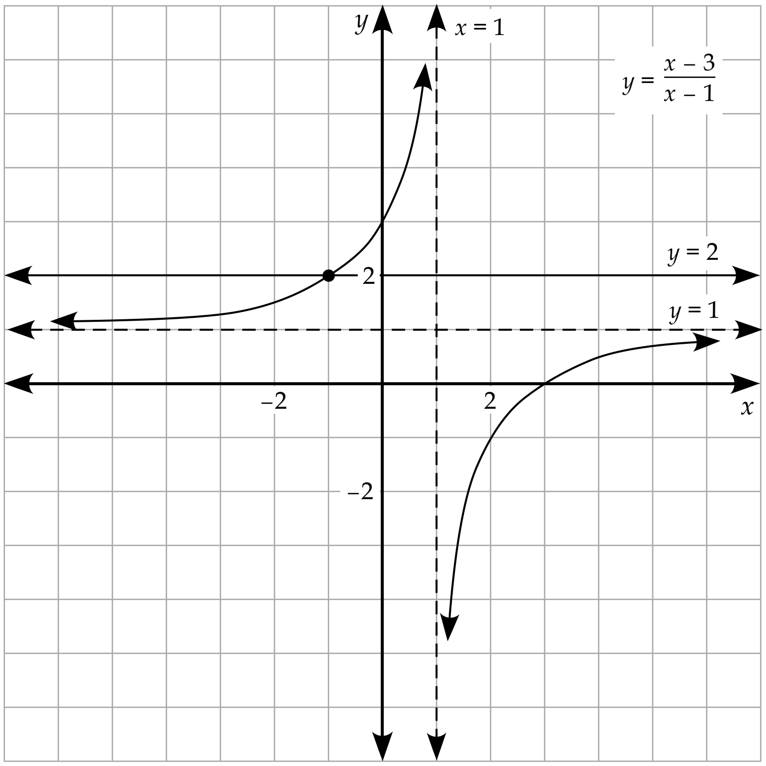
Horizontal asymptote at $y = 1$.

y-intercept at 3.

Points at $(-1, 2)$ and $(2, -1)$.

Also, sketch $y = 2$, a horizontal line.

Find the x-value at the intersection point: $x = -1$.

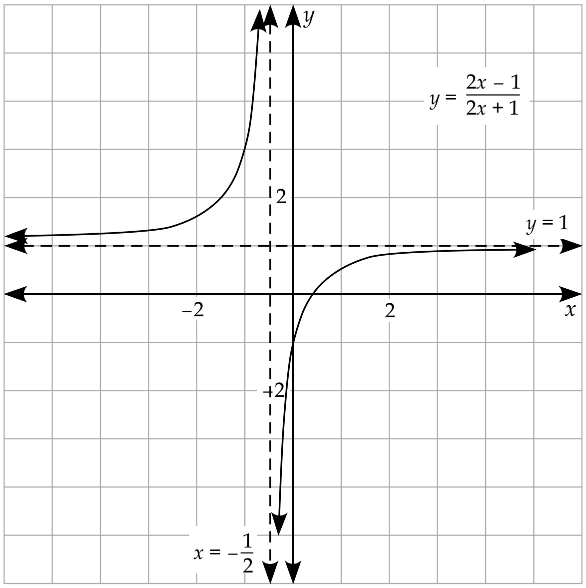


3. Approximate the roots to the equations using the related graph of $y = \frac{2x - 1}{2x + 1}$, as shown. Explain how you found your answer.

a) Solve: $\frac{2x - 1}{2x + 1} = 0$

b) Solve: $\frac{2x - 1}{2x + 1} = 2$

c) Solve: $\frac{2x - 1}{2x + 1} = 4$



Answers:

a) The root of the equation will be where the function (*y*-value) equals zero, which is the *x*-intercept. The *x*-intercept of the function is at 0.5 (approximately), so the root of the equation $\frac{2x - 1}{2x + 1} = 0$ is $x = 0.5$ (approximately).

b) The root of the equation will be where the function equals 2. Draw the horizontal line, $y = 2$, and find the *x*-value of the intersection point. The *x*-value and root of the equation $\frac{2x - 1}{2x + 1} = 2$ is approximately $x = -1.5$.

c) The root of the equation will be where the function equals 4. Draw the horizontal line, $y = 4$, and find the *x*-value of the intersection point. The *x*-value and root of the equation $\frac{2x - 1}{2x + 1} = 4$ is approximately $x = -0.75$.

Notes

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Glossary

GLOSSARY

absolute maximum

This is the largest y-value that a function $f(x)$ achieves over its entire domain.

absolute minimum

This is the smallest y-value that a function $f(x)$ achieves over its entire domain.

absolute value

The absolute value of an expression is denoted by vertical bars surrounding an expression and is a positive number. The absolute value of an expression is defined as

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}.$$

amplitude

This is the distance from the midpoint to the maxima or minima for the graph of a trigonometric function.

arc length

The length or distance measured along the circumference of a circle.

asymptote

An asymptote is a line that a function approaches, but never touches or crosses.

axis of symmetry

A line or axis on a graph that splits the entire graph in two; making a mirror image on each side of the line or axis.

binomial

This is a polynomial with two terms.

binomial theorem

This is a theorem that describes the terms in the expansion of any binomial $(x + y)^n$, where x and y can have any numeric or variable value as long as n is a natural number. The binomial theorem states:

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

coefficient

This is the numeric component of an algebraic term. In the term $3x^2$, 3 is the coefficient.

combination

This is an unordered arrangement of n objects being placed into r available positions and written as ${}_nC_r$.

common logarithm

$\log x$ is called a common logarithm and means $\log_{10} x$ (read "the log to base 10 of x).

composition

The composition of the functions f and g is $(f \circ g)(x) = f(g(x))$. This is when one function is represented as a function of another function.

conjugate

The conjugate of $a + b$ is $a - b$, and the conjugate of $a - b$ is $a + b$.

Example

The conjugate of $2 + \sqrt{3}$ is $2 - \sqrt{3}$.

constant

A constant is a term in an expression that has no variables being multiplied to it. It is either a number or a symbol that represents a number such as π.

constant function

A function, f, is constant in an interval if, for any x_1 and x_2 in the interval, $f(x_1) = f(x_2)$.

continuous

A function is continuous if its graph has no breaks. In other words, you can sketch the graph of a continuous function without lifting your pencil from the paper.

coterminal angle

This is an angle that terminates at the same point around the circle as a given angle; two coterminal angles share the same terminal arm of the angle.

decreasing function

A function, f, is decreasing in an interval if, for any x_1 and x_2 in the interval, $x_1 < x_2$, implies that $f(x_1) < f(x_2)$.

degree of a polynomial

This is the largest exponent of any term in a polynomial expression or equation.

domain

The domain is the set of all x-values for which a function is defined.

double root

If a polynomial $f(x)$ has a squared factor such as $(x - c)^2$, then $x = c$ is a double root of $f(x) = 0$. The graph of $y = f(x)$ is tangent to the x-axis at $x = c$. A double root is a root with multiplicity of 2.

end behaviour of a polynomial

This describes whether a polynomial is increasing or decreasing at its ends.

equation

This is any mathematical sentence with an equals (=) sign.

equidistant

Two points are said to be equidistant if they are the same distance away from a given location.

even function

A function $f(x)$ is even if $f(x) = f(-x)$.

exact value

This is a term used to evaluate an expression (or equation) that is correct to an infinite number of decimal places.

Example

π is an exact value, whereas 3.14 is a rounded estimate of π and would not be considered an exact value.

exponent

This is the number of times a number is multiplied together in a power. It is usually written as a superscript after the number.

Example

3 is the exponent in 4^3.

exponential form

This is an equation in the form $x = a^y$.

exponential function

This is a function of the form $y = a^x$, where $a > 0$, $a \neq 1$ and $\mathrm{x} \in \Re$. This is a function where the independent variable appears in the exponent.

expression

This is any mathematical sentence without an equals (=) sign.

Example

- $3x + 2 = 1$ is an equation
- $3x + 2$ is an expression

extraneous root

An extraneous root is defined as a number obtained when solving an equation that does not satisfy the initial restrictions on the variable.

factor theorem

If there exists a polynomial function $f(x)$ and a number a such that $f(a) = 0$, then $(x - a)$ is a factor of that polynomial and $x = a$ is a root of that polynomial.

factorial

The factorial of a positive integer n is the product of all of the positive integers less than or equal to n. In other words, for a natural number n, $n! = n(n - 1)(n - 2) \ldots 3 \cdot 2 \cdot 1$. $n!$ is read as "n factorial."

function

A function is a relation such that for each x-value, there is exactly one value of y.

fundamental counting principle

This principle represents the total number of choices needed to select independent terms. If a decision can be made in m different ways and a second decision can be made in n different ways, then the two decisions can be made in this order in mn different ways.

general solution

This is the solution to a trigonometric equation that includes all possible solutions over an infinite domain.

greatest common factor (GCF)

The GCF is the greatest number or term that is a factor of two or more numbers or terms.

Example

The greatest common factor of 12 and 18 is 6; the greatest common factor of $2xy$ and $2yz$ is $2y$.

half-life

This is the length of time required for a substance to decay to half of its original amount.

horizontal asymptote

The line $y = b$ is a horizontal asymptote of the graph of f if $f(x) \rightarrow b$ as $x \rightarrow +\infty$ or $x \rightarrow -\infty$. This line is an imaginary broken line that indicates an undefined y-value on the graph of a function.

horizontal line test

If a horizontal line intersects the graph of a function at not more than one point in any given location, then the function is said to be a one-to-one function.

improper fraction

An improper fraction is a fraction that is larger than one. In other words, the numerator is greater than the denominator.

increasing function

A function, f, is increasing in an interval if for any x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

invariant points

These are points that stay the same on the graph of $f(x)$ and the transformed graph of $f(x)$.

inverse function

A reflection of the function $y = f(x)$ over the line $y = x$, or an interchanging of the x- and y-coordinates of a function. The inverse of the function $y = f(x)$ is denoted as $y = f^{-1}(x)$.

leading coefficient

This is the coefficient of the term in a polynomial containing the highest power of x, or the coefficient of the leading term of the polynomial, when the polynomial is written in decreasing degree order.

like terms

These are terms that have the same combination of variables and exponents, but may have different coefficients.

Example

- $3ab^3$ and $2ab^3$ are like terms because the variables a and b have the same exponents of 1 and 3 in each expression.
- $3ab^3$ and $2a^3b$ are not like terms because one term has "a" and one term has "a^3."

linear function

A linear function is a function whose graph is a straight line. Its form is $f(x) = mx + b$.

logarithm

A logarithm of a number N to a base b is the exponent y to which the base must be raised to yield the number N.

logarithmic form

This is an equation in the form $y = \log_a x$.

logarithmic scales

This is a measurement that uses the logarithm of the quantity being studied instead of the quantity itself.

monomial

A monomial is a polynomial with one term.

multiplicity

The multiplicity of a root is related to the degree of the factor when a polynomial function is written in factored form.

Example

$f(x) = (x + 2)^3(x - 1)^2$

The multiplicity of the root –2 is 3 and the multiplicity of the root 1 is 2.

natural logarithm

$\ln x$ is called a natural logarithm and means $\log_e x$.

non-permissible values

The numbers excluded from the domain of a rational expression are often called the non-permissible values of the expression. These are values that make a rational function undefined because they make the denominator of the function equal to 0.

odd function

A function $f(x)$ is odd if $-f(x) = f(-x)$.

one-to-one function

A one-to-one function is one that passes both the vertical line test and the horizontal line test. A function is one-to-one if for every input value, there is one and only one output value. Also, only one input value can correspond to a given output value.

parabola

The graph of a quadratic function takes the shape of a parabola, or a U shape.

Pascal's triangle

The numerical coefficients in a binomial expansion of $(x + y)^n$ can be written in the form of a triangular array called Pascal's triangle.

perfect square trinomial

A quadratic expression is called a perfect square trinomial when it is the square of a binomial.

Example

$x^2 + 4x + 4$ is the square of $(x + 2)$ and is a perfect square trinomial.

period

A period is one complete cycle or revolution of a trigonometric function.

periodic function

A function $f(x)$ is a periodic function if there exists a number $p > 0$, such that for all x in the domain of f, $f(x + p) = f(x)$.

permutation

This is an ordered arrangement of n objects being placed into r available positions and written as ${}_nP_r$.

phase shift

This is the translation, either left or right, of a trigonometric function.

point of discontinuity

This is a point at which the graph of a function does not exist. These points generally occur when the numerator and the denominator of a rational function have a common factor.

point of tangency

This is the point at which a tangent line touches a circle.

polynomial

A polynomial is a mathematical expression with one or more than one term; an addition or subtraction of monomials.

polynomial function

A polynomial function is a function equation that can be written in the form $f(x) = a_nx^n + a_{(n-1)}x^{(n-1)} + \ldots + a_2x^2 + a_1x + a_0$, where n is a non-negative integer and the coefficients, a_n, $a_{(n-1)}$, . . ., a_2, a_1, and a_0, are real numbers.

quadrant

On a Cartesian plane, the x-axis and the y-axis divide the plane into four quadrants.

quadratic formula

The quadratic formula is used to solve for the zeros of a quadratic equation when it is not easily factorable. The quadratic formula is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ for the equation $ax^2 + bx + c = 0$.

quadratic function

A quadratic function is a function of degree 2 that can be written in the general form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$.

radian

One radian is defined as the measure of the central angle, θ, that intercepts an arc, s, equal in length to the radius, r, of the circle.

radical

A radical is another name for a *root* (such as a square or cube root).

radical equation

This is an equation containing at least one radical expression with a variable as part of the radicand.

radical functions

These functions are functions that contain a variable inside a radical, as part of a radicand.

radical symbol

$\sqrt{}$

radicand

The radicand is everything included under the radical symbol in a radical.

range

The range is the set of all y-values for which a function is defined.

rational equation

An equation involving rational expressions is a rational equation.

rational expression

$\frac{p}{q}$ is a rational expression if p and q are polynomials and $q \neq 0$.

rational function

A function of the form $f(x) = \frac{p(x)}{q(x)}$, where $q(x) \neq 0$ and where $p(x)$ and $q(x)$ are polynomials in x.

rational number

$\frac{a}{b}$ is a rational number if a and b are integers and $b \neq 0$.

rationalized denominator

A rationalized denominator does not contain an irrational number or expression in the denominator.

Example

- $\frac{1}{\sqrt{2}}$ does not have a rationalized denominator because $\sqrt{2}$ is an irrational number.
- $\frac{\sqrt{2}}{2}$ has a rationalized denominator because 2 is a rational number.

reciprocal

The reciprocal of any value is 1 over that value. In other words, a reciprocal is obtained by interchanging the numerator and the denominator. Therefore, the reciprocal of x, is $\frac{1}{x}$ and the reciprocal of a polynomial $f(x)$ is $\frac{1}{f(x)}$.

reference angle

A reference angle is the acute angle formed between the terminal arm of θ and the nearer portion of the x-axis. It is often denoted by θ_r.

reflection

This is the mirror image of an object over some line, point, or plane. A reflection through the x-axis of $y = f(x)$ would be $y = -f(x)$. A reflection through the y-axis of $y = f(x)$ would be $y = f(-x)$.

relation

A relation is a set of ordered pairs (x, y).

relative maximum

This is the largest y-value of a function $f(x)$ in a given interval.

relative minimum

This is the smallest y-value of a function $f(x)$ in a given interval.

remainder theorem

If there exists a polynomial $f(x)$ and a number a such that $f(a) = k$, then k is the remainder when the polynomial is divided by $(x - a)$.

root

This is the x-value where the graph of a function crosses the x-axis. A root is also called a zero or an x-intercept of the function.

sinusoidal function

Any sine or cosine curve is called a sinusoidal function.

standard position

An angle measured starting on the positive x-axis and turning counter-clockwise is said to be in standard position.

synthetic division

This is a method of dividing polynomials when the divisor is a first degree polynomial by using only the coefficients of the terms in the polynomial.

tangent

This is a line that intersects the circle at only one point.

terminal arm

This is the arm of an angle drawn from the origin through a specific point or through some angular measure.

translation

This is a transformation of a geometric figure in which every point is moved the same distance in the same direction, either horizontally or vertically.

trigonometric identities

These are trigonometric equations that are only true for certain values of the variable. They must be proven to be true by making one side of the equation equal to the other through a series of substitutions and algebraic processes.

trinomial

This is a polynomial with three terms.

triple root

If a polynomial $P(x)$ has a cubed factor such as $(x - c)^3$, then $x = c$ is a triple root of $P(x) = 0$. The graph of $y = P(x)$ flattens out around $(c, 0)$ and crosses the x-axis at this point. A triple root is a root with multiplicity 3.

unit circle

This is a circle with centre at the origin and with a radius of one unit.

variable

This is a letter or symbol that represents an unknown value. In the term $2x^2y^3$, both x and y are variables.

vertex

The turning point of a parabola is called the vertex. The vertex of the graph of a quadratic function is defined as the point where the graph changes from increasing to decreasing, or changes from decreasing to increasing.

vertical asymptote

The line $x = a$ is a vertical asymptote of the graph of f if $f(x) \to +\infty$ or $f(x) \to -\infty$, as $x \to a$ either from the right or from the left. This is an imaginary broken line that indicates an undefined x-value on the graph of a function. The asymptote usually occurs when the denominator of a rational function is 0.

vertical line test

If a vertical line intersects the graph of a relation at not more than one point in any given location, then the relation is said to be a function because it passes the vertical line test.

***x*-intercept**

The x-intercept of a function is the x-coordinate(s) of the point(s) where the graph of the function intersects the x-axis. The y-coordinate of this point is zero.

***y*-intercept**

The y-intercept of a function is the y-coordinate of the point where the graph of the function intersects the y-axis. The x-coordinate of this point is zero.

zero

This is the point at which the graph of a function crosses the x-axis. A zero is also called a root or x-intercept of a function.

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Graph Paper

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Midterm Practice Exam

GRADE 12 PRE-CALCULUS MATHEMATICS

Midterm Practice Exam

Name: ____________________________________

Student Number: ____________________________

Attending ☐ Non-Attending ☐

Phone Number: ____________________________

Address: __________________________________

____

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For Marker's Use Only

Date: ___________________________________

Final Mark: ________ /100 = ________ %

Comments:

Instructions

The midterm examination will be weighted as follows:

Modules 1–4	100%

Time allowed: 3 hours

Note: You are allowed to bring the following to the exam: pens/pencils (2 or 3 of each), blank paper, a ruler, a scientific calculator, and your Midterm Exam Resource Sheet. Your Midterm Exam Resource Sheet must be handed in with the exam. You will receive your Midterm Exam Resource Sheet back from your tutor/marker with the next module work that is submitted for marking.

Show all calculations and formulas used. Use all decimal places in your calculations and round the final answers to the correct number of decimal places. Include units where appropriate. Clearly state your final answer.

General Marking Principles

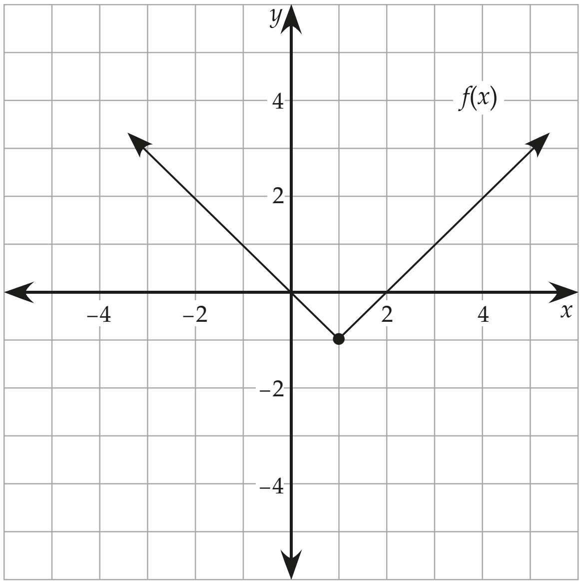
- Concepts learned in Grade 12 are worth 1 mark each. Concepts learned earlier (unless they are retaught under the curriculum, e.g., absolute value, reciprocals) are worth 0.5 mark each.
- Some errors are deducted only once (e.g., not putting arrowheads on graphs).
- Errors are followed through (e.g., if an arithmetic error is made in the first line, it is still possible for the student to receive nearly full marks).
- Many types of communication errors receive a 0.5 mark deduction, but 0.5 mark is the maximum communication error deduction for the entire exam.

Name: ____

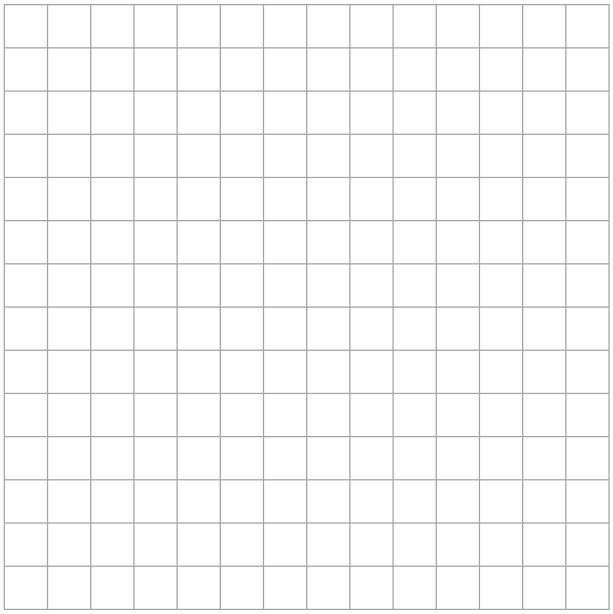
Answer all questions to the best of your ability. Show all your work.

