Grade 11 Pre-Calculus Mathematics (30S)

A Course for Independent Study

Field Validation Version

Manitoba

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A Course for Independent Study

Field Validation Version

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Grade 11 Pre-Calculus Mathematics (30S)

Introduction

Introduction

Overview

Welcome to Grade 11 Pre-Calculus Mathematics! This course is a continuation of the concepts you have studied in previous years, as well as an introduction to new topics. It builds upon the pre-calculus topics you were introduced to in *Grade 10 Introduction to Applied and Pre-Calculus Mathematics*. You will put to use many of the skills that you have already learned to solve problems and learn new skills along the way. This course helps you develop the skills, ideas, and confidence you will need to continue studying math in the future.

As a student enrolled in an independent study course, you have taken on a dual role—that of a student and a teacher. As a student, you are responsible for mastering the lessons and completing the learning activities and assignments. As a teacher, you are responsible for checking your work carefully, noting areas in which you need to improve and motivating yourself to succeed.

What Will You Learn in This Course?

Problem solving, communication, reasoning, and mental math are some of the themes you will discover in each module. You will engage in a variety of activities that promote the connections between symbolic math ideas and the world around you. There are three main areas that you will be exploring: Number, Patterns and Relations, and Shape and Space.

How Is This Course Organized?

The Grade 11 Pre-Calculus Mathematics course consists of the following eight modules:

- Module 1: Sequences and Series
- Module 2: Factoring and Rational Expressions
- Module 3: Quadratic Functions
- Module 4: Solving Quadratic and Rational Equations
- Module 5: Radicals
- Module 6: Systems of Equations and Inequalities
- Module 7: Trigonometry
- Module 8: Absolute Value and Reciprocal Functions

Each module in this course consists of several lessons, which contain the following components:

- **Lesson Focus:** The Lesson Focus at the beginning of each lesson identifies one or more specific learning outcomes (SLOs) that are addressed in the lesson. The SLOs identify the knowledge and skills you should have achieved by the end of the lesson.
- **Introduction:** Each lesson begins by outlining what you will be learning in that lesson.
- **Lesson:** The main body of the lesson consists of the content and processes that you need to learn. It contains information, explanations, diagrams, and completed examples.
- **Learning Activities:** Each lesson has a learning activity that focuses on the lesson content. Your responses to the questions in the learning activities will help you to practise or review what you have just learned. Once you have completed a learning activity, check your responses with those provided in the Learning Activity Answer Key found at the end of the applicable module. Do not send your learning activities to the Distance Learning Unit for assessment.
- **Assignments:** Assignments are found throughout each module within this course. At the end of each module, you will mail or electronically submit all your completed assignments from that module to the Distance Learning Unit for assessment. All assignments combined will be worth a total of 55 percent of your final mark in this course.
- **Summary:** Each lesson ends with a brief review of what you just learned.

There are two online resources for this course.

- Glossary
- Graph Paper

The online resources can be found in the learning management system (LMS). If you do not have access to the Internet, contact the Distance Learning Unit at 1-800-465-9915 to get a copy of these online resources.

What Resources Will You Need for This Course?

You do not need a textbook for this course. All of the content is provided directly within the course.

Required Resources

The **only** required resources for this course are a scientific calculator and graph paper. Graph paper is available as one of the online resources found in the learning management system (LMS).

You will also require access to an email account if you plan to

- communicate with your tutor/marker by email
- use the learning management system (LMS) to submit your completed assignments

Optional Resources

It would be helpful if you had access to the following resources:

- **A graphing calculator, a freeware graphing program, or a graphing calculator app:** None of these are required when writing either your midterm or final examination.
- **A computer with spreadsheet and graphing capabilities:** Access to a computer with spreadsheet software and graphing capabilities may be helpful to you for exploration and checking your understanding. However, none of these are required or allowed when writing either your midterm or final examination.
- **A computer with Internet access:** Use of the Internet may be suggested as a resource in some places, but if you do not have access to an online computer you can still complete the related learning activities and assignments without it.
- **A photocopier:** With access to a photocopier/scanner, you could make a copy of your assignments before submitting them so that if your tutor/marker wants to discuss an assignment with you over the phone, each of you will have a copy. It would also allow you to continue studying or to complete further lessons while your original work is with the tutor/marker. Photocopying or scanning your assignments will also ensure that you keep a copy in case the originals are lost.

Who Can Help You with This Course?

Taking an independent study course is different from taking a course in a classroom. Instead of relying on the teacher to tell you to complete a learning activity or an assignment, you must tell yourself to be responsible for your learning and for meeting deadlines. There are, however, two people who can help you be successful in this course: your tutor/marker and your learning partner.

Your Tutor/Marker

Tutor/markers are experienced educators who tutor Independent Study Option (ISO) students and mark assignments and examinations. When you are having difficulty with something in this course, contact your tutor/marker, who is there to help you. Your tutor/marker's name and contact information were sent to you with this course. You can also obtain this information in the learning management system (LMS).

Your Learning Partner

A learning partner is someone **you choose** who will help you learn. It may be someone who knows something about mathematics, but it doesn't have to be. A learning partner could be someone else who is taking this course, a teacher, a parent or guardian, a sibling, a friend, or anybody else who can help you. Most importantly, a learning partner should be someone with whom you feel comfortable and who will support you as you work through this course.

Your learning partner can help you keep on schedule with your coursework, read the course with you, check your work, look at and respond to your learning activities, or help you make sense of assignments. You may even study for your examination(s) with your learning partner. If you and your learning partner are taking the same course, however, your assignment work should not be identical.

One of the best ways that your learning partner can help you is by reviewing your midterm and final practice examinations with you. These are found in the learning management system (LMS), along with their answer keys. Your learning partner can administer your practice examination, check your answers with you, and then help you learn the things that you missed.

How Will You Know How Well You Are Learning?

You will know how well you are learning in this course by how well you complete the learning activities, assignments, and examinations.

Learning Activities

The learning activities in this course will help you to review and practise what you have learned in the lessons. You will not submit the completed learning activities to the Distance Learning Unit. Instead, you will complete the learning activities and compare your responses to those provided in the Learning Activity Answer Key found at the end of each module.

Each learning activity has two parts—Part A has BrainPower questions and Part B has questions related to the content in the lesson.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for you before trying the other questions. Each question should be completed quickly and without using a calculator, and most should be completed without using pencil and paper to write out the steps. Some of the questions will relate directly to content of the course. Some of the questions will review content from previous courses—content that you need to be able to answer efficiently.

Being able to do these questions in a few minutes will be helpful to you as you continue with your studies in mathematics. If you are finding it is taking you longer to do the questions, you can try one of the following:

- work with your learning partner to find more efficient strategies for completing the questions
- ask your tutor/marker for help with the questions
- search online for websites that help you practice the computations so you can become more efficient at completing the questions

None of the assignment questions or examination questions will require you to do the calculations quickly or without a calculator. However, it is for your benefit to complete the questions as they will help you in the course. Also, being able to successfully complete the BrainPower exercises will help build your confidence in mathematics. BrainPower questions are like a warm-up you would do before competing in a sporting event.

Part B: Course Content Questions

One of the easiest and fastest ways to find out how much you have learned is to complete Part B of the learning activities. These have been designed to let you assess yourself by comparing your answers with the answer keys at the end of each module. There is at least one learning activity in each lesson. You will need a notebook or loose-leaf pages to write your answers.

Make sure you complete the learning activities. Doing so will not only help you to practise what you have learned, but will also prepare you to complete your assignments and the examinations successfully. Many of the questions on the examinations will be similar to the questions in the learning activities. Remember that you **will not submit learning activities to the Distance Learning Unit**.

Assignments

Lesson assignments are located throughout the modules, and include questions similar to the questions in the learning activities of previous lessons. The assignments have space provided for you to write your answers on the question sheets. **You need to show all your steps as you work out your solutions, and make sure your answers are clear (include units, where appropriate).**

Once you have completed all the assignments in a module, you will submit them to the Distance Learning Unit for assessment. The assignments are worth a total of 55 percent of your final course mark. You must complete each assignment in order to receive a final mark in this course. **You will mail or electronically submit these assignments to the Distance Learning Unit along with the appropriate cover page once you complete each module.**

The tutor/marker will mark your assignments and return them to you. Remember to keep all marked assignments until you have finished the course so that you can use them to study for your examinations.

Resource Sheet

When you write your midterm and final examinations, you will be allowed to take an Examination Resource Sheet with you into the examination. This sheet will be one letter-sized page, 8½ " by 11", with both sides in your handwriting or typewritten. It is to be submitted with your examination. The Examination Resource Sheet is not worth any marks.

Creating your own resource sheet is an excellent way to review. It also provides you with a convenient reference and quick summary of the important facts of each module. Each student is asked to complete a resource sheet for each module to help with studying and reviewing.

The lesson summaries are written for you to use as a guide, as are the module summaries at the end of each module. Refer to these when you create your own resource sheet. Also, refer to the online glossary found in the learning management system (LMS) to check the information on your resource sheet.

After completing each module's resource sheet, you should summarize the sheets from all of the modules to prepare your Examination Resource Sheet. When preparing your Midterm Examination Resource Sheet, remember that your midterm examination is based on Modules 1 to 4. When preparing your Final Examination Resource Sheet, remember that your final examination is based on the entire course, Modules 1 to 8.

Midterm and Final Examinations

This course contains a midterm examination and a final examination.

- The **midterm examination** is based on Modules 1 to 4, and is worth 20 percent of your final course mark. You will write the midterm examination when you have completed Module 4.
- The **final examination** is cumulative, so it is based on Modules 1 to 8 and is worth 25 percent of your final course mark. The examination content will consist of 20% from Modules 1 to 4 and 80% from Modules 5 to 8. You will write the final examination when you have completed Module 8.

The two examinations are worth a total of 45 percent of your final course mark. You will write both examinations under supervision.

In order to do well on the examinations, you should review all of the work that you have completed from Modules 1 to 4 for your midterm examination and Modules 1 to 8 for your final examination, including all learning activities and assignments. You can use your Examination Resource Sheet to bring any formulas and other important information into the examination with you.

You will be required to bring the following supplies when you write both examinations: pens/pencils (2 or 3 of each), blank paper, a ruler, a scientific calculator or graphing calculator, and your Examination Resource Sheet. Both examinations are 2.5 hours in duration.

Practice Examinations and Answer Keys

To help you succeed in your examinations, you will have an opportunity to complete a Midterm Practice Examination and a Final Practice Examination. These examinations, along with the answer keys, are found in the learning management system (LMS). If you do not have access to the Internet, contact the Distance Learning Unit at 1-800-465-9915 to obtain a copy of the practice examinations.

These practice examinations are similar to the actual examinations you will be writing. The answer keys enable you to check your answers. This will give you the confidence you need to do well on your examinations.

Requesting Your Examinations

You are responsible for making arrangements to have the examinations sent to your proctor from the Distance Learning Unit. Please make arrangements before you finish Module 4 to write the midterm examination. Likewise, you should begin arranging for your final examination before you finish Module 8.

To write your examinations, you need to make the following arrangements:

- **If you are attending school,** your examination will be sent to your school as soon as all the applicable assignments have been submitted. You should make arrangements with your school's ISO school facilitator to determine a date, time, and location to write the examination.

- **If you are not attending school,** check the Examination Request Form for options available to you. Examination Request Forms can be found on the Distance Learning Unit's website, or look for information in the learning management system (LMS). Two weeks before you are ready to write the examination, fill in the Examination Request Form and mail, fax, or email it to

  Distance Learning Unit
  500–555 Main Street
  PO Box 2020
  Winkler MB R6W 4B8
  Fax: 204-325-1719
  Toll-Free Telephone: 1-800-465-9915
  Email: distance.learning@gov.mb.ca

How Much Time Will You Need to Complete This Course?

Learning through independent study has several advantages over learning in the classroom. You are in charge of how you learn and you can choose how quickly you will complete the course. You can read as many lessons as you wish in a single session. You do not have to wait for your teacher or classmates.

From the date of your registration, you have a maximum of **12 months** to complete the course, but the pace at which you proceed is up to you. Read the following suggestions on how to pace yourself.

Chart A: Semester 1

If you want to start this course in September and complete it in January, you can follow the timeline suggested below.

Module	Completion Date
Module 1	Middle of September
Module 2	End of September
Module 3	Middle of October
Module 4	End of October
Midterm Examination	Beginning of November
Module 5	Middle of November
Module 6	End of November
Module 7	Middle of December
Module 8	Middle of January
Final Examination	End of January

Chart B: Semester 2

If you want to start the course in February and complete it in May, you can follow the timeline suggested below.

Module	Completion Date
Module 1	Middle of February
Module 2	End of February
Module 3	Beginning of March
Module 4	Middle of March
Midterm Examination	End of March
Module 5	Beginning of April
Module 6	Middle of April
Module 7	End of April
Module 8	Beginning of May
Final Examination	Middle of May

Chart C: Full School Year (Not Semestered)

If you want to start the course in September and complete it in May, you can follow the timeline suggested below.

Module	Completion Date
Module 1	End of September
Module 2	End of October
Module 3	End of November
Module 4	End of December
Midterm Examination	Middle of January
Module 5	Middle of February
Module 6	Middle of March
Module 7	Beginning of April
Module 8	Beginning of May
Final Examination	Middle of May

Timelines

Do not wait until the last minute to complete your work, since your tutor/marker may not be available to mark it immediately. It may take a few weeks for your tutor/marker to assess your work and return it to you or your school.

If you need this course to graduate this school year, all coursework must be received by the Distance Learning Unit on or before the first Friday in May, and all examinations must be received by the Distance Learning Unit on or before the last Friday in May. Any coursework or examinations received after these deadlines may not be processed in time for a June graduation. Assignments or examinations submitted after these recommended deadlines will be processed and marked as they are received.

When and How Will You Submit Completed Assignments?

When to Submit Assignments

While working on this course, you will submit completed assignments to the Distance Learning Unit eight times. The following chart shows you exactly what assignments you will be submitting at the end of each module.

Submission of Assignments	
Submission	Assignments You Will Submit
1	**Module 1: Sequences and Series** Module 1 Cover Sheet Assignment 1.1: Arithmetic Sequences and Series Assignment 1.2: Geometric Sequences and Series Assignment 1.3: Finding the Sums of Infinite Geometric Series
2	**Module 2: Factoring and Rational Expressions** Module 2 Cover Sheet Assignment 2.1: Factoring Assignment 2.2: Multiplying, Dividing, and Simplifying Rational Expressions Assignment 2.3: Adding and Subtracting Rational Expressions
3	**Module 3: Quadratic Functions** Module 3 Cover Sheet Assignment 3.1: Quadratic Functions and Transformations Assignment 3.2: Graphing Quadratic Functions Using Transformations Assignment 3.3: Completing the Square and Finding the Vertex Assignment 3.4: Applications of Quadratic Functions
4	**Module 4: Solving Quadratic and Rational Equations** Module 4 Cover Sheet Assignment 4.1: Solving Quadratic Equations by Graphing and Taking Square Roots Assignment 4.2: Solving Quadratic Equations Assignment 4.3: The Discriminant and Solving Rational Equations

continued

Submission of Assignments (continued)	
Submission	Assignments You Will Submit
5	**Module 5: Radicals** Module 5 Cover Sheet Assignment 5.1: Simplifying Radical Expressions Assignment 5.2: Multiplying and Dividing Radical Expressions Assignment 5.3: Solving Radical Equations
6	**Module 6: Systems of Equations and Inequalities** Module 6 Cover Sheet Assignment 6.1: Solving Systems of Equations Assignment 6.2: Solving Linear Inequalities in Two Variables and Quadratic Inequalities in One Variable Assignment 6.3: Solving Quadratic Inequalities in Two Variables
7	**Module 7: Trigonometry** Module 7 Cover Sheet Assignment 7.1: Trigonometric Ratios, Reference Angles, and Trigonometric Equations Assignment 7.2: Patterns in Trigonometric Ratios and the Sine Law Assignment 7.3: The Cosine Law and the Ambiguous Case
8	**Module 8: Absolute Value and Reciprocal Functions** Module 8 Cover Sheet Assignment 8.1: Absolute Value Assignment 8.2: Absolute Value Equations and Graphing Reciprocal Functions: Part 1 Assignment 8.3: Reciprocal Functions: Part 2

How to Submit Assignments

In this course, you have the choice of submitting your assignments either by mail or electronically.

- **Mail:** Each time you **mail** something, you must include the print version of the applicable Cover Sheet (found at the end of this Introduction). Complete the information at the top of each Cover Sheet before submitting it along with your assignments.
- **Electronic submission:** You do not need to include a cover sheet when submitting assignments electronically.

Submitting Your Assignments by Mail

If you choose to mail your completed assignments, please photocopy/scan all the materials first so that you will have a copy of your work in case your package goes missing. You will need to place the applicable module Cover Sheet and assignment(s) in an envelope, and address it to

Distance Learning Unit
500–555 Main Street
PO Box 2020
Winkler MB R6W 4B8

Your tutor/marker will mark your work and return it to you by mail.

Submitting Your Assignments Electronically

Assignment submission options vary by course. Sometimes assignments can be submitted electronically and sometimes they must be submitted by mail. Specific instructions on how to submit assignments were sent to you with this course. In addition, this information is available in the learning management system (LMS).

If you are submitting assignments electronically, make sure you have saved copies of them before you send them. That way, you can refer to your assignments when you discuss them with your tutor/marker. Also, if the original hand-in assignments are lost, you are able to resubmit them.

Your tutor/marker will mark your work and return it to you electronically.

The Distance Learning Unit does not provide technical support for hardware-related issues. If troubleshooting is required, consult a professional computer technician.

What Are the Guide Graphics For?

Guide graphics are used throughout this course to identify and guide you in specific tasks. Each graphic has a specific purpose, as described below.

Lesson Introduction: The introduction sets the stage for the lesson. It may draw upon prior knowledge or briefly describe the organization of the lesson. It also lists the learning outcomes for the lesson. Learning outcomes describe what you will learn.

Learning Partner: Ask your learning partner to help you with this task.

Learning Activity: Complete a learning activity. This will help you to review or practise what you have learned and to prepare for an assignment or an examination. You will not submit learning activities to the Distance Learning Unit. Instead, you will compare your responses to those provided in the Learning Activity Answer Key found at the end of every module.

Assignment: Complete an assignment. You will submit your completed assignments to the Distance Learning Unit for assessment at the end of a given module.

Mail or Electronic Submission: Mail or electronically submit your completed assignments to the Distance Learning Unit for assessment at this time.

Phone Your Tutor/Marker: Telephone your tutor/marker.

Resource Sheet: Indicates material that may be valuable to include on your resource sheet.

Examination: Write your midterm or final examination at this time.

Note: Take note of and remember this important information or reminder.

Getting Started

Take some time right now to skim through the course material, locate your cover sheets, and familiarize yourself with how the course is organized. Get ready to learn!

Remember: If you have questions or need help at any point during this course, contact your tutor/marker or ask your learning partner for help.

Good luck with the course!

Math Goals

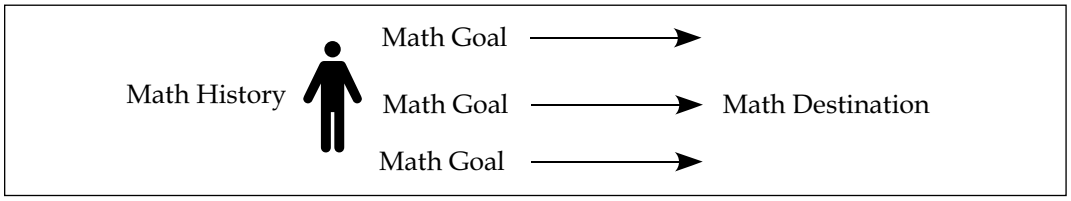
In Module 1, the first learning activity involves you having a conversation with your tutor/marker. Having this conversation with your tutor/marker has two important purposes. First, it introduces you to a very valuable resource – your tutor/marker. He or she is available for you to answer questions, explain concepts, and guide you through this course. You can discuss your math learning and progress. Feel free to contact your tutor/marker by phone or email at anytime during this course.

The second important purpose of this assignment is to get you thinking about your math goals. You may have a future career in mind and this course is getting you one step closer to it by completing a prerequisite for a future required course. There may be specific skills or topics you are interested in learning about and they are covered in this course.

If you are unsure of your math goals or why they are important, consider this:

- goals give you a sense of direction and purpose in taking this course
- goals help motivate you to learn and do your best, even when it's tough
- when you accomplish your goals, there is a great sense of achievement and success

Good goals need to be realistic, specific, and reflect what is important to you. They should give you direction and take you further down the path from where you have been to where you want to go.



Goals can be long term or short term, but they are the pathway that takes **you** from where you were/are, closer to where you want to go.

Notes

Grade 11 Pre-Calculus Mathematics (30S)

Module 1: Sequences and Series
Cover Sheet

Please complete this sheet and place it on top of your assignments to assist in proper recording of your work. Submit the package to:

Drop-off/Courier Address

Distance Learning Unit
555 Main Street
Winkler MB R6W 1C4

Mailing Address

Distance Learning Unit
500–555 Main Street
PO Box 2020
Winkler MB R6W 4B8

Contact Information

Legal Name: _____ Preferred Name: _____

Phone: _____ Email: _____

Mailing Address: _____

City/Town: _____ Postal Code: _____

Attending School: ❑ No ❑ Yes

School Name: _____

Has your contact information changed since you registered for this course? ❑ No ❑ Yes

Note: Please keep a copy of your assignments so that you can refer to them when you discuss them with your tutor/marker.

For Student Use	For Office Use Only	
Module 1 Assignments Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	**Attempt 1** _____ Date Received	**Attempt 2** _____ Date Received
❑ Assignment 1.1: Arithmetic Sequences and Series	_____ /40	_____ /40
❑ Assignment 1.2: Geometric Sequences and Series	_____ /54	_____ /54
❑ Assignment 1.3: Finding the Sums of Infinite Geometric Series	_____ /18	_____ /18
	Total: ____ /112	**Total: ____ /112**
For Tutor/Marker Use		
Remarks:		

Grade 11 Pre-Calculus Mathematics (30S)

Module 2: Factoring and Rational Expressions
Cover Sheet

Please complete this sheet and place it on top of your assignments to assist in proper recording of your work. Submit the package to:

Drop-off/Courier Address

Distance Learning Unit
555 Main Street
Winkler MB R6W 1C4

Mailing Address

Distance Learning Unit
500–555 Main Street
PO Box 2020
Winkler MB R6W 4B8

Contact Information

Legal Name: _____ Preferred Name: _____

Phone: _____ Email: _____

Mailing Address: _____

City/Town: _____ Postal Code: _____

Attending School: ❑ No ❑ Yes

School Name: _____

Has your contact information changed since you registered for this course? ❑ No ❑ Yes

Note: Please keep a copy of your assignments so that you can refer to them when you discuss them with your tutor/marker.

For Student Use	For Office Use Only	
Module 2 Assignments Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	**Attempt 1** _____ Date Received	**Attempt 2** _____ Date Received
❑ Assignment 2.1: Factoring	_____ /37	_____ /37
❑ Assignment 2.2: Multiplying, Dividing, and Simplifying Rational Expressions	_____ /51	_____ /51
❑ Assignment 2.3: Adding and Subtracting Rational Expressions	_____ /30	_____ /30
	Total: ____ /118	**Total: ____ /118**
For Tutor/Marker Use		
Remarks:		

Grade 11 Pre-Calculus Mathematics (30S)

Module 3: Quadratic Functions
Cover Sheet

Please complete this sheet and place it on top of your assignments to assist in proper recording of your work. Submit the package to:

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555 Main Street
Winkler MB R6W 1C4

Mailing Address

Distance Learning Unit
500–555 Main Street
PO Box 2020
Winkler MB R6W 4B8

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School Name: _____

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For Student Use	For Office Use Only	
Module 3 Assignments Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	**Attempt 1** _____ Date Received	**Attempt 2** _____ Date Received
❑ Assignment 3.1: Quadratic Functions and Transformations	_____ /58	_____ /58
❑ Assignment 3.2: Graphing Quadratic Functions Using Transformations	_____ /65	_____ /65
❑ Assignment 3.3: Completing the Square and Finding the Vertex	_____ /38	_____ /38
❑ Assignment 3.4: Applications of Quadratic Functions	_____ /19	_____ /19
	Total: ____ /180	**Total: ____ /180**
For Tutor/Marker Use		
Remarks:		

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Module 4: Solving Quadratic and Rational Functions
Cover Sheet

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Attending School: ❑ No ❑ Yes

School Name: _____

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Note: Please keep a copy of your assignments so that you can refer to them when you discuss them with your tutor/marker.

For Student Use	For Office Use Only	
Module 4 Assignments Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	**Attempt 1** _____ Date Received	**Attempt 2** _____ Date Received
❑ Assignment 4.1: Solving Quadratic Equations by Graphing and Taking Square Roots	_____ /33	_____ /33
❑ Assignment 4.2: Solving Quadratic Equations	_____ /49	_____ /49
❑ Assignment 4.3: The Discriminant and Solving Rational Equations	_____ /47	_____ /47
	Total: ____ /129	**Total: ____ /129**
For Tutor/Marker Use		
Remarks:		

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Module 5: Radicals
Cover Sheet

Please complete this sheet and place it on top of your assignments to assist in proper recording of your work. Submit the package to:

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Attending School: ❑ No ❑ Yes

School Name: _____

Has your contact information changed since you registered for this course? ❑ No ❑ Yes

Note: Please keep a copy of your assignments so that you can refer to them when you discuss them with your tutor/marker.

For Student Use	For Office Use Only	
Module 5 Assignments Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	**Attempt 1** _____ Date Received	**Attempt 2** _____ Date Received
❑ Assignment 5.1: Simplifying Radical Expressions	_____ /42	_____ /42
❑ Assignment 5.2: Multiplying and Dividing Radical Expressions	_____ /50	_____ /50
❑ Assignment 5.3: Solving Radical Equations	_____ /53	_____ /53
	Total: ____ /145	**Total: ____ /145**
For Tutor/Marker Use		
Remarks:		

Grade 11 Pre-Calculus Mathematics (30S)

Module 6: Systems of Equations and Inequalities
Cover Sheet

Please complete this sheet and place it on top of your assignments to assist in proper recording of your work. Submit the package to:

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Note: Please keep a copy of your assignments so that you can refer to them when you discuss them with your tutor/marker.

For Student Use	For Office Use Only	
Module 6 Assignments Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	**Attempt 1** _____ Date Received	**Attempt 2** _____ Date Received
❑ Assignment 6.1: Solving Systems of Equations	_____ /40	_____ /40
❑ Assignment 6.2: Solving Linear Inequalities in Two Variables and Quadratic Inequalities in One Variable	_____ /40	_____ /40
❑ Assignment 6.3: Solving Quadratic Inequalities in Two Variables	_____ /19	_____ /19
	Total: ____ /99	**Total: ____ /99**
For Tutor/Marker Use		
Remarks:		

Grade 11 Pre-Calculus Mathematics (30S)

Module 7: Trigonometry
Cover Sheet

Please complete this sheet and place it on top of your assignments to assist in proper recording of your work. Submit the package to:

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Attending School: ❑ No ❑ Yes

School Name: _____

Has your contact information changed since you registered for this course? ❑ No ❑ Yes

Note: Please keep a copy of your assignments so that you can refer to them when you discuss them with your tutor/marker.

For Student Use	For Office Use Only	
Module 7 Assignments Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	**Attempt 1** _____ Date Received	**Attempt 2** _____ Date Received
❑ Assignment 7.1: Trigonometric Ratios, Reference Angles, and Trigonometric Equations	_____ /58	_____ /58
❑ Assignment 7.2: Patterns in Trigonometric Ratios and the Sine Law	_____ /25	_____ /25
❑ Assignment 7.3: The Cosine Law and the Ambiguous Case	_____ /41	_____ /41
	Total: ____ /124	**Total: ____ /124**
For Tutor/Marker Use		
Remarks:		

Grade 11 Pre-Calculus Mathematics (30S)

Module 8: Absolute Value and Reciprocal Functions
Cover Sheet

Please complete this sheet and place it on top of your assignments to assist in proper recording of your work. Submit the package to:

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Attending School: ❑ No ❑ Yes

School Name: _____

Has your contact information changed since you registered for this course? ❑ No ❑ Yes

Note: Please keep a copy of your assignments so that you can refer to them when you discuss them with your tutor/marker.

For Student Use	For Office Use Only	
Module 8 Assignments Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	**Attempt 1** _____ Date Received	**Attempt 2** _____ Date Received
❑ Assignment 8.1: Absolute Value	_____ /42	_____ /42
❑ Assignment 8.2: Absolute Value Functions and Graphing Reciprocal Functions: Part 1	_____ /58	_____ /58
❑ Assignment 8.3: Reciprocal Functions: Part 2	_____ /20	_____ /20
	Total: ____ /120	**Total: ____ /120**
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Remarks:		

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Grade 11 Pre-Calculus Mathematics (30S)

Module 1
Sequences and Series

Module 1:
Sequences and Series

Introduction

In previous math courses, you have dealt with patterns in the context of graphs. In this module, you will be dealing with patterns a little differently. Patterns show up all around you in the natural world. The petals in flowers very often appear in a pattern. Pine cones form according to a pattern called a Fibonacci sequence.

Patterns also appear in lists of numbers. Whenever a pattern appears in a list of numbers, it is called a *sequence*. Sequences have many applications. An automobile will depreciate in value according to a sequence. Real estate and other investments will earn interest or gain in value according to a specific sequence. Loans will be paid according to a predetermined sequence. In this module, you will study two different types of sequences: arithmetic and geometric. You will also study two different types of series: arithmetic and geometric. Series are used when you are trying to find the sum of a large, even infinite, list of numbers. In this module you will learn when it is possible to find the sum of different types of series.

Assignments in Module 1

When you have completed the assignments for Module 1, submit your completed assignments to the Distance Learning Unit either by mail or electronically through the learning management system (LMS). The staff will forward your work to your tutor/marker.

Lesson	Assignment Number	Assignment Title
2	Assignment 1.1	Arithmetic Sequences and Series
4	Assignment 1.2	Geometric Sequences and Series
5	Assignment 1.3	Finding the Sums of Infinite Geometric Series

Resource Sheet

When you write your midterm examination, you are encouraged to take a Midterm Examination Resource Sheet with you into the examination. This sheet will be one letter-sized page, 8½ " by 11", with both sides in your handwriting or typewritten. You will submit it with your examination, but you do not receive any marks for it.

Many students have found that preparing a resource sheet is an excellent way to review. It provides you with a summary of the important facts of each module. You should complete a resource sheet for each module to help with your studying and reviewing. Lesson summaries and module summaries are included for you to use as a guide.

You may use the list of instructions provided below to help you with preparing your resource sheet for the material in Module 1. On this sheet, you should record math terms and definitions, formulas, sample questions, or a list of places where you often make mistakes. You should also identify special areas that require extra attention or review by writing the page numbers.

After you have completed each module's resource sheet, you may summarize the sheets from Modules 1, 2, 3, and 4 to prepare your Midterm Examination Resource Sheet. The midterm examination for this course is based on Modules 1 to 4.

Resource Sheet for Module 1

As you go through the lessons of this module, you may want to consider the following suggestions regarding the creation of a resource sheet.

1. List all the important math terms, and define them if necessary.
2. List all the formulas and perhaps a sample problem that shows how the formula is used.
3. If necessary, write the solutions to some problems, showing in detail how you did the calculations.
4. Copy any questions that represent the key points of the lesson, and perhaps include the solutions as well.
5. Identify the problems you found most difficult, and copy the page numbers onto the resource sheet so that you can review them before writing the examination. You may also copy the problems and the solutions onto your resource sheet, and later write them onto your Midterm Examination Resource Sheet.
6. Write any comments, ideas, shortcuts, or other reminders that may be helpful during an examination.

Lesson 1: Arithmetic Sequences

Lesson Focus

In this lesson, you will

- ❑ learn how to identify arithmetic sequences
- ❑ learn how to find the first term, the common difference, the number of terms, and the value of a specific term in an arithmetic sequence
- ❑ learn how arithmetic sequences are related to linear functions
- ❑ learn how to write a recursive formula
- ❑ learn how to derive a rule for finding the general term of an arithmetic sequence

Lesson Introduction

When you were younger, you were probably asked to count by ones from one to increasingly larger numbers. You were also probably asked to count by odd numbers, by even numbers, by fives, by tens, or by other values. These are all examples of arithmetic sequences.

Arithmetic sequences appear often in real-life situations. Hockey games and the formation of pyramids are another two examples where arithmetic sequences occur. Throughout this lesson, you will learn all about arithmetic sequences and their many applications.

Arithmetic Sequences

A sequence is an ordered list of terms. The **general form of a sequence** is: t_1, t_2, t_3, t_4, ... where each numerical subscript denotes the term in the sequence. This means that if you wanted to find the fifth term in a sequence, you would be looking for t_5. Also, t_{12} denotes the twelfth term in the sequence.

Sequences of numbers that follow the pattern of adding the same number (either positive or negative) to one term to get the next term are called **arithmetic sequences**. In other words, an **arithmetic sequence** is a list of numbers in which each succeeding term is found by adding or subtracting the same value from the previous term.

Include all definitions on your resource sheet.

Example 1

Which of these sequences are arithmetic sequences?

a) 2, 5, 8, 11, . . .

b) 1, 2, 4, 7, 11, . . .

c) 7, 3, −1, −5, . . .

d) 1, 1, 2, 3, 5, . . .

Solution

a) In this sequence, if you add '3' to the first term, you will get the second term. This works for any pair of consecutive terms (or terms that follow each other) in this sequence. Therefore, this is an arithmetic sequence.

b) In this sequence, if you add '1' to the first term, you will get the second term. However, to get to the third term, you need to add '2' to the second term. This is not an arithmetic sequence because you are not adding the same value to each term to get the next term.

c) In this sequence, if you add '−4' or subtract '4' from each term, you will get the next term. Therefore, this is an arithmetic sequence.

d) In this sequence, the first terms are the same. The rest of the terms are the sum of the previous two terms. However, the same number is not added to every term to get the next term. Therefore, this is not an arithmetic sequence. It is the Fibonaci sequence.

In an arithmetic sequence, the number that is added to each term to get the next term is called the **common difference**. Include this definition on your resource sheet.

Example 2

Find the common difference for each of these arithmetic sequences.

a) 5, 9, 13, 17, . . .

b) 1, 0, −1, −2, . . .

c) 2, 4, 6, 8, . . .

Solution

a) The number that is added to each term to get the next term is '4'. Therefore, the common difference of this arithmetic sequence is 4.

b) The number that is added to each term to get the next term is '−1'. Therefore, the common difference of this arithmetic sequence is −1.

c) The number that is added to each term to get the next term is '2'. Therefore, the common difference of this arithmetic sequence is 2.

Arithmetic Sequences and Linear Functions

Given a sequence such as 1, 6, 11, 16 … it may be easy for you to find the 10th or even the 20th term by just adding 5, the common difference, to each term until you have a sequence with 20 terms. What happens when you need to find the 600th term (t_{600})?

Calculating the 600th term in this way would take an extremely long time. In order to find terms that are increasingly farther away from the first term in a sequence, a formula can be used. For example, look at the arithmetic sequence:

2, 5, 8, 11, . . .

Each term in the sequence is 3 more than the preceding term:

2, 2 + 3 = 5, 5 + 3 = 8, 8 + 3 = 11, . . .

You can denote this pattern by stating that:

$t_1 = 2$ (where t_1 is the first term)

$t_2 = t_1 + 3$ (where t_2 is the second term)

$t_3 = t_2 + 3$

In general $t_n = t_{n-1} + 3$, $n \geq 2$ where the nth term is 3 more than the previous term. The variable, n, is used as a counter to count the number of terms. Its domain is {1, 2, 3, 4, ...}, which is the set of natural numbers.

The definition for this arithmetic sequence consists of two statements:

$$\begin{cases} t_1 = 2 \\ t_n = t_{n-1} + 3,\ n \geq 2 \end{cases}$$

In this case, n must be 2 or bigger because t_1 is already defined.

Both parts of the statement are required.

The definition states

a) the value of the first term is 2 as shown by the first statement, $t_1 = 2$

b) any other term after the first is 3 more than the previous term as shown by the second statement, $t_n = t_{n-1} + 3$, when $n \geq 2$

Note that 7, 10, 13, 16, . . . is another sequence in which each term after the first is 3 more than the previous term. However, it is a different sequence from the sequence 2, 5, 8, 11, . . . because the first terms are different.

You can denote the arithmetic sequence 7, 10, 13, 16, . . . by

$$\begin{cases} t_1 = 7 \\ t_n = t_{n-1} + 3,\ n \geq 2 \end{cases}$$

Also note that it is important to distinguish between the term number and the value of the term.

Example 3

Write the first five terms of the arithmetic sequence defined by

$$\begin{cases} t_1 = 4 \\ t_n = t_{n-1} + 5,\ n \geq 2 \end{cases}$$

Solution

Evaluating the formula can easily be organized by using a chart. The term number, n, is recorded in the first column; the value of the term is calculated in the second column.

$$\begin{cases} t_1 = 4 \\ t_n = t_{n-1} + 5,\ n \geq 2 \end{cases}$$

1	$t_1 = 4$	(from the first statement)
2	$t_2 = t_{2-1} + 5$ $= t_1 + 5$ $= 4 + 5$ $= 9$	(from the second statement) simplify substitute simplify
3	$t_3 = t_{3-1} + 5$ $= t_2 + 5$ $= 9 + 5$ $= 14$	
4	$t_4 = t_{4-1} + 5$ $= t_3 + 5$ $= 14 + 5$ $= 19$	
5	$t_5 = t_{5-1} + 5$ $= t_4 + 5$ $= 19 + 5$ $= 24$	

Therefore, the first five terms are 4, 9, 14, 19, 24.

In Example 3, after the first term, the succeeding terms are defined by the value of the preceding term and the common difference. If after the first term, or the first few terms, the remaining terms are defined in terms of the value(s) of the preceding term(s), the definition is called a **recursive definition** or recursive formula. It may be helpful for you to include this definition on your resource sheet.

Example 4

Write a recursive formula for the arithmetic sequence 7, 11, 15, 19, . . ., and graph it using both graph paper and a graphing calculator (if you have one).

Solution

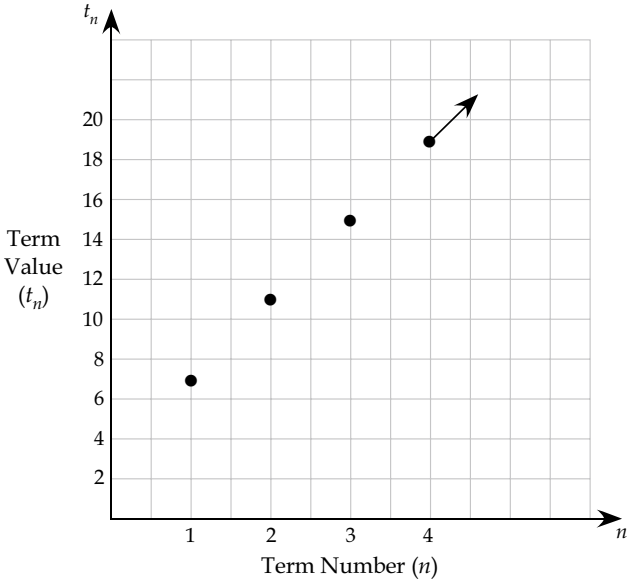
$t_1 = 7$

Since each term is 4 more than the previous term, it follows that the common difference is 4 and $t_2 = t_1 + 4$, $t_3 = t_2 + 4$, ... Therefore, the recursive formula is

$$\begin{cases} t_1 = 7 \\ t_n = t_{n-1} + 4,\ n \geq 2 \end{cases}$$

The points on the graph will be the set of ordered pairs where the first coordinates are the term numbers and the second coordinates are the corresponding term values. The coordinates are (1, 7), (2, 11), (3, 15), (4, 19), . . .

The graph is shown below. Notice that the points are not joined. The data is discrete, not continuous. Also note that there is no easy way to indicate the pattern of dots, which continues indefinitely. The arrow attached to (4, 19) points in only one direction.

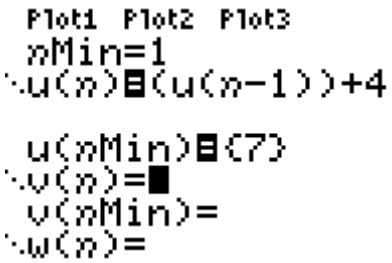


On a graphing calculator, use the sequence mode to graph the sequence using the recursion formula:

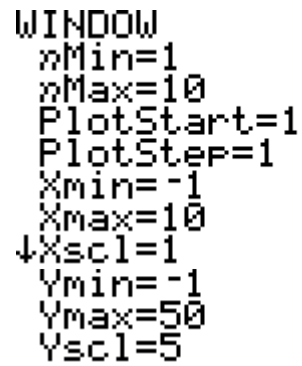
$$\begin{cases} t_1 = 7 \\ t_n = t_{n-1} + 4,\ n \geq 2 \end{cases}$$

and the first point in the sequence as (1, 7).

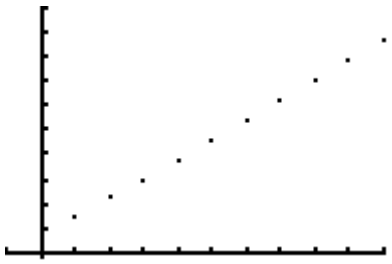
Using the Seq mode, enter your information into Y=.



Then set your window.



Finally, press GRAPH.



If you need instruction on how to use your graphing calculator for sequences, you might try searching the Internet for help. One useful site is www.mathbits.com.

From these examples, we could generalize the recursion formula for an arithmetic sequence to be:

$$\begin{cases} t_1 = \text{first term} \\ t_n = t_{n-1} + d,\ n \geq 2 \end{cases}$$

where d is the common difference.

Example 5

Find t_7 and t_{100} of the arithmetic sequence 50, 42, 34, 26, ...

Solution

The common difference is -8 and the recursive definition is:

$$\begin{cases} t_1 = 50 \\ t_n = t_{n-1} - 8,\ n \geq 2 \end{cases}$$

Listing the terms from t_2 to t_7, you get

$t_2 = 50 - 8 = 42$

$t_3 = 42 - 8 = 34$

$t_4 = 34 - 8 = 26$

$t_5 = 26 - 8 = 18$

$t_6 = 18 - 8 = 10$

$t_7 = 10 - 8 = 2$

Therefore, $t_7 = 2$.

When you are using a recursive formula or definition, you need to know t_{n-1} in order to find t_n. You need to know t_{99} in order to find t_{100}. Using recursion would be time consuming, as it would take a long time to list all the terms up to t_{100}. There is a different method to solve this problem.

It is easier to find t_{100} by relating the arithmetic sequence to a linear function. The answer is shown after Example 7.

A **linear function** is a function whose graph is an oblique or horizontal line. If the **domain** of a linear function is the set of **natural numbers**, the **range** is called an **arithmetic sequence**. Recall that the set of **natural numbers** is N = {1, 2, 3, 4, . . .}.

Include these definitions on your resource sheet.

Example 6

Which of the following rules defines a linear function? Graph the linear function(s).

a) $y = \dfrac{1}{x-2}$

b) $y = 3x + 5$

c) $y = \sqrt{x-1}$

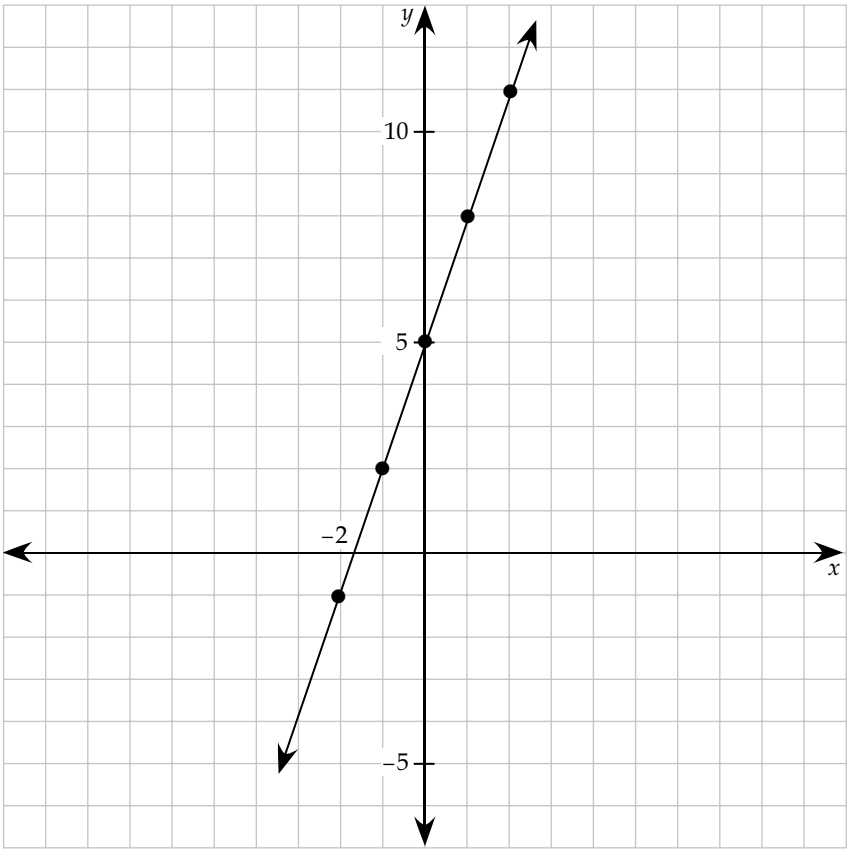
d) $y = x^2 + 1$

e) $y = 4 - x$

Solution

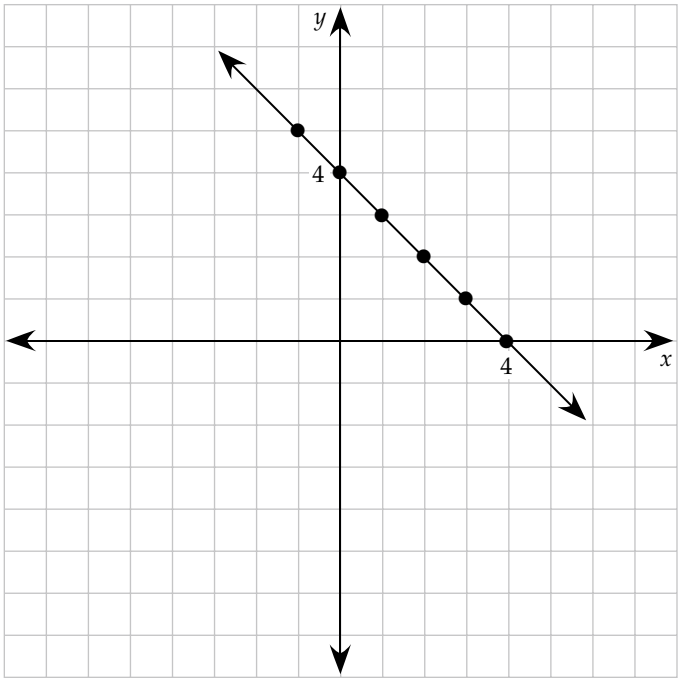
The only rules that produce straight line graphs are $y = 3x + 5$ and $y = -x + 4$. Hence (b) and (e) are linear functions. The graphs of the two linear functions are shown below. Notice their domains are the real numbers and the points are connected.

b)



$y = 3x + 5$
slope = 3
y-intercept = 5

e)



$y = 4 - x$
$y = -x + 4$
slope $= -1$
y-intercept $= 4$

Note: A straight line equation is of the form $y = mx + b$ where m is the slope of the line and b is the y-intercept. In example 6, part (e) above, the equation $y = 4 - x$ can be rearranged to the form $y = -x + 4$.

Example 7

Write the first five terms and find the common difference for each arithmetic sequence defined by the given function whose domain is the natural numbers:

a) $f(x) = 2x + 5$

b) $g(x) = -3x + 10$

Solution

Since the domain of an arithmetic sequence is the set of natural numbers, N = {1, 2, 3, 4, . . .}, the first five terms are as follows.

a) $f(x) = 2x + 5$

x	$f(x)$
1	$f(1) = 2(1) + 5 = 2 + 5 = 7$
2	$f(2) = 2(2) + 5 = 4 + 5 = 9$
3	$f(3) = 2(3) + 5 = 6 + 5 = 11$
4	$f(4) = 2(4) + 5 = 8 + 5 = 13$
5	$f(5) = 2(5) + 5 = 10 + 5 = 15$

The first five terms are 7, 9, 11, 13, 15.

The common difference is 2.

b) $g(x) = -3x + 10$

x	$g(x)$
1	$g(1) = -3(1) + 10 = -3 + 10 = 7$
2	$g(2) = -3(2) + 10 = -6 + 10 = 4$
3	$g(3) = -3(3) + 10 = -9 + 10 = 1$
4	$g(4) = -3(4) + 10 = -12 + 10 = -2$
5	$g(5) = -3(5) + 10 = -15 + 10 = -5$

The first five terms are 7, 4, 1, −2, −5.

The common difference is −3.

If you compare the common difference of each arithmetic sequence to its defining linear function, you may notice that the common difference has the same value as the slope. In other words, the slope of the line is the increase in term value as you proceed from one term to the next.

Note: It will be very helpful for you to remember that *common difference* = *slope* in an arithmetic sequence.

Now that you know how linear functions are related to arithmetic sequences, you can go back and find t_{100} in Example 5. You found the following pieces of information about the arithmetic sequence in Example 5.

$t_1 = 50$

$t_2 = 42$

$t_3 = 34$

$t_4 = 26$

$t_5 = 18$

$t_6 = 10$

$t_7 = 2$

The ordered pairs for the first seven terms of the function are:

(1, 50), (2, 42), (3, 34), (4, 26), (5, 18), (6, 10), (7, 2)

As every arithmetic sequence is a linear function, this set of points belongs to a straight line. Therefore, you can use the slope to find a defining linear function for this line to help you find t_{100}. Recall that the formula for slope given two points (x_1, y_1) and (x_2, y_2) on a line is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The equation for a line given its slope, m, and a point, (x_1, y_1), on the line is:

$y - y_1 = m(x - x_1)$

You can use any two points on the line to find its slope and equation.

If you use the first two points on the line, (1, 50) and (2, 42), then the slope is:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{42 - 50}{2 - 1} = \frac{-8}{1} = -8$$

The defining linear function is:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 50 &= -8(x - 1) \\ y &= -8x + 8 + 50 \\ y &= -8x + 58 \end{aligned}$$

The defining function is $f(x) = -8x + 58$, whose slope is the common difference for the sequence. The domain of $f(x)$ is the set of natural numbers.

Now, it is easier to find t_{100}. To find this value, replace x with 100.

$f(100) = -8(100) + 58$

$f(100) = -800 + 58$

$f(100) = -742$

Therefore, $t_{100} = -742$.

From this example, you can see that $f(x)$ represents the term value whereas x represents the number of the term.

Example 8

What is the defining linear function that produces the following first four terms of an arithmetic sequence?

a) $f(x)$: 1, 3, 5, 7, . . .

b) $g(x)$: 2, 4, 6, 8, . . .

Solution

a) Since the domain of the function is the set of natural numbers, the ordered pairs of the function are (1, 1), (2, 3), (3, 5), (4, 7), . . .

As each set of points belongs to a straight line, use the slope to find its equation.

Using $m = \frac{y_2 - y_1}{x_2 - x_1}$ and $y - y_1 = m(x - x_1)$, then, $m = \frac{3-1}{2-1} = \frac{2}{1} = 2$.

The equation is: $y - 1 = 2(x - 1)$

$y = 2x - 2 + 1$

$y = 2x - 1$

The defining function is $f(x) = 2x - 1$.

b) The ordered pairs are (1, 2), (2, 4), (3, 6), (4, 8), . . .

$m = \frac{4-2}{2-1} = \frac{2}{1} = 2$

The equation is: $y - 2 = 2(x - 1)$

$y = 2x - 2 + 2$

$y = 2x$

The defining function is $g(x) = 2x$.

General Term Formula

In general, the arithmetic sequence $t_1, t_2, t_3, \ldots$ can be written as $t_1, t_1 + d, t_1 + 2d, \ldots$ where d is the common difference between consecutive terms. As we have seen, d is the slope of the defining linear function. Notice the pattern for the number of ds in each term. In the first term, no ds; in the second term, one d; in the third term, two ds. Thus, there is always one less d than the term number. In general, $t_n = t_1 + (n - 1)d$. This formula is called the **general term of an arithmetic sequence**. It may be helpful for you to include this explanation and formula on your resource sheet.

Example 9

Use the formula $t_n = t_1 + (n - 1)d$ to find t_{10}, t_{21}, and t_{1000} of the arithmetic sequence 30, 36, 42, 48, ...

Solution

$t_1 = 30$ and $d = t_2 - t_1 = 36 - 30 = 6$ or

$d = t_3 - t_2 = 42 - 36 = 6$ or

$d = t_4 - t_3 = 48 - 42 = 6$

Therefore,

$t_{10} = t_1 + (10 - 1)d = 30 + 9(6) = 30 + 54 = 84$

$t_{21} = t_1 + (21 - 1)d = 30 + 20(6) = 30 + 120 = 150$

$t_{1000} = t_1 + (1000 - 1)d = 30 + 999(6) = 30 + 5994 = 6024$

The general term formula $t_n = t_1 + (n - 1)d$ can be used to solve questions about an arithmetic sequence such as finding the unknown values of t_n, t_1, n, or d.

Example 10

How many terms are in the arithmetic sequence 26, 28, 30, . . ., 1062?

Solution

Using the general term formula, $t_n = t_1 + (n - 1)d$, you have $t_n = 1062$, $t_1 = 26$, and $d = 2$. Now substitute these values into the formula and solve for n.

$1062 = 26 + (n - 1)(2)$

$1062 = 26 + 2n - 2$

$1062 = 24 + 2n$

$1038 = 2n$

$519 = n$

Therefore, $1062 = t_{519}$. Hence, there are 519 terms in the sequence.

You could also solve Example 10 by finding the defining linear function, using the points (1, 26) and (2, 28).

$$m = \frac{28 - 26}{2 - 1} = 2$$

$$y - 26 = 2(x - 1)$$

$$y = 2x - 2 + 26$$

$$y = 2x + 24 \text{ or}$$

$$f(x) = 2x + 24$$

Since you know the value of the term to be 1062, you know $f(x)$. Just substitute and solve for x, its corresponding term number.

Therefore, $1062 = 2x + 24$

$$1038 = 2x$$

$$519 = x$$

Hence, 519 terms are in the sequence.

Make sure you complete the following learning activity to practice what you have just learned in this lesson. Recall that learning activities have two parts. The first part is a BrainPower section that is intended as a warm-up activity to prepare your mental muscles for Part B. Part B contains questions that allow you to practice the material you have just learned in the lesson. Learning activities do not need to be handed in. However, completing every learning activity will give you an idea of what types of questions will be asked on upcoming assignments and the exams.

Learning Activity 1.1

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. At the end of a poker game, Alysha had $62. During the game she had won $250, lost $175, won $35, and lost $55. How much money did Alysha have at the start, in dollars?

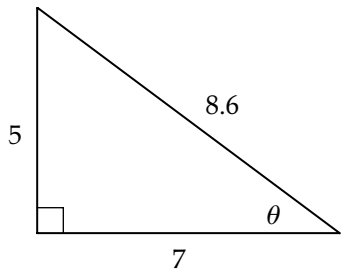
2. Complete the pattern: 1, 2, 5, 10, 17, ____ , ____ , ____

3. Evaluate when $z = -1$: $(3z + 1)(z - 5)$

4. Evaluate: $\left(\frac{8}{16}\right)(12)$

5. A clothing store is selling shirts for $30.00. They decided to raise the price of the shirt by 20%. A few days later they decided to raise the price another 20%. What is the new price of the shirt, in dollars?

Use the following right-angled triangle to answer the next three questions. Do not simplify your answer.



6. Identify the sine ratio for θ.

7. Identify the cosine ratio for θ.

8. Identify the tangent ratio for θ.

continued

Learning Activity 1.1 (continued)

Part B: Arithmetic Sequences

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Which of the following sequences are arithmetic?
   a) 2, 5, 8, 11, . . .
   b) –5, –1, 3, 7, . . .
   c) 2, 5, 7, 10, . . .
   d) $2, 2\frac{1}{2}, 1, 1\frac{1}{2}, 0, \ldots$

2. Write a recursive formula for each sequence. Indicate whether the sequence is arithmetic or not.
   a) 10, 7, 4, 1, . . .
   b) 2, 4, 8, 16, . . .
   c) 30, 41, 52, 63, . . .
   d) $3, -6, 12, -24, \ldots$
   e) 5, 15, 45, 135, . . .

3. Write the first four terms of each sequence.
   a) $\begin{cases} t_1 = 3 \\ t_n = t_{n-1} + 5,\ n \geq 2 \end{cases}$
   b) $\begin{cases} t_1 = 10 \\ t_n = t_{n-1} - 2,\ n \geq 2 \end{cases}$
   c) $\begin{cases} t_1 = 3.7 \\ t_n = t_{n-1} + 1.3,\ n \geq 2 \end{cases}$
   d) $\begin{cases} t_1 = 1 \\ t_2 = 1 \\ t_n = t_{n-1} + t_{n-2},\ n \geq 3 \end{cases}$

continued

e) $\begin{cases} t_1 = 3 \\ t_2 = -1 \\ t_n = t_{n-1} + t_{n-2},\ n \geq 3 \end{cases}$

4. Write the first three terms of the arithmetic sequence defined by the linear function.
   a) $f(x) = 2x + 4$
   b) $g(x) = 5x - 2$
   c) $h(x) = -3x + 2$
   d) $m(x) = x - 5$

5. Three terms of an arithmetic sequence (not necessarily the first three terms) are 27, 30, 33. Write two different linear functions that may have produced this sequence. The domain is the set of natural numbers.

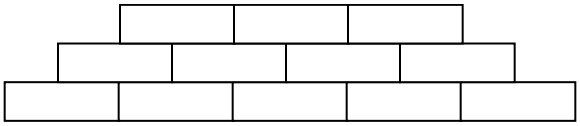
6. Find an equation of the linear function that produces the following first three terms of an arithmetic sequence.
   a) 5, 8, 11, . . .
   b) $6, 6\frac{1}{2}, 7, \ldots$
   c) 10, 8, 6, . . .
   d) $2, 2\frac{1}{10}, 2\frac{1}{5}, \ldots$

7. Find the 50th term of each of the following arithmetic sequences.
   a) $f(x)$: 7, 11, 15, . . .
   b) $g(x)$: 10, 15, 20, . . .
   c) $h(x)$: $4\frac{1}{2}, 5, 5\frac{1}{2}, \ldots$

8. The fifth term of an arithmetic sequence is -15 and the ninth term is -3. Find the value of the hundredth term.

continued

9. A pile of bricks is arranged in rows. The numbers of bricks in the rows form an arithmetic sequence. One such pile of bricks with 3 rows is shown below:



In another pile of bricks, there are 45 bricks in the fifth row and 33 bricks in the eleventh row.

a) How many bricks are in the first row?

b) How many bricks are in the nth row?

c) What is the maximum number of rows of bricks possible?

10. Find which term of the sequence 20, 23, 26, . . . is the value 245,

a) by using the general term formula

b) by using the defining linear function

11. Find the term indicated in each arithmetic sequence by using the general term formula.

a) 5, 3, 1, . . . t_{20}

b) 8, 11, 14, . . . t_{21}

c) 6, 3, 0, -3, . . . t_{57}

d) 18, 17, 16, 15, . . . t_{132}

e) 7, 8.3, 9.6, 10.9, . . . t_{172}

12. Joe bought a painting for $1800. After seven years, this painting sold for $10,550. If the painting appreciated arithmetically, determine the annual amount of appreciation.

13. For the arithmetic sequence 7, 11, 15, 19, . . ., find t_{29}.

14. Give an example of an arithmetic sequence, and then

a) prove your sequence is an arithmetic sequence

b) write the defining linear function of your sequence

c) write your arithmetic sequence using the general term formula

15. Graph the arithmetic sequence 7, 4, 1, −2, . . ., for the first six terms

a) on graph paper (A reproducible copy of a sheet of graph paper is included in Appendix B found on the downloads page.)

b) on the graphing calculator

Now that you have completed your first lesson and your first learning activity in Grade 11 Pre-Calculus Mathematics, it will be helpful for you to contact your tutor/marker. Part of Learning Activity 1.2 will be to do so. As mentioned in the introduction to this course, your tutor/marker is available if you need help learning any of the material. Also, your tutor/marker will be marking all of your assignments and examinations. Your tutor/marker's contact information is on the cover sheet that came with this course. If you are unable to find your cover sheet, phone the Distance Learning Unit at 1–800–465–9915 and someone will provide you with the information you need.

Learning Activity 1.2

This learning activity is the only one that doesn't include a BrainPower section, although it still has two parts.

Part A: Contacting Your Tutor/Marker

Your first task is to contact your tutor/marker by phone.

Fill in the following blanks using information provided with your course:

My tutor/marker's name is _____

I can phone my tutor/marker at 1-866-_____

My tutor/marker's email address is _____

Be ready to discuss the following topics and the reasons for your answers with your tutor/marker and learning partner. If you like, make some notes before you call so that you feel prepared. Feel free to add any other questions or comments that you may have.

1. I am taking this course by distance education because:

_____

_____

_____

_____

continued

Learning Activity 1.2 (continued)

2. What I like about math and can do mathematically is (include favourite topic, skill, where you use math, etc.):

3. What I dislike about math or have difficulty doing is:

4. Previous math experiences that influence the way I feel about math are:

5. The next math course I would like to take is:

continued

Learning Activity 1.2 (continued)

6. What I am hoping this course will help me accomplish and learn for the future:

_____

_____

_____

_____

7. What I am doing and how I organize things to help me succeed in this course is:

_____

_____

_____

_____

During your phone conversation, jot down a sentence or two about what you and your tutor/marker talk about, in the spaces above. For example, if you are taking this course because it doesn't fit into your schedule at school or because you travel with your basketball team and would miss a large number of classes, state that in the space below question 1.

continued

Learning Activity 1.2 (continued)

Part B: Your Math Pathway

Use the answers to the questions from the conversation with your tutor/marker as a starting point to fill in the following diagram. In the Math History box, jot down point-form notes about your prior experience and knowledge about math (questions 2, 3, and 4). In the Math Destination box, jot down what completing this course will help you accomplish in the future (questions 5 and 6).

In the Pathway box, write down what you will need to do to move along the pathway from your History to your Destination.

Math History	Pathway	Math Destination

For example, your destination might be receiving at least 75% in this course so you feel confident going into Grade 12 Pre-Calculus Mathematics so that you can take nursing after you graduate. Your destination could also be that you need to learn how to solve equations, and are trying to determine what will help you accomplish this. It may mean figuring out how you best learn and study math. It may mean setting up a schedule so you complete the assignments on time. You may need to find your calculator manual and figure out how to use it, set up regular appointments with your learning partner, research a topic on the Internet, or read a textbook about a certain math concept or skill. Your pathway is unique to you.

continued

Learning Activity 1.2 (continued)

As you move through this course and work on achieving your goals, self-assessment is important for you to determine whether you are getting closer to your destination. It helps you determine whether the steps along your pathway are taking you in the right direction. You will need to periodically ask yourself: Am I doing my assignments? Are my note-taking skills improving? How often have I contacted my tutor/marker or worked with my learning partner? Have I found useful homework websites if necessary? Is my schedule working? What do I need to change or adjust so I can get to my destination?

Repeatedly going through this cycle of looking at where you have been, where you want to go, and where you currently are is recommended. At any time, you may want to revise your goals or set new ones as you evaluate your own progress and learning.

- Look back/history. Reflect on what you know, how far you have come.
- Look around/pathway. Assess if you are achieving your goals, determine if new learning or understanding has occurred, and check your progress.
- Look forward/destination. Determine what you want to know, set goals.

Each time you go through these steps, you will become better at mathematics!

It is important that you keep this diagram handy, as you will revisit it at other points in this course.

Lesson Summary

In this lesson, you learned about arithmetic sequences and how they are formed. You also looked at how arithmetic sequences are related to linear functions. From these linear functions, you noticed that the slope of a linear function was equal to the common difference of an arithmetic sequence. In the next lesson, you will be building your knowledge about arithmetic sequences and learning about arithmetic series.

Notes

Lesson 2: Arithmetic Series

Lesson Focus

In this lesson, you will

- ❑ learn to distinguish between an arithmetic series and an arithmetic sequence
- ❑ learn how to determine the first term and the common difference in an arithmetic series
- ❑ learn how to derive a rule for finding the sum of a specific number of terms in an arithmetic series

Lesson Introduction

Last lesson you learned about arithmetic sequences. Arithmetic sequences are similar to arithmetic series, which you will be learning about in this lesson. To understand the difference between arithmetic sequences and arithmetic series, you need to understand the difference between sequences and series. Last lesson you saw that sequences were ordered lists of numbers. Series are sums of these ordered lists of numbers. Sums of large lists of numbers are useful when you are trying to count objects that have patterns, such as the number of seats in a stadium or the number of boxes in a store display. In this lesson, you will learn more about the differences between arithmetic sequences and series and why arithmetic series are useful in situations where arithmetic sequences are not.

Arithmetic Series

Arithmetic series are similar to arithmetic sequences. The only difference between the two is the presence of addition. If you added consecutive terms of an arithmetic sequence, you would form an **arithmetic series**.

- 2, 5, 8, 11, . . ., is an arithmetic sequence
- 2 + 5 + 8 + 11 + . . ., is the corresponding arithmetic series

It may be helpful if you include this definition and example on your resource sheet.

The First Term and the Common Difference of an Arithmetic Series

Finding the first term and the common difference of an arithmetic series is similar to finding the first term and the common difference of an arithmetic sequence. Consider the following example.

Example 1

Find the first term and the common difference of each of the following arithmetic series.

a) $3 + 5 + 7 + 9 + \ldots$

b) $-5 + (-10) + (-15) + (-20) + \ldots$

Solution

a) The first term of an arithmetic series is the first term in the addition. Therefore, the first term of this arithmetic series is 3. The common difference in an arithmetic series is the difference between each term, just like in an arithmetic sequence. The common difference in this arithmetic series is 2.

b) The first term of this arithmetic series is -5. The common difference in this arithmetic series is -5.

Finding the Sum of an Arithmetic Series

It is sometimes useful to find the sum of an arithmetic series. Using the following examples, try to develop a formula to find the sum of these arithmetic series. (**Hint:** Concentrate on the middle number and the number of terms in the series.)

a) $2 + 5 + 8 + 11 + 14 = 40$

b) $7 + 12 + 17 = 36$

c) $8 + 4 + 0 - 4 - 8 = 0$

d) $6 + 10 + 14 = 30$

By examining closely the four previous examples, (a) to (d), of arithmetic series, you may have realized that the sum is the product of the middle term and the number of terms in the series:

a) $8(5) = 40$

b) $12(3) = 36$

c) $0(5) = 0$

d) $10(3) = 30$

Example 2

What is the sum of the first 51 positive integers?

Solution

The arithmetic sequence is: 1, 2, 3, 4, 5, . . ., 51.

The corresponding arithmetic series is: 1 + 2 + 3 + 4 + . . . + 50 + 51.

Therefore, for the series 1 + 2 + 3 + 4 + . . . + 50 + 51, the middle number is 26 and the number of terms in the sequence is 51 so the sum is 26(51) = 1326.

Note: Since there are an odd number of numbers in the list, the middle number is $\frac{n+1}{2}$, where n is the number of numbers.

Why Does This Formula Work?

Consider the series 6 + 10 + 14.

- 14 is 4 more than 10.
- 6 is 4 less than 10.
- If you transfer the additional 4 (from the 14) to 6, you can look at 6 + 10 + 14 as 10 + 10 + 10.

$$6 + 10 + 4$$

$$\underbrace{+4 \qquad -4}_{\downarrow}$$

$$10 + 10 + 10$$

Overall, the series becomes: 10 + 10 + 10.