Long-Answer Questions *(100 marks)*

1. Given the sketch of $f(x)$ drawn below, show each transformation algebraically and graphically. State the domain and range of each function.



a) $y = \frac{1}{2} f(x) - 2$ *(3 marks)*



b) $y = f\left(\frac{1}{2}(x + 4)\right)$ *(3 marks)*

Name: ______________________________________

2. The y-intercept of the function $g(x)$ is 4. What would the new y-intercept be for each of the following? *(3 × 1 mark each = 3 marks)*

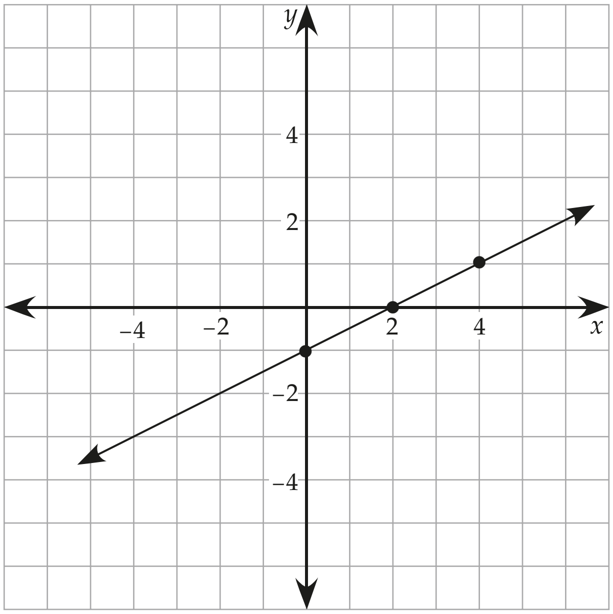
   a) $y = -2g(x)$

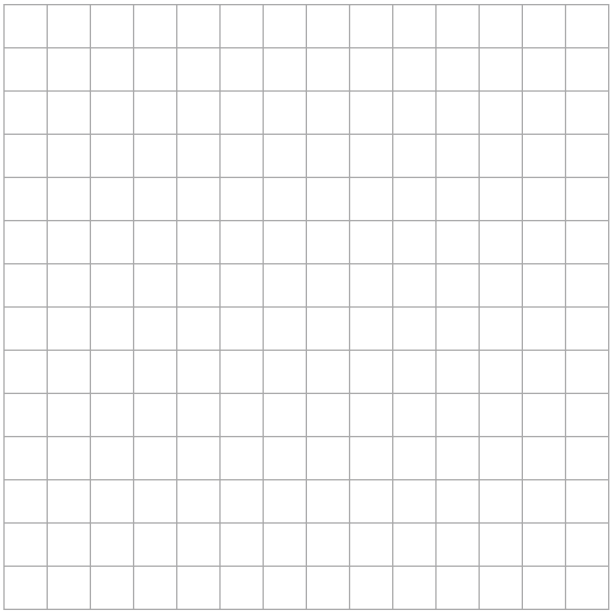
   b) $y = g(x) + 1$

   c) $y = g(3x)$

   d) $y = g(-x)$

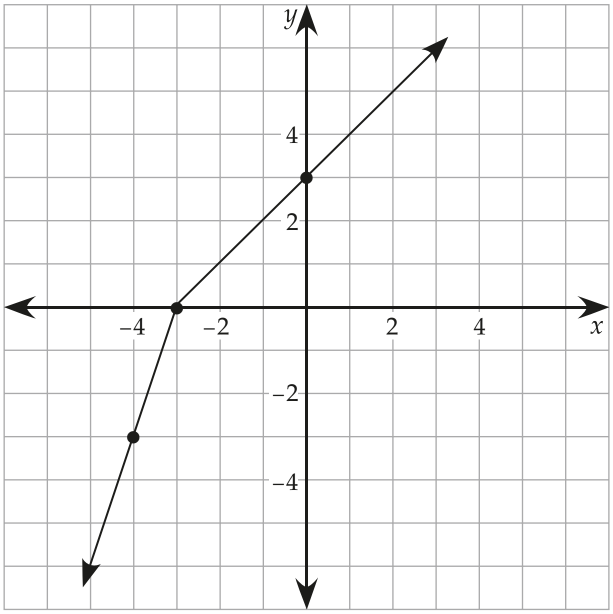
3. Use the graph of the function drawn below to sketch $y = \frac{1}{f(x)}$. *(2 marks)*



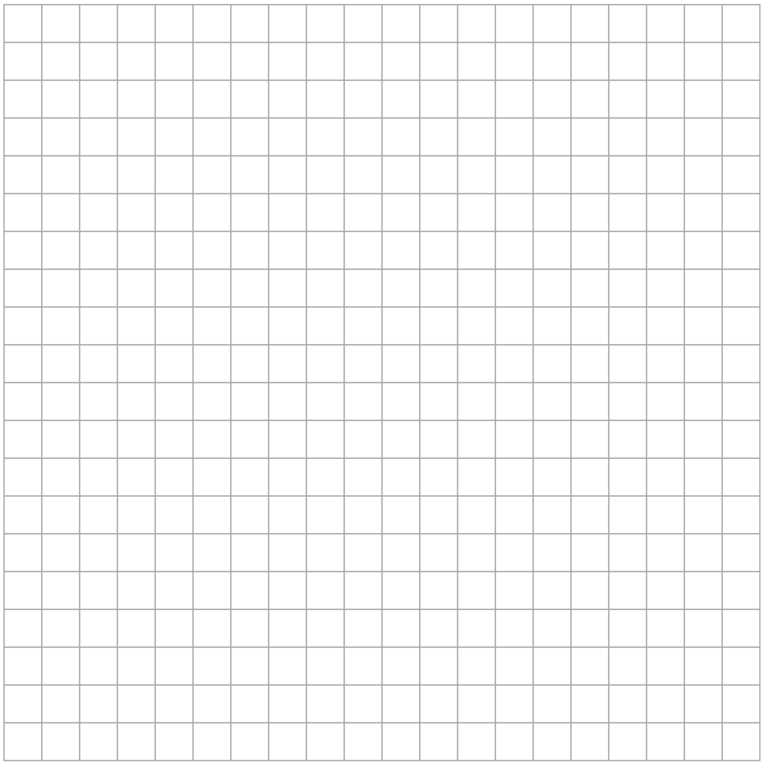


Name: ____

4. Using the sketch of $f(x)$, sketch the following.



a) $y = -3f(x)$ *(2 marks)*



b) $y = f(-2x)$ *(2 marks)*

c) $y = f^{-1}(x)$ *(2 marks)*

Name: ____________________

5. In how many ways can you order 13 songs in a playlist if
   a) there are no restrictions? *(1 mark)*

   b) your favourite song must be first? *(2 marks)*

6. How many distinct ways can 4 green cups, 2 blue cups, and 1 red cup be arranged on a shelf? *(2 marks)*

7. Four men and five women are on a parent council committee. In how many ways can a five-member subcommittee be formed if the women must have a majority on this subcommittee? *(4 marks)*

8. Evaluate each of the following using factorial notation. Show your work. *(4 × 2 marks each = 8 marks)*

   a) ${}_3P_2$

   b) ${}_5P_2$

   c) ${}_7C_3$

   d) ${}_6C_2$

Name: ____

9. Solve without using a calculator. *(3 marks)*

   ${}_{n+3}P_2 = 20$

10. Given the following row of Pascal’s Triangle, determine the next row. *(1 mark)*

1 11 55 165 330 462 462 330 165 55 11 1

11. Write and simplify the last term of the expansion of $\left(2+\frac{2}{x^2}\right)^4$. *(3 marks)*

Name: ____

12. Expand and simplify $(2x - 3)^4$ using the Binomial Theorem. *(4 marks)*

13. For the function $y = x^3 - 3x^2 - x + 3$, find the following:

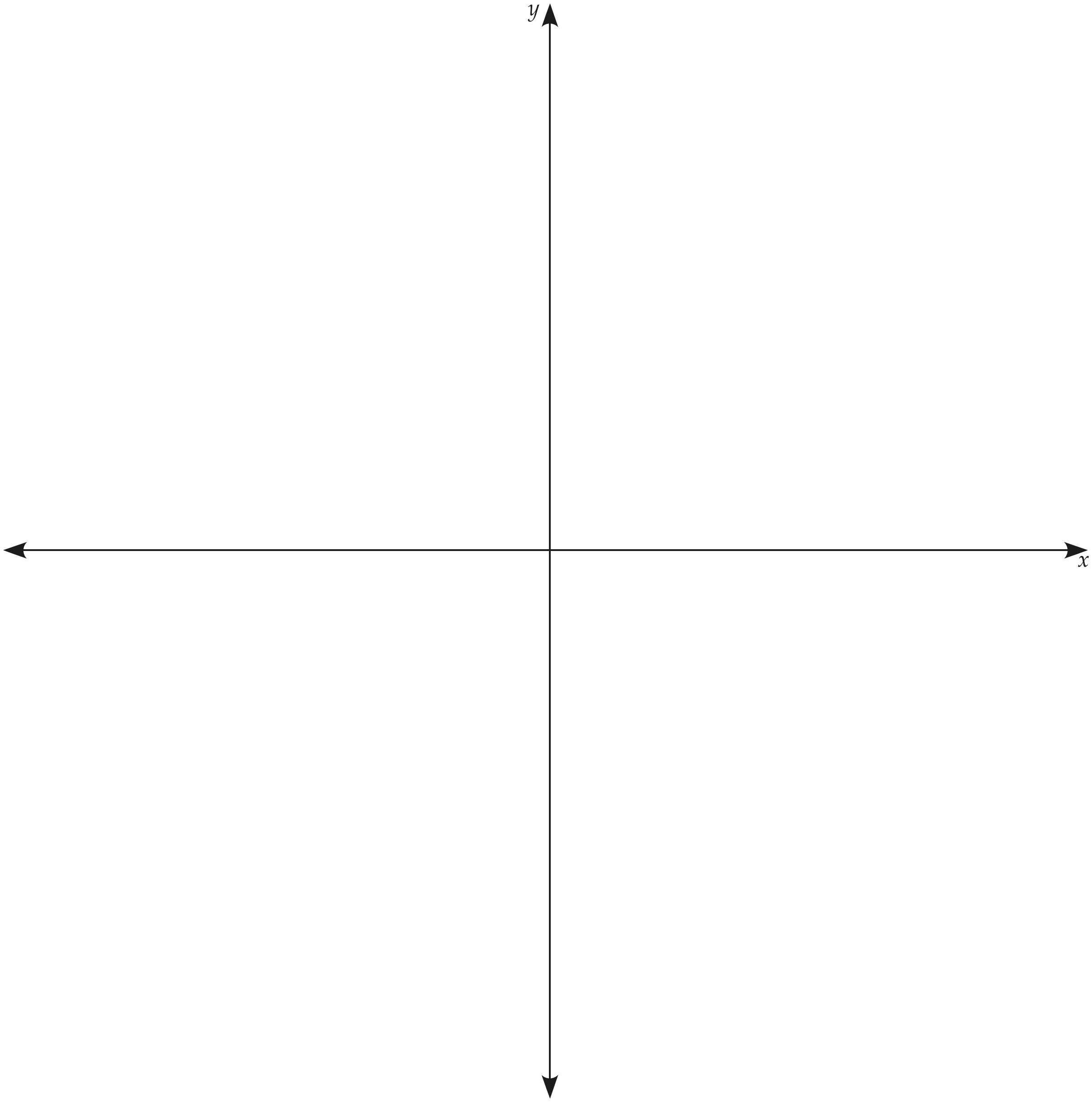
   a) the zeros of the function if you know that $(x + 1)$ is a factor of the polynomial *(4 marks)*

   b) left-right behaviour *(1 mark)*

   c) the y-intercept of the function *(1 mark)*

Name: ____________________

d) the sketch of the graph of the function *(2 marks)*

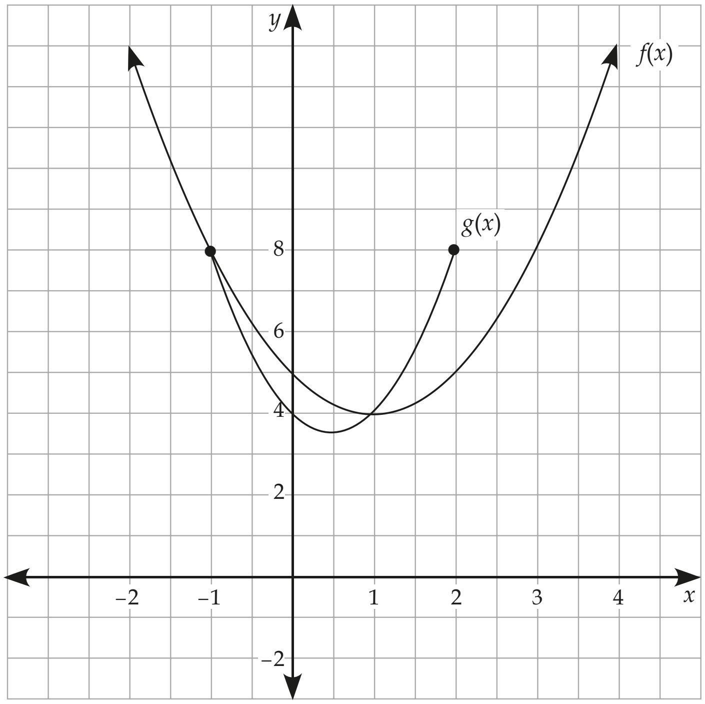


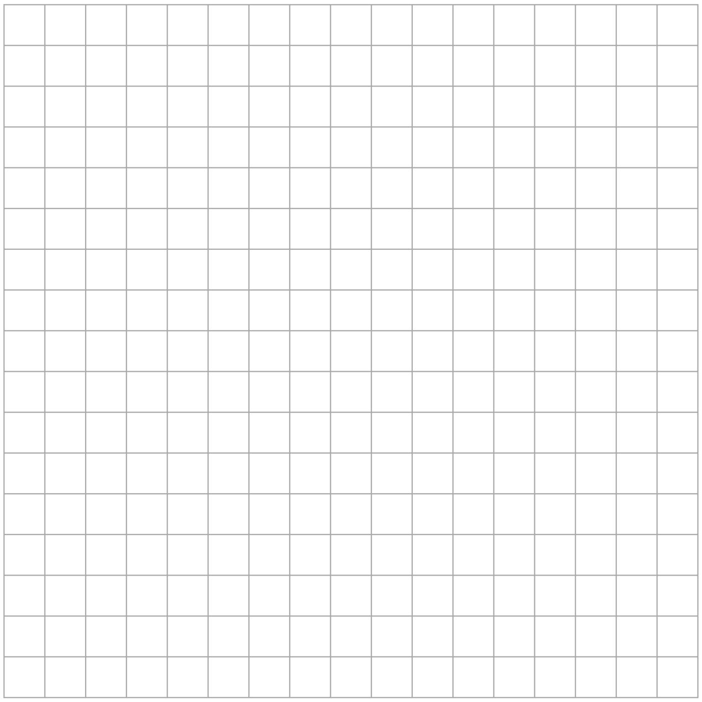
14. Graph a quartic that has roots of −1 and +2 and a root with a multiplicity 2 at +1. The function equation has a leading coefficient of −3. *(3 marks)*

Name: ____

15. If $f(x) = \dfrac{1}{x-3}$ and $g(x) = |x|$, determine the equation and graph of $h(x) = g(f(x))$.
State the domain and range. *(5 marks)*

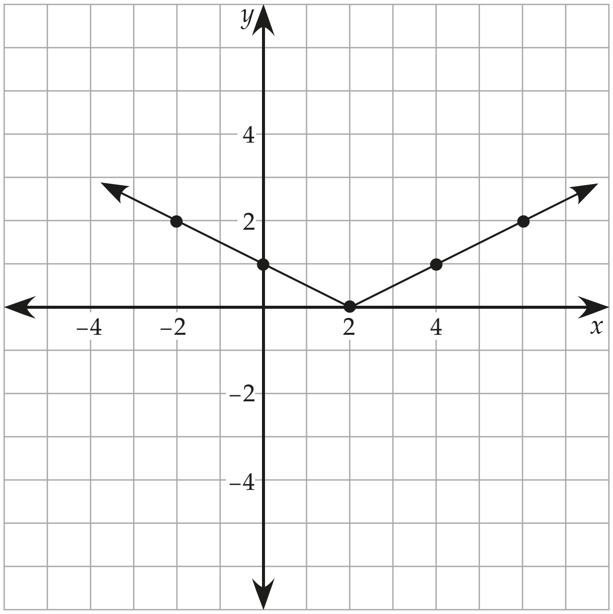
16. Consider the graphs of $f(x)$ and $g(x)$ below. Use these graphs to sketch $(g - f)(x)$. *(3 marks)*



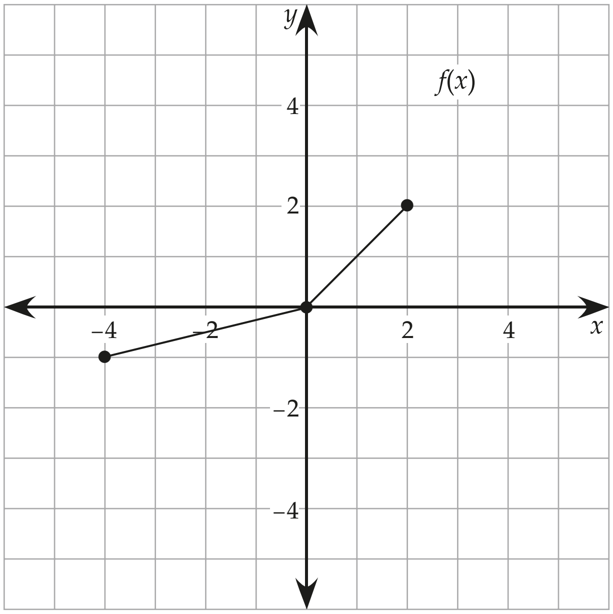


Name: ____

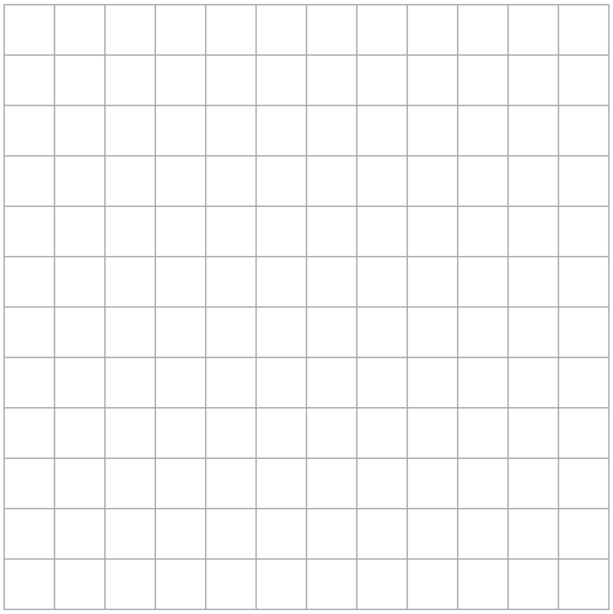
17. The following graph represents a transformation of $f(x) = |x|$. Write an expression for the new absolute value function. *(2 marks)*



18. The graph of a function $f(x)$ is drawn below.



a) Reflect the graph of $f(x)$ in the line $y = x$ to achieve the graph of $g(x)$. *(1 mark)*



b) Write the equation of the new function $g(x)$ in terms of $f(x)$. *(1 mark)*

c) Reflect the graph of $f(x)$ in the x-axis to achieve the graph of $h(x)$. *(1 mark)*

d) Write the equation of the new function $h(x)$ in terms of $f(x)$. *(1 mark)*

19. A function contains the ordered pairs $(-1, 0)$, $(0, 6)$, and $(-3, 7)$. What are the corresponding ordered pairs if this function is reflected through the following lines? *(3 × 1 mark each = 3 marks)*

a) $y = 0$ (x-axis)

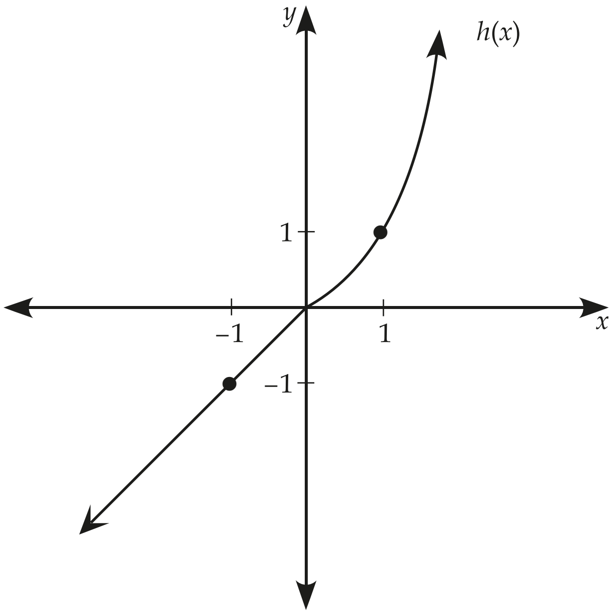
b) $x = 0$ (y-axis)

c) $y = x$

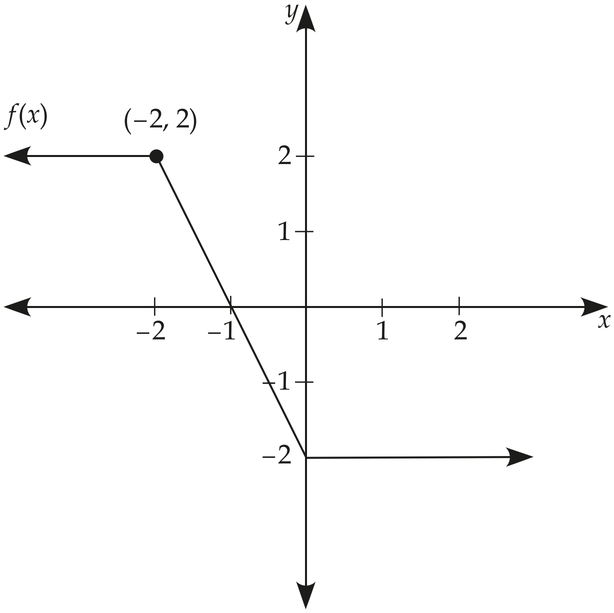
Name: ____

20. For each of the following relations, determine if they are one-to-one functions. Explain your reasoning. *(2 × 1 mark each = 2 marks)*

a)



b)