The sum of the series = 10 + 10 + 10 = (3)(10) = 30.

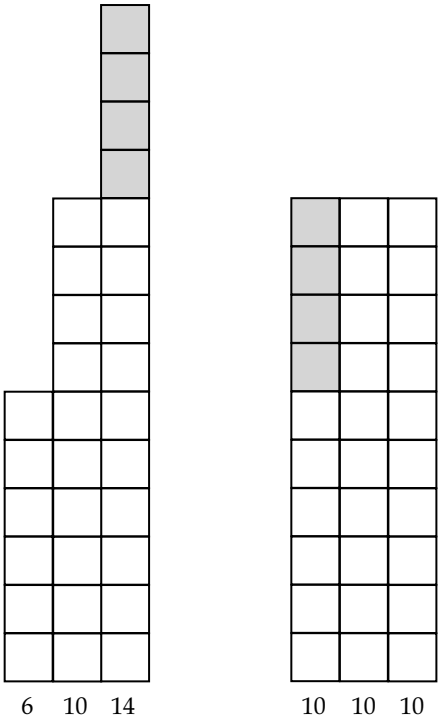
The formula for finding the sum of a series is:

Sum of series = (middle term)(number of terms)

Include this formula on your resource sheet.

Therefore, the sum of the series 6 + 10 + 14 = (middle term)(number of terms) = 10(3) = 30.

A diagram of this idea is shown below.



Consider the series 2 + 5 + 8 + 11 + 14.

- 11 is 3 more than 8.
- 5 is 3 less than 8.
- If you transfer the additional 3 (from the 11) to 5, you can look at 5 + 8 + 11 as

$$5 + 8 + 11$$
$$\underbrace{+3 \qquad -3}_{\downarrow}$$
$$8 + 8 + 8$$

Similarly,

- 14 is 6 more than 8.
- 2 is 6 less than 8.
- If you transfer the additional 6 (from the 14) to 2, you can look at 2 + 8 + 14 as 8 + 8 + 8.

$$2 + 8 + 14$$
$$\underbrace{+6 \qquad -6}_{\downarrow}$$
$$8 + 8 + 8$$

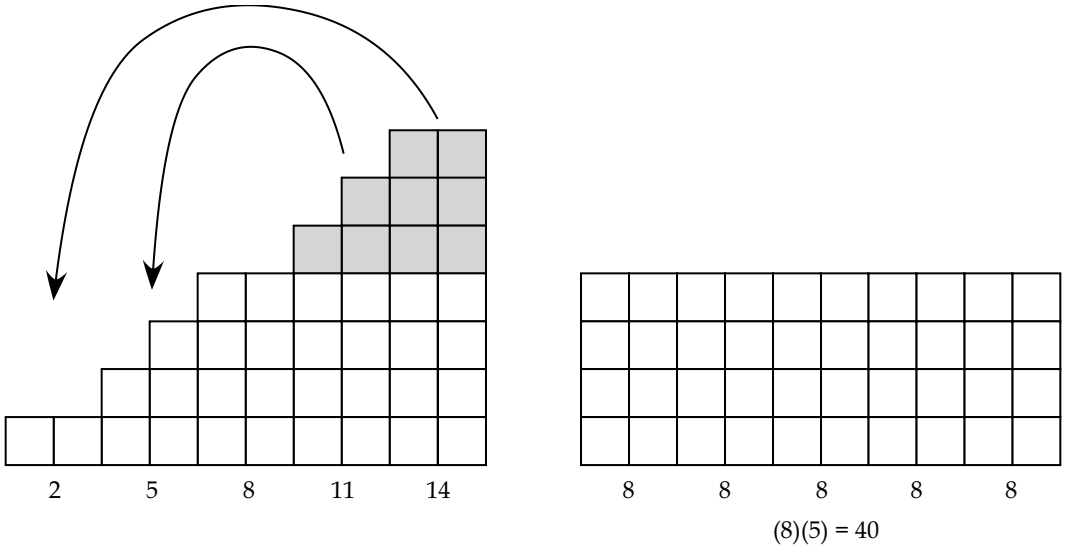
Overall, the series becomes: 8 + 8 + 8 + 8 + 8.

For an odd number of terms

Sum of series = (middle term)(number of terms)

Sum of series = 8(5) = 40.

A geometric representation of this idea is shown below.



Example 3

Find the sum of 7 + 15 + 23 + 31 + 39 + 47 + 55.

Solution

Sum = (middle term)(number of terms) = 31(7) = 217

Example 4

Develop a formula for the sum of 2 + 6 + 10 + 14.

Solution

There are an even number of terms in the series. This series has no middle term as one of its terms. Therefore, the sum formula developed previously will not work.

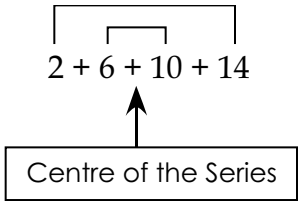
However, the mean of the series can be found: 2 + 6 + 10 + 14 = 32 and $\frac{32}{4}$ = 8. Hence, 8 is the middle value between 6 and 10, or the mean of 6 and 10.

Recall that the mean is the value obtained when the sum of the terms in a set is divided by the total number of terms in a set. Also note the following mean:

The mean of 6 and 10 = $\frac{6 + 10}{2} = 8.$

The mean of 2 and 14 = $\frac{2 + 14}{2} = 8.$

As you can see, the means of terms that are an equal distance away from the centre of the series are equal. Consider the following diagram.



Note that 6 and 10 are both the same distance away from the centre of the series. Also, 2 and 14 are the same distance away from the centre of the series. Therefore, the means of these two sets of numbers 6 and 10, 2 and 14 are equal.

The new formula for finding the sum of any arithmetic series $t_1 + t_2 + t_3 + \ldots + t_n$ is:

S_n = *(mean)(number of terms)*

$$S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$$

where: t_1 = the first term in the arithmetic series

t_n = the last term in the arithmetic series

n = the number of terms in the arithmetic series

You may want to include this formula on your resource sheet.

In any arithmetic series, t_1 and t_n are the same distance away from the centre. Therefore, the $\left(\frac{t_1 + t_n}{2}\right)$ part of the equation is representative of finding the mean value of the series, whether there are an even or an odd number of terms.

This formula can be used to find the sum of *any* arithmetic series. The previous formula, [Sum = (middle term)(number of terms)], can only be used to find the sum of arithmetic series with an *odd* number of terms.

Finally, find the sum of the series 2 + 6 + 10 + 14 using the new formula $S_n = \left(\frac{t_1 + t_n}{2}\right)n$.

$t_1 = 2$

$t_n = 14$

$n = 4$

$S_4 = \left(\frac{2 + 14}{2}\right)(4)$

$S_4 = \left(\frac{16}{2}\right)(4)$

$S_4 = (8)(4)$

$S_4 = 32$

Example 5

Find the sum of the following arithmetic series using the formula for S_n.

a) 2 + 5 + 8 + 11 + 14

b) 1 + 2 + 3 + . . . ++ 50 + 51

Solution

Using $S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$,

a) $S_5 = \left(\frac{2 + 14}{2}\right)(5) = \left(\frac{16}{2}\right)(5) = (8)(5) = 40$

b) $S_{51} = \left(\frac{1 + 51}{2}\right)(51) = \left(\frac{52}{2}\right)(51) = (26)(51) = 1326$

Example 6

Find the sum of all multiples of 7 between 50 and 300.

Solution

The first multiple of 7 greater than 50 is 56. The greatest multiple of 7 smaller than 300 is 294. Thus, the series is 56 + 63 + 70 + . . . + 294.

Before you use the sum of an arithmetic series formula, you first need to find how many terms are in the series.

Using the formula $t_n = t_1 + (n - 1)d$,

$294 = 56 + (n - 1)7$

$294 = 56 + 7n - 7$

$245 = 7n$

$35 = n$

Therefore, 35 terms are in the series.

Using $S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$

$$S_n = \left(\frac{56 + 294}{2}\right)(35) = 6125$$

Therefore, the sum of all multiples of 7 between 50 and 300 is 6125.

Finding the Number of Terms in an Arithmetic Series

It can also be useful to find the number of terms in an arithmetic series if you know the sum of the series. In the above example, you used the general term formula to find the number of terms in an arithmetic series. It is also possible to find the number of terms in an arithmetic series by using the sum of an arithmetic series formula. Consider the following example.

Example 7

How many terms are in the arithmetic series, 17 + 20 + 23 + . . . + 557, if the sum of the series is 51 660?

Solution

Using $S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$, you know that S_n is 51 660, t_1 is 17 and t_n is 557.

Therefore, it is possible to solve for n, the number of terms in the series.

$$S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$$

$$51\,660 = \left(\frac{17 + 557}{2}\right)(n)$$

$$51\,660 = 287n$$

$$\frac{51\,660}{287} = n$$

$$180 = n$$

Therefore, there are 180 terms in this series.

Learning Activity 1.3

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Continue the pattern: 0.125, 0.250, 0.375, _____ , _____
2. Andrew bought an mp3 player for $156.88. He gave the cashier eight $20 bills. The store was out of toonies and all they had were loonies, quarters, dimes, nickels, and pennies. What was the smallest number of coins he could have received as change?
3. If $4x - 3x + 8x + 2x = 121$, then $6x - 2 = ?$
4. A recipe calls for two tablespoons of sugar for every quarter cup of flour. If Santana wants to use 18 tablespoons of sugar, how many cups of flour does she need to make the recipe?
5. Calculate: $\left(\frac{5}{3}\right) - \left(\frac{2}{14}\right)$
6. Simplify: $2^8 \div 2^6$
7. Determine the distance between the following two points: (2, 6) and (−7, −4).
8. List the factors of 164.

continued

Learning Activity 1.3 (continued)

Part B: Arithmetic Series

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Find the indicated sum of each arithmetic series.
   a) $7 + 12 + 17 + \ldots, S_{20}$
   b) $-5 - 7 - 9 - 11 - \ldots, S_{20}$
   c) $40 + 35 + 30 + \ldots, S_{100}$
2. Find the sum of the first:
   a) 10 odd integers.
   b) 20 odd integers.
   c) 100 odd integers.
   d) n odd integers.
3. Find the sum of all multiples of 3 between 99 and 1000.
4. Find S_{100} of $1 + 4 + 7 + \ldots$.
5. How many terms are in the series $3 + 8 + 13 + \ldots + 248$ if the sum is 6275?
6. In an arithmetic sequence, $t_2 = -1$ and $t_{12} = 19$, find S_{13}.
7. An auditorium has 20 seats in the first row, and each row has two more seats than the row immediately in front of it. There are 26 rows in the auditorium. How many seats are there in the last row? What is the seating capacity of the auditorium?
8. Mary's annual salary ranges from $26,785 in the first year to $34,825 in the seventh year.
   a) If the salary range is an arithmetic sequence with seven terms, determine the raise Mary received each year.
   b) What was her salary in the fifth year?
   c) What is the first annual salary in this range that is greater than $30,000?
   d) What was the total amount Mary earned in the seven years?

continued

Learning Activity 1.3 (continued)

9. a) Show that the sum of n terms of an arithmetic series can be found using the formula $S_n = \frac{n}{2}\left[2t_1 + (n-1)d\right]$.

   b) Use this formula to find the sum of 10 terms of the series $4 + 9 + 14 + 19 + \ldots$

Lesson Summary

In this lesson, you built on your knowledge about arithmetic sequences and learned about arithmetic series. You learned how arithmetic series are the addition of terms in arithmetic sequences. You also learned how to find the sum of different kinds of arithmetic series. In the next lesson, you will be learning about geometric sequences.

Your next task is to complete a hand-in assignment found on the following pages. Keep all your Module 1 assignments until you finish the module and then send them to the Distance Learning Unit. Instructions for submitting your assignments can be found in the module summary located at the end of this module.

Before you begin, have you completed a resource sheet to use with your assignment?

Notes

Assignment 1.1

Arithmetic Sequences and Series

Total: 40 marks

1. How many multiples of 7 are there between 75 and 352? *(3 marks)*

2. Which term of the arithmetic sequence 45, 49, 53, . . . is the number 97? *(3 marks)*

continued

Assignment 1.1: Arithmetic Sequences and Series

3. In an arithmetic sequence $t_{10} = 29$ and $t_{14} = 41$, find the first term and the common difference. *(3 marks)*

4. Find the sum of the first
   a) 10 even natural numbers *(2 marks)*

   b) 100 even natural numbers *(2 marks)*

continued

5. Find the sum of the arithmetic series $3 + 7 + 11 + \ldots + 483$. *(4 marks)*

6. How many terms are in the series $2 + 11 + 20 + \ldots + 1116$ if the sum is 69 316? *(2 marks)*

continued

Assignment 1.1: Arithmetic Sequences and Series

7. One section of the MTS Centre has 10 seats in the first row. Each row has four more seats than the row immediately in front of it. There are 25 rows in this section of the MTS Centre.

   a) How many seats are there in the last row? *(2 marks)*

   b) What is the seating capacity of this section of the MTS Centre? *(2 marks)*

8. Write the first four terms of each arithmetic sequence.

   a) $\begin{cases} t_1 = 3 \\ t_n = t_n - 5,\ n \geq 2 \end{cases}$ *(1.5 marks)*

   b) $y = 2x - 5$, x is a natural number *(1.5 marks)*

continued

Assignment 1.1: Arithmetic Sequences and Series

9. What is the defining linear function of each arithmetic sequence?

   a) 6, 10, 14, . . . *(2 marks)*

   b) 3, 1, −1, . . . *(2 marks)*

10. Classify each of the following sequences as arithmetic or not arithmetic sequences. Identify the common difference of all the sequences you classify as arithmetic sequences. *(1.5 marks each, for a total of 6 marks)*

    a) 3, 6, 3, 6, 3

    b) 5, 1, −3, −7, −11

    c) 16, 8, 4, 2, 1

    d) 5, 7, 9, 11, 13

continued

Assignment 1.1: Arithmetic Sequences and Series

11. Write the recursive formula for the arithmetic sequences you found in Question 10. *(4 marks)*

Lesson 3: Geometric Sequences

Lesson Focus

In this lesson, you will

- ❑ learn the definition of a geometric sequence
- ❑ learn how to find the first term, the common ratio, and the number of terms in a geometric sequence
- ❑ learn how to use recursion with geometric sequences
- ❑ learn how to derive a formula for finding the general term of a geometric sequence
- ❑ learn how to find the value of a specific term of a geometric sequence

Lesson Introduction

You may be thinking that geometric sequences are similar to arithmetic sequences. Consider the following example.

On the first day of your new job, you are offered the choice of two salaries for 30 days' work. The first salary consists of \$0.01 on the first day, \$0.02 on the second day, \$0.04 on the third day, and so on. In other words, this salary doubles every day. The second salary consists of \$100 on the first day, \$200 on the second day, \$300 on the third day, and so on. In other words, you get paid an additional \$100 every day with this salary. Which salary would you choose? Charts comparing salaries after 30 days are shown on the next page.

Doubling	
Day	**Salary**
1	$0.01
2	$0.02
3	$0.04
4	$0.08
5	$0.16
6	$0.32
7	$0.64
8	$1.28
9	$2.56
10	$5.12
11	$10.24
12	$20.48
13	$40.96
14	$81.92
15	$163.84
16	$327.68
17	$655.36
18	$1,310.72
19	$2,621.44
20	$5,242.88
21	$10,485.76
22	$20,971.52
23	$41,943.04
24	$83,886.08
25	$167,772.16
26	$335,544.32
27	$671,088.64
28	$1,342,177.28
29	$2,684,354.56
30	$5,368,709.12

Adding	
Day	**Salary**
1	$100
2	$200
3	$300
4	$400
5	$500
6	$600
7	$700
8	$800
9	$900
10	$1000
11	$1,100
12	$1,200
13	$1,300
14	$1,400
15	$1,500
16	$1,600
17	$1,700
18	$1,800
19	$1,900
20	$2,000
21	$2,100
22	$2,200
23	$2,300
24	$2,400
25	$2,500
26	$2,600
27	$2,700
28	$2,800
29	$2,900
30	$3,000

Many people's first instinct is to choose the salary that starts at $100. However, after one month, the salary beginning at $0.01 will be much larger. Are you surprised?

The salary starting at $100 is an example of an arithmetic sequence. The salary starting at $0.01 is an example of a geometric sequence. As you can see, geometric sequences grow larger faster than arithmetic sequences do. This is one of the main differences between the two types of sequences.

Geometric Sequences

Arithmetic sequences are sequences where the differences between consecutive terms are equal. This value is called the common difference. **Geometric sequences** are sequences where the ratios of consecutive terms are equal. This ratio is called the **common ratio**. It is found by dividing any term by its preceding term.

Include these definitions on your resource sheet.

In arithmetic sequences, you go from one term to the next by adding the same value, the common difference, to each term to get the next term. In geometric sequences, you go from one term to the next by multiplying each term by the same value, the common ratio, to get the next term.

Consider the following sequence:

3, 6, 12, 24

The ratios of consecutive terms are equal:

$$\frac{6}{3} = 2$$

$$\frac{12}{6} = 2$$

$$\frac{24}{12} = 2$$

Therefore, this is a geometric sequence with a common ratio of 2.

Example 1

Which of the following sequences are geometric? Identify the common ratio if the sequence is geometric.

a) 2, 6, 18, 54, . . .

b) 2, −6, 18, −54, . . .

c) 63, 44, 25, 6, . . .

d) 64, 32, 16, 8, . . .

e) 1, −1, 1, −1, . . .

f) 2, 4, 6, 8, . . .

Solution

a) $\frac{6}{2} = 3; \frac{18}{6} = 3; \frac{54}{18} = 3$

This sequence is geometric because the ratios of consecutive terms are equal. The common ratio is 3.

b) $\frac{-6}{2} = -3; \frac{18}{-6} = -3; \frac{-54}{18} = -3$

This sequence is geometric. The common ratio is -3.

c) $\frac{44}{63} \approx 0.698; \frac{25}{44} \approx 0.568$ Note that $\approx$ is the symbol for "approximately."

Thus, $\frac{44}{63} \neq \frac{25}{44}$.

This sequence is not geometric because the ratios of consecutive terms are not equal.

d) $\frac{32}{64} = \frac{1}{2}; \frac{16}{32} = \frac{1}{2}; \frac{8}{16} = \frac{1}{2}$

This sequence is geometric. The common ratio is $\frac{1}{2}$.

e) $\frac{-1}{1} = -1; \frac{1}{-1} = -1; \frac{-1}{1} = -1$

This sequence is geometric. The common ratio is -1.

f) $\frac{4}{2} = 2; \frac{6}{4} = \frac{3}{2}$

Thus, $2 \neq \frac{3}{2}$.

This sequence is not geometric because the ratios of consecutive terms are not equal.

Recursive Formula

Just as arithmetic sequences could be written with a recursive formula if we know t_1 and d, so can geometric sequences. You may recall that arithmetic sequences were written in two parts, such as the following:

$$\begin{cases} t_1 = 5 \\ t_n = t_{n-1} - 4,\ n \geq 2 \end{cases}$$

Geometric sequences can also be written as a recursive formula consisting of two parts. Consider the geometric sequences 1, 2, 4, 8, 16, . . .

Each term in the sequence is double the preceding term:

$$\frac{2}{1} = 2;\ \frac{4}{2} = 2;\ \frac{8}{4} = 2;\ \frac{16}{8} = 2$$

You can denote this pattern by stating that:

$t_1 = 1$ (where t_1 is the first term)

$t_2 = 2t_1$ (where t_2 is double the first term)

$t_3 = 2t_2$ (where t_3 is double the second term)

In general $t_n = 2t_{n-1}$, $n \geq 2$ (where the nth term is double the previous term).

The definition for this geometric sequence consists of two statements:

$$\begin{cases} t_1 = 1 \\ t_n = 2t_{n-1},\ n \geq 2 \end{cases}$$

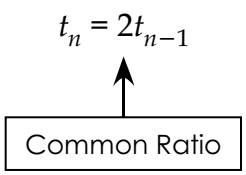
Both parts of the statement are required.

The definition states the following:

a) The first number is 1 ($t_1 = 1$)

b) Any other number after the first is double the previous number ($t_n = 2t_{n-1}$, when $n \geq 2$)

In the sequence 1, 2, 4, 8, 16, . . ., the common ratio is 2. If you consider the corresponding recursive formula, you will notice that the common ratio of '2' is being multiplied by t_{n-1} to get t_n.



Example 2

Write the first five terms of the geometric sequences defined by

a) $\begin{cases} t_1 = 3 \\ t_n = 5t_{n-1},\ n \geq 2 \end{cases}$

b) $\begin{cases} t_1 = 20 \\ t_n = \frac{1}{2}t_{n-1},\ n \geq 2 \end{cases}$

Solution

a)

n	t_n
1	$t_1 = 3$
2	$t_2 = 5(t_{2-1}) = 5t_1 = 5(3) = 15$
3	$t_3 = 5(t_{3-1}) = 5t_2 = 5(15) = 75$
4	$t_4 = 5(t_{4-1}) = 5t_3 = 5(75) = 375$
5	$t_5 = 5(t_{5-1}) = 5t_4 = 5(375) = 1875$

Therefore, the first five terms of this geometric sequence are 3, 15, 75, 375, and 1875.

b)

n	t_n
1	$t_1 = 20$
2	$t_2 = \frac{1}{2}t_{2-1} = \frac{1}{2}t_1 = \frac{1}{2}(20) = 10$
3	$t_3 = \frac{1}{2}t_{3-1} = \frac{1}{2}t_2 = \frac{1}{2}(10) = 5$
4	$t_4 = \frac{1}{2}t_{4-1} = \frac{1}{2}t_3 = \frac{1}{2}(5) = 2.5$
5	$t_5 = \frac{1}{2}t_{5-1} = \frac{1}{2}t_4 = \frac{1}{2}(2.5) = 1.25$

Therefore, the first five terms of this geometric sequence are 20, 10, 5, 2.5, and 1.25.

Example 3

Find the recursive formula for each of the following geometric sequences.

a) $27, 9, 3, 1, \frac{1}{3}, \ldots$

b) $2, 8, 32, 128, \ldots$

Solution

a) The first term is 27. After that, each term is being multiplied by $\frac{1}{3}$ to get the following term. Therefore, the recursive formula is:

$$\begin{cases} t_1 = 27 \\ t_n = \frac{1}{3} t_{n-1}, \; n \geq 2 \end{cases}$$

b) The first term is 2. After that, each term is being multiplied by 4 to get the following term. Therefore, the recursive formula is:

$$\begin{cases} t_1 = 2 \\ t_n = 4t_{n-1}, \; n \geq 2 \end{cases}$$

Note that, in general, a recursive formula to generate a geometric sequence is

$$\begin{cases} t_1 = \text{first term} \\ t_n = (r)(t_{n-1}), \; n \geq 2 \end{cases}$$

where r is the common ratio.

Defining Exponential Function and General Term Formula

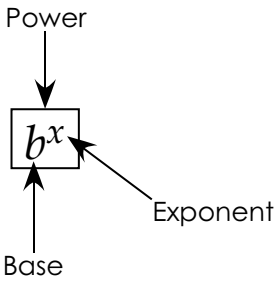
The recursive formula can be useful for finding terms near the beginning of the sequence. Consider the geometric sequence 1, 2, 4, 8, 16, ... If you wanted to find the 10th term, it would be possible to use the recursive formula and keep multiplying previous terms by '2' until you have 10 terms in the sequence. However, it would be difficult to find the 789th term in this way. To find terms increasingly farther away from the beginning of the sequence, a different method is needed. This new method is called *finding a defining exponential function*.

Note: An **exponential function** is a function in the form of:

$y = a(b^x), \; b \neq 0$

In this function, b represents the base. In all exponential functions, the independent variable appears as an exponent. The coefficient of b^x is a. It may be helpful for you to recall the basic form of a power.

Recall:



To develop a new method for finding the nth term of a geometric sequence, consider the following pattern. Remember each term is found by multiplying the previous term by the common ratio, r.

t_1

$t_2 = (t_1)(r) = t_1 r$

$t_3 = (t_1\, r)(r) = t_1 r^2$

$t_4 = (t_1\, r^2)(r) = t_1 r^3$

In general, $t_n = t_1 r^{n-1}$, because the exponent on r in any term is always one less than the term number.

This formula, $t_n = t_1 r^{n-1}$ is called the *defining exponential function for a geometric sequence* where $r \neq 0$, t_1 is the first term in the geometric sequence, n is the number of terms in the sequence, and r is the common ratio of the geometric sequence. This formula is called the *general term formula*. The value of n is a natural number.

Include this formula on your resource sheet.

Example 4

Identify the first term and the common ratio in each of the following geometric sequences.

a) $t_n = 4\left(\frac{1}{2}\right)^{n-1}$

b) $t_n = 3(5)^{n-1}$

Solution

a) $t_n = 4\left(\frac{1}{2}\right)^{n-1}$

n	t_n
1	$t_1 = 4\left(\frac{1}{2}\right)^{1-1} = 4\left(\frac{1}{2}\right)^{0} = 4(1) = 4$
2	$t_2 = 4\left(\frac{1}{2}\right)^{2-1} = 4\left(\frac{1}{2}\right)^{1} = 4\left(\frac{1}{2}\right) = 2$

Thus, $t_1 = 4$ and $r = \frac{2}{4} = \frac{1}{2}$.

Now compare these answers with the numbers in the formula. The coefficient, 4, is the value for t_1, the base, $\frac{1}{2}$, is the value for r, and the exponent of r is $n - 1$.

b) $t_n = 3(5)^{n-1}$

If $n = 1$, $t_1 = 3(5)^{1-1} = 3(5)^0 = 3(1) = 3$.
If $n = 2$, $t_2 = 3(5)^{2-1} = 3(5)^1 = 3(5) = 15$.

Thus, $r = \frac{15}{3} = 5$.

Again, by comparing the answers with the formula, we have the coefficient, 3, is t_1, the base, 5, is the common ratio, and the exponent on r is $n - 1$.

Example 5

Write the first three terms of the geometric sequence generated by each of the following defining exponential functions.

a) $f(x) = 3(5)^{x-1}$

b) $g(x) = 8\left(\frac{-1}{2}\right)^{x-1}$

Solution

The domain of the exponential function that yields a geometric sequence is the set of natural numbers N = {1, 2, 3, . . .}. Therefore, these are the values that you must substitute for x in order to find the first few terms of a geometric sequence.

a)

x	$f(x) = 3(5)^{x-1}$
1	$f(1) = 3(5)^{1-1} = 3(5)^0 = 3(1) = 3$
2	$f(2) = 3(5)^{2-1} = 3(5)^1 = 3(5) = 15$
3	$f(3) = 3(5)^{3-1} = 3(5)^2 = 3(25) = 75$

The first three terms are 3, 15, and 75.

b)

x	$g(x) = 8\left(\frac{-1}{2}\right)^{x-1}$
1	$g(1) = 8\left(\frac{-1}{2}\right)^{1-1} = 8\left(\frac{-1}{2}\right)^0 = 8(1) = 8$
2	$g(2) = 8\left(\frac{-1}{2}\right)^{2-1} = 8\left(\frac{-1}{2}\right)^1 = 8\left(\frac{-1}{2}\right) = -4$
3	$g(3) = 8\left(\frac{-1}{2}\right)^{3-1} = 8\left(\frac{-1}{2}\right)^2 = 8\left(\frac{1}{4}\right) = 2$

The first three terms are 8, -4, and 2.

Exponential functions can be defined in various ways. In this course, you will be using the type of exponential function $f(x) = t_1 r^{x-1}$, where the exponent is $(x - 1)$. This is the most common way of writing the exponential function to generate a geometric sequence. Also, the first term and the common ratio are easier to find in this form. The term number is x and the term value is $f(x)$.

The solution for the next example involves solving an exponential equation. In previous courses you have learned how to solve these types of equations. Recall that in order to solve equations where the variable is located in the exponent, you first need to express all powers in the equation as having the same base. Examples of this method are shown below.

a) $3^x = 9^4$

Solution

$3^x = (3^2)^4$	Change the right side of the equation to base 3
$3^x = 3^8$	Power rule for exponents
$x = 8$	Equate the exponents

b) $2^x(4^{3-2x}) = 16^{x+1}$

Solution

$2^x(2^2)^{3-2x} = (2^4)^{x+1}$	Change each power to a base of 2
$2^x 2^{2(3-2x)} = 2^{4(x+1)}$	Exponent rules
$2^{x+6-4x} = 2^{4x+4}$	Simplify
$-3x + 6 = 4x + 4$	Equate exponents
$6 - 4 = 4x + 3x$	Solve
$2 = 7x$	Solve
$\frac{2}{7} = x$	Solve

Now try an example using a geometric sequence.

Example 6

Which term of the geometric sequence defined by $h(n) = \left(\frac{1}{9}\right)(3)^{n-1}$ is 81?

Solution

In the formula, we are given the term value, $h(n)$, and are to solve for the term number, n. Substitute $h(n) = 81$ to get:

$$81 = \frac{1}{9}(3)^{n-1}$$

$(81)(9) = (9) \cdot \frac{1}{9}(3)^{n-1}$	Multiply both sides of the equation by 9
$729 = 3^{n-1}$	Simplify
$3^6 = 3^{n-1}$	Change 729 to 3^6 by using your calculator to guess and check
$6 = n - 1$	If bases are the same, you can equate the exponents
$7 = n$	Solve for n

Therefore, 81 is the seventh term, or $t_7 = 81$.

Using the Defining Exponential Function

Given a defining exponential function, you can find any required information about a geometric sequence. Reversing the problem, given some information about a geometric sequence, you should be able to develop the defining exponential function.

Example 7

Write a defining exponential function that generates each of the following geometric sequences.

a) 4, 8, 16, 32, 64, . . .

b) 81, 27, 9, 3, 1, . . .

Solution

Using the defining exponential function (also called the general term formula): $t_n = t_1 r^{n-1}$, $r \neq 0$.

a) $t_1 = 4$ and the common ratio, $r = 2$. Therefore, the function is $t_n = 4(2)^{n-1}$.

b) $t_1 = 81$ and the common ratio, $r = \frac{27}{81} = \frac{1}{3}$. Therefore, the function is

$$t_n = 81\left(\frac{1}{3}\right)^{n-1}.$$

Example 8

Write an expression for the 50th term of the geometric sequence 5, 10, 20, 40, 80, . . .

Solution

$$r = \frac{10}{5} = 2 \text{ and } t_1 = 5$$

The corresponding exponential function is $t_n = 5(2)^{n-1}$ and the fiftieth term, $t_{50} = 5(2)^{50-1} = 5(2)^{49} \approx 2.81 \times 10^{15}$, where "≈" is symbolic of the phrase "almost equal to."

Note:

- Since the domain of the exponential function that yields a geometric sequence is the set of natural numbers, the variable used in defining the function is usually n instead of x. However, it really does not matter which variable you use.
- To be able to write the defining exponential function of a geometric sequence, all you need is the first term, t_1, the common ratio r, and the formula $t_n = t_1(r)^{n-1}$, $r \neq 0$.

Example 9

In a geometric sequence, $t_3 = 50$ and $t_6 = 6250$.

a) Write an expression for t_{15}.

b) Which term of the sequence is 156 250?

Solution

a) Using the general term formula, $t_n = t_1 r^{n-1}$, $r \neq 0$, set up two equations from the given information.

If $n = 6$, $t_6 = 6250$. Write $t_1 r^{6-1} = 6250$, or $t_1 r^5 = 6250$.

If $n = 3$, $t_3 = 50$. Write $t_1 r^{3-1} = 50$ or $t_1 r^2 = 50$.

To solve this system of equations, it is possible to use one of two methods—the division method or the substitution method—to isolate for one of the variables.

Division Method:

The two equations are:

$t_1 r^5 = 6250$

$t_1 r^2 = 50$

Divide the equations:

$$\frac{t_1 r^5}{t_1 r^2} = \frac{6250}{50}$$

$\frac{r^5}{r^2} = 125$	Cancel t_1
$r^{5-2} = 125$	Exponent rule for division
$r^3 = 125$	Simplify
$r = 5$	Cube root

Substitution Method

The two equations are

$t_1 r^5 = 6250$

$t_1 r^2 = 50$

Isolating for t_1 in the first equation gives: $t_1 = \frac{6250}{r^5}$

Substituting this value in for t_1 in the second equation gives

$$\frac{6250}{r^5} \cdot r^2 = 50$$

$\frac{6250}{r^3} = 50$	Simplify
$6250 = 50r^3$	Multiply by r^3
$\frac{6250}{50} = r^3$	Divide by 50
$125 = r^3$	Simplify
$5 = r$	Cube root

After finding r using either the division method or the substitution method, it is possible to solve for t_1.

Substitute $r = 5$ into either equation.

$t_1 r^2 = 50$

$t_1(5)^2 = 50$

$t_1(25) = 50$

$t_1 = 2$

Therefore, the general formula is $t_n = 2(5)^{n-1}$.

Now you can find t_{15}.

$t_{15} = 2(5)^{15-1} = 2(5)^{14} \approx 1.22 \times 10^{10}$

b) You are given the value of the term and need to solve for the term number. Thus, substitute 156 250 for t_n.

$$t_n = 2(5)^{n-1}$$

$$156\ 250 = 2(5)^{n-1} \quad \text{(Substitute.)}$$

$$\frac{156\ 250}{2} = \frac{2(5)^{n-1}}{2} \quad \text{(Divide by 2.)}$$

$$78\ 125 = 5^{n-1} \quad \text{(Simplify.)}$$

$$5^7 = 5^{n-1} \quad \text{(Change 78 125 into a power of 5 using your calculator to guess and check.)}$$

$$7 = n - 1 \quad \text{(Since bases are now the same, you can equate the exponents.)}$$

$$8 = n \quad \text{(Solve for } n\text{.)}$$

Therefore, 156 250 is the eighth term.

Note: As soon as you find t_1 and r, you can state the defining exponential function and answer any question pertaining to the geometric sequence.

Example 10

Consider the example from the introduction to this lesson. The first salary option consists of the following sequence of numbers:

0.01, 0.02, 0.04, . . .

The second salary option consists of the following sequence of numbers:

100, 200, 300, . . .

a) Write the defining linear function for the arithmetic sequence.
b) Write the defining exponential function for the geometric sequence.
c) Find the 30th term of the arithmetic sequence algebraically.
d) Find the 30th term of the geometric sequence algebraically.
e) Is the 30th term larger in the geometric sequence or in the arithmetic sequence? Explain.

Solution

a) The arithmetic sequence for the second option is: 100, 200, 300, . . .

The function consists of the following ordered pairs: (1, 100), (2, 200), (3, 300), ... Therefore, you can use the slope to find the defining linear function.

$$m = \frac{200 - 100}{2 - 1} = 100$$

The equation is: $y - y_1 = m(x - x_1)$

$$y - 100 = 100(x - 1)$$

$$y = 100x - 100 + 100$$

$$y = 100x$$

The defining function is $f(x) = 100x$, where x represents how many days on the job and $f(x)$ represents your salary for that day.

b) The geometric sequence for the first option is: 0.01, 0.02, 0.04, . . .

$t_1 = 0.01$

$$r = \frac{0.02}{0.01} = 2$$

Defining exponential function: $t_n = t_1 r^{n-1}$

$$t_n = (0.01)(2)^{n-1}$$

where n represents the number of days on the job and t_n represents the salary for that day.

c) $f(x) = 100x$

$f(30) = (100)(30) = \$3000$

Therefore, the 30th term of the arithmetic sequence is \$3000. In other words, your salary on the 30th day would be \$3000 if you chose this option.

d) $t_n = (0.01)(2)^{n-1}$

$t_{30} = (0.01)(2)^{30-1}$

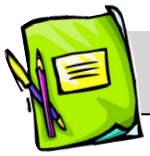
$t_{30} = (0.01)(2)^{29}$

$t_{30} = 5{,}368{,}709.12$

Therefore, the 30th term of the geometric sequence is \$5,368,709.12. In other words, your salary on the 30th day would be \$5,368,709.12 if you chose this option.

e) The 30th term in the geometric sequence is significantly larger. This is because geometric sequences, even when beginning at smaller values, grow significantly larger than arithmetic sequences.

Now that you have learned all about geometric sequences, it is time to complete the following learning activity. This learning activity will help you to practice the skills you have just learned and help you do well on your next assignment.



Learning Activity 1.4

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. On the first day of your marathon training, you run 1 km. If you double that distance every day, on which day will you run over 20 km?
2. Complete the pattern: 1, 2, 4, 7, _____ , _____ , _____
3. You and two friends want to go to a movie but you are arguing over the seating arrangements. If you all sit beside each other in a row, how many different ways can you be seated?
4. State the common difference of the following arithmetic sequence: 7, 5, 3, 1, −1, ...
5. Estimate the answer: 204 × 712
6. How many inches are in 6.5 feet?
7. Determine the volume of a rectangular prism with base dimensions of 12 cm by 10 cm and a height of 6 cm.
8. Write this decimal as a percent: 0.0091

continued

Learning Activity 1.4 (continued)

Part B: Geometric Sequences

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. For each sequence below, indicate whether it is geometric, arithmetic, or neither. If it is geometric, state the value of r, the common ratio; if it is arithmetic, state the value of d, the common difference.
   a) 4, 6, 8, 10, . . .
   b) 3, 6, 12, 24, . . .
   c) 18, 15, 12, 9, . . .
   d) 80, 40, 20, 10, . . .
   e) 1, −5, 25, −125, . . .

2. Write the first four terms of each sequence. If it is geometric, state the value of r, the common ratio; if it is arithmetic, state the value of d, the common difference.
   a) $\begin{cases} t_1 = 2 \\ t_n = 2t_{n-1},\ n \geq 2 \end{cases}$
   b) $\begin{cases} t_1 = 2 \\ t_n = t_{n-1} + 2,\ n \geq 2 \end{cases}$
   c) $\begin{cases} t_1 = 2 \\ t_n = 2t_{n-1} + 2,\ n \geq 2 \end{cases}$
   d) $\begin{cases} t_1 = 2 \\ t_n = \left(t_{n-1}\right)^2,\ n \geq 2 \end{cases}$
   e) $\begin{cases} t_1 = 5 \\ t_n = (-1)^n t_{n-1},\ n \geq 2 \end{cases}$
   f) $\begin{cases} t_1 = 4 \\ t_n = t_{n-1} - 5,\ n \geq 2 \end{cases}$

continued

Learning Activity 1.4 (continued)

3. Write the first three terms of the geometric sequence generated by each of the following exponential functions. Notice none of them are written in the usual form, $t_n = t_1 r^{n-1}$, but all of them generate a geometric sequence.
   a) $f(x) = 2^x$
   b) $t_n = 4^{n-1}$
   c) $f(x) = 2(3^x)$
   d) $t_n = 16\left(\frac{1}{2}\right)^n$

4. Write a recursive formula and a defining exponential function to generate each of the following geometric sequences.
   a) 4, 8, 16, 32, 64, . . .
   b) 6, 18, 54, 162, 486, . . .
   c) 1, −2, 4, −8, 16, . . .
   d) 40, 20, 10, 5, $\frac{5}{2}$, . . .

5. Write an expression for t_{20} of the sequence 3, 6, 12, 24, ... Do not simplify.

6. Which term of the geometric sequence 4, 2, 1, . . . has the value $\frac{1}{1024}$?

7. In a geometric sequence, $t_2 = 6$ and $t_4 = 54$, find the value of t_1.

8. What is the value of x if the sequence, $x + 2$, $3x - 2$, $5x + 2$ is geometric?

9. Insert three numbers a, b, c between 5 and 80, so that 5, a, b, c, 80 forms a geometric sequence.

10. A store is conducting a Dutch auction. It will take 10% off the cost of an item at the end of each day the item doesn't sell. If an item originally costs \$150, find its cost for each of the next five days.

continued

Learning Activity 1.4 (continued)

11. A geometric sequence is defined by the exponential function $t_n = 5(2)^{n-1}$
    a) What is the first term?
    b) What is the common ratio?
    c) What is the value of the 10th term?
    d) What is the value of t_7?
    e) What position in the sequence has the value 163 840?
    f) Is the number 4100 a member of this geometric sequence? Justify your answer.
    g) Graph the sequence using graph paper or your graphing calculator.

Lesson Summary

In this lesson, you learned about geometric sequences. You learned how geometric sequences grow larger faster than arithmetic sequences. Also, geometric sequences have a common ratio while arithmetic sequences have a common difference. Just as arithmetic sequences have corresponding arithmetic series, geometric sequences have corresponding geometric series. In the next lesson, you will be learning about geometric series.

Lesson 4: The Sum of a Finite Geometric Series

Lesson Focus

In this lesson, you will

- ❑ learn the difference between a geometric sequence and a geometric series
- ❑ learn how to use sigma notation to represent the sum of a geometric series
- ❑ learn how to derive a rule for finding the sum of a finite geometric series
- ❑ learn how to find the first term, the common ratio, and the number of terms in a geometric series

Lesson Introduction

Geometric series appear in nature, economics, and chain emails. Each of these situations involves increasing growth that spreads farther and farther with time. This growth, however, could end. For example, you could decide to withdraw all of the money from a bank account. In this case, the economic growth you created by depositing money has now stopped. A simpler example in which growth stops is if you receive a chain email and decide to delete it rather than forwarding it along. A geometric series can have an infinite number of terms. In this lesson, you will be dealing with situations where the growth has stopped, or where you have a finite geometric series.

In the last lesson, you learned how geometric sequences grow larger and faster than arithmetic sequences. As a result, the sum of a geometric series—even a finite geometric series—is usually significantly larger than the sum of an arithmetic series with similar initial conditions.

Geometric Series

The only difference between a geometric sequence and a geometric series is the presence of addition.

A **geometric sequence** is the *range* of an exponential function, $y = a(b^x)$, with domain N where $N = \{1, 2, 3, 4, \ldots\}$. The ratio of any two consecutive values is a constant. A **geometric series** is the *sum* of the terms of a geometric sequence. For example,

1, 2, 4, 8, . . . is a geometric sequence.

1 + 2 + 4 + 8 + . . . is the corresponding geometric series.

It may be helpful for you to include these definitions and examples on your resource sheet.

Sigma Notation

When you were finding the sum of an arithmetic series, you used "S_n" to represent the sum of n terms of a series. The Greek capital letter Σ, read as sigma, can also be used as a symbol for sum.

The sum of the first n terms of the sequence $t_1, t_2, t_3, \ldots, t_n$ is written as

$$S_n = \sum_{i=1}^{n} t_i = t_1 + t_2 + t_3 + \ldots + t_n$$

The symbol $\sum_{i=1}^{n} t_i$ is referred to as **sigma notation**.

The series, $t_1 + t_2 + t_3 + \ldots + t_n$, is the **expanded form** of the sigma notation.

Include these definitions on your resource sheet.

Notes:

1. The symbol $\sum_{i=1}^{4} i^2$ is read as "the sum of the squares from $i = 1$ to $i = 4$" and indicates that the series is to start with $i = 1$ and end with $i = 4$. The symbol i is called the index.
2. The index can begin and end at any term. It does not have to start at the value 1.
3. The letter used for the index is arbitrary. The most commonly used letters for the index are i and k.

Example 1

Write in expanded form and find the sum.

a) $\sum_{i=1}^{4} i^2$

b) $\sum_{i=1}^{3} 3(2)^{i-1}$

c) $\sum_{i=4}^{6} (2i + 3)$

Solution

a)

i	i^2
1	$(1)^2 = 1$
2	$(2)^2 = 4$
3	$(3)^2 = 9$
4	$(4)^2 = 16$

$$\sum_{i=1}^{4} i^2 = 1 + 4 + 9 + 16 = 30$$

b)

i	$3(2)^{i-1}$
1	$3(2)^{1-1} = 3(2)^0 = 3(1) = 3$
2	$3(2)^{2-1} = 3(2)^1 = 3(2) = 6$
3	$3(2)^{3-1} = 3(2)^2 = 3(4) = 12$

$$\sum_{i=1}^{3} 3(2)^{i-1} = 3 + 6 + 12 = 21$$

c)

i	$2i + 3$
4	$2(4) + 3 = 8 + 3 = 11$
5	$2(5) + 3 = 10 + 3 = 13$
6	$2(6) + 3 = 12 + 3 = 15$

$$\sum_{i=4}^{6} (2i + 3) = 11 + 13 + 15 = 39$$

Reversing the notation has more steps and requires you to generate a defining formula for the sequence.

Example 2

Write the sum using sigma notation. Find the first term and the common ratio.

a) $2 + 4 + 8 + 16 + 32$

b) $2 + 6 + 18 + 54$

Solution

a) $t_1 = 2$

$r = 2$

Using the formula for the general term, $t_n = t_1 r^{n-1}$,

$t_n = 2(2)^{n-1} = 2^1 \cdot 2^{n-1} = 2^{1+n-1} = 2^n$.

Now use the formula and the sigma notation for 5 terms:

$$\sum_{k=1}^{5} 2^k \text{ or } \sum_{k=1}^{5} (2)\left(2^{k-1}\right)$$

b) $t_1 = 2$

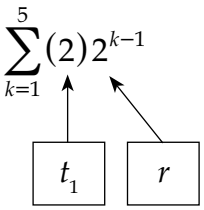
$r = 3$

Use the general term formula, $t_n = 2(3)^{n-1}$, to write the sigma notation for 4 terms.

$$\sum_{k=1}^{4} (2)\left(3^{k-1}\right)$$

Note: When you write a geometric series in sigma notation and the exponent is in the form "$k - 1$" or "$i - 1$," it is easy to recognize the first term and the common ratio of the geometric series.

In (a), the geometric series written in sigma notation was $\sum_{k=1}^{5} (2)2^{k-1}$. In this form, the coefficient is the first term and is the value being multiplied by the power. Also, the common ratio is the base of the exponent. Consider the following diagram.



In (b), the geometric series written in sigma notation was $\sum_{k=1}^{4}(2)3^{k-1}$. In this form, the first term, 2, is the value being multiplied by the power. Also, the common ratio, 3, is the base of the exponent.

Note: When the exponent is just "k" or "i," you may notice that the same analysis won't work. The common ratio, r, will always be the base of the exponent. However, t_1 will be harder to find when the exponent is "k" instead of "$k - 1$" or "i" instead of "$i - 1$."

Sum of a Finite Geometric Series

Expanding and finding the sum of the geometric series defined by $\sum_{k=1}^{5}3(4)^k$ can be done relatively quickly, but finding the sum of $\sum_{k=1}^{50}3(4)^k$ takes much more time. If a series is geometric, a formula can be derived to find these sums. First, let's look at an example so that the steps of the formula proof make sense.

To find the sum of $3 + 6 + 12 + 24 + 48$:

$S_5 = 3 + 6 + 12 + 24 + 48$	$r = 2, n = 5$
$2S_5 = \quad 6 + 12 + 24 + 48 + 96$	Multiply each term by $r = 2$ and adjust where you write your answer.
$S_5 - 2S_5 = 3 - 96$	Subtract; the terms on the right side cancel out except for the first and last.
$-S_5 = -93$	Simplify.
$S_5 = 93$	Divide by (–1).

93 is the sum of the geometric series $3 + 6 + 12 + 24 + 48$.

The multiplication by r, followed by the subtraction, eliminates all terms except the difference of the first term and the product of r times the last term. Finding the sum of the series in this way is easier than adding up individual terms.

You can follow the same steps to develop a general formula.

Formula for Finding the Sum of a Finite Geometric Series

The sum of the first n terms of a geometric series is

$$S_n = \frac{t_1\left(1 - r^n\right)}{1 - r}, \text{ provided } r \neq 1$$

Proof:

$S_n = t_1 + t_2 + t_3 + \ldots + t_n$	Geometric series
$S_n = t_1 + t_1 r + t_1 r^2 + \ldots + t_1 r^{n-2} + t_1 r^{n-1}$	Write each term with only t_1 and r
$rS_n = \quad t_1 r + t_1 r^2 + t_1 r^3 + \ldots + t_1 r^{n-1} + t_1 r^n$	Multiply by r
$S_n - rS_n = t_1 \quad - t_1 r^n$	Subtract
$S_n(1 - r) = t_1\left(1 - r^n\right)$	Factor
$S_n = \frac{t_1\left(1 - r^n\right)}{(1 - r)}$, where $r \neq 1$	Divide

where: t_1 is the first term in the series

n is the number of terms

r is the common ratio

S_n is the sum of the first n terms

Include this formula on your resource sheet.

Note: The formula for the sum of a geometric series includes a restriction on the common ratio. Because division by zero is not allowed, the formula needs the restriction so it can be evaluated.

What would happen if a geometric series had a common ratio of 1? An example would be 4 + 4 + 4 + 4 + 4.

Note that $t_1 = 4$, $r = 1$, and $n = 5$. To find the sum, you could just multiply $t_1 \cdot n$ or $4 \cdot 5 = 20$.

Example 3

Use the formula to find the sum of the following geometric series.

a) $3 + 6 + 12 + 24 + 48$

b) $\sum_{k=1}^{50} 3(4)^k$

Solution

Using the formula $S_n = \dfrac{t_1\left(1 - r^n\right)}{1 - r}$, $r \neq 1$, you get:

a) $S_n = \dfrac{3\left(1 - (2)^5\right)}{1 - 2} = \dfrac{3(-31)}{-1} = 93,$

which agrees with the answer in the previous example.

b) From the sigma notation $\sum_{k=1}^{50} 3(4)^k$, you notice that the common ratio is 4,

as it is the base being raised to the exponent "k" in this sigma notation. If you did not recognize 4 as the common ratio, it is also possible to find two terms of the series and find their common ratio.

$t_1 = 3(4)^1 = 12$

$t_2 = 3(4)^2 = 3(16) = 48$

$r = \dfrac{48}{12} = 4$

Therefore, the common ratio is 4.

$n = 50$

$\therefore S_{50} = \dfrac{t_1\left(1 - r^n\right)}{1 - r} = \dfrac{12\left(1 - 4^{50}\right)}{1 - 4} \approx 5.07 \times 10^{30}$

This is a very large positive number!

However, you were able to find the sum quickly with the use of a formula.

Example 4

Find the sum of the first 10 terms of the geometric series
$3 - 6 + 12 - 24 + \ldots$.

Solution

Using the formula $S_n = \dfrac{t_1\left(1 - r^n\right)}{1 - r}$, you get

$$S_{10} = \frac{3\left(1 - (-2)^{10}\right)}{1 - (-2)} = \frac{3(1 - 1024)}{3} = -1023$$

Using the formula $S_n = \frac{t_1\left(1 - r^n\right)}{1 - r}$, it is possible to solve for any of the values for S_n, t_1, r, and n when given enough information. Consider the following example.

Example 5

The sum of a geometric series with $t_1 = 5$ is 635. If $r = 2$, how many terms are in this geometric series?

Solution

You have the following information:

$S_n = 635$

$t_1 = 5$

$r = 2$

Using the formula $S_n = \frac{t_1\left(1 - r^n\right)}{1 - r}$, it is possible to solve for n, the number of terms in this series.

$$635 = \frac{5\left(1 - 2^n\right)}{1 - 2}$$

$$635 = \frac{5\left(1 - 2^n\right)}{-1}$$

$$-635 = 5\left(1 - 2^n\right)$$

$$\frac{-635}{5} = \frac{5\left(1 - 2^n\right)}{5}$$

$$-127 = 1 - 2^n$$

$$-127 - 1 = -2^n$$

$$-128 = -2^n$$

$$(-1)(-128) = (-1)\left(-2^n\right)$$

$$128 = 2^n$$

$$2^7 = 2^n$$

$$n = 7$$

Therefore, there are 7 terms in this series.

Learning Activity 1.5

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. For Josh's workout, he runs $\frac{1}{9}$ of the football field, sprints $\frac{7}{9}$ of the field, and then jogs the rest of the way. What fraction of the field does Josh jog?
2. What is the sum of the digits from 1 to 10?
3. Calculate the perimeter of a square garden with an area of 25 m^2.
4. Express $\frac{6}{20}$ as a decimal.
5. Find the common difference in the following pattern: $-2, -9, -16, -23, \ldots$
6. Estimate: $\sqrt{96}$
7. Simplify: $(2^4)(3^2)$
8. Find the slope of the line defined by the equation, $3y - 2x + 7 = 0$.

continued

Part B: Sums of Geometric Series

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. a) Give an example of a geometric sequence.
   b) Write the corresponding geometric series.

2. Write in expanded form and find the sum.

   a) $\sum_{k=1}^{4} k^3$ b) $\sum_{k=1}^{4} 3^k$

   c) $\sum_{k=1}^{3} 3k$ d) $\sum_{k=1}^{4} 3^{k-1}$

   e) $\sum_{k=2}^{4} 3^{k+1}$ f) $\sum_{k=2}^{4} 5 \cdot 3^{k-1}$

   g) $\sum_{k=2}^{5} (2k - 5)$ h) $\sum_{k=3}^{6} 2(-3)^{k-1}$

3. Use sigma notation to write each geometric sum.
   a) $4 + 12 + 36 + 108$
   b) $24 + 4 + \frac{2}{3} + \frac{1}{9} + \frac{1}{54}$
   c) $2 + 4 + 8 + \ldots + 1024$

4. Consider the sequence 3, 6, 12, 24, . . .
   a) Find the eighth term.
   b) Find the sum of the first eight terms, using the sum formula.

continued

Learning Activity 1.5 (continued)

5. Find the sum of the first 10 terms of each series. If the numerical answer is not an integer, state it correct to two decimal places.
   a) $120 + 60 + 30 + 15 + \ldots$
   b) $8 + 12 + 18 + 27 + \ldots$
   c) $128 - 64 + 32 - 16 + \ldots$
   d) $24 + 8 + \frac{8}{3} + \frac{8}{9} + \ldots$
   e) $x^2 + x^4 + x^6 + x^8 + \ldots$
   f) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

6. In a geometric sequence, $t_4 = \frac{1}{4}$, and $t_{11} = 32$. Calculate S_{11}. Express your answer as an exact value.

7. The sum of a geometric series with $t_1 = 7$ is 889. If $r = 2$, how many terms are in this geometric series?

8. The sum of a geometric series is 98 301. If there are 15 terms in this sequence and the common ratio is 2, what is the value for t_1?

Lesson Summary

In this lesson, you learned how geometric series are related to geometric sequences. You also learned how to find the sum of a finite geometric series. What happens if you want to find the sum of an infinite geometric series? In the next lesson, you will learn when infinite geometric series have a sum and how to find that sum.

Your next task is to complete a hand-in assignment found on the following pages. Keep all your Module 1 assignments until you finish the module and then send them to the Distance Learning Unit. Instructions for submitting your assignments can be found in the module summary located at the end of this module.

Before you begin, have you completed a resource sheet to use with your assignment?

Notes

Assignment 1.2

Geometric Sequences and Series

Total: 54 marks

1. Write the first five terms of each of these geometric sequences. State the common ratio for each geometric sequence. *(3 marks each, for a total of 9 marks)*

   a) $f(x) = 3^x$

   b) $t_n = 4(2)^{n-1}$

   c) $\begin{cases} t_1 = 7 \\ t_n = 3t_{n-1},\ n \geq 2 \end{cases}$

continued

Assignment 1.2: Geometric Sequences and Series (continued)

2. Write the recursive formula and the defining exponential formula for each geometric sequence. *(2 marks each, for a total of 8 marks)*

   a) 3, 6, 12, 24, . . .

   b) 2, 4, 8, 16, . . .

   c) 3, −9, 27, −81, . . .

   d) 8, −4, 2, −1, . . .

3. Find the value of t_7 in each of the geometric sequences in question 2, using the exponential function formula. *(1 mark each, for a total of 4 marks)*

continued

Assignment 1.2: Geometric Sequences and Series (continued)

4. A geometric sequence is defined by the exponential function $t_n = 16\left(\frac{1}{2}\right)^{n-1}$.

   a) What is the first term? *(1 mark)*

   b) What is the common ratio? *(1 mark)*

   c) What is the value of the 5th term? *(1 mark)*

   d) What is the value of t_7? *(1 mark)*

   e) What position in the sequence is occupied by the value $\frac{1}{32}$? *(2 marks)*

   f) Is the number $\frac{1}{1024}$ a member of this geometric sequence? Justify your answer. *(2 marks)*

continued

Assignment 1.2: Geometric Sequences and Series (continued)

5. Write an expression for t_{25} for each of the following geometric sequences. Simplify your answers and write as an exact value using exponents. *(2 marks each, for a total of 8 marks)*

   a) $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots$

   b) $100, 20, 4, \frac{4}{5}, \frac{4}{25}, \ldots$

   c) $\frac{9}{4}, \frac{3}{2}, 1, \frac{2}{3}, \frac{4}{9}, \ldots$

   d) $3, -2, \frac{4}{3}, \frac{-8}{9}, \frac{16}{27}, \ldots$

continued

Assignment 1.2: Geometric Sequences and Series (continued)

6. Write in expanded form. Do not find the sum. *(1 mark each, for a total of 2 marks)*

   a) $\sum_{k=2}^{6} 3(2)^k$

   b) $\sum_{k=1}^{4} 2\left(\frac{1}{3}\right)^k$

7. Use sigma notation to write each geometric sum. State the values of t_1 and r. *(3 marks each, for a total of 6 marks)*

   a) $\frac{1}{4} + \frac{1}{2} + 1 + 2$

   b) $9 + 3 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27}$

continued

Assignment 1.2: Geometric Sequences and Series (continued)

8. Consider the geometric sequence with $t_1 = 1000$ and $r = 1.05$.
   a) Write the first three terms. *(1 mark)*

   b) Find the sum of the first 20 terms correct to two decimal places. *(2 marks)*

9. Albert sent a chain email to five of his friends. Each of them sent a copy of the email to five different friends. These friends repeated the process. If no person receives more than one email, how many different people have been contacted after seven levels of mailings, counting Albert's own email as the first? *(3 marks)*

10. The sum of a geometric series is 1 398 101. If there are 11 terms in this sequence and the common ratio is 4, what is the value for t_1? *(3 marks)*

Lesson 5: The Sum of an Infinite Geometric Series

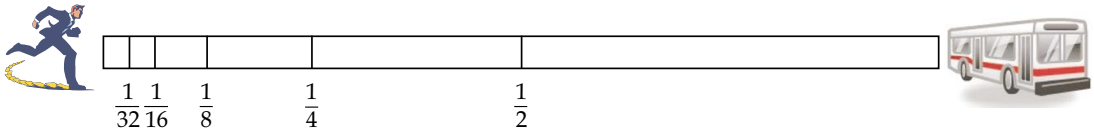
Lesson Focus

In this lesson, you will

- ❑ learn how to distinguish between a finite and an infinite geometric series
- ❑ learn how to tell if a series converges or diverges
- ❑ learn how to derive a rule for finding the sum of an infinite geometric series
- ❑ learn how to determine the sum of a converging infinite geometric series

Lesson Introduction

In the last lesson, you learned how to find the sum of *finite* geometric series. It is also possible to find the sum of *infinite* geometric series. Consider one of Zeno's paradoxes. In this paradox, a man is trying to catch a bus that is stopped at the bus stop. The man realizes he must run the whole distance between where he is standing and the bus. If so, then to catch this bus, the man must also run halfway there. In order to run halfway to the bus, the man must also run a quarter of the way to the bus. In order to run a quarter of the way to the bus, the man must run an eighth of a way to the bus, and so on. Therefore, this man must complete an infinite number of tasks in order to catch the bus. Zeno believed that this was an impossibility. What do you think?



In some situations, finding the sum of an infinite geometric series is impossible. This is not, however, the case in this example. Clearly, finding the sum of an infinite geometric series is not impossible. Zeno's man will catch the bus. Therefore, there must be a way to add up this infinite series of numbers, $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$.

Infinite Geometric Series

In previous lessons, you have found the sums of various arithmetic series and geometric series. These series all had a last term. Therefore, you were finding the sums of **finite series. Infinite series** are series that go on forever and do not end. They have an unlimited number of terms.

$1 + 2 + 3 + \ldots + 100$ is a **finite series**. This series has a last term of 100.

$1 + 2 + 3 + \ldots$ is an **infinite series**. This series does not have a last term.

Include these definitions and examples on your resource sheet.

Consider Zeno's infinite series:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$$

and the *partial sums* associated with this series.

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$$

As you keep adding additional terms, the partial sums will form a pattern.

$$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \ldots, \frac{2^n - 1}{2^n}, \ldots \text{ where } n \text{ is the power of 2}$$

In this pattern, the denominator is always one more than the numerator. Therefore, each term in the pattern gets closer and closer to the number 1. Since the series is infinite, we say "as the number of terms approaches infinity, the limit of this series converges to the number 1" and we write

$$\sum_{k=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{k-1} = 1,$$

where ∞ is the symbol for infinity.

This notation for the sum is frequently written as $S = 1$ or $S_\infty = 1$.

Note: There are two notations used to denote an infinite sum, S and S_∞.

Convergent Series

Because the partial sums of this series form a sequence that gets closer and closer to a specific value, the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ is a **convergent series**.

A **convergent series** is a series with an infinite number of terms in which its sequence of partial sums approaches a fixed value. In other words, a convergent series is a series with an infinite number of terms where S or S_∞ has an answer.

Divergent Series

However, not all infinite geometric series converge. For example, the partial sums of the series

$2 + 4 + 8 + 16 + \ldots$ are

2, 6, 14, 30, . . .

and are not getting closer and closer to any number.

The series $2 + 4 + 8 + \ldots$ is said to diverge and cannot be represented by any number. A **divergent series** is a series with an infinite number of terms in which its sequence of its partial sums does not approach a fixed value. In other words, a divergent series is a series with an infinite number of terms where S or S_∞ does not exist. All series that are not convergent are divergent.

Example 1

Classify these series as convergent or divergent series.

a) $1 + (-1) + 1 + (-1) + 1 \ldots$

b) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \ldots$

c) $2 + 6 + 18 + 54 + \ldots$

d) $9 + 3 + 1 + \frac{1}{3} + \frac{1}{9} + \ldots$

Solution

Convergent series are series where their partial sums get closer and closer to a specific value. Divergent series are series where their partial sums do not get closer and closer to a specific value.

a) Consider the partial sums of this series:

$S_1 = 1$

$S_2 = 1 - 1 = 0$

$S_3 = 1 - 1 + 1 = 1$

$S_4 = 1 - 1 + 1 - 1 = 0$

The sequence of partial sums (1, 0, 1, 0, . . .) is not getting closer to any value. Therefore, the series, $1 + (-1) + 1 + (-1) + 1 + \ldots$, is a divergent series.

b) Consider the partial sums of this series rounded to 5 decimal places:

$S_1 = 1$

$S_2 = 1 + \frac{1}{3} = \frac{4}{3}$

$S_3 = 1 + \frac{1}{3} + \frac{1}{9} = \frac{13}{9} \approx 1.44444$

$S_4 = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} = \frac{40}{27} \approx 1.48148$

$S_5 = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} = \frac{121}{81} \approx 1.49383$

It appears as if the sequence of partial sums, $1, \frac{4}{3}, \frac{13}{9}, \frac{40}{27}, \frac{121}{81}, \ldots,$ is getting closer and closer to 1.5. Therefore, the series $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$ is a convergent series.

c) Consider the partial sums of this series:

$S_1 = 2$

$S_2 = 2 + 6 = 8$

$S_3 = 2 + 6 + 18 = 26$

$S_4 = 2 + 6 + 18 + 54 = 80$

The sequence of partial sums, 2, 8, 26, 80, . . ., is not getting closer to any particular value. Therefore, the series, $2 + 6 + 18 + 54 + \ldots$, is a divergent series.

d) Consider the partial sums of this series:

$S_1 = 9$

$S_2 = 9 + 3 = 12$

$S_3 = 9 + 3 + 1 = 13$

$S_4 = 9 + 3 + 1 + \frac{1}{3} = \frac{40}{3} = 13.33333$

$S_5 = 9 + 3 + 1 + \frac{1}{3} + \frac{1}{9} = \frac{121}{9} = 13.44444$

The sequence of partial sums, $9, 12, 13, \frac{40}{3}, \frac{121}{9}, \ldots,$ appears to be getting closer and closer to 13.5. Therefore, the series, $9 + 3 + 1 + \frac{1}{3} + \ldots,$ is a convergent series.

Converging Infinite Geometric Series

An infinite geometric series will converge only if r is a positive or negative proper fraction. We write this restriction as $-1 < r < 1$, or as $|r| < 1$.

Include this definition on your resource sheet.

Why is this the restriction? Look at the sum of a geometric series formula.

$$S_n = \frac{t_1\left(1 - r^n\right)}{1 - r}$$

and write it in the following fashion by first distributing and then separating into two fractions.

$$S_n = \frac{t_1\left(1 - r^n\right)}{1 - r}$$

$$= \frac{t_1 - t_1 r^n}{1 - r}$$

$$= \frac{t_1}{1 - r} - \frac{t_1 r^n}{1 - r}$$

Assume $-1 < r < 1$. Consider r^n as n gets larger and larger. As n, the exponent, gets larger, the value of r^n will get smaller and much, much smaller, until it becomes virtually zero.

You can try this on your calculator by taking higher and higher powers of an r-value which is a proper fraction, say, -0.25.

Consider the following example when $r = -0.25$.

n	r^n
1	$(-0.25)^1 = -0.25$
2	$(-0.25)^2 = 0.0625$
5	$(-0.25)^5 = -0.000\ 976\ 562\ 5$
10	$(-0.25)^{10} = 0.000\ 000\ 953\ 674\ 316\ 4$

As you can see, r^n is getting closer and closer to zero.

For values of n = one million, or one billion, the entire term $\frac{t_1 r^n}{1-r}$ will become $\frac{t_1(0)}{1-r}$ or zero and can be ignored. Therefore, the sum gets closer and closer to the number $\frac{t_1}{1-r}$ when n gets larger and larger, or as n approaches infinity (∞).

Sum of an Infinite Geometric Series

The infinite geometric series, $t_1 + t_2 + t_3 + \ldots$, converges to the sum S or S_∞, given by the formula

$$S_\infty \text{ or } S = \frac{t_1}{1-r}, \text{ provided } |r| < 1.$$

Note: The restriction on r is very important. If r is not in this range, $-1 < r < 1$, there is no point in using this formula. It will not give the correct answer because the series diverges.

Include this formula on your resource sheet.

Example 2

Find the sum of the infinite geometric series, if possible.

a) $4 + 2 + 1 + \frac{1}{2} + \ldots$

b) $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \ldots$

c) $27 + 36 + 48 + 64 + \ldots$

d) $\sum_{k=1}^{\infty} 3\left(\frac{2}{3}\right)^{k-1}$

e) $\sum_{k=1}^{\infty} \left(\frac{1}{64}\right)(2)^{k-1}$

f) $\sum_{k=1}^{\infty} 25(-2)^{k-1}$

Solution

a) $r = \frac{1}{2}$

Therefore, r satisfies the condition of $-1 < r < 1$.

Therefore, $S = \frac{4}{1 - 0.5} = 8.$

b) $r = -\frac{1}{3}$ and $-1 < -\frac{1}{3} < 1$

Therefore, $S = \frac{1}{1 - \left(-\frac{1}{3}\right)} = \frac{3}{4}.$

c) $r = \frac{36}{27} = \frac{4}{3}$ and $r > 1$

The series diverges. Note that if you use the formula, you get an answer, but it just won't be the correct answer.

$$S_{\infty} = \frac{27}{1 - \frac{4}{3}}$$

$$S_{\infty} = -81$$

The answer doesn't make sense. The terms of the series are all positive. Their sums cannot be negative.

d) $r = \frac{2}{3}$ and $-1 < \frac{2}{3} < 1$

Therefore, $S = \frac{3}{1-\left(\frac{2}{3}\right)} = 9.$

e) $r = 2 > 1$. The series diverges.

f) $r = -2 < -1$. The series diverges.

Example 3

Find the values of x for which the following infinite geometric series converges.

$$1 + (x-1) + (x-1)^2 + (x-1)^3 + \ldots$$

Solution

For a geometric series to converge, the value of r is restricted to numbers between –1 and 1. The value for r is $\frac{t_2}{t_1} = \frac{x-1}{1} = x - 1$. Thus,

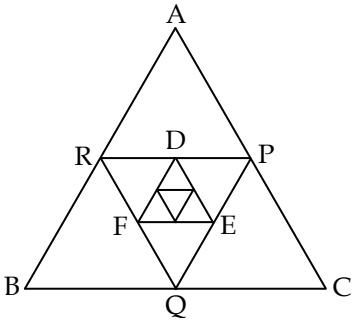
$-1 < r$	< 1	rule
$-1 < x - 1$	< 1	substitute
$-1 + 1 < x - 1 + 1$	$< 1 + 1$	add 1 to each term
$0 < x$	< 2	simplify

This interval, $0 < x < 2$, is called the **interval of convergence**. Therefore, the series $1 + (x - 1) + (x - 1)^2 + (x - 1)^3 + \ldots$ will converge when $0 < x < 2$.

So far, you have seen how to calculate the sum of infinite geometric sequences in artificial situations. Sometimes real-life problems can be solved by finding the sum of an infinite geometric sequence. Consider the following example.

Example 4

Equilateral triangle ABC has a perimeter equal to 12 cm. A new triangle, PQR, is formed by joining the midpoints of the sides of ΔABC. The midpoints of triangle PQR are joined to form another triangle. If this process is carried on indefinitely, what is the sum of the perimeters of all these triangles?



Solution

Since ΔABC is equilateral, the angles in all the triangles will be equal, and all triangles will be equilateral. By making a chart, you can develop a pattern and the formula for the sum of the perimeters.

The first triangle, ΔABC, has a perimeter of 12 cm, so each side length is 4. The second triangle has three equal sides, RP, PQ, and RQ. Since RP is half of BC, then RP = 2 and the perimeter is 2 + 2 + 2 = 6. In each successive triangle, both the side length and the perimeter are half as much as the previous amount. The chart below summarizes this information.

Triangle Name	Triangle Number	Length of Side	Perimeter
ΔABC	1	$\frac{12}{3} = 4$ cm	12 cm
ΔRPQ	2	2 cm	6 cm
ΔDEF	3	1 cm	3 cm

Therefore, the perimeters of these triangles form the infinite geometric series:

$12 + 6 + 3 + 1.5 + \ldots$

Since $r = \frac{6}{12} = \frac{1}{2}$, it follows that $S = \frac{t_1}{1 - r} = \frac{12}{1 - 0.5} = 24$ cm.

The pattern developed in this example can be generalized for all triangles as follows:

> The segment joining the midpoints of two sides of a triangle is one-half the length of the third side. It follows that the perimeter of each newly formed triangle is one-half the perimeter of its "parent" triangle.

Make sure you complete the following learning activity. This learning activity focuses on finding sums of infinite geometric series and will prepare you for the assignment.

Learning Activity 1.6

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Find the common ratio in the following pattern: 189, 63, 21, 7, . . .
2. Which of these values would come next in this pattern:

   One, four, three, eleven, fifteen, thirteen, . . .

   Fourteen

   Five

   Seven

   Seventeen
3. If your favourite television show starts at 7 pm and your mom tells you that your show is still 48 minutes away, what time is it now?
4. Your school ordered 56 boxes of cookies that each contain 25 cookies. If there are 700 students at your school, will you have enough cookies to give each student two cookies?
5. Calculate: $\frac{3}{16} + \frac{2}{4}$

continued

Learning Activity 1.6 (continued)

6. Solve the inequality $5x - 2 < 0$.
7. Simplify: $(2x^2 + 3x - 5) - (9x^2 - 4x + 6)$
8. Simplify the following fraction: $\frac{144}{180}$

Part B: Sums of Infinite Geometric Series

If you haven't already, you may want to go back and include all important definitions, terms, or examples that you have found more challenging on your resource sheet.

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Find the sum of each of the following infinite geometric series. Express your answers as exact values.

   a) $8 + \frac{8}{3} + \frac{8}{9} + \frac{8}{27} + \ldots$

   b) $6 - 3 + \frac{3}{2} - \frac{3}{4} + \ldots$

   c) $6 + 4 + \frac{8}{3} + \frac{16}{9} + \ldots$

   d) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$

   e) $\sum_{k=1}^{\infty}\left(\frac{16}{2^k}\right)$

   f) $\sum_{k=2}^{\infty}\left(\frac{18}{3^k}\right)$

2. Consider a geometric sequence where $t_3 = 16$ and $t_5 = 10$.

   a) Find the common ratio r, rounded to two decimal places.

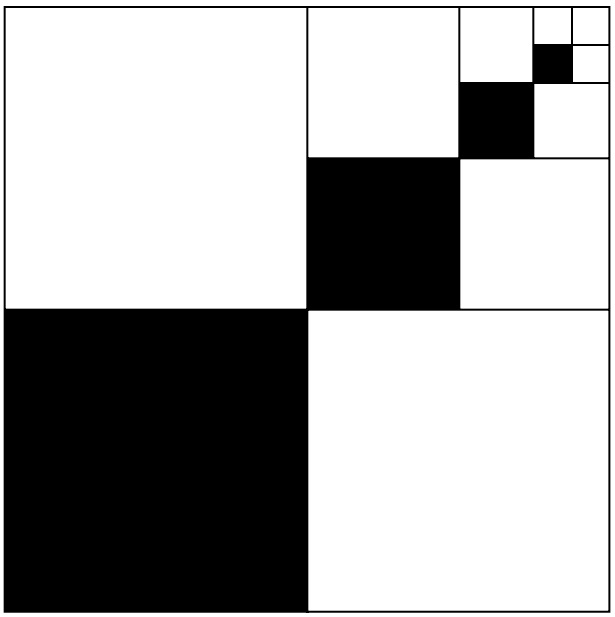
   b) Find $\sum_{k=1}^{\infty} t_k$. State your answer rounded to two decimal places.

continued

Learning Activity 1.6 (continued)

3. The sum to infinity of an infinite geometric series is 12. If the common ratio is 0.25, what is the first term of this series?

4. A 4-by-4 square is divided into four congruent squares, as shown in the diagram. The bottom left square is shaded. The top right square is subdivided into four congruent squares again, and the bottom left square is shaded. This process is repeated indefinitely. What is the total shaded area?



5. A ball is dropped from a height of two metres. Each time it strikes the ground, it rebounds three-quarters of the distance it has fallen.
   a) How far does it rise after it strikes the ground the third time?
   b) How far does it travel before it comes to rest?

6. What is the value of x if the geometric series $1 + 3x + 9x^2 + 27x^3 + \ldots$ converges to 4?

7. Recall Zeno's paradox from the introduction to this lesson. Zeno's paradox stated that there is no sum to the infinite geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$. Prove that Zeno is incorrect.

Lesson Summary

In this lesson, you learned how to find the sum of an infinite geometric series. You also learned that in some cases it is impossible to find the sum of an infinite geometric series. However, in the case of Zeno's paradox, it is possible to find the sum of an infinite geometric series.

Be sure to complete the final assignment in this module before you continue on to the next module.

Your next task is to complete a hand-in assignment found on the following pages. Keep all your Module 1 assignments until you finish the module and then send them to the Distance Learning Unit. Instructions for submitting your assignments can be found in the module summary located at the end of this module.

Before you begin, have you completed a resource sheet to use with your assignment?

Notes

Assignment 1.3

Finding the Sums of Infinite Geometric Series

Total: 18 marks

1. Consider the series $4 + 2 + 1 + \frac{1}{2} + \ldots$

   a) Find the sixth term. Express your answer as an exact value. *(2 marks)*

   b) Use the formula to find the sum of six terms. Express your answer as an exact value. *(1 mark)*

   c) Find, to four decimal places, the sum of 10 terms. *(1 mark)*

   d) Find the infinite sum. Express your answer as an exact value. *(2 marks)*

continued

Assignment 1.3: Finding the Sums of Infinite Geometric Series (continued)

2. The sum to infinity of an infinite geometric series is -24. If possible, find the common ratio if the first term of this series is 20. *(2 marks)*

3. For what values of x does the following infinite geometric series converge (i.e., find the interval of convergence)? *(2 marks each, for a total of 4 marks)*

   a) $1 + (x + 1) + (x + 1)^2 + (x + 1)^3 + \ldots$

   b) $1 + (x - 2) + (x - 2)^2 + (x - 2)^3 + \ldots$

continued

Assignment 1.3: Finding the Sums of Infinite Geometric Series (continued)

4. Find the sum of each of the following infinite geometric series. Express your answers as exact values. *(1 mark each, for a total of 6 marks)*

   a) $7 + \frac{7}{3} + \frac{7}{9} + \frac{7}{27} + \ldots$

   b) $12 - 6 + 3 - \frac{3}{2} + \ldots$

   c) $10 + 8 + \frac{32}{5} + \frac{128}{25} + \ldots$

   d) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

continued

Assignment 1.3: Finding the Sums of Infinite Geometric Series (continued)

e) $\sum_{k=1}^{\infty} \frac{14}{3^k}$

f) $\sum_{k=3}^{\infty} \left(\frac{8}{4^k} \right)$

Module 1 Summary

You have now completed the first module in Grade 11 Pre-Calculus Mathematics. In this module, you learned about sequences and series. You learned how arithmetic sequences are created by adding the common difference to each term to get the next term. You also learned how geometric sequences are created by multiplying the common ratio by the term to get the next term. Because the mathematical operations of multiplication and addition are different, geometric sequences grow exponentially faster than arithmetic sequences. Also, it is possible to find the sum of different types of geometric and arithmetic series.

In the next module, you will shift your focus to learn about factoring. You will also learn how factoring relates to rational expressions. Patterns are important to factoring. Recognizing patterns in numbers can be very useful when asked to factor a difficult or simple polynomial.

Submitting Your Assignments

It is now time for you to submit Assignments 1.1 to 1.3 to the Distance Learning Unit so that you can receive some feedback on how you are doing in this course. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 1 assignments and organize your material in the following order:

- ❑ Module 1 Cover Sheet (found at the end of the course Introduction)
- ❑ Assignment 1.1: Arithmetic Sequences and Series
- ❑ Assignment 1.2: Geometric Sequences and Series
- ❑ Assignment 1.3: Finding the Sums of Infinite Geometric Series

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Notes

Grade 11 Pre-Calculus Mathematics (30S)

Module 1
Sequences and Series

Learning Activity Answer Keys

Module 1:
Sequences and Series

Learning Activity 1.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. At the end of a poker game, Alysha had $62. During the game she had won $250, lost $175, won $35, and lost $55. How much money did Alysha have at the start, in dollars?

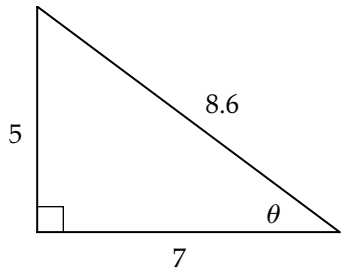
2. Complete the pattern: 1, 2, 5, 10, 17, ____ , ____ , ____

3. Evaluate when $z = -1$: $(3z + 1)(z - 5$

4. Evaluate: $\left(\frac{8}{16}\right)(12)$

5. A clothing store is selling shirts for $30.00. They decided to raise the price of the shirt by 20%. A few days later they decided to raise the price another 20%. What is the new price of the shirt, in dollars?

Use the following right-angled triangle to answer the next three questions. Do not simplify your answer.



6. Identify the sine ratio for θ.

7. Identify the cosine ratio for θ.

8. Identify the tangent ratio for θ.

Answers:

1. Alysha started with $7.

   (x + \$250 − \$175 + \$35 − \$55 = \$62

   x + \$55 = \$62

   x = \$62 − \$55

   x = \$7)

2. 26, 37, 50 (You are adding the next consecutive odd integer each time, add 1, add 3, add 5, etc.)

3. 12 ((3z + 1)(z − 5) = (3(−1) + 1)(−1 − 5) = (−3 + 1)(−6) = (−2)(−6))

4. $6\left(\left(\frac{1}{2}\right)(12)\right)$

5. \$43.20

   (10% of \$30 = \$3

   20% of \$30 = \$6

   Therefore, after the first increase, the price of the shirt is \$36.

   10% of \$36 = \$3.60

   20% of \$36 = \$7.20

   The new price of the shirt is \$36.00 + \$7.20 = \$43.20.)

6. $\frac{5}{8.6}$

7. $\frac{7}{8.6}$

8. $\frac{5}{7}$

Part B: Arithmetic Sequences

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Which of the following sequences are arithmetic?
   a) $2, 5, 8, 11, \ldots$
   b) $-5, -1, 3, 7, \ldots$
   c) $2, 5, 7, 10, \ldots$
   d) $2, 2\frac{1}{2}, 1, 1\frac{1}{2}, 0, \ldots$

   Answers:
   a) $5 - 2 = 3, 8 - 5 = 3, 11 - 8 = 3$; arithmetic sequence
   b) $-1 - (-5) = 4, 3 - (-1) = 4, 7 - 3 = 4$; arithmetic sequence
   c) $5 - 2 = 3, 7 - 5 = 2$; not an arithmetic sequence, $3 \neq 2$
   d) $2\frac{1}{2} - 2 = \frac{1}{2}, 1 - 2\frac{1}{2} = -1\frac{1}{2}$; not an arithmetic sequence, $-1\frac{1}{2} \neq \frac{1}{2}$

2. Write a recursive formula for each sequence. Indicate whether the sequence is arithmetic or not.
   a) $10, 7, 4, 1, \ldots$
   b) $2, 4, 8, 16, \ldots$
   c) $30, 41, 52, 63, \ldots$
   d) $3, -6, 12, -24, \ldots$
   e) $5, 15, 45, 135, \ldots$

   Answers:
   a) $\begin{cases} t_1 = 10 \\ t_n = t_{n-1} - 3, \; n \geq 2 \end{cases}$

   Arithmetic sequence with common difference $= -3$.

   b) $\begin{cases} t_1 = 2 \\ t_n = 2t_{n-1}, \; n \geq 2 \end{cases}$

   Not an arithmetic sequence.

c) $\begin{cases} t_1 = 30 \\ t_n = t_{n-1} + 11,\ n \geq 2 \end{cases}$

Arithmetic sequence with common difference = 11.

d) $\begin{cases} t_1 = 3 \\ t_n = -2t_{n-1},\ n \geq 2 \end{cases}$

Not an arithmetic sequence.

e) $\begin{cases} t_1 = 5 \\ t_n = 3t_{n-1},\ n \geq 2 \end{cases}$

Not an arithmetic sequence.

3. Write the first four terms of each sequence.

a) $\begin{cases} t_1 = 3 \\ t_n = t_{n-1} + 5,\ n \geq 2 \end{cases}$

Answer:

n	$t_n = t_{n-1} + 5$
1	$t_1 = 3$
2	$t_2 = t_{2-1} + 5 = t_1 + 5 = 3 + 5 = 8$
3	$t_3 = t_{3-1} + 5 = t_2 + 5 = 8 + 5 = 13$
4	$t_4 = t_{4-1} + 5 = t_3 + 5 = 13 + 5 = 18$

The first four terms are 3, 8, 13, 18.

b) $\begin{cases} t_1 = 10 \\ t_n = t_{n-1} - 2,\ n \geq 2 \end{cases}$

Answer:

n	$t_n = t_{n-1} - 2$
1	$t_1 = 10$
2	$t_2 = t_{2-1} - 2 = t_1 - 2 = 10 - 2 = 8$
3	$t_3 = t_{3-1} - 2 = t_2 - 2 = 8 - 2 = 6$
4	$t_4 = t_{4-1} - 2 = t_3 - 2 = 6 - 2 = 4$

The first four terms are 10, 8, 6, 4.

c) $\begin{cases} t_1 = 3.7 \\ t_n = t_{n-1} + 1.3,\ n \geq 2 \end{cases}$

Answer:

n	$t_n = t_{n-1} + 1.3$
1	$t_1 = 3.7$
2	$t_2 = t_{2-1} + 1.3 = t_1 + 1.3 = 3.7 + 1.3 = 5$
3	$t_3 = t_{3-1} + 1.3 = t_2 + 1.3 = 5 + 1.3 = 6.3$
4	$t_4 = t_{4-1} + 1.3 = t_3 + 1.3 = 6.3 + 1.3 = 7.6$

The first four terms are 3.7, 5, 6.3, 7.6

d) $\begin{cases} t_1 = 1 \\ t_2 = 1 \\ t_n = t_{n-1} + t_{n-2},\ n \geq 3 \end{cases}$

Answer:

n	$t_n = t_{n-1} + 1.3$
1	$t_1 = 1$
2	$t_2 = 1$
3	$t_3 = t_{3-1} + t_{3-2} = t_2 + t_1 = 1 + 1 = 2$
4	$t_4 = t_{4-1} + t_{4-2} = t_3 + t_2 = 2 + 1 = 3$

The first four terms are 1, 1, 2, 3.

e) $\begin{cases} t_1 = 3 \\ t_2 = -1 \\ t_n = t_{n-1} + t_{n-2},\ n \geq 3 \end{cases}$

Answer:

n	$t_n = t_{n-1} + 1.3$
1	$t_1 = 3$
2	$t_2 = -1$
3	$t_3 = t_{3-1} + t_{3-2} = t_2 + t_1 = -1 + 3 = 2$
4	$t_4 = t_{4-1} + t_{4-2} = t_3 + t_2 = 2 + (-1) = 1$

The first four terms are 3, −1, 2, 1.

4. Write the first three terms of the arithmetic sequence defined by the linear function.

   a) $f(x) = 2x + 4$

   Answer:

   $f(1) = 2(1) + 4 = 2 + 4 = 6$

   $f(2) = 2(2) + 4 = 4 + 4 = 8$

   $f(3) = 2(3) + 4 = 6 + 4 = 10$

   Therefore, the first three terms are 6, 8, and 10.

   b) $g(x) = 5x - 2$

   Answer:

   $g(1) = 5(1) - 2 = 5 - 2 = 3$

   $g(2) = 5(2) - 2 = 10 - 2 = 8$

   $g(3) = 5(3) - 2 = 15 - 2 = 13$

   Therefore, the first three terms are 3, 8, and 13.

   c) $h(x) = -3x + 2$

   Answer:

   $h(1) = -3(1) + 2 = -3 + 2 = -1$

   $h(2) = -3(2) + 2 = -6 + 2 = -4$

   $h(3) = -3(3) + 2 = -9 + 2 = -7$

   Therefore, the first three terms are -1, -4, and -7.

   d) $m(x) = x - 5$

   Answer:

   $m(1) = 1 - 5 = -4$

   $m(2) = 2 - 5 = -3$

   $m(3) = 3 - 5 = -2$

   Therefore, the first three terms are -4, -3, and -2.

5. Three terms of an arithmetic sequence (not necessarily the first three terms) are 27, 30, 33. Write two different linear functions that may have produced this sequence.

   Answer:

   Since the common difference between terms is 3, it follows that $m = 3$. Possible linear functions are $y = 3x$, $y = 3x - 3$. In general, you could write $y = 3x + b$ where b is any real number.

6. Find an equation of the linear function that produces the following first three terms of an arithmetic sequence.

   a) 5, 8, 11, . . .

   Answer:

   The common difference in this sequence is three. Therefore, the slope of the line, m, is 3. A point on the line is (1, 5), or (2, 8), or (3, 11).

   $y - y_1 = m(x - x_1)$

   $y - 5 = 3(x - 1)$

   $y = 3x - 3 + 5$

   $y = 3x + 2$

   b) $6, 6\frac{1}{2}, 7, \ldots$

   Answer:

   The common difference in this sequence is $\frac{1}{2}$. Therefore, the slope of the line, m, is $\frac{1}{2}$. A point on the line is $(1, 6)$, or $\left(2, 6\frac{1}{2}\right)$, or $(3, 7)$.

   $y - y_1 = m(x - x_1)$

   $y - 6 = \frac{1}{2}(x - 1)$

   $y = \frac{1}{2}x - \frac{1}{2} + 6$

   $y = \frac{1}{2}x + 5\frac{1}{2}$

c) 10, 8, 6, . . .

Answer:

The common difference in this sequence is −2. Therefore, the slope of the line, m, is −2. A point on the line is (1, 10), or (2, 8), or (3, 6).

$y - y_1 = m(x - x_1)$

$y - 10 = -2(x - 1)$

$y = -2x + 2 + 10$

$y = -2x + 12$

d) $2, 2\frac{1}{10}, 2\frac{1}{5}, \ldots$

Answer:

The common difference in this sequence is $\frac{1}{10}$. Therefore, the slope of the line, m, is $\frac{1}{10}$. A point on the line is $(1, 2)$, or $\left(2, 2\frac{1}{10}\right)$, or $\left(3, 2\frac{1}{5}\right)$.

$y - y_1 = m(x - x_1)$

$y - 2 = \frac{1}{10}(x - 1)$

$y = \left(\frac{1}{10}\right)x - \frac{1}{10} + 2$

$y = \left(\frac{1}{10}\right)x + 1\frac{9}{10}$

7. Find the 50th term of each of the following arithmetic sequences.

a) $f(x)$: 7, 11, 15, . . .

Answer:

$y - 7 = 4(x - 1)$

$y = 4x - 4 + 7$

$y = 4x + 3$

$\therefore f(x) = 4x + 3$

and $f(50) = 4(50) + 3$

$= 200 + 3$

$= 203$

Thus, the 50th term is 203.

b) $g(x)$: 10, 15, 20, . . .

Answer:

$y - 10 = 5(x - 1)$

$y = 5x - 5 + 10$

$y = 5x + 5$

$\therefore g(x) = 5x + 5$

and $g(50) = 5(50) + 5$

$= 250 + 5$

$= 255$

Thus, the 50th term is 255.

c) $h(x)$: $4\frac{1}{2}$, 5, $5\frac{1}{2}$, . . .

Answer:

$$y - 4\frac{1}{2} = \frac{1}{2}(x - 1)$$

$$y = \frac{1}{2}x - \frac{1}{2} + 4\frac{1}{2}$$

$$y = \frac{1}{2}x + 4$$

$$\therefore h(x) = \frac{1}{2}x + 4$$

$$\text{and } h(50) = \frac{1}{2}(50) + 4$$

$$= 25 + 4$$

$$= 29$$

8. The fifth term of an arithmetic sequence is -15 and the ninth term is -3. Find the value of the hundredth term.

Answer:

(5, -15) and (9, -3) are points on the line.

$$m = \frac{-3-(-15)}{9-5} = \frac{12}{4} = 3$$

$$y-(-15) = 3(x-5)$$

$$y+15 = 3x-15$$

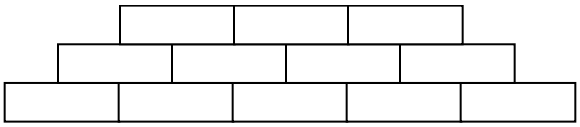
$$y = 3x-15-15$$

$$y = 3x-30$$

$$f(x) = 3x-30$$

$$f(100) = 3(100)-30 = 270 = t_{100}$$

9. A pile of bricks is arranged in rows. The numbers of bricks in the rows form an arithmetic sequence. One such pile of bricks with 3 rows is shown below:



In another pile of bricks, there are 45 bricks in the fifth row and 33 bricks in the eleventh row.

a) How many bricks are in the first row?

Answer:

Let x represent the row number and let $f(x)$ represent the number of bricks in the row.

The linear function contains the points with coordinates (5, 45) and (11, 33).

$$m = \frac{33-45}{11-5} = \frac{-12}{6} = -2$$

$$y-45 = -2(x-5)$$

$$y = -2x+10+45$$

$$y = -2x+55 \text{ or}$$

$$f(x) = -2x+55 \text{ is the defining linear function}$$

Now find the first term.

$$f(1) = -2(1)+55 = 53$$

There are 53 bricks in the first row.

b) How many bricks are in the nth row?

Answer:

$f(n) = -2n + 55$

There are $-2n + 55$ bricks in the nth row.

c) What is the maximum number of rows of bricks possible?

Answer:

Since the number of bricks cannot be negative or equal to zero, the top row of bricks may have as few as one brick in it. You need to solve the following equation:

$$1 = -2n + 55$$
$$-54 = -2n$$
$$27 = n$$

The maximum number of rows is 27.

10. Find which term of the sequence 20, 23, 26, . . ., is the value 245,

a) by using the general term formula

Answer:

Using the formula $t_n = t_1 + (n - 1)d$, you have

$$245 = 20 + (n - 1)3$$
$$245 = 20 + 3n - 3$$
$$245 = 3n + 17$$
$$245 - 17 = 3n$$
$$228 = 3n$$
$$76 = n$$

Therefore, $245 = t_{76}$.

Hence, the 76th term of the sequence has a value of 245.

b) by using the defining linear function

Answer:

The defining linear function can be found using the points (1, 20) and (2, 23).

$$m = \frac{23 - 20}{2 - 1} = 3$$

$y - 20 = 3(x - 1)$

$y = 3x - 3 + 20$

$y = 3x + 17$

Now solve for x for the point $(x, 245)$.

Therefore, $245 = 3x + 17$

$228 = 3x$

$76 = x$

Hence, the 76th term of the sequence has a value of 245.

11. Find the term indicated in each arithmetic sequence by using the general term formula.

a) 5, 3, 1, . . . t_{20}

Answer:

$d = 3 - 5 = -2$

$t_1 = 5$

$n = 20$

$t_n = t_1 + (n - 1)d$

$t_{20} = 5 + (20 - 1)(-2)$

$t_{20} = -33$

b) 8, 11, 14, . . . t_{21}

Answer:

$d = 11 - 8 = 3$

$t_1 = 8$

$n = 21$

$t_n = t_1 + (n - 1)d$

$t_{21} = 8 + (21 - 1)(3)$

$t_{21} = 68$

c) 6, 3, 0, −3, . . . t_{57}

Answer:

$d = 3 - 6 = -3$

$t_1 = 6$

$n = 57$

$t_n = t_1 + (n - 1)d$

$t_{57} = 6 + (57 - 1)(-3)$

$t_{57} = -162$

d) 18, 17, 16, 15, . . . t_{132}

Answer:

$d = 17 - 18 = -1$

$t_1 = 18$

$n = 132$

$t_n = t_1 + (n - 1)d$

$t_{132} = 18 + (132 - 1)(-1)$

$t_{132} = -113$

e) 7, 8.3, 9.6, 10.9, . . . t_{172}

Answer:

$d = 8.3 - 7 = 1.3$

$t_1 = 7$

$n = 172$

$t_n = t_1 + (n - 1)d$

$t_{172} = 7 + (172 - 1)(1.3)$

$t_{172} = 229.3$

12. Joe bought a painting for \$1800. After seven years, this painting sold for \$10,550. If the painting appreciated arithmetically, determine the annual amount of appreciation.

Answer:

Method 1:

The sequence is \$1800, \$1800 + d, \$1800 + 2d, . . ., \$1800 + 7d

After 7 years, the painting is worth \$10,550.

Therefore, \$10,550 = \$1800 + 7d

\$8750 = 7d

\$1250 = d

The painting appreciated \$1250 annually.

Method 2:

Use the general term formula with

$t_1 = 1800$

$n = 8$

$t_8 = 10\ 550$

$d = ?$

$10\ 550 = 1800 + (8 - 1)d$

$10\ 550 = 1800 + 7d$

The rest of the solution is the same. Note that in this method $n = 8$ because you want the amount **after** 7 years, not **in** 7 years.

13. For the arithmetic sequence 7, 11, 15, 19, . . ., find t_{29}.

Answer:

$d = 11 - 7 = 4$

$t_1 = 7$

$n = 29$

$t_n = t_1 + (n - 1)d$

$t_{29} = 7 + (29 - 1)4$

$t_{29} = 119$

14. Give an example of an arithmetic sequence, and then

    a) prove your sequence is an arithmetic sequence

    Answer:

    Answers may vary. However, all arithmetic sequences must increase or decrease by the same value (the common difference) between each term.

    b) write the defining linear function of your sequence

    Answer:

    Answers may vary. However, answers must be in the form, $y = mx + b$.

    c) write your arithmetic sequence using the general term formula

    Answer:

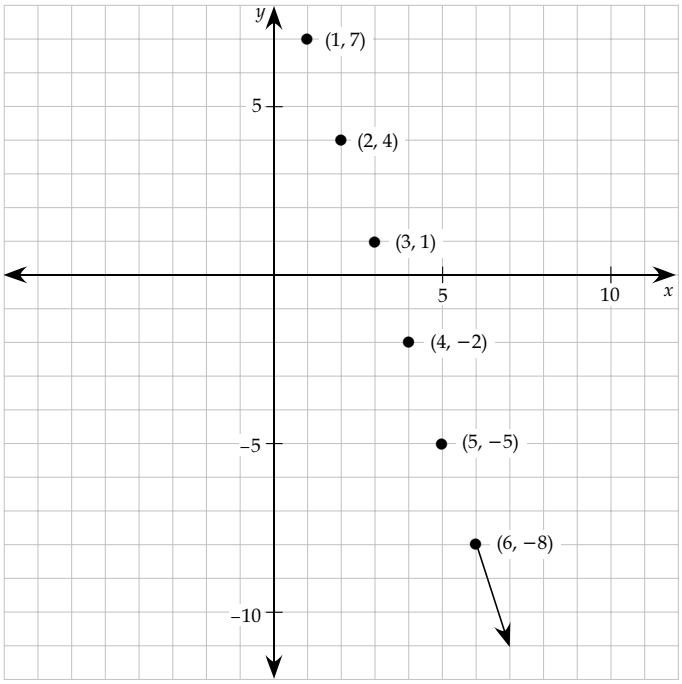
    Answers may vary. However, answers must be in the form, $t_n = t_1 + (n - 1)d$.

15. Graph the arithmetic sequence 7, 4, 1, −2, . . ., for the first six terms

    a) on graph paper (A reproducible copy of a sheet of graph paper is included in Appendix B found on the downloads page.)

    Answer:

    The points are (1, 7), (2, 4), (3, 1), (4, −2), (5, −5), (6, −8). The graph is shown below.



b) on the graphing calculator

Answer:

The common difference is -3. The first term is 7 so the first point is (1, 7). The recursion formula is:

$$\begin{cases} t_1 = 7 \\ t_n = t_{n-1} - 3,\ n \geq 2 \end{cases}$$

Set your calculator to sequence mode and graph the sequence. The screen should look something like the one below.

First, enter your information into Y=

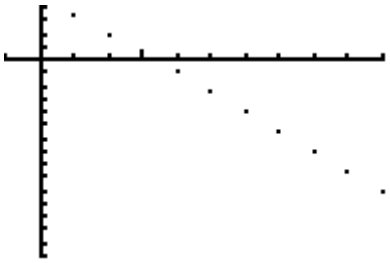
```
 Plot1 Plot2 Plot3
 nMin=1
u(n)=(u(n-1))-3

 u(nMin)=(7)
v(n)=
 v(nMin)=
w(n)=
```

Then, set your window.

```
WINDOW
 nMin=1
 nMax=10
 PlotStart=1
 PlotStep=1
 Xmin=-1
 Xmax=10
↓Xscl=1
Ymin=-30
Ymax=8
Yscl=2
```

Press GRAPH.



Learning Activity 1.2

There is no answer key for this learning activity.

Learning Activity 1.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Continue the pattern: 0.125, 0.250, 0.375, _____ , _____
2. Andrew bought an mp3 player for $156.88. He gave the cashier eight $20 bills. The store was out of toonies and all they had were loonies, quarters, dimes, nickels, and pennies. What was the smallest number of coins he could have received as change?
3. If $4x - 3x + 8x + 2x = 121$, then $6x - 2 =$
4. A recipe calls for two tablespoons of sugar for every quarter cup of flour. If Santana wants to use 18 tablespoons of sugar, how many cups of flour does she need to make the recipe?
5. Calculate: $\left(\frac{5}{3}\right) - \left(\frac{2}{14}\right)$
6. Simplify: $2^8 \div 2^6$
7. Determine the distance between the following two points: (2, 6) and (−7, −4).
8. List the factors of 164.

Answers:

1. 0.5, 0.625
2. 3 loonies, 1 dime, and 2 pennies for a total of 6 coins ($8 \times \$20 = \$160$; $\$160 - \$156.88 = \$3.12$; Remember, the store was out of toonies.)
3. 64 ($11x = 121$, so $x = 11$; $6(11) - 2 = 66 - 2 = 64$)
4. $2\frac{1}{4}$ $\left(\text{Santana increased the recipe by a factor of nine; } 9 \times \frac{1}{4} = \frac{9}{4}\right)$
5. $\frac{32}{21}$ $\left(\frac{5}{3} - \frac{1}{7} = \frac{35}{21} - \frac{3}{21}\right)$
6. 4 $\left(2^8 \div 2^6 = \frac{2^8}{2^6} = 2^2 = 4\right)$

7. $\sqrt{181}$

$\left(d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\right.$

$\left. d = \sqrt{(-9)^2 + (-10)^2} = \sqrt{81 + 100} = \sqrt{181}\right)$

8. 1, 2, 4, 41, 82, and 164

Part B: Arithmetic Series

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Find the indicated sum of each arithmetic series.

   a) $7 + 12 + 17 + \ldots, S_{20}$

   Answer:

   The common difference = 5

   Use the formula $t_n = t_1 + (n - 1)d$ to find t_n

   $t_{20} = 7 + (20 - 1)(5)$

   $t_{20} = 102$

   $S_{20} = \left(\frac{7 + 102}{2}\right)(20)$

   $S_{20} = 1090$

   b) $-5 - 7 - 9 - 11 - \ldots, S_{20}$

   Answer:

   The common difference = -2

   $t_{20} = -5 + (20 - 1)(-2)$

   $t_{20} = -43$

   $S_{20} = \left(\frac{-5 + (-43)}{2}\right)(20)$

   $S_{20} = -480$

c) $40 + 35 + 30 + \ldots, S_{100}$

Answer:

The common difference = -5

$t_{100} = 40 + (100 - 1)(-5)$

$t_{100} = -455$

$S_{100} = \left(\frac{40 + (-455)}{2}\right)(100)$

$S_{100} = -20\ 750$

2. Find the sum of the first:

a) 10 odd integers.

Answer:

The series is $1 + 3 + 5 + \ldots$ and $n = 10$

$t_n = t_1 + (n - 1)d$

$t_{10} = 1 + (10 - 1)(2)$

$t_{10} = 19$

The sum of 10 terms is

$S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$

$S_{10} = \left(\frac{t_1 + t_{10}}{2}\right)(10)$

$S_{10} = \left(\frac{1 + 19}{2}\right)(10)$

$S_{10} = 100$

b) 20 odd integers.

Answer:

The series is $1 + 3 + 5 + \ldots$ and $n = 20$

$t_n = t_1 + (n - 1)d$

$t_{20} = 1 + (20 - 1)(2)$

$t_{20} = 39$

The sum of 20 terms is

$$S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$$

$$S_{20} = \left(\frac{1 + 39}{2}\right)(20)$$

$$S_{20} = 400$$

c) 100 odd integers.

Answer:

The series is $1 + 3 + 5 + \ldots$ and $n = 100$

$t_n = t_1 + (n - 1)d$

$t_{100} = 1 + (100 - 1)(2)$

$t_{100} = 199$

The sum of 100 terms is

$$S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$$

$$S_{100} = \left(\frac{1 + 199}{2}\right)(100)$$

$$S_{100} = 10\ 000$$

d) n odd integers.

Answer:

The series is $1 + 3 + 5 + \ldots$ with n terms

The nth term is $t_n = t_1 + (n - 1)d$

Thus, $t_n = 1 + (n - 1)(2)$

$= 1 + 2n - 2$

$= 2n - 1$

The sum of n terms is

$$S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$$

$$S_n = \left(\frac{1 + 2n - 1}{2}\right)(n)$$

$$S_n = \left(\frac{2n}{2}\right)(n)$$

$$S_n = (n)(n)$$

$$S_n = n^2$$

Note that the formula you have developed might be worth memorizing. Can you show that the sum of the first n even natural numbers is $n(n + 1)$? Can you show that the sum of the first n natural numbers is $\frac{n(n+1)}{2}$?

3. Find the sum of all multiples of 3 between 99 and 1000.

Answer:

The first multiple of 3 greater than 99 is 102. The first term is not 99 because the multiples must lie between 99 and 1000, not including 99 nor 1000.

The greatest multiple of 3 smaller than 1000 is 999.

The series is $102 + 105 + 108 + \ldots + 999$.

To find the sum, the value of n must be found using the general term formula and $d = 3$.

$t_n = t_1 + (n - 1)d$

$999 = 102 + (n - 1)(3)$

$897 = (n - 1)(3)$

$299 = n - 1$

$300 = n$

The number of terms is 300.

The sum of the series is

$$S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$$

$$S_{300} = \left(\frac{102 + 999}{2}\right)(300)$$

$$S_{300} = 165\ 150$$

4. Find S_{100} of $1 + 4 + 7 + \ldots$.

Answer:

The series is $1 + 4 + 7 + \ldots$ $n = 100$, and $d = 3$.

To find the sum of 100 terms, the last term, t_{100}, needs to be calculated using the general term formula and $d = 3$.

$$t_n = t_1 + (n - 1)d$$

$$t_{100} = 1 + (100 - 1)(3)$$

$$t_{100} = 298$$

The sum of 100 terms is

$$S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$$

$$S_{100} = \left(\frac{1 + 298}{2}\right)(100)$$

$$S_{100} = 14\ 950$$

5. How many terms are in the series $3 + 8 + 13 + \ldots + 248$ if the sum is 6275?

Answer:

The series is $3 + 8 + 13 + \ldots + 248$.

To find the number of terms, use the formula

$$S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$$

where $S_n = 6275$ (given)

$t_1 = 3$

$t_n = 248$

Substitute and solve for n:

$$6275 = \left(\frac{3 + 248}{2}\right)(n)$$

$$6275 = 125.5n$$

$$50 = n$$

There are 50 terms in the series.

6. In an arithmetic sequence, $t_2 = -1$ and $t_{12} = 19$, find S_{13}.

Answer:

Method 1:

Since $t_2 = -1$ and $t_{12} = 19$, two points on the line determined by the defining linear function are (2, −1) and (12, 19).

By finding the function, t_1 and t_{13} can be found.

$$\text{Slope} = \frac{19 - (-1)}{12 - 2} = \frac{20}{10} = 2$$

Equation of line is:

$$y - y_1 = m(x - x_1)$$

$$y - (-1) = 2(x - 2)$$

$$y + 1 = 2x - 4$$

$$y = 2x - 5$$

The defining linear function is $f(x) = 2x - 5$ and

$t_1 = f(1) = 2(1) - 5 = 2 - 5 = -3$

$t_{13} = f(13) = 2(13) - 5 = 26 - 5 = 21$

The sum of the series is

$$S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$$

$$S_{13} = \left(\frac{-3 + 21}{2}\right)(13)$$

$$S_{13} = 117$$

Method 2:

Since the slope was calculated in Method 1 to be 2, then $d = 2$ for the series.

Since $t_2 = 1$

$t_1 = t_2 - d = -1 - 2 = -3$

and $t_{12} = 19$

$t_{13} = t_{12} + d = 19 + 2 = 21$

then the sum can be calculated as in Method 1.

7. An auditorium has 20 seats in the first row, and each row has two more seats than the row immediately in front of it. There are 26 rows in the auditorium. How many seats are there in the last row? What is the seating capacity of the auditorium?

Answer:

The arithmetic sequence is 20, 22, 24, . . . t_{26}. To calculate t_{26}, use the general term formula.

$t_n = t_1 + (n - 1)d$

$t_{26} = 20 + (26 - 1)(2)$

$t_{26} = 70$

There are 70 seats in the last row.

To find the seating capacity means to find how many seats there are altogether. The series is

$20 + 22 + 24 + \ldots + 70$

and $n = 26$

The sum is

$$S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$$

$$S_{26} = \left(\frac{20 + 70}{2}\right)(26)$$

$S_{26} = 1170$

The seating capacity is 1170 seats.

8. Mary's annual salary ranges from \$26,785 in the first year to \$34,825 in the seventh year.

   a) If the salary range is an arithmetic sequence with seven terms, determine the raise Mary received each year.

   Answer:

   The sequence is \$26,785, . . ., \$34,825, with $n = 7$.

   Use the general term formula to find d.

   $$t_n = t_1 + (n - 1)d$$
   $$34\,825 = 26\,785 + (7 - 1)(d)$$
   $$8040 = 6d$$
   $$1340 = d$$

   Mary received a raise of \$1340 each year.

   b) What was her salary in the fifth year?

   Answer:

   To find Mary's salary in the fifth year, we need to find t_5 using the general term formula and $n = 5$.

   $t_n = t_1 + (n - 1)d$

   $t_5 = 26\,785 + (5 - 1)(1340)$

   $t_5 = \$32{,}145$

   Mary's salary in the fifth year is \$32,145.

   c) What is the first annual salary in this range that is greater than \$30,000?

   Answer:

   From part (b), we know $t_5 = \$32{,}145$.

   Thus, $t_4 = 32\,145 - d$

   $= 32\,145 - 1340$

   $= \$30{,}805$

   and $t_3 = 30\,805 - 1340$

   $= \$29{,}465$

   In the fourth year, Mary's salary is over \$30,000.

d) What was the total amount Mary earned in the seven years?

Answer:

To find the total amount Mary earned in seven years, we find the sum of the series with 7 terms.

$\$26{,}785 + \ldots + \$34{,}825$

$$S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$$

$$S_7 = \left(\frac{26\,785 + 34\,825}{2}\right)(7)$$

$$S_7 = \$215{,}635$$

Mary earns \$215,635 in 7 years.

9. a) Show that the sum of n terms of an arithmetic series can be found using the formula $S_n = \frac{n}{2}\left[2t_1 + (n-1)d\right]$.

Answer:

Use the formula: $S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$

Rearrange to get: $S_n = \frac{n}{2}\left[t_1 + t_n\right]$

Use the general term formula to replace t_n and simplify.

$t_n = t_1 + (n-1)d$

$\therefore\ S_n = \frac{n}{2}\left[t_1 + \left(t_1 + (n-1)d\right)\right]$

$S_n = \frac{n}{2}\left[2t_1 + (n-1)d\right]$

b) Use this formula to find the sum of 10 terms of the series $4 + 9 + 14 + 19 + \ldots$

Answer:

$4 + 9 + 14 + 19 + \ldots$

$t_1 = 4$

$n = 10$

$d = 5$

$$S_n = \frac{n}{2}\left[2t_1 + (n-1)d\right]$$

$$S_n = \frac{10}{2}\left[2(4) + (10-1)(5)\right]$$

$$S_n = 5\left[8 + 9(5)\right]$$

$$S_{10} = 265$$

Note: You now can use either formula to find the sum of an arithmetic series.

$$S_n = \frac{n}{2}\left[2t_1 + (n-1)d\right]$$

$$S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$$

Learning Activity 1.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. On the first day of your marathon training, you run 1 km. If you double that distance every day, on which day will you run over 20 km?
2. Complete the pattern: 1, 2, 4, 7, _____ , _____ , _____
3. You and two friends want to go to a movie but you are arguing over the seating arrangements. If you all sit beside each other in a row, how many different ways can you be seated?
4. State the common difference of the following arithmetic sequence: 7, 5, 3, 1, −1, . . .
5. Estimate the answer: 204 × 712
6. How many inches are in 6.5 feet?
7. Determine the volume of a rectangular prism with base dimensions of 12 cm by 10 cm and a height of 6 cm.
8. Write this decimal as a percent: 0.0091

Answers:

1. Day 6 (Day 1 = 1 km; Day 2 = 2 km; Day 3 = 4 km; Day 4 = 8 km; Day 5 = 16; km; Day 6 = 32 km)
2. 11, 16, 22 (add 1, add 2, add 3 is the pattern, so add 4, then 5, then 6 to get the next three terms)
3. 6 ways (3 choices for the frist seat, 2 choices for the second, and 1 choice for the last seat means 3 × 2 × 1 = 6 ways)
4. −2 (5 − 7)
5. 140 000 (a good estimate is 200 × 700 = 140 000)
6. 78 inches (There are 12 inches in 1 foot and 6.5 × 12 = 78 inches.)
7. 720 cm^3 (The formula for the volume of a rectangular prism is V = (area of the base)(height); V = (12 × 10) × 6 = 120 × 6 = 720 cm^3)
8. 0.91% (multiply by 100)

Part B: Geometric Sequences

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. For each sequence below, indicate whether it is geometric, arithmetic, or neither. If it is geometric, state the value of r, the common ratio; if it is arithmetic, state the value of d, the common difference.

   a) 4, 6, 8, 10, . . .

   Answer:

   $6 - 4 = 2$; $8 - 6 = 2$; $10 - 8 = 2$

   Therefore, the sequence is arithmetic with $d = 2$.

   b) 3, 6, 12, 24, . . .

   Answer:

   $\frac{6}{3} = 2$; $\frac{12}{6} = 2$; $\frac{24}{12} = 2$

   Therefore, the sequence is geometric with $r = 2$.

   c) 18, 15, 12, 9, . . .

   Answer:

   $15 - 18 = -3$; $12 - 15 = -3$; $9 - 12 = -3$

   Therefore, the sequence is arithmetic with $d = -3$.

   d) 80, 40, 20, 10, . . .

   Answer:

   $\frac{40}{80} = \frac{1}{2}$; $\frac{20}{40} = \frac{1}{2}$; $\frac{10}{20} = \frac{1}{2}$

   Therefore, the sequence is geometric with $r = \frac{1}{2}$.

   e) 1, −5, 25, −125, . . .

   Answer:

   $\frac{-5}{1} = -5$; $\frac{25}{-5} = -5$; $\frac{-125}{25} = -5$

   Therefore, the sequence is geometric with $r = -5$.

2. Write the first four terms of each sequence. If it is geometric, state the value of r, the common ratio; if it is arithmetic, state the value of d, the common difference.

a) $\begin{cases} t_1 = 2 \\ t_n = 2t_{n-1},\ n \geq 2 \end{cases}$

Answer:

n	t_n
1	$t_1 = 2$
2	$t_2 = 2t_{2-1} = 2t_1 = 2(2) = 4$
3	$t_3 = 2t_{3-1} = 2t_2 = 2(4) = 8$
4	$t_4 = 2t_{4-1} = 2t_3 = 2(8) = 16$

$\therefore$ geometric, with $r = 2$

b) $\begin{cases} t_1 = 2 \\ t_n = t_{n-1} + 2,\ n \geq 2 \end{cases}$

Answer:

n	t_n
1	$t_1 = 2$
2	$t_2 = t_{2-1} + 2 = t_1 + 2 = 2 + 2 = 4$
3	$t_3 = t_{3-1} + 2 = t_2 + 2 = 4 + 2 = 6$
4	$t_4 = t_{4-1} + 2 = t_3 + 2 = 6 + 2 = 8$

$\therefore$ arithmetic, with $d = 2$

c) $\begin{cases} t_1 = 2 \\ t_n = 2t_{n-1} + 2,\ n \geq 2 \end{cases}$

Answer:

n	t_n
1	$t_1 = 2$
2	$t_2 = 2t_{2-1} + 2 = 2t_1 + 2 = 2(2) + 2 = 4 + 2 = 6$
3	$t_3 = 2t_{3-1} + 2 = 2t_2 + 2 = 2(6) + 2 = 12 + 2 = 14$
4	$t_4 = 2t_{4-1} + 2 = 2t_3 + 2 = 2(14) + 2 = 28 + 2 = 30$

$\therefore$ neither

d) $\begin{cases} t_1 = 2 \\ t_n = (t_{n-1})^2,\ n \geq 2 \end{cases}$

Answer:

n	t_n
1	$t_1 = 2$
2	$t_2 = (t_{2-1})^2 = (t_1)^2 = 2^2 = 4$
3	$t_3 = (t_{3-1})^2 = (t_2)^2 = 4^2 = 16$
4	$t_4 = (t_{4-1})^2 = (t_3)^2 = 16^2 = 256$

$\therefore$ neither

e) $\begin{cases} t_1 = 5 \\ t_n = (-1)^n t_{n-1},\ n \geq 2 \end{cases}$

Answer:

n	t_n
1	$t_1 = 5$
2	$t_2 = (-1)^2 t_{2-1} = (-1)^2 t_1 = (1)(5) = 5$
3	$t_3 = (-1)^3 t_{3-1} = (-1)^3 t_2 = (-1)(5) = -5$
4	$t_4 = (-1)^4 t_{4-1} = (-1)^4 t_3 = (1)(-5) = -5$

$\therefore$ neither

f) $\begin{cases} t_1 = 4 \\ t_n = t_{n-1} - 5,\ n \geq 2 \end{cases}$

Answer:

n	t_n
1	$t_1 = 4$
2	$t_2 = t_{2-1} - 5 = t_1 - 5 = 4 - 5 = -1$
3	$t_3 = t_{3-1} - 5 = t_2 - 5 = -1 - 5 = -6$
4	$t_4 = t_{4-1} - 5 = t_3 - 5 = -6 - 5 = -11$

$\therefore$ arithmetic, with $d = -5$

3. Write the first three terms of the geometric sequence generated by each of the following exponential functions. Notice none of them are written in the usual form, $t_n = t_1 r^{n-1}$, but all of them generate a geometric sequence.

   a) $f(x) = 2^x$

   Answer:

n	$t_n = 2^x$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$

   b) $t_n = 4^{n-1}$

   Answer:

n	$t_n = 4^{n-1}$
1	$4^0 = 1$
2	$4^1 = 4$
3	$4^2 = 16$

   c) $f(x) = 2(3^x)$

   Answer:

x	$f(x)$
1	$2(3^1) = 6$
2	$2(3^2) = 18$
3	$2(3^3) = 54$

d) $t_n = 16\left(\frac{1}{2}\right)^n$

Answer:

n	$f(x)$
1	$16\left(\frac{1}{2}\right) = 8$
2	$16\left(\frac{1}{2}\right)^2 = 16\left(\frac{1}{4}\right) = 4$
3	$16\left(\frac{1}{2}\right)^3 = 16\left(\frac{1}{8}\right) = 2$

4. Write a recursive formula and a defining exponential function to generate each of the following geometric sequences.

a) 4, 8, 16, 32, 64, . . .

Answer:

Recursive formula: $\begin{cases} t_1 = 4 \\ t_n = 2t_{n-1},\ n \geq 2 \end{cases}$

Exponential function: $t_n = 4(2)^{n-1}$

b) 6, 18, 54, 162, 486, . . .

Answer:

Recursive formula: $\begin{cases} t_1 = 6 \\ t_n = 3t_{n-1},\ n \geq 2 \end{cases}$

Exponential function: $t_n = 6(3)^{n-1}$

c) 1, −2, 4, −8, 16, . . .

Answer:

Recursive formula: $\begin{cases} t_1 = 1 \\ t_n = -2t_{n-1},\ n \geq 2 \end{cases}$

Exponential function: $t_n = (-2)^{n-1}$

d) 40, 20, 10, 5, $\frac{5}{2}$, . . .

Answer:

Recursive formula: $\begin{cases} t_1 = 40 \\ t_n = \left(\frac{1}{2}\right)\left(t_{n-1}\right),\ n \geq 2 \end{cases}$

Exponential function: $t_n = 40\left(\frac{1}{2}\right)^{n-1}$

5. Write an expression for t_{20} of the sequence 3, 6, 12, 24, ... Do not simplify.

Answer:

The defining exponential function is $t_n = 3(2)^{n-1}$. Therefore, $t_{20} = 3(2)^{20-1} = 3(2)^{19}$.

6. Which term of the geometric sequence 4, 2, 1, . . . has the value $\frac{1}{1024}$?

Answer:

$r = \frac{2}{4} = \frac{1}{2}$

The defining exponential function is $t_n = 4\left(\frac{1}{2}\right)^{n-1}$.

Hence, $\frac{1}{1024} = 4\left(\frac{1}{2}\right)^{n-1}$

$\frac{1}{2^{10}} = 2^2\left(\frac{1}{2}\right)^{n-1}$

$\frac{1}{2^{10}2^2} = \left(\frac{1}{2}\right)^{n-1}$

$\left(\frac{1}{2}\right)^{12} = \left(\frac{1}{2}\right)^{n-1}$

$n - 1 = 12$

$n = 13$

Therefore, $\frac{1}{1024}$ is the 13th term.

7. In a geometric sequence, $t_2 = 6$ and $t_4 = 54$, find the value of t_1.

   Answer:

   Substitute the coordinates of the points (2, 6) and (4, 54) into the exponential function $t_n = t_1 r^{n-1}$ and solve the two equations for the two unknowns.

   The point (2, 6) yields $6 = t_1 r$ and (4, 54) yields $54 = t_1 r^3$. Using division method:

$$\frac{t_1 r^3}{t_1 r} = \frac{54}{6}$$

$$r^2 = 9$$

$$r = \pm 3$$

   Two possible sequences exist, one with $r = 3$ and the other with $r = -3$.

   Case 1: If $r = 3$, then $6 = t_1(3)$ or $t_1 = 2$.

   Case 2: If $r = -3$, then $6 = t_1(-3)$ or $t_1 = -2$.

8. What is the value of x if the sequence, $x + 2, 3x - 2, 5x + 2$ is geometric?

   Answer:

   If the sequence is geometric, the ratios, $\frac{t_2}{t_1}$ and $\frac{t_3}{t_2}$, must be equal.

$$\frac{3x - 2}{x + 2} = \frac{5x + 2}{3x - 2}$$

$$(3x - 2)(3x - 2) = (x + 2)(5x + 2) \qquad \text{(cross-multiply)}$$

$$9x^2 - 12x + 4 = 5x^2 + 12x + 4$$

$$4x^2 - 24x = 0$$

$$4x(x - 6) = 0$$

$$x = 6 \text{ or } x = 0$$

   If you are interested in seeing the sequences, substitute the values of x into the terms.

   When $x = 6$, the sequence is 8, 16, 32.

   When $x = 0$, the sequence is 2, −2, 2.

   Both are geometric sequences with common ratio 2 and −1, respectively.

9. Insert three numbers a, b, c between 5 and 80, so that 5, a, b, c, 80 forms a geometric sequence.

   Answer:

   If r is the common ratio then the terms of the geometric sequence are 5, $5r$, $5r^2$, $5r^3$, $5r^4$.

   Therefore, $5r^4 = 80$.

   $r^4 = 16$

   $r = \pm 2$

   Case 1: If $r = 2$, then $a = 10$, $b = 20$, $c = 40$.

   Case 2: If $r = -2$, then $a = -10$, $b = 20$, $c = -40$.

10. A store is conducting a Dutch auction. It will take 10% off the cost of an item at the end of each day the item doesn't sell. If an item originally costs \$150, find its cost for each of the next five days.

    Answer:

    If the cost of the item is being reduced by 10%, then the sale price is 100% − 10% or 90% of the original price. Thus, the cost of the item at the end of the next five days is calculated as follows.

    first day: (\$150.00)(0.90) = \$135.00

    second day: (\$135.00)(0.90) = \$121.50

    third day: (\$121.50)(0.90) = \$109.35

    fourth day: (\$109.35)(0.90) = \$98.42

    fifth day: (\$98.42)(0.90) = \$88.58

    Note that if you use the formula $t_6 = 150(0.9)^{6-1}$ to find your answer for the end of the fifth day, you get \$88.57. Since the deductions are made daily, the correct answer is one cent different and is \$88.58.

11. A geometric sequence is defined by the exponential function $t_n = 5(2)^{n-1}$

    a) What is the first term?

       Answer:

       $t_1 = 5$, the coefficient

    b) What is the common ratio?

       Answer:

       $r = 2$, the base

c) What is the value of the 10th term?

Answer:

$t_{10} = 5(2)^{10-1} = 5(2)^9 = 2560$

d) What is the value of t_7?

Answer:

$t_7 = 5(2)^{7-1} = 5(2)^6 = 320$

e) What position in the sequence has the value 163 840?

Answer:

$$t_n = 5(2)^{n-1}$$

$$163\,840 = 5(2)^{n-1}$$

$$\frac{163\,840}{5} = \frac{5(2)^{n-1}}{5} \qquad \text{Divide by 5}$$

$$32\,768 = 2^{n-1} \qquad \text{Simplify}$$

$$2^{15} = 2^{n-1} \qquad \text{Use guess and check}$$

$$15 = n - 1 \qquad \text{Equate exponents}$$

$$16 = n \qquad \text{Solve}$$

f) Is the number 4100 a member of this geometric sequence? Justify your answer.

Answer:

$$t_n = 5(2)^{n-1}$$

$$4100 = 5(2)^{n-1}$$

$$\frac{4100}{5} = \frac{5(2)^{n-1}}{5}$$

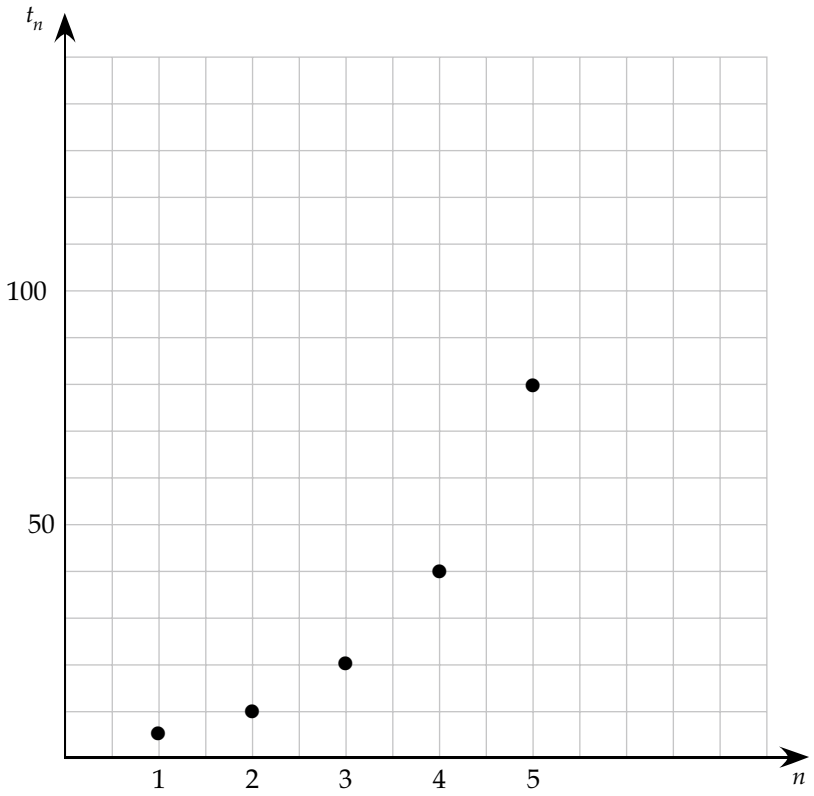
$$820 = 2^{n-1}$$

Powers of 2 are 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, . . ., and 820 is not one of them. Thus, 4100 is not a term of the sequence.

g) Graph the sequence using graph paper or your graphing calculator.

Answer:

The graph of the function is shown below.



The shape is a curve and not a straight line. The points are not joined because the domain is the set of natural numbers.

Learning Activity 1.5

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. For Josh's workout, he runs $\frac{1}{9}$ of the football field, sprints $\frac{7}{9}$ of the field, and then jogs the rest of the way. What fraction of the field does Josh jog?
2. What is the sum of the digits from 1 to 10?
3. Calculate the perimeter of a square garden with an area of 25 m^2.
4. Express $\frac{6}{20}$ as a decimal.
5. Find the common difference in the following pattern: −2, −9, −16, −23, . . .
6. Estimate: $\sqrt{96}$
7. Simplify: $(2^4)(3^2)$
8. Find the slope of the line defined by the equation, $3y - 2x + 7 = 0$.

Answers:

1. $\frac{1}{9}\left(1-\left(\frac{1}{9}+\frac{7}{9}\right)=\frac{9}{9}-\frac{8}{9}=\frac{1}{9}\right)$
2. 55 (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10; combining digits that sum to 10 (1 and 9, 2 and 8, etc.) gives 10 + 10 + 10 + 10 + 10 + 5 = 55)
3. 20 m (Length of the side = $\sqrt{25} = 5$; Perimeter = 5 × 4)
4. $0.3\left(\frac{6}{20}=\frac{3}{10}=0.3\right)$
5. −7
6. 9.8 (96 is between 81 and 100; therefore, $\sqrt{96}$ is between 9 and 10 and is closer to 10)
7. 144 ($2^4 \times 3^2 = 16 \times 9$)
8. $\frac{2}{3}\left(3y = 2x - 7;\ y = \frac{2}{3}x - \frac{7}{3}\right)$

Part B: Sums of Geometric Series

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. a) Give an example of a geometric sequence.

   Answer:

   Answers will vary. Be sure every term is a multiple of the term before.

   b) Write the corresponding geometric series.

   Answer:

   Answers will vary. Answers must be an addition of the terms in the corresponding geometric sequence.

Check with your learning partner or review examples in the lesson to check your work.

2. Write in expanded form and find the sum.

   a) $\sum_{k=1}^{4} k^3$

   b) $\sum_{k=1}^{4} 3^k$

   c) $\sum_{k=1}^{3} 3k$

   d) $\sum_{k=1}^{4} 3^{k-1}$

   e) $\sum_{k=2}^{4} 3^{k+1}$

   f) $\sum_{k=2}^{4} 5 \cdot 3^{k-1}$

   g) $\sum_{k=2}^{5} (2k-5)$

   h) $\sum_{k=3}^{6} 2(-3)^{k-1}$

   Answers:

   a) $\sum_{k=1}^{4} k^3 = (1)^3 + (2)^3 + (3)^3 + (4)^3 = 1 + 8 + 27 + 64 = 100$

   b) $\sum_{k=1}^{4} 3^k = 3^1 + 3^2 + 3^3 + 3^4 = 3 + 9 + 27 + 81 = 120$

   c) $\sum_{k=1}^{3} 3k = 3(1) + 3(2) + 3(3) = 3 + 6 + 9 = 18$

d) $\sum_{k=1}^{4} 3^{k-1} = 3^{1-1} + 3^{2-1} + 3^{3-1} + 3^{4-1} = 3^0 + 3^1 + 3^2 + 3^3$

$= 1 + 3 + 9 + 27 = 40$

e) $\sum_{k=2}^{4} 3^{k+1} = 3^{2+1} + 3^{3+1} + 3^{4+1} = 3^3 + 3^4 + 3^5 = 27 + 81 + 243 = 351$

f) $\sum_{k=2}^{4} 5 \cdot 3^{k-1} = 5 \cdot 3^{2-1} + 5 \cdot 3^{3-1} + 5 \cdot 3^{4-1} = 5 \cdot 3^1 + 5 \cdot 3^2 + 5 \cdot 3^3$

$= 15 + 45 + 135 = 195$

g) $\sum_{k=2}^{5} (2k - 5) = (2(2) - 5) + (2(3) - 5) + (2(4) - 5) + (2(5) - 5)$

$= -1 + 1 + 3 + 5 = 8$

h) $\sum_{k=3}^{6} 2(-3)^{k-1} = 2(-3)^{3-1} + 2(-3)^{4-1} + 2(-3)^{5-1} + 2(-3)^{6-1}$

$= 2(-3)^2 + 2(-3)^3 + 2(-3)^4 + 2(-3)^5$

$= 2(9) + 2(-27) + 2(81) + 2(-243)$

$= 18 - 54 + 162 - 486 = -360$

3. Use sigma notation to write each geometric sum.

a) $4 + 12 + 36 + 108$

Answer:

$r = 3$

$t_1 = 4$

$n = 4$

$\sum_{k=1}^{4} 4(3)^{k-1}$

b) $24 + 4 + \frac{2}{3} + \frac{1}{9} + \frac{1}{54}$

Answer:

$r = \frac{4}{24} = \frac{1}{6}$

$t_1 = 24$

$n = 5$

$\sum_{k=1}^{5} 24\left(\frac{1}{6}\right)^{k-1}$

c) $2 + 4 + 8 + \ldots + 1024$

Answer:

$r = 2$

$t_1 = 2$

$n = 10$, since $2^{10} = 1024$

$\sum_{k=1}^{10} 2(2)^{k-1}$ or $\sum_{k=1}^{10} 2^k$

4. Consider the sequence 3, 6, 12, 24, . . .

a) Find the eighth term.

Answer:

$t_n = 3(2)^{n-1}$

Therefore, $t_8 = 3(2)^{8-1} = 384$.

b) Find the sum of the first eight terms, using the sum formula.

Answer:

Using $S_n = \frac{t_1\left(1 - r^n\right)}{1 - r}$, $S_8 = \frac{3\left(1 - (2)^8\right)}{1 - 2} = \frac{3(-255)}{-1} = 765$.

5. Find the sum of the first 10 terms of each series. If the numerical answer is not an integer, state it correct to two decimal places.

a) $120 + 60 + 30 + 15 + \ldots$

Answer:

$$\text{Using } S_n = \frac{t_1\left(1 - r^n\right)}{1 - r},\ S_{10} = \frac{120\left(1 - (0.5)^{10}\right)}{1 - 0.5} = 239.77.$$

b) $8 + 12 + 18 + 27 + \ldots$

Answer:

$$\text{Using } S_n = \frac{t_1\left(1 - r^n\right)}{1 - r} \text{ and } r = \frac{12}{8} = \frac{3}{2} = 1.5,\ S_{10} = \frac{8\left(1 - (1.5)^{10}\right)}{1 - 1.5} = 906.64.$$

c) $128 - 64 + 32 - 16 + \ldots$

Answer:

$$\text{Using } S_n = \frac{t_1\left(1 - r^n\right)}{1 - r},\ S_{10} = \frac{128\left(1 - (-0.5)^{10}\right)}{1 - (-0.5)} = 85.25.$$

d) $24 + 8 + \frac{8}{3} + \frac{8}{9} + \ldots$

Answer:

$$\text{Using } S_n = \frac{t_1\left(1 - r^n\right)}{1 - r},\ S_{10} = \frac{24\left(1 - \left(\frac{1}{3}\right)^{10}\right)}{1 - \frac{1}{3}} = 36.00.$$

e) $x^2 + x^4 + x^6 + x^8 + \ldots$

Answer:

$$\text{Using } S_n = \frac{t_1\left(1 - r^n\right)}{1 - r},\ S_{10} = \frac{x^2\left(1 - \left(x^2\right)^{10}\right)}{1 - x^2} = \frac{x^2\left(1 - x^{20}\right)}{1 - x^2}.$$

f) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

Answer:

Using $S_n = \frac{t_1\left(1 - r^n\right)}{1 - r}$, $S_{10} = \frac{1\left(1 - (0.5)^{10}\right)}{1 - 0.5} = 2.00.$

6. In a geometric sequence, $t_4 = \frac{1}{4}$, and $t_{11} = 32$. Calculate S_{11}. Express your answer as an exact value.

Answer:

Use the general term formula to write two equations. Then solve using division method.

$t_4 = \frac{1}{4} = t_1 r^3$ equation (a)

$t_{11} = 32 = t_1 r^{10}$ equation (b)

$$\frac{32}{\frac{1}{4}} = \frac{t_1 r^{10}}{t_1 r^3}$$

$$128 = r^7$$

$$2 = r$$

Now substitute $r = 2$ into equation (a) and solve for t_1.

$$\frac{1}{4} = t_1 2^3$$

$$\frac{1}{4} = t_1 (8)$$

$$\frac{\frac{1}{4}}{8} = \frac{t_1 (8)}{8}$$

$$\frac{1}{32} = t_1$$

Now find S_{11}.

$$S_n = \frac{t_1\left(1 - r^n\right)}{1 - r}$$

$$S_{11} = \frac{\left(\frac{1}{32}\right)\left(1 - 2^{11}\right)}{1 - 2}$$

$$S_{11} = \frac{2047}{32}$$

Note that you can use the $a\frac{b}{c}$ button on your scientific calculator or Math Frac on your graphing calculator to find the exact answers when using fractions.

7. The sum of a geometric series with $t_1 = 7$ is 889. If $r = 2$, how many terms are in this geometric series?

Answer:

You have the following information:

$S_n = 889$

$t_1 = 7$

$r = 2$

Using the formula $S_n = \frac{t_1\left(1 - r^n\right)}{1 - r}$, it is possible to solve for n, the number of terms in this series.

$$889 = \frac{7\left(1 - 2^n\right)}{1 - 2}$$

$$889 = -7\left(1 - 2^n\right)$$

$$\frac{889}{-7} = \frac{-7\left(1 - 2^n\right)}{-7}$$

$$-127 = 1 - 2^n$$

$$-127 - 1 = -2^n$$

$$-128 = -2^n$$

$$128 = 2^n$$

$$2^7 = 2^n$$

$$n = 7$$

Therefore, there are 7 terms in this series.