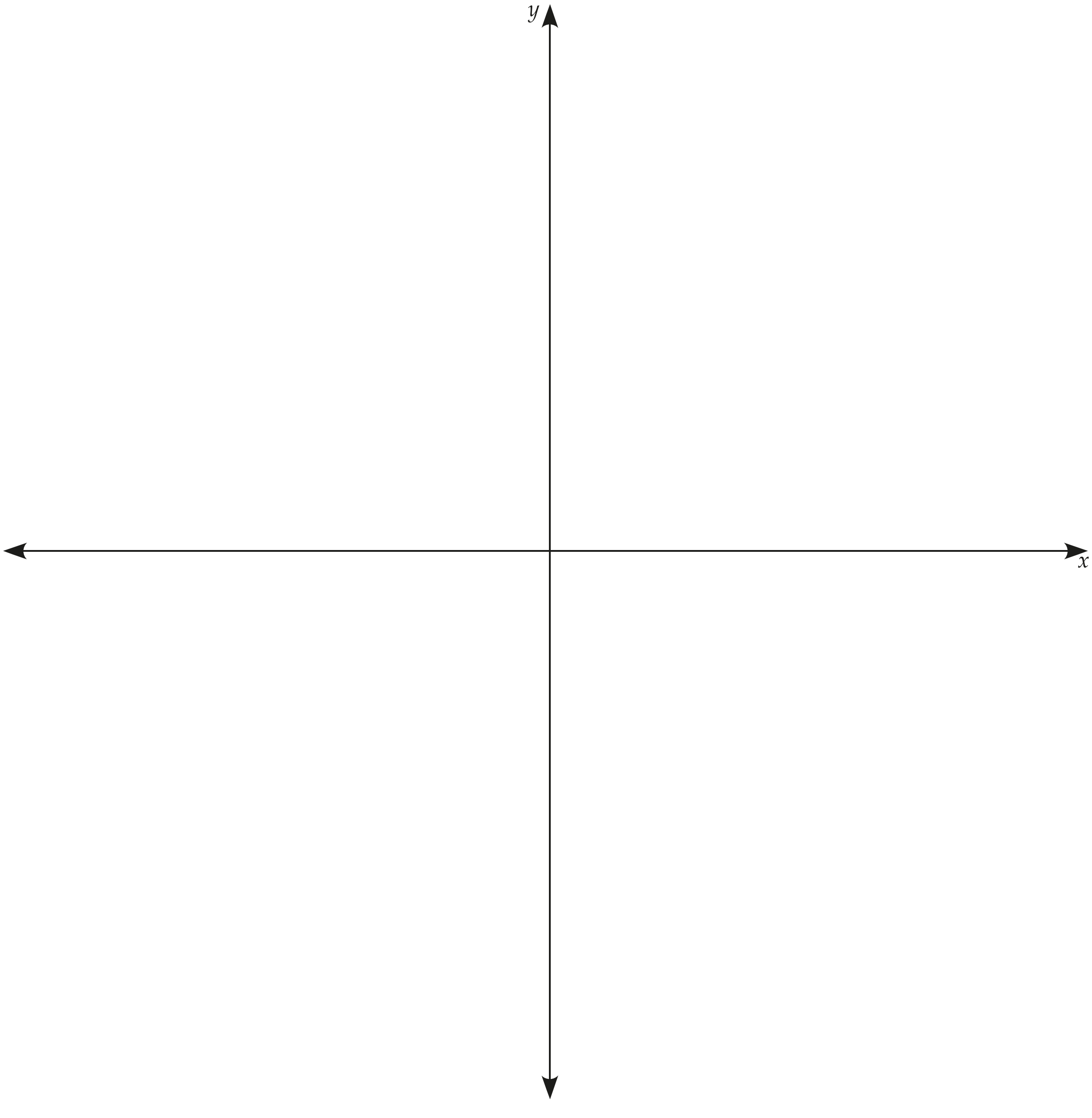
21. Show algebraically that the functions f and g are inverses of each other. *(2 marks)*

$$f(x)=\sqrt{4x-1} \qquad g(x)=\frac{x^2+1}{4}$$

Name: ____

22. Find $f^{-1}(x)$ algebraically. Graph $f^{-1}(x)$. Consider the domain and range of $f^{-1}(x)$. *(3 marks)*

    $f(x) = (x - 6)^2,\ x \leq 6$



23. Given that $f(x) = |2x - 1|$ and $g(x) = x^3 - 1$, find the following.
(3 × 1 mark each = 3 marks)

a) $f(f(x))$

b) $g(g(-1))$

c) $f(g(2))$

24. Given: $f(x) = x^3$ and $g(x) = x - 2$

a) Determine $f(g(x))$ and describe the graph of $f(g(x))$ in terms of a transformation of $f(x)$.
(2 marks)

b) Determine $g(f(\mathrm{x}))$ and describe the graph of $g(f(x))$ in terms of a transformation of $f(x)$.
(2 marks)

Name: ____

25. For each of the following polynomials, determine whether it is divisible by $x - 2$. Show your work.*(2 × 2 marks each = 4 marks)*

    a) $f(x) = -x^4 + x^3 - 8x^2 + 6$

    b) $g(x) = -x^3 + x^2 - 5x + 14$

26. Divide, using long division or synthetic division, and write in the form given by the division algorithm. *(3 marks)*

    $(2x^3 - 4x^2 - 12x - 14) \div (x - 4)$

27. Factor $g(x) = x^4 + 2x^3 - 20x^2 - 66x - 45$ completely. *(5 marks)*

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Midterm Practice Exam
Answer Key

Grade 12 Pre-Calculus Mathematics

Midterm Practice Exam
Answer Key

Answer Key

Name: ______________________________

Student Number: ______________________________

Attending ☐ Non-Attending ☐

Phone Number: ______________________________

Address: ______________________________

____

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For Marker's Use Only

Date: ______________________________

[illegible]al Mar[illegible] ______ [illegible]00 = ______ %

[illegible]omments:

Instructions

The midterm examination will be weighted as follows:

Modules 1–4 100%

Time allowed: 3 hours

Note: You are allowed to bring the following to the exam: pens/pencils (2 or 3 of each), blank paper, a ruler, a scientific calculator, and your Midterm Exam Resource Sheet. Your Midterm Exam Resource Sheet must be handed in with the exam. You will receive your Midterm Exam Resource Sheet back from your tutor/marker with the next module work that is submitted for marking.

Show all calculations and formulas used. Use all decimal places in your calculations and round the final answers to the correct number of decimal places. Include units where appropriate. Clearly state your final answer.

General Marking Principles

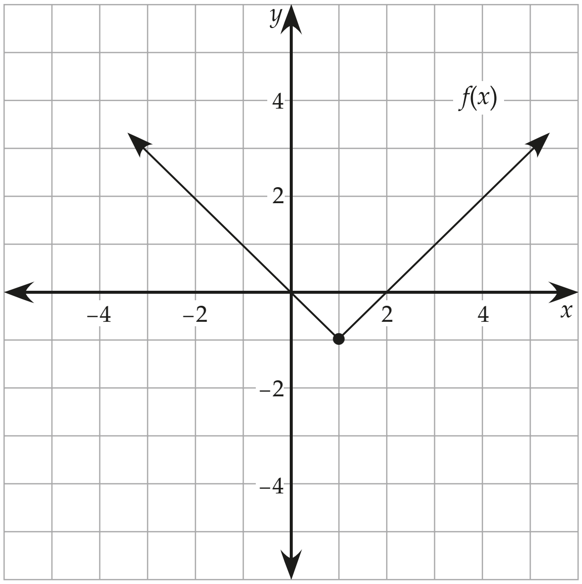
- Concepts learned in Grade 12 are worth 1 mark each. Concepts learned earlier (unless they are retaught under the curriculum, e.g., absolute value, reciprocals) are worth 0.5 mark each.
- Some errors are deducted only once (e.g., not putting arrowheads on graphs).
- Errors are followed through (e.g., if an arithmetic error is made in the first line, it is still possible for the student to receive nearly full marks).
- Many types of communication errors receive a 0.5 mark deduction, but 0.5 mark is the maximum communication error deduction for the entire exam.

Name: ____

Answer all questions to the best of your ability. Show all your work.

Long-Answer Questions *(100 marks)*

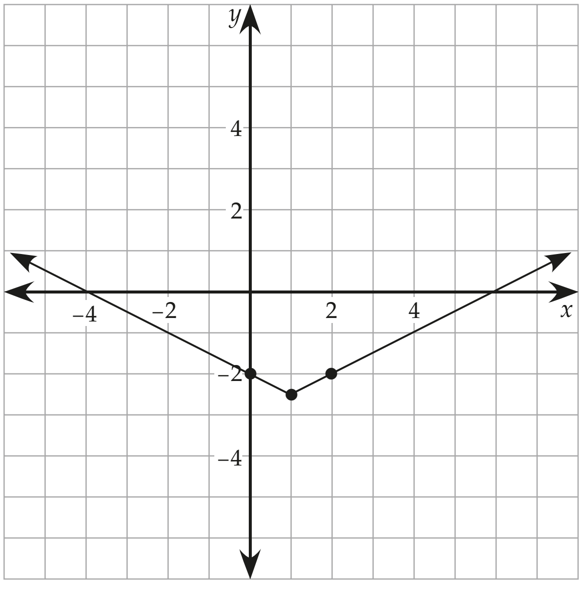
1. Given the sketch of $f(x)$ drawn below, show each transformation algebraically and graphically. State the domain and range of each function.



a) $y = \frac{1}{2} f(x) - 2$ *(3 marks)* (Module 2, Lesson 3)

Answer:

Graphically:



Algebraically:

$$(x, y) \to \left(x, \frac{1}{2}y - 2\right)$$

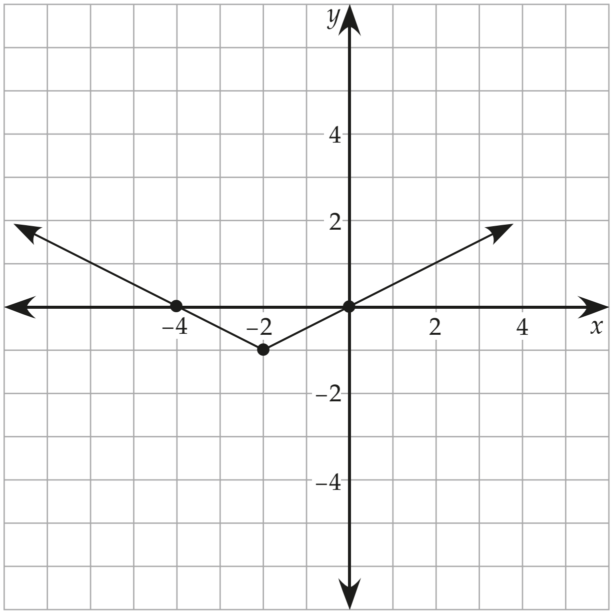
Domain: $\{x \in \Re\}$
Range: $\{y \geq -2.5\}$

(1 mark for algebraic notation)
(1 mark for graph)
(0.5 mark for domain)
(0.5 mark for range)

b) $y = f\left(\frac{1}{2}(x + 4)\right)$ *(3 marks)* (Module 2, Lesson 3)

Answer:

Graphically:



Domain: $\{x \in \Re\}$

Range: $\{y \geq -1\}$

Algebraically:

$(x, y) \rightarrow (2x - 4, y)$

(1 mark for algebraic notation)
(1 mark for graph)
(0.5 mark for domain)
(0.5 mark for range)

Name: ____

2. The y-intercept of the function $g(x)$ is 4. What would the new y-intercept be for each of the following functions? *(4 × 1 mark each = 4 marks)*

   a) $y = -2g(x)$ (Module 3, Lesson 1)

   Answer:

   Vertical reflection and stretch by 2.

   $4 \rightarrow -8$

   b) $y = g(x) + 1$ (Module 2, Lesson 1)

   Answer:

   Vertical shift up 1.

   $4 \rightarrow 5$

   c) $y = g(3x)$ (Module 2, Lesson 2)

   Answer:

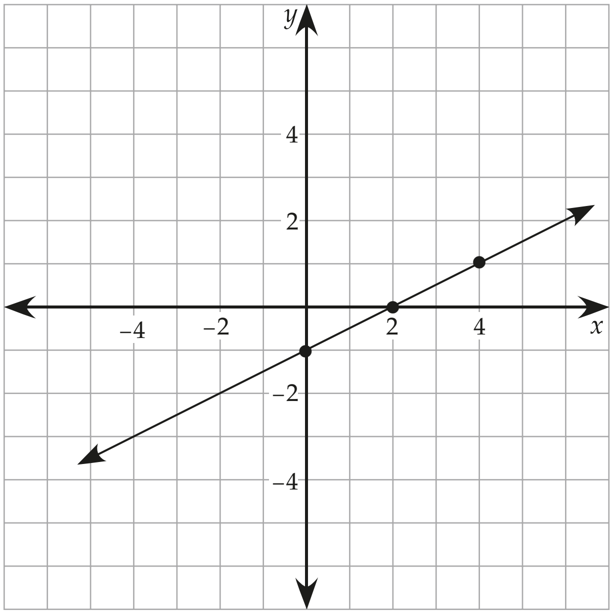
   Horizontal stretch would not affect the y-intercept; it would be an invariant point at (0, 4).

   d) $y = g(-x)$ (Module 3, Lesson 2)

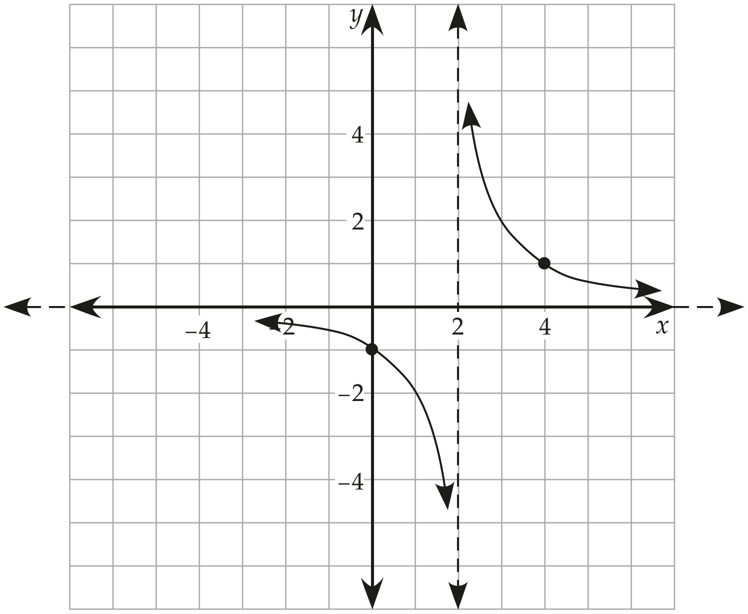
   Answer:

   Horizontal reflection would not affect the y-intercept; it would be an invariant point at (0, 4).

3. Use the graph of the function drawn below to sketch $y = \frac{1}{f(x)}$. *(2 marks)* (Module 2, Lesson 5)



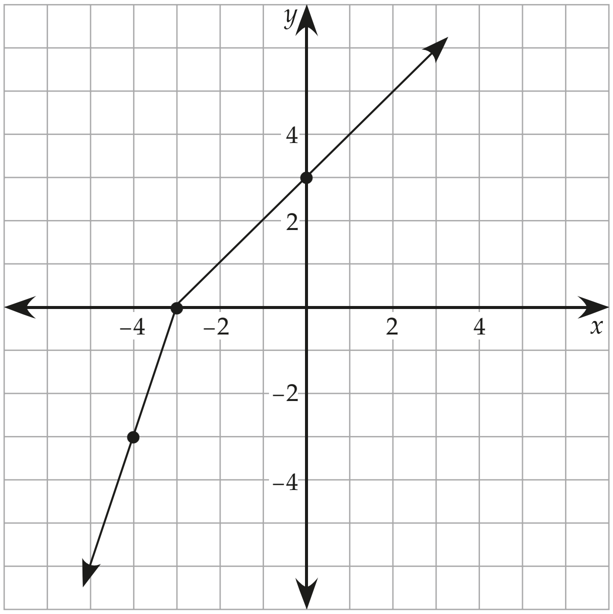
Answer:



(0.5 mark for asymptote at y = 0)
(0.5 mark for asymptote at x = 2)
(0.5 mark for graph on right side of x = 2)
(0.5 mark for graph on left side of x = 2)

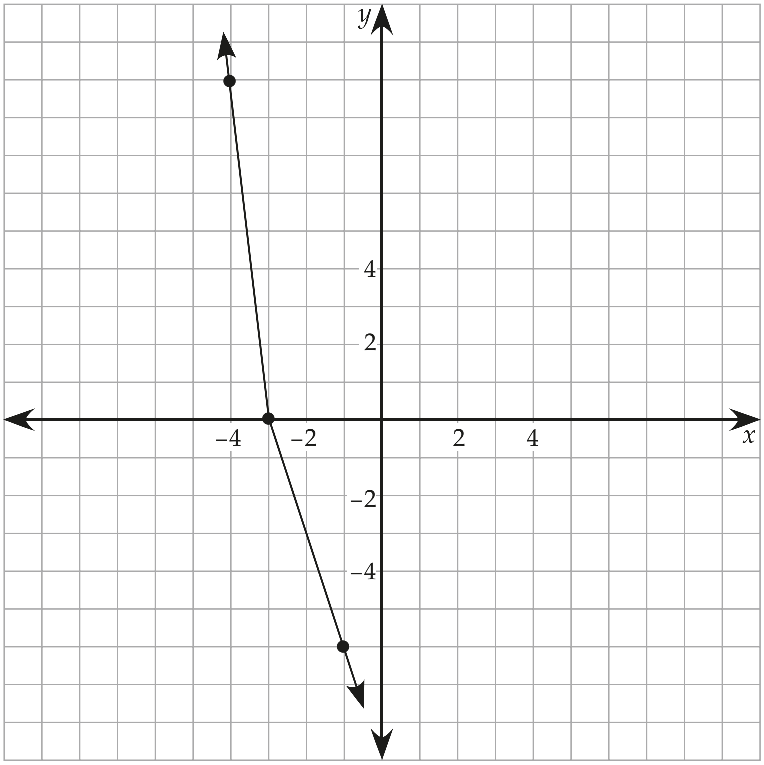
Name: ____

4. Using the sketch of $f(x)$, sketch the following.



a) $y = -3f(x)$ *(2 marks)* (Module 3, Lesson 1)

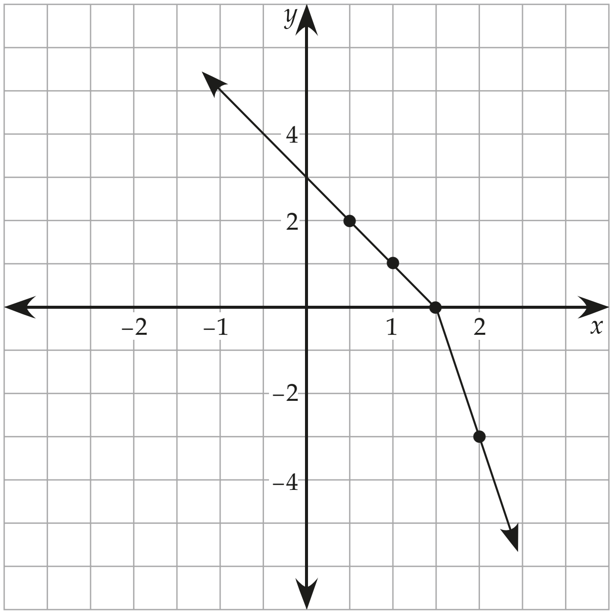
Answer:



(1 mark for vertical stretch)
(1 mark for reflection over x-axis)

b) $y = f(-2x)$ *(2 marks)* (Module 3, Lesson 2)

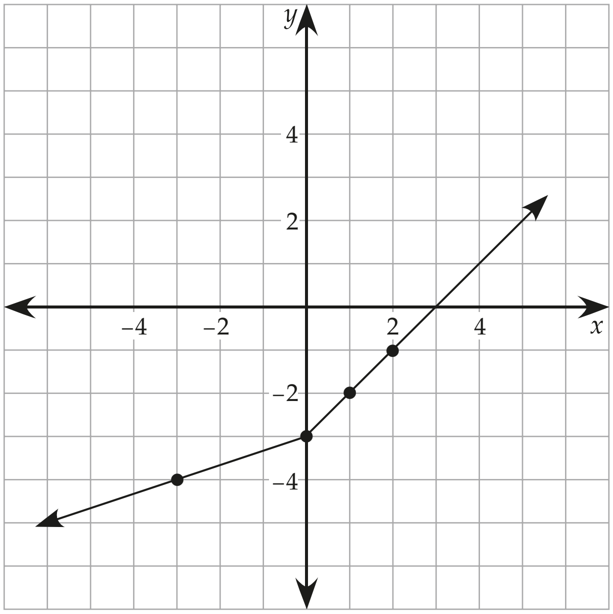
Answer:



(1 mark for horizontal compression)
(1 mark for horizontal reflection)

c) $y = f^{-1}(x)$ *(2 marks)* (Module 3, Lesson 3)

Answer:



(1 mark for correct points after switching x- and y-coordinates)
(1 mark for correct shape)

Name: ________________________________

5. In how many ways can you order 13 songs in a playlist if
   a) there are no restrictions? *(1 mark)* (Module 1, Lesson 2)

      Answer:

      13! = 6,227,020,800

   b) your favourite song must be first? *(2 marks)* (Module 1, Lesson 2)

      Answer:

      $1 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 479{,}001{,}600$

(1 mark for 12!)
(1 mark for restricting first place)

6. How many distinct ways can 4 green cups, 2 blue cups, and 1 red cup be arranged on a shelf? *(2 marks)* (Module 1, Lesson 3)

   Answer:

   $$\frac{7!}{4!2!} = 105$$

(0.5 mark for 7!)
(0.5 mark for 4! in denominator)
(0.5 mark for 2! in denominator)
(0.5 mark for final answer)

7. Four men and five women are on a parent council committee. In how many ways can a five-member subcommittee be formed if the women must have a majority on this subcommittee? *(4 marks)* (Module 1, Lesson 4)

Answer:

Case 1: 5 women $_5C_5 = 1$
Case 2: 4 women and 1 man $_5C_4 \cdot {_4C_3} = 5(4) = 20$
Case 3: 3 women and 2 men $_5C_3 \cdot {_4C_2} = 10(6) = 60$
$1 + 20 + 60 = 81$

(1 mark for each case × 3)
(1 mark for addition of cases)

8. Evaluate each of the following using factorial notation. Show your work. *(4 × 2 marks each = 8 marks)*

a) $_3P_2$ (Module 1, Lesson 2)

Answer:

$$\frac{3!}{(3-2)!} = \frac{3!}{1!} = 6 \quad \text{or} \quad \underline{\ 3\ } \cdot \underline{\ 2\ } = 6$$

Mark break down for all:
(1 mark for set up)
(1 mark for answer)

b) $_5P_2$

Answer:

$$\frac{5!}{(5-2)!} = \frac{5!}{3!} = 5 \cdot 4 = 20 \quad \text{or} \quad \underline{\ 5\ } \cdot \underline{\ 4\ } = 20$$

c) $_7C_3$ (Module 1, Lesson 4)

Answer:

$$\frac{7!}{(7-3)!3!} = \frac{7!}{4!3!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$$

d) $_6C_2$

Answer:

$$\frac{6!}{(6-2)!2!} = \frac{6!}{4!2!} = \frac{6 \cdot 5}{2 \cdot 1} = 15$$

Name: ____

9. Solve without using a calculator. *(3 marks)* (Module 1, Lesson 4)

   $_{n+3}P_2 = 20$

   Answer:

   $_{n+3}P_2 = 20$

   Use $_nP_r = \dfrac{n!}{(n-r)!}$

   where $n = n + 3$

   $r = 2$

   $$\frac{(n+3)!}{(n+3-2)!} = 20$$

   Simplify the denominator:

   $$\frac{(n+3)!}{(n+1)!} = 20$$

   Expand the factorials and simplify: *(0.5 mark for expanding factorials)*

   $$\frac{(n+3)(n+2)\cancel{(n+1)!}}{\cancel{(n+1)!}} = 20$$

   $$(n+3)(n+2) = 20$$

   Expand and simplify: *(1 mark for expanding and simplifying)*

   $$n^2 + 5n + 6 = 20$$

   $$n^2 + 5n + 6 - 20 = 0$$

   $$n^2 + 5n - 14 = 0$$

   Factor and solve: *(1 mark for factoring and solving)*

   $$(n+7)(n-2) = 0$$

   $$n = -7$$

   $$n = 2$$

   n cannot equal -7 since you cannot take the factorial of a negative number

   $n = 2$ *(0.5 mark for solving for n)*

10. Given the following row of Pascal’s Triangle, determine the next row. *(1 mark)*
(Module 1, Lesson 5)

1 11 55 165 330 462 462 330 165 55 11 1

Answer:

1 12 66 220 495 792 924 792 495 220 66 12 1

To determine the next row of Pascal’s Triangle, start by adding a 1 as the first and last term in the row. Then, determine the middle terms by adding the two values that are diagonally above the term. For example, the second term in this row of Pascal’s Triangle is found by adding 1 to 11. The third term in this row is found by adding 11 to 55, and so on.

11. Write and simplify the last term of the expansion of $\left(2+\frac{2}{x^2}\right)^4$. *(3 marks)*
(Module 1, Lesson 5)

Answer:

There are 5 terms.

$$t_5 = \binom{4}{4}(2)^0\left(\frac{2}{x^2}\right)^4 = 1(1)\left(\frac{16}{x^8}\right) = \frac{16}{x^8}$$

(1 mark for $n = 4$, $k = 4$)

(0.5 mark for 2^0)

(0.5 mark for $\left(\frac{2}{x^2}\right)^4$)

(0.5 mark for final coefficient)

(0.5 mark for final variable)

Name: ____

12. Expand and simplify $(2x - 3)^4$ using the Binomial Theorem. *(4 marks)*
(Module 1, Lesson 5)

Answer:

$$(2x-3)^4$$

$$=\binom{4}{0}(2x)^4(-3)^0+\binom{4}{1}(2x)^3(-3)^1+\binom{4}{2}(2x)^2(-3)^2+\binom{4}{3}(2x)^1(-3)^3+\binom{4}{4}(2x)^0(-3)^4$$

$$=1\left(16x^4\right)(1)+4\left(8x^3\right)(-3)+6\left(4x^2\right)(9)+4(2x)(-27)+1(1)(81)$$

$$=16x^4-96x^3+216x^2-216x+81$$

(2 marks for the first line
0.5 mark for pattern of powers
0.5 mark for $_nC_r$ *pattern*
1 mark for five terms)
(1 mark for exponent laws)
(1 mark for correct coefficients)

13. For the function $y = x^3 - 3x^2 - x + 3$, find the following:

a) the zeros of the function if you know that $(x + 1)$ is a factor of the polynomial *(4 marks)* (Module 4, Lesson 3)

Answer:

$y = x^3 - 3x^2 - x + 3$

$(x + 1)$ is a factor so use synthetic division for $x = -1$ *(1 mark for Factor Theorem)*

$$\begin{array}{r|rrr|r} -1 & 1 & -3 & -1 & 3 \\ & & -1 & 4 & -3 \\ \hline & 1 & -4 & 3 & 0 \end{array}$$

(1 mark for synthetic division)

$\therefore y = x^3 - 3x^2 - x + 3 = (x + 1)(x^2 - 4x + 3) = (x + 1)(x - 3)(x - 1)$

$y = (x + 1)(x - 3)(x - 1)$ *(1 mark for correct factors)*

Zeros: $x = -1$, 3, and 1 *(1 mark for correct zeros)*

b) left-right behaviour *(1 mark)* (Module 4, Lesson 1)

Answer:

Left/Right Behaviour: Points down to the left and points up to the right

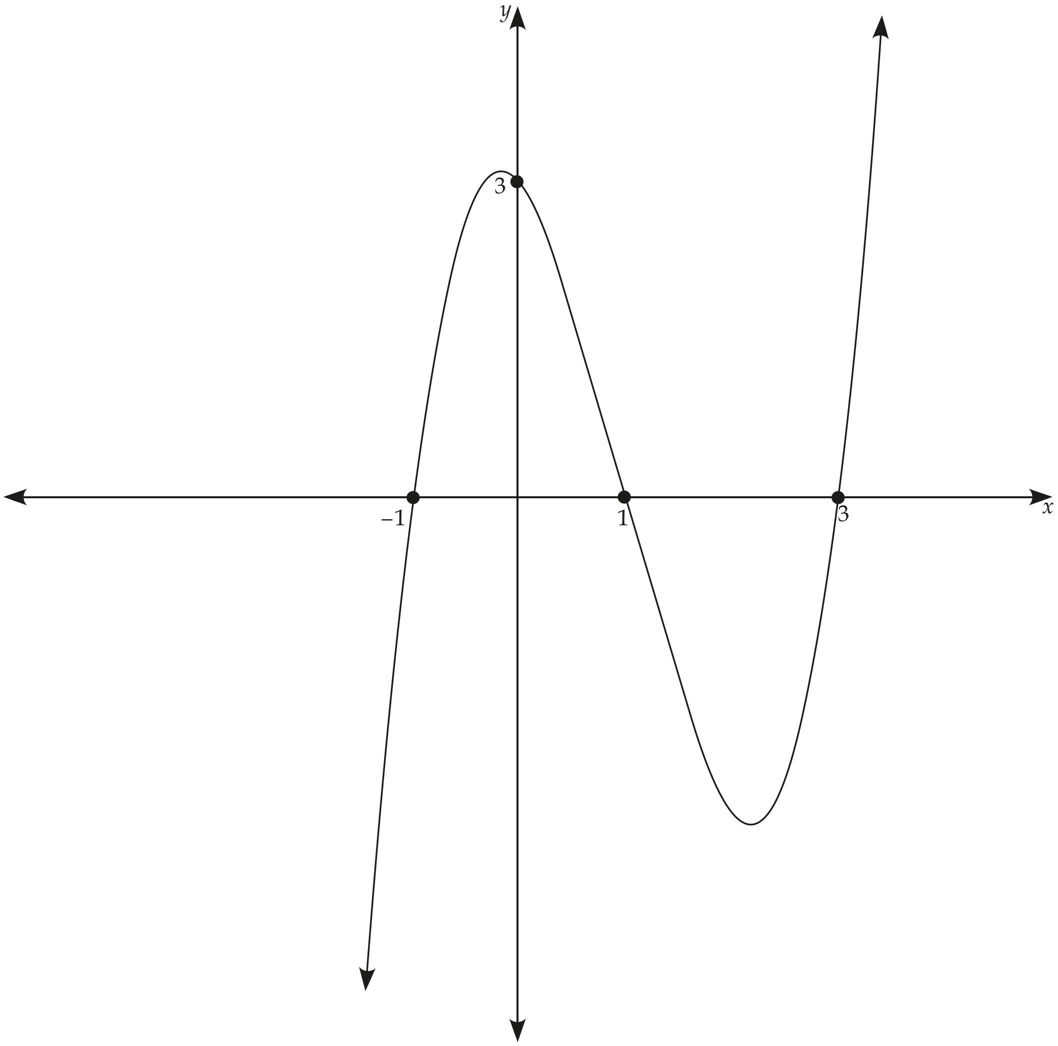
c) the y-intercept of the function *(1 mark)* (Module 4, Lesson 1)

Answer:

y-intercept: 3

d) the sketch of the graph of the function *(2 marks)* (Module 4, Lesson 4)

Answer:



(0.5 mark for positive polynomial cubic or cubic consistent with previous work (end behaviour))

(1 mark for correct x-intercepts)

(0.5 mark for y-intercept)

14. Graph a quartic that has roots of −1 and +2 and a root with multiplicity 2 at +1. The function equation has a leading coefficient of −3. *(3 marks)* (Module 4, Lesson 4)

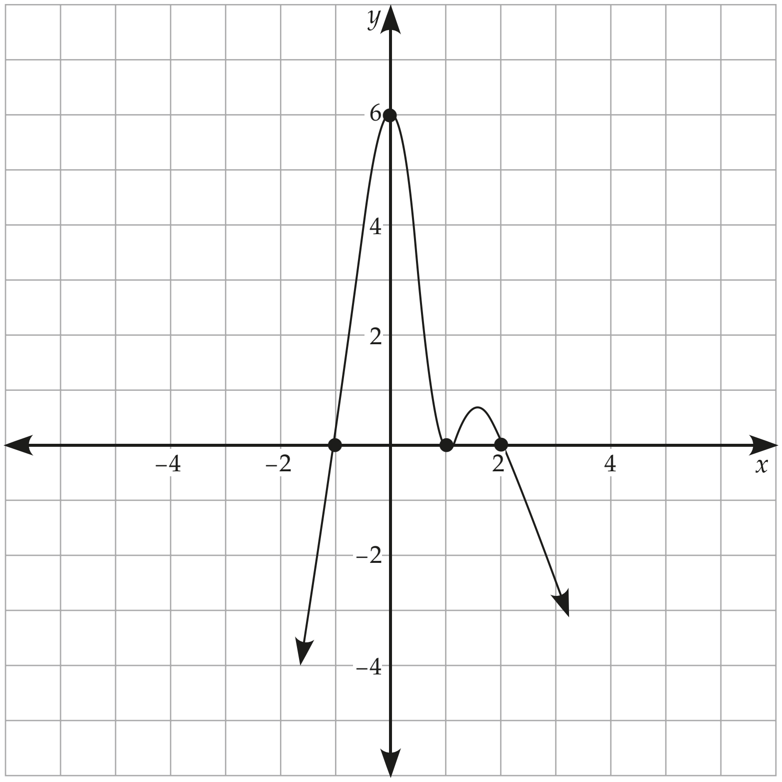
Answer:

x-intercepts at −1, +2, +1. Then, $y = -3(x + 1)(x - 2)(x - 1)^2$.

Graph falls in Quadrants III and IV.

Graph is tangent to x-axis at $x = +1$.

$$\begin{aligned} y\text{-intercept at } & (-3)(+1)(-2)(-1)^2 \\ &= (-3)(-2) \\ &= +6 \end{aligned}$$



(1 mark for end behaviour)
(1 mark for behaviour at x-intercept)
(1 mark for y-intercept)

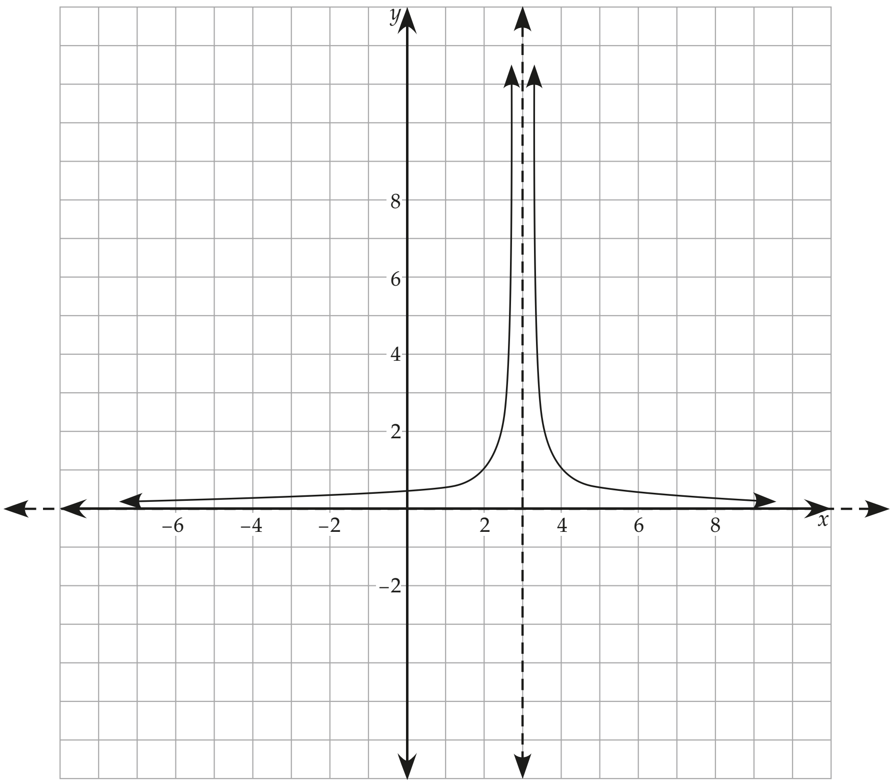
Name: ____

15. If $f(x) = \dfrac{1}{x-3}$ and $g(x) = |x|$, determine the equation and graph of $h(x) = g(f(x))$.

State the domain and range. *(5 marks)* (Module 2, Lesson 5)

Answer:

$h(x) = g\left(\dfrac{1}{x-3}\right) = \left|\dfrac{1}{x-3}\right|$ *(1 mark for correct h(x))*



(marks for correct graph:

- *1 mark for understanding the effect of absolute value;*
- *2 marks for* $f(x) = \dfrac{1}{x-3}$*:*

  (0.5 mark for vertical asymptote at x = 3;

  0.5 mark for horizontal asymptote at y = 0;

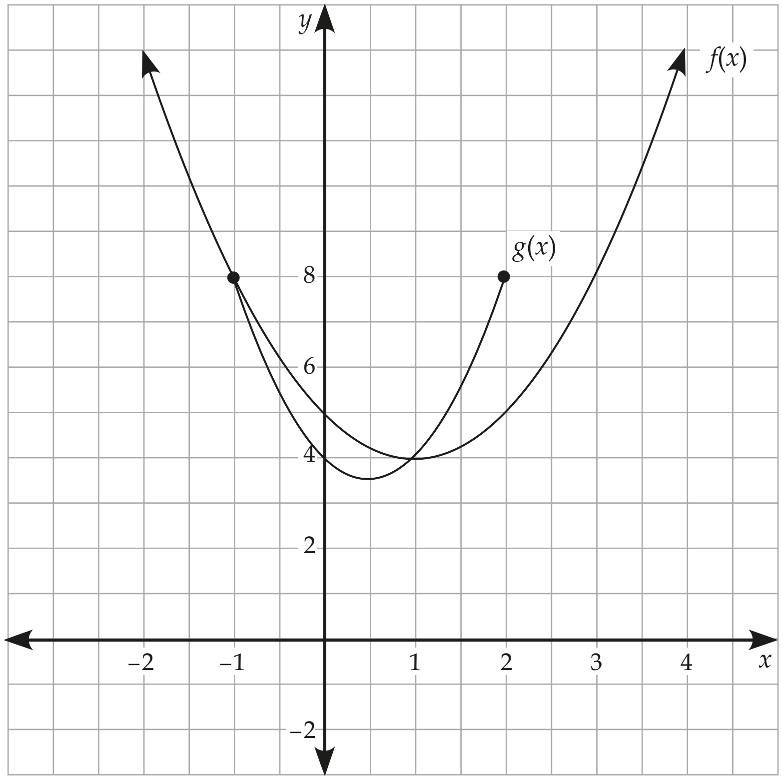
  0.5 mark for right side;

  0.5 mark for left side)

Domain: $\{x \mid x \neq 3, x \in \Re\}$ *(0.5 mark)*

Range: $\{y \mid y > 0, y \in \Re\}$ *(0.5 mark)*

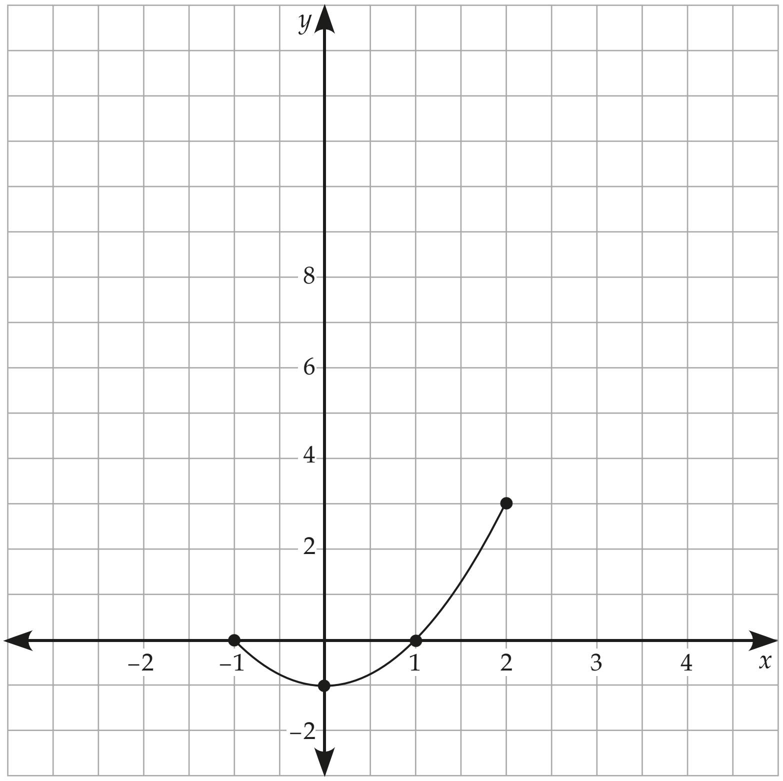
16. Consider the graphs of $f(x)$ and $g(x)$ below. Use these graphs to sketch $(g - f)(x)$. *(3 marks)* (Module 2, Lesson 4)



Answer:

x	$f(x)$	$g(x)$	$(g - f)(x)$
−1	8	8	0
0	5	4	−1
1	4	4	0
2	5	8	3

Name: ______________________________

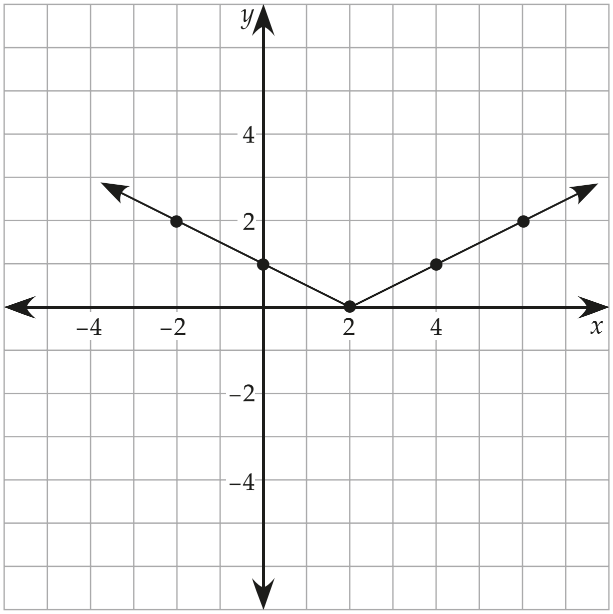


(1 mark for restricting the domain)

(1 mark for (g− f)(x) values)

(1 mark for correct graph)

17. The following graph represents a transformation of $f(x) = |x|$. Write an equation for the new absolute value function. *(2 marks)* (Module 2, Lesson 3)



Answer:

This function has been shifted 2 units to the right and either stretched vertically by a factor of $\frac{1}{2}$ or compressed horizontally by a factor of $\frac{1}{2}$. The resulting function could be

$$y = f\left(\frac{1}{2}(x-2)\right) = \left|\frac{1}{2}(x-2)\right| \text{ or } y = \frac{1}{2}f(x-2) = \frac{1}{2}|x-2|$$

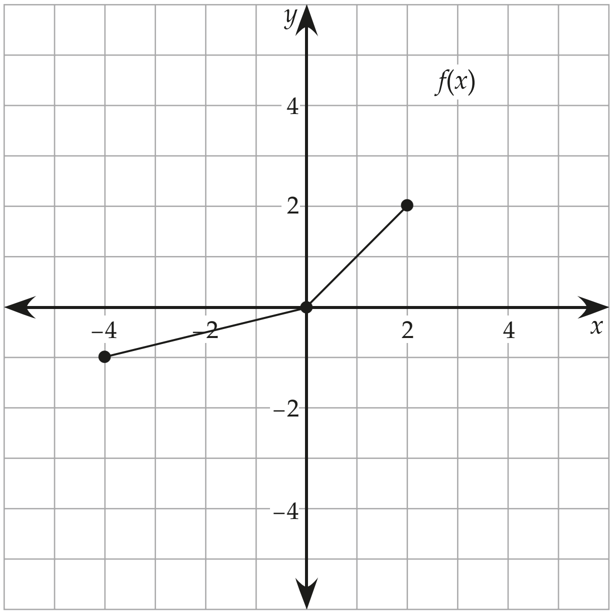
(1 mark for horizontal compression value of $\frac{1}{2}$ or vertical stretch by $\frac{1}{2}$)

(0.5 mark for correct horizontal shift)

(0.5 mark for correct placement of absolute value)

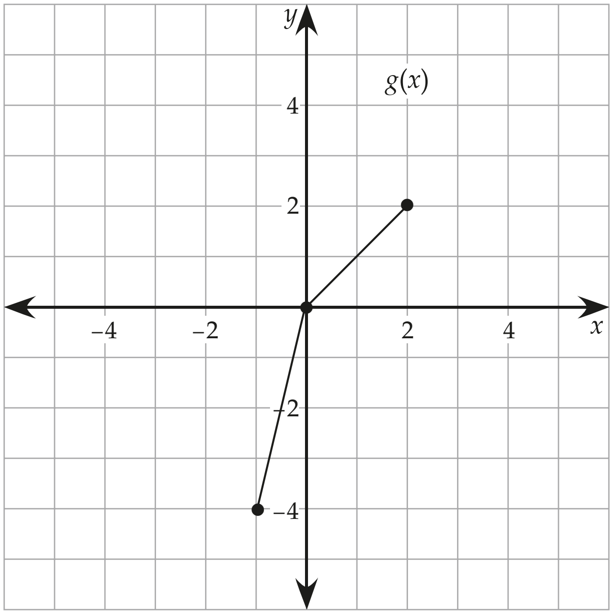
Name: ____

18. The graph of a function $f(x)$ is drawn below.



a) Reflect the graph of $f(x)$ in the line $y = x$ to achieve the graph of $g(x)$. *(1 mark)* (Module 3, Lesson 3)

Answer:



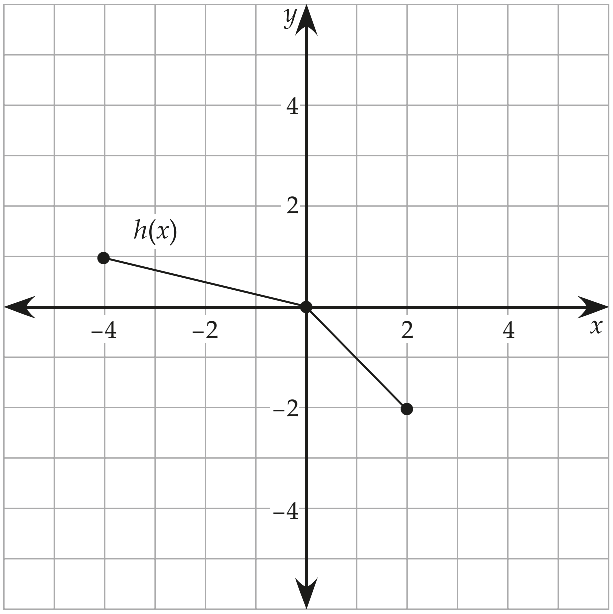
b) Write the equation of the new function $g(x)$ in terms of $f(x)$. *(1 mark)*
(Module 3, Lesson 3)

Answer:

$g(x) = f^{-1}(x)$

c) Reflect the graph of $f(x)$ in the x-axis to achieve the graph of $h(x)$. *(1 mark)*
(Module 3, Lesson 1)

Answer:



d) Write the equation of the new function $h(x)$ in terms of $f(x)$. *(1 mark)*
(Module 3, Lesson 1)

Answer:

$h(x) = -f(x)$

Name: ______________________________________

19. A function contains the ordered pairs $(-1, 0)$, $(0, 6)$, and $(-3, 7)$. What are the corresponding ordered pairs if this function is reflected through the following lines? *(3 × 1 mark each = 3 marks)*

    a) $y = 0$ (x-axis) (Module 3, Lesson 1)

    Answer:

    $(-1, 0)$, $(0, -6)$, $(-3, -7)$

    b) $x = 0$ (y-axis) (Module 3, Lesson 2)

    Answer:

    $(1, 0)$, $(0, 6)$, $(3, 7)$

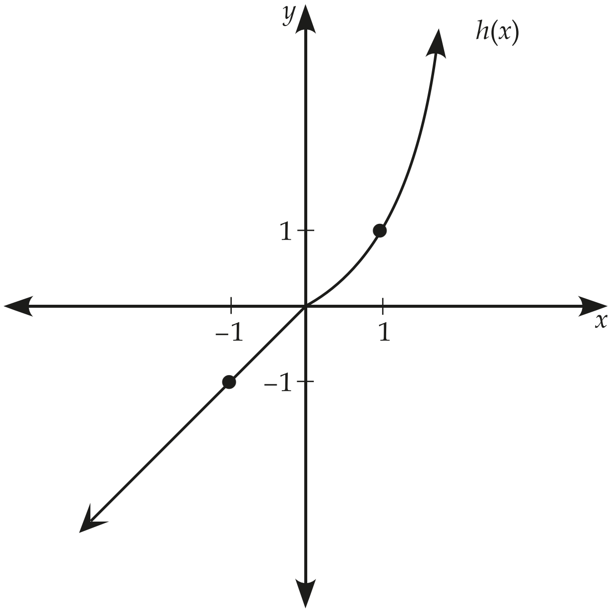
    c) $y = x$ (Module 3, Lesson 3)

    Answer:

    $(0, -1)$, $(6, 0)$, $(7, -3)$

20. For each of the following relations, determine if they are one-to-one functions. Explain your reasoning. *(2 × 1 mark each = 2 marks)*
(Module 3, Lesson 4)

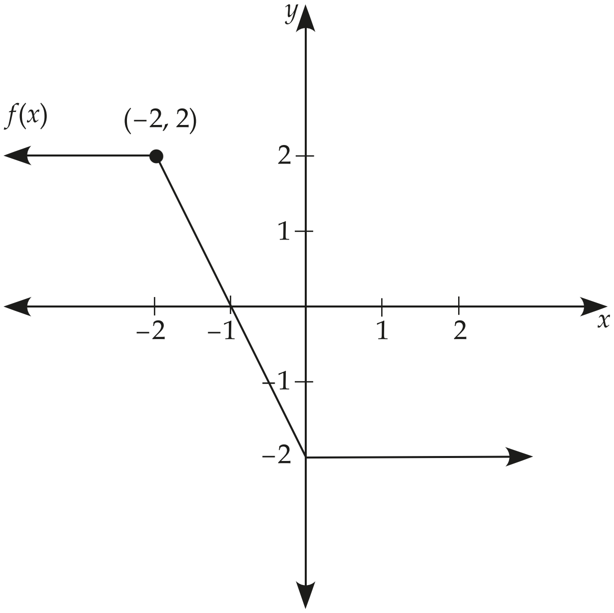
a)



Answer:

Yes, $f(x)$ is a function because it passes the vertical line test. The inverse of $f(x)$, $f^{-1}(x)$ is also a function because $f(x)$ is one-to-one.

b)



Answer:

Yes, $f(x)$ is a function because it passes the vertical line test. The inverse of $f(x)$ is not a function because it does not pass the horizontal line test. If you restrict the domain of $f(x)$ to be $\{x \mid -2 \le x \le 0\}$, then $f^{-1}(x)$ would be a function.

Name: ____________________________________

21. Show algebraically that the functions f and g are inverses of each other. *(2 marks)* (Module 3, Lesson 4)

$$f(x) = \sqrt{4x-1} \qquad g(x) = \frac{x^2+1}{4}$$

Answer:

Method 1

$$\begin{aligned} f(g(x)) &= \sqrt{4\left(\frac{x^2+1}{4}\right)-1} \\ &= \sqrt{x^2+1-1} \\ &= \sqrt{x^2} \\ &= x \end{aligned}$$

$$\begin{aligned} g(f(x)) &= \frac{\left(\sqrt{4x-1}\right)^2+1}{4} \\ &= \frac{4x-1+1}{4} \\ &= \frac{4x}{4} \\ &= x \end{aligned}$$

(0.5 mark for writing both compositions)

(1 mark for simplifying)

As $f(g(x))$ and $g(f(x))$ both equal x, $f(x)$ and $g(x)$ are inverses of each other. *(0.5 mark)*

Method 2

$$y = \sqrt{4x-1}$$

$$x = \sqrt{4y-1}$$ *(0.5 mark for switching x and y)*

$$x^2 = 4y-1$$

$$x^2+1 = 4y$$

$$\frac{x^2+1}{4} = y$$ *(1 mark for solving for y)*

$\therefore$ $f(x)$ and $g(x)$ are inverses of each other. *(0.5 mark)*

22. Find $f^{-1}(x)$ algebraically. Graph $f^{-1}(x)$. Consider the domain and range of $f^{-1}(x)$. *(3 marks)*

$f(x) = (x - 6)^2, x \leq 6$

Answer: (Module 3, Lesson 4)

$y = (x - 6)^2, \quad x \leq 6$ and $y \geq 0$

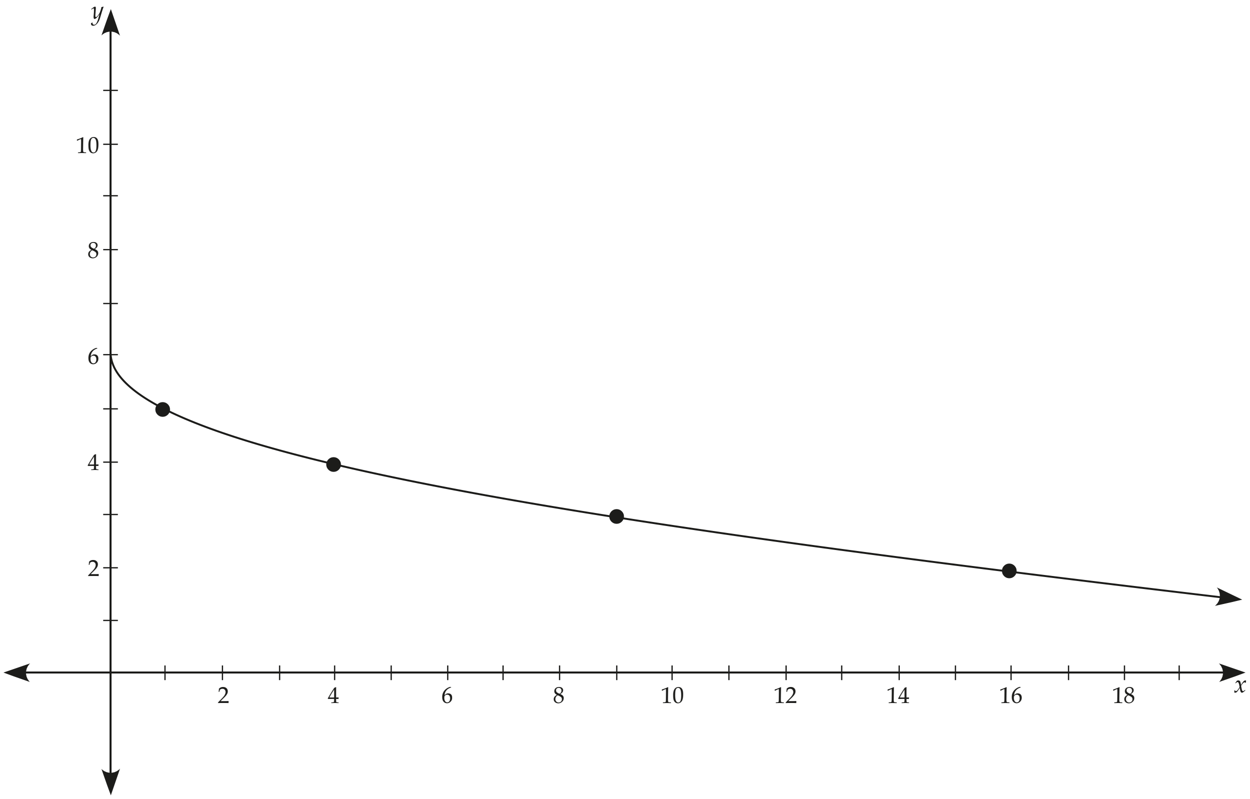
Represent inverse as:

$x = (y - 6)^2$ *(0.5 mark for switching x and y)*

$\pm\sqrt{x} = y - 6$ *(0.5 mark for solving)*

$\pm\sqrt{x} + 6 = y$

$f^{-1}(x) = -\sqrt{x} + 6, x \geq 0$ and $y \leq 6$ *(1 mark for correct sign of radical considering restriction on range)*



(1 mark for the graph

Name: ____

23. Given that $f(x) = |2x - 1|$ and $g(x) = x^3 - 1$, find the following.
(3 × 1 mark each = 3 marks) (Module 2, Lesson 5)

a) $f(f(x))$

Answer:

$$f\left(f(x)\right) = \left|2\left(|2x-1|\right)-1\right|$$

b) $g(g(-1))$

Answer:

$$g\left(g(-1)\right) = g\left((-1)^3-1\right) = g(-1-1) = g(-2) = (-2)^3 - 1 = -9$$

c) $f(g(2))$

Answer:

$$\left(g(2)\right) = f\left(\left(2^3\right)-1\right) = f(8-1) = f(7) = |2(7)-1| = |14-1| = 13$$

24. Given: $f(x) = x^3$ and $g(x) = x - 2$ (Module 2, Lesson 5)

a) Determine $f(g(x))$ and describe the graph of $f(g(x))$ in terms of a transformation of $f(x)$. *(2 marks)*

Answer:

$f(g(x)) = (x - 2)^3$ *(1 mark for equation)*

It is the graph of $f(x)$ shifted 2 units to the right. *(1 mark for description)*

b) Determine $g(f(\mathrm{x}))$ and describe the graph of $g(f(x))$ in terms of a transformation of $f(x)$. *(2 marks)*

Answer:

$g(f(x)) = x^3 - 2$ *(1 mark for equation)*

It is the graph of $f(x)$ shifted 2 units down. *(1 mark for description)*

25. For each of the following polynomials, determine whether it is divisible by $x - 2$. Show your work. *(2 × 2 marks each = 4 marks)* (Module 4, Lesson 3)

a) $f(x) = -x^4 + x^3 - 8x^2 + 6$

Answer:

$f(2) = (2)^4 + (2)^3 - 8(2)^2 + 6$ *(1 mark for strategy)*

$= -16 + 8 - 32 + 6$

$= -34$

This polynomial is not divisible by $x - 2$, as the remainder does not equal zero.

(1 mark for correct, consistent conclusion)

b) $g(x) = -x^3 + x^2 - 5x + 14$

Answer:

$g(2) = -(2)^3 + (2)^2 - 5(2) + 14$ *(1 mark for strategy)*

$= -8 + 4 - 10 + 14$

$= 0$

This polynomial is divisible by $x - 2$, as the remainder equals zero.

(1 mark for conclusion)

26. Divide, using long division or synthetic division, and write in the form given by the division algorithm. *(3 marks)* (Module 4, Lesson 2)

$(2x^3 - 4x^2 - 12x - 14) \div (x - 4)$

Answer:

$$\begin{array}{r|rrr|r} 4 & 2 & -4 & -12 & -14 \\ & & 8 & 16 & 16 \\ \hline & 2 & 4 & 4 & 2 \end{array}$$

(1 mark for correct strategy)

(1 mark for correct answer)

$\therefore 2x^3 - 4x^2 - 12x - 14 = (2x^2 + 4x + 4)(x - 4) + 2$ *(1 mark for final division statement)*

27. Factor $g(x) = x^4 + 2x^3 - 20x^2 - 66x - 45$ completely. *(5 marks)* (Module 4, Lesson 3)

Answer:

$g(x) = x^4 + 2x^3 - 20x^2 - 66x - 45$

Possible rational roots: ±1, ±3, ±5, ±9, ±15, ±45 *(1 mark for strategy)*

Test $x = -1$:
$$\begin{aligned} g(-1) &= (-1)^4 + 2(-1)^3 - 20(-1)^2 - 66(-1) - 45 \\ &= 1 - 2 - 20 + 66 - 45 \\ &= 0 \end{aligned}$$

$\therefore x = -1$ is a root and $(x + 1)$ is a factor

$\therefore g(x) = (x + 1)(q(x))$

$$\begin{array}{r|rrrr|r} -1 & 1 & 2 & -20 & -66 & -45 \\ & & -1 & -1 & 21 & 45 \\ \hline & 1 & 1 & -21 & -45 & 0 \end{array}$$

(1 mark for synthetic division)

$\therefore q(x) = x^3 + x^2 - 21x - 45$ and

$g(x) = (x + 1)(x^3 + x^2 - 21x - 45)$

Test $x = -3$ in $q(x)$:
$$\begin{aligned} q(-3) &= (-3)^3 + (-3)^2 - 21(-3) - 45 \\ &= -27 + 9 + 63 - 45 \\ &= 0 \end{aligned}$$

(1 mark)

$\therefore x = -3$ is a root and $(x + 3)$ is a factor

$\therefore g(x) = (x + 1)(x + 3)(p(x))$

$$\begin{array}{r|rrr|r} -3 & 1 & 1 & -21 & -45 \\ & & -3 & 6 & 45 \\ \hline & 1 & -2 & -15 & 0 \end{array}$$

(1 mark for synthetic division)

$\therefore p(x) = x^2 - 2x - 15$ and

$g(x) = (x + 1)(x + 3)(x^2 - 2x - 15)$

$\therefore g(x) = (x + 1)(x + 3)(x + 3)(x - 5)$ or

$g(x) = (x + 1)(x + 3)^2(x - 5)$

(1 mark)

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Final Practice Exam

GRADE 12 PRE-CALCULUS MATHEMATICS

Final Practice Exam

Name: ____________________________________

Student Number: ____________________________

Attending ☐ Non-Attending ☐

Phone Number: _____________________________

Address: __________________________________

____

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For Marker's Use Only

Date: ________________________________

Final Mark: ________ /100 = ________ %

Comments:

Instructions

The final examination will be weighted as follows:

Modules 1–8	100%

Time allowed: 3 hours

Note: You are allowed to bring the following to the exam: pens/pencils (2 or 3 of each), blank paper, a ruler, a scientific calculator, and your Final Exam Resource Sheet. Your Final Exam Resource Sheet must be handed in with the exam.

Show all calculations and formulas used. Use all decimal places in your calculations and round the final answers to the correct number of decimal places. Include units where appropriate. Clearly state your final answer.

General Marking Principles

- Concepts learned in Grade 12 are worth 1 mark each. Concepts learned earlier (unless they are retaught under the curriculum, e.g., absolute value, reciprocals) are worth 0.5 mark each.
- Some errors are deducted only once (e.g., not putting arrow heads on graphs).
- Errors are followed through (e.g., if an arithmetic error is made in the first line, it is still possible for the student to receive nearly full marks).
- Many types of communication errors receive a 0.5 mark deduction, but 0.5 mark is the maximum communication error deduction for the entire exam.

Name: ____

Answer all questions to the best of your ability. Show all your work.

Long-Answer Questions *(100 marks)*

1. Given that $f(x) = \left|\frac{1}{x}\right|$ and $g(x) = x^3 + 6x - 3$, find the following.

   (2 × 1 mark each = 2 marks)

   a) $f(f(x))$

   b) $g(f(-2))$

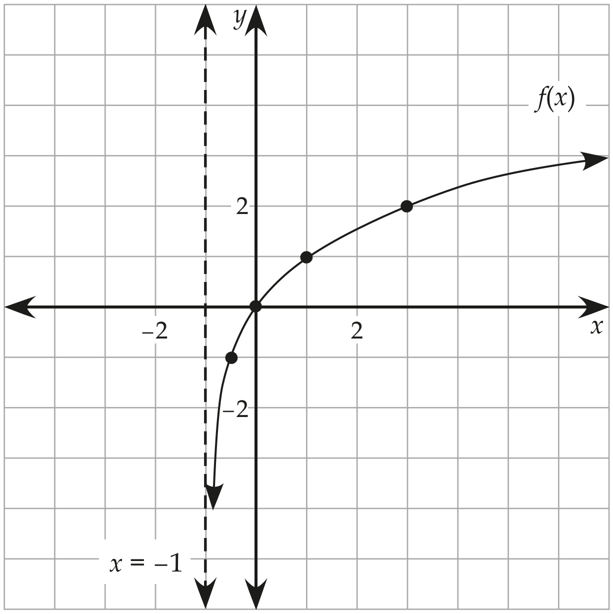
2. Divide, using long division or synthetic division, and write in the form given by the division algorithm. *(3 marks)*

   $(-x^3 - 4x^2 + 7x + 4) \div (x + 3)$

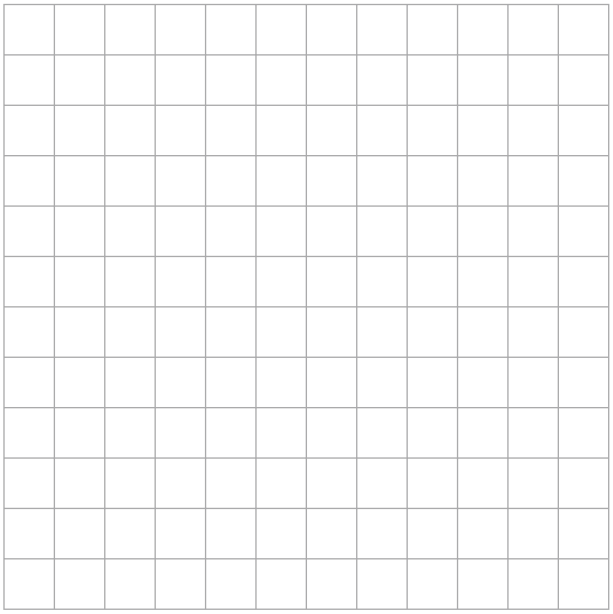
3. Rewrite $2 \log x + \frac{1}{2} \log 5 - \frac{1}{3} \log (x + 2)$ as a single logarithm statement. *(3 marks)*

Name: ______________________________

4. Given the sketch of *f*(*x*) drawn below, sketch the following functions.



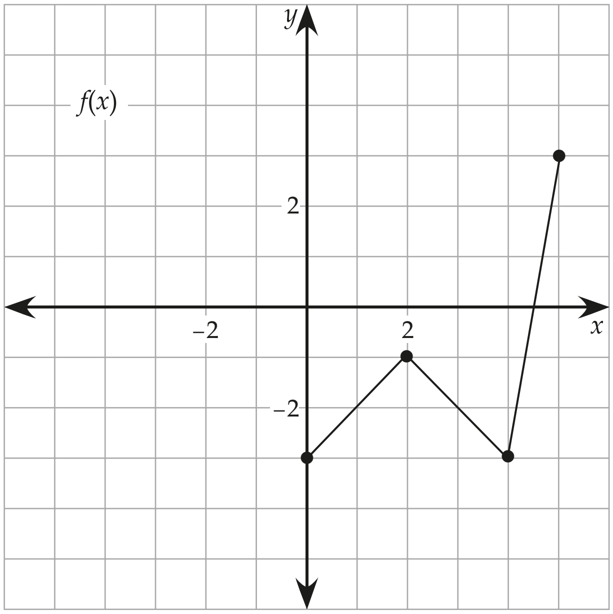
a) $y = f(x - 2)$ *(1 mark)*



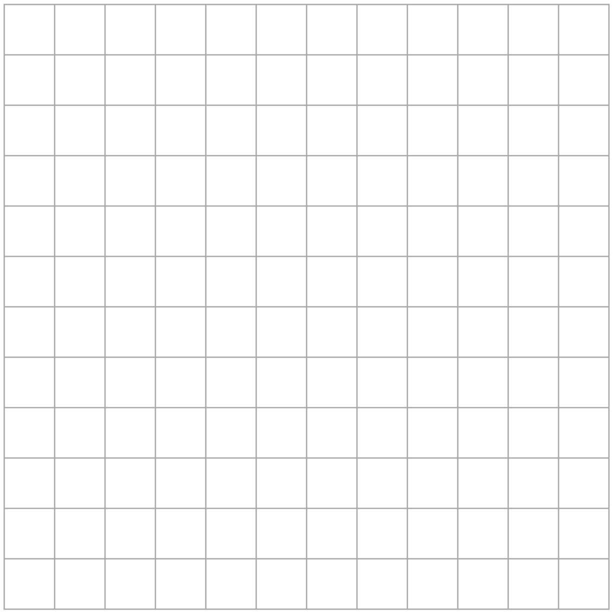
b) $y = |f(x)| + 2$ *(1 mark)*

Name: ______________________________

5. Using the sketch of $f(x)$, sketch the following. Express the transformation algebraically or in words.



a) $y = -\frac{1}{2} f(x)$ *(2 marks)*



b) $y = f(-2x)$ *(2 marks)*

c) $y = f^{-1}(x)$ *(2 marks)*

Name: ____

6. Write and simplify the fifth term of the expansion of $(x + 1)^8$. *(2 marks)*

7. There are 9 boys and 11 girls in a Grade 12 English class. In how many ways can 5 students be chosen for a group project if the group must have 3 female members and 2 male members? *(2 marks)*