8. The sum of a geometric series is 98 301. If there are 15 terms in this sequence and the common ratio is 2, what is the value for t_1?

   Answer:

   You have the following information:

   $S_n = 98\ 301$

   $r = 2$

   $n = 15$

   Using the formula $S_n = \frac{t_1\left(1 - r^n\right)}{1 - r}$, it is possible to solve for t_1.

$$98\ 301 = \frac{t_1\left(1 - 2^{15}\right)}{1 - 2}$$

$$98\ 301 = t_1\left(32\ 767\right)$$

$$\frac{98\ 301}{32\ 767} = \frac{t_1\left(32\ 767\right)}{32\ 767}$$

$$3 = t_1$$

Learning Activity 1.6

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Find the common ratio in the following pattern: 189, 63, 21, 7, . . .
2. Which of these values would come next in this pattern:

   One, four, three, eleven, fifteen, thirteen, . . .

   Fourteen

   Five

   Seven

   Seventeen
3. If your favourite television show starts at 7 pm and your mom tells you that your show is still 48 minutes away, what time is it now?
4. Your school ordered 56 boxes of cookies that each contain 25 cookies. If there are 700 students at your school, will you have enough cookies to give each student two cookies?
5. Calculate: $\frac{3}{16} + \frac{2}{4}$
6. Solve the inequality $5x - 2 < 0$.
7. Simplify: $(2x^2 + 3x - 5) - (9x^2 - 4x + 6)$
8. Simplify the following fraction: $\frac{144}{180}$

Answers:

1. $\frac{1}{3}$
2. Seventeen (Each number has one more letter in the spelling of the word than the previous number.)
3. 6:12 pm.
4. You will have exactly enough cookies. ($56 \times 25 = 50 \times 25 + 6 \times 25 = 1250 + 150 = 1400 = 700 \times 2$)
5. $\frac{11}{16}$

6. $5x < 2$

$$x < \frac{2}{5}$$

7. $-7x^2 + 7x - 11$
$[(2x^2 + 3x - 5) - (9x^2 - 4x + 6) = 2x^2 + 3x - 5 - 9x^2 + 4x - 6]$

8. $\frac{4}{5}\left(\frac{144}{180} = \frac{72}{90} = \frac{36}{45}\right)$

Part B: Sums of Infinite Geometric Series

If you haven't already, you may want to go back and include all important definitions, terms, or examples that you have found more challenging on your resource sheet.

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Find the sum of each of the following infinite geometric series. Express your answers as exact values.

   a) $8 + \frac{8}{3} + \frac{8}{9} + \frac{8}{27} + \ldots$

   Answer :

   $$r = \frac{1}{3},\ S = \frac{t_1}{1-r} = \frac{8}{1-\frac{1}{3}} = 12$$

   b) $6 - 3 + \frac{3}{2} - \frac{3}{4} + \ldots$

   Answer :

   $$r = -\frac{1}{2},\ S = \frac{t_1}{1-r} = \frac{6}{1-\left(-\frac{1}{2}\right)} = 4$$

c) $6 + 4 + \frac{8}{3} + \frac{16}{9} + \ldots$

Answer :

$$r = \frac{2}{3}, \quad S = \frac{t_1}{1-r} = \frac{6}{1-\frac{2}{3}} = 18$$

d) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$

Answer:

$$r = -\frac{1}{2}, \quad S = \frac{t_1}{1-r} = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3}$$

e) $\sum_{k=1}^{\infty}\left(\frac{16}{2^k}\right)$

Answer:

$$\sum_{k=1}^{\infty}\left(\frac{16}{2^k}\right) = \frac{16}{2^1} + \frac{16}{2^2} + \frac{16}{2^3} + \ldots$$

$$= 8 + 4 + 2 + \ldots$$

$$r = \frac{1}{2}, \quad S = \frac{t_1}{1-r} = \frac{8}{1-\frac{1}{2}} = 16$$

f) $\sum_{k=2}^{\infty}\left(\frac{18}{3^k}\right)$

Answer:

$$\sum_{k=2}^{\infty}\left(\frac{18}{3^k}\right) = \frac{18}{3^2} + \frac{18}{3^3} + \frac{18}{3^4} + \ldots$$

$$= \frac{18}{9} + \frac{18}{27} + \frac{18}{81} + \ldots$$

$$= 2 + \frac{2}{3} + \frac{2}{9} + \ldots$$

$$r = \frac{1}{3}, \quad S = \frac{t_1}{1-r} = \frac{2}{1-\left(\frac{1}{3}\right)} = 3$$

2. Consider a geometric sequence where $t_3 = 16$ and $t_5 = 10$.

   a) Find the common ratio r, rounded to two decimal places.

   Answer:

   Use the formula for the general term to find two equations. Then use division method to solve.

$$t_3 = 16 = t_1 r^2$$

$$t_5 = 10 = t_1 r^4$$

$$\frac{t_1 r^4}{t_1 r^2} = \frac{10}{16}$$

$$r^2 = 0.625$$

$$r = \pm 0.79$$

   b) Find $\sum_{k=1}^{\infty} t_k$. State your answer rounded to two decimal places.

   Answer:

   Substitute the values for r into either equation to find t_1.

$$16 = t_1 r^2$$

$$16 = t_1 (\pm 0.79)^2$$

$$16 = t_1 (0.625)$$

$$t_1 = \frac{16}{0.625}$$

$$t_1 = 25.6$$

   Note that both values for r meet the restriction: $|r| < 1$.

Case 1:	Case 2:
$r = 0.79$	$r = -0.79$
$S_\infty = \frac{t_1}{1 - r}$	$S_\infty = \frac{t_1}{1 - r}$
$S_\infty = \frac{25.6}{1 - 0.79}$	$S_\infty = \frac{25.6}{1 - (-0.79)}$
$S_\infty = 121.90$ or 122.24	$S_\infty = 14.30$

   Note the answer for S_∞ can be obtained using the calculator answer for r before rounding. If you used $r = 0.79$, $S_\infty = 121.90$, for $r = -0.79$, $S_\infty = 14.30$. Both 121.90 and 122.24 would be considered correct.

3. The sum to infinity of an infinite geometric series is 12. If the common ratio is 0.25, what is the first term of this series?

Answer:

$$S = \frac{t_1}{1 - r}$$

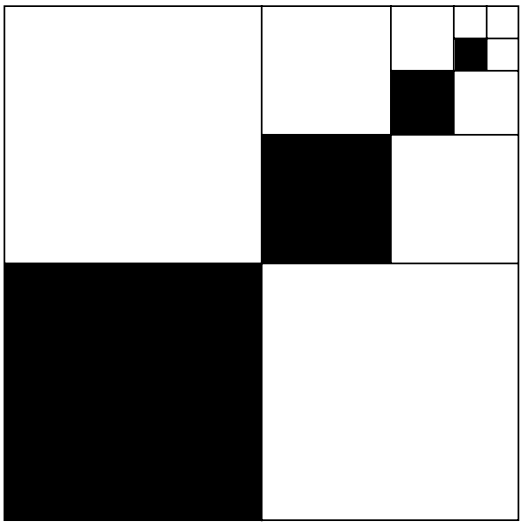
Therefore, $12 = \frac{t_1}{1 - 0.25}$

$$12 = \frac{t_1}{0.75}$$

$$t_1 = 12(0.75)$$

$$t_1 = 9$$

4. A 4-by-4 square is divided into four congruent squares, as shown in the diagram. The bottom left square is shaded. The top right square is subdivided into four congruent squares again, and the bottom left square is shaded. This process is repeated indefinitely. What is the total shaded area?



Answer:

$2^2,\ 1^2,\ \left(\frac{1}{2}\right)^2, \ldots$ or $4, 1, \frac{1}{4}, \ldots$

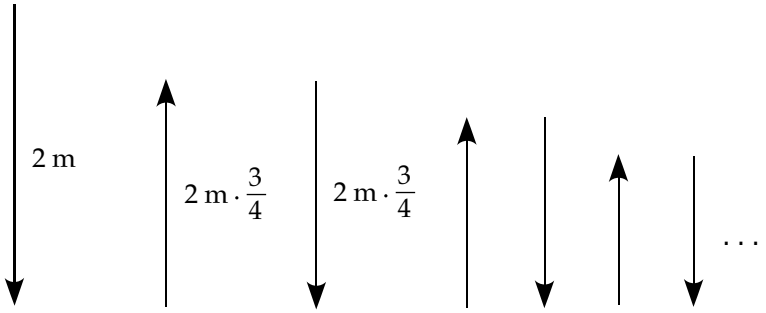
$$S = \frac{t_1}{1 - r} = \frac{4}{1 - \frac{1}{4}} = \frac{4}{\frac{3}{4}} = 4 \times \frac{4}{3} = \frac{16}{3}$$

5. A ball is dropped from a height of two metres. Each time it strikes the ground, it rebounds three-quarters of the distance it has fallen.

   a) How far does it rise after it strikes the ground the third time?

   Answer:

   Draw a diagram to get a sense of what is happening:



   First bounce $= 2\left(\frac{3}{4}\right) = \frac{3}{2}$; second bounce $= \frac{3}{2}\left(\frac{3}{4}\right) = \frac{9}{8}$;

   third bounce $= \frac{9}{8}\left(\frac{3}{4}\right) = \frac{27}{32}$.

   b) How far does it travel before it comes to rest?

   Answer:

   From the diagram, the total downward distance series is

$$2 + \frac{3}{2} + \frac{9}{8} + \ldots = \frac{t_1}{1-r} = \frac{2}{1-\frac{3}{4}} = 8.$$

   The total upward distance = downward distance − original drop = $8 - 2 = 6$.

   The distance travelled is $8 + 6 = 14$ metres.

6. What is the value of x if the geometric series $1 + 3x + 9x^2 + 27x^3 + \ldots$ converges to 4?

   Answer:

   $t_1 = 1$ and $r = 3x$

   Furthermore, the sum is 4, so solving for x you get

$$S = \frac{t_1}{1 - r}$$

$$4 = \frac{1}{1 - 3x}$$

$$4(1 - 3x) = \frac{1}{1 - 3x}(1 - 3x)$$

$$4 - 12x = 1$$

$$3 = 12x$$

$$x = \frac{3}{12} = \frac{1}{4}$$

7. Recall Zeno's paradox from the introduction to this lesson. Zeno's paradox stated that there is no sum to the infinite geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$. Prove that Zeno is incorrect.

   Answer:

   S_∞ or $S = \frac{t_1}{1 - r}$, provided $|r| < 1$

   In this example, $r = \frac{1}{2} < 1$.

   Therefore, you can use the sum formula.

$$S = \frac{t_1}{1 - r} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

   Therefore, there is a sum to this geometric series, and it is possible to catch the bus.

Notes

Grade 11 Pre-Calculus Mathematics (30S)

Module 1
Sequences and Series

Learning Activity Answer Keys

Module 1:
Sequences and Series

Learning Activity 1.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. At the end of a poker game, Alysha had $62. During the game she had won $250, lost $175, won $35, and lost $55. How much money did Alysha have at the start, in dollars?

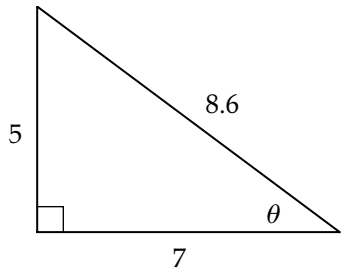
2. Complete the pattern: 1, 2, 5, 10, 17, ____ , ____ , ____

3. Evaluate when $z = -1$: $(3z + 1)(z - 5$

4. Evaluate: $\left(\frac{8}{16}\right)(12)$

5. A clothing store is selling shirts for $30.00. They decided to raise the price of the shirt by 20%. A few days later they decided to raise the price another 20%. What is the new price of the shirt, in dollars?

Use the following right-angled triangle to answer the next three questions. Do not simplify your answer.



6. Identify the sine ratio for θ.

7. Identify the cosine ratio for θ.

8. Identify the tangent ratio for θ.

Answers:

1. Alysha started with $7.

   (x + \$250 − \$175 + \$35 − \$55 = \$62

   x + \$55 = \$62

   x = \$62 − \$55

   x = \$7)

2. 26, 37, 50 (You are adding the next consecutive odd integer each time, add 1, add 3, add 5, etc.)

3. 12 ((3z + 1)(z − 5) = (3(−1) + 1)(−1 − 5) = (−3 + 1)(−6) = (−2)(−6))

4. $6\left(\left(\frac{1}{2}\right)(12)\right)$

5. \$43.20

   (10% of \$30 = \$3

   20% of \$30 = \$6

   Therefore, after the first increase, the price of the shirt is \$36.

   10% of \$36 = \$3.60

   20% of \$36 = \$7.20

   The new price of the shirt is \$36.00 + \$7.20 = \$43.20.)

6. $\frac{5}{8.6}$

7. $\frac{7}{8.6}$

8. $\frac{5}{7}$

Part B: Arithmetic Sequences

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Which of the following sequences are arithmetic?
   a) $2, 5, 8, 11, \ldots$
   b) $-5, -1, 3, 7, \ldots$
   c) $2, 5, 7, 10, \ldots$
   d) $2, 2\frac{1}{2}, 1, 1\frac{1}{2}, 0, \ldots$

   Answers:
   a) $5 - 2 = 3, 8 - 5 = 3, 11 - 8 = 3$; arithmetic sequence
   b) $-1 - (-5) = 4, 3 - (-1) = 4, 7 - 3 = 4$; arithmetic sequence
   c) $5 - 2 = 3, 7 - 5 = 2$; not an arithmetic sequence, $3 \neq 2$
   d) $2\frac{1}{2} - 2 = \frac{1}{2}, 1 - 2\frac{1}{2} = -1\frac{1}{2}$; not an arithmetic sequence, $-1\frac{1}{2} \neq \frac{1}{2}$

2. Write a recursive formula for each sequence. Indicate whether the sequence is arithmetic or not.
   a) $10, 7, 4, 1, \ldots$
   b) $2, 4, 8, 16, \ldots$
   c) $30, 41, 52, 63, \ldots$
   d) $3, -6, 12, -24, \ldots$
   e) $5, 15, 45, 135, \ldots$

   Answers:
   a) $\begin{cases} t_1 = 10 \\ t_n = t_{n-1} - 3, \; n \geq 2 \end{cases}$

   Arithmetic sequence with common difference $= -3$.

   b) $\begin{cases} t_1 = 2 \\ t_n = 2t_{n-1}, \; n \geq 2 \end{cases}$

   Not an arithmetic sequence.

c) $\begin{cases} t_1 = 30 \\ t_n = t_{n-1} + 11,\ n \geq 2 \end{cases}$

Arithmetic sequence with common difference = 11.

d) $\begin{cases} t_1 = 3 \\ t_n = -2t_{n-1},\ n \geq 2 \end{cases}$

Not an arithmetic sequence.

e) $\begin{cases} t_1 = 5 \\ t_n = 3t_{n-1},\ n \geq 2 \end{cases}$

Not an arithmetic sequence.

3. Write the first four terms of each sequence.

a) $\begin{cases} t_1 = 3 \\ t_n = t_{n-1} + 5,\ n \geq 2 \end{cases}$

Answer:

n	$t_n = t_{n-1} + 5$
1	$t_1 = 3$
2	$t_2 = t_{2-1} + 5 = t_1 + 5 = 3 + 5 = 8$
3	$t_3 = t_{3-1} + 5 = t_2 + 5 = 8 + 5 = 13$
4	$t_4 = t_{4-1} + 5 = t_3 + 5 = 13 + 5 = 18$

The first four terms are 3, 8, 13, 18.

b) $\begin{cases} t_1 = 10 \\ t_n = t_{n-1} - 2,\ n \geq 2 \end{cases}$

Answer:

n	$t_n = t_{n-1} - 2$
1	$t_1 = 10$
2	$t_2 = t_{2-1} - 2 = t_1 - 2 = 10 - 2 = 8$
3	$t_3 = t_{3-1} - 2 = t_2 - 2 = 8 - 2 = 6$
4	$t_4 = t_{4-1} - 2 = t_3 - 2 = 6 - 2 = 4$

The first four terms are 10, 8, 6, 4.

c) $\begin{cases} t_1 = 3.7 \\ t_n = t_{n-1} + 1.3,\ n \geq 2 \end{cases}$

Answer:

n	$t_n = t_{n-1} + 1.3$
1	$t_1 = 3.7$
2	$t_2 = t_{2-1} + 1.3 = t_1 + 1.3 = 3.7 + 1.3 = 5$
3	$t_3 = t_{3-1} + 1.3 = t_2 + 1.3 = 5 + 1.3 = 6.3$
4	$t_4 = t_{4-1} + 1.3 = t_3 + 1.3 = 6.3 + 1.3 = 7.6$

The first four terms are 3.7, 5, 6.3, 7.6

d) $\begin{cases} t_1 = 1 \\ t_2 = 1 \\ t_n = t_{n-1} + t_{n-2},\ n \geq 3 \end{cases}$

Answer:

n	$t_n = t_{n-1} + 1.3$
1	$t_1 = 1$
2	$t_2 = 1$
3	$t_3 = t_{3-1} + t_{3-2} = t_2 + t_1 = 1 + 1 = 2$
4	$t_4 = t_{4-1} + t_{4-2} = t_3 + t_2 = 2 + 1 = 3$

The first four terms are 1, 1, 2, 3.

e) $\begin{cases} t_1 = 3 \\ t_2 = -1 \\ t_n = t_{n-1} + t_{n-2},\ n \geq 3 \end{cases}$

Answer:

n	$t_n = t_{n-1} + 1.3$
1	$t_1 = 3$
2	$t_2 = -1$
3	$t_3 = t_{3-1} + t_{3-2} = t_2 + t_1 = -1 + 3 = 2$
4	$t_4 = t_{4-1} + t_{4-2} = t_3 + t_2 = 2 + (-1) = 1$

The first four terms are 3, −1, 2, 1.

4. Write the first three terms of the arithmetic sequence defined by the linear function.

   a) $f(x) = 2x + 4$

   Answer:

   $f(1) = 2(1) + 4 = 2 + 4 = 6$

   $f(2) = 2(2) + 4 = 4 + 4 = 8$

   $f(3) = 2(3) + 4 = 6 + 4 = 10$

   Therefore, the first three terms are 6, 8, and 10.

   b) $g(x) = 5x - 2$

   Answer:

   $g(1) = 5(1) - 2 = 5 - 2 = 3$

   $g(2) = 5(2) - 2 = 10 - 2 = 8$

   $g(3) = 5(3) - 2 = 15 - 2 = 13$

   Therefore, the first three terms are 3, 8, and 13.

   c) $h(x) = -3x + 2$

   Answer:

   $h(1) = -3(1) + 2 = -3 + 2 = -1$

   $h(2) = -3(2) + 2 = -6 + 2 = -4$

   $h(3) = -3(3) + 2 = -9 + 2 = -7$

   Therefore, the first three terms are -1, -4, and -7.

   d) $m(x) = x - 5$

   Answer:

   $m(1) = 1 - 5 = -4$

   $m(2) = 2 - 5 = -3$

   $m(3) = 3 - 5 = -2$

   Therefore, the first three terms are -4, -3, and -2.

5. Three terms of an arithmetic sequence (not necessarily the first three terms) are 27, 30, 33. Write two different linear functions that may have produced this sequence.

   Answer:

   Since the common difference between terms is 3, it follows that $m = 3$. Possible linear functions are $y = 3x$, $y = 3x - 3$. In general, you could write $y = 3x + b$ where b is any real number.

6. Find an equation of the linear function that produces the following first three terms of an arithmetic sequence.

   a) 5, 8, 11, . . .

   Answer:

   The common difference in this sequence is three. Therefore, the slope of the line, m, is 3. A point on the line is (1, 5), or (2, 8), or (3, 11).

   $y - y_1 = m(x - x_1)$

   $y - 5 = 3(x - 1)$

   $y = 3x - 3 + 5$

   $y = 3x + 2$

   b) $6, 6\frac{1}{2}, 7, \ldots$

   Answer:

   The common difference in this sequence is $\frac{1}{2}$. Therefore, the slope of the line, m, is $\frac{1}{2}$. A point on the line is $(1, 6)$, or $\left(2, 6\frac{1}{2}\right)$, or $(3, 7)$.

   $y - y_1 = m(x - x_1)$

   $y - 6 = \frac{1}{2}(x - 1)$

   $y = \frac{1}{2}x - \frac{1}{2} + 6$

   $y = \frac{1}{2}x + 5\frac{1}{2}$

c) 10, 8, 6, . . .

Answer:

The common difference in this sequence is −2. Therefore, the slope of the line, m, is −2. A point on the line is (1, 10), or (2, 8), or (3, 6).

$y - y_1 = m(x - x_1)$

$y - 10 = -2(x - 1)$

$y = -2x + 2 + 10$

$y = -2x + 12$

d) $2, 2\frac{1}{10}, 2\frac{1}{5}, \ldots$

Answer:

The common difference in this sequence is $\frac{1}{10}$. Therefore, the slope of the line, m, is $\frac{1}{10}$. A point on the line is $(1, 2)$, or $\left(2, 2\frac{1}{10}\right)$, or $\left(3, 2\frac{1}{5}\right)$.

$y - y_1 = m(x - x_1)$

$y - 2 = \frac{1}{10}(x - 1)$

$y = \left(\frac{1}{10}\right)x - \frac{1}{10} + 2$

$y = \left(\frac{1}{10}\right)x + 1\frac{9}{10}$

7. Find the 50th term of each of the following arithmetic sequences.

a) $f(x)$: 7, 11, 15, . . .

Answer:

$y - 7 = 4(x - 1)$

$y = 4x - 4 + 7$

$y = 4x + 3$

$\therefore f(x) = 4x + 3$

and $f(50) = 4(50) + 3$

$= 200 + 3$

$= 203$

Thus, the 50th term is 203.

b) $g(x)$: 10, 15, 20, . . .

Answer:

$y - 10 = 5(x - 1)$

$y = 5x - 5 + 10$

$y = 5x + 5$

$\therefore g(x) = 5x + 5$

and $g(50) = 5(50) + 5$

$= 250 + 5$

$= 255$

Thus, the 50th term is 255.

c) $h(x)$: $4\frac{1}{2}$, 5, $5\frac{1}{2}$, . . .

Answer:

$$y - 4\frac{1}{2} = \frac{1}{2}(x - 1)$$

$$y = \frac{1}{2}x - \frac{1}{2} + 4\frac{1}{2}$$

$$y = \frac{1}{2}x + 4$$

$$\therefore h(x) = \frac{1}{2}x + 4$$

$$\text{and } h(50) = \frac{1}{2}(50) + 4$$

$$= 25 + 4$$

$$= 29$$

8. The fifth term of an arithmetic sequence is -15 and the ninth term is -3. Find the value of the hundredth term.

Answer:

(5, -15) and (9, -3) are points on the line.

$$m = \frac{-3-(-15)}{9-5} = \frac{12}{4} = 3$$

$$y-(-15) = 3(x-5)$$

$$y+15 = 3x-15$$

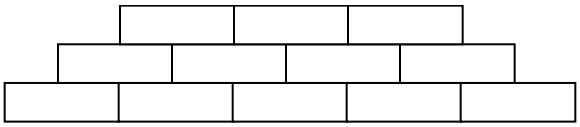
$$y = 3x-15-15$$

$$y = 3x-30$$

$$f(x) = 3x-30$$

$$f(100) = 3(100)-30 = 270 = t_{100}$$

9. A pile of bricks is arranged in rows. The numbers of bricks in the rows form an arithmetic sequence. One such pile of bricks with 3 rows is shown below:



In another pile of bricks, there are 45 bricks in the fifth row and 33 bricks in the eleventh row.

a) How many bricks are in the first row?

Answer:

Let x represent the row number and let $f(x)$ represent the number of bricks in the row.

The linear function contains the points with coordinates (5, 45) and (11, 33).

$$m = \frac{33-45}{11-5} = \frac{-12}{6} = -2$$

$$y-45 = -2(x-5)$$

$$y = -2x+10+45$$

$$y = -2x+55 \text{ or}$$

$$f(x) = -2x+55 \text{ is the defining linear function}$$

Now find the first term.

$$f(1) = -2(1)+55 = 53$$

There are 53 bricks in the first row.

b) How many bricks are in the nth row?

Answer:

$f(n) = -2n + 55$

There are $-2n + 55$ bricks in the nth row.

c) What is the maximum number of rows of bricks possible?

Answer:

Since the number of bricks cannot be negative or equal to zero, the top row of bricks may have as few as one brick in it. You need to solve the following equation:

$$1 = -2n + 55$$
$$-54 = -2n$$
$$27 = n$$

The maximum number of rows is 27.

10. Find which term of the sequence 20, 23, 26, . . ., is the value 245,

a) by using the general term formula

Answer:

Using the formula $t_n = t_1 + (n - 1)d$, you have

$$245 = 20 + (n - 1)3$$
$$245 = 20 + 3n - 3$$
$$245 = 3n + 17$$
$$245 - 17 = 3n$$
$$228 = 3n$$
$$76 = n$$

Therefore, $245 = t_{76}$.

Hence, the 76th term of the sequence has a value of 245.

b) by using the defining linear function

Answer:

The defining linear function can be found using the points (1, 20) and (2, 23).

$$m = \frac{23 - 20}{2 - 1} = 3$$

$y - 20 = 3(x - 1)$

$y = 3x - 3 + 20$

$y = 3x + 17$

Now solve for x for the point $(x, 245)$.

Therefore, $245 = 3x + 17$

$228 = 3x$

$76 = x$

Hence, the 76th term of the sequence has a value of 245.

11. Find the term indicated in each arithmetic sequence by using the general term formula.

a) 5, 3, 1, . . . t_{20}

Answer:

$d = 3 - 5 = -2$

$t_1 = 5$

$n = 20$

$t_n = t_1 + (n - 1)d$

$t_{20} = 5 + (20 - 1)(-2)$

$t_{20} = -33$

b) 8, 11, 14, . . . t_{21}

Answer:

$d = 11 - 8 = 3$

$t_1 = 8$

$n = 21$

$t_n = t_1 + (n - 1)d$

$t_{21} = 8 + (21 - 1)(3)$

$t_{21} = 68$

c) 6, 3, 0, −3, . . . t_{57}

Answer:

$d = 3 - 6 = -3$

$t_1 = 6$

$n = 57$

$t_n = t_1 + (n - 1)d$

$t_{57} = 6 + (57 - 1)(-3)$

$t_{57} = -162$

d) 18, 17, 16, 15, . . . t_{132}

Answer:

$d = 17 - 18 = -1$

$t_1 = 18$

$n = 132$

$t_n = t_1 + (n - 1)d$

$t_{132} = 18 + (132 - 1)(-1)$

$t_{132} = -113$

e) 7, 8.3, 9.6, 10.9, . . . t_{172}

Answer:

$d = 8.3 - 7 = 1.3$

$t_1 = 7$

$n = 172$

$t_n = t_1 + (n - 1)d$

$t_{172} = 7 + (172 - 1)(1.3)$

$t_{172} = 229.3$

12. Joe bought a painting for \$1800. After seven years, this painting sold for \$10,550. If the painting appreciated arithmetically, determine the annual amount of appreciation.

Answer:

Method 1:

The sequence is \$1800, \$1800 + d, \$1800 + 2d, . . ., \$1800 + 7d

After 7 years, the painting is worth \$10,550.

Therefore, \$10,550 = \$1800 + 7d

\$8750 = 7d

\$1250 = d

The painting appreciated \$1250 annually.

Method 2:

Use the general term formula with

$t_1 = 1800$

$n = 8$

$t_8 = 10\ 550$

$d = ?$

$10\ 550 = 1800 + (8 - 1)d$

$10\ 550 = 1800 + 7d$

The rest of the solution is the same. Note that in this method $n = 8$ because you want the amount **after** 7 years, not **in** 7 years.

13. For the arithmetic sequence 7, 11, 15, 19, . . ., find t_{29}.

Answer:

$d = 11 - 7 = 4$

$t_1 = 7$

$n = 29$

$t_n = t_1 + (n - 1)d$

$t_{29} = 7 + (29 - 1)4$

$t_{29} = 119$

14. Give an example of an arithmetic sequence, and then

    a) prove your sequence is an arithmetic sequence

    Answer:

    Answers may vary. However, all arithmetic sequences must increase or decrease by the same value (the common difference) between each term.

    b) write the defining linear function of your sequence

    Answer:

    Answers may vary. However, answers must be in the form, $y = mx + b$.

    c) write your arithmetic sequence using the general term formula

    Answer:

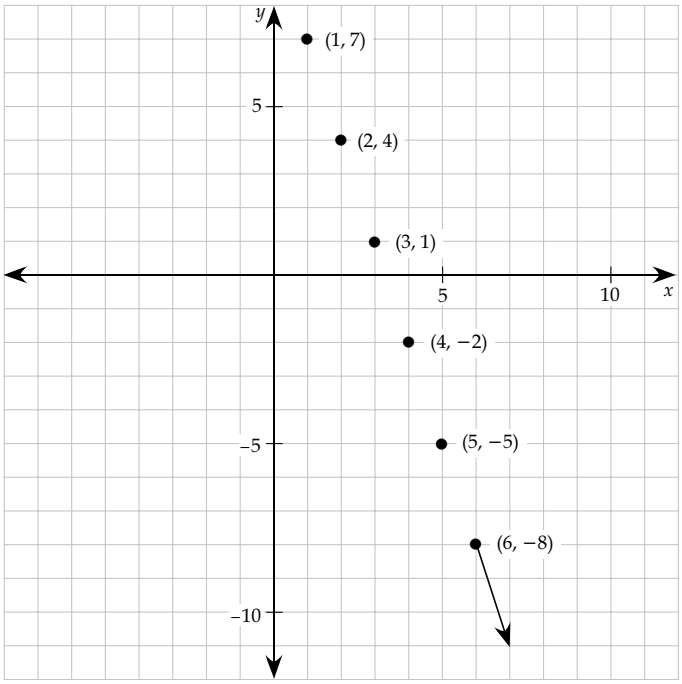
    Answers may vary. However, answers must be in the form, $t_n = t_1 + (n - 1)d$.

15. Graph the arithmetic sequence 7, 4, 1, −2, . . ., for the first six terms

    a) on graph paper (A reproducible copy of a sheet of graph paper is included in Appendix B found on the downloads page.)

    Answer:

    The points are (1, 7), (2, 4), (3, 1), (4, −2), (5, −5), (6, −8). The graph is shown below.



b) on the graphing calculator

Answer:

The common difference is -3. The first term is 7 so the first point is (1, 7). The recursion formula is:

$$\begin{cases} t_1 = 7 \\ t_n = t_{n-1} - 3,\ n \geq 2 \end{cases}$$

Set your calculator to sequence mode and graph the sequence. The screen should look something like the one below.

First, enter your information into Y=

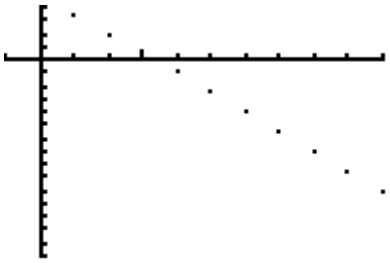
```
 Plot1 Plot2 Plot3
 nMin=1
u(n)=(u(n-1))-3

 u(nMin)=(7)
v(n)=
 v(nMin)=
w(n)=
```

Then, set your window.

```
WINDOW
 nMin=1
 nMax=10
 PlotStart=1
 PlotStep=1
 Xmin=-1
 Xmax=10
↓Xscl=1
Ymin=-30
Ymax=8
Yscl=2
```

Press GRAPH.



Learning Activity 1.2

There is no answer key for this learning activity.

Learning Activity 1.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Continue the pattern: 0.125, 0.250, 0.375, _____ , _____
2. Andrew bought an mp3 player for $156.88. He gave the cashier eight $20 bills. The store was out of toonies and all they had were loonies, quarters, dimes, nickels, and pennies. What was the smallest number of coins he could have received as change?
3. If $4x - 3x + 8x + 2x = 121$, then $6x - 2 =$
4. A recipe calls for two tablespoons of sugar for every quarter cup of flour. If Santana wants to use 18 tablespoons of sugar, how many cups of flour does she need to make the recipe?
5. Calculate: $\left(\frac{5}{3}\right) - \left(\frac{2}{14}\right)$
6. Simplify: $2^8 \div 2^6$
7. Determine the distance between the following two points: (2, 6) and (−7, −4).
8. List the factors of 164.

Answers:

1. 0.5, 0.625
2. 3 loonies, 1 dime, and 2 pennies for a total of 6 coins ($8 \times \$20 = \$160$; $\$160 - \$156.88 = \$3.12$; Remember, the store was out of toonies.)
3. 64 ($11x = 121$, so $x = 11$; $6(11) - 2 = 66 - 2 = 64$)
4. $2\frac{1}{4}$ $\left(\text{Santana increased the recipe by a factor of nine; } 9 \times \frac{1}{4} = \frac{9}{4}\right)$
5. $\frac{32}{21}$ $\left(\frac{5}{3} - \frac{1}{7} = \frac{35}{21} - \frac{3}{21}\right)$
6. 4 $\left(2^8 \div 2^6 = \frac{2^8}{2^6} = 2^2 = 4\right)$

7. $\sqrt{181}$

$\left(d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\right.$

$\left. d = \sqrt{(-9)^2 + (-10)^2} = \sqrt{81 + 100} = \sqrt{181}\right)$

8. 1, 2, 4, 41, 82, and 164

Part B: Arithmetic Series

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Find the indicated sum of each arithmetic series.

   a) $7 + 12 + 17 + \ldots, S_{20}$

   Answer:

   The common difference = 5

   Use the formula $t_n = t_1 + (n - 1)d$ to find t_n

   $t_{20} = 7 + (20 - 1)(5)$

   $t_{20} = 102$

   $S_{20} = \left(\frac{7 + 102}{2}\right)(20)$

   $S_{20} = 1090$

   b) $-5 - 7 - 9 - 11 - \ldots, S_{20}$

   Answer:

   The common difference = -2

   $t_{20} = -5 + (20 - 1)(-2)$

   $t_{20} = -43$

   $S_{20} = \left(\frac{-5 + (-43)}{2}\right)(20)$

   $S_{20} = -480$

c) $40 + 35 + 30 + \ldots, S_{100}$

Answer:

The common difference = -5

$t_{100} = 40 + (100 - 1)(-5)$

$t_{100} = -455$

$S_{100} = \left(\frac{40 + (-455)}{2}\right)(100)$

$S_{100} = -20\ 750$

2. Find the sum of the first:

a) 10 odd integers.

Answer:

The series is $1 + 3 + 5 + \ldots$ and $n = 10$

$t_n = t_1 + (n - 1)d$

$t_{10} = 1 + (10 - 1)(2)$

$t_{10} = 19$

The sum of 10 terms is

$S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$

$S_{10} = \left(\frac{t_1 + t_{10}}{2}\right)(10)$

$S_{10} = \left(\frac{1 + 19}{2}\right)(10)$

$S_{10} = 100$

b) 20 odd integers.

Answer:

The series is $1 + 3 + 5 + \ldots$ and $n = 20$

$t_n = t_1 + (n - 1)d$

$t_{20} = 1 + (20 - 1)(2)$

$t_{20} = 39$

The sum of 20 terms is

$$S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$$

$$S_{20} = \left(\frac{1 + 39}{2}\right)(20)$$

$$S_{20} = 400$$

c) 100 odd integers.

Answer:

The series is $1 + 3 + 5 + \ldots$ and $n = 100$

$t_n = t_1 + (n - 1)d$

$t_{100} = 1 + (100 - 1)(2)$

$t_{100} = 199$

The sum of 100 terms is

$$S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$$

$$S_{100} = \left(\frac{1 + 199}{2}\right)(100)$$

$$S_{100} = 10\ 000$$

d) n odd integers.

Answer:

The series is $1 + 3 + 5 + \ldots$ with n terms

The nth term is $t_n = t_1 + (n - 1)d$

Thus, $t_n = 1 + (n - 1)(2)$

$= 1 + 2n - 2$

$= 2n - 1$

The sum of n terms is

$$S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$$

$$S_n = \left(\frac{1 + 2n - 1}{2}\right)(n)$$

$$S_n = \left(\frac{2n}{2}\right)(n)$$

$$S_n = (n)(n)$$

$$S_n = n^2$$

Note that the formula you have developed might be worth memorizing. Can you show that the sum of the first n even natural numbers is $n(n + 1)$? Can you show that the sum of the first n natural numbers is $\frac{n(n+1)}{2}$?

3. Find the sum of all multiples of 3 between 99 and 1000.

Answer:

The first multiple of 3 greater than 99 is 102. The first term is not 99 because the multiples must lie between 99 and 1000, not including 99 nor 1000.

The greatest multiple of 3 smaller than 1000 is 999.

The series is $102 + 105 + 108 + \ldots + 999$.

To find the sum, the value of n must be found using the general term formula and $d = 3$.

$t_n = t_1 + (n - 1)d$

$999 = 102 + (n - 1)(3)$

$897 = (n - 1)(3)$

$299 = n - 1$

$300 = n$

The number of terms is 300.

The sum of the series is

$$S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$$

$$S_{300} = \left(\frac{102 + 999}{2}\right)(300)$$

$$S_{300} = 165\ 150$$

4. Find S_{100} of $1 + 4 + 7 + \ldots$.

Answer:

The series is $1 + 4 + 7 + \ldots$ $n = 100$, and $d = 3$.

To find the sum of 100 terms, the last term, t_{100}, needs to be calculated using the general term formula and $d = 3$.

$$t_n = t_1 + (n - 1)d$$

$$t_{100} = 1 + (100 - 1)(3)$$

$$t_{100} = 298$$

The sum of 100 terms is

$$S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$$

$$S_{100} = \left(\frac{1 + 298}{2}\right)(100)$$

$$S_{100} = 14\ 950$$

5. How many terms are in the series $3 + 8 + 13 + \ldots + 248$ if the sum is 6275?

Answer:

The series is $3 + 8 + 13 + \ldots + 248$.

To find the number of terms, use the formula

$$S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$$

where $S_n = 6275$ (given)

$t_1 = 3$

$t_n = 248$

Substitute and solve for n:

$$6275 = \left(\frac{3 + 248}{2}\right)(n)$$

$$6275 = 125.5n$$

$$50 = n$$

There are 50 terms in the series.

6. In an arithmetic sequence, $t_2 = -1$ and $t_{12} = 19$, find S_{13}.

Answer:

Method 1:

Since $t_2 = -1$ and $t_{12} = 19$, two points on the line determined by the defining linear function are (2, −1) and (12, 19).

By finding the function, t_1 and t_{13} can be found.

$$\text{Slope} = \frac{19 - (-1)}{12 - 2} = \frac{20}{10} = 2$$

Equation of line is:

$$y - y_1 = m(x - x_1)$$

$$y - (-1) = 2(x - 2)$$

$$y + 1 = 2x - 4$$

$$y = 2x - 5$$

The defining linear function is $f(x) = 2x - 5$ and

$t_1 = f(1) = 2(1) - 5 = 2 - 5 = -3$

$t_{13} = f(13) = 2(13) - 5 = 26 - 5 = 21$

The sum of the series is

$$S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$$

$$S_{13} = \left(\frac{-3 + 21}{2}\right)(13)$$

$$S_{13} = 117$$

Method 2:

Since the slope was calculated in Method 1 to be 2, then $d = 2$ for the series.

Since $t_2 = 1$

$t_1 = t_2 - d = -1 - 2 = -3$

and $t_{12} = 19$

$t_{13} = t_{12} + d = 19 + 2 = 21$

then the sum can be calculated as in Method 1.

7. An auditorium has 20 seats in the first row, and each row has two more seats than the row immediately in front of it. There are 26 rows in the auditorium. How many seats are there in the last row? What is the seating capacity of the auditorium?

Answer:

The arithmetic sequence is 20, 22, 24, . . . t_{26}. To calculate t_{26}, use the general term formula.

$t_n = t_1 + (n - 1)d$

$t_{26} = 20 + (26 - 1)(2)$

$t_{26} = 70$

There are 70 seats in the last row.

To find the seating capacity means to find how many seats there are altogether. The series is

$20 + 22 + 24 + \ldots + 70$

and $n = 26$

The sum is

$$S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$$

$$S_{26} = \left(\frac{20 + 70}{2}\right)(26)$$

$S_{26} = 1170$

The seating capacity is 1170 seats.

8. Mary's annual salary ranges from \$26,785 in the first year to \$34,825 in the seventh year.

   a) If the salary range is an arithmetic sequence with seven terms, determine the raise Mary received each year.

   Answer:

   The sequence is \$26,785, . . ., \$34,825, with $n = 7$.

   Use the general term formula to find d.

   $$t_n = t_1 + (n - 1)d$$
   $$34\,825 = 26\,785 + (7 - 1)(d)$$
   $$8040 = 6d$$
   $$1340 = d$$

   Mary received a raise of \$1340 each year.

   b) What was her salary in the fifth year?

   Answer:

   To find Mary's salary in the fifth year, we need to find t_5 using the general term formula and $n = 5$.

   $t_n = t_1 + (n - 1)d$

   $t_5 = 26\,785 + (5 - 1)(1340)$

   $t_5 = \$32{,}145$

   Mary's salary in the fifth year is \$32,145.

   c) What is the first annual salary in this range that is greater than \$30,000?

   Answer:

   From part (b), we know $t_5 = \$32{,}145$.

   Thus, $t_4 = 32\,145 - d$

   $= 32\,145 - 1340$

   $= \$30{,}805$

   and $t_3 = 30\,805 - 1340$

   $= \$29{,}465$

   In the fourth year, Mary's salary is over \$30,000.

d) What was the total amount Mary earned in the seven years?

Answer:

To find the total amount Mary earned in seven years, we find the sum of the series with 7 terms.

$\$26{,}785 + \ldots + \$34{,}825$

$$S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$$

$$S_7 = \left(\frac{26\,785 + 34\,825}{2}\right)(7)$$

$$S_7 = \$215{,}635$$

Mary earns \$215,635 in 7 years.

9. a) Show that the sum of n terms of an arithmetic series can be found using the formula $S_n = \frac{n}{2}\left[2t_1 + (n-1)d\right]$.

Answer:

Use the formula: $S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$

Rearrange to get: $S_n = \frac{n}{2}\left[t_1 + t_n\right]$

Use the general term formula to replace t_n and simplify.

$t_n = t_1 + (n-1)d$

$\therefore\ S_n = \frac{n}{2}\left[t_1 + \left(t_1 + (n-1)d\right)\right]$

$S_n = \frac{n}{2}\left[2t_1 + (n-1)d\right]$

b) Use this formula to find the sum of 10 terms of the series $4 + 9 + 14 + 19 + \ldots$

Answer:

$4 + 9 + 14 + 19 + \ldots$

$t_1 = 4$

$n = 10$

$d = 5$

$$S_n = \frac{n}{2}\left[2t_1 + (n-1)d\right]$$

$$S_n = \frac{10}{2}\left[2(4) + (10-1)(5)\right]$$

$$S_n = 5\left[8 + 9(5)\right]$$

$$S_{10} = 265$$

Note: You now can use either formula to find the sum of an arithmetic series.

$$S_n = \frac{n}{2}\left[2t_1 + (n-1)d\right]$$

$$S_n = \left(\frac{t_1 + t_n}{2}\right)(n)$$

Learning Activity 1.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. On the first day of your marathon training, you run 1 km. If you double that distance every day, on which day will you run over 20 km?
2. Complete the pattern: 1, 2, 4, 7, _____ , _____ , _____
3. You and two friends want to go to a movie but you are arguing over the seating arrangements. If you all sit beside each other in a row, how many different ways can you be seated?
4. State the common difference of the following arithmetic sequence: 7, 5, 3, 1, −1, . . .
5. Estimate the answer: 204 × 712
6. How many inches are in 6.5 feet?
7. Determine the volume of a rectangular prism with base dimensions of 12 cm by 10 cm and a height of 6 cm.
8. Write this decimal as a percent: 0.0091

Answers:

1. Day 6 (Day 1 = 1 km; Day 2 = 2 km; Day 3 = 4 km; Day 4 = 8 km; Day 5 = 16; km; Day 6 = 32 km)
2. 11, 16, 22 (add 1, add 2, add 3 is the pattern, so add 4, then 5, then 6 to get the next three terms)
3. 6 ways (3 choices for the frist seat, 2 choices for the second, and 1 choice for the last seat means 3 × 2 × 1 = 6 ways)
4. −2 (5 − 7)
5. 140 000 (a good estimate is 200 × 700 = 140 000)
6. 78 inches (There are 12 inches in 1 foot and 6.5 × 12 = 78 inches.)
7. 720 cm^3 (The formula for the volume of a rectangular prism is V = (area of the base)(height); V = (12 × 10) × 6 = 120 × 6 = 720 cm^3)
8. 0.91% (multiply by 100)

Part B: Geometric Sequences

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. For each sequence below, indicate whether it is geometric, arithmetic, or neither. If it is geometric, state the value of r, the common ratio; if it is arithmetic, state the value of d, the common difference.

   a) 4, 6, 8, 10, . . .

   Answer:

   $6 - 4 = 2$; $8 - 6 = 2$; $10 - 8 = 2$

   Therefore, the sequence is arithmetic with $d = 2$.

   b) 3, 6, 12, 24, . . .

   Answer:

   $\frac{6}{3} = 2$; $\frac{12}{6} = 2$; $\frac{24}{12} = 2$

   Therefore, the sequence is geometric with $r = 2$.

   c) 18, 15, 12, 9, . . .

   Answer:

   $15 - 18 = -3$; $12 - 15 = -3$; $9 - 12 = -3$

   Therefore, the sequence is arithmetic with $d = -3$.

   d) 80, 40, 20, 10, . . .

   Answer:

   $\frac{40}{80} = \frac{1}{2}$; $\frac{20}{40} = \frac{1}{2}$; $\frac{10}{20} = \frac{1}{2}$

   Therefore, the sequence is geometric with $r = \frac{1}{2}$.

   e) 1, −5, 25, −125, . . .

   Answer:

   $\frac{-5}{1} = -5$; $\frac{25}{-5} = -5$; $\frac{-125}{25} = -5$

   Therefore, the sequence is geometric with $r = -5$.

2. Write the first four terms of each sequence. If it is geometric, state the value of r, the common ratio; if it is arithmetic, state the value of d, the common difference.

a) $\begin{cases} t_1 = 2 \\ t_n = 2t_{n-1},\ n \geq 2 \end{cases}$

Answer:

n	t_n
1	$t_1 = 2$
2	$t_2 = 2t_{2-1} = 2t_1 = 2(2) = 4$
3	$t_3 = 2t_{3-1} = 2t_2 = 2(4) = 8$
4	$t_4 = 2t_{4-1} = 2t_3 = 2(8) = 16$

$\therefore$ geometric, with $r = 2$

b) $\begin{cases} t_1 = 2 \\ t_n = t_{n-1} + 2,\ n \geq 2 \end{cases}$

Answer:

n	t_n
1	$t_1 = 2$
2	$t_2 = t_{2-1} + 2 = t_1 + 2 = 2 + 2 = 4$
3	$t_3 = t_{3-1} + 2 = t_2 + 2 = 4 + 2 = 6$
4	$t_4 = t_{4-1} + 2 = t_3 + 2 = 6 + 2 = 8$

$\therefore$ arithmetic, with $d = 2$

c) $\begin{cases} t_1 = 2 \\ t_n = 2t_{n-1} + 2,\ n \geq 2 \end{cases}$

Answer:

n	t_n
1	$t_1 = 2$
2	$t_2 = 2t_{2-1} + 2 = 2t_1 + 2 = 2(2) + 2 = 4 + 2 = 6$
3	$t_3 = 2t_{3-1} + 2 = 2t_2 + 2 = 2(6) + 2 = 12 + 2 = 14$
4	$t_4 = 2t_{4-1} + 2 = 2t_3 + 2 = 2(14) + 2 = 28 + 2 = 30$

$\therefore$ neither

d) $\begin{cases} t_1 = 2 \\ t_n = (t_{n-1})^2,\ n \geq 2 \end{cases}$

Answer:

n	t_n
1	$t_1 = 2$
2	$t_2 = (t_{2-1})^2 = (t_1)^2 = 2^2 = 4$
3	$t_3 = (t_{3-1})^2 = (t_2)^2 = 4^2 = 16$
4	$t_4 = (t_{4-1})^2 = (t_3)^2 = 16^2 = 256$

$\therefore$ neither

e) $\begin{cases} t_1 = 5 \\ t_n = (-1)^n t_{n-1},\ n \geq 2 \end{cases}$

Answer:

n	t_n
1	$t_1 = 5$
2	$t_2 = (-1)^2 t_{2-1} = (-1)^2 t_1 = (1)(5) = 5$
3	$t_3 = (-1)^3 t_{3-1} = (-1)^3 t_2 = (-1)(5) = -5$
4	$t_4 = (-1)^4 t_{4-1} = (-1)^4 t_3 = (1)(-5) = -5$

$\therefore$ neither

f) $\begin{cases} t_1 = 4 \\ t_n = t_{n-1} - 5,\ n \geq 2 \end{cases}$

Answer:

n	t_n
1	$t_1 = 4$
2	$t_2 = t_{2-1} - 5 = t_1 - 5 = 4 - 5 = -1$
3	$t_3 = t_{3-1} - 5 = t_2 - 5 = -1 - 5 = -6$
4	$t_4 = t_{4-1} - 5 = t_3 - 5 = -6 - 5 = -11$

$\therefore$ arithmetic, with $d = -5$

3. Write the first three terms of the geometric sequence generated by each of the following exponential functions. Notice none of them are written in the usual form, $t_n = t_1 r^{n-1}$, but all of them generate a geometric sequence.

   a) $f(x) = 2^x$

   Answer:

n	$t_n = 2^x$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$

   b) $t_n = 4^{n-1}$

   Answer:

n	$t_n = 4^{n-1}$
1	$4^0 = 1$
2	$4^1 = 4$
3	$4^2 = 16$

   c) $f(x) = 2(3^x)$

   Answer:

x	$f(x)$
1	$2(3^1) = 6$
2	$2(3^2) = 18$
3	$2(3^3) = 54$

d) $t_n = 16\left(\frac{1}{2}\right)^n$

Answer:

n	$f(x)$
1	$16\left(\frac{1}{2}\right) = 8$
2	$16\left(\frac{1}{2}\right)^2 = 16\left(\frac{1}{4}\right) = 4$
3	$16\left(\frac{1}{2}\right)^3 = 16\left(\frac{1}{8}\right) = 2$

4. Write a recursive formula and a defining exponential function to generate each of the following geometric sequences.

a) 4, 8, 16, 32, 64, . . .

Answer:

Recursive formula: $\begin{cases} t_1 = 4 \\ t_n = 2t_{n-1},\ n \geq 2 \end{cases}$

Exponential function: $t_n = 4(2)^{n-1}$

b) 6, 18, 54, 162, 486, . . .

Answer:

Recursive formula: $\begin{cases} t_1 = 6 \\ t_n = 3t_{n-1},\ n \geq 2 \end{cases}$

Exponential function: $t_n = 6(3)^{n-1}$

c) 1, −2, 4, −8, 16, . . .

Answer:

Recursive formula: $\begin{cases} t_1 = 1 \\ t_n = -2t_{n-1},\ n \geq 2 \end{cases}$

Exponential function: $t_n = (-2)^{n-1}$

d) 40, 20, 10, 5, $\frac{5}{2}$, . . .

Answer:

Recursive formula: $\begin{cases} t_1 = 40 \\ t_n = \left(\frac{1}{2}\right)\left(t_{n-1}\right),\ n \geq 2 \end{cases}$

Exponential function: $t_n = 40\left(\frac{1}{2}\right)^{n-1}$

5. Write an expression for t_{20} of the sequence 3, 6, 12, 24, ... Do not simplify.

Answer:

The defining exponential function is $t_n = 3(2)^{n-1}$. Therefore, $t_{20} = 3(2)^{20-1} = 3(2)^{19}$.

6. Which term of the geometric sequence 4, 2, 1, . . . has the value $\frac{1}{1024}$?

Answer:

$r = \frac{2}{4} = \frac{1}{2}$

The defining exponential function is $t_n = 4\left(\frac{1}{2}\right)^{n-1}$.

Hence, $\frac{1}{1024} = 4\left(\frac{1}{2}\right)^{n-1}$

$\frac{1}{2^{10}} = 2^2\left(\frac{1}{2}\right)^{n-1}$

$\frac{1}{2^{10}2^2} = \left(\frac{1}{2}\right)^{n-1}$

$\left(\frac{1}{2}\right)^{12} = \left(\frac{1}{2}\right)^{n-1}$

$n - 1 = 12$

$n = 13$

Therefore, $\frac{1}{1024}$ is the 13th term.

7. In a geometric sequence, $t_2 = 6$ and $t_4 = 54$, find the value of t_1.

   Answer:

   Substitute the coordinates of the points (2, 6) and (4, 54) into the exponential function $t_n = t_1 r^{n-1}$ and solve the two equations for the two unknowns.

   The point (2, 6) yields $6 = t_1 r$ and (4, 54) yields $54 = t_1 r^3$. Using division method:

$$\frac{t_1 r^3}{t_1 r} = \frac{54}{6}$$

$$r^2 = 9$$

$$r = \pm 3$$

   Two possible sequences exist, one with $r = 3$ and the other with $r = -3$.

   Case 1: If $r = 3$, then $6 = t_1(3)$ or $t_1 = 2$.

   Case 2: If $r = -3$, then $6 = t_1(-3)$ or $t_1 = -2$.

8. What is the value of x if the sequence, $x + 2, 3x - 2, 5x + 2$ is geometric?

   Answer:

   If the sequence is geometric, the ratios, $\frac{t_2}{t_1}$ and $\frac{t_3}{t_2}$, must be equal.

$$\frac{3x - 2}{x + 2} = \frac{5x + 2}{3x - 2}$$

$$(3x - 2)(3x - 2) = (x + 2)(5x + 2) \qquad \text{(cross-multiply)}$$

$$9x^2 - 12x + 4 = 5x^2 + 12x + 4$$

$$4x^2 - 24x = 0$$

$$4x(x - 6) = 0$$

$$x = 6 \text{ or } x = 0$$

   If you are interested in seeing the sequences, substitute the values of x into the terms.

   When $x = 6$, the sequence is 8, 16, 32.

   When $x = 0$, the sequence is 2, −2, 2.

   Both are geometric sequences with common ratio 2 and −1, respectively.

9. Insert three numbers a, b, c between 5 and 80, so that 5, a, b, c, 80 forms a geometric sequence.

   Answer:

   If r is the common ratio then the terms of the geometric sequence are 5, $5r$, $5r^2$, $5r^3$, $5r^4$.

   Therefore, $5r^4 = 80$.

   $r^4 = 16$

   $r = \pm 2$

   Case 1: If $r = 2$, then $a = 10$, $b = 20$, $c = 40$.

   Case 2: If $r = -2$, then $a = -10$, $b = 20$, $c = -40$.

10. A store is conducting a Dutch auction. It will take 10% off the cost of an item at the end of each day the item doesn't sell. If an item originally costs \$150, find its cost for each of the next five days.

    Answer:

    If the cost of the item is being reduced by 10%, then the sale price is 100% − 10% or 90% of the original price. Thus, the cost of the item at the end of the next five days is calculated as follows.

    first day: (\$150.00)(0.90) = \$135.00

    second day: (\$135.00)(0.90) = \$121.50

    third day: (\$121.50)(0.90) = \$109.35

    fourth day: (\$109.35)(0.90) = \$98.42

    fifth day: (\$98.42)(0.90) = \$88.58

    Note that if you use the formula $t_6 = 150(0.9)^{6-1}$ to find your answer for the end of the fifth day, you get \$88.57. Since the deductions are made daily, the correct answer is one cent different and is \$88.58.

11. A geometric sequence is defined by the exponential function $t_n = 5(2)^{n-1}$

    a) What is the first term?

       Answer:

       $t_1 = 5$, the coefficient

    b) What is the common ratio?

       Answer:

       $r = 2$, the base

c) What is the value of the 10th term?

Answer:

$t_{10} = 5(2)^{10-1} = 5(2)^9 = 2560$

d) What is the value of t_7?

Answer:

$t_7 = 5(2)^{7-1} = 5(2)^6 = 320$

e) What position in the sequence has the value 163 840?

Answer:

$$t_n = 5(2)^{n-1}$$

$$163\,840 = 5(2)^{n-1}$$

$$\frac{163\,840}{5} = \frac{5(2)^{n-1}}{5} \qquad \text{Divide by 5}$$

$$32\,768 = 2^{n-1} \qquad \text{Simplify}$$

$$2^{15} = 2^{n-1} \qquad \text{Use guess and check}$$

$$15 = n - 1 \qquad \text{Equate exponents}$$

$$16 = n \qquad \text{Solve}$$

f) Is the number 4100 a member of this geometric sequence? Justify your answer.

Answer:

$$t_n = 5(2)^{n-1}$$

$$4100 = 5(2)^{n-1}$$

$$\frac{4100}{5} = \frac{5(2)^{n-1}}{5}$$

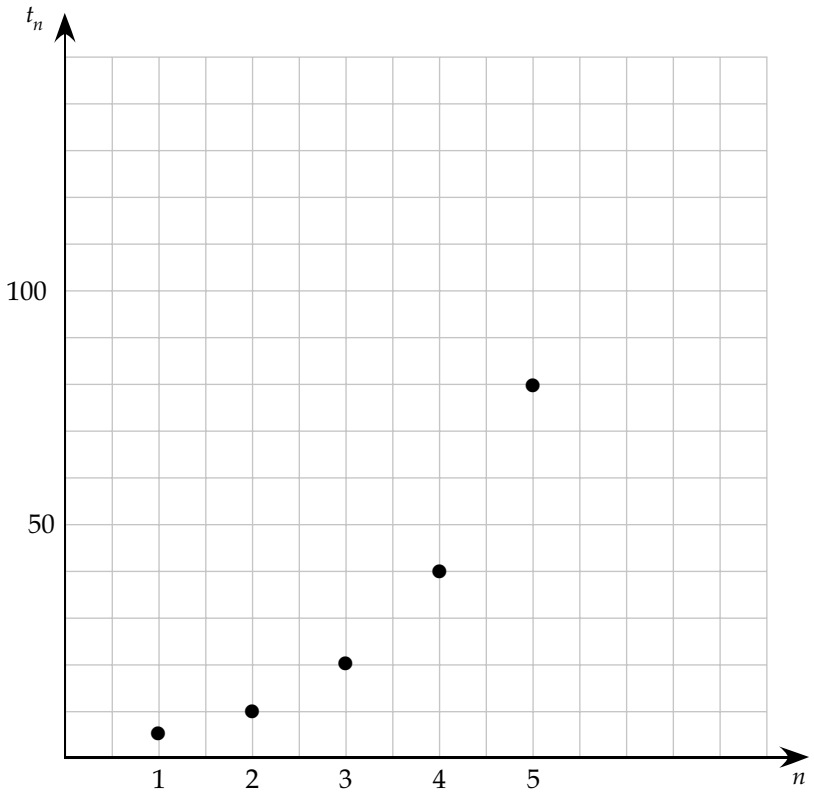
$$820 = 2^{n-1}$$

Powers of 2 are 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, . . ., and 820 is not one of them. Thus, 4100 is not a term of the sequence.

g) Graph the sequence using graph paper or your graphing calculator.

Answer:

The graph of the function is shown below.



The shape is a curve and not a straight line. The points are not joined because the domain is the set of natural numbers.

Learning Activity 1.5

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. For Josh's workout, he runs $\frac{1}{9}$ of the football field, sprints $\frac{7}{9}$ of the field, and then jogs the rest of the way. What fraction of the field does Josh jog?
2. What is the sum of the digits from 1 to 10?
3. Calculate the perimeter of a square garden with an area of 25 m^2.
4. Express $\frac{6}{20}$ as a decimal.
5. Find the common difference in the following pattern: −2, −9, −16, −23, . . .
6. Estimate: $\sqrt{96}$
7. Simplify: $(2^4)(3^2)$
8. Find the slope of the line defined by the equation, $3y - 2x + 7 = 0$.

Answers:

1. $\frac{1}{9}\left(1-\left(\frac{1}{9}+\frac{7}{9}\right)=\frac{9}{9}-\frac{8}{9}=\frac{1}{9}\right)$
2. 55 (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10; combining digits that sum to 10 (1 and 9, 2 and 8, etc.) gives 10 + 10 + 10 + 10 + 10 + 5 = 55)
3. 20 m (Length of the side = $\sqrt{25} = 5$; Perimeter = 5 × 4)
4. $0.3\left(\frac{6}{20}=\frac{3}{10}=0.3\right)$
5. −7
6. 9.8 (96 is between 81 and 100; therefore, $\sqrt{96}$ is between 9 and 10 and is closer to 10)
7. 144 ($2^4 \times 3^2 = 16 \times 9$)
8. $\frac{2}{3}\left(3y = 2x - 7;\ y = \frac{2}{3}x - \frac{7}{3}\right)$

Part B: Sums of Geometric Series

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. a) Give an example of a geometric sequence.

   Answer:

   Answers will vary. Be sure every term is a multiple of the term before.

   b) Write the corresponding geometric series.

   Answer:

   Answers will vary. Answers must be an addition of the terms in the corresponding geometric sequence.

Check with your learning partner or review examples in the lesson to check your work.

2. Write in expanded form and find the sum.

   a) $\sum_{k=1}^{4} k^3$

   b) $\sum_{k=1}^{4} 3^k$

   c) $\sum_{k=1}^{3} 3k$

   d) $\sum_{k=1}^{4} 3^{k-1}$

   e) $\sum_{k=2}^{4} 3^{k+1}$

   f) $\sum_{k=2}^{4} 5 \cdot 3^{k-1}$

   g) $\sum_{k=2}^{5} (2k-5)$

   h) $\sum_{k=3}^{6} 2(-3)^{k-1}$

   Answers:

   a) $\sum_{k=1}^{4} k^3 = (1)^3 + (2)^3 + (3)^3 + (4)^3 = 1 + 8 + 27 + 64 = 100$

   b) $\sum_{k=1}^{4} 3^k = 3^1 + 3^2 + 3^3 + 3^4 = 3 + 9 + 27 + 81 = 120$

   c) $\sum_{k=1}^{3} 3k = 3(1) + 3(2) + 3(3) = 3 + 6 + 9 = 18$

d) $\sum_{k=1}^{4} 3^{k-1} = 3^{1-1} + 3^{2-1} + 3^{3-1} + 3^{4-1} = 3^0 + 3^1 + 3^2 + 3^3$

$= 1 + 3 + 9 + 27 = 40$

e) $\sum_{k=2}^{4} 3^{k+1} = 3^{2+1} + 3^{3+1} + 3^{4+1} = 3^3 + 3^4 + 3^5 = 27 + 81 + 243 = 351$

f) $\sum_{k=2}^{4} 5 \cdot 3^{k-1} = 5 \cdot 3^{2-1} + 5 \cdot 3^{3-1} + 5 \cdot 3^{4-1} = 5 \cdot 3^1 + 5 \cdot 3^2 + 5 \cdot 3^3$

$= 15 + 45 + 135 = 195$

g) $\sum_{k=2}^{5} (2k - 5) = (2(2) - 5) + (2(3) - 5) + (2(4) - 5) + (2(5) - 5)$

$= -1 + 1 + 3 + 5 = 8$

h) $\sum_{k=3}^{6} 2(-3)^{k-1} = 2(-3)^{3-1} + 2(-3)^{4-1} + 2(-3)^{5-1} + 2(-3)^{6-1}$

$= 2(-3)^2 + 2(-3)^3 + 2(-3)^4 + 2(-3)^5$

$= 2(9) + 2(-27) + 2(81) + 2(-243)$

$= 18 - 54 + 162 - 486 = -360$

3. Use sigma notation to write each geometric sum.

a) $4 + 12 + 36 + 108$

Answer:

$r = 3$

$t_1 = 4$

$n = 4$

$\sum_{k=1}^{4} 4(3)^{k-1}$

b) $24 + 4 + \frac{2}{3} + \frac{1}{9} + \frac{1}{54}$

Answer:

$r = \frac{4}{24} = \frac{1}{6}$

$t_1 = 24$

$n = 5$

$\sum_{k=1}^{5} 24\left(\frac{1}{6}\right)^{k-1}$

c) $2 + 4 + 8 + \ldots + 1024$

Answer:

$r = 2$

$t_1 = 2$

$n = 10$, since $2^{10} = 1024$

$\sum_{k=1}^{10} 2(2)^{k-1}$ or $\sum_{k=1}^{10} 2^k$

4. Consider the sequence 3, 6, 12, 24, . . .

a) Find the eighth term.

Answer:

$t_n = 3(2)^{n-1}$

Therefore, $t_8 = 3(2)^{8-1} = 384$.

b) Find the sum of the first eight terms, using the sum formula.

Answer:

Using $S_n = \frac{t_1\left(1 - r^n\right)}{1 - r}$, $S_8 = \frac{3\left(1 - (2)^8\right)}{1 - 2} = \frac{3(-255)}{-1} = 765$.

5. Find the sum of the first 10 terms of each series. If the numerical answer is not an integer, state it correct to two decimal places.

a) $120 + 60 + 30 + 15 + \ldots$

Answer:

$$\text{Using } S_n = \frac{t_1\left(1 - r^n\right)}{1 - r},\ S_{10} = \frac{120\left(1 - (0.5)^{10}\right)}{1 - 0.5} = 239.77.$$

b) $8 + 12 + 18 + 27 + \ldots$

Answer:

$$\text{Using } S_n = \frac{t_1\left(1 - r^n\right)}{1 - r} \text{ and } r = \frac{12}{8} = \frac{3}{2} = 1.5,\ S_{10} = \frac{8\left(1 - (1.5)^{10}\right)}{1 - 1.5} = 906.64.$$

c) $128 - 64 + 32 - 16 + \ldots$

Answer:

$$\text{Using } S_n = \frac{t_1\left(1 - r^n\right)}{1 - r},\ S_{10} = \frac{128\left(1 - (-0.5)^{10}\right)}{1 - (-0.5)} = 85.25.$$

d) $24 + 8 + \frac{8}{3} + \frac{8}{9} + \ldots$

Answer:

$$\text{Using } S_n = \frac{t_1\left(1 - r^n\right)}{1 - r},\ S_{10} = \frac{24\left(1 - \left(\frac{1}{3}\right)^{10}\right)}{1 - \frac{1}{3}} = 36.00.$$

e) $x^2 + x^4 + x^6 + x^8 + \ldots$

Answer:

$$\text{Using } S_n = \frac{t_1\left(1 - r^n\right)}{1 - r},\ S_{10} = \frac{x^2\left(1 - \left(x^2\right)^{10}\right)}{1 - x^2} = \frac{x^2\left(1 - x^{20}\right)}{1 - x^2}.$$

f) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

Answer:

Using $S_n = \frac{t_1\left(1 - r^n\right)}{1 - r}$, $S_{10} = \frac{1\left(1 - (0.5)^{10}\right)}{1 - 0.5} = 2.00.$

6. In a geometric sequence, $t_4 = \frac{1}{4}$, and $t_{11} = 32$. Calculate S_{11}. Express your answer as an exact value.

Answer:

Use the general term formula to write two equations. Then solve using division method.