8. Convert 1265° to radians. Write the exact answer. *(1 mark)*

9. You know that $\sin \alpha = -\frac{2}{7}$ and $\pi < \alpha < \frac{3\pi}{2}$. You also know that $P(\beta)$ is in Quadrant IV and $\cos \beta = \frac{4}{5}$. Find $\sin(\alpha + \beta)$.*(3 marks)*

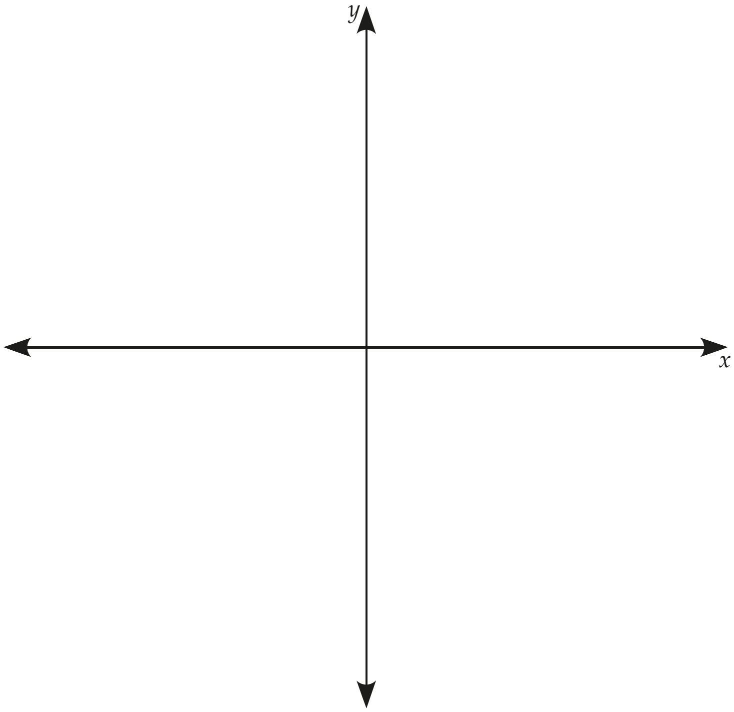
Name: ______________________________________

10. Consider the function $f(x) = -(x - 1)(x + 3)(x - 7)$.

   a) Determine the end behaviour of the function. *(1 mark)*

   b) Find all x- and y-intercept(s). *(2 marks)*

   c) Sketch the function. *(2 marks)*



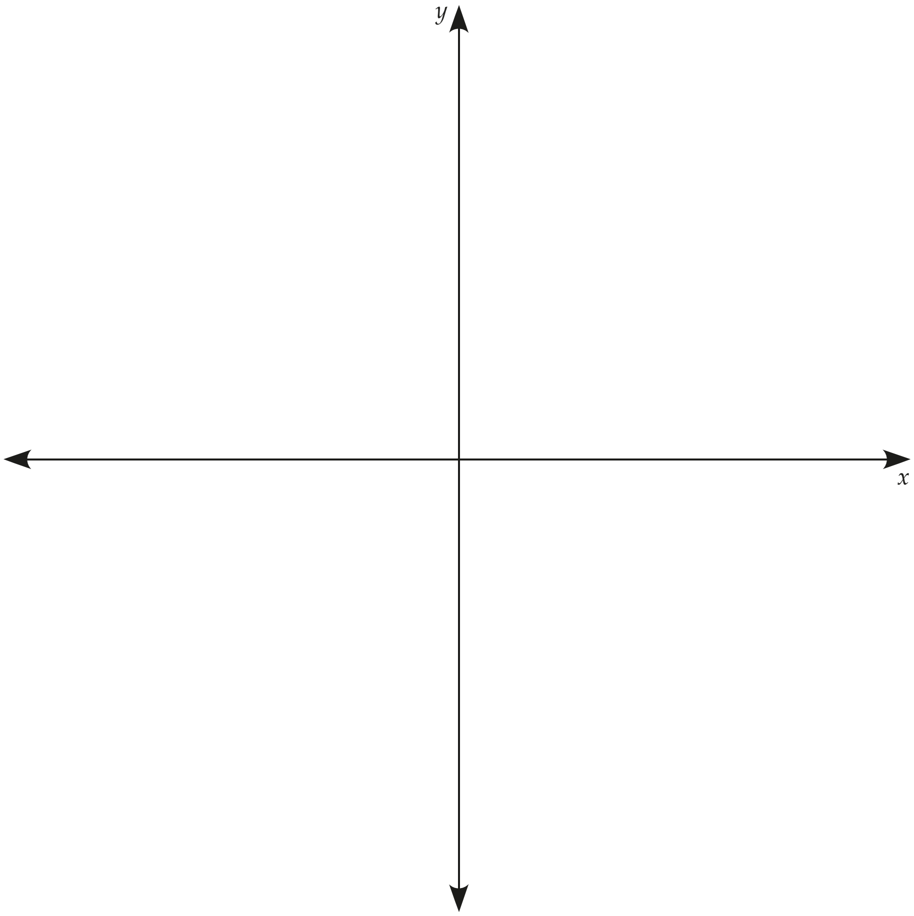
11. Sketch the function $g(x) = 9 - 3^x$ and state its range, x-intercept, and equation of asymptote. *(4 marks)*

12. Sketch the function $f(x) = \log_3 x - 2$. *(1 mark)*

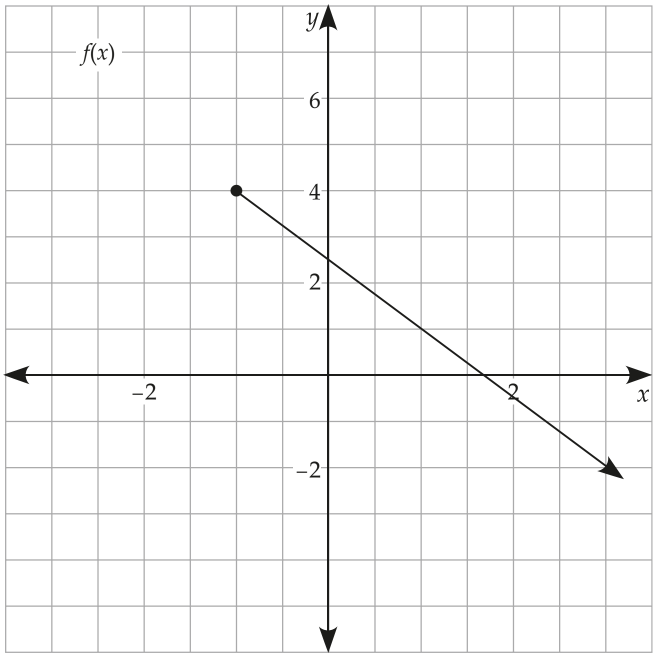
Name: ____

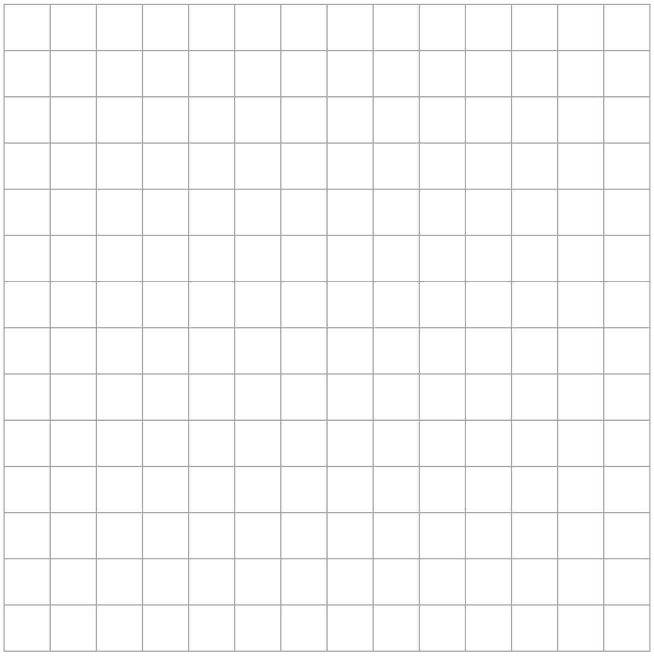
13. Graph the following function using transformations. State the domain and range of the function. *(3 marks)*

$g(x) = 2\sqrt{x} + 4$



14. Given the following graph of $f(x)$, graph $\sqrt{f(x)}$. *(2 marks)*





Name: ______________________________

15. Solve the following radical equation for x algebraically. Check your solution for extraneous roots. *(2 marks)*

$$0 = \frac{1}{2}\sqrt{(x+2)} - 1$$

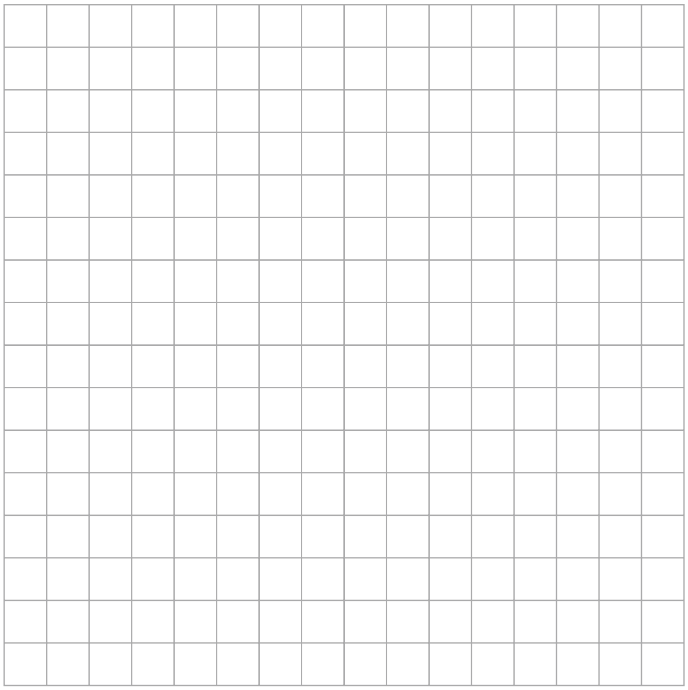
16. Graph the following function. Pay attention to whether the graph should have a point of discontinuity or a vertical asymptote. *(3 marks)*

$$y = \frac{4x}{x-1}$$

Name: ____

17. Graph the following function. Pay attention to whether the graph should have a point of discontinuity or a vertical asymptote. *(5 marks)*

$$y = \frac{x+1}{x^2 - 4x - 5}$$



18. Solve for the variable in the equation $\log_{\sqrt{2}} 64 = x$. *(2 marks)*

19. Solve the exponential equation. Round the final answer to the nearest thousandth. *(3 marks)*

$$e^{3x+2} = 5^{x+1}$$

Name: ____

20. Solve the following equations. Your answer should be exact, whenever possible. Otherwise, round to two decimal places.

   a) $5(3^x) = e^{x-1}$ *(3 marks)*

   b) $\log_2 (x - 4) + \log_2 (x - 3) = 1$ *(3 marks)*

21. A bacteria culture is growing according to the formula $y = 1000e^{0.6t}$, where t is the time in days.

   a) Determine the number of bacteria after 7 days. *(1 mark)*

   b) How long will it take the bacteria culture to triple in size? *(3 marks)*

22. Determine all of the angles that are coterminal with $\frac{2\pi}{3}$ over the domain $[-2\pi, 4\pi]$. *(2 marks)*

Name: ____

23. Determine all of the angles that are coterminal with −261° in general form. *(1 mark)*

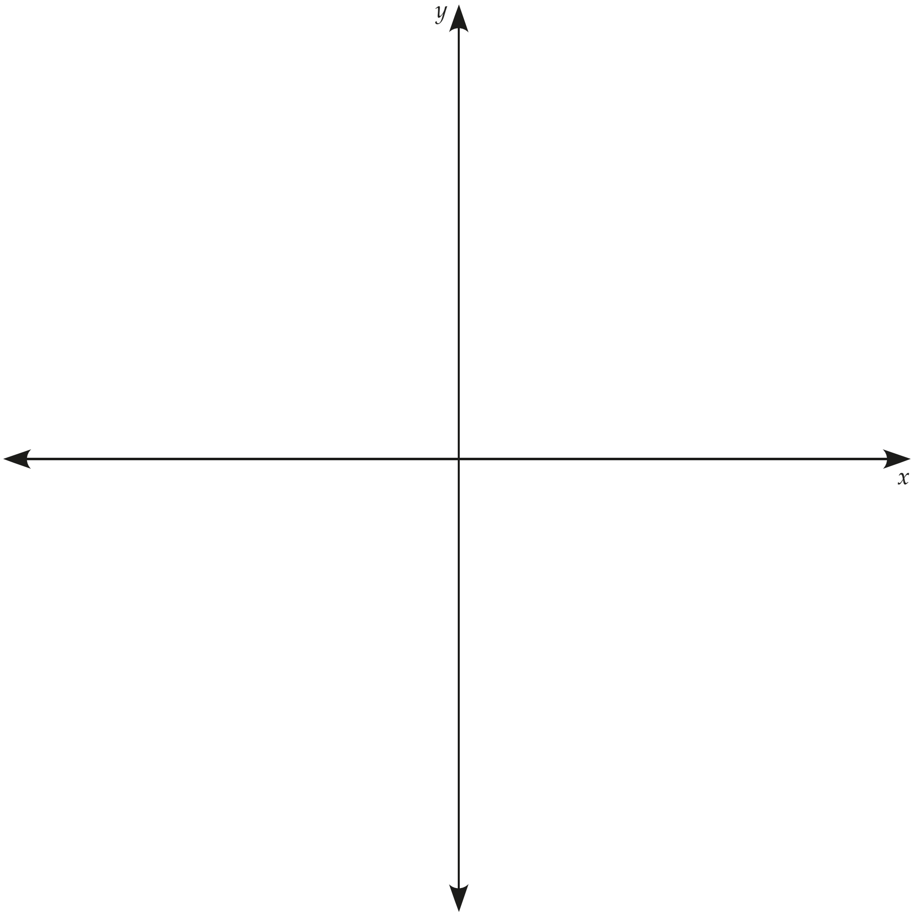
24. Determine the exact value of each of the following expressions. State the coterminal values, where necessary. Show all work. *(2 × 2 marks each = 4 marks)*

    a) $\sec\left(-\frac{2\pi}{3}\right)$

    b) $\cot(630^\circ)$

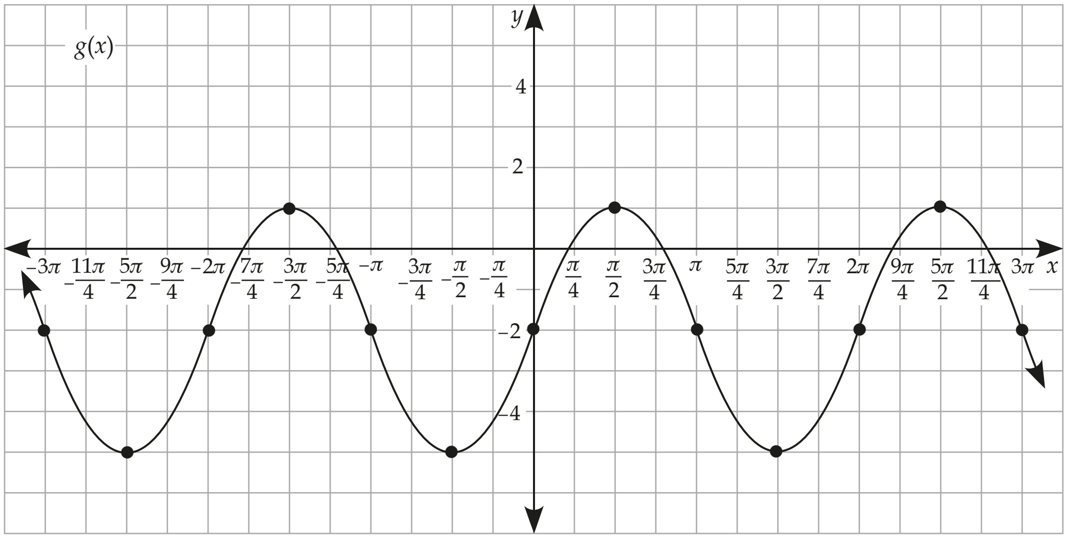
25. State the amplitude, phase shift, period, domain, range, and y-intercept of the function. Sketch the following curve. *(6 marks)*

$$y = -\frac{1}{2}\sin\left(\frac{\pi}{2}x\right) - 1$$



Name: ____

26 Find the equation of the following graph as a cosine function. *(2 marks)*



27. Solve the following equations over the indicated intervals. Provide exact answers wherever possible. Write answers as exact values.

a) $2\sin^2\theta + 3\cos\theta = 3$, where $-2\pi \leq \theta \leq 0$ *(5 marks)*

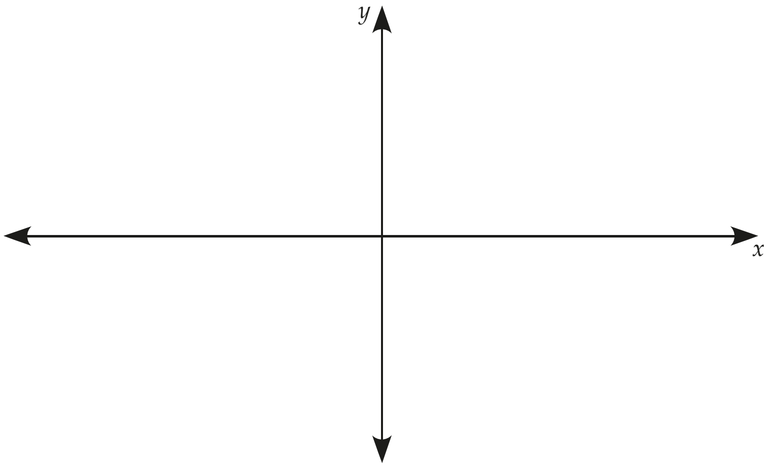
b) $6\sin^2 x + 5\sin x + 1 = 0$, where $0 \leq x < 2\pi$. *(4 marks)*

Name: ______________________________________

28 Explain the difference between a trigonometric identity and a trigonometric equation. *(1 mark)*

29 Consider the equation $\sin(2x) = -\cos\left(2\left(x + \frac{\pi}{4}\right)\right)$.

a) Graph $y = \sin(2x)$ and $y = -\cos\left(2\left(x + \frac{\pi}{4}\right)\right)$ on the coordinate grids below. *(4 marks)*



b) Using the graph you created in (b), do you believe this demonstrates an identity? *(1 mark)*

30. Prove the identity $\frac{2 \sin x}{\sin 2x} = \sec x$. *(2 marks)*

31 Prove the identity $\cos 2\theta + 2 \sin^2 \theta = 1$. *(2 marks)*

32. Find the exact value of $\frac{\tan 80° + \tan 55°}{1 - \tan 80° \cdot \tan 55°}$. *(2 marks)*

GRADE 12 PRE-CALCULUS MATHEMATICS (40S)

Final Practice Exam
Answer Key

Grade 12 Pre-Calculus Mathematics

Final Practice Exam
Answer Key

Answer Key

Name: ______________________________

Student Number: ______________________________

Attending ☐ Non-Attending ☐

Phone Number: ______________________________

Address: ______________________________

____

____

For Marker's Use Only

Date: ______________________________

Final Mark: ________ /100 = ________ %

Comments:

Instructions

The final examination will be weighted as follows:

Modules 1–8	100%

Time allowed: 3 hours

Note: You are allowed to bring the following to the exam: pens/pencils (2 or 3 of each), blank paper, a ruler, a scientific calculator, and your Final Exam Resource Sheet. Your Final Exam Resource Sheet must be handed in with the exam.

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General Marking Principles

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- Errors are followed through (e.g., if an arithmetic error is made in the first line, it is still possible for the student to receive nearly full marks).
- Many types of communication errors receive a 0.5 mark deduction, but 0.5 mark is the maximum communication error deduction for the entire exam.

Name: ____

Answer all questions to the best of your ability. Show all your work.

Long-Answer Questions *(100 marks)*

1. Given that $f(x)=\left|\frac{1}{x}\right|$ and $g(x) = x^3 + 6x - 3$, find the following.

   (2 × 1 mark each = 2 marks) (Module 2, Lesson 5)

   a) $f(f(x))$

   Answer:

   $$f(f(x))=\left|\frac{1}{\left|\frac{1}{x}\right|}\right|$$

   b) $g(f(-2))$

   Answer:

   $$g(f(-2))=g\left(\frac{1}{2}\right)=\frac{1}{8}$$

2. Divide, using long division or synthetic division, and write in the form given by the division algorithm. *(3 marks)* (Module 4, Lesson 2)

   $(-x^3 - 4x^2 + 7x + 4) \div (x + 3)$

   Answer:

   $$\begin{array}{r|rrr|r} -3 & -1 & -4 & 7 & 4 \\ & & 3 & 3 & -30 \\ \hline & -1 & -1 & 10 & -26 \end{array}$$

   (1 mark for set up)

   (1 mark for division)

   $\therefore -x^3 - 4x^2 + 7x + 4 = \left(-x^2 - x + 10\right)\left(x + 3\right) - 26$ *(1 mark for division statement)*

3. Rewrite $2\log x + \frac{1}{2}\log 5 - \frac{1}{3}\log\left(x + 2\right)$ as a single logarithm statement. *(3 marks)*

   (Module 7, Lesson 2)

   Answer:

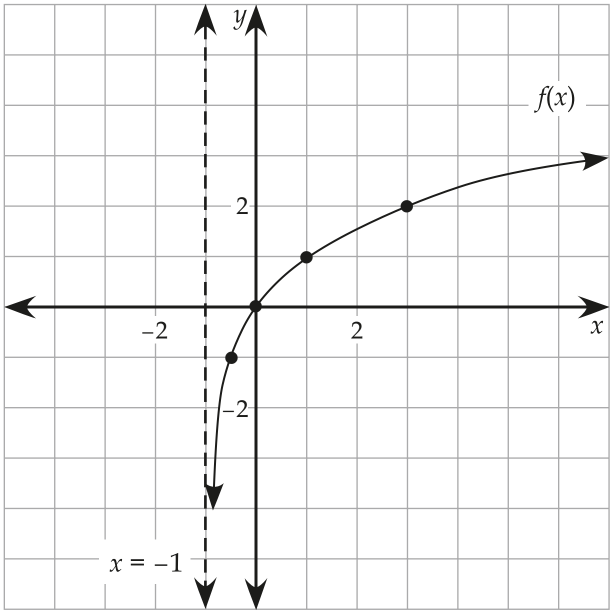
   $\log x^2 + \log\sqrt{5} - \log\left(\sqrt[3]{x + 2}\right)$

   $= \log\dfrac{x^2\sqrt{5}}{\sqrt[3]{x + 2}}$

   (1 mark for product law)
   (1 mark for quotient law)
   (1 mark for power law)

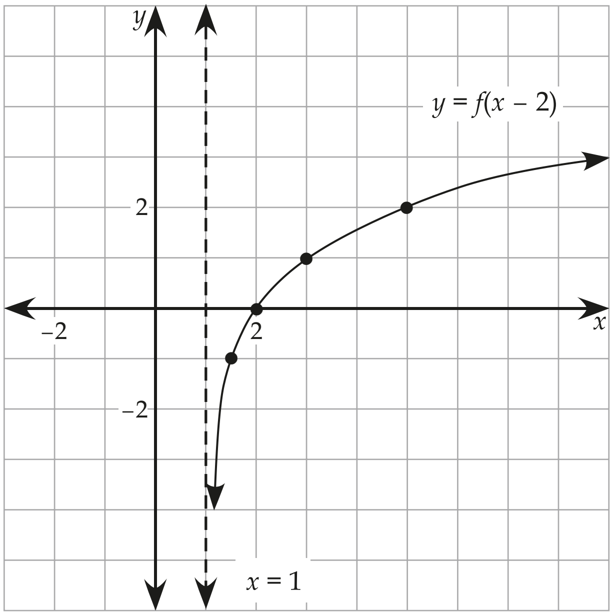
Name: ____

4. Given the sketch of $f(x)$ drawn below, sketch the following functions.



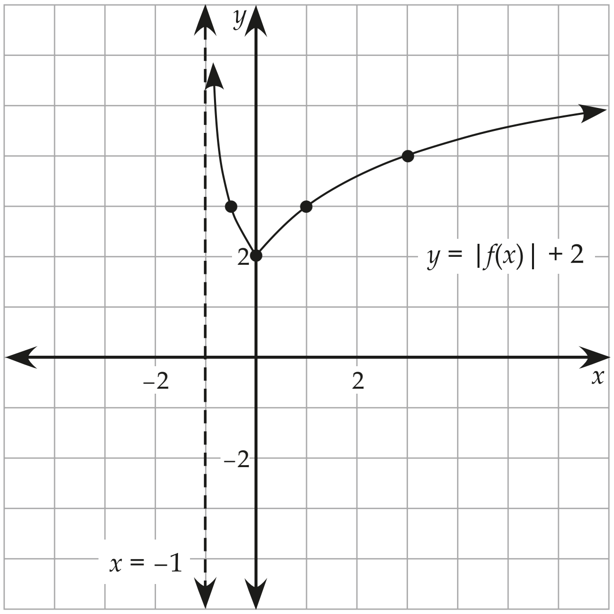
a) $y = f(x - 2)$ *(1 mark)* (Module 2, Lesson 1)

Answer:



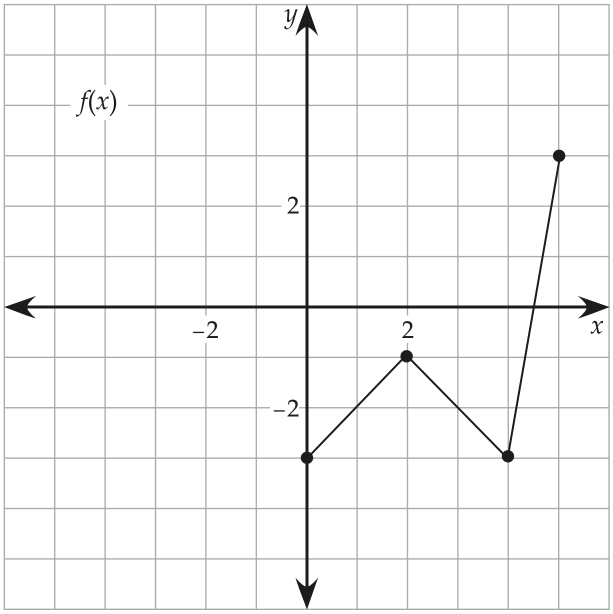
b) $y = |f(x)| + 2$ *(1 mark)* (Module 2, Lesson 5)

Answer:



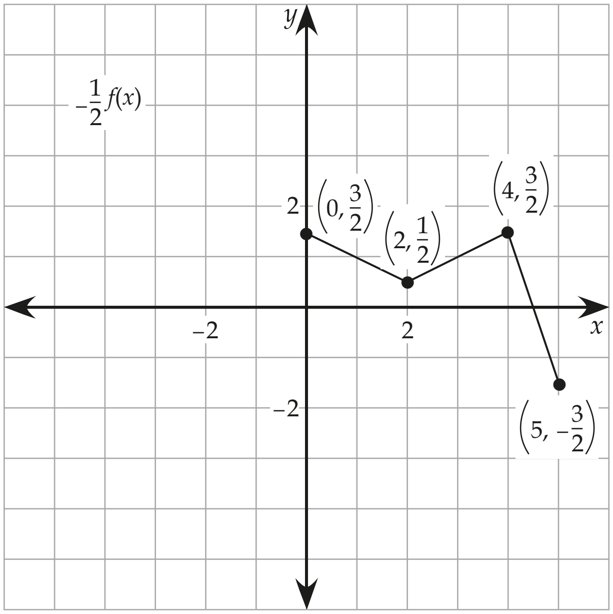
Name: ____

5. Using the sketch of $f(x)$, sketch the following. Express the transformation algebraically or in words.



a) $y=-\frac{1}{2}f(x)$ *(2 marks)* (Module 3, Lesson 1)

Answer:



Algebraically:

$$(x, y) \rightarrow \left(x, -\frac{1}{2}y\right)$$

Words:

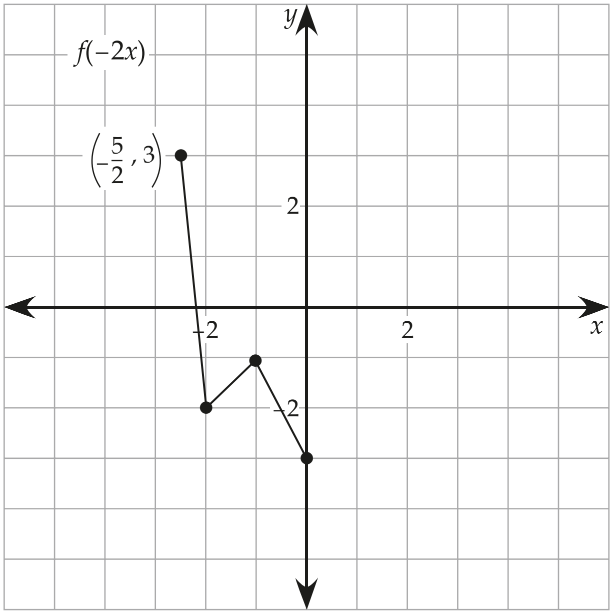
Reflected over x-axis.

Vertical compression by 2.

(1 mark for expressing transformation)
(1 mark for graph)

b) $y = f(-2x)$ *(2 marks)* (Module 3, Lesson 2)

Answer:



Algebraically:

$$(x, y) \rightarrow \left(\frac{x}{-2}, y\right)$$

Words:

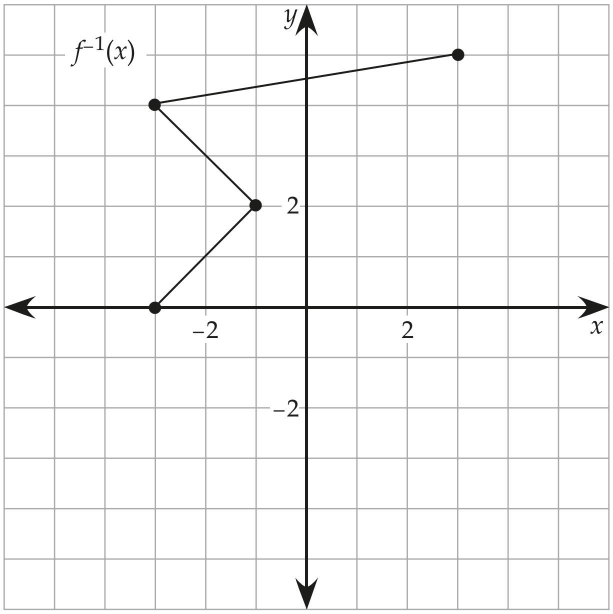
Reflected over y-axis.

Horizontal compression by 2.

(1 mark for expressing transformation)
(1 mark for graph)

c) $y = f^{-1}(x)$ *(2 marks)* (Module 3, Lesson 3)

Answer:



Algebraically:
$(x, y) \rightarrow (y, x)$

Words:
Reflect over line $y = x$.

(1 mark for expressing transformation)
(1 mark for graph)

Name: ____

6. Write and simplify the fifth term of the expansion of $(x + 1)^8$. *(2 marks)* (Module 1, Lesson 5)

Answer:

$$t_5 = \binom{8}{4} x^4 (1)^4 = 70x^4$$

$n = 8$

$k = 4$

$t_5 = {}_8C_4(x)^4(1)^4$

$t_5 = 70x^4$

(0.5 mark for ${}_8C_4$)
(0.5 mark for consistent factors)
(1 mark for consistent answer)

7. There are 9 boys and 11 girls in a Grade 12 English class. In how many ways can 5 students be chosen for a group project if the group must have 3 female members and 2 male members? *(2 marks)* (Module 1, Lesson 4)

Answer:

The committee must have 3 female members and 2 male members.

$$\binom{9}{2}\binom{11}{3} = 5940$$

There are 5940 possible group combinations for this project.

${}_9C_2 \times {}_{11}C_3$

5940 ways

(0.5 mark for ${}_9C_2$)
(0.5 mark for ${}_{11}C_3$)
(1 mark for multiplying the cases)

8. Convert 1265° to radians. Write the exact answer. *(1 mark)* (Module 5, Lesson 1)

   Answer:

$$(1265^\circ)\left(\frac{\pi}{180}\right) = \frac{1265\pi}{180} = \frac{253\pi}{36}$$

9. You know that $\sin \alpha = -\frac{2}{7}$ and $\pi < \alpha < \frac{3\pi}{2}$. You also know that P($\beta$) is in Quadrant IV and $\cos \beta = \frac{4}{5}$. Find $\sin(\alpha + \beta)$.*(3 marks)* (Module 6, Lesson 4)

   Answer:

   The angle that satisfies these requirements is located in Quadrant III.

$$\sin \alpha = -\frac{2}{7} \text{ and } \pi < \alpha < \frac{3\pi}{2} \text{ means } \mathrm{P}(\alpha) \text{ is in Quadrant III}$$

$$\therefore \cos \alpha < 0$$

$$\sin^2 \alpha + \cos^2 \alpha = 1$$

$$\left(-\frac{2}{7}\right)^2 + \cos^2 \alpha = 1$$

$$\cos^2 \alpha = \frac{45}{49}$$

$$\cos \alpha = -\frac{3\sqrt{5}}{7}$$

(0.5 mark for value)
(0.5 mark for sign consistent with quadrant)

$$\cos \beta = \frac{4}{5} \text{ and } \mathrm{P}(\beta) \text{ is in Quadrant IV means } \sin\beta < 0$$

$$\sin^2 \beta + \cos^2 \beta = 1$$

$$\sin^2 \beta + \left(\frac{4}{5}\right)^2 = 1$$

$$\sin^2 \beta = \frac{9}{25}$$

$$\sin \beta = -\frac{3}{5}$$

(0.5 mark for value)
(0.5 mark for sign consistent with quadrant)
(1 mark for solving)

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$= \left(-\frac{2}{7}\right)\left(\frac{4}{5}\right) + \left(-\frac{3\sqrt{5}}{7}\right)\left(-\frac{3}{5}\right) = -\frac{8}{35} + \frac{9\sqrt{5}}{35} = -\frac{8 + 9\sqrt{5}}{35}$$

10. Consider the function $f(x) = -(x - 1)(x + 3)(x - 7)$.

a) Determine the end behaviour of the function. *(1 mark)* (Module 4, Lesson 1)

Answer:

The graph points up to the left and down on the right. In other words, the graph begins in Quadrant II and ends in Quadrant IV.

b) Find all x- and y-intercept(s). *(2 marks)* (Module 4, Lesson 1)

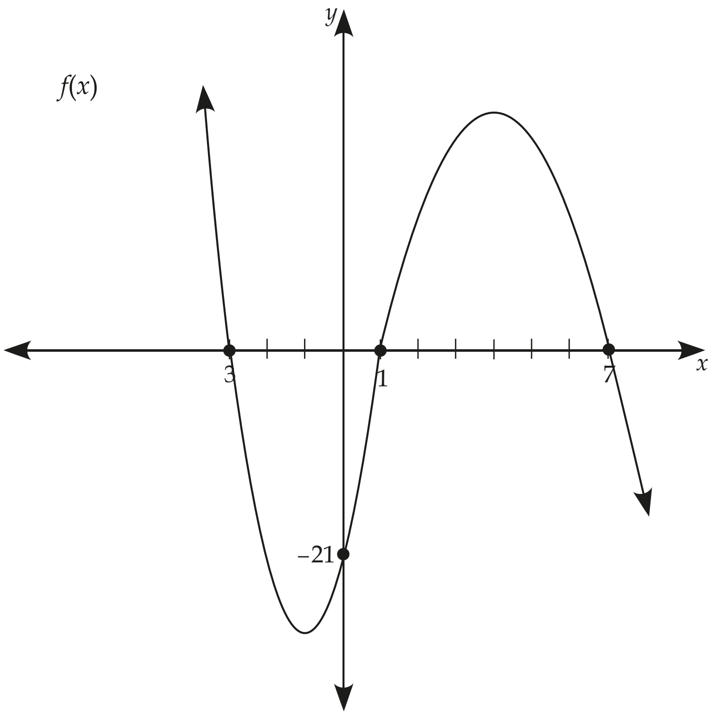
Answer:

$x = \{1, -3, 7\}$ *(1 mark)*

$y = -21$ *(1 mark)*

c) Sketch the function. *(2 marks)* (Module 4, Lesson 4)

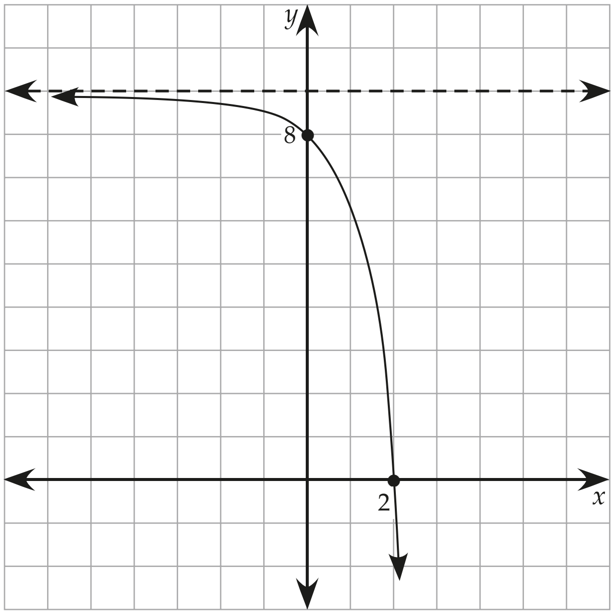
Answer:



(0.5 mark for consistent y-intercept)
(0.5 mark for consistent x-intercepts)
(0.5 mark for correct shape)
(0.5 mark for consistent end behaviour)

11. Sketch the function $g(x) = 9 - 3^x$ and state its range, x-intercept, and equation of asymptote. *(4 marks)* (Module 7, Lesson 1)

Answer:



Range: $(-\infty, 9)$ *(0.5 mark)*

x-intercept: 2 *(1 mark)*

$0 = 9 - 3^x$

$3^x = 9$

$3^x = 3^2$

$x = 2$

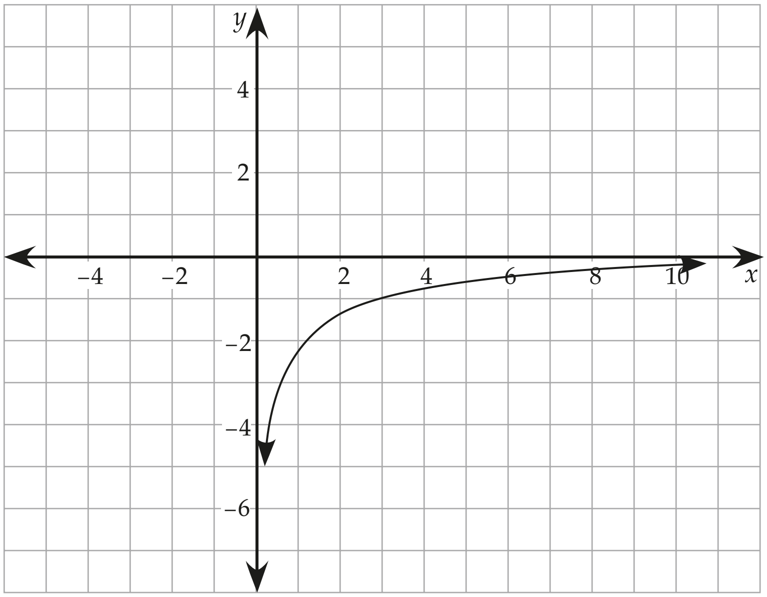
Equation of asymptote: $y = 9$ *(0.5 mark)*

Graph

(0.5 mark for asymptote)
(0.5 mark for shape)
(1 mark for intercepts)

12. Sketch the function $f(x) = \log_3 x - 2$. *(1 mark)* (Module 7, Lesson 4)

Answer:



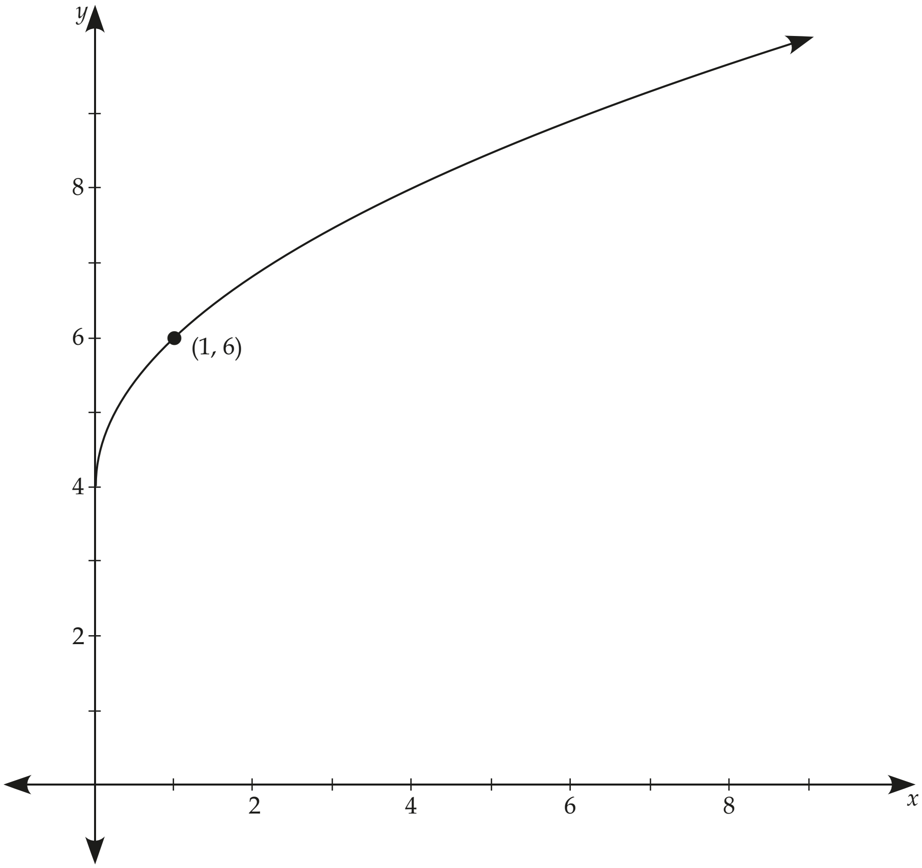
(0.5 mark for asymptote)
(0.5 mark for shape)

Name: ____

13. Graph the following function using transformations. State the domain and range of the function. *(3 marks)* (Module 8, Lesson 1)

$g(x) = 2\sqrt{x} + 4$

Answer:

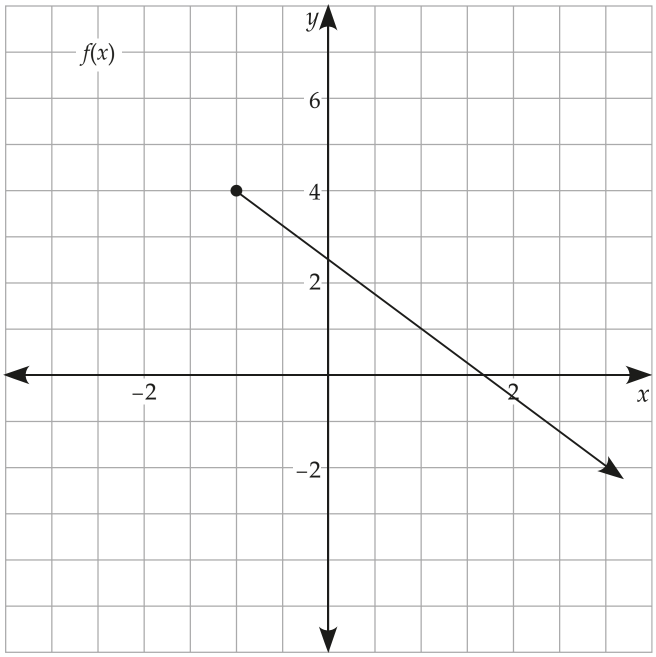


Domain: $\{x \mid x \geq 0\}$

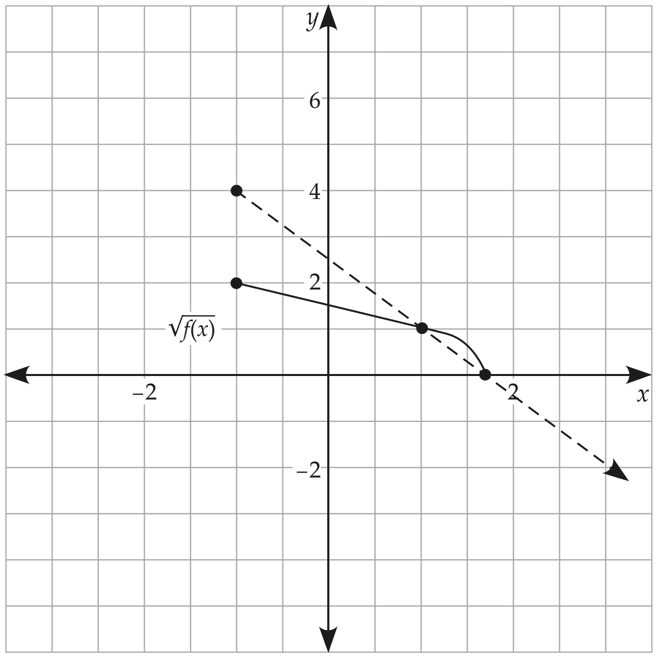
Range: $\{y \mid y \geq 4\}$

(0.5 mark for domain)
(0.5 mark for range)
(0.5 mark for correct radical shape)
(0.5 mark for vertical shift)
(0.5 mark for vertical stretch)
(0.5 mark for end point)

14. Given the following graph of $f(x)$, graph $\sqrt{f(x)}$. *(2 marks)* (Module 8, Lesson 2)



Answer:



(0.5 mark for invariant point)
(0.5 mark for restricted domain)
(0.5 mark for $\sqrt{f(x)} > f(x)$ *when*
$0 < f(x) < 1$)
(0.5 mark for endpoint at $(-1, -2)$)

Name: ____________________

15. Solve the following radical equation for x algebraically. Check your solution for extraneous roots. *(2 marks)* (Module 8, Lesson 3)

$$0 = \frac{1}{2}\sqrt{(x+2)} - 1$$

Answer:

$$0 = \frac{1}{2}\sqrt{(x+2)} - 1$$

$$+1 = \frac{1}{2}\sqrt{(x+2)}$$

$$(2)^2 = \left(\sqrt{(x+2)}\right)^2$$

$$4 = x + 2$$

$$4 - 2 = x$$

$$2 = x$$

Check:

LHS	RHS
0	$\frac{1}{2}\sqrt{(2+2)} - 1$
	$\frac{1}{2}\sqrt{4} - 1$
	$\frac{1}{2}(2) - 1$
	$1 - 1$

LHS = RHS

So, $x = 2$ is a solution.