$t_4 = \frac{1}{4} = t_1 r^3$ equation (a)

$t_{11} = 32 = t_1 r^{10}$ equation (b)

$$\frac{32}{\frac{1}{4}} = \frac{t_1 r^{10}}{t_1 r^3}$$

$$128 = r^7$$

$$2 = r$$

Now substitute $r = 2$ into equation (a) and solve for t_1.

$$\frac{1}{4} = t_1 2^3$$

$$\frac{1}{4} = t_1 (8)$$

$$\frac{\frac{1}{4}}{8} = \frac{t_1 (8)}{8}$$

$$\frac{1}{32} = t_1$$

Now find S_{11}.

$$S_n = \frac{t_1\left(1 - r^n\right)}{1 - r}$$

$$S_{11} = \frac{\left(\frac{1}{32}\right)\left(1 - 2^{11}\right)}{1 - 2}$$

$$S_{11} = \frac{2047}{32}$$

Note that you can use the $a\frac{b}{c}$ button on your scientific calculator or Math Frac on your graphing calculator to find the exact answers when using fractions.

7. The sum of a geometric series with $t_1 = 7$ is 889. If $r = 2$, how many terms are in this geometric series?

Answer:

You have the following information:

$S_n = 889$

$t_1 = 7$

$r = 2$

Using the formula $S_n = \frac{t_1\left(1 - r^n\right)}{1 - r}$, it is possible to solve for n, the number of terms in this series.

$$889 = \frac{7\left(1 - 2^n\right)}{1 - 2}$$

$$889 = -7\left(1 - 2^n\right)$$

$$\frac{889}{-7} = \frac{-7\left(1 - 2^n\right)}{-7}$$

$$-127 = 1 - 2^n$$

$$-127 - 1 = -2^n$$

$$-128 = -2^n$$

$$128 = 2^n$$

$$2^7 = 2^n$$

$$n = 7$$

Therefore, there are 7 terms in this series.

8. The sum of a geometric series is 98 301. If there are 15 terms in this sequence and the common ratio is 2, what is the value for t_1?

   Answer:

   You have the following information:

   $S_n = 98\ 301$

   $r = 2$

   $n = 15$

   Using the formula $S_n = \frac{t_1\left(1 - r^n\right)}{1 - r}$, it is possible to solve for t_1.

$$98\ 301 = \frac{t_1\left(1 - 2^{15}\right)}{1 - 2}$$

$$98\ 301 = t_1\left(32\ 767\right)$$

$$\frac{98\ 301}{32\ 767} = \frac{t_1\left(32\ 767\right)}{32\ 767}$$

$$3 = t_1$$

Learning Activity 1.6

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Find the common ratio in the following pattern: 189, 63, 21, 7, . . .
2. Which of these values would come next in this pattern:

   One, four, three, eleven, fifteen, thirteen, . . .

   Fourteen

   Five

   Seven

   Seventeen
3. If your favourite television show starts at 7 pm and your mom tells you that your show is still 48 minutes away, what time is it now?
4. Your school ordered 56 boxes of cookies that each contain 25 cookies. If there are 700 students at your school, will you have enough cookies to give each student two cookies?
5. Calculate: $\frac{3}{16} + \frac{2}{4}$
6. Solve the inequality $5x - 2 < 0$.
7. Simplify: $(2x^2 + 3x - 5) - (9x^2 - 4x + 6)$
8. Simplify the following fraction: $\frac{144}{180}$

Answers:

1. $\frac{1}{3}$
2. Seventeen (Each number has one more letter in the spelling of the word than the previous number.)
3. 6:12 pm.
4. You will have exactly enough cookies. ($56 \times 25 = 50 \times 25 + 6 \times 25 = 1250 + 150 = 1400 = 700 \times 2$)
5. $\frac{11}{16}$

6. $5x < 2$

$$x < \frac{2}{5}$$

7. $-7x^2 + 7x - 11$
$[(2x^2 + 3x - 5) - (9x^2 - 4x + 6) = 2x^2 + 3x - 5 - 9x^2 + 4x - 6]$

8. $\frac{4}{5}\left(\frac{144}{180} = \frac{72}{90} = \frac{36}{45}\right)$

Part B: Sums of Infinite Geometric Series

If you haven't already, you may want to go back and include all important definitions, terms, or examples that you have found more challenging on your resource sheet.

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Find the sum of each of the following infinite geometric series. Express your answers as exact values.

   a) $8 + \frac{8}{3} + \frac{8}{9} + \frac{8}{27} + \ldots$

   Answer :

   $$r = \frac{1}{3},\ S = \frac{t_1}{1-r} = \frac{8}{1-\frac{1}{3}} = 12$$

   b) $6 - 3 + \frac{3}{2} - \frac{3}{4} + \ldots$

   Answer :

   $$r = -\frac{1}{2},\ S = \frac{t_1}{1-r} = \frac{6}{1-\left(-\frac{1}{2}\right)} = 4$$

c) $6 + 4 + \frac{8}{3} + \frac{16}{9} + \ldots$

Answer :

$$r = \frac{2}{3}, \quad S = \frac{t_1}{1-r} = \frac{6}{1-\frac{2}{3}} = 18$$

d) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$

Answer:

$$r = -\frac{1}{2}, \quad S = \frac{t_1}{1-r} = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3}$$

e) $\sum_{k=1}^{\infty}\left(\frac{16}{2^k}\right)$

Answer:

$$\sum_{k=1}^{\infty}\left(\frac{16}{2^k}\right) = \frac{16}{2^1} + \frac{16}{2^2} + \frac{16}{2^3} + \ldots$$

$$= 8 + 4 + 2 + \ldots$$

$$r = \frac{1}{2}, \quad S = \frac{t_1}{1-r} = \frac{8}{1-\frac{1}{2}} = 16$$

f) $\sum_{k=2}^{\infty}\left(\frac{18}{3^k}\right)$

Answer:

$$\sum_{k=2}^{\infty}\left(\frac{18}{3^k}\right) = \frac{18}{3^2} + \frac{18}{3^3} + \frac{18}{3^4} + \ldots$$

$$= \frac{18}{9} + \frac{18}{27} + \frac{18}{81} + \ldots$$

$$= 2 + \frac{2}{3} + \frac{2}{9} + \ldots$$

$$r = \frac{1}{3}, \quad S = \frac{t_1}{1-r} = \frac{2}{1-\left(\frac{1}{3}\right)} = 3$$

2. Consider a geometric sequence where $t_3 = 16$ and $t_5 = 10$.

   a) Find the common ratio r, rounded to two decimal places.

   Answer:

   Use the formula for the general term to find two equations. Then use division method to solve.

$$t_3 = 16 = t_1 r^2$$

$$t_5 = 10 = t_1 r^4$$

$$\frac{t_1 r^4}{t_1 r^2} = \frac{10}{16}$$

$$r^2 = 0.625$$

$$r = \pm 0.79$$

   b) Find $\sum_{k=1}^{\infty} t_k$. State your answer rounded to two decimal places.

   Answer:

   Substitute the values for r into either equation to find t_1.

$$16 = t_1 r^2$$

$$16 = t_1 (\pm 0.79)^2$$

$$16 = t_1 (0.625)$$

$$t_1 = \frac{16}{0.625}$$

$$t_1 = 25.6$$

   Note that both values for r meet the restriction: $|r| < 1$.

Case 1:	Case 2:
$r = 0.79$	$r = -0.79$
$S_\infty = \frac{t_1}{1 - r}$	$S_\infty = \frac{t_1}{1 - r}$
$S_\infty = \frac{25.6}{1 - 0.79}$	$S_\infty = \frac{25.6}{1 - (-0.79)}$
$S_\infty = 121.90$ or 122.24	$S_\infty = 14.30$

   Note the answer for S_∞ can be obtained using the calculator answer for r before rounding. If you used $r = 0.79$, $S_\infty = 121.90$, for $r = -0.79$, $S_\infty = 14.30$. Both 121.90 and 122.24 would be considered correct.

3. The sum to infinity of an infinite geometric series is 12. If the common ratio is 0.25, what is the first term of this series?

Answer:

$$S = \frac{t_1}{1 - r}$$

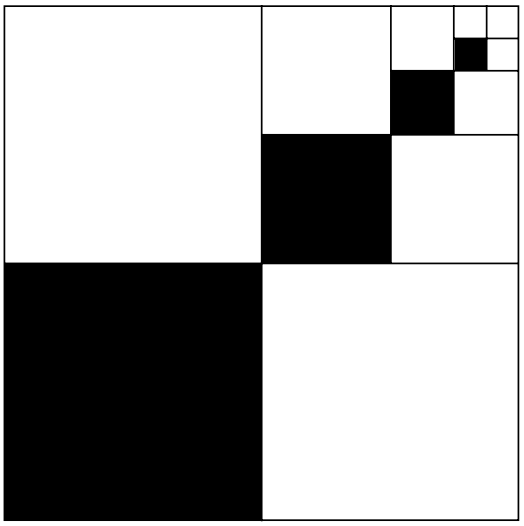
Therefore, $12 = \frac{t_1}{1 - 0.25}$

$$12 = \frac{t_1}{0.75}$$

$$t_1 = 12(0.75)$$

$$t_1 = 9$$

4. A 4-by-4 square is divided into four congruent squares, as shown in the diagram. The bottom left square is shaded. The top right square is subdivided into four congruent squares again, and the bottom left square is shaded. This process is repeated indefinitely. What is the total shaded area?



Answer:

$2^2,\ 1^2,\ \left(\frac{1}{2}\right)^2, \ldots$ or $4, 1, \frac{1}{4}, \ldots$

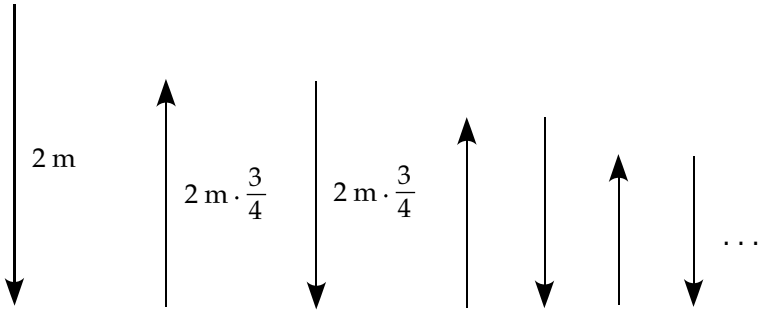
$$S = \frac{t_1}{1 - r} = \frac{4}{1 - \frac{1}{4}} = \frac{4}{\frac{3}{4}} = 4 \times \frac{4}{3} = \frac{16}{3}$$

5. A ball is dropped from a height of two metres. Each time it strikes the ground, it rebounds three-quarters of the distance it has fallen.

   a) How far does it rise after it strikes the ground the third time?

   Answer:

   Draw a diagram to get a sense of what is happening:



   First bounce $= 2\left(\frac{3}{4}\right) = \frac{3}{2}$; second bounce $= \frac{3}{2}\left(\frac{3}{4}\right) = \frac{9}{8}$;

   third bounce $= \frac{9}{8}\left(\frac{3}{4}\right) = \frac{27}{32}$.

   b) How far does it travel before it comes to rest?

   Answer:

   From the diagram, the total downward distance series is

$$2 + \frac{3}{2} + \frac{9}{8} + \ldots = \frac{t_1}{1-r} = \frac{2}{1-\frac{3}{4}} = 8.$$

   The total upward distance = downward distance − original drop = $8 - 2 = 6$.

   The distance travelled is $8 + 6 = 14$ metres.

6. What is the value of x if the geometric series $1 + 3x + 9x^2 + 27x^3 + \ldots$ converges to 4?

   Answer:

   $t_1 = 1$ and $r = 3x$

   Furthermore, the sum is 4, so solving for x you get

$$S = \frac{t_1}{1 - r}$$

$$4 = \frac{1}{1 - 3x}$$

$$4(1 - 3x) = \frac{1}{1 - 3x}(1 - 3x)$$

$$4 - 12x = 1$$

$$3 = 12x$$

$$x = \frac{3}{12} = \frac{1}{4}$$

7. Recall Zeno's paradox from the introduction to this lesson. Zeno's paradox stated that there is no sum to the infinite geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$. Prove that Zeno is incorrect.

   Answer:

   S_∞ or $S = \frac{t_1}{1 - r}$, provided $|r| < 1$

   In this example, $r = \frac{1}{2} < 1$.

   Therefore, you can use the sum formula.

$$S = \frac{t_1}{1 - r} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

   Therefore, there is a sum to this geometric series, and it is possible to catch the bus.

Notes

Grade 11 Pre-Calculus Mathematics (30S)

Module 2
Factoring and Rational Expressions

Module 2:
Factoring and Rational Expressions

Introduction

In Module 1, you discovered patterns in the form of arithmetic and geometric sequences and series. In this module, you will build on your knowledge of patterns by looking at patterns found when factoring polynomials. You have learned the basics of factoring in previous courses. However, in this module, you will extend your knowledge of factoring one step further to see how factoring relates to rational expressions. Rational expressions are fractions containing polynomials in the numerator and denominator.

Assignments in Module 2

When you have completed the assignments for Module 2, submit your completed assignments to the Distance Learning Unit either by mail or electronically through the learning management system (LMS). The staff will forward your work to your tutor/marker.

Lesson	Assignment Number	Assignment Title
2	Assignment 2.1	Factoring
4	Assignment 2.2	Multiplying, Dividing, and Simplifying Rational Expressions
5	Assignment 2.3	Adding and Subtracting Rational Expressions

Resource Sheet

When you write your midterm examination, you are encouraged to take a Midterm Examination Resource Sheet with you into the examination. This sheet will be one letter-sized page, 8½ " by 11", with both sides in your handwriting or typewritten. You will submit it with your examination, but you do not receive any marks for it.

Many students have found that preparing a resource sheet is an excellent way to review. It provides you with a summary of the important facts of each module. You should complete a resource sheet for each module to help with your studying and reviewing. Lesson summaries and module summaries are included for you to use as a guide.

You may use the list of instructions provided below to help you with preparing your resource sheet for the material in Module 2. On this sheet, you should record math terms and definitions, formulas, sample questions, or a list of places where you often make mistakes. You should also identify special areas that require extra attention or review by writing the page numbers.

After you have completed each module's resource sheet, you may summarize the sheets from Modules 1, 2, 3, and 4 to prepare your Midterm Examination Resource Sheet. The midterm examination for this course is based on Modules 1 to 4.

Resource Sheet for Module 2

As you go through the lessons of this module, you may want to consider the following suggestions regarding the creation of a resource sheet.

1. List all the important math terms, and define them if necessary.
2. List all the formulas and perhaps a sample problem that shows how the formula is used.
3. If necessary, write the solutions to some problems, showing in detail how you did the calculations.
4. Copy any questions that represent the key points of the lesson, and perhaps include the solutions as well.
5. Identify the problems you found most difficult, and copy the page numbers onto the resource sheet so that you can review them before writing the examination. You may also copy the problems and the solutions onto your resource sheet, and later write them onto your Midterm Examination Resource Sheet.
6. Write any comments, ideas, shortcuts, or other reminders that may be helpful during an examination.

Lesson 1: Factoring Review

Lesson Focus

In this lesson, you will

- ❑ review how to find the greatest common factor of a polynomial
- ❑ review how to factor a trinomial
- ❑ review how to factor a difference of squares

Lesson Introduction

In previous courses, you have learned how to factor different forms of polynomial expressions. Factoring is one topic that may appear to have no immediate applications. However, when it comes to dealing with rational expressions, which you will be learning about in later lessons, factoring is extremely useful. Factoring can reduce a complex equation to a more simple equation. This can make seemingly complex equations easier to work with. Factoring can even help manipulate formulas, which allows you to calculate the different variables included in the formula more easily.

Factoring

Factoring is the inverse of multiplication. Therefore, when you factor a polynomial, you are essentially undoing the multiplication. Because going backwards is usually more difficult than going forwards, some people find factoring to be more complicated than multiplication. If you need additional help with the material in this lesson, your tutor/marker is always available to help.

The following definitions may be familiar to you from previous courses. The terms will be helpful to know not only for this lesson, but for the rest of the lessons in this module and for the entire course. Add these terms to your resource sheet if you feel it would be beneficial to do so.

- **Term:** a number or the product of a number and a variable or variables. The exponent on the variable must be a natural number.
- **Polynomial:** a mathematical expression with one or more terms separated by addition or subtraction.
- **Greatest common factor (GCF):** the greatest divisor that is a factor of two or more terms (e.g., the greatest common factor of 12 and 18 is 6; the greatest common factor of $2xy$ and $2yz$ is $2y$).
- **Coefficient:** the number multiplying the variable(s) in a term (e.g., 7 is the coefficient of 7×3).
- **Constant term:** a term in an expression that has no variables. It is either a number or a symbol that represents a number, such as π.

Multiplication of two Binomials

In *Grade 10 Introduction to Applied and Pre-Calculus Mathematics*, you learned how to multiply two binomials together by using the distributive property, which ensures you multiply each term in the first binomial by each term in the second binomial. You also may be familiar with other methods, such as FOIL or area of rectangles.

Factoring polynomials is the inverse of multiplying polynomials. Therefore, you need to be comfortable multiplying polynomials before you continue to learn about factoring polynomials. Consider the following example.

Example 1

Multiply.

a) $(x + 1)(x - 3)$, using the distributive property

b) $(2x - 1)(x^2 + 3)$, using FOIL

c) $(x^2 + x - 1)(x + 4)$, using area of a rectangle

Solution

a) $(x + 1)(x - 3)$, using the distributive property

$= x(x - 3) + 1(x - 3)$

$= x^2 - 3x + 1x - 3$

$= x^2 - 2x - 3$

b) $(2x - 1)(x^2 + 3)$, using FOIL

F multiply the **first** terms

$(2x)(x^2) = 2x^3$

O multiply the **outside** terms

$(2x)(3) = 6x$

I multiply the **inside** terms

$(-1)(x^2) = -x^2$

L multiply the **last** terms

$(-1)(3) = -3$

Now add all your answers:

$$(2x - 1)(x^2 + 3) = 2x^3 + 6x + (-x^2) + (-3)$$
$$= 2x^3 - x^2 + 6x - 3$$

Note: FOIL only works for a binomial times a binomial, and not for other polynomial products.

c) $(x^2 + x - 1)(x + 4)$, using the area of a rectangle

	x	4
x^2	area = $(x)(x^2)$ = x^3	area = $4(x^2)$ = $4x^2$
x	area = $x(x)$ = x^2	area = $4(x)$ = $4x$
-1	area = $(x)(-1)$ = $-x$	area = $4(-1)$ = -4

Now add all the areas to get the answer.

$$(x^2 + x - 1)(x + 4) = x^3 + x^2 + (-x) + 4x^2 + 4x + (-4)$$
$$= x^3 + 5x^2 + 3x - 4$$

Greatest Common Factors

When factoring a polynomial expression, you should always factor out the greatest common factor first to make the remaining factoring easier. Consider the two terms in the binomial $3x^2 - 6x$.

What factors do they have in common?

Both terms have the variable x in common. Also, both terms are divisible by 3.

Therefore, $3x$ is the greatest common factor of the binomial $3x^2 - 6x$.

When you divide $3x$ out of the polynomial, you are left with $x - 2$. Therefore, $3x$ and $x - 2$ are the two factors of this polynomial.

We write $3x^2 - 6x = 3x(x - 2)$.

$3x(x - 2)$ is called the *factored form* of the polynomial $3x^2 - 6x$.

Example 2

Factor the following polynomials.

a) $5x^2 + 10x$

b) $2x - 8$

c) $3x^3 - 6x^2 + 9x$

Solution

a) The terms $5x^2$ and $10x$ both have the variable x in common and are divisible by 5. Therefore, the greatest common factor of this polynomial is $5x$. The factored form of this polynomial is $5x(x + 2)$.

   Always check your answer by multiplying the factors to see if you get the original expression.

b) The terms $2x$ and 8 are both divisible by 2. Therefore, the greatest common factor of this polynomial is 2. The factored form of this polynomial is $2(x - 4)$.

   Check your work by distributing.

c) The terms $3x^3$, $-6x^2$, and $9x$ all have the variable x in common and are divisible by 3. Therefore, the greatest common factor is $3x$. The factored form of this polynomial is $3x(x^2 - 2x + 3)$.

There might not be a greatest common factor to factor out of a polynomial. However, looking for the greatest common factor in a polynomial is the first step to take when factoring a polynomial.

Factoring Trinomials

Consider a polynomial in the form $ax^2 + bx + c$, where a and b are coefficients and c is the constant term. This polynomial is called a **trinomial** because it has three terms.

Factoring a trinomial where $a = 1$

If $a = 1$, you can use the following method to factor the trinomial.

To factor trinomials of this form, you need to look for two numbers that will give you a sum of "b" and a product of "c." Consider the following examples.

Example 3

Factor the following trinomials.

a) $x^2 + 8x + 12$

b) $x^2 + 20x + 100$

c) $x^2 + 2x - 8$

Solution

In all of the above examples, $a = 1$. In other words, the coefficient of x^2 is equal to one.

a) Compare the trinomial, $x^2 + 8x + 12$, to the form $ax^2 + bx + c$. In this example, $b = 8$, and $c = 12$. Therefore, you are looking for two numbers that will sum to 8 and multiply to 12.

   The factors of 12 are:

   1, 12

   2, 6

   3, 4

   Out of these factors, the two numbers that sum to 8 are 2 and 6. Therefore, the factored form of $x^2 + 8x + 12$ is $(x + 2)(x + 6)$.

   To check that you have factored this polynomial correctly, you can multiply the factors together to see if you arrive at the original polynomial.

   Check using FOIL:

   $(x + 2)(x + 6)$

   $= x^2 + 6x + 2x + 12$

   $= x^2 + 8x + 12$

   This answer is the same as the original polynomial. Thus, the answer is $(x + 2)(x + 6)$.

b) Compare the trinomial, $x^2 + 20x + 100$, to the form $ax^2 + bx + c$. In this example, $b = 20$ and $c = 100$. Therefore, you are looking for two numbers that will sum to 20 and multiply to 100.

The factors of 100 are

1, 100

2, 50

4, 25

5, 20

10, 10

Out of these factors, the two numbers that sum to 20 are 10 and 10. Therefore, the factored form of $x^2 + 20x + 100$ is $(x + 10)(x + 10) = (x + 10)^2$. Check your work by multiplying the two factors to see if you get the original polynomial.

c) Compare the trinomial, $x^2 + 2x - 8$, to the form $ax^2 + bx + c$. In this example, $b = 2$ and $c = -8$. Therefore, you are looking for two numbers that will sum to 2 and multiply to -8.

The factors of -8 are

1, -8

-1, 8

2, -4

-2, 4

Out of these factors, the two numbers that sum to 2 are -2 and 4. Therefore, the factored form of $x^2 + 2x - 8$ is $(x - 2)(x + 4)$.

Factoring a trinomial where $a \neq 1$

What happens if the coefficient of x^2 does not equal one? In this case, you can use a similar method to factor the trinomial. However, there is one additional process.

Consider a polynomial of the form $ax^2 + bx + c$. In Example 2, you factored the trinomials by finding two numbers where their sum was "b" and their product was "c." This will not work when the coefficient of the x^2 term is not 1. Therefore, a new method is needed. You will still need to find two numbers that sum to the middle term, or b. However, their product needs to be equal to the product of "a" and "c."

Example 4

Factor the following trinomials.

a) $6x^2 + x - 2$

b) $2x^2 - 9x + 4$

c) $5x^2 + 9x + 4$

d) $\frac{1}{2}x^2 - \frac{1}{2}x - 3$

Solution

a) $6x^2 + x - 2$

The product of "a" and "c" is $(6)(-2) = -12$. The value of b is 1.

Therefore, you need to find two numbers whose sum is 1 and whose product is -12. The factors of -12 are:

$1, -12$

$-1, 12$

$2, 6$

$-2, 6$

$3, -4$

$-3, 4$

Out of these factors, the two numbers that sum to 1 are -3 and 4. Now, instead of writing the factored form of this trinomial as before, you need to break up the original trinomial and then factor by grouping.

$6x^2 + x - 2$

The original middle term, x, is separated into two terms using the factors -3 and 4. We write $1x$ as $-3x + 4x$ or as $4x - 3x$, the order doesn't matter. The polynomial now has four terms instead of three. We write

$6x^2 - 3x + 4x - 2$

Consider just the first two terms of the polynomial and determine if they have a common factor. If they do, factor it out.

$6x^2 - 3x + 4x - 2 = 3x(2x - 1) + 4x - 2$

Now, consider just the last two terms of the polynomial. Determine if they have a common factor and, if they do, factor it out.

$3x(2x - 1) + 4x - 2 = 3x(2x - 1) + 2(2x - 1)$

You now have only two terms.

The expression in the brackets is the same in each term. It is a common factor and therefore you can factor it out.

$3x(2x - 1) + 2(2x - 1) = (2x - 1)(3x + 2)$

Thus, $(2x - 1)(3x + 2)$ is the factored form of $6x^2 + x - 2$, or $6x^2 + x - 2 = (2x - 1)(3x + 2)$. Always check your work by multiplying the two binomials.

b) $2x^2 - 9x + 4$

The product of "a" and "c" is (2)(4) = 8. The value of b is -9. Two numbers whose product is 8 and whose sum is -9 are -1 and -8.

Splitting up the middle term accordingly:

$$2x^2 - x - 8x + 4$$

Taking a common factor out of the first two terms:

$$x(2x - 1) - 8x + 4$$

Taking a common factor out of the last two terms (notice the sign change inside the second bracket):

$$x(2x - 1) - 4(2x - 1)$$

Taking the common factor out of the binomial:

$$(2x - 1)(x - 4)$$

The answer is

$$2x^2 - 9x + 4 = (2x - 1)(x - 4)$$

Don't forget to check your work by multiplying the factors together.

c) $5x^2 + 9x + 4$

The product of "a" and "c" is (5)(4) = 20. The value of b is 9. Two numbers whose product is 20 and whose sum is 9 are 5 and 4.

Splitting up the middle term:

$$5x^2 + 5x + 4x + 4$$

Taking a common factor out of the first two terms:

$$5x(x + 1) + 4x + 4$$

Taking a common factor out of the last two terms:

$$5x(x + 1) + 4(x + 1)$$

Taking the common factor out of the binomial:

$$(x + 1)(5x + 4)$$

The answer is

$$5x^2 + 9x + 4 = (x + 1)(5x + 4)$$

d) $\frac{1}{2}x^2 - \frac{1}{2}x - 3$

This is a trinomial with a fractional coefficient for x^2. Thus, factor $\frac{1}{2}$ out of each term by dividing by $\frac{1}{2}$. The result is $\frac{1}{2}\left(x^2 - x - 6\right)$, which is factored further to get $\frac{1}{2}(x - 3)(x + 2)$.

Difference of Squares

What happens if the polynomial has only two terms? Is it still possible to factor this polynomial?

Yes. It may be possible to factor this polynomial, but only if the polynomial is a difference of squares. The polynomial must be in the form $A^2 - B^2$ in order to be a difference of squares.

The factors of the difference of squares are:

$A^2 - B^2 = (A - B)(A + B)$

When you check the answer by multiplying, the sum of the inner product and the outer product is $-AB + (AB)$ and, when simplified, is zero. Only the first and last terms are left as shown below:

$(A - B)(A + B)$
$= A(A + B) - B(A + B)$
$= A^2 + AB - AB - B^2$
$= A^2 - B^2$

Example 5

Factor each difference of squares.

a) $x^2 - 9$

b) $4x^2 - 25$

c) $16x^2 - 9$

Solution

Each polynomial has two terms subtracted, and each term is a perfect square. Thus, they can be factored as a difference of squares.

a) Compare $x^2 - 9$ with the form $A^2 - B^2$. In this example, $A = x$ and $B = 3$. Therefore, you can factor the difference of squares using the form $(A - B)(A + B)$. This gives $(x - 3)(x + 3)$ as the factored form of $x^2 - 9$.

b) $4x^2 - 25 = (2x)^2 - (5)^2$. Therefore, the factored form of this difference of squares is $(2x - 5)(2x + 5)$.

c) $16x^2 - 9 = (4x)^2 - (3)^2$. Therefore, the factored form of this difference of squares is $(4x - 3)(4x + 3)$.

Don't forget to check your answer by multiplying the two factors together.

Note that binomials that are a sum or difference of cubes may also be factored but the method is shown in a later course.

Use the following learning activity to brush up on your factoring skills so you will be prepared to factor more complicated expressions in the next lesson. If any of the factoring techniques seem difficult to you, it would be an excellent time to ask your tutor/marker for additional help. Your learning partner is also available for help, should you need it.

Learning Activity 2.1

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Carla has a shoe addiction. She owns 13 pairs of boots, 18 pairs of sneakers, 23 pairs of high heels, and 19 pairs of sandals. How many pairs of shoes does she have in total?
2. What are two numbers that sum to 11 and multiply to 30?
3. Multiply: $(2x + 1)(x - 5)$
4. Evaluate: $\dfrac{\frac{9}{21}}{\frac{3}{63}}$
5. Factor: $x^2 + 7x + 12$
6. How many metres are in 2.3 km?
7. Simplify: $(2x^2 - 6x + 11) - (-14x^2 - 2x + 4)$
8. Is xy^2 the greatest common factor of $6x^2y^2 + 8xy^3$?

continued

Learning Activity 2.1 (continued)

Part B: Factoring Review

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Factor out the greatest common factor.
   a) $5x + 20$
   b) $5x^3 + 20x^2$
   c) $30x^2y - 24xy^3$
   d) $x^3 + x^2$

2. Factor each trinomial completely.
   a) $x^2 + 7x - 8$
   b) $x^2 - 2x - 8$
   c) $x^2 - 11x + 28$
   d) $3x^3 + 30x^2 + 63x$
   e) $2x^2 + 7x - 15$
   f) $4x^2 - 12x + 9$
   g) $\frac{2}{3}x^2 - 4x + 6$

3. Factor each binomial completely.
   a) $x^2 - 100$
   b) $9x^2 - 49$
   c) $9x^2 - 16y^2$
   d) $2x^2 - 50y^2z^2$

4. Create a graphic organizer of the steps to take when factoring a polynomial. Include descriptions for how to complete each step.

Lesson Summary

Now that you have reviewed some basic factoring techniques, it will be easier to factor more complicated expressions. In the next lesson, you will be learning how to factor more difficult polynomials using the same techniques you reviewed in this lesson.

Lesson 2: Quadratic Patterns; Is a Binomial a Factor of a Polynomial?

Lesson Focus

In this lesson, you will

- ❑ learn how to factor polynomial expressions with a trinomial quadratic pattern
- ❑ learn how to factor polynomial expressions with a difference of squares pattern
- ❑ learn how to determine if a binomial is a factor of a polynomial

Lesson Introduction

In Lesson 1, you reviewed your knowledge about factoring. You may notice that different patterns are used when factoring polynomials. For example, there are patterns or rules to follow when factoring a difference of squares and when factoring trinomials where $a = 1$. You can also use these patterns to factor more difficult expressions, like those with a quadratic pattern.

Factoring

The quadratic patterns in trinomials and differences of squares allow you to factor complex polynomials. You use similar methods to those you used to factor less complex polynomials.

Factoring Polynomials of the Form $a(f(x))^2 + b(f(x)) + c$

Consider the polynomial $(x + 2)^2 + 6(x + 2) + 8$. This polynomial is in the form $a(f(x))^2 + b(f(x)) + c$ where $f(x) = (x + 2)$ and $a = 1$, $b = 6$, $c = 8$.

How many different methods are there to factor polynomials of this form?

Consider the following two methods.

Method 1: Multiplying then Factoring

You may decide the best way to factor this polynomial is to first multiply to eliminate the brackets and then simplify. The resulting polynomial, therefore, will be in the form $ax^2 + bx + c$, and you have just reviewed how to factor polynomials of this form in Lesson 1.

$(x + 2)^2 + 6(x + 2) + 8$	
$= x^2 + 2x + 2x + 4 + 6x + 12 + 8$	Multiply
$= x^2 + 10x + 24$	Combine like terms
$= (x + 4)(x + 6)$	Factor the trinomial with $a = 1$

Method 2: Factoring by Replacement—the R Method

Instead of multiplying out the entire polynomial, you may recognize a pattern in this polynomial. Comparing $a(f(x))^2 + b(f(x)) + c$ to the form $ax^2 + bx + c$, you may notice that x is being replaced by $f(x)$. In the polynomial, $(x + 2)^2 + 6(x + 2) + 8$, x is being replaced by $f(x) = (x + 2)$.

$(x + 2)^2 + 6(x + 2) + 8 = (f(x))^2 + 6(f(x)) + 8$, where $f(x) = (x + 2)$

It is possible to replace $(x + 2)$ with a different variable—say w. Then you will be able to factor the polynomial as you normally would. After you complete your factoring, you need to substitute $(x + 2)$ back in for the variable w and simplify.

$(x + 2)^2 + 6(x + 2) + 8$	
$= w^2 + 6w + 8$	Substitute w for $(x + 2)$
$= (w + 4)(w + 2)$	Factor the trinomial with $a = 1$
$= (x + 2 + 4)(x + 2 + 2)$	Substitute $(x + 2)$ back in for w
$= (x + 6)(x + 4)$	Simplify

As you can see, both methods arrive at the same answer. Which method do you prefer?

Some students prefer to use Method 1 and multiply out the entire expression. However, using Method 1 can sometimes become too complicated. Therefore, Method 2 is more easily used. Using Method 2 and then Method 1 to see if your answer is correct is an excellent idea.

How to use Method 2 to factor a polynomial of the form $a(f(x))^2 + b(f(x)) + c$:

1. Replace $f(x)$ with a single variable—say w.
2. Factor the resulting polynomial.
3. Replace the substituted variable, w, by $f(x)$.
4. Simplify the expression.
5. Factor further, if necessary.

Include these steps on your resource sheet if you feel they will help you remember how to factor a polynomial with a quadratic pattern.

Example 1

Factor the following polynomials completely.

a) $(x + 4)^2 - 6(x + 4) - 16$

b) $6(2x - 8)^2 - 21(2x - 8) + 15$

c) $2(x^2 + 3x - 2)^2 - 18(x^2 + 3x - 2) + 36$

d) $(x^2 + 12x + 11)^2 + 25(x^2 + 12x + 11) + 144$

Solution

a) $(x + 4)^2 - 6(x + 4) - 16$

Comparing $(x + 4)^2 - 6(x + 4) - 16$ with $a(f(x))^2 + b(f(x)) + c$, you notice that $f(x) = (x + 4)$. Therefore, $f(x) = (x + 4)$ can be replaced with a variable to reduce this polynomial to one that is easier to factor.

Replace $(x + 4)$ with w:

$$w^2 - 6w - 16$$

Factor:

$$(w + 2)(w - 8)$$

Replace w with $(x + 4)$:

$$(x + 4 + 2)(x + 4 - 8)$$

Simplify to get

$$(x + 6)(x - 4)$$

b) $6(2x - 8)^2 - 21(2x - 8) + 15$

In this expression, $f(x) = (2x - 8)$.

Replace $(2x - 8)$ with w:

$6w^2 - 21w + 15$

Factor out the GCF first:

$3(2w^2 - 7w + 5)$

$3(2w^2 - 5w - 2w + 5)$

$3[w(2w - 5) - 1(2w - 5)]$

$3[(2w - 5)(w - 1)]$

Replace w with $2x - 8$:

$3\{[2(2x - 8) - 5][2x - 8 - 1]\}$

$3\{(4x - 16 - 5)(2x - 9)\}$

Final answer:

$3(4x - 21)(2x - 9)$

c) $2(x^2 + 3x - 2)^2 - 18(x^2 + 3x - 2) + 36$

In this expression, $f(x) = (x^2 + 3x - 2)$

Replace $(x^2 + 3x - 2)$ with w:

$2w^2 - 18w + 36$

Factor out the GCF:

$2(w^2 - 9w + 18)$

Factor the remaining trinomial:

$2(w - 6)(w - 3)$

Replace w with $(x^2 + 3x - 2)$:

$2(x^2 + 3x - 2 - 6)(x^2 + 3x - 2 - 3)$

Simplify:

$2(x^2 + 3x - 8)(x^2 + 3x - 5)$

Neither of these trinomials factors further.

d) $(x^2 + 12x + 11)^2 + 25(x^2 + 12x + 11) + 144$

In this expression, $f(x) = (x^2 + 12x + 11)$.

Replace $(x^2 + 12x + 11)$ with w:

$w^2 + 25w + 144$

Factor:

$(w + 9)(w + 16)$

Replace w with $(x^2 + 12x + 11)$:

$(x^2 + 12x + 11 + 9)(x^2 + 12x + 11 + 16)$

Simplify:

$(x^2 + 12x + 20)(x^2 + 12x + 27)$

You may consider yourself to be done. However, both polynomials are still factorable. Therefore, there is more factoring for you to do.

Factor again to get the final answer:

$(x + 2)(x + 10)(x + 3)(x + 9)$

Factoring Polynomials of the Form $a^2(f(x))^2 - b^2(g(y))^2$

Consider the expression $a^2(f(x))^2 - b^2(g(y))^2$. Does this expression seem familiar to you? This expression is simply a more complex form of differences of squares. Just as differences of squares $A^2 - B^2$ could be factored as $(A - B)(A + B)$, so can $a^2(f(x))^2 - b^2(g(y))^2$.

The factored form of $a^2(f(x))^2 - b^2(g(y))^2$ is $[a(f(x)) - b(g(y))][a(f(x)) + b(g(y))]$.

Include this on your resource sheet.

Why does this work?

Consider $a^2(f(x))^2 - b^2(g(y))^2$ and compare it to $A^2 - B^2$.

$a^2(f(x))^2 - b^2(g(y))^2 = [a(f(x))]^2 - [b(g(y))]^2$

Therefore, $A = a(f(x))$ and $B = b(g(y))$.

Now, you can substitute these values in for A and B in the equation $(A - B)(A + B)$ to find the factored form of $a^2(f(x))^2 - b^2(g(y))^2$.

$a^2(f(x))^2 - b^2(g(y))^2 = [a(f(x)) - b(g(y))][a(f(x)) + b(g(y))]$

Example 2

Factor the following polynomials completely.

a) $64x^2 - 81y^2$

b) $4(x+2)^2 - 25(3y+5)^2$

c) $36(2x-5)^2 - 49(y-1)^2$

d) $\left(\frac{1}{100}\right)(2x-9)^2 - \left(\frac{1}{25}\right)(5y-7)^4$

Solution

a) Compare $64x^2 - 81y^2$ to the form $[a(f(x))]^2 - [b(g(y))]^2$.

$64x^2 - 81y^2$

$= (8x)^2 - (9y)^2$

Therefore, $a(f(x)) = 8x$ and $b(g(y)) = 9y$.

Using the formula $a^2(f(x))^2 - b^2(g(y))^2 = [a(f(x)) - b(g(y))][a(f(x)) + b(g(y))]$,
$64x^2 - 81y^2 = (8x - 9y)(8x + 9y)$

Now check your work by multiplying the factors together to get the original polynomial.

b) **Method 1: Factor Directly**

Compare $4(x+2)^2 - 25(3y+5)^2$ to the form $[a(f(x))]^2 - [b(g(y))]^2$.

$4(x+2)^2 - 25(3y+5)^2$

$= [2(x+2)]^2 - [5(3y+5)]^2$

Therefore, $a(f(x)) = 2(x+2)$ and $b(g(y)) = 5(3y+5)$.

$4(x+2)^2 - 25(3y+5)^2 = [2(x+2) - 5(3y+5)][2(x+2) + 5(3y+5)]$

After you arrive at this step, you still have to simplify your answer.

$(2x + 4 - 15y - 25)(2x + 4 + 15y + 25)$

$= (2x - 15y - 21)(2x + 15y + 29)$

Method 2: Replacement

Substitute:

let $w = x + 2$ and $p = 3y + 5$

Then $4(x + 2)^2 - 25(3y + 5)^2$ becomes

$4w^2 - 25p^2$

Factor:

$(2w - 5p)(2w + 5p)$

Back substituting:

$[2(x + 2) - 5(3y + 5)][2(x + 2) + 5(3y + 5)]$

Simplify:

$(2x + 4 - 15y - 25)(2x + 4 + 15y + 25)$

$(2x - 15y - 21)(2x + 15y + 29)$

c) **Method 1: Factor Directly**

Compare $36(2x - 5)^2 - 49(y - 1)^2$ to the form $[a(f(x))]^2 - [b(g(y))]^2$.

$36(2x - 5)^2 - 49(y - 1)^2$

$= [6(2x - 5)]^2 - [7(y - 1)]^2$

Therefore, $a(f(x)) = 6(2x - 5)$ and $b(g(y)) = 7(y - 1)$.

$36(2x - 5)^2 - 49(y - 1)^2 = [6(2x - 5) - 7(y - 1)][6(2x - 5) + 7(y - 1)]$

Simplify:

$(12x - 30 - 7y + 7)(12x - 30 + 7y - 7)$

$= (12x - 7y - 23)(12x + 7y - 37)$

Method 2: Replacement

Let $w = 2x - 5$ and $p = y - 1$

Then $36(2x - 5)^2 - 49(y - 1)^2$ becomes

$36w^2 - 49p^2$

Factor:

$(6w - 7p)(6w + 7p)$

Back substituting:

$[6(2x - 5) - 7(y - 1)][6(2x - 5) + 7(y - 1)]$

Simplify:

$(12x - 30 - 7y + 7)(12x - 30 + 7y - 7)$

$(12x - 7y - 23)(12x + 7y - 37)$

d) **Method 1: Factor Directly**

Compare $\left(\frac{1}{100}\right)(2x-9)^2 - \left(\frac{1}{25}\right)(5y-7)^4$ to the form $[a(f(x))]^2 - [b(g(y))]^2$.

$$\left(\frac{1}{100}\right)(2x-9)^2 - \left(\frac{1}{25}\right)(5y-7)^4$$

$$= \left[\left(\frac{1}{10}\right)(2x-9)\right]^2 - \left[\left(\frac{1}{5}\right)(5y-7)^2\right]^2$$

Recall that when an exponent is being raised to a power, you can multiply the exponent by the power: $(x^2)^2 = x^{(2)(2)} = x^4$. Therefore $(5y-7)^4$ is the same as $((5y-7)^2)^2$.

Thus:

$$a(f(x)) = \left(\frac{1}{10}\right)(2x-9) \text{ and } b(g(y)) = \left(\frac{1}{5}\right)(5y-7)^2$$

$$\left(\frac{1}{100}\right)(2x-9) - \left(\frac{1}{25}\right)(5y-7)^4$$

$$= \left[\left(\frac{1}{10}\right)(2x-9) - \left(\frac{1}{5}\right)(5y-7)^2\right]\left[\left(\frac{1}{10}\right)(2x-9) + \left(\frac{1}{5}\right)(5y-7)^2\right]$$

Simplify:

$$\left[\frac{2}{10}x - \frac{9}{10} - \frac{1}{5}(25y^2 - 70y + 49)\right]\left[\frac{2}{10}x - \frac{9}{10} + \frac{1}{5}(25y^2 - 70y + 49)\right]$$

$$= \left[\frac{2}{10}x - \frac{9}{10} - 5y^2 + 14y - \frac{49}{5}\right]\left[\frac{2}{10}x - \frac{9}{10} + 5y^2 - 14y + \frac{49}{5}\right]$$

$$= \left[\frac{1}{5}x - 5y^2 + 14y - \frac{107}{10}\right]\left[\frac{1}{5}x + 5y^2 - 14y + \frac{89}{10}\right]$$

Method 2: Replacement

Substitution:

Let $w = 2x - 9$ and $t = 5y - 7$

Then $\frac{1}{100}(2x-9)^2 - \frac{1}{25}(5y-7)^4$ becomes:

$$\frac{1}{100}w^2 - \frac{1}{25}t^4$$

Common factor of $\frac{1}{100}$

$$\frac{1}{100}\left(w^2 - 4t^4\right)$$

Note: $\frac{1}{25} \div \frac{1}{100} = \frac{1}{25} \times \frac{100}{1} = 4$

Factor as a difference of squares:

$$\frac{1}{100}\left(w - 2t^2\right)\left(w + 2t^2\right)$$

Back substitute:

$$\frac{1}{100}\left[2x - 9 - 2(5y-7)^2\right]\left[2x - 9 + 2(5y-7)^2\right]$$

Simplify:

$$\frac{1}{100}\left[2x - 9 - 2\left(25y^2 - 70y + 49\right)\right]\left[2x - 9 + 2\left(25y^2 - 70y + 49\right)\right]$$

Simplify:

$$\frac{1}{100}\left[2x - 9 - 50y^2 + 140y - 98\right]\left[2x - 9 + 50y^2 - 140y + 98\right]$$

$$\frac{1}{100}\left(2x - 50y^2 + 140y - 107\right)\left(2x + 50y^2 - 140y + 89\right)$$

This answer is slightly different than the one in Method 1 because the fractional common factor was factored out in Method 2.

Determining Whether a Binomial is a Factor of a Polynomial

You have now learned how to factor various forms of polynomials. However, what happens if someone tells you that a certain binomial is a factor of a polynomial? How can you know if that person is correct?

Consider the polynomial $x^2 + 3x + 2$.

The factored form is $(x + 1)(x + 2)$.

What happens when you equate each factor to zero?

$x + 1 = 0$ or $x + 2 = 0$

$x = -1$ $x = -2$

You now have two values for x. What happens when you substitute these values for x back into the factored form of the polynomial?

$x = -1$

$(-1 + 1)(-1 + 2) = (0)(1) = 0$

$x = -2$

$(-2 + 1)(-2 + 2) = (-1)(0) = 0$

In both situations, the answer is zero. What happens when you substitute these values for x back into the original polynomial?

$x = -1$

$(-1)^2 + 3(-1) + 2 = 1 - 3 + 2 = 0$

$x = -2$

$(-2)^2 + 3(-2) + 2 = 4 - 6 + 2 = 0$

Both of these equations equal zero again! When you equate a factor of a polynomial to zero and substitute the resulting value into the original polynomial, you always get an answer of zero. This is because the factored form of the polynomial is equal to the polynomial.

$x^2 + 3x + 2 = (x + 1)(x + 2)$

Therefore, if one of the factors of a polynomial equals zero, the whole expression equals zero. Recall: Anything multiplied by zero is zero.

For that reason, in order to check if a binomial is a factor of a polynomial, you can use the following steps. Include these steps on your resource sheet.

1. Equate the binomial to zero and solve for the variable.
2. Substitute the value you calculated for the variable into the original polynomial.

3. Evaluate to see if the value of the polynomial is equal to 0.
   - If the evaluated polynomial equals zero, the binomial is a factor.
   - If the evaluated polynomial does not equal zero, the binomial is not a factor.

Example 3

Determine whether $(x + 5)$ is a factor of the following polynomials.

a) $x^2 + 6x + 5$

b) $2x^2 - 6x + 7$

c) $3x^2 - 4x + 1$

Solution

Step 1 Equate the binomial to zero and solve for the variable.

$$x + 5 = 0$$
$$x = -5$$

Step 2: Substitute $x = -5$ into each polynomial.

a) $x^2 + 6x + 5$
$= (-5)^2 + 6(-5) + 5$
$= 25 - 30 + 5$
$= 0$

b) $2x^2 - 6x + 7$
$= 2(-5)^2 - 6(-5) + 7$
$= 50 + 30 + 7$
$= 87$

c) $3x^2 - 4x + 1$
$= 3(-5)^2 - 4(-5) + 1$
$= 75 + 20 + 1$
$= 96$

Step 3: Check to see if the evaluated polynomial equals zero.

The evaluated polynomial in (a) equals zero. Therefore, $(x + 5)$ is a factor of the polynomial in (a).

The evaluated polynomials in (b) and (c) do not equal zero. Therefore, $(x + 5)$ is not a factor of the polynomials in (b) and (c).

Example 4

Determine if $(2x - 5)$ is a factor of the following polynomials.

a) $2x^2 - x - 10$

b) $4x^2 - x + 7$

c) $4x^2 - 20x + 25$

Solution

$2x - 5 = 0$

$2x = 5$

$x = \frac{5}{2}$

a) $2x^2 - x - 10$

$= 2\left(\frac{5}{2}\right)^2 - \frac{5}{2} - 10$

$= \frac{25}{2} - \frac{5}{2} - 10$

$= 0$

b) $4x^2 - x + 7$

$= 4\left(\frac{5}{2}\right)^2 - \frac{5}{2} + 7$

$= 25 - \frac{5}{2} + 7$

$= 29.5$

c) $4x^2 - 20x + 25$

$= 4\left(\frac{5}{2}\right)^2 - 20\left(\frac{5}{2}\right) + 25$

$= 25 - 50 + 25$

$= 0$

The evaluated polynomials in (a) and (c) both equal zero. Therefore, $(2x - 5)$ is a factor of the polynomials in (a) and (c).

The evaluated polynomial in (b) does not equal zero. Therefore, $(2x - 5)$ is not a factor of the polynomial in (b).

Learning Activity 2.2

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. There are a group of 80 people in a shareholder's meeting making a decision about whether or not to acquire a new company. The decision needs to pass with an 80% majority. How many people must agree in order for the decision to pass?
2. Evaluate: $5 \times 17 \times 20$
3. Katrina is flying from Winnipeg to Belize. She leaves Winnipeg at 7:00 am and takes a 3-hour flight to Minneapolis where she has a two and a half hour layover. From Minneapolis, she takes a direct flight to Belize that lasts for 7.25 hours. What time does Katrina arrive in Belize?
4. What is the greatest common factor of 48 and 144?
5. Convert 6.3 kilometres to metres.
6. Simplify: $\dfrac{2^8}{4^2}$
7. Find the slope of the line parallel to the line, $-2 = 4x + 6y$.
8. Determine the total surface area of a cube with a volume of 8 cm^3.

Part B: Factoring Expressions with Quadratic Patterns

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Add to the factoring graphic organizer you completed in the last learning activity (Learning Activity 2.1). Include all of the new techniques you learned about factoring in this lesson.

continued

Learning Activity 2.2 (continued)

2. Factor completely.
   a) $(7x - 6)^2 - 8(7x - 6) + 15$
   b) $9(2x - 1)^2 - 42(2x - 1) + 49$
   c) $(x^2 - x)^2 - 7(x^2 - x) + 6$
   d) $(x - 1)^2 + 4(x - 1) - 32$
3. Factor each expression.
   a) $50(2x - 7)^2 - 162(3y - 4)^2$
   b) $25x^2 - 4y^2$
   c) $16(7x - 6)^2 - 121(4y - 7)^2$
   d) $144(x + 3)^4 - 729(y - 1)^4$
4. Determine if $(x - 2)$ is a factor of each of the following polynomials.
   a) $8x^2 - 30x + 28$
   b) $(x - 2)^2 + 4(x - 2) - 32$
   c) $-15x^2 - 196x - 572$
5. Determine if $(7x - 6)$ is a factor of each of the following polynomials.
   a) $(7x - 6)^2 - 8(7x - 6) + 15$
   b) $7x^2 + 22x - 24$
   c) $16(7x - 6)^2 - 121(4x - 7)^2$
   d) $35x^2 - 58x + 24$
6. Janique was asked to factor the trinomial $4x^2 + 2x - 12$ completely. Her work is shown below.

   $4x^2 + 2x - 12$
   $= 4x^2 + 8x - 6x - 12$
   $= 4x(x + 2) - 6(x + 2)$
   $= (4x - 6)(x + 2)$

   Determine if her work is correct, and explain why or why not.

Lesson Summary

In this lesson, you built on your knowledge of factoring from Lesson 1 and from previous courses. You learned how to factor polynomials that follow a quadratic pattern. You also discovered that there are ways to check if a binomial is a factor of a polynomial. In the next lesson, you will be discovering how factoring is related to rational expressions.

Assignment 2.1

Factoring

Total: 37 marks

1. Factor the following expressions completely.

   a) $4x^2 + 13x + 9$ *(2 marks)*

   b) $4x^2 - 16x$ *(1 mark)*

   c) $x^2 - y^2$ *(1 mark)*

   d) $3abc^3 - 27bc^2$ *(1 mark)*

   e) $x^2 - 6x - 27$ *(1 mark)*

   f) $25 - x^2$ *(1 mark)*

continued

Assignment 2.1: Factoring (continued)

2. Factor completely.

   a) $(x + 2)^2 - 6(x + 2) - 16$ *(3 marks)*

   b) $3(2x - 1)^2 + 19(2x - 1) + 16$ *(3 marks)*

   c) $12(2x - 5)^2 + 17(2x - 5) - 5$ *(4 marks)*

continued

Assignment 2.1: Factoring (continued)

3. Factor completely.

   a) $25(x-7)^2-(y-1)^2$ *(3 marks)*

   b) $81(x-2)^2-36(3y-7)^2$ *(3 marks)*

   c) $4(4x-1)^4-49(2y)^2$ *(4 marks)*

continued

Assignment 2.1: Factoring (continued)

4. Determine if $(x - 5)$ is a factor of the following polynomials. *(4 marks)*

   a) $2x^2 - 17x + 35$ *(2 marks)*

   b) $2(x - 4)^2 - 2(x - 5)^2$ *(2 marks)*

continued

Assignment 2.1: Factoring (continued)

5. Determine if $(2x - 7)$ is a factor of the following polynomials.

   a) $3(2x - 7)^2 - 4(2x - 6)^2$ *(2 marks)*

   b) $4x^2 - 28x + 49$ *(2 marks)*

continued

Assignment 2.1: Factoring (continued)

6. Jacob was asked to factor the trinomial $2(x-2)^2 + 8(x-2) - 10$ completely. His work is shown below. Is his work correct? Explain why or why not. *(2 marks)*

   $2(x-2)^2 + 8(x-2) - 10$

   Let $w = (x-2)$

   $= 2w^2 + 8w - 10$

   $= 2w^2 - 2w + 10w - 10$

   $= 2w(w-1) + 10(w-1)$

   $= (2w + 10)(w-1)$

   $= 2(w + 5)(w-1)$

Lesson 3: Equivalent Rational Expressions

Lesson Focus

In this lesson, you will

- ❑ learn the definition of a non-permissible value
- ❑ learn how to find non-permissible values of rational expressions
- ❑ learn how to simplify a rational expression by factoring
- ❑ learn how to create equivalent rational expressions
- ❑ learn how to find errors in your solution

Lesson Introduction

In the first two lessons of this module, you reviewed some basic factoring techniques and used these techniques to factor more complex polynomials. Factoring is very useful when dealing with rational expressions. Rational expressions are similar to fractions. However, rational expressions contain polynomials in the numerator and the denominator. In this lesson, you will discover how factoring and fractions are related to rational expressions.

Rational Expressions

You have studied fractions, which can also be called rational numbers, in previous mathematics courses. You know how to add, subtract, multiply, and divide rational numbers. Furthermore, you know how to reduce fractions to their lowest form.

In the next few lessons, you will learn how to perform these operations with the algebraic equivalents of rational numbers, called **rational expressions**.

- $\frac{a}{b}$ is a **rational number** if a and b are integers and $b \neq 0$.
- $\frac{p}{q}$ is a **rational expression** if p and q are polynomials and $q \neq 0$, and q contains a variable.

A **rational number** is a ratio of integers, while a **rational expression** is a ratio of polynomials.

- $\frac{2}{3}, \frac{-5}{6}, \frac{4}{1}$ are rational numbers
- $\frac{x^2 - x + 1}{x + 2}, \frac{-(x+1)}{x}, \frac{3}{x-1}$ are rational expressions

Include all of the previous definitions and examples on your resource sheet.

Non-Permissible Values

In all fractions, the denominator cannot equal zero. This is because there is no answer when dividing by zero. Also, in all rational expressions, the denominator cannot equal zero.

The **domain** of a rational expression is the set of all real numbers for which the denominator of the rational expression is not zero. The numbers excluded from the domain of a rational expression are called the **non-permissible values** of the expression.

Example 1

Find the domain and the non-permissible values of the following rational expressions.

a) $\frac{x-1}{x-2}$

b) $\frac{5}{x}$

c) $\frac{x^2 - 1}{x - 7}$

d) $\frac{1}{x^2 - 2x + 1}$

Solution

To find the non-permissible values of each rational expression, you need to find the values at which the denominator of the rational expression equals zero.

a) $\dfrac{x-1}{x-2}$

In this rational expression, the denominator equals zero when $x = 2$.

Therefore, the non-permissible value of this rational expression is $x = 2$.

The domain is all real numbers except $x = 2$.

D: $\{x \mid x \neq 2, x \in \Re\}$

Recall: This statement is read as "x is any value such that x does not equal two and x is an element of the real number system."

b) $\dfrac{5}{x}$

In this rational expression, the denominator equals zero when $x = 0$.

Therefore, the non-permissible value of this rational expression is $x = 0$.

The domain is all real numbers except $x = 0$.

D: $\{x \mid x \neq 0, x \in \Re\}$

c) $\dfrac{x^2-1}{x-7}$

In this rational expression, the denominator equals zero when $x = 7$.

Therefore, the non-permissible value of this rational expression is $x = 7$.

The domain is all real numbers except $x = 7$.

D: $\{x \mid x \neq 7, x \in \Re\}$

d) $\dfrac{1}{x^2-2x+1}$

In this rational expression, the way to discover when the denominator equals zero is to first factor the denominator. You will revisit this method in more detail in Modules 3 and 4.

$x^2 - 2x + 1 = (x - 1)(x - 1)$

Now, you have to equate each factor of the denominator to zero. As both factors of this polynomial are the same, you only have one non-permissible value.

$x - 1 = 0$

$x = 1$

Therefore, the denominator equals zero when $x = 1$.

Recall from last lesson that the factored form of a polynomial is equal to the polynomial itself. Therefore, when one factor of the polynomial equals zero, the whole polynomial equals zero.

The domain is all real numbers except $x = 1$.

D: $\{x \mid x \neq 1, x \in \Re\}$

Simplifying Rational Expressions

What is the value of any rational expression in which the numerator and the denominator are the same non-zero polynomial?

- $\frac{7}{7} = 1$
- $\frac{4}{4} = 1$
- $\frac{x-4}{x-4} = 1$, except when $x = 4$
- $\frac{x^2 + x + 7}{x^2 + x + 7} = 1$

When there are no variables in the denominator of an expression, any expression divided by itself is one. However, when variables are located in the denominator of an expression, non-permissible values can sometimes appear. These non-permissible values can make an expression that would normally equal one, such as $\frac{x-4}{x-4}$, be undefined at that value. Because of this, you need to be careful when you are reducing or simplifying rational expressions.

Note: $x^2 + x + 7$ does not factor. No value of x will make $x^2 + x + 7$ equal zero.

You can reduce rational numbers by factoring to form equivalent fractions.

$$\frac{4}{6} = \frac{\cancel{2}(2)}{\cancel{2}(3)} = \frac{2}{3}$$

Thus, $\frac{4}{6}$ and $\frac{2}{3}$ are equivalent fractions.

Similarly, you can reduce a rational expression by factoring to produce an equivalent rational expression.

$$\frac{x^2-1}{x^2+x} = \frac{(x-1)\cancel{(x+1)}}{x\cancel{(x+1)}} = \frac{x-1}{x}$$

The rational expressions $\frac{x^2-1}{x^2+1}$ and $\frac{x-1}{x}$ are said to be equivalent.

However, the non-permissible values must be determined as well. These values are calculated according to how the original expression is defined, not the reduced expression. The easiest way is to use the factored form of the original before you reduce it. Thus, the non-permissible values for $\frac{x^2-1}{x^2+1} = \frac{(x-1)(x+1)}{x(x+1)}$ are where the denominator is zero.

These are at the values:

$x = 0$ or $x + 1 = 0$

$x = -1$

Thus, the non-permissible values are $x = 0, -1$.

Consider the following table of values which compares $\frac{x^2-1}{x^2+x}$ with its reduced form $\frac{x-1}{x}$ for various values of x.

value of x	value of $\frac{x^2-1}{x^2+x}$	value of the reduced form of $\frac{x^2-1}{x^2+x}$ or $\frac{x-1}{x}$
-2	$\frac{x^2-1}{x^2+x} = \frac{(-2)^2-1}{(-2)^2+(-2)}$ $= \frac{3}{2}$	$\frac{x-1}{x} = \frac{(-2)-1}{(-2)}$ $= \frac{-3}{-2}$ $= \frac{3}{2}$

(continued)

−1	$\frac{x^2-1}{x^2+x} = \frac{(-1)^2-1}{(-1)^2+(-1)}$ $= \frac{0}{0}$ = indeterminate	$\frac{x-1}{x} = \frac{(-1)-1}{(-1)}$ $= \frac{-2}{-1}$ $= 2$
	The expression $\frac{0}{0}$ is an indeterminate form that is studied in more detail in calculus. You know that any number over itself is 1, so you might think $\frac{0}{0}$ is 1. However, you also know that 0 in the numerator of any fraction is zero, so you might think $\frac{0}{0}$ is zero. Finally, you know that any number divided by a very small number is infinitely large, so you might think $\frac{0}{0}$ approaches infinity. The reason that $\frac{0}{0}$ is called indeterminate is because its value cannot be determined. If you were to graph $y = \frac{x^2-1}{x^2+x}$, the graph would be identical to $y = \frac{x-1}{x}$, except there would be a "hole" in the graph where $x = 1$ and $y = 2$.	
0	$\frac{x^2-1}{x^2+x} = \frac{0-1}{0+0}$ $= \frac{-1}{0}$ = no answer Division by zero is not possible.	$\frac{x-1}{x} = \frac{0-1}{0}$ $= \frac{-1}{0}$ = no answer Division by zero is not possible.
1	$\frac{x^2-1}{x^2+x} = \frac{1-1}{1+1}$ $= \frac{0}{2}$ $= 0$	$\frac{x-1}{x} = \frac{1-1}{1}$ $= \frac{0}{1}$ $= 0$
2	$\frac{x^2-1}{x^2+x} = \frac{2^2-1}{2^2+2}$ $= \frac{3}{6}$ $= \frac{1}{2}$	$\frac{x-1}{x} = \frac{2-1}{2}$ $= \frac{1}{2}$

The non-permissible values for the original expression are $x = -1$ and $x = 0$, where the expression has a denominator of 0. As you can see from the chart, the values for $\frac{x^2 - 1}{x^2 + x}$ and its reduced form $\frac{x - 1}{x}$ are equivalent except for $x = -1$. The reduced expression only shows one non-permissible value where the denominator is zero, at $x = 0$. Thus, when finding non-permissible values always consider the original expression. At the non-permissible values of x, the two expressions are not equivalent or are not defined. At all other values, they are equivalent.

The answer of "indeterminate" will be explained in more detail in future math courses. It is included here for comparison purposes. It is not on your examinations.

Steps for simplifying rational expressions:

1. Factor the numerator and denominator completely.
2. Find the non-permissible values.
3. Cancel out factors that are common to both the numerator and the denominator.
4. Write the reduced equivalent rational expression including the non-permissible values. The answer may be written in factored form.

It may be helpful for you to include these steps on your resource sheet.

Example 2

Simplify each rational expression. State the non-permissible values.

a) $\frac{x^2 - 9}{x^2 - x - 6}$

b) $\frac{x^2 - 3x}{x}$

c) $\frac{6x^2 + 3x}{3x^2}$

d) $\frac{x - 3}{3 - x}$

e) $\frac{125x^3y^4}{5x^5y^2}$

Solution

a) $\dfrac{x^2 - 9}{x^2 - x - 6}$

First, you need to factor the numerator and denominator of this expression.

$$\frac{(x-3)(x+3)}{(x-3)(x+2)}$$

Before you simplify this rational expression, you need to state the non-permissible values. Recall that the non-permissible values of the variable occur when the denominator of the original rational expression equals zero.

$x - 3 = 0$ $\qquad$ $x + 2 = 0$

$x = 3$ $\qquad$ $x = -2$

Therefore, the non-permissible values are 3 and -2.

Now you can go ahead and simplify the rational expression.

$$\frac{\cancel{(x-3)}\,(x+3)}{\cancel{(x-3)}\,(x+2)} = \frac{x+3}{x+2}, \text{ where } x \neq 3, -2$$

b) $\dfrac{x^2 - 3x}{x}$

The non-permissible value is 0. Now factor.

$$\frac{x(x-3)}{x} = x - 3$$

Thus, the answer is $x - 3$, $x \neq 0$.

c) $\dfrac{6x^2 + 3x}{3x^2}$

$$= \frac{3x(2x+1)}{3x(x)}$$

The non-permissible value is 0 (since $3x = 0$ when $x = 0$ and $x = 0$ are the factors).

$$\frac{\cancel{3x}\,(2x+1)}{\cancel{3x}\,(x)} = \frac{2x+1}{x}, \; x \neq 0$$

d) $\dfrac{x-3}{3-x}$

At first, you may think it is not possible to reduce this rational expression. However, $(3 - x) = -(x - 3)$.

Therefore, you can rewrite the rational expression as follows.

$$\frac{x-3}{3-x} = \frac{\cancel{x-3}}{-\left(\cancel{x-3}\right)} = \frac{1}{-1} = -1$$

The non-permissible value of this rational expression is $x = 3$.

e) $\dfrac{125x^3y^4}{5x^5y^2}$

The non-permissible values of this rational expression are $x = 0$ and $y = 0$.

When you see an expression in this form, it may be helpful for you to expand the expression as follows:

$$\frac{125x^3y^4}{5x^5y^2} = \frac{(5)(5)(5)(x)(x)(x)(y)(y)(y)(y)}{(5)(x)(x)(x)(x)(x)(y)(y)}$$

It is now easier to see what is going on, and what factors you can cancel from both the numerator and denominator.

$$\frac{\cancel{(5)}\,(5)(5)\,\cancel{(x)}\,\cancel{(x)}\,\cancel{(x)}\,\cancel{(y)}\,\cancel{(y)}\,(y)(y)}{\cancel{(5)}\,\cancel{(x)}\,\cancel{(x)}\,\cancel{(x)}\,(x)(x)\,\cancel{(y)}\,\cancel{(y)}} = \frac{(5)(5)(y)(y)}{(x)(x)} = \frac{25y^2}{x^2},\ x \neq 0,\ y \neq 0$$

Example 3

Is the domain of $\dfrac{x^2 + x - 12}{x^2 - 9}$ all real numbers except $x = -3$?

Solution

$$\frac{x^2 + x - 12}{x^2 - 9} = \frac{(x+4)(x-3)}{(x-3)(x+3)} = \frac{x+4}{x+3}$$

$x = -3$ is a non-permissible value of this rational expression. However, in order to find all non-permissible values of a rational expression, you need to find the values at which the denominator of the original expression equals zero.

Therefore, the non-permissible values of this expression are $x = 3$ and $x = -3$. The domain of this rational expression is all real numbers except 3 and -3.

Equivalent Rational Expressions

Just as it is possible to reduce fractions, you can also find equivalent fractions by multiplying the denominator and numerator of a fraction by the same value. In other words, you are multiplying by 1 in some form. Multiplying by 1 does not change the value of an expression.

$$\frac{2}{3} = \frac{2(5)}{3(5)} = \frac{10}{15}$$

$\frac{2}{3}$ and $\frac{10}{15}$ are equivalent fractions. Here you multiplied by 1 in the form of $\frac{5}{5}$.

It is also possible to create equivalent rational expressions. However, instead of just multiplying the numerator and the denominator by the same numerical value, you can also multiply them by a polynomial as long as its value is not zero, as shown below.

$$\frac{x+3}{x-2} = \frac{(x+3)(x+1)}{(x-2)(x+1)} = \frac{x^2+4x+3}{x^2-x-2}$$

$\frac{x+3}{x-2}$ and $\frac{x^2+4x+3}{x^2-x-2}$ are equivalent fractions for $x \neq 2$ and $x \neq -1$, and you multiplied by 1 in the form $\frac{x+1}{x+1}$.

Finding Non-Permissible Values of Equivalent Rational Expressions

The non-permissible values of equivalent rational expressions are always determined from the polynomial with more factors in its denominator. When you are creating equivalent rational expressions by multiplying the numerator and denominator by the same polynomial, the non-permissible values are found by equating the denominator of your new rational expression to zero.

Example 4

Create equivalent rational expressions for each of the following rational expressions. State the non-permissible values of the new equivalent rational expression.

a) $\dfrac{x+6}{x+2}$

b) $\dfrac{x^2+3}{x-7}$

Solution

a) To create an equivalent rational expression for $\dfrac{x+6}{x+2}$, you can multiply the numerator and denominator by any polynomial, say $(x + 8)$.

$$\frac{x+6}{x+2} = \frac{(x+6)(x+8)}{(x+2)(x+8)} = \frac{x^2+14x+48}{x^2+10x+16}$$

The non-permissible values of this new rational expression are determined where its denominator equals zero.

$x + 2 = 0$ $\qquad$ $x + 8 = 0$

$x = -2$ $\qquad$ $x = -8$

Therefore, the non-permissible values of the rational expression are -2 and -8.

There are other possible answers as well.

b) To create an equivalent rational expression equal to, $\dfrac{x^2+3}{x-7}$, you can multiply the numerator and denominator by any polynomial, say $(x^2 + 1)$.

$$\frac{x^2+3}{x-7} = \frac{\left(x^2+3\right)\left(x^2+1\right)}{(x-7)\left(x^2+1\right)} = \frac{x^4+4x^2+3}{x^3-7x^2+x-7}$$

The non-permissible value of this new rational expression is determined where its denominator equals zero.

$x - 7 = 0$

$x = 7$

Note: $x^2 + 1$ never equals zero for real numbers.

The non-permissible value is $x = 7$.

Finding Errors in Simplifications of Rational Expressions

Many students find simplifying rational expressions requires greater concentration. Because of this, there are mistakes that students could make. Being aware of these mistakes makes you less likely to make them in the future. Consider the following examples.

Example 5

Explain why the simplification of the following rational expressions is incorrect.

a) $\dfrac{4-\cancel{8}x}{\cancel{8}} = \dfrac{4-x}{1} = 4-x$

b) $\dfrac{\cancel{(x+1)}+6}{3-\cancel{(x+1)}} = \dfrac{6}{3} = 2$

Solution

a) It is only possible to cancel out factors of a rational expression. The value "8" is not a factor of the rational expression as it only occurs in one term in the numerator. Therefore, it is not possible to cancel out this value. However, it is possible to first factor the numerator and denominator and then cancel out common factors.

$$\frac{4-8x}{8} = \frac{\cancel{4}(1-2x)}{\cancel{4}(2)} = \frac{1-2x}{2}$$

b) Again, it is only possible to cancel out factors of a rational expression. Since $(x + 1)$ does not appear in both terms in the numerator and the denominator, the expression needs to be factored before factors of the numerator and denominator can be cancelled out. However, there are no common factors between the denominator and the numerator. Thus, this rational expression cannot be reduced any further. It can be simplified by first removing the brackets.

$$\frac{(x+1)+6}{3-(x+1)} = \frac{x+1+6}{3-x-1} = \frac{x+7}{2-x}$$

The following learning activity will allow you to practice factoring and simplifying rational expressions. Be sure to complete this learning activity, as it will allow you to know whether or not you have learned the material. If you check your answers in the answer key and discover you have answered many questions incorrectly, you should go back and relearn the material in the lesson.

Learning Activity 2.3

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Factor: $x^2 - 5x - 6$
2. Find the common ratio in the following pattern: 8, 16, 32, 64, . . .
3. The air you breathe is a mixture of different gases including nitrogen, carbon dioxide, oxygen, and argon. If the air is 78.1% nitrogen, 0.1% carbon dioxide, 20.9% oxygen, and 0.7% argon, what percentage of the air is composed of other gases?
4. State the reciprocal of $\frac{4}{3}$.
5. Your grandma is making a turkey dinner for the family and needs you to figure out when to start putting everything in the oven. If the turkey takes five hours to cook, the ham takes three hours to cook, and the meatballs take one hour to cook (they cannot go in the oven at the same time), what time does your grandma need to start putting food in the oven to have dinner ready by 2:00 pm?

For questions 6, 7, and 8, consider the following values:
3, 4, 4, 4, 6, 7, 9, 10, 16, 17.

6. Find the mode.
7. Find the mean.
8. Find the median.

continued

Part B: Equivalent Forms of Rational Expressions

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. State the domain and the non-permissible values of each rational expression.

   a) $\dfrac{(x+3)(x-2)}{(x+2)(x-2)}$

   b) $\dfrac{x(x-1)}{(x+2)}$

   c) $\dfrac{x(x-2)}{(x+3)(x-3)}$

   d) $\dfrac{x(x-1)(x+2)}{(x-1)(x+1)(x+2)}$

   e) $\dfrac{x(x-1)}{(x-1)x}$

2. Simplify each rational expression. State the non-permissible values of x for each expression. Your final answer may be in factored form.

   a) $\dfrac{2x^2-18}{x^2-x-12}$

   b) $\dfrac{x^3-2x^2-15x}{x^3-25x}$

   c) $\dfrac{2x^3+8x}{2x^2}$

   d) $\dfrac{3x+6}{x^2-4}$

   e) $\dfrac{2x-2}{1-x}$

   f) $\dfrac{100x^3y^2}{25x^3y^3}$

continued

Learning Activity 2.3 (continued)

3. Is the following method an acceptable way to reduce fractions? Explain.

   a) $\dfrac{\cancel{5}}{x+\cancel{5}} = \dfrac{1}{x+1}$

   b) $\dfrac{\cancel{6}x^3 - 24\cancel{x}}{\cancel{6x}} = x^3 - 24$

   c) $\dfrac{5+10x}{5x} = \dfrac{5(1+2x)}{(5)(x)} = \dfrac{1+2x}{x}$

4. Write the polynomial A if $\dfrac{A}{x^2-9}$ reduced to $\dfrac{x}{x-3}$.

5. Explain why 7 may not be the only non-permissible value for the following rational expression: $\dfrac{x}{x-7}$.

6. Create a rational expression that has two non-permissible values: $x = -1$ and $x = 5$.

7. The distance, d, can be found using the formula $d = rt$, where r is the speed and t is the time.

   a) If the distance is represented by $2x^2 + 3x - 2$ and the speed is represented by $4x + 8$, what is an expression for the time? Simplify your answer.

   b) State any non-permissible values of the expression in part (a) above.

8. Create equivalent rational expressions for each of the following rational expressions by multiplying by a form of 1. State the non-permissible values for each rational expression.

   a) $\dfrac{x+1}{x-2}$

   b) $\dfrac{x-7}{x^2}$

Lesson Summary

In this lesson, you learned how factoring is related to simplifying, or reducing, rational expressions. You also discovered how to find the non-permissible values of rational expressions. Multiplying the numerator and denominator by the same polynomial to create equivalent rational expressions was another mathematical skill that you learned.

In addition, you related rational expressions to rational numbers, or fractions. Just as it is possible to multiply and divide rational numbers, it is also possible to multiply and divide rational expressions. In the next lesson, you will learn how to multiply and divide rational expressions using similar rules to those you would use when multiplying and dividing fractions.

Lesson 4: Multiplication and Division of Rational Expressions

Lesson Focus

In this lesson, you will

- ❑ learn how to find and simplify the product of rational expressions
- ❑ learn how to find and simplify the quotient of rational expressions

Lesson Introduction

In the last lesson, you learned how to simplify rational expressions. This knowledge becomes very useful when you are asked to multiply and divide rational expressions. These expressions can sometimes become very complex. However, knowing how to simplify these expressions makes multiplication and division much easier.

Just as simplifying rational expressions is related to simplifying fractions, multiplying and dividing rational expressions is similar to multiplying and dividing fractions. In this lesson, you will learn about these similarities. These similarities make multiplying and dividing rational expressions similar to calculations you have completed before.

Remember, multiplying or dividing fractions does **not** require finding a common denominator.

Operations on Rational Expressions

You know how to simplify products of rational numbers. For example,

$$\frac{4}{9}\times\frac{3}{7}=\frac{(2)(2)}{(3)(3)}\times\frac{3}{7}=\frac{4}{3(7)}=\frac{4}{21}$$

To simplify products of rational numbers, you use the following steps:

1. Factor the numerators and denominators
2. Cancel out factors common to the numerator and denominator
3. Multiply the numerators
4. Multiply the denominators

Example 1

Find the product of $\frac{4}{3} \times \frac{9}{2}$.

Solution

$$\frac{4}{3} \times \frac{9}{2} = \frac{2(2)}{3} \times \frac{3(3)}{2}$$

Now, in the expression $\frac{2(2)}{3} \times \frac{3(3)}{2}$, 2 is common to both the numerator and denominator. Therefore, you are able to cancel out a factor of 2.

$$\frac{2\cancel{(2)}}{3} \times \frac{3(3)}{\cancel{2}}$$

Also, 3 is common to both the numerator and denominator. Thus, you can cancel out a factor of 3 as well.

$$\frac{2\cancel{(2)}}{\cancel{3}} \times \frac{3\cancel{(3)}}{\cancel{2}}$$

The new fraction thus becomes:

$$\frac{2(3)}{1} = 6$$

Another way to calculate the answer is to multiply first, then cancel:

$$\frac{4}{3} \times \frac{9}{2} = \frac{36}{6} = 6$$

However, with algebraic fractions, this method becomes too cumbersome. It is easier to factor first, then cancel and write your answer.

Multiplying Rational Expressions

Multiplying rational expressions is identical to multiplying fractions. The following are the steps to multiplying rational expressions:

1. Factor the numerators and denominators.
2. Identify non-permissible values.
3. Cancel out factors that are common to both the numerator and denominator.
4. Multiply the numerators. Leave in factored form.
5. Multiply the denominators. Leave in factored form.

It may be helpful for you to include these steps on your resource sheet.

Example 2

Simplify the following rational expressions. Identify the non-permissible values.

a) $\dfrac{x^2-4}{x+3} \times \dfrac{x^2+6x+9}{x+2}$

b) $\dfrac{3x^2}{5y^3} \times \dfrac{10y}{9x^2}$

c) $\dfrac{x-1}{x^2-4} \times \dfrac{x^2-x-6}{2x^2-2}$

Solution

a) $\dfrac{x^2-4}{x+3} \times \dfrac{x^2+6x+9}{x+2}$

Factoring:

$$= \frac{(x-2)(x+2)}{(x+3)} \times \frac{(x+3)(x+3)}{(x+2)}$$

The non-permissible values are $x = -3$ and $x = -2$.

Cancelling out like terms:

$$= \frac{(x-2)\cancel{(x+2)}}{\cancel{(x+3)}} \times \frac{\cancel{(x+3)}(x+3)}{\cancel{(x+2)}}$$

Multiplying the numerators and denominators to get the answer:

$= (x-2)(x+3), x \neq -3, x \neq -2$

Notice the product may be left in factored form.

b) $\dfrac{3x^2}{5y^3} \times \dfrac{10y}{9x^2} = \dfrac{\cancel{3}\,\cancel{(x)}\,\cancel{(x)}}{\cancel{5}\,(y)(y)\,\cancel{(y)}} \times \dfrac{2\,\cancel{(5)}\,\cancel{(y)}}{\cancel{3}\,(3)\,\cancel{(x)}\,\cancel{(x)}} = \dfrac{2}{3y^2}$

The non-permissible values are $x = 0$ and $y = 0$.

c) $\dfrac{x-1}{x^2-4} \times \dfrac{x^2-x-6}{2x^2-2} = \dfrac{\cancel{x-1}}{(x-2)\,\cancel{(x+2)}} \times \dfrac{(x-3)\,\cancel{(x+2)}}{2\,\cancel{(x-1)}\,(x+1)} = \dfrac{x-3}{2(x-2)(x+1)}$

The non-permissible values are $x = 2, -2, 1$, and -1.

Dividing Rational Expressions

Just as you know how to simplify the product of rational numbers, you can simplify quotients of rational numbers. Consider the following example.

Example 3

Find the quotient for $\frac{4}{3} \div \frac{6}{5}$.

Solution

$$\frac{4}{3} \div \frac{6}{5} = \frac{4}{3} \times \frac{5}{6} = \frac{\cancel{2}(2)}{3} \times \frac{5}{\cancel{2}(3)} = \frac{2(5)}{3(3)} = \frac{10}{9}$$

To divide fractions, you use the following steps:

1. Multiply by the reciprocal of the second fraction.
2. Factor the numerators and denominators.
3. Cancel out factors common to the numerator and denominator.
4. Multiply the numerators.
5. Multiply the denominators.

Finding the quotient of rational expressions is similar. The following are the steps for dividing rational expressions:

1. Write the reciprocal of the second fraction and change the division to multiplication.
2. Factor the numerators and denominators.
3. Identify non-permissible values.
4. Cancel out factors that are common to both the numerator and the denominator.
5. Multiply the numerators. Leave in factored form.
6. Multiply the denominators. Leave in factored form.

Note: Because rational expressions involve non-permissible values, there is an extra step or two when performing operations with rational expressions.

It may be helpful for you to include these steps on your resource sheet.

Example 4

Simplify the following rational expressions. Identify the non-permissible values.

a) $\dfrac{x^2-4}{x+3} \div \dfrac{x+2}{x^2-9}$

b) $\dfrac{x+y}{x^2} \div \dfrac{x^2-y^2}{4x}$

c) $\dfrac{x^2+3x}{x^2+x-6} \div \dfrac{x^2-2x}{x-4}$

d) $\dfrac{\dfrac{x^2-36}{x^2-49}}{\dfrac{x^2-12x+36}{x^2+14x+49}}$

Solution

a) $\dfrac{x^2-4}{x+3} \div \dfrac{x+2}{x^2-9}$

Multiplying by the reciprocal of the second fraction:

$$\frac{x^2-4}{x+3} \times \frac{x^2-9}{x+2}$$

Factoring:

$$\frac{(x-2)(x+2)}{(x+3)} \times \frac{(x-3)(x+3)}{(x+2)}$$

The non-permissible values are $x = -3$, 3, and -2.

Cancelling out common factors:

$$\frac{(x-2)\cancel{(x+2)}}{\cancel{(x+3)}} \times \frac{(x-3)\cancel{(x+3)}}{\cancel{(x+2)}}$$

Multiplying the numerators and denominators:

$$\frac{(x-2)(x-3)}{1} = (x-2)(x-3),\ x \neq -3,\ 3,\ -2$$

Note: In the original expression, $x^2 - 9$ is in the denominator of the second fraction. Therefore, $x^2 - 9$, or $(x - 3)(x + 3)$ cannot equal zero. The non-permissible values thus include $x = -3$ and $x = 3$. Thus, there are three places in the original expression where you need to check for non-permissible values: both denominators and the numerator of the fraction after the division sign.

b) $$\frac{x+y}{x^2} \div \frac{x^2-y^2}{4x} = \frac{x+y}{x^2} \times \frac{4x}{x^2-y^2} = \frac{\cancel{x+y}}{(x)\cancel{(x)}} \times \frac{4\cancel{x}}{(x-y)\cancel{(x+y)}} = \frac{4}{x(x-y)}$$

The non-permissible values of x are $x = 0$, y, and $-y$, as shown below.

Note: $(x - y)$ and $(x + y)$ are factors in the denominator. Therefore, they cannot equal zero.

$$x - y = 0 \qquad x + y = 0$$
$$x = y \qquad x = -y$$

Thus, $-y$ and y become non-permissible values of x.

c) $$\frac{x^2+3x}{x^2+x-6} \div \frac{x^2-2x}{x-4}$$

$$= \frac{x^2+3x}{x^2+x-6} \times \frac{x-4}{x^2-2x} = \frac{\cancel{x}\,\cancel{(x+3)}}{\cancel{(x+3)}\,(x-2)} \times \frac{x-4}{\cancel{x}\,(x-2)} = \frac{x-4}{(x-2)(x-2)}$$

$$= \frac{x-4}{(x-2)^2}$$

The non-permissible values are $x = -3$, $x = 2$, $x = 0$, and $x = 4$.

d) $$\frac{\dfrac{x^2-36}{x^2-49}}{\dfrac{x^2-12x+36}{x^2+14x+49}} = \frac{x^2-36}{x^2-49} \div \frac{x^2-12x+36}{x^2+14x+49}$$

$$= \frac{x^2-36}{x^2-49} \times \frac{x^2+14x+49}{x^2-12x+36}$$

$$= \frac{\cancel{(x-6)}\,(x+6)}{(x-7)\,\cancel{(x+7)}} \times \frac{\cancel{(x+7)}\,(x+7)}{\cancel{(x-6)}\,(x-6)}$$

$$= \frac{(x+6)(x+7)}{(x-7)(x-6)}$$

The non-permissible values are $x = 7$, -7, and 6.

Example 5

For what value(s) of x is the following expression not defined? Simplify the expression.

$$\frac{x}{x^2-25} \times \frac{2x+10}{x+2} \div \frac{5x}{x-8}$$

Solution

$$\frac{x}{x^2-25} \times \frac{2x+10}{x+2} \div \frac{5x}{x-8}$$

$$= \frac{\cancel{x}}{(x-5)\cancel{(x+5)}} \times \frac{2\cancel{(x+5)}}{x+2} \times \frac{x-8}{5\cancel{x}} = \frac{2(x-8)}{5(x-5)(x+2)}$$

Since no denominators can have a value of 0, it follows that the non-permissible values are $x = 5, -5, -2, 0$, and 8, since $x - 8$ is a denominator in the original expression, and x is a denominator in the multiplication step.

Make sure you complete the following learning activity and check your answers in the Module 2 Learning Activity Answer Keys. If you do well on this learning activity, you are likely to do well on Assignment 2.2, as similar questions will appear there.

Learning Activity 2.4

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. What is the perimeter of a regular hexagon if one side measures 4 cm?
2. You and your friends want to drive to the beach but you want to know how far away it is. The map you are using has a scale of 1 cm = 100 km. On the map, if the beach is 1.4 cm away from your house, how many kilometres are you from the beach?
3. Calculate: $\frac{8}{9} - \frac{1}{5}$
4. Factor: $2x + 4xy$
5. Simplify: $(4^2)^2$
6. Solve the inequality $10x + 4 \leq 3x - 1$.
7. List the factors of 180.
8. What is the sum of the reciprocals of the factors of 4?

continued

Learning Activity 2.4 (continued)

Part B: Multiplying and Dividing Rational Expressions

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Perform the indicated operation. Identify any non-permissible values.

a) $\dfrac{x+1}{x} \times \dfrac{x^2}{x^2-1}$

b) $\dfrac{x-2}{x} \div \dfrac{x+2}{x^2}$

c) $\dfrac{x^2+2x+1}{5-x} \times \dfrac{x^2-25}{x^2+6x+5}$

d) $\dfrac{3x^2+10x+3}{x^2-9} \div \dfrac{3x+1}{x-3}$

e) $\dfrac{\dfrac{3}{2}}{\dfrac{}{x}}$

f) $\dfrac{\dfrac{2x+6}{x+1}}{\dfrac{x+3}{x^2-1}}$

g) $\dfrac{x^2}{y} \div \dfrac{x}{y^2}$

h) $\dfrac{x^2-9}{3x^2+12} \times \dfrac{x^2+4}{x+3}$

i) $\dfrac{x^4-y^4}{x^2-y^2} \div \dfrac{x^2-x-6}{x+2}$

j) $\dfrac{2x-6}{x^2y} \div \dfrac{3x-9}{xy}$

k) $\dfrac{x^2+2x-8}{x^2-16} \times \dfrac{x^2-8x+16}{x^2-6x+8}$

l) $\dfrac{5a-a^2}{a^2-4a+3} \times \dfrac{a^2-5a+4}{a^2-9a+20} \div \dfrac{a^2-7a}{a^2-10a+21}$

m) $\dfrac{m^2-m-20}{m^2-25} \div \dfrac{m^2+2m-8}{m^2-m-2} \div \dfrac{m^2+5m+4}{m^2+5m}$

continued

Learning Activity 2.4 (continued)

2. Stephanie was asked to simplify the following expression completely. Her work is shown below.

$$\frac{4x+16}{3x^2} \div \frac{5x+20}{4x}$$

$$= \frac{4x+16}{3x^2} \times \frac{5x+20}{4x}$$

$$= \frac{\cancel{4}(x+4)}{3x^2} \times \frac{5(x+4)}{\cancel{4}x}$$

$$= \frac{5(x+4)(x+4)}{3x^2}$$

Is Stephanie's work correct? Explain why or why not. If Stephanie's answer is incorrect, find the correct answer.

3. Charles' Law relates the temperature and volume of one gas to the temperature and volume of another gas at a constant pressure according to the formula:

$$\frac{V_1}{V_2} = \frac{T_1}{T_2}$$

If $V_1 = \frac{2x^2 + x - 10}{x^2 + x - 6}$, $V_2 = \frac{x(x-1)}{x^2 - 1}$, and $T_2 = \frac{x^3 - 9x}{2x^2 - x - 15}$, find T_1.

Express your answer in simplest form.

Lesson Summary

In this lesson, you learned how you can multiply and divide rational expressions using similar rules to those you use to multiply and divide fractions. You also used your factoring skills to simplify these new expressions. It is also possible to add and subtract rational expressions, which you will be learning about in the next lesson.

Assignment 2.2

Multiplying, Dividing, and Simplifying Rational Expressions

Total: 51 marks

1. Simplify the following rational expressions. State the non-permissible values. *(3 marks each, for a total of 18 marks)*

   a) $\dfrac{(x+2)(x-3)}{3-x}$

   b) $\dfrac{x(x+5)(x-5)}{(5-x)(5+x)}$

   c) $\dfrac{x^2-4}{x^2+4x+4}$

continued

Assignment 2.2: Multiplying, Dividing, and Simplifying Rational Expressions (continued)

d) $\dfrac{4-x}{4x^2-64}$

e) $\dfrac{1-x^2}{x^2-1}$

f) $\dfrac{x^2-6x-7}{x^2-9x+14}$

continued

Assignment 2.2: Multiplying, Dividing, and Simplifying Rational Expressions (continued)

2. Are the following simplifications of rational expressions correct? Explain why or why not. *(2 marks each, for a total of 4 marks)*

   a) $\dfrac{x+\cancel{7}}{\cancel{7}} = x+1$

   b) $\dfrac{x-1}{1-x} = 1$

3. Create equivalent rational expressions for each of the following expressions. Your answer may be left in factored form. *(1 mark each, for a total of 2 marks)*

   a) $\dfrac{x-1}{x^2+5}$

   b) $\dfrac{3}{(x+7)(x-2)}$

continued

Assignment 2.2: Multiplying, Dividing, and Simplifying Rational Expressions (continued)

4. Create two different rational expressions with non-permissible values of $x = -1$ and $x = 9$. *(4 marks)*

5. Perform the indicated operation. Identify any non-permissible values.

   a) $\dfrac{x^2 + 3x + 2}{x + 1} \times \dfrac{x + 3}{x^2 + 5x + 6}$ *(4 marks)*

   b) $\dfrac{x^2 - 7x + 10}{x^2 - 2x - 15} \times \dfrac{x^2 - 25}{x^2 - 4} \times \dfrac{x + 2}{x + 5}$ *(6 marks)*

continued

Assignment 2.2: Multiplying, Dividing, and Simplifying Rational Expressions (continued)

c) $\dfrac{4x^2 - 9}{12x + 18} \div \dfrac{4x - 6}{6x + 12}$ *(5 marks)*

d) $\dfrac{x^2 - 5x - 14}{x^2 - 49} \times \dfrac{x^2 + 7x}{5}$ *(3 marks)*

e) $\dfrac{x^2}{x - 3} \div \dfrac{x}{x^2 - 4x + 3}$ *(3 marks)*

continued

Assignment 2.2: Multiplying, Dividing, and Simplifying Rational Expressions (continued)

6. Jeremiah was asked to simplify the following rational expression completely and state the non-permissible values. His work is shown below.

$$\frac{x-x^2}{10x+8} \div \frac{x^2-2x+1}{5x^2+4x}$$

$$= \frac{x-x^2}{10x+8} \times \frac{5x^2+4x}{x^2-2x+1}$$

$$= \frac{x(1-x)}{2(5x+4)} \times \frac{x(5x+4)}{(x-1)(x-1)}$$

$$= \frac{(-x)\cancel{(x-1)}}{2\cancel{(5x+4)}} \times \frac{\cancel{x(5x+4)}}{\cancel{(x-1)}(x-1)}$$

$$= \frac{-x^2}{2(x-1)}$$

The non-permissible values are $x = 1, \frac{-4}{5}$.

Is Jeremiah's work correct? Explain why or why not. If Jeremiah's work is incorrect, find the correct answer. *(2 marks)*

Lesson 5: Addition and Subtraction of Rational Expressions

Lesson Focus

In this lesson, you will

- ❑ learn how determine the sum or difference of two or more rational expressions with the same denominator
- ❑ learn how to determine the sum or difference of two or more rational expressions with different denominators

Lesson Introduction

In the last lesson, you learned how to multiply and divide rational expressions. In this lesson, you will learn how to add and subtract rational expressions. Adding and subtracting rational expressions is similar to adding and subtracting fractions. As with fractions, you need to have the same denominator before you can add or subtract rational expressions. This makes adding and subtracting rational expressions more complicated than just multiplying or dividing them.

Operations on Rational Expressions

When you are adding and subtracting fractions, there are two situations that can occur. The first situation involves fractions with the same denominator. If this occurs, you can just add or subtract the numerators and keep the denominator the same. The second situation involves fractions with different denominators. In this situation you need to first find a common denominator before you add or subtract. The same is true for adding and subtracting rational expressions.

Adding and Subtracting Rational Expressions with the Same Denominator

Recall how to add and subtract fractions with the same denominator.

$$\frac{2}{3}+\frac{5}{3}=\frac{2+5}{3}=\frac{7}{3} \text{ or}$$

$$\frac{2}{3}-\frac{5}{3}=\frac{2-5}{3}=\frac{-3}{3}=-1$$

If the rational expressions contain the same denominator, you can add or subtract the numerators the same way you would if you were dealing with fractions. Consider the steps for adding or subtracting rational expressions that have a common denominator.

Steps for adding or subtracting rational expressions that have a common denominator:

1. Add or subtract the terms in the numerators.
2. Keep the denominator the same.
3. Reduce the resulting expression, if possible, by factoring.

Include these steps on your resource sheet.

Example 1

Perform the operation and simplify your answer. From now on, you do not need to write the non-permissible values unless you are asked to do so.

a) $\frac{x+3}{2}+\frac{x+1}{2}$

b) $\frac{x+3}{2}-\frac{x+1}{2}$

c) $\frac{x-1}{x+1}+\frac{3x+1}{x+1}$

d) $\frac{x^2+6x-7}{x^2}-\frac{3x-8}{x^2}$

Solution

a) $\frac{x+3}{2}+\frac{x+1}{2}$

$$=\frac{(x+3)+(x+1)}{2}=\frac{x+3+x+1}{2}=\frac{2x+4}{2}=\frac{2(x+2)}{2}=x+2$$

b) $\dfrac{x+3}{2} - \dfrac{x+1}{2}$

$= \dfrac{(x+3)-(x+1)}{2} = \dfrac{x+3-x-1}{2} = \dfrac{2}{2} = 1$

c) $\dfrac{x-1}{x+1} + \dfrac{3x+1}{x+1}$

$= \dfrac{(x-1)+(3x+1)}{x+1} = \dfrac{x-1+3x+1}{x+1} = \dfrac{4x}{x+1}$

d) $\dfrac{x^2+6x-7}{x^2} - \dfrac{3x-8}{x^2}$

$= \dfrac{\left(x^2+6x-7\right)-(3x-8)}{x^2} = \dfrac{x^2+6x-7-3x+8}{x^2} = \dfrac{x^2+3x+1}{x^2}$

Adding and Subtracting Rational Expressions with Different Denominators

When rational expressions do not have the same denominator, adding or subtracting them means you need to do an extra step. Before you can add or subtract these rational expressions, you need to find the lowest common denominator (LCD) for both rational expressions and then find equivalent fractions using the LCD.

Finding the Lowest Common Denominator (LCD)

Steps for finding equivalent rational expressions having a specified LCD:

1. Factor each denominator completely.
2. Form the least common multiple of these denominators in factored form. This is your LCD.
3. Complete the following steps for each rational expression to find the equivalent fraction.
   - Find the factors that do not occur in the original denominator but appear in the LCD.
   - Multiply the numerator and denominator by these factors to create an equivalent rational expression.

Note: You should now have equivalent rational expressions all with the same common denominator.

Write these steps down on your resource sheet. The following examples will help clarify how to find the lowest common denominator for rational expressions by finding the least common multiple of the factored denominators.

Example 2

Find the least common multiple of the following polynomials.

a) x and $(x - 4)$

b) x^2 and $2x$

c) $(x^2 - 9)$ and $(x - 3)$

d) $(x^2 - 4x + 4)$ and $(x^2 - 4)$

Solution

You first need to factor each of the polynomials. Remember, the lowest common multiple (LCM) needs to be common. You need to include each of the factors at least once and you need to use each factor at most the greatest number of times it occurs in any of the polynomials. The LCM is the product of the required number of factors.

	Polynomials	**Factored Polynomials**	**Maximum Occurrence in Any One Polynomial**	**Least Common Multiple (LCM)**
a)	x and $(x - 4)$	x and $(x - 4)$	x once $x - 4$ once	$x(x - 4)$
b)	x^2 and $2x$	xx and $2x$	x twice 2 once	$2xx$ or, simply $2x^2$
c)	$(x^2 - 9)$ and $(x - 3)$	$(x - 3)(x + 3)$ and $(x - 3)$	$x - 3$ once $x + 3$ once	$(x - 3)(x + 3)$
d)	$(x^2 - 4x + 4)$ and $(x^2 - 4)$	$(x - 2)(x - 2)$ and $(x - 2)(x + 2)$	$x - 2$ twice $x + 2$ once	$(x - 2)(x - 2)(x + 2)$ or, simply $(x - 2)^2(x + 2)$

The **lowest common denominator (LCD)** of two or more rational expressions is the least common multiple (LCM) of the denominators of the rational expressions.

Example 3

Find the lowest common denominator for each set of fractions. Write equivalent fractions for each set.

a) $\frac{5}{4}, \frac{7}{12}$

b) $\frac{7}{36}, \frac{29}{27}$

Solution

a) First, you need to factor each denominator completely.

$4 = (2)(2)$

$12 = (3)(2)(2)$

To find the least common multiple, you need to find the greatest number of times each factor appears in a denominator. "2" appears twice in both denominators and "3" appears once in one denominator. Therefore, the least common multiple of these factors is (2)(2)(3) = 12. This means that 12 is the smallest value that can be used for the denominator in order to make equivalent fractions for $\frac{5}{4}$ and $\frac{7}{12}$ and 12 is called the LCD.

Therefore, each fraction needs to have a denominator of 12.

Now, for each fraction, you need to find the factors that do not occur in the original denominator but appear in the LCD.

Consider $\frac{5}{4}$.

The factor that does not occur in 4 but occurs in the LCD of 12 is 3.

Therefore, $\frac{5}{4}$ needs to be multiplied by $\frac{3}{3}$.

$$\frac{5}{4} \times \frac{3}{3} = \frac{(5)(3)}{(4)(3)} = \frac{15}{12}$$

Now, equivalent fractions having the same LCD of 12 are $\frac{15}{12}$ and $\frac{7}{12}$.

b) First, you need to factor each denominator completely.

$36 = (2)(2)(3)(3)$

$27 = (3)(3)(3)$

The least common multiple of these factors is $(2)(2)(3)(3)(3) = 108$.

Notice that to find the least common multiple, you first need to find the greatest number of times each factor occurs in each denominator. In this example, "3" occurs only twice in 36, but 3 times in 27. Therefore, (3)(3)(3) is necessary in the least common multiple. Also, "2" occurs twice in 36 but not at all in 27. Therefore, (2)(2) are necessary factors of the least common multiple. The least common multiple is $(3)(3)(3)(2)(2) = 108$.

Thus, the LCD is 108.

Now find the equivalent fractions.

Consider $\frac{7}{36}$.

The factor that does not occur in 36 but occurs in the LCD of 108 is 3.

Therefore, $\frac{7}{36}$ needs to be multiplied by $\frac{3}{3}$.

$$\frac{7}{36} \times \frac{3}{3} = \frac{(7)(3)}{(36)(3)} = \frac{21}{108}$$

Consider $\frac{29}{27}$.

The factors that do not occur in 27 but occur in the LCD of 108 are 2 and 2, or $(2)(2) = 4$. Therefore, $\frac{29}{27}$ needs to be multiplied by $\frac{4}{4}$.

$$\frac{29}{27} \times \frac{4}{4} = \frac{(29)(4)}{(27)(4)} = \frac{116}{108}$$

Now, the equivalent fractions written with a common denominator of 108 are $\frac{21}{108}$ and $\frac{116}{108}$.

Example 4

For each set of rational expressions, find their lowest common denominator. Then write equivalent fractions using the LCD.

a) $\dfrac{x+1}{(x-1)(x+1)}, \dfrac{x}{(x-2)(x-1)}$

b) $\dfrac{x-2}{(x+3)^2(x-7)}, \dfrac{x-1}{x+3}$

Solution

a) The first step is to factor each denominator completely. This is already completed. Now you can form the least common multiple of these factors.

The least common multiple = $(x-1)(x+1)(x-2)$. This is the LCD for the set of fractions.

Note: $(x-1)$ only appears once in the LCD because it only occurs once in each denominator.

Now write equivalent fractions using the LCD.

Consider. $\dfrac{x+1}{(x-1)(x+1)}$.

The factor that does not occur in $(x-1)(x+1)$ but appears in the LCD is $(x-2)$. Therefore, $\dfrac{x+1}{(x-1)(x+1)}$ needs to be multiplied by $\dfrac{x-2}{x-2}$.

$$\frac{x+1}{(x-1)(x+1)} \times \frac{x-2}{x-2} = \frac{(x+1)(x-2)}{(x-1)(x+1)(x-2)}$$

Consider $\dfrac{x}{(x-2)(x-1)}$.

The factor that does not occur in $(x-2)(x-1)$ but appears in the LCD is $(x+1)$. Therefore, $\dfrac{x}{(x-2)(x-1)}$ needs to be multiplied by $\dfrac{x+1}{x+1}$.

$$\frac{x}{(x-2)(x-1)} \times \frac{x+1}{x+1} = \frac{(x)(x+1)}{(x-2)(x-1)(x+1)}$$

Thus, equivalent fractions having the same denominator are

$\dfrac{(x+1)(x-2)}{(x-1)(x+1)(x-2)}$ and $\dfrac{(x)(x+1)}{(x-2)(x-1)(x+1)}$.

b) Both of the denominators have already been factored. Therefore, you can find the LCD right away.

LCD = $(x + 3)^2(x - 7)$

The first rational expression already has the LCD as its denominator. Therefore, only the second rational expression needs to be changed into an equivalent fraction. The two factors present in the LCD that do not occur in the denominator of the second rational expression are $(x + 3)$ and $(x - 7)$.

$$\frac{x-1}{x+3} = \frac{(x-1)}{(x+3)} \times \frac{(x+3)(x-7)}{(x+3)(x-7)} = \frac{(x-1)(x+3)(x-7)}{(x+3)^2(x-7)}$$

Now, equivalent fractions having the same denominator are $\frac{x-2}{(x+3)^2(x-7)}$ and $\frac{(x-1)(x+3)(x-7)}{(x+3)^2(x-7)}$.

Now that you know how to change rational expressions into equivalent fractions having a common denominator, you can learn how to add or subtract rational expressions that do not have a common denominator.

Steps for adding or subtracting rational expressions that do not have a common denominator:

1. Factor each denominator.
2. Find the lowest common denominator (LCD).
3. Find equivalent rational expressions all having the same LCD.
4. Add or subtract the numerators.
5. Keep the denominator the same.
6. Reduce the resulting expression, if possible, by factoring.

Include these steps on your resource sheet.

Consider the following example that compares finding equivalent fractions to finding equivalent rational expressions.

In Arithmetic

$$\frac{5}{6}+\frac{3}{20}$$

$$=\frac{5}{2(3)}+\frac{3}{2(2)(5)}$$

$$\left[\begin{array}{l}\text{LCD}=2(3)(2)(5)\\ \text{Note: } \frac{3}{20} \text{ requires}\\ \text{two factors of 2.}\end{array}\right]$$

$$=\frac{5}{2(3)}\cdot\frac{2(5)}{2(5)}+\frac{3}{2(2)(5)}\cdot\frac{3}{3}$$

$$=\frac{50+9}{60}$$

$$=\frac{59}{60}$$

In Algebra

$$\frac{2}{x^2-1}+\frac{3x}{x^2+2x+1}$$

$$=\frac{2}{(x-1)(x+1)}+\frac{3x}{(x+1)(x+1)}$$

$$\left[\begin{array}{l}\text{LCD}=(x-1)(x+1)(x+1)\\ \text{Note: } \frac{3x}{x^2+2x+1} \text{ requires}\\ \text{two factors of } (x+1).\end{array}\right]$$

$$=\frac{2}{(x-1)(x+1)}\cdot\frac{x+1}{x+1}+$$

$$\frac{3x}{(x+1)(x+1)}\cdot\frac{x-1}{x-1}$$

$$=\frac{2(x+1)+3x(x-1)}{(x-1)(x+1)(x+1)}$$

$$=\frac{2x+2+3x^2-3x}{(x-1)(x+1)(x+1)}$$

$$=\frac{3x^2-x+2}{(x-1)(x+1)(x+1)}$$

Example 5

Perform the operation and simplify your answer. Indicate the non-permissible values for each solution.

a) $\frac{2}{3x}+\frac{5}{2x}$

b) $\frac{2}{x^2-x}+\frac{3}{x^2-1}$

c) $\frac{3}{x+3}+\frac{2}{x-3}-\frac{4x}{x^2-9}$

d) $\frac{2}{x^2-4}+\frac{1}{2x-x^2}$

Solution

a) $\dfrac{2}{3x} + \dfrac{5}{2x}$

$\text{LCD} = (3)(2)(x) = 6x$

$= \left(\dfrac{2}{3x} \times \dfrac{2}{2}\right) + \left(\dfrac{5}{2x} \times \dfrac{3}{3}\right)$

$= \dfrac{4}{6x} + \dfrac{15}{6x}$

$= \dfrac{19}{6x}$

The non-permissible value is $x = 0$.

b) $\dfrac{2}{x^2 - x} + \dfrac{3}{x^2 - 1}$

$= \dfrac{2}{x(x-1)} + \dfrac{3}{(x-1)(x+1)}$

$\therefore \text{LCD} = x(x-1)(x+1)$

$\therefore \dfrac{2}{x^2 - x} + \dfrac{3}{x^2 - 1} = \dfrac{2}{x(x-1)} + \dfrac{3}{(x-1)(x+1)}$

$= \dfrac{2}{x(x-1)} \cdot \dfrac{x+1}{x+1} + \dfrac{3}{(x-1)(x+1)} \cdot \dfrac{x}{x}$

$= \dfrac{2(x+1) + 3x}{x(x-1)(x+1)}$

$= \dfrac{2x + 2 + 3x}{x(x-1)(x+1)}$

$= \dfrac{5x + 2}{x(x-1)(x+1)}$

The non-permissible values are $x = 0$, 1, and -1.

c) $\dfrac{3}{x+3}+\dfrac{2}{x-3}-\dfrac{4x}{x^2-9}$

$=\dfrac{3}{x+3}+\dfrac{2}{x-3}-\dfrac{4x}{(x-3)(x+3)}$

$\therefore \text{LCD} = (x+3)(x-3)$

$\dfrac{3}{x+3}+\dfrac{2}{x-3}-\dfrac{4x}{x^2-9}$

$=\dfrac{3}{x+3}\cdot\dfrac{x-3}{x-3}+\dfrac{2}{x-3}\cdot\dfrac{x+3}{x+3}-\dfrac{4x}{(x-3)(x+3)}$

$=\dfrac{3(x-3)+2(x+3)-4x}{(x+3)(x-3)}$

$=\dfrac{3x-9+2x+6-4x}{(x+3)(x-3)}$

$=\dfrac{\cancel{x-3}}{(x+3)\cancel{(x-3)}}$

$=\dfrac{1}{x+3}$

The non-permissible values are $x = 3$ and -3.

d) $\dfrac{2}{x^2-4}+\dfrac{1}{2x-x^2}$

$=\dfrac{2}{(x-2)(x+2)}+\dfrac{1}{-x(x-2)}$

$=\dfrac{2}{(x-2)(x+2)}-\dfrac{1}{x(x-2)}$

$\therefore \text{LCD}=(x-2)(x+2)x$

$\dfrac{2}{x^2-4}+\dfrac{1}{2x-x^2}$

$=\dfrac{2}{(x-2)(x+2)}\cdot\dfrac{x}{x}-\dfrac{1}{x(x-2)}\cdot\dfrac{x+2}{x+2}$

$=\dfrac{2x-(x+2)}{x(x-2)(x+2)}$

$=\dfrac{2x-x-2}{x(x-2)(x+2)}$

$=\dfrac{\cancel{x-2}}{x\cancel{(x-2)}(x+2)}$

$=\dfrac{1}{x(x+2)}$

The non-permissible values are $x = 0$, 2, and -2.

Learning Activity 2.5

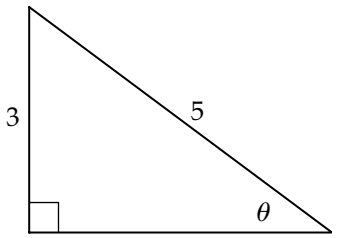
Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. What are two numbers that sum to 8 and multiply to -65?
2. Camille has $200 to buy a new outfit. She wants to buy a sweater for $55.95, a shirt for $44.99, jeans for $49.99, and new shoes for $54.99. Ignoring tax, does Camille have enough money to buy her outfit?
3. Evaluate: $\left(\frac{3}{7}\right) \times \left(\frac{14}{6}\right)$
4. On Monday it rained 5 mm, Tuesday it rained 1.3 cm, Wednesday it rained 4 mm, and on Thursday it rained 1.7 cm. How many millimetres did it rain in total?
5. Factor: $2x^2 - x - 6$

Use the following right-angled triangle to answer the next three questions.



6. Identify the sine ratio for θ.
7. Identify the cosine ratio for θ.
8. Identify the tangent ratio for θ.

continued

Learning Activity 2.5 (continued)

Part B: Operations on Rational Expressions

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Add and subtract as indicated.

   a) $\frac{x}{4} + \frac{5x}{4}$

   b) $\frac{5x}{2} - \frac{x}{2}$

   c) $\frac{5a}{a+b} + \frac{5b}{a+b}$

   d) $\frac{3d}{c+d} - \frac{d}{c+d}$

   e) $\frac{2a^2 + 4a + 17}{(a+2)(a+3)} - \frac{a^2 - 4a + 2}{(a+2)(a+3)}$

2. Find the lowest common denominator for each of the following sets of rational expressions.

   a) $\frac{4}{x-8}, \frac{3x}{2x-16}, \frac{4x}{x+1}$

   b) $\frac{x+1}{(x+2)(x-3)}, \frac{x-1}{(x-2)(x-6)(x-3)^2}$

continued

Learning Activity 2.5 (continued)

3. Add and subtract as indicated. Identify the non-permissible values.

   a) $\dfrac{4x}{x^2+x-6} - \dfrac{3x}{x^2+5x+6}$

   b) $\dfrac{3}{x^2-4x+3} + \dfrac{4}{x^2-5x+6}$

   c) $\dfrac{2x}{x^2-6x-7} - \dfrac{3}{x+1} + \dfrac{3}{x-7}$

   d) $\dfrac{2x}{x^2+5xy} - \dfrac{x}{x^2+8xy+15y^2}$

   e) $\dfrac{y-4}{y-3} - \dfrac{y-7}{y+8} + \dfrac{y+8}{y^2+5y-24}$

   f) $\dfrac{7}{y+2} - \dfrac{5}{2-y} + \dfrac{y+18}{y^2-4}$

   g) $\dfrac{2x}{x-2} + \dfrac{2-3x}{x^2-x-2} - \dfrac{x+2}{x+1}$

4. Simplify the following rational expressions.

   a) $\dfrac{\dfrac{x(x+2)}{5} - \dfrac{x}{2(x-1)}}{\dfrac{5}{x+2} + \dfrac{7}{x-1}}$

   b) $\dfrac{\dfrac{1}{x+2} + \dfrac{1}{x-2}}{\dfrac{x}{x^2-4} + \dfrac{1}{x+2}}$

   c) $\dfrac{x}{2x-4} + \left(\dfrac{x-3}{x^2-3x+2} \div \dfrac{x^2-9}{x-1}\right)$

continued

Learning Activity 2.5 (continued)

5. Kahlia is trying to simplify a rational expression. However, the answer she received does not match the answer at the back of the textbook. Kahlia's work is shown below.

$$\frac{x-1}{(x+2)(x-3)} - \frac{2x}{(x+2)}$$

$$= \frac{x-1-2x}{(x+2)(x-3)}$$

$$= \frac{-x-1}{(x+2)(x-3)}$$

   Explain Kahlia's mistake and simplify the rational expression correctly.

6. A swimmer swam 50 laps at a constant speed. For the remaining 50 laps, he reduced his speed by 3 km/h. Write a rational expression for the total time of this swimmer's workout. You do not need to simplify your answer. **Note:** distance = (speed)(time), or $d = rt$.

7. A rational expression puzzle is shown on the following page. Work with your learning partner to cut out each puzzle piece and reassemble them all in a 4×4 configuration so that each expression matches its factored form or its simplified form. If you are stuck, you can look in the Module 2 Answer Key to find helpful hints.

Lesson Summary

In this lesson, you learned how to add and subtract rational expressions with the same denominator and with different denominators. As with fractions, rational expressions need to have the same denominator before you can add or subtract them. Therefore, you learned how to find the lowest common denominator for rational expressions.

The following Assignment 2.3 will test your knowledge of material covered in this lesson. Make sure you complete the preceding learning activity and check your answers in the Module 2 Answer Key before you complete this assignment. This is the last assignment in Module 2.

Rational Expression Puzzle

$(x+7)(x+2)$ **A** $4x^2-7x+3$	$\frac{x^2-x-2}{x^2-4}$ **B** $(x+2)(x+1)$	$3(3x-5)(6x-13)$ **C** $2(x-6)^2+10(x-6)-48$	$2(x+2)(x-9)$ **D** $x+3$
$1-100x^2$ **I** $(10x-1)(10x+1)$	$\frac{x+1}{8x}\times\frac{2x}{x^2-1}$ **J** $\frac{x}{2}$	$(4x-3)(x-1)$ **F** $(4x+3)(x+1)$	$\frac{x}{x-3}+\frac{2}{x+4}$ **G** $25x^2+20x+4$
x^2-36 **E** $9(3x-4)^2-21(3x-4)+15$	$3x^2(2x+1)$ **K** $\frac{5}{x}+\frac{4}{x+1}$	$(x+5)(x-5)$ **L** $\frac{x}{2}+\frac{x}{4}$	$\frac{5}{x}+\frac{4}{x+1}$ **H** $\frac{3}{x}-\frac{x}{2}$
$x+1$ **M** $(x+3)^2-6(x+3)-16$	$(x+3)(x-1)$ **N** $(x-6)(x+6)$	$(2x+9)(6x+13)$ **O** $(3x+4)^2-(2y-1)^2$	$\frac{a}{2c}$ **P** $3(2x+5)^2+10(2x+5)-8$
x^2+25 **Q** $\frac{(x+2)}{(x+1)}$	$\frac{x-3}{x}$ **R** $\frac{(x+2)}{(x+3)}$	$\frac{x}{x+7}-\frac{5x}{x+2}$ **S** $2y(3xy-1)$	$100x^2-1$ **T** $6xy^2-2y$
$(x-3)(x+1)$ **U** $\frac{1}{4(x-1)}$	$\frac{3x}{4}$ **V** $\frac{x}{x-1}+\frac{3}{(x+2)(x+3)}$	$\frac{-x(4x+33)}{(x+7)(x+2)}$ **W** x^2+2x-3	$\frac{5x}{x+7}+\frac{x}{x+2}$ **X** $6x^3+3x^2$
$\frac{x^2-4}{(5x+2)^2}$ **Y** $\frac{(x-3)}{(x+1)}$	$\frac{a^2b^2c}{8m}\div\frac{ab^2c^2}{4m}$ **Z** $\frac{2x+5}{2}$	$\frac{1}{4(x+1)}$ **AA** $\frac{x^2+7x+12x}{x^2+4x}$	$\frac{3x+6}{2x^2-x-10}\times\frac{4x^2-25}{6}$ **BB** $\frac{5x^2+13x+18}{(x-1)(x+2)(x+3)}$
$\frac{1}{4(x-1)}$ **CC** $\frac{a}{4c}$	$x-2$ **DD** $\frac{x}{x-3}+\frac{2}{x+4}$	$\frac{3}{x-1}+\frac{2x}{(x+2)(x+3)}$ **EE** $\frac{x^2+5x+6}{x^2+x-2}\div\frac{x+3}{x^2-1}$	$x+3$ **FF** $\frac{a^2b^2c}{8m}\div\frac{ab^2c^2}{4m}$
$\frac{x-10}{x(x-2)}$ **GG** $\frac{2x+5}{2}$	$\frac{5}{x}-\frac{4}{x-2}$ **HH** $\frac{3}{x}-\frac{x}{2}$	$\frac{x}{2}+\frac{x}{4}$ **II** $\frac{3x}{4}$	$\frac{a}{2c}$ **JJ** $\frac{x^2+7x+12x}{x^2+4x}$
$(x+3)(x-1)$ **KK** $\frac{x^2-36}{(x-3)(x+4)}$	$\frac{(x+1)}{(x+2)}$ **LL** $\frac{x-3}{x}$	$\frac{x}{x+7}-\frac{5x}{x+2}$ **MM** $\frac{(x+3)}{(x+1)}$	$\frac{x^2-4}{(5x+2)^2}$ **NN** $6xy^2-2y$

Assignment 2.3

Adding and Subtracting Rational Expressions

Total: 30 marks

1. Add and subtract as indicated. Identify the non-permissible values. *(3 marks each, for a total of 12 marks)*

   a) $\dfrac{2}{x-1} - \dfrac{3}{x^2+x-2}$

   b) $\dfrac{2}{x^2-4} + \dfrac{3}{x^2+3x-10}$

continued

Assignment 2.3: Adding and Subtracting Rational Expressions (continued)

c) $\dfrac{2}{x-5} - \dfrac{x+1}{x^2-7x+10}$

d) $\dfrac{2x^2}{x-5} - \dfrac{-5x+3}{5-x}$

continued

Assignment 2.3: Adding and Subtracting Rational Expressions (continued)

2. Find the lowest common denominator for each of the following sets of rational expressions. *(2 marks each, for a total of 4 marks)*

   a) $\frac{5}{x^2 - 36}, \frac{5x + 1}{x^2 - 6x}$

   b) $\frac{2x + 1}{x^2 + 6x + 8}, \frac{2}{x^2 - x - 20}$

continued

Assignment 2.3: Adding and Subtracting Rational Expressions (continued)

3. Simplify the following rational expressions. *(4 marks each, for a total of 8 marks)*

a) $$\frac{\left(\frac{x+2}{x^2+3x+2}\right)\left(\frac{x+1}{x+5}\right)}{\frac{x+5}{x^2}}$$

b) $$\frac{x}{x+7}+\left(\frac{x^2+3x-40}{x^2+2x-35}\right)\left(\frac{x^2+3x-18}{x^2+2x-48}\right)$$

continued

Assignment 2.3: *Adding and Subtracting Rational Expressions (continued)*

4. Kenny is trying to simplify the following expression. His work is shown below.

$$\frac{5x+2}{(x+1)(x+2)} - \frac{10x^2-4x+4}{(x+1)(x+2)}$$

$$= \frac{5x+2-10x^2-4x+4}{(x+1)(x+2)}$$

$$= \frac{-10x^2+x+6}{(x+1)(x+2)}$$

Is Kenny's work correct? Explain why or why not. Correct Kenny's work if his work is incorrect. *(2 marks)*

continued

Assignment 2.3: Adding and Subtracting Rational Expressions (continued)

5 A triathlete is preparing to compete in a triathlon. During her workout, she can swim 1 km at a constant speed. After swimming, she can bike for 10 km at 15 km/h faster than she can swim. After biking, she can run for 5 km at 5 km/h faster than she can swim. Write a rational expression for the total time of this triathlete's workout. Do not simplify your answer. Recall: $d = rt$. *(4 marks)*

Notes

Module 2 Summary

In this module, you studied factoring and rational expressions. You reviewed your previous knowledge of factoring and extended this knowledge to factor more complex polynomials. You then related factoring to rational expressions. You learned how rational expressions are similar to fractions. You can add, subtract, multiply, divide, and simplify rational expressions in similar ways to those you would use to add, subtract, multiply, divide, and simplify fractions. Having this knowledge of factoring and rational expressions will be very useful in Module 3, where you will be learning about quadratic functions. Quadratic equations can sometimes be solved by factoring. You will learn how to do this in Module 4. However, before you learn how to solve quadratic equations by factoring, you need to understand what a quadratic function is and the patterns these functions follow.

Submitting Your Assignments

It is now time for you to submit Assignments 2.1 to 2.3 to the Distance Learning Unit so that you can receive some feedback on how you are doing in this course. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 2 assignments and organize your material in the following order:

- ❑ Module 2 Cover Sheet (found at the end of the course Introduction)
- ❑ Assignment 2.1: Factoring
- ❑ Assignment 2.2: Multiplying, Dividing, and Simplifying Rational Expressions
- ❑ Assignment 2.3: Adding and Subtracting Rational Expressions

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Notes

Grade 11 Pre-Calculus Mathematics (30S)

Module 2
Factoring and Rational Expressions

Learning Activity Answer Keys

MODULE 2:
FACTORING AND RATIONAL EXPRESSIONS

Learning Activity 2.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Carla has a shoe addiction. She owns 13 pairs of boots, 18 pairs of sneakers, 23 pairs of high heels, and 19 pairs of sandals. How many pairs of shoes does she have in total?
2. What are two numbers that sum to 11 and multiply to 30?
3. Multiply: $(2x + 1)(x - 5)$
4. Evaluate: $\dfrac{\frac{9}{21}}{\frac{3}{63}}$
5. Factor: $x^2 + 7x + 12$
6. How many metres are in 2.3 km?
7. Simplify: $(2x^2 - 6x + 11) - (-14x^2 - 2x + 4)$
8. Is xy^2 the greatest common factor of $6x^2y^2 + 8xy^3$?

Answers:

1. 73 (13 + 18 + 23 + 19)
2. 5 and 6
3. $2x^2 - 9x - 5$
4. $9 \left(\dfrac{\frac{9}{21}}{\frac{3}{63}} = \left(\frac{9}{21}\right)\left(\frac{63}{3}\right) = \left(\frac{3}{7}\right)\left(\frac{21}{1}\right) = \frac{63}{7} \right)$
5. $(x + 3)(x + 4)$
6. 2300 metres (2.3×1000)
7. $16x^2 - 4x + 7$ ($2x^2 - 6x + 11 + 14x^2 + 2x - 4$)
8. No, the greatest common factor is $2xy^2$. ($6x^2y^2 + 8xy^3 = 2xy^2(3x + 4y)$)

Part B: Factoring Review

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Factor out the greatest common factor.
   a) $5x + 20$
   b) $5x^3 + 20x^2$
   c) $30x^2y - 24xy^3$
   d) $x^3 + x^2$

   Answers:

   a) $5(x + 4)$
   b) $5x^2(x + 4)$
   c) $6xy(5x - 4y^2)$
   d) $x^2(x + 1)$

2. Factor each trinomial completely.
   a) $x^2 + 7x - 8-$
   b) $x^2 - 2x - 8$
   c) $x^2 - 11x + 28$
   d) $3x^3 + 30x^2 + 63x$
   e) $2x^2 + 7x - 15$
   f) $4x^2 - 12x + 9$

   Answers:

   a) $(x + 8)(x - 1)$
   b) $(x - 4)(x + 2)$
   c) $(x - 7)(x - 4)$
   d) First factor out the GCF. Then factor the remaining trinomial.
      $3x(x^2 + 10x + 21) = 3x(x + 7)(x + 3)$
   e) $2x^2 - 3x + 10x - 15$
      $= x(2x - 3) + 5(2x - 3)$
      $= (2x - 3)(x + 5)$

f) $4x^2 - 6x - 6x + 9$

$= 2x(2x - 3) - 3(2x - 3)$

$= (2x - 3)(2x - 3)$

$= (2x - 3)^2$

g) $\frac{2}{3}x^2 - 4x + 6$

$= \frac{2}{3}\left(x^2 - 6x + 9\right)$

$= \frac{2}{3}(x - 3)(x - 3)$

$= \frac{2}{3}(x - 3)^2$

3. Factor each binomial completely.

   a) $x^2 - 100$

   b) $9x^2 - 49$

   c) $9x^2 - 16y^2$

   d) $2x^2 - 50y^2z^2$

   Answers:

   a) $(x - 10)(x + 10)$

   b) $(3x - 7)(3x + 7)$

   c) $(3x - 4y)(3x + 4y)$

   d) First factor out the GCF. Then factor the remaining binomial as a difference of squares.

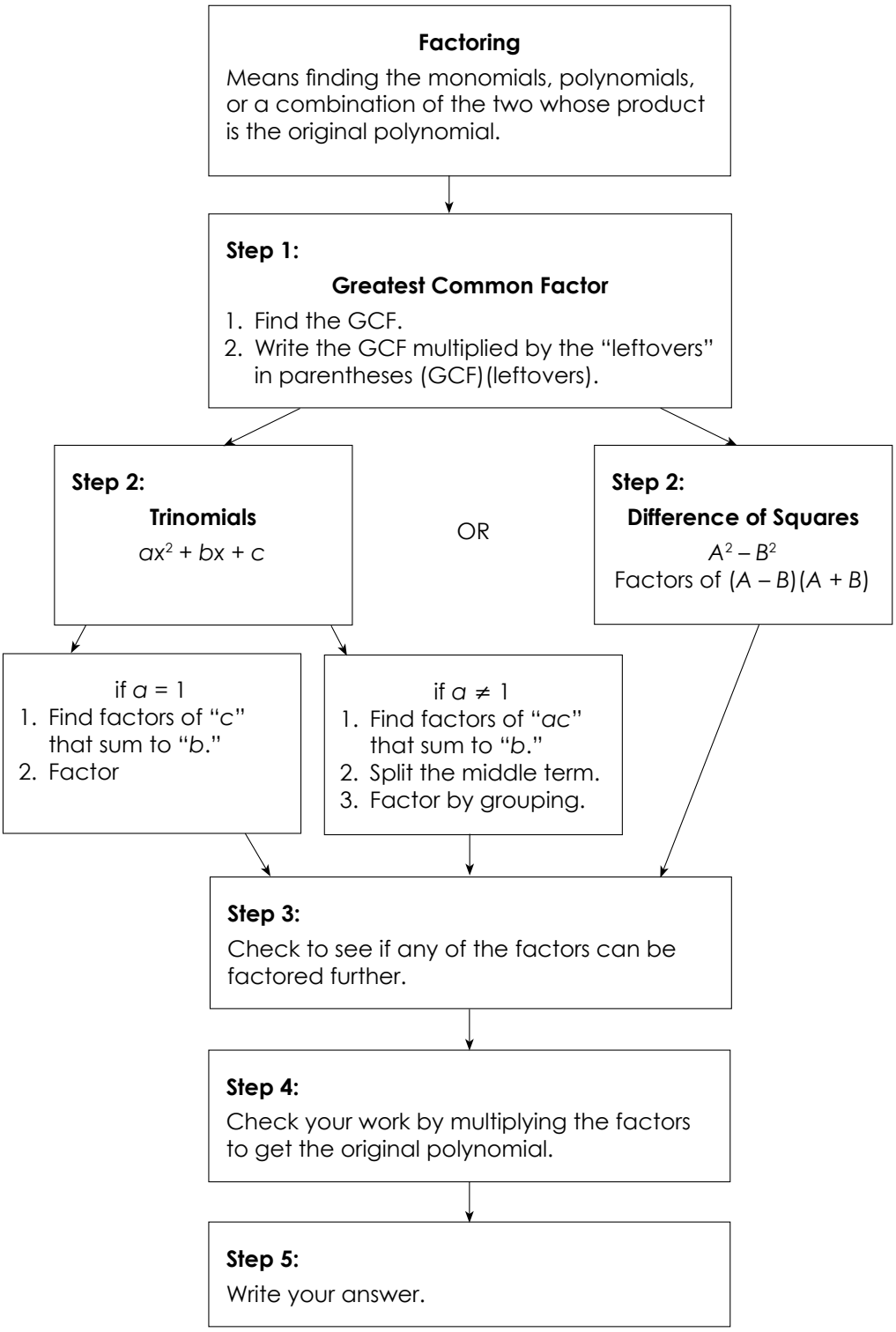
   $= 2(x^2 - 25y^2z^2)$

   $= 2(x - 5yz)(x + 5yz)$

4. Create a graphic organizer of the steps to take when factoring a polynomial. Include descriptions for how to complete each step.

   Answer:

   See the following page for one example of the graphic organizer.


Factoring
Means finding the monomials, polynomials, or a combination of the two whose product is the original polynomial.
Step 1:
Greatest Common Factor
1. Find the GCF.
2. Write the GCF multiplied by the "leftovers" in parentheses (GCF)(leftovers).
Step 2:
Trinomials
$ax^2 + bx + c$
OR
Step 2:
Difference of Squares
$A^2 - B^2$
Factors of $(A - B)(A + B)$
if $a = 1$
1. Find factors of "c" that sum to "b."
2. Factor
if $a \neq 1$
1. Find factors of "ac" that sum to "b."
2. Split the middle term.
3. Factor by grouping.
Step 3:
Check to see if any of the factors can be factored further.
Step 4:
Check your work by multiplying the factors to get the original polynomial.
Step 5:
Write your answer.

Learning Activity 2.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. There are a group of 80 people in a shareholder's meeting making a decision about whether or not to acquire a new company. The decision needs to pass with an 80% majority. How many people must agree in order for the decision to pass?
2. Evaluate: $5 \times 17 \times 20$
3. Katrina is flying from Winnipeg to Belize. She leaves Winnipeg at 7:00 am and takes a 3-hour flight to Minneapolis where she has a two and a half hour layover. From Minneapolis, she takes a direct flight to Belize that lasts for 7.25 hours. What time does Katrina arrive in Belize?
4. What is the greatest common factor of 48 and 144?
5. Convert 6.3 kilometres to metres.
6. Simplify: $\frac{2^8}{4^2}$
7. Find the slope of the line parallel to the line, $-2 = 4x + 6y$.
8. Determine the total surface area of a cube with a volume of 8 cm^3.

Answers:

1. 64 people (50% of 80 = 40; 10% of 80 = 8; 80% of 80 = 40 + 8 + 8 + 8)
2. 1700 ($5 \times 17 \times 20 = 5 \times 20 \times 17 = 100 \times 17$)
3. 7:45 pm (7 am + 3 hours = 10 am; 10 am + 2.5 hours = 12:30; 12:30 + 7.25 hours = 7:45 pm)
4. 48
5. 6300 metres (There are 1000 metres in 1 kilometre, so 6.3×1000)
6. 16 $\left(\frac{2^8}{\left(2^2\right)^2} = \frac{2^8}{2^4} = 2^4 \right)$
7. $-\frac{2}{3}$ $\left(y = -\frac{2}{3}x - \frac{1}{3}\left(6y = -4x - 2;\ y = -\frac{4}{6}x - \frac{2}{6} \right) \right)$

   Recall: Parallel lines have the same slope.

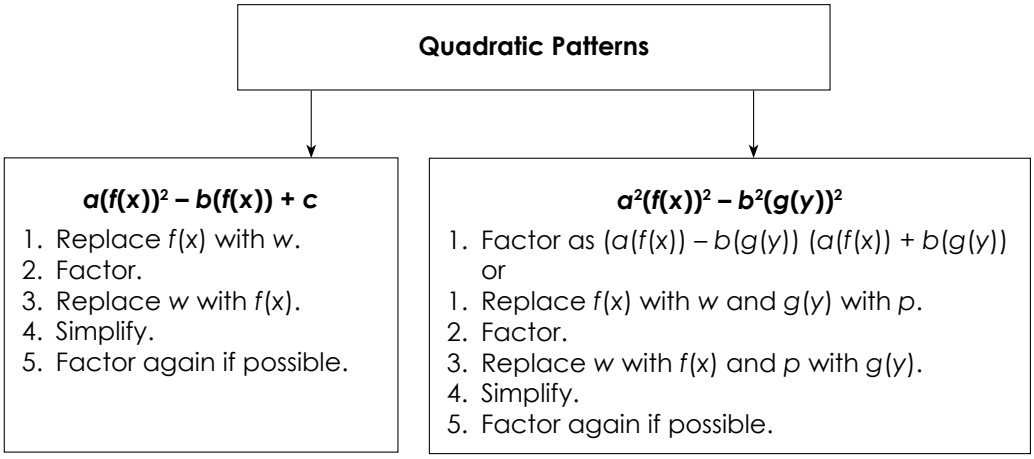
8. 8 cm^3. (The sides of a cube are equal in length. Therefore, each side length must be 2 cm. The total surface area is $(2 \times 2) \times 6 = 24$ cm^2, as each side is 2 cm × 2 cm and there are 6 equal sides on a cube.)

Part B: Factoring Expressions with Quadratic Patterns

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Add to the factoring graphic organizer you completed in the last learning activity (Learning Activity 2.1). Include all of the new techniques you learned about factoring in this lesson.

   Answer:



2. Factor completely.

a) $(7x - 6)^2 - 8(7x - 6) + 15$

Answer:

Replace $(7x - 6)$ by w:

$(w)^2 - 8(w) + 15$

Factor:

$(w - 5)(w - 3)$

Replace w with $(7x - 6)$:

$(7x - 6 - 5)(7x - 6 - 3)$

Simplify:

$(7x - 11)(7x - 9)$

b) $9(2x - 1)^2 - 42(2x - 1) + 49$

Answer:

Replace $(2x - 1)$ by w:

$9w^2 - 42w + 49$

Factor:

$9w^2 - 21w - 21w + 49$

$= 3w(3w - 7) - 7(3w - 7)$

$= (3w - 7)(3w - 7)$

$= (3w - 7)^2$

Replace w with $(2x - 1)$:

$[3(2x - 1) - 7]^2$

Simplify:

$(6x - 3 - 7)^2$

$= (6x - 10)^2$

$= (6x - 10)(6x - 10)$

$= 2(3x - 5)(2)(3x - 5)$

$= 4(3x - 5)^2$

c) $(x^2 - x)^2 - 7(x^2 - x) + 6$

Answer:

Replace $(x^2 - x)$ by w:

$w^2 - 7w + 6$

Factor:

$(w - 6)(w - 1)$

Replace w with $(x^2 - x)$:

$(x^2 - x - 6)(x^2 - x - 1)$

Factor again:

$(x - 3)(x + 2)(x^2 - x - 1)$

d) $(x - 1)^2 + 4(x - 1) - 32$

Answer:

Replace $(x - 1)$ by w:

$w^2 + 4w - 32$

Factor:

$(w + 8)(w - 4)$

Replace w with $(x - 1)$:

$(x - 1 + 8)(x - 1 - 4)$

Simplify:

$(x + 7)(x - 5)$

3. Factor each expression.

a) $50(2x - 7)^2 - 162(3y - 4)^2$

Answer:

$50(2x - 7)^2 - 162(3y - 4)^2$

Let $w = 2x - 7$

$p = 3y - 4$

$50(2x - 7)^2 - 162(3y - 4)^2$

$= 50w^2 - 162p^2$

$=2(25w^2 - 81p^2)$

$= 2(5w - 9p)(5w + 9p)$

Substitute:

$= 2[5(2x - 7) - 9(3y - 4)][5(2x - 7) + 9(3y - 4)$

$= 2[10x - 35 - 27y + 36][10x - 35 + 27y - 36]$

$= 2(10x - 27y + 1)(10x + 27y - 71)$

b) $25x^2 - 4y^2$

Answer:

$25x^2 - 4y^2$

$= (5x)^2 - (2y)^2$

$a(f(x)) = 5x$

$b(g(y)) = 2y$

$= (5x - 2y)(5x + 2y)$

c) $16(7x - 6)^2 - 121(4y - 7)^2$

Answer:

$16(7x - 6)^2 - 121(4y - 7)^2$

Let $w = 7x - 6$

$p = 4y - 7$

$16(7x - 6)^2 - 121(4y - 7)^2$

$= 16w^2 - 121p^2$

$= (4w - 11p)(4w + 11p)$

$= [4(7x - 6) - 11(4y - 7)][4(7x - 6) + 11(4y - 7)]$

$= [28x - 24 - 44y + 77][28x - 24 + 44y - 77]$

$= (28x - 44y + 53)(28x + 44y - 101)$

d) $144(x + 3)^4 - 729(y - 1)^4$

Answer:

$144(x + 3)^4 - 729(y - 1)^4$

Let $w = x + 3$

$p = y - 1$

$144(x + 3)^4 - 729(y - 1)^4$

$= 144w^4 - 729p^4$

$= 9[16w^4 - 81p^4]$

$= 9(4w^2 - 9p^2)(4w^2 + 9p^2)$

$= 9(2w - 3p)(2w + 3p)(4w^2 + 9p^2)$

$= 9[2(x + 3) - 3(y - 1)][2(x + 3) + 3(y - 1)][4(x + 3)^2 + 9(y - 1)^2]$

$= 9(2x + 6 - 3y + 3][2x + 6 + 3y - 3][4(x^2 + 6x + 9) + 9(y^2 - 2y + 1)]$

$= 9[2x - 3y + 9][2x + 3y + 3][4x^2 + 24x + 36 + 9y^2 - 18y + 9]$

$= 9(2x - 3y + 9)(2x + 3y + 3)(4x^2 + 9y^2 + 24x - 18y + 45)$

4. Determine if $(x - 2)$ is a factor of each of the following polynomials.

a) $8x^2 - 30x + 28$

Answer:

$x - 2 = 0$

$x = 2$

$8x^2 - 30x + 28$

$= 8(2)^2 - 30(2) + 28$

$= 32 - 60 + 28$

$= 0$

This evaluated polynomial equals zero. Therefore, $(x - 2)$ is a factor of this polynomial.

b) $(x - 2)^2 + 4(x - 2) - 32$

Answer:

$x - 2 = 0$

$x = 2$

$(x - 2)^2 + 4(x - 2) - 32$

$= (2 - 2)^2 + 4(2 - 2) - 32$

$= 0 + 0 - 32$

$= -32$

This evaluated polynomial does not equal zero. Therefore, $(x - 2)$ is not a factor of this polynomial.

c) $-15x^2 - 196x - 572$

Answer:

$x - 2 = 0$

$x = 2$

$-15x^2 - 196x - 572$

$= -15(2)^2 - 196(2) - 572$

$= -60 - 392 - 572$

$= -1024$

This evaluated polynomial does not equal zero. Therefore, $(x - 2)$ is not a factor of this polynomial.