(0.5 mark for isolating root)
(0.5 mark for squaring both sides)
(0.5 mark for solving)
(0.5 mark for checking for extraneous roots)

16. Graph the following function. Pay attention to whether the graph should have a point of discontinuity or a vertical asymptote. *(3 marks)* (Module 8, Lesson 5)

$$y = \frac{4x}{x - 1}$$

Answer:

Non-permissible value at $x = 1$

The non-permissible value at $x = 1$ represents a vertical asymptote.

Horizontal asymptote at $y = 4$.

Find points on either side of $x = 1$.

$x = 2$

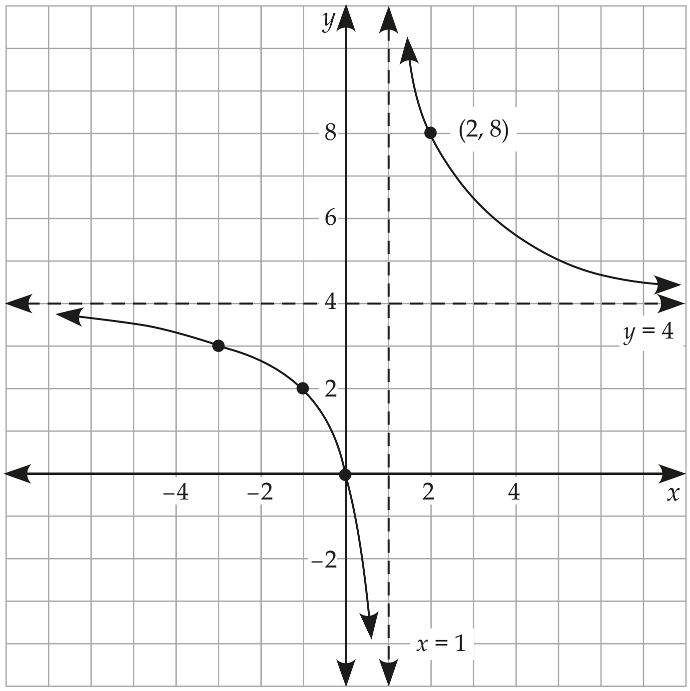
$$y = \frac{4(2)}{2 - 1} = 8$$

Point: (2, 8)

$x = 0$

$$y = \frac{4 \cdot 0}{0 - 1} = 0$$

Point: (0, 0)



(1 mark for vertical asymptote)
(1 mark for correct horizontal asymptote)
(0.5 mark for graph right of asymptote with a point)
(0.5 mark for graph left of asymptote with a point)

Name: ____

17. Graph the following function. Pay attention to whether the graph should have a point of discontinuity or a vertical asymptote. *(5 marks)* (Module 8, Lesson 5)

$$y = \frac{x+1}{x^2 - 4x - 5}$$

Answer:

$$y = \frac{x+1}{(x+1)(x-5)}$$

Non-permissible values at $x = -1$ and $x = +5$.

Since $x + 1$ is also in the numerator, $x = -1$ represents a point of discontinuity, which occurs at $y = -1$.

$$y = \frac{1}{(-1) - 5} = -\frac{1}{6}$$ *(1 mark for simplifying)*

$\therefore$ point of discontinuity is $\left(-1, -\frac{1}{6}\right)$.

$x = 5$ is vertical asymptote.

The horizontal asymptote is $y = 0$, since the degree of the denominator is larger than the numerator.

Graph $y = \frac{1}{x-5}$ with a hole at $\left(-1, -\frac{1}{6}\right)$.

Points on each side of asymptote:

$x = 6$	$y = 1$
$x = 4$	$y = -1$

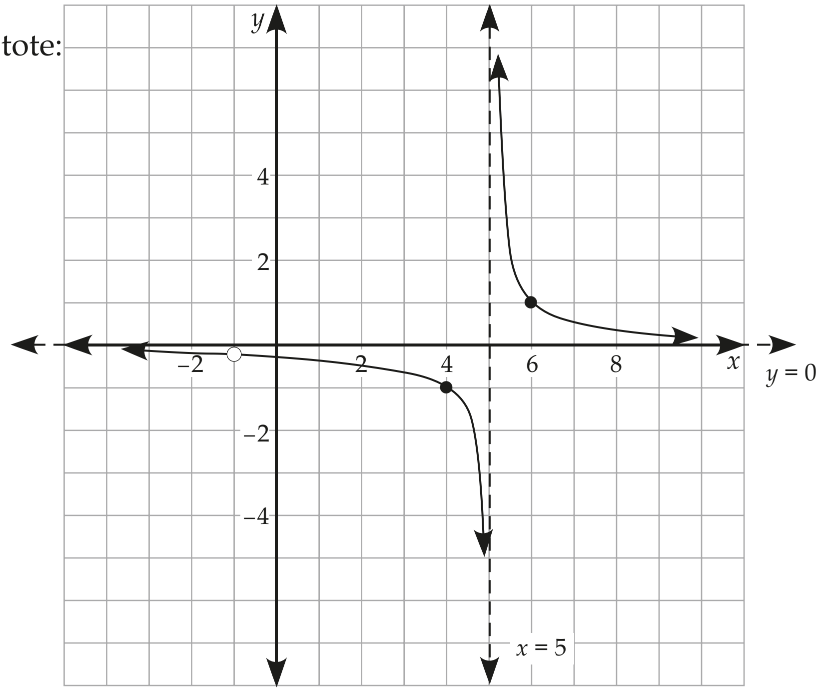
(1 mark for vertical asymptote)

(1 mark for horizontal asymptote)

(1 mark for point $\left(-1, -\frac{1}{6}\right)$ *as a point of discontinuity)*

(0.5 mark for graph right of vertical asymptote)

(0.5 mark for graph left of vertical asymptote)



18. Solve for the variable in the equation $\log_{\sqrt{2}} 64 = x$. *(2 marks)* (Module 7, Lesson 2)

Answer:

$$\left(\sqrt{2}\right)^x = 64$$

$$2^{\frac{1}{2}x} = 2^6$$

$$\frac{1}{2}x = 6$$

$$x = 12$$

(0.5 mark for in exponential form)
(1 mark for base 2)
(0.5 mark for solution)

19. Solve the exponential equation. Round the final answer to the nearest thousandth. *(3 marks)* (Module 7, Lesson 5)

$e^{3x+2} = 5^{x+1}$

Answer:

$$\ln e^{3x+2} = \ln 5^{x+1}$$

$$(3x + 2)\ln e = (x + 1)\ln 5$$

$$3x + 2 = x\ln 5 + \ln 5$$

$$3x - x\ln 5 = \ln 5 - 2$$

$$x(3 - \ln 5) = \ln 5 - 2$$

$$x = \frac{\ln 5 - 2}{3 - \ln 5}$$

$$x = -0.280866\ldots$$

$$x = -0.281$$

(0.5 mark for log both sides)
(1 mark for power law)
(0.5 mark for collecting like terms)
(0.5 mark for isolating x)
(0.5 mark for solving)

Name: ____

20. Solve the following equations. Your answer should be exact, whenever possible. Otherwise, round to two decimal places.

a) $5(3^x) = e^{x-1}$ *(3 marks)* (Module 7, Lesson 5)

Answer:

$\ln\left(5 \cdot 3^x\right) = \ln\left(e^{x-1}\right)$ *(0.5 mark for applying logs to both sides)*

$\ln 5 + x \ln 3 = (x-1)\ln e$ *(1 mark for Product Log Law; 1 mark for Power Log Law)*

$\ln 5 + x \ln 3 = x \ln e - \ln e$

$x \ln 3 - x \ln e = -\ln e - \ln 5$ *(0.5 mark for collecting like terms)*

$x(\ln 3 - 1) = -1 - \ln 5$ *(0.5 mark for isolating for x)*

$x = \frac{-1-\ln 5}{\ln 3 - 1}$ *(0.5 mark for correct evaluation of a quotient of logs)*

$x = \frac{-2.609437912}{0.098612288}$

$x = -26.46$

b) $\log_2 (x - 4) + \log_2 (x - 3) = 1$ *(3 marks)* (Module 7, Lesson 6)

Answer:

$\log_2 \left[(x-4)(x-3)\right] = 1$ *(1 mark for writing as a single log)*

$x^2 - 7x + 12 = 2^1$ *(0.5 mark for exponential form)*

$x^2 - 7x + 10 = 0$

$(x-5)(x-2) = 0$

$x = 2 \qquad x = 5$ *(1 mark for solving the equation)*

$x = 2$ is extraneous *(0.5 mark for rejecting extraneous root)*

$\therefore x = 5$

21. A bacteria culture is growing according to the formula $y = 1000e^{0.6t}$, where t is the time in days.

a) Determine the number of bacteria after 7 days. *(1 mark)* (Module 7, Lesson 7)

Answer:

$y = 1000e^{0.6(7)} = 1000e^{4.2} = 66\ 686$ bacteria

b) How long will it take the bacteria culture to triple in size? *(3 marks)* (Module 7, Lesson 7)

Answer:

$$3x = xe^{0.6t}$$
$$3 = e^{0.6t}$$
$$\ln 3 = \ln e^{0.6t}$$
$$\ln 3 = 0.6t(\ln e)$$
$$t = \frac{\ln 3}{0.6} = 1.83 \text{ days}$$

(0.5 mark for substitution)
(0.5 mark for log of both sides)
(1 mark for Power Law)
(0.5 mark for isolating t)
(0.5 mark for evaluating)

22. Determine all of the angles that are coterminal with $\frac{2\pi}{3}$ over the domain $[-2\pi, 4\pi]$.

(2 marks) (Module 5, Lesson 1)

Answer:

$$\frac{2\pi}{3} - \frac{6\pi}{3} = -\frac{4\pi}{3}$$
$$\frac{2\pi}{3} + \frac{6\pi}{3} = \frac{8\pi}{3}$$

The two angles that are coterminal with $\frac{2\pi}{3}$ over the domain $[-2\pi, 4\pi]$ are

$-\frac{4\pi}{3}$ and $\frac{8\pi}{3}$.

Name: ______________________________________

23. Determine all of the angles that are coterminal with −261° in general form. *(1 mark)* (Module 5, Lesson 1)

Answer:

$-261° + 360°n,\ n \in I$

24. Determine the exact value of each of the following expressions. State the coterminal values, where necessary. Show all work. *(2 × 2 marks each = 4 marks)* (Module 5, Lesson 4)

a) $\sec\left(-\frac{2\pi}{3}\right)$

Answer:

$$\sec\left(-\frac{2\pi}{3}\right) = \sec\left(\frac{4\pi}{3}\right) = \frac{1}{\cos\left(\frac{4\pi}{3}\right)} = \frac{1}{-\frac{1}{2}} = -2$$

(1 mark for coterminal angle)
(1 mark for exact answer)

b) $\cot(630°)$

Answer:

$$\cot(630°) = \cot(270°) = \frac{\cos(270°)}{\sin(270°)} = \frac{0}{-1} = 0$$

(1 mark for coterminal angle)
(1 mark for exact answer)

25. State the amplitude, phase shift, period, domain, range, and y-intercept of the function. Sketch the following curve. *(6 marks)* (Module 5, Lesson 6)

$$y = -\frac{1}{2}\sin\left(\frac{\pi}{2}x\right) - 1$$

Answer:

Amplitude: $\frac{1}{2}$

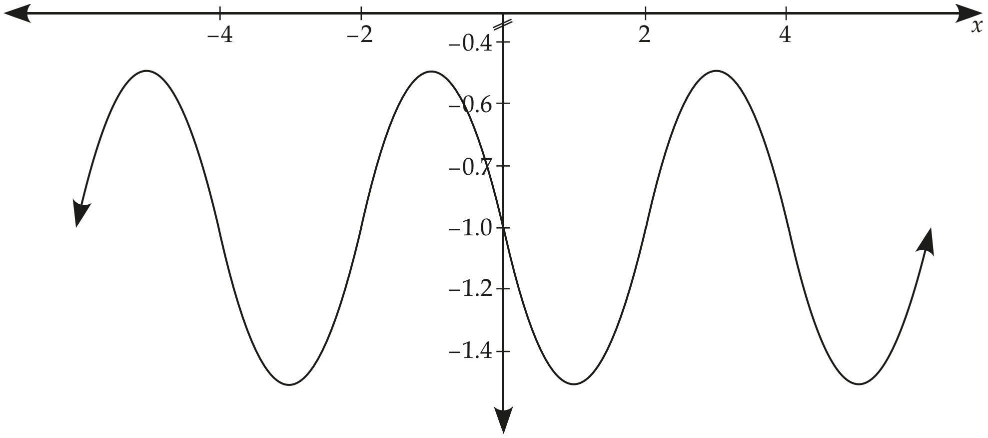
Phase Shift: none

Period: $\frac{2\pi}{\left(\frac{\pi}{2}\right)} = 2\pi\left(\frac{2}{\pi}\right) = 4$

Domain: $(-\infty, \infty)$

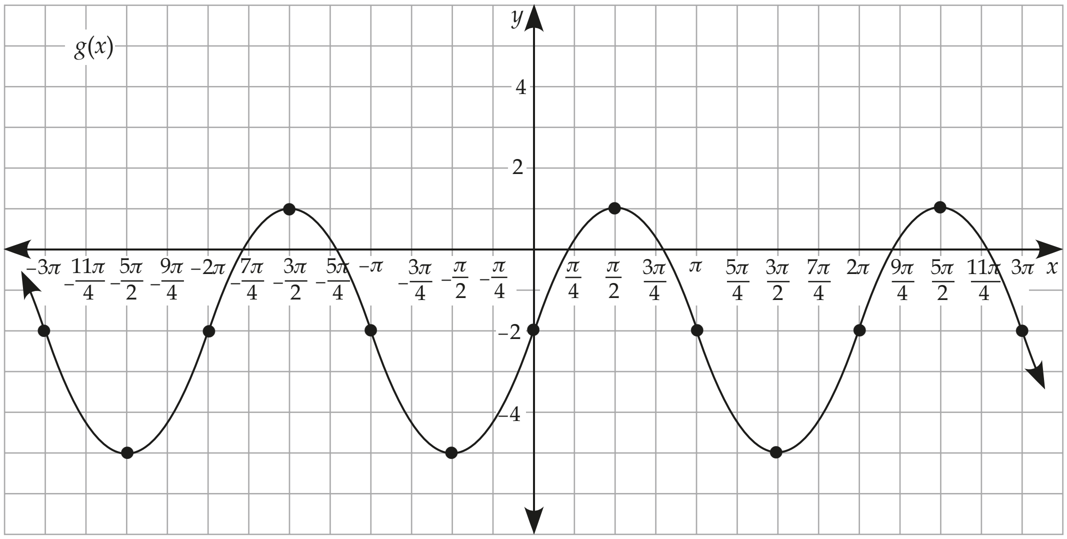
Range: $[-1.5, -0.5]$

y-intercept: -1



(1 mark for sinusoidal shape)
(0.5 mark for correct amplitude)
(1 mark for correct period)
(0.5 mark for phase shift)
(0.5 mark for domain)
(1 mark for range)
(0.5 mark for y-intercept)
(1 mark for graph consistent with properties)

26 Find the equation of the following graph as a cosine function. *(2 marks)* (Module 5, Lesson 6)



Answer:

Here is one possible answer for a trigonometric equation of the form $y = A\cos(B(x - C)) + D$ (the value of C could be $-\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}$, etc.):

$$g(x) = 3\cos\left(x - \frac{\pi}{2}\right) - 2$$

(0.5 mark for correct value of A)
(1 mark for correct value of C)
(0.5 mark for correct value of D)

27. Solve the following equations over the indicated intervals. Provide exact answers wherever possible. Write answers as exact values.

a) $2 \sin^2 \theta + 3 \cos \theta = 3$, where $-2\pi \leq \theta \leq 0$ *(5 marks)* (Module 6, Lesson 3)

Answer:

Note: If final answers are correct but related angle is **not** stated, full marks will still be awarded.

$$2\left(1 - \cos^2\theta\right) + 3 \cos \theta = 3$$

$$2 - 2 \cos^2\theta + 3 \cos \theta - 3 = 0$$

$$-2 \cos^2 \theta + 3 \cos \theta - 1 = 0$$

$$2 \cos^2 \theta - 3 \cos \theta + 1 = 0$$

$$\left(2 \cos \theta - 1\right)\left(\cos \theta - 1\right) = 0$$

$\cos \theta = \frac{1}{2}$ $\quad$ $\cos \theta = 1$

(1 mark for identity)
(1 mark for simplification)
(1 mark for factor and solve)

Related: $\theta_r = \frac{\pi}{3}$ or $\theta_r = 0$ *(0.5 mark)*

Final answers: $\theta = -\frac{\pi}{3}$ $\quad$ $\theta = 0$ *(1.5 marks)*

$\theta = -\frac{5\pi}{3}$

b) $6 \sin^2 x + 5 \sin x + 1 = 0$, where $0 \leq x < 2\pi$. *(4 marks)* (Module 6, Lesson 1)

Answer:

Note: If final answers are correct but related angle is **not** stated, full marks will still be awarded.

$$\left(3 \sin x + 1\right)\left(2 \sin x + 1\right) = 0$$

$\sin x = -\frac{1}{3}$ $\quad$ $\sin x = -\frac{1}{2}$

Related arc $= \sin^{-1} = 0.3398$ and $x = \frac{7\pi}{6}, \frac{11\pi}{6}$

$$x = \pi + 0.3398,\ 2\pi - 0.3398,\ \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$x = 3.48,\ 5.94,\ \frac{7\pi}{6}, \frac{11\pi}{6}$$

(1 mark for factor and solve)
(1 mark for related angles)
(2 marks for answer)

Name: ____

28 Explain the difference between a trigonometric identity and a trigonometric equation. *(1 mark)* (Module 6, Lesson 2)

Answer:

A trigonometric equation is only true for certain values of the variable, while a trigonometric identity is true for *all* values of the variable that are in its domain.

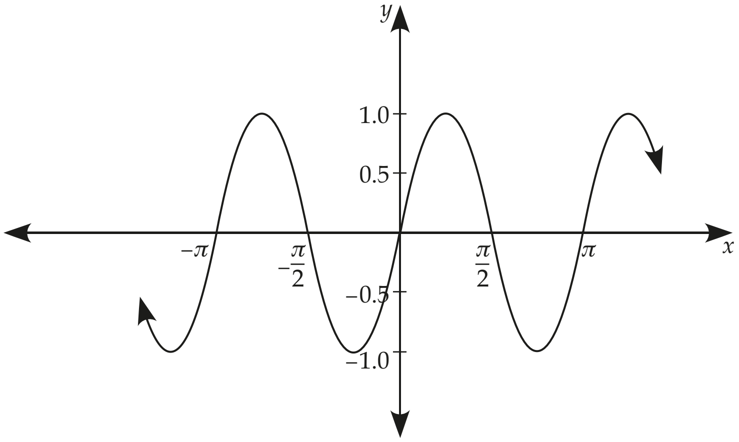
29 Consider the equation $\sin(2x) = -\cos\left(2\left(x + \frac{\pi}{4}\right)\right)$.

a) Graph $y = \sin(2x)$ and $y = -\cos\left(2\left(x + \frac{\pi}{4}\right)\right)$ on the coordinate grids below. *(4 marks)*

(Module 5, Lesson 6 and Module 6, Lesson 5)

Answer:

Note: Both graphs look like the one shown below.



b) Using the graph you created in (b), do you believe this demonstrates an identity? *(1 mark)* (Module 6, Lesson 5)

Answer:

Yes, because the two graphs are identical on the interval graphed.

30. Prove the identity $\dfrac{2\sin x}{\sin 2x} = \sec x$. *(2 marks)* (Module 6, Lesson 5)

Answer:

$$\begin{aligned} \text{LHS} &= \frac{2\sin x}{\sin(2x)} \\ &= \frac{2\sin x}{2\sin x\cos x} \\ &= \frac{1}{\cos x} \\ &= \sec x \\ &= \text{RHS} \end{aligned}$$

(1 mark for identity)
(1 mark for simplification)

31 Prove the identity $\cos 2\theta + 2\sin^2\theta = 1$. *(2 marks)* (Module 6, Lesson 5)

Answer:

$$\begin{aligned} \text{LHS} &= \cos 2\theta + 2\sin^2\theta \\ &= 1 - 2\sin^2\theta + 2\sin^2\theta \\ &= 1 \\ &= \text{RHS} \end{aligned}$$

(1 mark for identity)
(1 mark for simplification)

32. Find the exact value of $\dfrac{\tan 80° + \tan 55°}{1 - \tan 80° \cdot \tan 55°}$. *(2 marks)* (Module 6, Lesson 4)

Answer:

$$\begin{aligned} \tan(80° + 55°) &= \tan 135° \\ &= -1 \end{aligned}$$

(1 mark for identity)
(0.5 mark for simplification)
(0.5 mark for solving)