5. Determine if $(7x - 6)$ is a factor of each of the following polynomials.

a) $(7x - 6)^2 - 8(7x - 6) + 15$

Answer:

$7x - 6 = 0$

$7x = 6$

$x = \frac{6}{7}$

$(7x-6)^2 - 8(7x-6) + 15$

$= \left(7 \cdot \left(\frac{6}{7}\right) - 6\right)^2 - 8\left(7 \cdot \left(\frac{6}{7}\right) - 6\right) + 15$

$= (6-6)^2 - 8(6-6)^2 + 15$

$= 0 - 0 + 15$

$= 15$

This evaluated polynomial does not equal zero. Therefore, $(7x - 6)$ is not a factor of this polynomial.

b) $7x^2 + 22x - 24$

Answer:

$7x - 6 = 0$

$7x = 6$

$x = \frac{6}{7}$

$7x^2 + 22x - 24$

$= 7\left(\frac{6}{7}\right)^2 + 22\left(\frac{6}{7}\right) - 24$

$= \frac{36}{7} + \frac{132}{7} - 24$

$= \frac{168}{7} - \frac{168}{7}$

$= 0$

This evaluated polynomial equals zero. Therefore, ($7x$ – 6) is a factor of this polynomial.

c) $16(7x - 6)^2 - 121(4x - 7)^2$

Answer:

$7x - 6 = 0$

$7x = 6$

$x = \frac{6}{7}$

For this question, you might use the calculator to manipulate the fractions:

$16(7x - 6)^2 - 121(4x - 7)^2$

$= 16\left[7\left(\frac{6}{7}\right) - 6\right]^2 - 121\left[4\left(\frac{6}{7}\right) - 7\right]^2$

$= \frac{-75\ 625}{49}$

This evaluated polynomial does not equal zero. Therefore, ($7x$ – 6) is not a factor of this polynomial.

d) $35x^2 - 58x + 24$

Answer:

$7x - 6 = 0$

$7x = 6$

$x = \frac{6}{7}$

$35x^2 - 58x + 24$

$= 35\left(\frac{6}{7}\right)^2 - 58\left(\frac{6}{7}\right) + 24$

$= 0$

This evaluated polynomial equals zero. Therefore, $(7x - 6)$ is a factor of this polynomial.

6. Janique was asked to factor the trinomial $4x^2 + 2x - 12$ completely. Her work is shown below.

$4x^2 + 2x - 12$

$= 4x^2 + 8x - 6x - 12$

$= 4x(x + 2) - 6(x + 2)$

$= (4x - 6)(x + 2)$

Determine if her work is correct, and explain why or why not.

Answer:

Janique's work is correct so far. However, she has not factored the polynomial completely. The expression $(4x - 6)$ has a common factor of 2 that Janique can factor out. Therefore, the completely factored expression is:

$4x^2 + 2x - 12 = 2(2x - 3)(x + 2)$

Also, Janique could have factored out the common factor of 2 at the beginning of her factoring steps.

$4x^2 + 2x - 12$

$2(2x^2 + x - 6)$

$2(2x^2 + 4x - 3x - 6)$

$2[2x(x + 2) - 3(x + 2)]$

$2[(x + 2)(2x - 3)]$

Learning Activity 2.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Factor: $x^2 - 5x - 6$
2. Find the common ratio in the following pattern: 8, 16, 32, 64, . . .
3. The air you breathe is a mixture of different gases including nitrogen, carbon dioxide, oxygen, and argon. If the air is 78.1% nitrogen, 0.1% carbon dioxide, 20.9% oxygen, and 0.7% argon, what percentage of the air is composed of other gases?
4. State the reciprocal of $\frac{4}{3}$.
5. Your grandma is making a turkey dinner for the family and needs you to figure out when to start putting everything in the oven. If the turkey takes five hours to cook, the ham takes three hours to cook, and the meatballs take one hour to cook (they cannot go in the oven at the same time), what time does your grandma need to start putting food in the oven to have dinner ready by 2:00 pm?

For questions 6, 7, and 8, consider the following values:

3, 4, 4, 4, 6, 7, 9, 10, 16, 17.

6. Find the mode.
7. Find the mean.
8. Find the median.

Answers:

1. $(x - 6)(x + 1)$
2. 2
3. 0.2% (100% − (78.1% + 0.1% + 20.9% + 0.7%) = 100% − 99.8%)
4. $\frac{3}{4}$
5. 5:00 am (You need to use the oven for 9 hours; 9 hours before 2 pm is 5 am.)

6. 4 (The mode is the most frequently occurring number.)

7. $8\left(\frac{3+4+4+4+6+7+9+10+16+17}{10}=\frac{80}{10}\right)$

   (Look for friendly pairs of numbers to add without a calculator.)

8. $6.5\ \left(\frac{6+7}{2}=\frac{13}{2}=6.5\right)$

   (Since there is an even number of numbers, the median will be the sum of the two middle values divided by two.)

Part B: Equivalent Forms of Rational Expressions

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. State the domain and the non-permissible values of each rational expression.

   a) $\frac{(x+3)(x-2)}{(x+2)(x-2)}$

   Answer:

   The non-permissible values are $x = -2$ and $x = 2$.

   The domain is $\{x \mid x \neq \pm 2, x \in \Re\}$.

   b) $\frac{x(x-1)}{(x+2)}$

   Answer:

   The non-permissible value is $x = -2$.

   The domain is $\{x \mid x \neq -2, x \in \Re\}$.

   c) $\frac{x(x-2)}{(x+3)(x-3)}$

   Answer:

   The non-permissible values are $x = -3$ and $x = 3$.

   The domain is $\{x \mid x \neq \pm 3, x \in \Re\}$.

d) $\dfrac{x(x-1)(x+2)}{(x-1)(x+1)(x+2)}$

Answer:

The non-permissible values are $x = -1$, $x = 1$, and $x = -2$.

The domain is $\{x \mid x \neq -1, 1, -2, x \in \Re\}$.

e) $\dfrac{x(x-1)}{(x-1)x}$

Answer:

The non-permissible values are $x = 1$ and $x = 0$.

The domain is $\{x \mid x \neq 1, 0, x \in \Re\}$.

2. Simplify each rational expression. State the non-permissible values of x for each expression. Your final answer may be in factored form.

a) $\dfrac{2x^2 - 18}{x^2 - x - 12}$

Answer:

$$\frac{2\left(x^2 - 9\right)}{(x-4)(x+3)} = \frac{2(x-3)(x+3)}{(x-4)(x+3)} = \frac{2(x-3)}{x-4}$$

The non-permissible values are $x = 4$ and $x = -3$.

b) $\dfrac{x^3 - 2x^2 - 15x}{x^3 - 25x}$

Answer:

$$\frac{x\left(x^2 - 2x - 15\right)}{x\left(x^2 - 25\right)} = \frac{x(x-5)(x+3)}{x(x-5)(x+5)} = \frac{x+3}{x+5}$$

The non-permissible values are $x = 0$, $x = -5$, and $x = 5$.

c) $\dfrac{2x^3 + 8x}{2x^2}$

Answer:

$$\frac{2x\left(x^2 + 4\right)}{2x^2} = \frac{x^2 + 4}{x}$$

The non-permissible value is $x = 0$.

d) $\dfrac{3x + 6}{x^2 - 4}$

Answer:

$$\frac{3(x + 2)}{(x - 2)(x + 2)} = \frac{3}{x - 2}$$

The non-permissible values are $x = -2$ and $x = 2$.

e) $\dfrac{2x - 2}{1 - x}$

Answer:

$$\frac{2(x - 1)}{-(x - 1)} = -2$$

The non-permissible value is $x = 1$.

f) $\dfrac{100x^3y^2}{25x^3y^3}$

Answer:

$$\frac{100x^3y^2}{25x^3y^3} = \frac{\cancel{(25)}\,(4)\,\cancel{(x)}\,\cancel{(x)}\,\cancel{(x)}\,\cancel{(y)}\,\cancel{(y)}}{\cancel{(25)}\,\cancel{(x)}\,\cancel{(x)}\,\cancel{(x)}\,\cancel{(y)}\,\cancel{(y)}\,(y)} = \frac{4}{y}$$

The non-permissible values are $x = 0$ and $y = 0$.

3. Is the following method an acceptable way to reduce fractions? Explain.

a) $\dfrac{\cancel{5}}{x+\cancel{5}} = \dfrac{1}{x+1}$

Answer:

No. The value "5" is not a factor of the denominator. Therefore, even though "5" is a factor of the numerator, it is not possible to cancel out a "5" from the numerator and denominator. The original expression cannot be reduced.

b) $\dfrac{\cancel{6}x^3 - 24\cancel{x}}{\cancel{6x}} = x^3 - 24$

Answer:

No. The value "6" is a factor of the numerator. However, you would also have to factor out a 6 from 24 before you cancel the factor "6" from the numerator and denominator. Again, x is a factor of the numerator and the denominator. However, you would have to factor out an x from $6x^3$ before you are able to cancel the factor "x" from both the numerator and the denominator. The correct answer is $\dfrac{6x\left(x^2-4\right)}{6x} = x^2 - 4$.

c) $\dfrac{5+10x}{5x} = \dfrac{5(1+2x)}{(5)(x)} = \dfrac{1+2x}{x}$

Answer:

Yes. This is a correct reduction as 5 is a factor of both the numerator and denominator and thus can be eliminated from the rational expression.

4. Write the polynomial A if $\frac{A}{x^2-9}$ reduced to $\frac{x}{x-3}$.

   Answer:

   If $\frac{A}{x^2-9}$ reduces to $\frac{x}{x-3}$, it follows that the rational expressions are equivalent.

   $$\frac{A}{x^2-9} = \frac{x}{x-3}$$

   $$\frac{A}{(x-3)(x+3)} = \frac{x}{x-3}$$

   Now solve for A.

   $$A = \frac{x(x-3)(x+3)}{(x-3)} = x(x+3),\ x \neq \pm 3$$

5. Explain why 7 may not be the only non-permissible value for the following rational expression: $\frac{x}{x-7}$.

   Answer:

   This rational expression may be the simplified form of another rational expression. Therefore, another non-permissible value from the original rational expression may have been factored out. For example, the original expression may have been $\frac{x}{x-7} \cdot \frac{x+3}{x+3} = \frac{x(x+3)}{(x-7)(x+3)}$, in which case $x = -3$ is also a non-permissible value. Thus, $x = 7$ may not be the only non-permissible value of this rational expression.

6. Create a rational expression that has two non-permissible values: $x = -1$ and $x = 5$.

   Answer:

   $(x + 1)$ and $(x - 5)$ must be factors of the denominator in order to have non-permissible values of $x = -1$ and $x = 5$. Therefore, any rational expression with $(x + 1)$ and $(x - 5)$ in the denominator would have non-permissible values of $x = -1$ and $x = 5$. An example is:

   $$\frac{(x+2)}{(x+1)(x-5)}$$

7. The distance, d, can be found using the formula $d = rt$, where r is the speed and t is the time.

   a) If the distance is represented by $2x^2 + 3x - 2$ and the speed is represented by $4x + 8$, what is an expression for the time? Simplify your answer.

   Answer:

   $d = rt$

   $2x^2 + 3x - 2 = (4x + 8)t$

   $\frac{2x^2 + 3x - 2}{4x + 8} = \frac{(4x + 8)t}{4x + 8}$

   $\frac{(2x - 1)\cancel{(x + 2)}}{4\cancel{(x + 2)}} = t$

   $t = \frac{2x - 1}{4}$

   b) State any non-permissible values of the expression in part (a) above.

   Answer:

   The non-permissible value is $x = -2$. Thus, the expression for time is written as $t = \frac{2x - 1}{4}$, $x \neq -2$.

8. Create equivalent rational expressions for each of the following rational expressions by multiplying by a form of 1. State the non-permissible values for each rational expression.

a) $\frac{x+1}{x-2}$

Answer:

Answers will vary. An example is:

$$\frac{x+1}{x-2} = \frac{(x+1)(x+1)}{(x-2)(x+1)} = \frac{x^2+2x+1}{x^2-x-2}$$

The non-permissible values are $x = 2$ and $x = -1$.

Note: $x = 2$ will be one of the non-permissible values no matter what equivalent expression you create. Note also that your answer may be left in factored form as $\frac{(x+1)(x+1)}{(x-2)(x+1)}$.

b) $\frac{x-7}{x^2}$

Answer:

Answers will vary. An example is:

$$\frac{x-7}{x^2} = \frac{(x-7)(x-5)}{x^2(x-5)} = \frac{x^2-12x+35}{x^3-5x^2}$$

The non-permissible values are $x = 0$ and $x = 5$.

Note: $x = 0$ will be one of the non-permissible values no matter what equivalent expression you create.

Learning Activity 2.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. What is the perimeter of a regular hexagon if one side measures 4 cm?
2. You and your friends want to drive to the beach but you want to know how far away it is. The map you are using has a scale of 1 cm = 100 km. On the map, if the beach is 1.4 cm away from your house, how many kilometres are you from the beach?
3. Calculate: $\frac{8}{9} - \frac{1}{5}$
4. Factor: $2x + 4xy$
5. Simplify: $(4^2)^2$
6. Solve the inequality $10x + 4 \leq 3x - 1$.
7. List the factors of 180.
8. What is the sum of the reciprocals of the factors of 4?

Answers:

1. 24 cm (6×4; a hexagon has 6 sides)
2. The beach is 140 km away ($1.4 \times 100 = 140$ km).
3. $\frac{31}{45}\left(\frac{40}{45} - \frac{9}{45}\right)$
4. $2x(1 + 2y)$
5. 256 [$(4^2)^2 = 4^4 = 256$ or $16^2 = 256$]
6. $x \leq -\frac{5}{7}$ $(7x \leq -5)$
7. The factors of 180 are 1, 2, 3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 30, 36, 45, 60, 90, and 180.
8. $\frac{7}{4}\left(\begin{array}{l}\text{The factors of 4 are 1, 2, and 4. Their reciprocals are 1, } \frac{1}{2}\text{, and } \frac{1}{4}. \\ \\ \text{The sum is } 1 + \frac{1}{2} + \frac{1}{4} = \frac{4}{4} + \frac{2}{4} + \frac{1}{4} = \frac{7}{4}.\end{array}\right)$

Part B: Multiplying and Dividing Rational Expressions

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Perform the indicated operation. Identify any non-permissible values.

a) $\frac{x+1}{x} \times \frac{x^2}{x^2-1}$

Answer:

$\frac{x+1}{x} \cdot \frac{x^2}{(x-1)(x+1)} = \frac{x}{x-1}$

$x \neq 0, 1, -1$

b) $\frac{x-2}{x} \div \frac{x+2}{x^2}$

Answer:

$\frac{x-2}{x} \cdot \frac{x^2}{x+2} = \frac{(x-2)x}{x+2}$

$x \neq 0, -2$

c) $\frac{x^2+2x+1}{5-x} \times \frac{x^2-25}{x^2+6x+5}$

Answer:

$\frac{(x+1)(x+1)}{(-1)(x-5)} \cdot \frac{(x-5)(x+5)}{(x+5)(x+1)}$

$= -(x+1)$ or $-x-1$

$x \neq 5, -5, -1$

d) $\dfrac{3x^2+10x+3}{x^2-9} \div \dfrac{3x+1}{x-3}$

Answer:

$\dfrac{(3x+1)(x+3)}{(x-3)(x+3)} \cdot \dfrac{x-3}{3x+1} = 1$

$x \neq 3, -3, -\dfrac{1}{3}$

e) $\dfrac{3}{\dfrac{2}{x}}$

Answer:

$3 \div \dfrac{2}{x}$

$= 3 \cdot \dfrac{x}{2} = \dfrac{3x}{2}$

$x \neq 0$

f) $\dfrac{\dfrac{2x+6}{x+1}}{\dfrac{x+3}{x^2-1}}$

Answer:

$\dfrac{2x+6}{x+1} \div \dfrac{x+3}{x^2-1}$

$= \dfrac{2(x+3)}{x+1} \cdot \dfrac{(x-1)(x+1)}{x+3} = 2(x-1)$

$x \neq -1, 1, -3$

g) $\dfrac{x^2}{y} \div \dfrac{x}{y^2}$

Answer:

$$\frac{x^2}{y} \cdot \frac{y^2}{x} = xy$$

$x \neq 0,\ y \neq 0$

h) $\dfrac{x^2 - 9}{3x^2 + 12} \times \dfrac{x^2 + 4}{x + 3}$

Answer:

$$\frac{(x-3)(x+3)}{3\left(x^2+4\right)} \cdot \frac{x^2+4}{x+3} = \frac{x-3}{3}$$

$x \neq -3$

i) $\dfrac{x^4 - y^4}{x^2 - y^2} \div \dfrac{x^2 - x - 6}{x + 2}$

Answer:

$$\frac{\left(x^2-y^2\right)\left(x^2+y^2\right)}{x^2-y^2} \cdot \frac{x+2}{(x-3)(x+2)}$$

$$= \frac{x^2+y^2}{x-3}$$

$x \neq 3, -2;\ x \neq \pm y$

j) $\dfrac{2x - 6}{x^2 y} \div \dfrac{3x - 9}{xy}$

Answer:

$$\frac{2(x-3)}{x^2 y} \cdot \frac{xy}{3(x-3)} = \frac{2}{3x}$$

$x \neq 0, 3;\ y \neq 0$

k) $\dfrac{x^2+2x-8}{x^2-16} \times \dfrac{x^2-8x+16}{x^2-6x+8}$

Answer:

$$\frac{(x+4)(x-2)}{(x-4)(x+4)} \cdot \frac{(x-4)(x-4)}{(x-4)(x-2)} = 1$$

$x \neq 4, -4, 2$

l) $\dfrac{5a-a^2}{a^2-4a+3} \times \dfrac{a^2-5a+4}{a^2-9a+20} \div \dfrac{a^2-7a}{a^2-10a+21}$

Answer:

$$\frac{(-a)(a-5)}{(a-3)(a-1)} \cdot \frac{(a-4)(a-1)}{(a-5)(a-4)} \cdot \frac{(a-3)(a-7)}{a(a-7)} = -1$$

$a \neq 3, 1, 5, 4, 0, 7$

m) $\dfrac{m^2-m-20}{m^2-25} \div \dfrac{m^2+2m-8}{m^2-m-2} \div \dfrac{m^2+5m+4}{m^2+5m}$

Answer:

$$\frac{(m-5)(m+4)}{(m-5)(m+5)} \cdot \frac{(m-2)(m+1)}{(m+4)(m-2)} \cdot \frac{m(m+5)}{(m+4)(m+1)} = \frac{m}{m+4}$$

$m \neq 5, -5, -4, 2, -1, 0$

2. Stephanie was asked to simplify the following expression completely. Her work is shown below.

$$\frac{4x+16}{3x^2} \div \frac{5x+20}{4x}$$

$$= \frac{4x+16}{3x^2} \times \frac{5x+20}{4x}$$

$$= \frac{\cancel{4}(x+4)}{3x^2} \times \frac{5(x+4)}{\cancel{4}x}$$

$$= \frac{5(x+4)(x+4)}{3x^2}$$

Is Stephanie's work correct? Explain why or why not. If Stephanie's answer is incorrect, find the correct answer.

Answer:

Stephanie's work is not correct. Stephanie forgot to write the reciprocal of the second fraction before she multiplied. She also did not find the non-permissible values and her answer should have a denominator of $3x^3$.

The correct solution is:

$$\frac{4x+16}{3x^2} \div \frac{5x+20}{4x}$$

$$= \frac{4x+16}{3x^2} \times \frac{4x}{5x+20}$$

$$= \frac{4\cancel{(x+4)}}{3(x)\cancel{(x)}} \times \frac{4\cancel{x}}{5\cancel{(x+4)}}$$

$$= \frac{4(4)}{3(5)(x)}$$

$$= \frac{16}{15x}, x \neq 0, -4$$

3. Charles' Law relates the temperature and volume of one gas to the temperature and volume of another gas at a constant pressure according to the formula:

$$\frac{V_1}{V_2} = \frac{T_1}{T_2}$$

If $V_1 = \dfrac{2x^2 + x - 10}{x^2 + x - 6}$, $V_2 = \dfrac{x(x-1)}{x^2 - 1}$, and $T_2 = \dfrac{x^3 - 9x}{2x^2 - x - 15}$, find T_1.

Express your answer in simplest form.

Answer:

$$\frac{\dfrac{2x^2 + x - 10}{x^2 + x - 6}}{\dfrac{x(x-1)}{x^2 - 1}} = \frac{T_1}{\dfrac{x^3 - 9x}{2x^2 - x - 15}}$$

Multiply both sides by $\dfrac{x^3 - 9x}{2x^2 - x - 15}$.

$$T_1 = \frac{2x^2 + x - 10}{x^2 + x - 6} \div \frac{x(x-1)}{x^2 - 1} \times \frac{x^3 - 9x}{2x^2 - x - 15}$$

$$T_1 = \frac{2x^2 + x - 10}{x^2 + x - 6} \times \frac{x^2 - 1}{x(x-1)} \times \frac{x^3 - 9x}{2x^2 - x - 15}$$

$$T_1 = \frac{(2x+5)(x-2)}{(x+3)(x-2)} \times \frac{(x-1)(x+1)}{x(x-1)} \times \frac{x(x-3)(x+3)}{(2x+5)(x-3)}$$

$$T_1 = x + 1,\ x \neq -3,\ 2,\ 0,\ 1,\ -\frac{5}{2},\ 3,\ -1$$

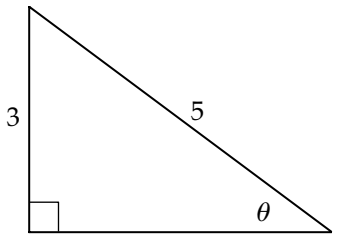
Learning Activity 2.5

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. What are two numbers that sum to 8 and multiply to –65?
2. Camille has $200 to buy a new outfit. She wants to buy a sweater for $55.95, a shirt for $44.99, jeans for $49.99, and new shoes for $54.99. Ignoring tax, does Camille have enough money to buy her outfit?
3. Evaluate: $\left(\frac{3}{7}\right) \times \left(\frac{14}{6}\right)$
4. On Monday it rained 5 mm, Tuesday it rained 1.3 cm, Wednesday it rained 4 mm, and on Thursday it rained 1.7 cm. How many millimetres did it rain in total?
5. Factor: $2x^2 - x - 6$

Use the following right-angled triangle to answer the next three questions.



6. Identify the sine ratio for θ.
7. Identify the cosine ratio for θ.
8. Identify the tangent ratio for θ.

Answers:

1. 13 and −5
2. No ($200 − ($55.95 + $44.99 + $49.99 + $54.99) = $200 − $205.92 or 200 − (56 + 45 + 50 + 55) = 200 − 206)
3. $1 \left(\left(\frac{\cancel{3}^{1}}{\cancel{7}_{1}} \right) \left(\frac{\cancel{14}^{2}}{\cancel{6}_{2}} \right) = \frac{2}{2} \right)$
4. 39 mm (5 mm + 13 mm + 4 mm + 17 mm)

5. $2x^2 - 4x + 3x - 6$

   $2x(x - 2) + 3(x - 2)$

   $(2x + 3)(x - 2)$

6. $\frac{3}{5}$

7. $\frac{4}{5}$

8. $\frac{3}{4}$

Part B: Operations on Rational Expressions

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Add and subtract as indicated.

   a) $\frac{x}{4} + \frac{5x}{4}$

   Answer:

   $$\frac{x + 5x}{4} = \frac{6x}{4} = \frac{3x}{2}$$

   b) $\frac{5x}{2} - \frac{x}{2}$

   Answer:

   $$\frac{5x - x}{2} = \frac{4x}{2} = 2x$$

   c) $\frac{5a}{a + b} + \frac{5b}{a + b}$

   Answer:

   $$\frac{5a + 5b}{a + b} = \frac{5(a + b)}{a + b} = 5$$

d) $\dfrac{3d}{c+d} - \dfrac{d}{c+d}$

Answer:

$$\frac{3d-d}{c+d} = \frac{2d}{c+d}$$

e) $\dfrac{2a^2+4a+17}{(a+2)(a+3)} - \dfrac{a^2-4a+2}{(a+2)(a+3)}$

Answer:

$$\frac{2a^2+4a+17-\left(a^2-4a+2\right)}{(a+2)(a+3)}$$

$$= \frac{2a^2+4a+17-a^2+4a-2}{(a+2)(a+3)}$$

$$= \frac{a^2+8a+15}{(a+2)(a+3)}$$

$$= \frac{(a+5)(a+3)}{(a+2)(a+3)}$$

$$= \frac{a+5}{a+2}$$

2. Find the lowest common denominator for each of the following sets of rational expressions.

a) $\dfrac{4}{x-8}, \dfrac{3x}{2x-16}, \dfrac{4x}{x+1}$

Answer:

Factor the denominator: $\dfrac{4}{x-8}, \dfrac{3x}{2(x-8)}, \dfrac{4x}{x+1}$

$\therefore$ LCD = $2(x-8)(x+1)$

b) $\dfrac{x+1}{(x+2)(x-3)}, \dfrac{x-1}{(x-2)(x-6)(x-3)^2}$

Answer:

LCD: $(x+2)(x-3)^2(x-2)(x-6)$

3. Add and subtract as indicated. Identify the non-permissible values.

a) $\dfrac{4x}{x^2+x-6} - \dfrac{3x}{x^2+5x+6}$

Answer:

$$\frac{4x}{(x+3)(x-2)} - \frac{3x}{(x+3)(x+2)} \quad \text{LCD} = (x+3)(x-2)(x+2)$$

$$= \frac{4x}{(x+3)(x-2)} \cdot \frac{x+2}{x+2} - \frac{3x}{(x+3)(x+2)} \cdot \frac{x-2}{x-2}$$

$$= \frac{4x^2+8x-3x^2+6x}{(x+3)(x-2)(x+2)}$$

$$= \frac{x^2+14x}{(x+3)(x-2)(x+2)}$$

Non-permissible values: $x = -3$, 2, and –2.

b) $\dfrac{3}{x^2-4x+3} + \dfrac{4}{x^2-5x+6}$

Answer:

$$\frac{3}{(x-3)(x-1)} + \frac{4}{(x-2)(x-3)} \quad \text{LCD} = (x-3)(x-1)(x-2)$$

$$= \frac{3}{(x-3)(x-1)} \cdot \frac{x-2}{x-2} + \frac{4}{(x-2)(x-3)} \cdot \frac{x-1}{x-1}$$

$$= \frac{3x-6+4x-4}{(x-3)(x-1)(x-2)}$$

$$= \frac{7x-10}{(x-3)(x-1)(x-2)}$$

Non-permissible values: $x = 3$, 1, and 2.

c) $\dfrac{2x}{x^2-6x-7}-\dfrac{3}{x+1}+\dfrac{3}{x-7}$

Answer:

$$\frac{2x}{(x-7)(x+1)}-\frac{3}{(x+1)}+\frac{3}{x-7} \quad \text{LCD} = (x-7)(x+1)$$

$$=\frac{2x}{(x-7)(x+1)}-\frac{3}{x+1}\cdot\frac{x-7}{x-7}+\frac{3}{x-7}\cdot\frac{x+1}{x+1}$$

$$=\frac{2x-3x+21+3x+3}{(x-7)(x+1)}$$

$$=\frac{2x+24}{(x-7)(x+1)}$$

Non-permissible values: $x = 7$ and -1.

d) $\dfrac{2x}{x^2+5xy}-\dfrac{x}{x^2+8xy+15y^2}$

Answer:

$$\frac{2x}{x(x+5y)}-\frac{x}{(x+3y)(x+5y)}$$

$$=\frac{2}{x+5y}-\frac{x}{(x+3y)(x+5y)} \qquad \text{LCD} = (x+3y)(x+5y)$$

$$=\frac{2}{x+5y}\cdot\frac{x+3y}{x+3y}-\frac{x}{(x+3y)(x+5y)}$$

$$=\frac{2x+6y-x}{(x+3y)(x+5y)}$$

$$=\frac{x+6y}{(x+3y)(x+5y)}$$

Non-permissible values: $x = 0$, $-5y$, and $-3y$.

e) $\dfrac{y-4}{y-3} - \dfrac{y-7}{y+8} + \dfrac{y+8}{y^2+5y-24}$

Answer:

$$\frac{y-4}{y-3} - \frac{y-7}{y+8} + \frac{y+8}{(y+8)(y-3)} \qquad \text{LCD} = (y-3)(y+8)$$

$$= \frac{y-4}{y-3} \cdot \frac{y+8}{y+8} - \frac{y-7}{y+8} \cdot \frac{y-3}{y-3} + \frac{y+8}{(y+8)(y-3)}$$

$$= \frac{y^2+4y-32-(y^2-10y+21)+y+8}{(y-3)(y+8)}$$

$$= \frac{y^2+4y-32-y^2+10y-21+y+8}{(y-3)(y+8)}$$

$$= \frac{15y-45}{(y-3)(y+8)}$$

$$= \frac{15(y-3)}{(y-3)(y+8)}$$

$$= \frac{15}{y+8}$$

Non-permissible values: $y = 3$ and -8.

f) $\dfrac{7}{y+2} - \dfrac{5}{2-y} + \dfrac{y+18}{y^2-4}$

Answer:

$$\frac{7}{y+2} - \frac{5}{(-1)(y-2)} + \frac{y+18}{(y-2)(y+2)}$$

$$= \frac{7}{y+2} + \frac{5}{y-2} + \frac{y+18}{(y-2)(y+2)} \qquad \text{LCD} = (y+2)(y-2)$$

$$= \frac{7}{y+2} \cdot \frac{y-2}{y-2} + \frac{5}{y-2} \cdot \frac{y+2}{y+2} + \frac{y+18}{(y-2)(y+2)}$$

$$= \frac{7y - 14 + 5y + 10 + y + 18}{(y-2)(y+2)}$$

$$= \frac{13y + 14}{(y-2)(y+2)}$$

Non-permissible values: $y = -2$ and 2.

g) $\dfrac{2x}{x-2} + \dfrac{2-3x}{x^2-x-2} - \dfrac{x+2}{x+1}$

Answer:

$$\frac{2x}{x-2} + \frac{2-3x}{(x-2)(x+1)} - \frac{x+2}{x+1}$$

$$= \frac{2x}{x-2} \cdot \frac{x+1}{x+1} + \frac{2-3x}{(x-2)(x+1)} - \frac{x+2}{x+1} \cdot \frac{x-2}{x-2} \qquad \text{LCD} = (x-2)(x+1)$$

$$= \frac{2x^2 + 2x + 2 - 3x - \left(x^2 - 4\right)}{(x-2)(x+1)}$$

$$= \frac{2x^2 - x + 2 - x^2 + 4}{(x-2)(x+1)}$$

$$= \frac{x^2 - x + 6}{(x-2)(x+1)}$$

Non-permissible values: $x = 2$ and -1.

4. Simplify the following rational expressions.

a) $$\frac{\frac{x(x+2)}{5}-\frac{x}{2(x-1)}}{\frac{5}{x+2}+\frac{7}{x-1}}$$

Answer:

$$\left(\frac{x(x+2)}{5}-\frac{x}{2(x-1)}\right)\div\left(\frac{5}{x+2}+\frac{7}{x-1}\right)$$ Rewrite

LCD of first expression: $(5)(2)(x-1)$

LCD of second expression: $(x+2)(x-1)$

Make common denominators:

$$\left(\frac{x(x+2)(2)(x-1)}{5(2)(x-1)}-\frac{5x}{5(2)(x-1)}\right)\div\left(\frac{5(x-1)}{(x+2)(x-1)}+\frac{7(x+2)}{(x-1)(x+2)}\right)$$

$$=\frac{2(x)\left(x^2+x-2\right)-5x}{5(2)(x-1)}\div\frac{5x-5+7x+14}{(x+2)(x-1)}$$ Simplify the numerators

$$=\frac{2x^3+2x^2-4x-5x}{5(2)(x-1)}\div\frac{5x-5+7x+14}{(x+2)(x-1)}$$ Simplify the numerators

$$=\frac{2x^3+2x^2-9x}{5(2)(x-1)}\div\frac{12x+9}{(x+2)(x-1)}$$ Simplify the numerators

Multiply by the reciprocal and factor

$$=\frac{x\left(2x^2+2x-9\right)}{5(2)(x-1)}\times\frac{(x+2)(x-1)}{3(4x+3)}$$

$$=\frac{x\left(2x^2+2x-9\right)(x+2)}{30(4x+3)}$$ Multiply and cancel

b) $\dfrac{\dfrac{1}{x+2}+\dfrac{1}{x-2}}{\dfrac{x}{x^2-4}+\dfrac{1}{x+2}}$

Answer:

$$\left(\frac{1}{x+2}+\frac{1}{x-2}\right)\div\left(\frac{x}{x^2-4}+\frac{1}{x+2}\right)$$

$$\left(\frac{1}{x+2}+\frac{1}{x-2}\right)\div\left(\frac{x}{(x-2)(x+2)}+\frac{1}{x+2}\right)$$

LCD of first expression: $(x-2)(x+2)$

LCD of second expression: $(x-2)(x+2)$

$$\left(\frac{x-2}{(x+2)(x-2)}+\frac{x+2}{(x+2)(x-2)}\right)\div\left(\frac{x}{(x-2)(x+2)}+\frac{x-2}{(x+2)(x-2)}\right)$$

$$=\frac{x-2+x+2}{(x+2)(x-2)}\div\frac{x+x-2}{(x-2)(x+2)}$$

$$=\frac{2x}{(x+2)(x-2)}\div\frac{2x-2}{(x-2)(x+2)}$$

$$=\frac{2x}{(x+2)(x-2)}\times\frac{(x-2)(x+2)}{2(x-1)}$$

$$=\frac{x}{x-1}$$

c) $\dfrac{x}{2x-4}+\left(\dfrac{x-3}{x^2-3x+2}\div\dfrac{x^2-9}{x-1}\right)$

Answer:

$$\frac{x}{2x-4}+\left(\frac{x-3}{x^2-3x+2}\div\frac{x^2-9}{x-1}\right)$$

$$=\frac{x}{2(x-2)}+\left(\frac{x-3}{(x-2)(x-1)}\times\frac{x-1}{(x+3)(x-3)}\right)$$ Factor and multiply by the reciprocal

$$=\frac{x}{2(x-2)}+\frac{1}{(x-2)(x+3)}$$ Cancel to reduce

$$\text{LCD}=2(x-2)(x+3)$$

$$=\frac{x(x+3)}{2(x-2)(x+3)}+\frac{2}{2(x-2)(x+3)}$$

$$=\frac{x^2+3x+2}{2(x-2)(x+3)}\text{ or }\frac{(x+2)(x+1)}{2(x-2)(x+3)}$$

5. Kahlia is trying to simplify a rational expression. However, the answer she received does not match the answer at the back of the textbook. Kahlia's work is shown below.

$$\frac{x-1}{(x+2)(x-3)} - \frac{2x}{(x+2)}$$

$$= \frac{x-1-2x}{(x+2)(x-3)}$$

$$= \frac{-x-1}{(x+2)(x-3)}$$

Explain Kahlia's mistake and simplify the rational expression correctly.

Answer:

The two original rational expressions do not have the same denominator. Therefore, these rational expressions need to be changed to equivalent rational expressions that have the same denominator before you can subtract them.

$$\frac{x-1}{(x+2)(x-3)} - \frac{2x}{(x+2)}$$ LCD: $(x+2)(x-3)$

$$= \frac{x-1}{(x+2)(x-3)} - \frac{2x(x-3)}{(x+2)(x-3)}$$

$$= \frac{x-1}{(x+2)(x-3)} - \frac{2x^2-6x}{(x+2)(x-3)}$$

$$= \frac{x-1-2x^2+6x}{(x+2)(x-3)}$$

$$= \frac{-2x^2+7x-1}{(x+2)(x-3)}$$

6. A swimmer swam 50 laps at a constant speed. For the remaining 50 laps, he reduced his speed by 3 km/h. Write a rational expression for the total time of this swimmer's workout. You do not need to simplify your answer. **Note:** distance = (speed)(time), or $d = rt$.

   Answer:

   $d = rt$

   $t = \frac{d}{r}$

   Let x = original constant speed

   Time of first 50 laps: $\frac{50}{x}$

   Time of second 50 laps: $\frac{50}{x-3}$

   Total time of 100 laps: $\frac{50}{x} + \frac{50}{x-3}$ hours

7. A rational expression puzzle is shown on the following page. Work with your learning partner to cut out each puzzle piece and reassemble them all in a 4×4 configuration so that each expression matches its factored form or its simplified form. If you are stuck, you can look in the Module 2 Answer Key to find helpful hints.

Rational Expression Puzzle Solution

$(x-3)(x+1)$	$(4x+3)(x+1)$	x^2+25	$(x-3)(x-1)$
A	**F**	**M**	**K**
$(x+7)(x+2)$	$(4x-3)(x-1)$	$x+1$	$(x+5)(x-5)$
$4x^2-7x+3$	$\dfrac{x}{x-3}+\dfrac{2}{x+4}$	$(x+3)(x-1)$	$\dfrac{x^2+7x+12x}{x^2+4x}$
$\dfrac{x^2-x-2}{x^2-4}$	$\dfrac{x^2+5x+6}{x^2+x-2}\div\dfrac{x+3}{x^2-1}$	$(x+3)^2-6(x+3)-16$	$\dfrac{5}{x}+\dfrac{4}{x+1}$
$\dfrac{(x+2)}{(1+x)}$	$\dfrac{(x-3)(x+4)}{x^2+6x-6}$	x^2+2x-3	$x+3$
N	**E**	**C**	**D**
$\dfrac{x-3}{x}$	$3x^2(2x+1)$	$\dfrac{-x(4x+33)}{(x+7)(x+2)}$	$\dfrac{3x+6}{2x^2-x-10}\times\dfrac{4x^2-25}{6}$
$(x-6)(x+6)$	x^2-36	$3(3x-5)(6x-13)$	$2(x+2)(x-9)$
$\dfrac{5}{x}-\dfrac{4}{x-2}$	$6(3x-4)^2-21(3x-4)+15$	$2(x-6)^2+10(x-6)-48$	$\dfrac{x}{x+7}+\dfrac{5x}{x+2}$
$\dfrac{x-10}{x(x-2)}$	$6x^3+3x^2$	$\dfrac{x}{x+7}-\dfrac{5x}{x+2}$	$\dfrac{2x+5}{2}$
I	**L**	**O**	**H**
$1-100x^2$	$\dfrac{a}{2c}$	$2y(3xy-1)$	$\dfrac{5x^2+13x+18}{(x-1)(x+2)(x+3)}$
$(10x-1)(10x+1)$	$100x^2-1$	$\dfrac{3x}{4}$	$(3x+2y+3)(3x-2y+5)$
$\dfrac{x+1}{8x}\times\dfrac{2x}{x^2-1}$	$\dfrac{x}{2}+\dfrac{x}{4}$	$(3x+4)^2-(2y-1)^2$	$\dfrac{3}{x}-\dfrac{x}{2}$
$\dfrac{1}{4(x-1)}$	$25x^2+20x+4$	$3(2x+5)^2+10(2x+5)-8$	$\dfrac{3}{x-1}+\dfrac{2x}{(x+2)(x+3)}$
B	**G**	**P**	**J**
$\dfrac{a}{4c}$	$\dfrac{a^2bc}{8m}\div\dfrac{abc^2}{4m}$	$6xy^2-2y$	$\dfrac{x}{2}$
$(x+2)(x+1)$	x^2+3x+2	$(5x+2)^2$	$(2x+9)(6x+13)$
$x-2$	$\dfrac{1}{4(x+1)}$	x^2-4	$x+3$

Notes

Grade 11 Pre-Calculus Mathematics (30S)

Module 2
Factoring and Rational Expressions

Learning Activity Answer Keys

MODULE 2:
FACTORING AND RATIONAL EXPRESSIONS

Learning Activity 2.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Carla has a shoe addiction. She owns 13 pairs of boots, 18 pairs of sneakers, 23 pairs of high heels, and 19 pairs of sandals. How many pairs of shoes does she have in total?
2. What are two numbers that sum to 11 and multiply to 30?
3. Multiply: $(2x + 1)(x - 5)$
4. Evaluate: $\dfrac{\frac{9}{21}}{\frac{3}{63}}$
5. Factor: $x^2 + 7x + 12$
6. How many metres are in 2.3 km?
7. Simplify: $(2x^2 - 6x + 11) - (-14x^2 - 2x + 4)$
8. Is xy^2 the greatest common factor of $6x^2y^2 + 8xy^3$?

Answers:

1. 73 (13 + 18 + 23 + 19)
2. 5 and 6
3. $2x^2 - 9x - 5$
4. $9 \left(\dfrac{\frac{9}{21}}{\frac{3}{63}} = \left(\frac{9}{21}\right)\left(\frac{63}{3}\right) = \left(\frac{3}{7}\right)\left(\frac{21}{1}\right) = \frac{63}{7} \right)$
5. $(x + 3)(x + 4)$
6. 2300 metres (2.3×1000)
7. $16x^2 - 4x + 7$ ($2x^2 - 6x + 11 + 14x^2 + 2x - 4$)
8. No, the greatest common factor is $2xy^2$. ($6x^2y^2 + 8xy^3 = 2xy^2(3x + 4y)$)

Part B: Factoring Review

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Factor out the greatest common factor.
   a) $5x + 20$
   b) $5x^3 + 20x^2$
   c) $30x^2y - 24xy^3$
   d) $x^3 + x^2$

   Answers:

   a) $5(x + 4)$
   b) $5x^2(x + 4)$
   c) $6xy(5x - 4y^2)$
   d) $x^2(x + 1)$

2. Factor each trinomial completely.
   a) $x^2 + 7x - 8-$
   b) $x^2 - 2x - 8$
   c) $x^2 - 11x + 28$
   d) $3x^3 + 30x^2 + 63x$
   e) $2x^2 + 7x - 15$
   f) $4x^2 - 12x + 9$

   Answers:

   a) $(x + 8)(x - 1)$
   b) $(x - 4)(x + 2)$
   c) $(x - 7)(x - 4)$
   d) First factor out the GCF. Then factor the remaining trinomial.
      $3x(x^2 + 10x + 21) = 3x(x + 7)(x + 3)$
   e) $2x^2 - 3x + 10x - 15$
      $= x(2x - 3) + 5(2x - 3)$
      $= (2x - 3)(x + 5)$

f) $4x^2 - 6x - 6x + 9$

$= 2x(2x - 3) - 3(2x - 3)$

$= (2x - 3)(2x - 3)$

$= (2x - 3)^2$

g) $\frac{2}{3}x^2 - 4x + 6$

$= \frac{2}{3}\left(x^2 - 6x + 9\right)$

$= \frac{2}{3}(x - 3)(x - 3)$

$= \frac{2}{3}(x - 3)^2$

3. Factor each binomial completely.

   a) $x^2 - 100$

   b) $9x^2 - 49$

   c) $9x^2 - 16y^2$

   d) $2x^2 - 50y^2z^2$

   Answers:

   a) $(x - 10)(x + 10)$

   b) $(3x - 7)(3x + 7)$

   c) $(3x - 4y)(3x + 4y)$

   d) First factor out the GCF. Then factor the remaining binomial as a difference of squares.

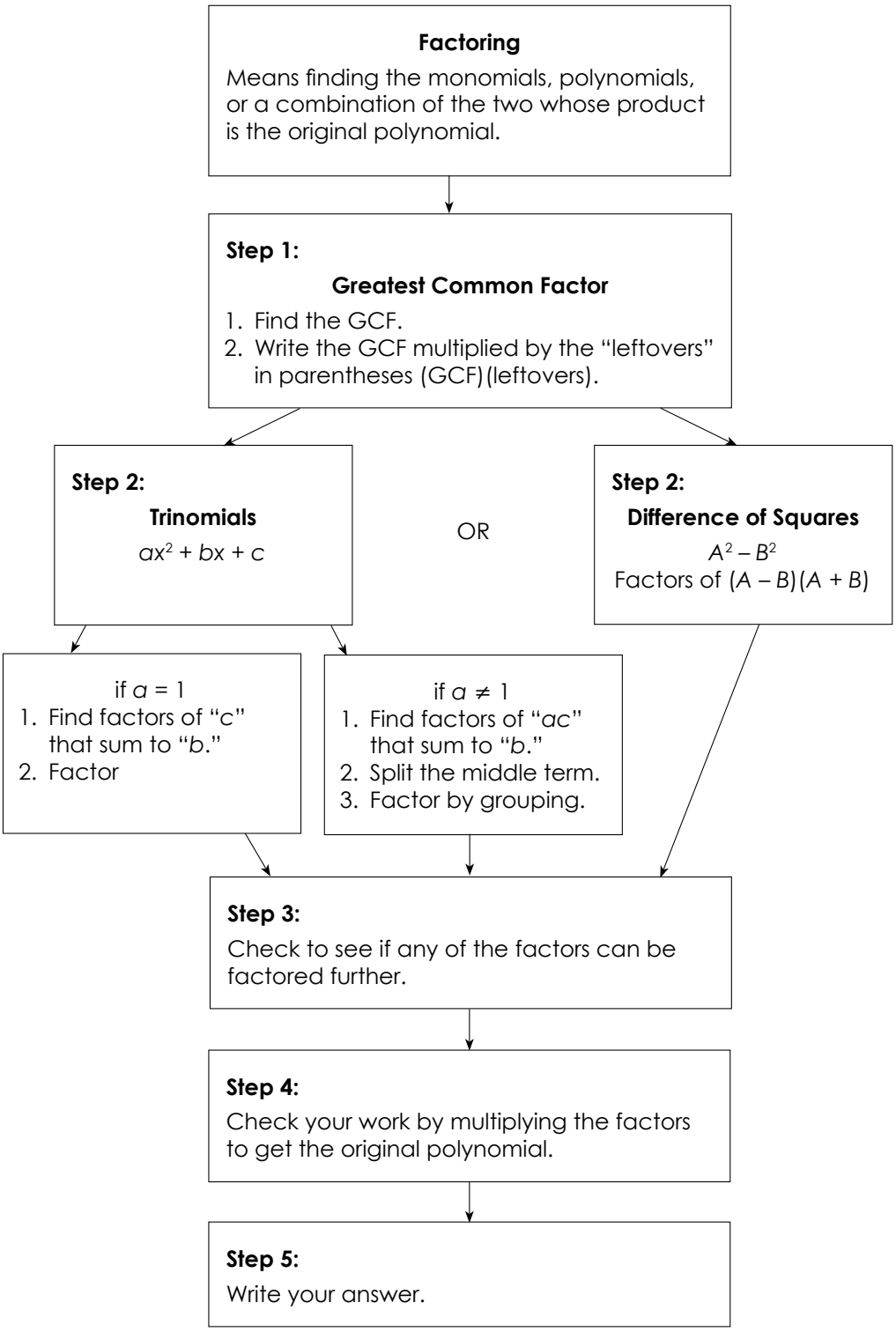
   $= 2(x^2 - 25y^2z^2)$

   $= 2(x - 5yz)(x + 5yz)$

4. Create a graphic organizer of the steps to take when factoring a polynomial. Include descriptions for how to complete each step.

   Answer:

   See the following page for one example of the graphic organizer.


Factoring
Means finding the monomials, polynomials, or a combination of the two whose product is the original polynomial.
Step 1:
Greatest Common Factor
1. Find the GCF.
2. Write the GCF multiplied by the "leftovers" in parentheses (GCF)(leftovers).
Step 2:
Trinomials
$ax^2 + bx + c$
OR
Step 2:
Difference of Squares
$A^2 - B^2$
Factors of $(A - B)(A + B)$
if $a = 1$
1. Find factors of "c" that sum to "b."
2. Factor
if $a \neq 1$
1. Find factors of "ac" that sum to "b."
2. Split the middle term.
3. Factor by grouping.
Step 3:
Check to see if any of the factors can be factored further.
Step 4:
Check your work by multiplying the factors to get the original polynomial.
Step 5:
Write your answer.

Learning Activity 2.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. There are a group of 80 people in a shareholder's meeting making a decision about whether or not to acquire a new company. The decision needs to pass with an 80% majority. How many people must agree in order for the decision to pass?
2. Evaluate: $5 \times 17 \times 20$
3. Katrina is flying from Winnipeg to Belize. She leaves Winnipeg at 7:00 am and takes a 3-hour flight to Minneapolis where she has a two and a half hour layover. From Minneapolis, she takes a direct flight to Belize that lasts for 7.25 hours. What time does Katrina arrive in Belize?
4. What is the greatest common factor of 48 and 144?
5. Convert 6.3 kilometres to metres.
6. Simplify: $\frac{2^8}{4^2}$
7. Find the slope of the line parallel to the line, $-2 = 4x + 6y$.
8. Determine the total surface area of a cube with a volume of 8 cm^3.

Answers:

1. 64 people (50% of 80 = 40; 10% of 80 = 8; 80% of 80 = 40 + 8 + 8 + 8)
2. 1700 ($5 \times 17 \times 20 = 5 \times 20 \times 17 = 100 \times 17$)
3. 7:45 pm (7 am + 3 hours = 10 am; 10 am + 2.5 hours = 12:30; 12:30 + 7.25 hours = 7:45 pm)
4. 48
5. 6300 metres (There are 1000 metres in 1 kilometre, so 6.3×1000)
6. 16 $\left(\frac{2^8}{\left(2^2\right)^2} = \frac{2^8}{2^4} = 2^4 \right)$
7. $-\frac{2}{3}$ $\left(y = -\frac{2}{3}x - \frac{1}{3}\left(6y = -4x - 2;\ y = -\frac{4}{6}x - \frac{2}{6} \right) \right)$

   Recall: Parallel lines have the same slope.

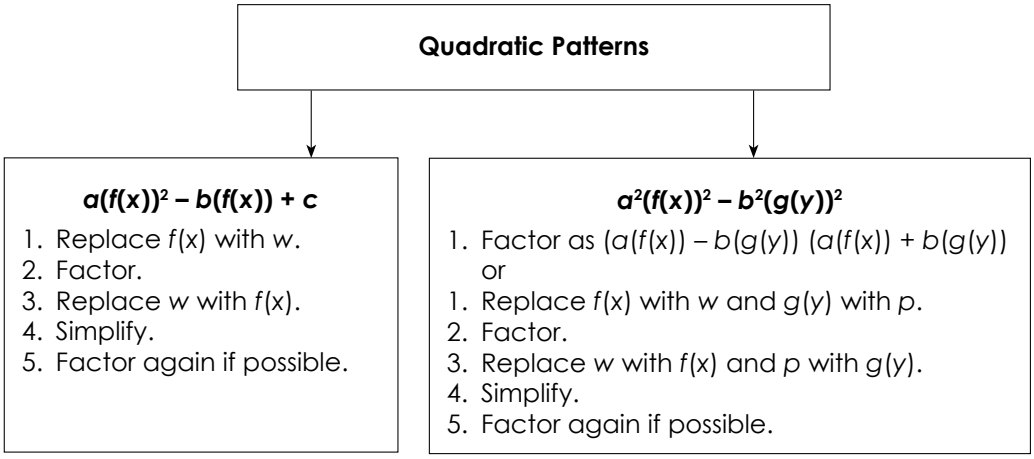
8. 8 cm^3. (The sides of a cube are equal in length. Therefore, each side length must be 2 cm. The total surface area is $(2 \times 2) \times 6 = 24$ cm^2, as each side is 2 cm × 2 cm and there are 6 equal sides on a cube.)

Part B: Factoring Expressions with Quadratic Patterns

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Add to the factoring graphic organizer you completed in the last learning activity (Learning Activity 2.1). Include all of the new techniques you learned about factoring in this lesson.

   Answer:



2. Factor completely.

a) $(7x - 6)^2 - 8(7x - 6) + 15$

Answer:

Replace $(7x - 6)$ by w:

$(w)^2 - 8(w) + 15$

Factor:

$(w - 5)(w - 3)$

Replace w with $(7x - 6)$:

$(7x - 6 - 5)(7x - 6 - 3)$

Simplify:

$(7x - 11)(7x - 9)$

b) $9(2x - 1)^2 - 42(2x - 1) + 49$

Answer:

Replace $(2x - 1)$ by w:

$9w^2 - 42w + 49$

Factor:

$9w^2 - 21w - 21w + 49$

$= 3w(3w - 7) - 7(3w - 7)$

$= (3w - 7)(3w - 7)$

$= (3w - 7)^2$

Replace w with $(2x - 1)$:

$[3(2x - 1) - 7]^2$

Simplify:

$(6x - 3 - 7)^2$

$= (6x - 10)^2$

$= (6x - 10)(6x - 10)$

$= 2(3x - 5)(2)(3x - 5)$

$= 4(3x - 5)^2$

c) $(x^2 - x)^2 - 7(x^2 - x) + 6$

Answer:

Replace $(x^2 - x)$ by w:

$w^2 - 7w + 6$

Factor:

$(w - 6)(w - 1)$

Replace w with $(x^2 - x)$:

$(x^2 - x - 6)(x^2 - x - 1)$

Factor again:

$(x - 3)(x + 2)(x^2 - x - 1)$

d) $(x - 1)^2 + 4(x - 1) - 32$

Answer:

Replace $(x - 1)$ by w:

$w^2 + 4w - 32$

Factor:

$(w + 8)(w - 4)$

Replace w with $(x - 1)$:

$(x - 1 + 8)(x - 1 - 4)$

Simplify:

$(x + 7)(x - 5)$

3. Factor each expression.

a) $50(2x - 7)^2 - 162(3y - 4)^2$

Answer:

$50(2x - 7)^2 - 162(3y - 4)^2$

Let $w = 2x - 7$

$p = 3y - 4$

$50(2x - 7)^2 - 162(3y - 4)^2$

$= 50w^2 - 162p^2$

$=2(25w^2 - 81p^2)$

$= 2(5w - 9p)(5w + 9p)$

Substitute:

$= 2[5(2x - 7) - 9(3y - 4)][5(2x - 7) + 9(3y - 4)$

$= 2[10x - 35 - 27y + 36][10x - 35 + 27y - 36]$

$= 2(10x - 27y + 1)(10x + 27y - 71)$

b) $25x^2 - 4y^2$

Answer:

$25x^2 - 4y^2$

$= (5x)^2 - (2y)^2$

$a(f(x)) = 5x$

$b(g(y)) = 2y$

$= (5x - 2y)(5x + 2y)$

c) $16(7x - 6)^2 - 121(4y - 7)^2$

Answer:

$16(7x - 6)^2 - 121(4y - 7)^2$

Let $w = 7x - 6$

$p = 4y - 7$

$16(7x - 6)^2 - 121(4y - 7)^2$

$= 16w^2 - 121p^2$

$= (4w - 11p)(4w + 11p)$

$= [4(7x - 6) - 11(4y - 7)][4(7x - 6) + 11(4y - 7)]$

$= [28x - 24 - 44y + 77][28x - 24 + 44y - 77]$

$= (28x - 44y + 53)(28x + 44y - 101)$

d) $144(x + 3)^4 - 729(y - 1)^4$

Answer:

$144(x + 3)^4 - 729(y - 1)^4$

Let $w = x + 3$

$p = y - 1$

$144(x + 3)^4 - 729(y - 1)^4$

$= 144w^4 - 729p^4$

$= 9[16w^4 - 81p^4]$

$= 9(4w^2 - 9p^2)(4w^2 + 9p^2)$

$= 9(2w - 3p)(2w + 3p)(4w^2 + 9p^2)$

$= 9[2(x + 3) - 3(y - 1)][2(x + 3) + 3(y - 1)][4(x + 3)^2 + 9(y - 1)^2]$

$= 9(2x + 6 - 3y + 3][2x + 6 + 3y - 3][4(x^2 + 6x + 9) + 9(y^2 - 2y + 1)]$

$= 9[2x - 3y + 9][2x + 3y + 3][4x^2 + 24x + 36 + 9y^2 - 18y + 9]$

$= 9(2x - 3y + 9)(2x + 3y + 3)(4x^2 + 9y^2 + 24x - 18y + 45)$

4. Determine if $(x - 2)$ is a factor of each of the following polynomials.

a) $8x^2 - 30x + 28$

Answer:

$x - 2 = 0$

$x = 2$

$8x^2 - 30x + 28$

$= 8(2)^2 - 30(2) + 28$

$= 32 - 60 + 28$

$= 0$

This evaluated polynomial equals zero. Therefore, $(x - 2)$ is a factor of this polynomial.

b) $(x - 2)^2 + 4(x - 2) - 32$

Answer:

$x - 2 = 0$

$x = 2$

$(x - 2)^2 + 4(x - 2) - 32$

$= (2 - 2)^2 + 4(2 - 2) - 32$

$= 0 + 0 - 32$

$= -32$

This evaluated polynomial does not equal zero. Therefore, $(x - 2)$ is not a factor of this polynomial.

c) $-15x^2 - 196x - 572$

Answer:

$x - 2 = 0$

$x = 2$

$-15x^2 - 196x - 572$

$= -15(2)^2 - 196(2) - 572$

$= -60 - 392 - 572$

$= -1024$

This evaluated polynomial does not equal zero. Therefore, $(x - 2)$ is not a factor of this polynomial.

5. Determine if $(7x - 6)$ is a factor of each of the following polynomials.

a) $(7x - 6)^2 - 8(7x - 6) + 15$

Answer:

$7x - 6 = 0$

$7x = 6$

$x = \frac{6}{7}$

$(7x-6)^2 - 8(7x-6) + 15$

$= \left(7 \cdot \left(\frac{6}{7}\right) - 6\right)^2 - 8\left(7 \cdot \left(\frac{6}{7}\right) - 6\right) + 15$

$= (6-6)^2 - 8(6-6)^2 + 15$

$= 0 - 0 + 15$

$= 15$

This evaluated polynomial does not equal zero. Therefore, $(7x - 6)$ is not a factor of this polynomial.

b) $7x^2 + 22x - 24$

Answer:

$7x - 6 = 0$

$7x = 6$

$x = \frac{6}{7}$

$7x^2 + 22x - 24$

$= 7\left(\frac{6}{7}\right)^2 + 22\left(\frac{6}{7}\right) - 24$

$= \frac{36}{7} + \frac{132}{7} - 24$

$= \frac{168}{7} - \frac{168}{7}$

$= 0$

This evaluated polynomial equals zero. Therefore, ($7x$ – 6) is a factor of this polynomial.

c) $16(7x - 6)^2 - 121(4x - 7)^2$

Answer:

$7x - 6 = 0$

$7x = 6$

$x = \frac{6}{7}$

For this question, you might use the calculator to manipulate the fractions:

$16(7x - 6)^2 - 121(4x - 7)^2$

$= 16\left[7\left(\frac{6}{7}\right) - 6\right]^2 - 121\left[4\left(\frac{6}{7}\right) - 7\right]^2$

$= \frac{-75\ 625}{49}$

This evaluated polynomial does not equal zero. Therefore, ($7x$ – 6) is not a factor of this polynomial.

d) $35x^2 - 58x + 24$

Answer:

$7x - 6 = 0$

$7x = 6$

$x = \frac{6}{7}$

$35x^2 - 58x + 24$

$= 35\left(\frac{6}{7}\right)^2 - 58\left(\frac{6}{7}\right) + 24$

$= 0$

This evaluated polynomial equals zero. Therefore, $(7x - 6)$ is a factor of this polynomial.

6. Janique was asked to factor the trinomial $4x^2 + 2x - 12$ completely. Her work is shown below.

$4x^2 + 2x - 12$

$= 4x^2 + 8x - 6x - 12$

$= 4x(x + 2) - 6(x + 2)$

$= (4x - 6)(x + 2)$

Determine if her work is correct, and explain why or why not.

Answer:

Janique's work is correct so far. However, she has not factored the polynomial completely. The expression $(4x - 6)$ has a common factor of 2 that Janique can factor out. Therefore, the completely factored expression is:

$4x^2 + 2x - 12 = 2(2x - 3)(x + 2)$

Also, Janique could have factored out the common factor of 2 at the beginning of her factoring steps.

$4x^2 + 2x - 12$

$2(2x^2 + x - 6)$

$2(2x^2 + 4x - 3x - 6)$

$2[2x(x + 2) - 3(x + 2)]$

$2[(x + 2)(2x - 3)]$

Learning Activity 2.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Factor: $x^2 - 5x - 6$
2. Find the common ratio in the following pattern: 8, 16, 32, 64, . . .
3. The air you breathe is a mixture of different gases including nitrogen, carbon dioxide, oxygen, and argon. If the air is 78.1% nitrogen, 0.1% carbon dioxide, 20.9% oxygen, and 0.7% argon, what percentage of the air is composed of other gases?
4. State the reciprocal of $\frac{4}{3}$.
5. Your grandma is making a turkey dinner for the family and needs you to figure out when to start putting everything in the oven. If the turkey takes five hours to cook, the ham takes three hours to cook, and the meatballs take one hour to cook (they cannot go in the oven at the same time), what time does your grandma need to start putting food in the oven to have dinner ready by 2:00 pm?

For questions 6, 7, and 8, consider the following values:

3, 4, 4, 4, 6, 7, 9, 10, 16, 17.

6. Find the mode.
7. Find the mean.
8. Find the median.

Answers:

1. $(x - 6)(x + 1)$
2. 2
3. 0.2% (100% − (78.1% + 0.1% + 20.9% + 0.7%) = 100% − 99.8%)
4. $\frac{3}{4}$
5. 5:00 am (You need to use the oven for 9 hours; 9 hours before 2 pm is 5 am.)

6. 4 (The mode is the most frequently occurring number.)

7. $8\left(\frac{3+4+4+4+6+7+9+10+16+17}{10}=\frac{80}{10}\right)$

   (Look for friendly pairs of numbers to add without a calculator.)

8. $6.5\ \left(\frac{6+7}{2}=\frac{13}{2}=6.5\right)$

   (Since there is an even number of numbers, the median will be the sum of the two middle values divided by two.)

Part B: Equivalent Forms of Rational Expressions

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. State the domain and the non-permissible values of each rational expression.

   a) $\frac{(x+3)(x-2)}{(x+2)(x-2)}$

   Answer:

   The non-permissible values are $x = -2$ and $x = 2$.

   The domain is $\{x \mid x \neq \pm 2, x \in \Re\}$.

   b) $\frac{x(x-1)}{(x+2)}$

   Answer:

   The non-permissible value is $x = -2$.

   The domain is $\{x \mid x \neq -2, x \in \Re\}$.

   c) $\frac{x(x-2)}{(x+3)(x-3)}$

   Answer:

   The non-permissible values are $x = -3$ and $x = 3$.

   The domain is $\{x \mid x \neq \pm 3, x \in \Re\}$.

d) $\dfrac{x(x-1)(x+2)}{(x-1)(x+1)(x+2)}$

Answer:

The non-permissible values are $x = -1$, $x = 1$, and $x = -2$.

The domain is $\{x \mid x \neq -1, 1, -2, x \in \Re\}$.

e) $\dfrac{x(x-1)}{(x-1)x}$

Answer:

The non-permissible values are $x = 1$ and $x = 0$.

The domain is $\{x \mid x \neq 1, 0, x \in \Re\}$.

2. Simplify each rational expression. State the non-permissible values of x for each expression. Your final answer may be in factored form.

a) $\dfrac{2x^2 - 18}{x^2 - x - 12}$

Answer:

$$\frac{2\left(x^2 - 9\right)}{(x-4)(x+3)} = \frac{2(x-3)(x+3)}{(x-4)(x+3)} = \frac{2(x-3)}{x-4}$$

The non-permissible values are $x = 4$ and $x = -3$.

b) $\dfrac{x^3 - 2x^2 - 15x}{x^3 - 25x}$

Answer:

$$\frac{x\left(x^2 - 2x - 15\right)}{x\left(x^2 - 25\right)} = \frac{x(x-5)(x+3)}{x(x-5)(x+5)} = \frac{x+3}{x+5}$$

The non-permissible values are $x = 0$, $x = -5$, and $x = 5$.

c) $\dfrac{2x^3 + 8x}{2x^2}$

Answer:

$$\frac{2x\left(x^2 + 4\right)}{2x^2} = \frac{x^2 + 4}{x}$$

The non-permissible value is $x = 0$.

d) $\dfrac{3x + 6}{x^2 - 4}$

Answer:

$$\frac{3(x + 2)}{(x - 2)(x + 2)} = \frac{3}{x - 2}$$

The non-permissible values are $x = -2$ and $x = 2$.

e) $\dfrac{2x - 2}{1 - x}$

Answer:

$$\frac{2(x - 1)}{-(x - 1)} = -2$$

The non-permissible value is $x = 1$.

f) $\dfrac{100x^3y^2}{25x^3y^3}$

Answer:

$$\frac{100x^3y^2}{25x^3y^3} = \frac{\cancel{(25)}\,(4)\,\cancel{(x)}\,\cancel{(x)}\,\cancel{(x)}\,\cancel{(y)}\,\cancel{(y)}}{\cancel{(25)}\,\cancel{(x)}\,\cancel{(x)}\,\cancel{(x)}\,\cancel{(y)}\,\cancel{(y)}\,(y)} = \frac{4}{y}$$

The non-permissible values are $x = 0$ and $y = 0$.

3. Is the following method an acceptable way to reduce fractions? Explain.

a) $\dfrac{\cancel{5}}{x+\cancel{5}} = \dfrac{1}{x+1}$

Answer:

No. The value "5" is not a factor of the denominator. Therefore, even though "5" is a factor of the numerator, it is not possible to cancel out a "5" from the numerator and denominator. The original expression cannot be reduced.

b) $\dfrac{\cancel{6}x^3 - 24\cancel{x}}{\cancel{6x}} = x^3 - 24$

Answer:

No. The value "6" is a factor of the numerator. However, you would also have to factor out a 6 from 24 before you cancel the factor "6" from the numerator and denominator. Again, x is a factor of the numerator and the denominator. However, you would have to factor out an x from $6x^3$ before you are able to cancel the factor "x" from both the numerator and the denominator. The correct answer is $\dfrac{6x\left(x^2-4\right)}{6x} = x^2 - 4$.

c) $\dfrac{5+10x}{5x} = \dfrac{5(1+2x)}{(5)(x)} = \dfrac{1+2x}{x}$

Answer:

Yes. This is a correct reduction as 5 is a factor of both the numerator and denominator and thus can be eliminated from the rational expression.

4. Write the polynomial A if $\frac{A}{x^2-9}$ reduced to $\frac{x}{x-3}$.

   Answer:

   If $\frac{A}{x^2-9}$ reduces to $\frac{x}{x-3}$, it follows that the rational expressions are equivalent.

   $$\frac{A}{x^2-9} = \frac{x}{x-3}$$

   $$\frac{A}{(x-3)(x+3)} = \frac{x}{x-3}$$

   Now solve for A.

   $$A = \frac{x(x-3)(x+3)}{(x-3)} = x(x+3),\ x \neq \pm 3$$

5. Explain why 7 may not be the only non-permissible value for the following rational expression: $\frac{x}{x-7}$.

   Answer:

   This rational expression may be the simplified form of another rational expression. Therefore, another non-permissible value from the original rational expression may have been factored out. For example, the original expression may have been $\frac{x}{x-7} \cdot \frac{x+3}{x+3} = \frac{x(x+3)}{(x-7)(x+3)}$, in which case $x = -3$ is also a non-permissible value. Thus, $x = 7$ may not be the only non-permissible value of this rational expression.

6. Create a rational expression that has two non-permissible values: $x = -1$ and $x = 5$.

   Answer:

   $(x + 1)$ and $(x - 5)$ must be factors of the denominator in order to have non-permissible values of $x = -1$ and $x = 5$. Therefore, any rational expression with $(x + 1)$ and $(x - 5)$ in the denominator would have non-permissible values of $x = -1$ and $x = 5$. An example is:

   $$\frac{(x+2)}{(x+1)(x-5)}$$

7. The distance, d, can be found using the formula $d = rt$, where r is the speed and t is the time.

   a) If the distance is represented by $2x^2 + 3x - 2$ and the speed is represented by $4x + 8$, what is an expression for the time? Simplify your answer.

   Answer:

   $d = rt$

   $2x^2 + 3x - 2 = (4x + 8)t$

   $\frac{2x^2 + 3x - 2}{4x + 8} = \frac{(4x + 8)t}{4x + 8}$

   $\frac{(2x - 1)\cancel{(x + 2)}}{4\cancel{(x + 2)}} = t$

   $t = \frac{2x - 1}{4}$

   b) State any non-permissible values of the expression in part (a) above.

   Answer:

   The non-permissible value is $x = -2$. Thus, the expression for time is written as $t = \frac{2x - 1}{4}$, $x \neq -2$.

8. Create equivalent rational expressions for each of the following rational expressions by multiplying by a form of 1. State the non-permissible values for each rational expression.

a) $\frac{x+1}{x-2}$

Answer:

Answers will vary. An example is:

$$\frac{x+1}{x-2} = \frac{(x+1)(x+1)}{(x-2)(x+1)} = \frac{x^2+2x+1}{x^2-x-2}$$

The non-permissible values are $x = 2$ and $x = -1$.

Note: $x = 2$ will be one of the non-permissible values no matter what equivalent expression you create. Note also that your answer may be left in factored form as $\frac{(x+1)(x+1)}{(x-2)(x+1)}$.

b) $\frac{x-7}{x^2}$

Answer:

Answers will vary. An example is:

$$\frac{x-7}{x^2} = \frac{(x-7)(x-5)}{x^2(x-5)} = \frac{x^2-12x+35}{x^3-5x^2}$$

The non-permissible values are $x = 0$ and $x = 5$.

Note: $x = 0$ will be one of the non-permissible values no matter what equivalent expression you create.

Learning Activity 2.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. What is the perimeter of a regular hexagon if one side measures 4 cm?
2. You and your friends want to drive to the beach but you want to know how far away it is. The map you are using has a scale of 1 cm = 100 km. On the map, if the beach is 1.4 cm away from your house, how many kilometres are you from the beach?
3. Calculate: $\frac{8}{9} - \frac{1}{5}$
4. Factor: $2x + 4xy$
5. Simplify: $(4^2)^2$
6. Solve the inequality $10x + 4 \leq 3x - 1$.
7. List the factors of 180.
8. What is the sum of the reciprocals of the factors of 4?

Answers:

1. 24 cm (6×4; a hexagon has 6 sides)
2. The beach is 140 km away ($1.4 \times 100 = 140$ km).
3. $\frac{31}{45}\left(\frac{40}{45} - \frac{9}{45}\right)$
4. $2x(1 + 2y)$
5. 256 [$(4^2)^2 = 4^4 = 256$ or $16^2 = 256$]
6. $x \leq -\frac{5}{7}$ $(7x \leq -5)$
7. The factors of 180 are 1, 2, 3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 30, 36, 45, 60, 90, and 180.
8. $\frac{7}{4}\left(\begin{array}{l}\text{The factors of 4 are 1, 2, and 4. Their reciprocals are 1, } \frac{1}{2}\text{, and } \frac{1}{4}. \\ \\ \text{The sum is } 1 + \frac{1}{2} + \frac{1}{4} = \frac{4}{4} + \frac{2}{4} + \frac{1}{4} = \frac{7}{4}.\end{array}\right)$

Part B: Multiplying and Dividing Rational Expressions

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Perform the indicated operation. Identify any non-permissible values.

a) $\frac{x+1}{x} \times \frac{x^2}{x^2-1}$

Answer:

$\frac{x+1}{x} \cdot \frac{x^2}{(x-1)(x+1)} = \frac{x}{x-1}$

$x \neq 0, 1, -1$

b) $\frac{x-2}{x} \div \frac{x+2}{x^2}$

Answer:

$\frac{x-2}{x} \cdot \frac{x^2}{x+2} = \frac{(x-2)x}{x+2}$

$x \neq 0, -2$

c) $\frac{x^2+2x+1}{5-x} \times \frac{x^2-25}{x^2+6x+5}$

Answer:

$\frac{(x+1)(x+1)}{(-1)(x-5)} \cdot \frac{(x-5)(x+5)}{(x+5)(x+1)}$

$= -(x+1)$ or $-x-1$

$x \neq 5, -5, -1$

d) $\dfrac{3x^2+10x+3}{x^2-9} \div \dfrac{3x+1}{x-3}$

Answer:

$\dfrac{(3x+1)(x+3)}{(x-3)(x+3)} \cdot \dfrac{x-3}{3x+1} = 1$

$x \neq 3, -3, -\dfrac{1}{3}$

e) $\dfrac{3}{\dfrac{2}{x}}$

Answer:

$3 \div \dfrac{2}{x}$

$= 3 \cdot \dfrac{x}{2} = \dfrac{3x}{2}$

$x \neq 0$

f) $\dfrac{\dfrac{2x+6}{x+1}}{\dfrac{x+3}{x^2-1}}$

Answer:

$\dfrac{2x+6}{x+1} \div \dfrac{x+3}{x^2-1}$

$= \dfrac{2(x+3)}{x+1} \cdot \dfrac{(x-1)(x+1)}{x+3} = 2(x-1)$

$x \neq -1, 1, -3$

g) $\dfrac{x^2}{y} \div \dfrac{x}{y^2}$

Answer:

$$\frac{x^2}{y} \cdot \frac{y^2}{x} = xy$$

$x \neq 0,\ y \neq 0$

h) $\dfrac{x^2 - 9}{3x^2 + 12} \times \dfrac{x^2 + 4}{x + 3}$

Answer:

$$\frac{(x-3)(x+3)}{3\left(x^2+4\right)} \cdot \frac{x^2+4}{x+3} = \frac{x-3}{3}$$

$x \neq -3$

i) $\dfrac{x^4 - y^4}{x^2 - y^2} \div \dfrac{x^2 - x - 6}{x + 2}$

Answer:

$$\frac{\left(x^2-y^2\right)\left(x^2+y^2\right)}{x^2-y^2} \cdot \frac{x+2}{(x-3)(x+2)}$$

$$= \frac{x^2+y^2}{x-3}$$

$x \neq 3, -2;\ x \neq \pm y$

j) $\dfrac{2x - 6}{x^2 y} \div \dfrac{3x - 9}{xy}$

Answer:

$$\frac{2(x-3)}{x^2 y} \cdot \frac{xy}{3(x-3)} = \frac{2}{3x}$$

$x \neq 0, 3;\ y \neq 0$

k) $\dfrac{x^2+2x-8}{x^2-16} \times \dfrac{x^2-8x+16}{x^2-6x+8}$

Answer:

$$\frac{(x+4)(x-2)}{(x-4)(x+4)} \cdot \frac{(x-4)(x-4)}{(x-4)(x-2)} = 1$$

$x \neq 4, -4, 2$

l) $\dfrac{5a-a^2}{a^2-4a+3} \times \dfrac{a^2-5a+4}{a^2-9a+20} \div \dfrac{a^2-7a}{a^2-10a+21}$

Answer:

$$\frac{(-a)(a-5)}{(a-3)(a-1)} \cdot \frac{(a-4)(a-1)}{(a-5)(a-4)} \cdot \frac{(a-3)(a-7)}{a(a-7)} = -1$$

$a \neq 3, 1, 5, 4, 0, 7$

m) $\dfrac{m^2-m-20}{m^2-25} \div \dfrac{m^2+2m-8}{m^2-m-2} \div \dfrac{m^2+5m+4}{m^2+5m}$

Answer:

$$\frac{(m-5)(m+4)}{(m-5)(m+5)} \cdot \frac{(m-2)(m+1)}{(m+4)(m-2)} \cdot \frac{m(m+5)}{(m+4)(m+1)} = \frac{m}{m+4}$$

$m \neq 5, -5, -4, 2, -1, 0$

2. Stephanie was asked to simplify the following expression completely. Her work is shown below.

$$\frac{4x+16}{3x^2} \div \frac{5x+20}{4x}$$

$$= \frac{4x+16}{3x^2} \times \frac{5x+20}{4x}$$

$$= \frac{\cancel{4}(x+4)}{3x^2} \times \frac{5(x+4)}{\cancel{4}x}$$

$$= \frac{5(x+4)(x+4)}{3x^2}$$

Is Stephanie's work correct? Explain why or why not. If Stephanie's answer is incorrect, find the correct answer.

Answer:

Stephanie's work is not correct. Stephanie forgot to write the reciprocal of the second fraction before she multiplied. She also did not find the non-permissible values and her answer should have a denominator of $3x^3$.

The correct solution is:

$$\frac{4x+16}{3x^2} \div \frac{5x+20}{4x}$$

$$= \frac{4x+16}{3x^2} \times \frac{4x}{5x+20}$$

$$= \frac{4\cancel{(x+4)}}{3(x)\cancel{(x)}} \times \frac{4\cancel{x}}{5\cancel{(x+4)}}$$

$$= \frac{4(4)}{3(5)(x)}$$

$$= \frac{16}{15x}, x \neq 0, -4$$

3. Charles' Law relates the temperature and volume of one gas to the temperature and volume of another gas at a constant pressure according to the formula:

$$\frac{V_1}{V_2} = \frac{T_1}{T_2}$$

If $V_1 = \dfrac{2x^2 + x - 10}{x^2 + x - 6}$, $V_2 = \dfrac{x(x-1)}{x^2 - 1}$, and $T_2 = \dfrac{x^3 - 9x}{2x^2 - x - 15}$, find T_1.

Express your answer in simplest form.

Answer:

$$\frac{\dfrac{2x^2 + x - 10}{x^2 + x - 6}}{\dfrac{x(x-1)}{x^2 - 1}} = \frac{T_1}{\dfrac{x^3 - 9x}{2x^2 - x - 15}}$$

Multiply both sides by $\dfrac{x^3 - 9x}{2x^2 - x - 15}$.

$$T_1 = \frac{2x^2 + x - 10}{x^2 + x - 6} \div \frac{x(x-1)}{x^2 - 1} \times \frac{x^3 - 9x}{2x^2 - x - 15}$$

$$T_1 = \frac{2x^2 + x - 10}{x^2 + x - 6} \times \frac{x^2 - 1}{x(x-1)} \times \frac{x^3 - 9x}{2x^2 - x - 15}$$

$$T_1 = \frac{(2x+5)(x-2)}{(x+3)(x-2)} \times \frac{(x-1)(x+1)}{x(x-1)} \times \frac{x(x-3)(x+3)}{(2x+5)(x-3)}$$

$$T_1 = x + 1,\ x \neq -3,\ 2,\ 0,\ 1,\ -\frac{5}{2},\ 3,\ -1$$

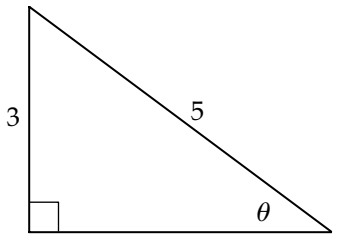
Learning Activity 2.5

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. What are two numbers that sum to 8 and multiply to –65?
2. Camille has $200 to buy a new outfit. She wants to buy a sweater for $55.95, a shirt for $44.99, jeans for $49.99, and new shoes for $54.99. Ignoring tax, does Camille have enough money to buy her outfit?
3. Evaluate: $\left(\frac{3}{7}\right) \times \left(\frac{14}{6}\right)$
4. On Monday it rained 5 mm, Tuesday it rained 1.3 cm, Wednesday it rained 4 mm, and on Thursday it rained 1.7 cm. How many millimetres did it rain in total?
5. Factor: $2x^2 - x - 6$

Use the following right-angled triangle to answer the next three questions.



6. Identify the sine ratio for θ.
7. Identify the cosine ratio for θ.
8. Identify the tangent ratio for θ.

Answers:

1. 13 and −5
2. No ($200 − ($55.95 + $44.99 + $49.99 + $54.99) = $200 − $205.92 or 200 − (56 + 45 + 50 + 55) = 200 − 206)
3. $1 \left(\left(\frac{\cancel{3}^{1}}{\cancel{7}_{1}} \right) \left(\frac{\cancel{14}^{2}}{\cancel{6}_{2}} \right) = \frac{2}{2} \right)$
4. 39 mm (5 mm + 13 mm + 4 mm + 17 mm)

5. $2x^2 - 4x + 3x - 6$

   $2x(x - 2) + 3(x - 2)$

   $(2x + 3)(x - 2)$

6. $\frac{3}{5}$

7. $\frac{4}{5}$

8. $\frac{3}{4}$

Part B: Operations on Rational Expressions

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Add and subtract as indicated.

   a) $\frac{x}{4} + \frac{5x}{4}$

   Answer:

   $$\frac{x + 5x}{4} = \frac{6x}{4} = \frac{3x}{2}$$

   b) $\frac{5x}{2} - \frac{x}{2}$

   Answer:

   $$\frac{5x - x}{2} = \frac{4x}{2} = 2x$$

   c) $\frac{5a}{a + b} + \frac{5b}{a + b}$

   Answer:

   $$\frac{5a + 5b}{a + b} = \frac{5(a + b)}{a + b} = 5$$

d) $\dfrac{3d}{c+d} - \dfrac{d}{c+d}$

Answer:

$$\frac{3d-d}{c+d} = \frac{2d}{c+d}$$

e) $\dfrac{2a^2+4a+17}{(a+2)(a+3)} - \dfrac{a^2-4a+2}{(a+2)(a+3)}$

Answer:

$$\frac{2a^2+4a+17-\left(a^2-4a+2\right)}{(a+2)(a+3)}$$

$$= \frac{2a^2+4a+17-a^2+4a-2}{(a+2)(a+3)}$$

$$= \frac{a^2+8a+15}{(a+2)(a+3)}$$

$$= \frac{(a+5)(a+3)}{(a+2)(a+3)}$$

$$= \frac{a+5}{a+2}$$

2. Find the lowest common denominator for each of the following sets of rational expressions.

a) $\dfrac{4}{x-8}, \dfrac{3x}{2x-16}, \dfrac{4x}{x+1}$

Answer:

Factor the denominator: $\dfrac{4}{x-8}, \dfrac{3x}{2(x-8)}, \dfrac{4x}{x+1}$

$\therefore$ LCD = $2(x-8)(x+1)$

b) $\dfrac{x+1}{(x+2)(x-3)}, \dfrac{x-1}{(x-2)(x-6)(x-3)^2}$

Answer:

LCD: $(x+2)(x-3)^2(x-2)(x-6)$

3. Add and subtract as indicated. Identify the non-permissible values.

a) $\dfrac{4x}{x^2+x-6} - \dfrac{3x}{x^2+5x+6}$

Answer:

$$\frac{4x}{(x+3)(x-2)} - \frac{3x}{(x+3)(x+2)} \quad \text{LCD} = (x+3)(x-2)(x+2)$$

$$= \frac{4x}{(x+3)(x-2)} \cdot \frac{x+2}{x+2} - \frac{3x}{(x+3)(x+2)} \cdot \frac{x-2}{x-2}$$

$$= \frac{4x^2+8x-3x^2+6x}{(x+3)(x-2)(x+2)}$$

$$= \frac{x^2+14x}{(x+3)(x-2)(x+2)}$$

Non-permissible values: $x = -3$, 2, and –2.

b) $\dfrac{3}{x^2-4x+3} + \dfrac{4}{x^2-5x+6}$

Answer:

$$\frac{3}{(x-3)(x-1)} + \frac{4}{(x-2)(x-3)} \quad \text{LCD} = (x-3)(x-1)(x-2)$$

$$= \frac{3}{(x-3)(x-1)} \cdot \frac{x-2}{x-2} + \frac{4}{(x-2)(x-3)} \cdot \frac{x-1}{x-1}$$

$$= \frac{3x-6+4x-4}{(x-3)(x-1)(x-2)}$$

$$= \frac{7x-10}{(x-3)(x-1)(x-2)}$$

Non-permissible values: $x = 3$, 1, and 2.

c) $\dfrac{2x}{x^2-6x-7}-\dfrac{3}{x+1}+\dfrac{3}{x-7}$

Answer:

$$\frac{2x}{(x-7)(x+1)}-\frac{3}{(x+1)}+\frac{3}{x-7} \quad \text{LCD} = (x-7)(x+1)$$

$$=\frac{2x}{(x-7)(x+1)}-\frac{3}{x+1}\cdot\frac{x-7}{x-7}+\frac{3}{x-7}\cdot\frac{x+1}{x+1}$$

$$=\frac{2x-3x+21+3x+3}{(x-7)(x+1)}$$

$$=\frac{2x+24}{(x-7)(x+1)}$$

Non-permissible values: $x = 7$ and -1.

d) $\dfrac{2x}{x^2+5xy}-\dfrac{x}{x^2+8xy+15y^2}$

Answer:

$$\frac{2x}{x(x+5y)}-\frac{x}{(x+3y)(x+5y)}$$

$$=\frac{2}{x+5y}-\frac{x}{(x+3y)(x+5y)} \qquad \text{LCD} = (x+3y)(x+5y)$$

$$=\frac{2}{x+5y}\cdot\frac{x+3y}{x+3y}-\frac{x}{(x+3y)(x+5y)}$$

$$=\frac{2x+6y-x}{(x+3y)(x+5y)}$$

$$=\frac{x+6y}{(x+3y)(x+5y)}$$

Non-permissible values: $x = 0$, $-5y$, and $-3y$.

e) $\dfrac{y-4}{y-3} - \dfrac{y-7}{y+8} + \dfrac{y+8}{y^2+5y-24}$

Answer:

$$\frac{y-4}{y-3} - \frac{y-7}{y+8} + \frac{y+8}{(y+8)(y-3)} \qquad \text{LCD} = (y-3)(y+8)$$

$$= \frac{y-4}{y-3} \cdot \frac{y+8}{y+8} - \frac{y-7}{y+8} \cdot \frac{y-3}{y-3} + \frac{y+8}{(y+8)(y-3)}$$

$$= \frac{y^2+4y-32-(y^2-10y+21)+y+8}{(y-3)(y+8)}$$

$$= \frac{y^2+4y-32-y^2+10y-21+y+8}{(y-3)(y+8)}$$

$$= \frac{15y-45}{(y-3)(y+8)}$$

$$= \frac{15(y-3)}{(y-3)(y+8)}$$

$$= \frac{15}{y+8}$$

Non-permissible values: $y = 3$ and -8.

f) $\dfrac{7}{y+2} - \dfrac{5}{2-y} + \dfrac{y+18}{y^2-4}$

Answer:

$$\frac{7}{y+2} - \frac{5}{(-1)(y-2)} + \frac{y+18}{(y-2)(y+2)}$$

$$= \frac{7}{y+2} + \frac{5}{y-2} + \frac{y+18}{(y-2)(y+2)} \qquad \text{LCD} = (y+2)(y-2)$$

$$= \frac{7}{y+2} \cdot \frac{y-2}{y-2} + \frac{5}{y-2} \cdot \frac{y+2}{y+2} + \frac{y+18}{(y-2)(y+2)}$$

$$= \frac{7y - 14 + 5y + 10 + y + 18}{(y-2)(y+2)}$$

$$= \frac{13y + 14}{(y-2)(y+2)}$$

Non-permissible values: $y = -2$ and 2.

g) $\dfrac{2x}{x-2} + \dfrac{2-3x}{x^2-x-2} - \dfrac{x+2}{x+1}$

Answer:

$$\frac{2x}{x-2} + \frac{2-3x}{(x-2)(x+1)} - \frac{x+2}{x+1}$$

$$= \frac{2x}{x-2} \cdot \frac{x+1}{x+1} + \frac{2-3x}{(x-2)(x+1)} - \frac{x+2}{x+1} \cdot \frac{x-2}{x-2} \qquad \text{LCD} = (x-2)(x+1)$$

$$= \frac{2x^2 + 2x + 2 - 3x - \left(x^2 - 4\right)}{(x-2)(x+1)}$$

$$= \frac{2x^2 - x + 2 - x^2 + 4}{(x-2)(x+1)}$$

$$= \frac{x^2 - x + 6}{(x-2)(x+1)}$$

Non-permissible values: $x = 2$ and -1.

4. Simplify the following rational expressions.

a) $$\frac{\frac{x(x+2)}{5}-\frac{x}{2(x-1)}}{\frac{5}{x+2}+\frac{7}{x-1}}$$

Answer:

$$\left(\frac{x(x+2)}{5}-\frac{x}{2(x-1)}\right)\div\left(\frac{5}{x+2}+\frac{7}{x-1}\right)$$ Rewrite

LCD of first expression: $(5)(2)(x-1)$

LCD of second expression: $(x+2)(x-1)$

Make common denominators:

$$\left(\frac{x(x+2)(2)(x-1)}{5(2)(x-1)}-\frac{5x}{5(2)(x-1)}\right)\div\left(\frac{5(x-1)}{(x+2)(x-1)}+\frac{7(x+2)}{(x-1)(x+2)}\right)$$

$$=\frac{2(x)\left(x^2+x-2\right)-5x}{5(2)(x-1)}\div\frac{5x-5+7x+14}{(x+2)(x-1)}$$ Simplify the numerators

$$=\frac{2x^3+2x^2-4x-5x}{5(2)(x-1)}\div\frac{5x-5+7x+14}{(x+2)(x-1)}$$ Simplify the numerators

$$=\frac{2x^3+2x^2-9x}{5(2)(x-1)}\div\frac{12x+9}{(x+2)(x-1)}$$ Simplify the numerators

Multiply by the reciprocal and factor

$$=\frac{x\left(2x^2+2x-9\right)}{5(2)(x-1)}\times\frac{(x+2)(x-1)}{3(4x+3)}$$

$$=\frac{x\left(2x^2+2x-9\right)(x+2)}{30(4x+3)}$$ Multiply and cancel

b) $\dfrac{\dfrac{1}{x+2}+\dfrac{1}{x-2}}{\dfrac{x}{x^2-4}+\dfrac{1}{x+2}}$

Answer:

$$\left(\frac{1}{x+2}+\frac{1}{x-2}\right)\div\left(\frac{x}{x^2-4}+\frac{1}{x+2}\right)$$

$$\left(\frac{1}{x+2}+\frac{1}{x-2}\right)\div\left(\frac{x}{(x-2)(x+2)}+\frac{1}{x+2}\right)$$

LCD of first expression: $(x-2)(x+2)$

LCD of second expression: $(x-2)(x+2)$

$$\left(\frac{x-2}{(x+2)(x-2)}+\frac{x+2}{(x+2)(x-2)}\right)\div\left(\frac{x}{(x-2)(x+2)}+\frac{x-2}{(x+2)(x-2)}\right)$$

$$=\frac{x-2+x+2}{(x+2)(x-2)}\div\frac{x+x-2}{(x-2)(x+2)}$$

$$=\frac{2x}{(x+2)(x-2)}\div\frac{2x-2}{(x-2)(x+2)}$$

$$=\frac{2x}{(x+2)(x-2)}\times\frac{(x-2)(x+2)}{2(x-1)}$$

$$=\frac{x}{x-1}$$

c) $\dfrac{x}{2x-4}+\left(\dfrac{x-3}{x^2-3x+2}\div\dfrac{x^2-9}{x-1}\right)$

Answer:

$$\frac{x}{2x-4}+\left(\frac{x-3}{x^2-3x+2}\div\frac{x^2-9}{x-1}\right)$$

$$=\frac{x}{2(x-2)}+\left(\frac{x-3}{(x-2)(x-1)}\times\frac{x-1}{(x+3)(x-3)}\right)$$ Factor and multiply by the reciprocal

$$=\frac{x}{2(x-2)}+\frac{1}{(x-2)(x+3)}$$ Cancel to reduce

$$\text{LCD}=2(x-2)(x+3)$$

$$=\frac{x(x+3)}{2(x-2)(x+3)}+\frac{2}{2(x-2)(x+3)}$$

$$=\frac{x^2+3x+2}{2(x-2)(x+3)}\text{ or }\frac{(x+2)(x+1)}{2(x-2)(x+3)}$$

5. Kahlia is trying to simplify a rational expression. However, the answer she received does not match the answer at the back of the textbook. Kahlia's work is shown below.

$$\frac{x-1}{(x+2)(x-3)} - \frac{2x}{(x+2)}$$

$$= \frac{x-1-2x}{(x+2)(x-3)}$$

$$= \frac{-x-1}{(x+2)(x-3)}$$

Explain Kahlia's mistake and simplify the rational expression correctly.

Answer:

The two original rational expressions do not have the same denominator. Therefore, these rational expressions need to be changed to equivalent rational expressions that have the same denominator before you can subtract them.

$$\frac{x-1}{(x+2)(x-3)} - \frac{2x}{(x+2)}$$ LCD: $(x+2)(x-3)$

$$= \frac{x-1}{(x+2)(x-3)} - \frac{2x(x-3)}{(x+2)(x-3)}$$

$$= \frac{x-1}{(x+2)(x-3)} - \frac{2x^2-6x}{(x+2)(x-3)}$$

$$= \frac{x-1-2x^2+6x}{(x+2)(x-3)}$$

$$= \frac{-2x^2+7x-1}{(x+2)(x-3)}$$

6. A swimmer swam 50 laps at a constant speed. For the remaining 50 laps, he reduced his speed by 3 km/h. Write a rational expression for the total time of this swimmer's workout. You do not need to simplify your answer. **Note:** distance = (speed)(time), or $d = rt$.

   Answer:

   $d = rt$

   $t = \frac{d}{r}$

   Let x = original constant speed

   Time of first 50 laps: $\frac{50}{x}$

   Time of second 50 laps: $\frac{50}{x-3}$

   Total time of 100 laps: $\frac{50}{x} + \frac{50}{x-3}$ hours

7. A rational expression puzzle is shown on the following page. Work with your learning partner to cut out each puzzle piece and reassemble them all in a 4×4 configuration so that each expression matches its factored form or its simplified form. If you are stuck, you can look in the Module 2 Answer Key to find helpful hints.

Rational Expression Puzzle Solution

$(x-3)(x+1)$	$(4x+3)(x+1)$	x^2+25	$(x-3)(x-1)$
A	**F**	**M**	**K**
$(x+7)(x+2)$	$(4x-3)(x-1)$	$x+1$	$(x+5)(x-5)$
$4x^2-7x+3$	$\dfrac{x}{x-3}+\dfrac{2}{x+4}$	$(x+3)(x-1)$	$\dfrac{x^2+7x+12x}{x^2+4x}$
$\dfrac{x^2-x-2}{x^2-4}$	$\dfrac{x^2+5x+6}{x^2+x-2}\div\dfrac{x+3}{x^2-1}$	$(x+3)^2-6(x+3)-16$	$\dfrac{5}{x}+\dfrac{4}{x+1}$
$\dfrac{(x+2)}{(1+x)}$	$\dfrac{(x-3)(x+4)}{x^2+6x-6}$	x^2+2x-3	$x+3$
N	**E**	**C**	**D**
$\dfrac{x-3}{x}$	$3x^2(2x+1)$	$\dfrac{-x(4x+33)}{(x+7)(x+2)}$	$\dfrac{3x+6}{2x^2-x-10}\times\dfrac{4x^2-25}{6}$
$(x-6)(x+6)$	x^2-36	$3(3x-5)(6x-13)$	$2(x+2)(x-9)$
$\dfrac{5}{x}-\dfrac{4}{x-2}$	$6(3x-4)^2-21(3x-4)+15$	$2(x-6)^2+10(x-6)-48$	$\dfrac{x}{x+7}+\dfrac{5x}{x+2}$
$\dfrac{x-10}{x(x-2)}$	$6x^3+3x^2$	$\dfrac{x}{x+7}-\dfrac{5x}{x+2}$	$\dfrac{2x+5}{2}$
I	**L**	**O**	**H**
$1-100x^2$	$\dfrac{a}{2c}$	$2y(3xy-1)$	$\dfrac{5x^2+13x+18}{(x-1)(x+2)(x+3)}$
$(10x-1)(10x+1)$	$100x^2-1$	$\dfrac{3x}{4}$	$(3x+2y+3)(3x-2y+5)$
$\dfrac{x+1}{8x}\times\dfrac{2x}{x^2-1}$	$\dfrac{x}{2}+\dfrac{x}{4}$	$(3x+4)^2-(2y-1)^2$	$\dfrac{3}{x}-\dfrac{x}{2}$
$\dfrac{1}{4(x-1)}$	$25x^2+20x+4$	$3(2x+5)^2+10(2x+5)-8$	$\dfrac{3}{x-1}+\dfrac{2x}{(x+2)(x+3)}$
B	**G**	**P**	**J**
$\dfrac{a}{4c}$	$\dfrac{a^2bc}{8m}\div\dfrac{abc^2}{4m}$	$6xy^2-2y$	$\dfrac{x}{2}$
$(x+2)(x+1)$	x^2+3x+2	$(5x+2)^2$	$(2x+9)(6x+13)$
$x-2$	$\dfrac{1}{4(x+1)}$	x^2-4	$x+3$

Notes

Grade 11 Pre-Calculus Mathematics (30S)

Module 3
Quadratic Functions

Module 3:
Quadratic Functions

Introduction

Quadratic functions have many applications in everyday life. Quadratic functions appear in many sports including football, basketball, soccer, and even golf. Quadratic functions can also model situations including satellite dishes, suspension bridges, and maximizing the profit of a business.

In this module, you will look at quadratic functions, their important properties, and how to graph them. This will allow you to learn how to create quadratic functions that model a certain situation. By the end of this module, you will be able to solve application problems such as those modelling the situations listed above.

Assignments in Module 3

When you have completed the assignments for Module 3, submit your completed assignments to the Distance Learning Unit either by mail or electronically through the learning management system (LMS). The staff will forward your work to your tutor/marker.

Lesson	Assignment Number	Assignment Title
2	Assignment 3.1	Quadratic Functions and Transformations
4	Assignment 3.2	Graphing Quadratic Functions Using Transformations
6	Assignment 3.3	Completing the Square and Finding the Vertex
7	Assignment 3.4	Applications of Quadratic Functions

Resource Sheet

When you write your midterm examination, you are encouraged to take a Midterm Examination Resource Sheet with you into the examination. This sheet will be one letter-sized page, 8½ " by 11", with both sides in your handwriting or typewritten. You will submit it with your examination, but you do not receive any marks for it.

Many students have found that preparing a resource sheet is an excellent way to review. It provides you with a summary of the important facts of each module. You should complete a resource sheet for each module to help with your studying and reviewing. Lesson summaries and module summaries are included for you to use as a guide.

You may use the list of instructions provided below to help you with preparing your resource sheet for the material in Module 3. On this sheet, you should record math terms and definitions, formulas, sample questions, or a list of places where you often make mistakes. You should also identify special areas that require extra attention or review by writing the page numbers.

After you have completed each module's resource sheet, you may summarize the sheets from Modules 1, 2, 3, and 4 to prepare your Midterm Examination Resource Sheet. The midterm examination for this course is based on Modules 1 to 4.

Resource Sheet for Module 3

As you go through the lessons of this module, you may want to consider the following suggestions regarding the creation of a resource sheet.

1. List all the important math terms, and define them if necessary.
2. List all the formulas and perhaps a sample problem that shows how the formula is used.
3. If necessary, write the solutions to some problems, showing in detail how you did the calculations.
4. Copy any questions that represent the key points of the lesson, and perhaps include the solutions as well.
5. Identify the problems you found most difficult, and copy the page numbers onto the resource sheet so that you can review them before writing the examination. You may also copy the problems and the solutions onto your resource sheet, and later write them onto your Midterm Examination Resource Sheet.
6. Write any comments, ideas, shortcuts, or other reminders that may be helpful during an examination.

Lesson 1: What Is a Quadratic Function?

Lesson Focus

In this lesson, you will

- ❑ learn how to distinguish between linear and quadratic functions
- ❑ learn how to find the vertex, domain and range, direction of opening, minimum and maximum values, axis of symmetry, and intercepts of a quadratic function in general form
- ❑ learn how to graph quadratic functions given the general form

Lesson Introduction

In *Grade 10 Introduction to Applied and Pre-Calculus Mathematics*, you studied linear functions. You learned how linear functions, when graphed, are straight lines. You also learned how to find the slope of these lines. In this lesson, you will begin to learn about quadratic functions. Quadratic functions are *not* straight lines when graphed; instead, they are curved. More specifically, quadratic functions have *one* turn. This turn, called the **vertex**, is just one of the important properties of quadratic functions that you will be learning about in this lesson.

Functions

Before you begin to look at quadratic functions, it will be helpful for you to review your knowledge of linear functions.

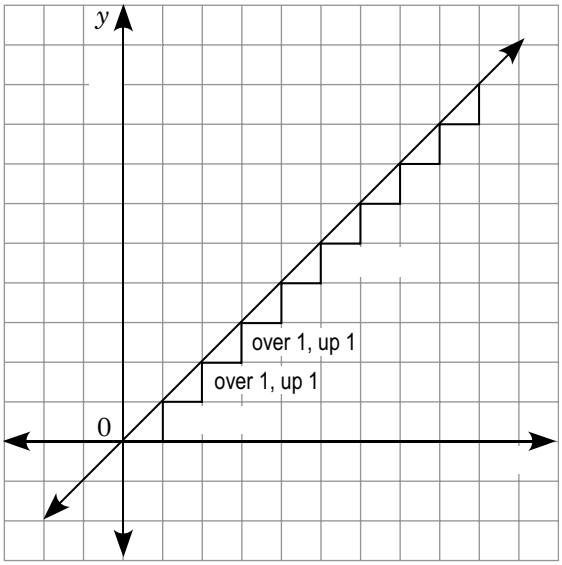
Linear Functions

Recall the definition of a linear function. A **linear function** is a function whose graph is a straight line. Its form is $f(x) = mx + b$. Remember that a straight line equation is of the form $y = mx + b$.

This first section contains many definitions. These definitions should be familiar to you from previous courses. However, you can copy these definitions on your resource sheet if you need to.

Remember, on a line, the rate of change of y with respect to x is a constant called the slope (m).

For example:



The change in y-value of this line is 1. Also, the change in the x-value of this line is 1. Therefore, the slope of this line is $m = \frac{\text{rise}}{\text{run}} = \frac{1}{1} = 1$.

What are the x- and y-intercepts of this function?

Recall:

- The ***x*-intercept** of a function is the x-coordinate(s) of the point(s) where the graph of the function intersects the x-axis. The y-coordinate of this point is zero.
- The ***y*-intercept** of a function is the y-coordinate of the point where the graph of the function intersects the y-axis. The x-coordinate of this point is zero.

In the graph above, when the y-coordinate is 0, the x-coordinate is also 0. Therefore, the x-intercept is 0.

As well, when the x-coordinate is 0, the y-coordinate is also 0. Therefore, the y-intercept is 0.

Note: In the linear equation $y = mx + b$, the y-intercept is symbolized using the letter b.

Example 1

Find the x- and y-intercepts of the function $f(x) = -2x + 3$. Then use these intercepts to graph the function.

Solution

To find the x-intercept, let $f(x) = 0$ since $f(x)$ is always 0 for any point on the x-axis. Then solve for x.

$$-2x + 3 = 0$$

$$-2x = -3$$

$$x = \frac{3}{2}$$

The x-intercept is $\frac{3}{2}$.

To find the y-intercept, let $x = 0$ since the x-coordinate is always zero for any point on the y-axis. Then solve for y.

$$y = -2x + 3$$

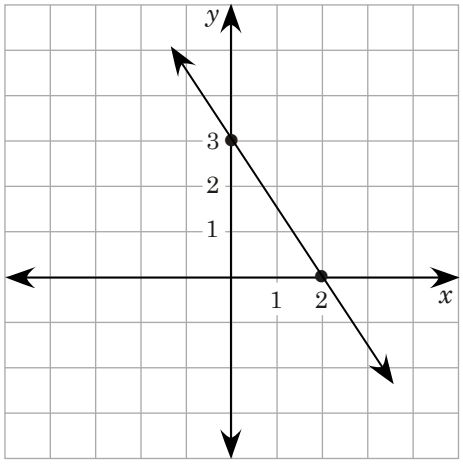
$$y = -2(0) + 3$$

$$y = 3$$

The y-intercept is 3.

Note that the x-intercept or the y-intercept is only one number and not the coordinates of a point on the graph.

The graph would be:



When you are given the graph of a linear function, you should be able to write its rule or equation.

Example 2

Find the equation of the straight line that passes through the points (0, 2) and (−1, 0).

Solution

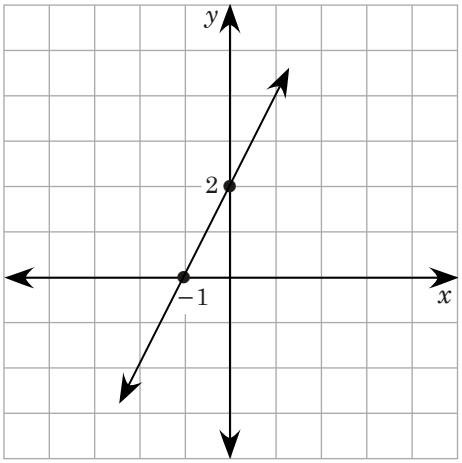
Recall:

$$\text{Slope} = m = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of this line = $\frac{0-2}{-1-0} = \frac{-2}{-1} = 2$. Its y-intercept is 2. You know this because you are given the point (0, 2).

The equation of the linear function is $f(x) = 2x + 2$.

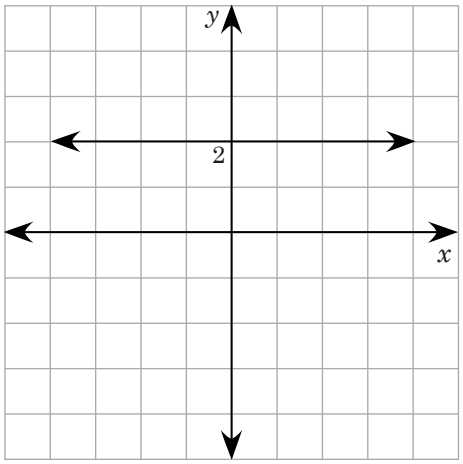
The graph of this linear function is shown below.



Example 3

A special case of the linear function results when the slope of the function is 0.

In this diagram, $b = 2$ and the slope is 0. Find its equation.



Solution

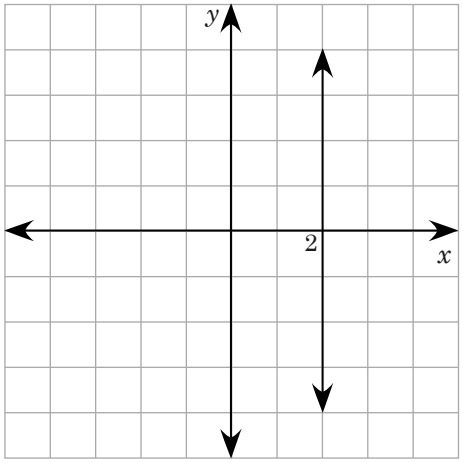
The equation is $f(x) = 0x + 2$. Therefore, $f(x) = 2$ or $y = 2$.

Every point on this horizontal line has a y-value of 2. The y-intercept of this line is 2. There is no x-intercept. The domain is $\{x \mid x \in \Re\}$ and the range is $\{2\}$.

Example 4

A vertical straight line is said to have a slope that is undefined. The line has rise but no run. The slope formula becomes $m = \dfrac{\text{rise}}{\text{run}} = \dfrac{\text{rise}}{0} =$ no answer.

The equation of the line shown below is $x = 2$. Is this a function or a relation?



Solution

Recall: A **relation** is a set of ordered pairs (x, y). A **function** is a relation such that for each x-value there is exactly one value of y.

Therefore, a vertical line is not a linear function but it is a relation. Two points on the line would be (2, 0) and (2, 1), both having the same x-value.

Every point on this vertical line has an x-value of 2. Therefore, the x-value of 2 has more than one value of y associated with it. Notice the x-intercept of the line is also 2. There is no y-intercept.

Not every pattern is linear. By studying quadratic functions and relations, you greatly expand the applications that you can model in science, engineering, business, sports, and in many other fields.

Quadratic Functions

Quadratic functions take the shape of a **parabola** when graphed. A parabola is basically a 'U' shape. The shape of a parabola is created because the quadratic functions contain a squared term (for example, $y = x^2$ or $y = 5x^2$).

Remember that if you multiply two linear functions together you get a squared term.

For example, if $f(x) = x$ and $g(x) = x + 3$, then $[f(x)][g(x)] = x(x + 3) = x^2 + 3x$.

A **quadratic function** is a function that can be written in the general form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$. Note that in this course, a, b, and c will be rational numbers only.

Include all definitions from this section on your resource sheet.

Quadratic functions can be written in various forms:

$f(x) = ax^2 + bx + c, a \neq 0$

$f: x \rightarrow ax^2 + bx + c, a \neq 0$

$y = ax^2 + bx + c, a \neq 0$

Example 5

Here are six examples of quadratic functions. For each of the six examples, determine *a*, *b*, and *c*. Remember that *a* is the coefficient of the squared term, x^2; *b* is the coefficient of the linear term, *x*; and *c* is the constant term.

a) $f:x \to 2x^2 + x + 1$

b) $f(x) = x^2$

c) $y = 3x^2 - 1$

d) $f(x) = 2 + \frac{1}{2}x^2$

e) $g(x) = 7x - x^2$

f) $y = x - x^2$

Solution

	a	*b*	*c*
a)	2	1	1
b)	1	0	0
c)	3	0	−1
d)	$\frac{1}{2}$	0	2
e)	−1	7	0
f)	−1	1	0

Example 6

Determine whether the following are quadratic functions.

a) $f(x) = 3x^2$

b) $x^2 + \frac{1}{2}y = 2$

c) $y = 2x + 5$

d) $y^2 = x$

Solution

a) Yes, $a = 3$, $b = 0$, and $c = 0$.

b) Yes. Re-arrange to get

$$\frac{1}{2}y = -x^2 + 2$$

$$y = -2x^2 + 4 \quad \text{(multiply by 2)}$$

with $a = -2$, $b = 0$, and $c = 4$.

c) Since there is no squared term, it is not a quadratic function.

d) The relation $y^2 = x$ is not a quadratic function because y is squared rather than x. Instead, it is a quadratic relation.

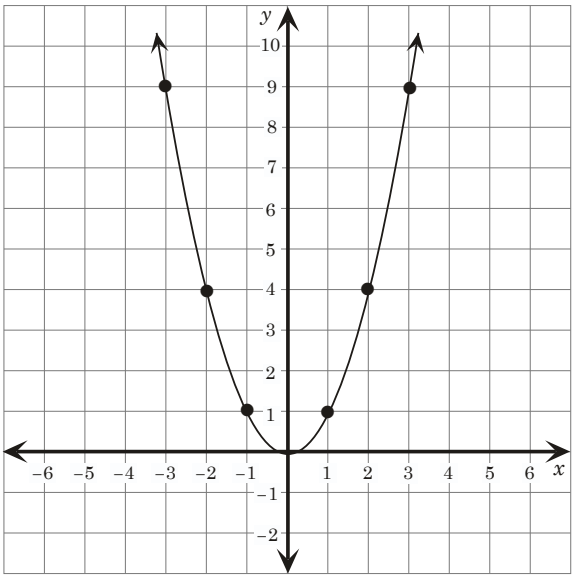
In the following example, you will graph a quadratic function where $a = 1$, $b = 0$, and $c = 0$. This graph will give you the basic quadratic function graph. Its table of values and its graph should be memorized. This quadratic function is used extensively in the discussion of transformations in Lessons 2, 3, and 4.

Example 7

Given the function $f(x) = x^2$, draw the graph of f. Make a table of values for $-3 \leq x \leq 3$.

Solution

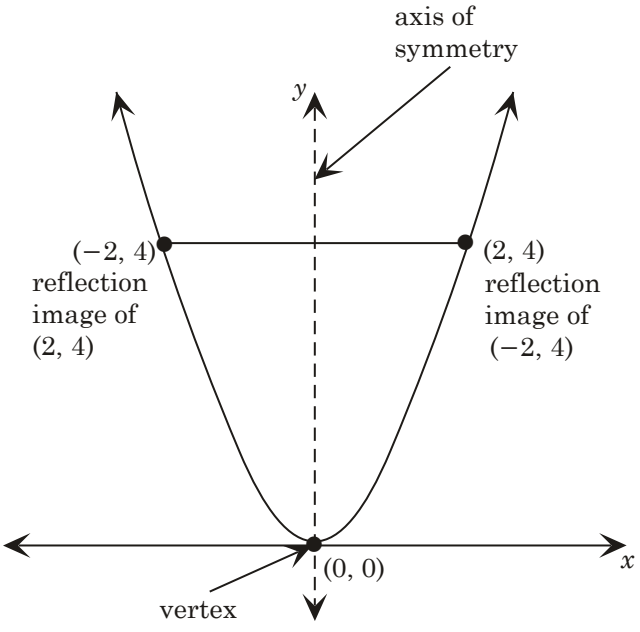
x	$f(x) = x^2$
−3	9
−2	4
−1	1
0	0
1	1
2	4
3	9



From the table of values, the points are plotted. A smooth curve is drawn through the points.

From the above graph of $f(x) = x^2$, note the following:

- The domain is all real numbers or in set notation is $\{x \mid x \in \Re\}$. Recall that the **domain** is the set of all x-values for which a function is defined.
- The range is $\{y \mid y \geq 0, y \in \Re\}$. Recall that the **range** is the set of all y-values for which a function is defined.
- The turning point of a parabola is called the **vertex**. The vertex of the graph of a quadratic function is defined as the point where the graph has its maximum y-value if the parabola opens down and where it has its minimum y-value if the parabola opens up. In this example, the vertex is (0, 0).
- The parabola opens upward. When a parabola opens upward, its minimum y-value is at its vertex. This function is said to have a **minimum value** of 0. The minimum value is the y-coordinate of the vertex when the parabola opens up.
- An imaginary vertical line passing through the vertex is the **axis of symmetry**. This line divides a parabola into two symmetrical parts, which are mirror images of each other, using the axis of symmetry as the mirror. Every point on the parabola except its vertex has a reflection image with respect to the line of symmetry. In this example, the y-axis is the axis of symmetry. Its equation is $x = 0$.



- The **zeros** of a quadratic function are the value(s) of x which make the quadratic function zero. These values of x also are called the x-intercepts of the graph and occur where the graph crosses or touches the x-axis. In this example, the quadratic function is zero when $x = 0$. Notice that this is the same definition as an x-intercept. Thus, zero of a function and the x-intercept of the graph mean the same thing.

Make sure you include the definitions for domain, range, vertex, minimum value, axis of symmetry, x-intercept, and zeros on your resource sheet.

Example 8

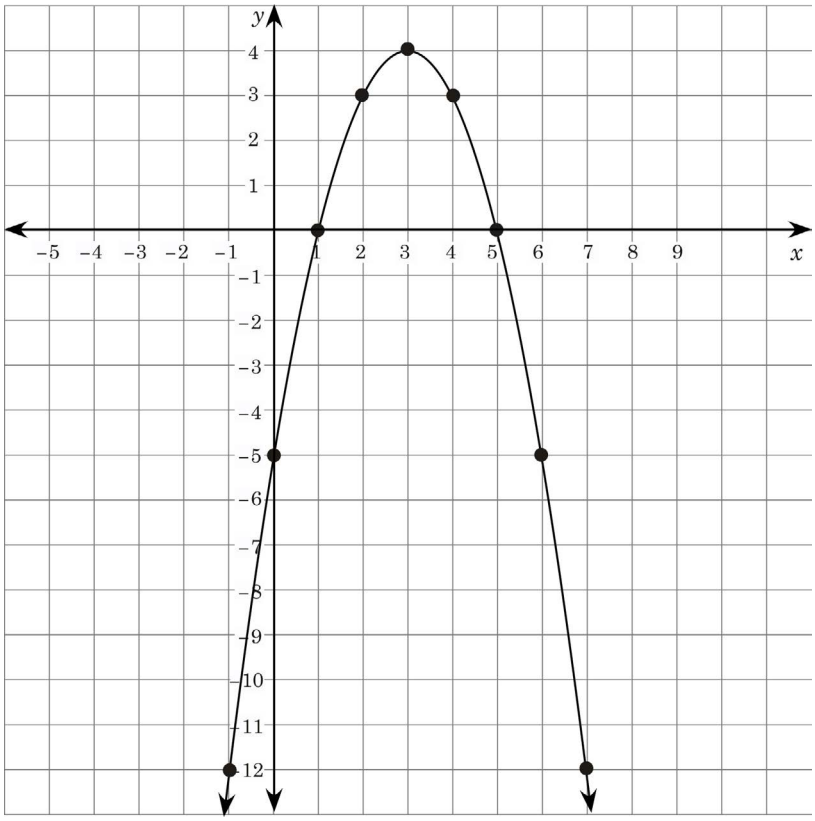
Draw the graph of $g(x) = -x^2 + 6x - 5$. Use a table of values for $-1 \leq x \leq 7$.

Write the domain, range, coordinates of the vertex, the equation of the axis of symmetry, x-intercept(s), y-intercept, and zeros.

Solution

Table of values

x	-1	0	1	2	3	4	5	6	7
$g(x)$	-12	-5	0	3	4	3	0	-5	-12



Note: You can use any x-values for the table of values. The table of values above used x-values that were easy to calculate and easy to graph.

Domain: $\{x \mid x \in \Re\}$

Range: $\{y \mid y \leq 4, y \in \Re\}$ }

Vertex: (3, 4)

Equation of axis of symmetry: $x = 3$

x-intercepts: 1, 5

y-intercept: -5

Zeros: 1, 5

This function is said to have a **maximum value** when $y = 4$. The graphs of quadratic functions have a **maximum value** when the graphs open downward. This maximum value is the y-coordinate of the vertex when the parabola opens downward.

Include the definition for maximum value on your resource sheet.

Learning Activity 3.1

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Evaluate for $t = 5$: $3t^2 - 6t + 8$
2. The daily recommended intake of water is around 2 litres. If Ashton drinks 250 mL of water for breakfast, 550 mL of water for lunch, 620 mL of water during basketball practice, and 300 mL during dinner, how much water does he still need drink to reach the recommended amount?
3. Solve for x: $\frac{7}{8} + 4 = \frac{x}{8}$
4. Ms. Sparling's class has 24 students. If there are 3 girls for every boy, how many boys are in the class?

continued

Learning Activity 3.1 (continued)

5. What is the greatest common factor of 3^4 and 3^5?

6. Express as a radical: $16^{\frac{4}{3}}$. Do not simplify.

7. Rationalize the denominator: $\dfrac{9\sqrt{2}}{4\sqrt{5}}$

8. Find the slope of the line perpendicular to the line $2x + y = 1$.

Part B: Quadratic Functions

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Draw the graph of the following functions:

a) $y = 3x + 2$

b) $2x - y + 1 = 0$

c) Join these points with a smooth curve.

x	-3	-2	0	2	3
y	10	5	1	5	10

2. Find the slope and intercepts for the linear functions in Question 1.

3. Which of the following describe a quadratic function?

   a) $f : x \to 3x^2$

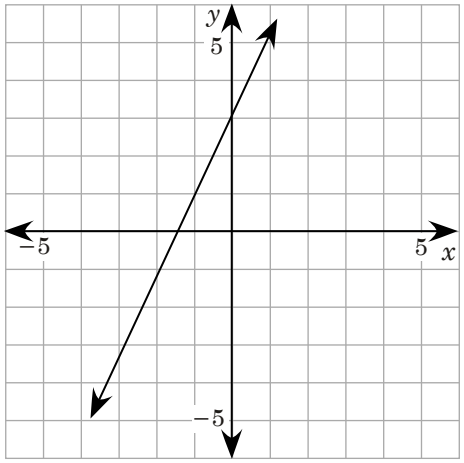
   b) $g : x \to 2x^3 - 5$

   c) $y = 2x^2 - x + 1$

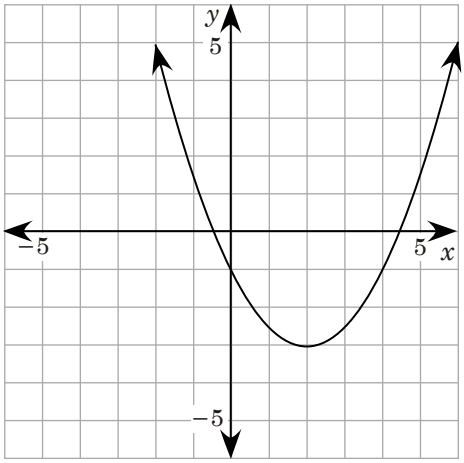
   d) $f(x) = 3x - \dfrac{1}{x}$

continued

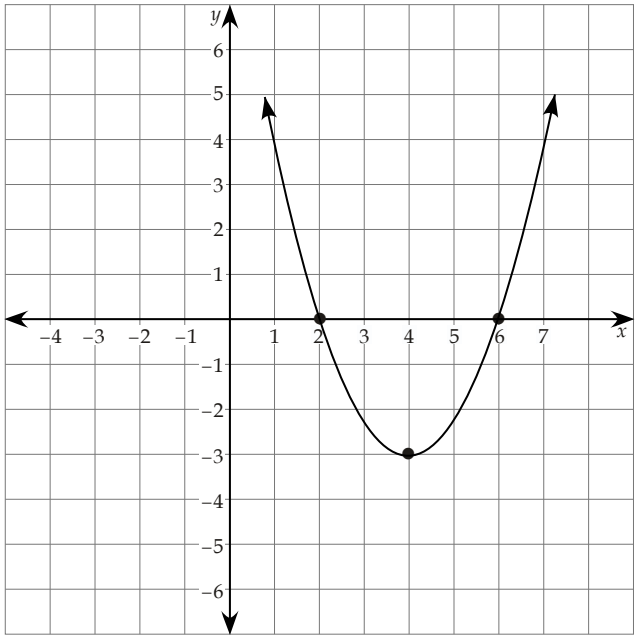
e)



f)



4. A graph of a quadratic function is shown below.



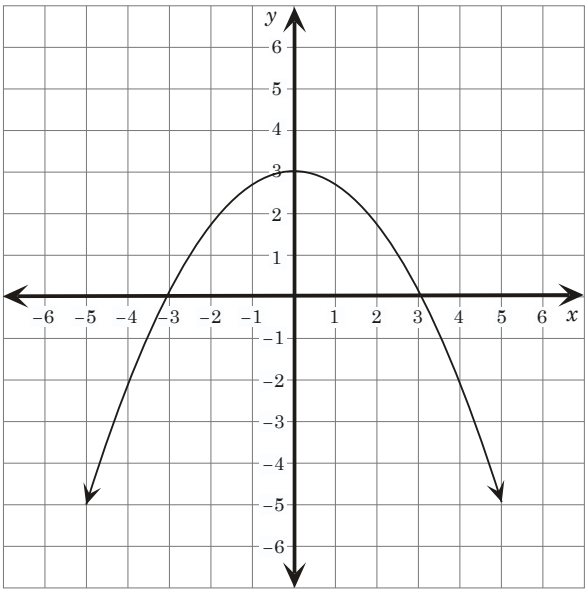
a) What is the domain of the graph?
b) What is the range of the graph?
c) What are the coordinates of the vertex?
d) What is the equation of the axis of symmetry?
e) What are the zeros of the function?
f) What are the *x*-intercepts?
g) Does the curve have a maximum? What is its value?
h) Does the curve have a minimum? What is its value?

continued

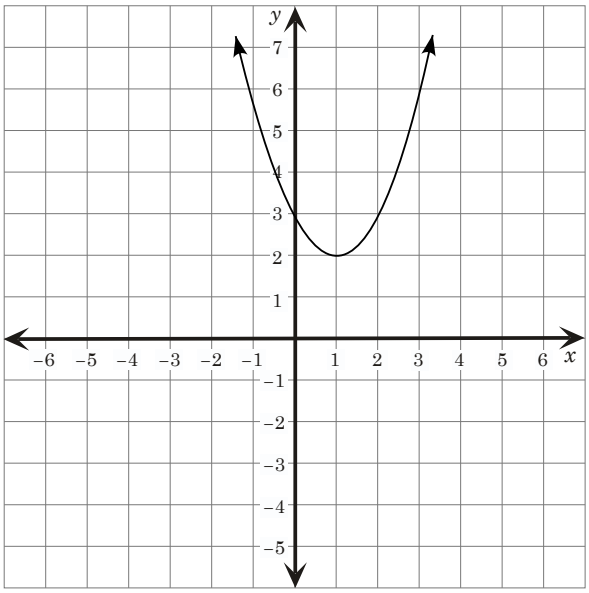
Learning Activity 3.1 (continued)

5. For each of the following, write the domain, range, coordinates of the vertex, equation of the axis of symmetry, y-intercept, and maximum or minimum value.

a)



b)



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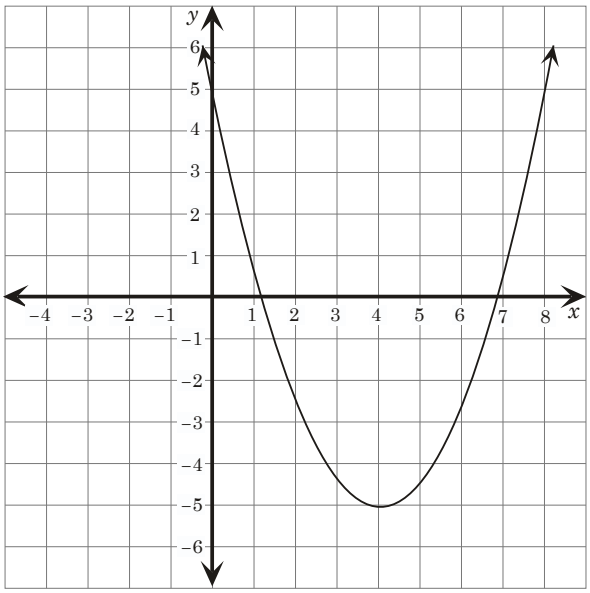
c)



6. Sketch the graphs of the following quadratic functions. Use a suitable table of values. Find the domain, range, coordinates of vertex, equation of axis of symmetry, zeros, x-intercept(s), y-intercept, and the maximum value or minimum value.

   a) $f(x) = x^2 - 4x - 5$

   b) $f : x \to -x^2 - 2x + 8$

7. Match the equation with its graph. Use a table of values to graph each function if necessary.

   a) $y = x^2 - 2$

   b) $y = x^2 + 4x - 1$

   c) $y = -x^2 - 4x - 3$

   d) $y = x^2 + 2$

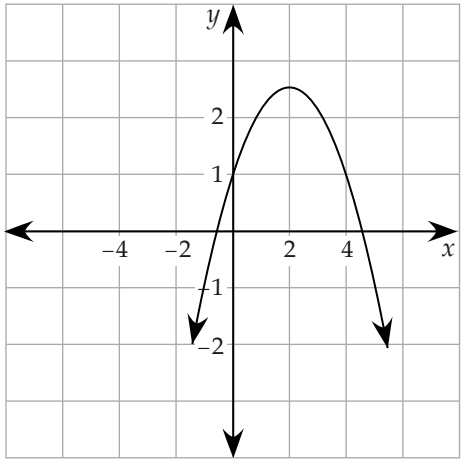
   e) $y = -x^2 + 4x + 1$

   f) $y = -x^2 + 2$

continued

Learning Activity 3.1 (continued)

i)



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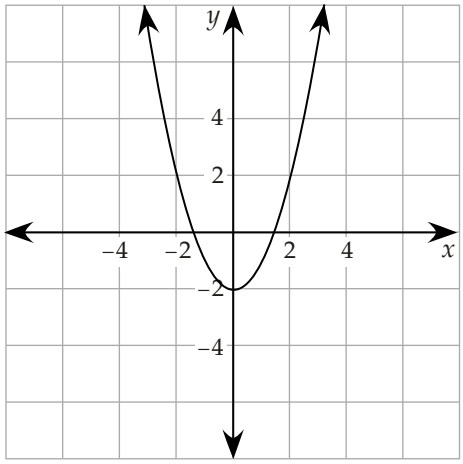

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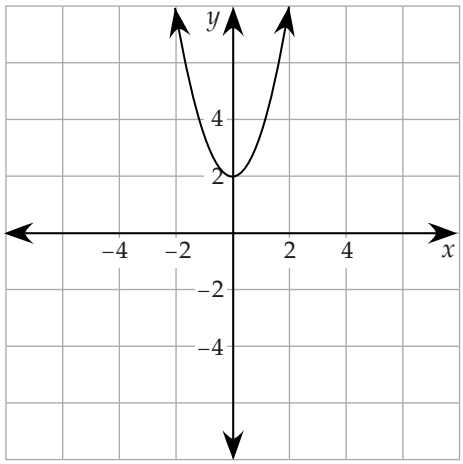

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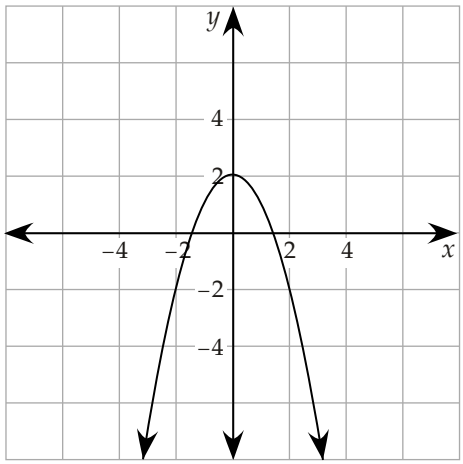

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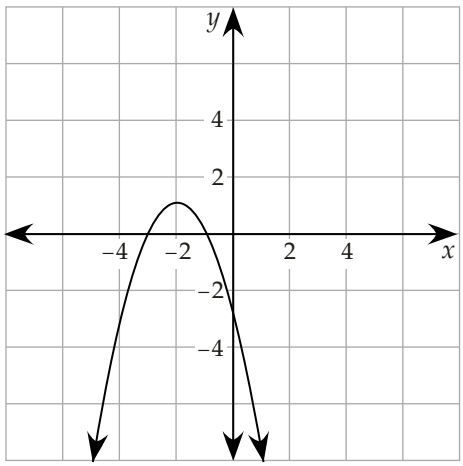

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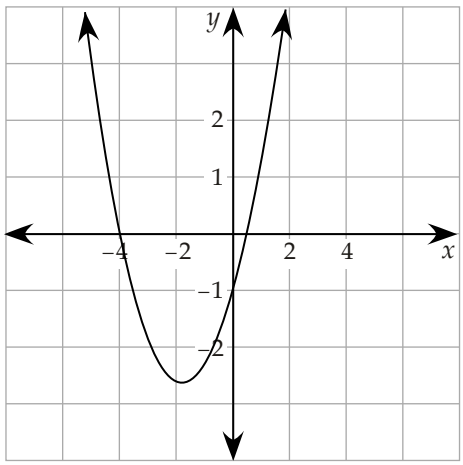


continued

Learning Activity 3.1 (continued)

8. a) The point (4, 3) is on the graph of $f(x) = 2x^2 - 8x + 3$. Use symmetry to find the coordinates of another point on the parabola if the axis of symmetry is $x = 2$.

   b) A parabola has a vertex at (2, 3) and passes through the point P(−1, 6). Find the coordinates of a third point on the parabola.

   c) Given certain reflection image points (0, 2) and (4, 2) on a parabola, find the equation of the line of symmetry.

Lesson Summary

In this lesson, you first considered linear functions. Linear functions are functions that produce a straight line when graphed. You then looked at quadratic functions. Quadratic functions are functions that produce a parabola when graphed. You learned how to find certain characteristics of quadratic functions including the vertex and axis of symmetry. You also learned how to graph quadratic functions using a table of values. In the next lesson, you will begin to look at transformations of quadratic functions.

Notes

Lesson 2: Quadratic Functions $y = ax^2$ and $y = ax^2 + q$

Lesson Focus

In this lesson, you will

- ❑ learn how to determine basic characteristics of the quadratic functions $y = ax^2$ and $y = ax^2 + q$
- ❑ learn how to use translations, stretches, compressions, and reflections with respect to the quadratic function $y = x^2$ to obtain a new quadratic function of the form $y = ax^2 + q$

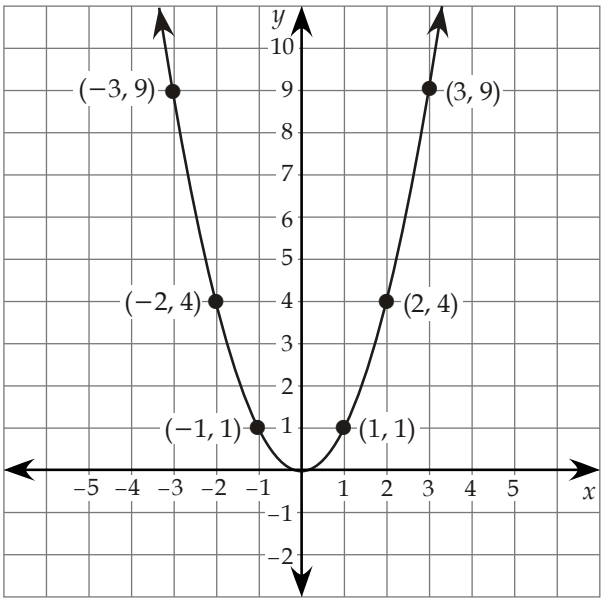
Lesson Introduction

In Lesson 1, you looked at quadratic functions in general form: $y = ax^2 + bx + c$. In this lesson, you will begin to look at quadratic functions in *vertex* form: $y = a(x - p)^2 + q$. These two forms of the quadratic function are useful in different situations. The general form of the quadratic function is useful when you are trying to determine the zeros of a quadratic function using the quadratic formula or by factoring. You will learn how to do this in Module 4. The *vertex* form of the quadratic function is useful when you are finding the graph of a quadratic function.

Quadratic Functions

You have learned how to find characteristics of the graphs of quadratic functions that were written in the form $f(x) = ax^2 + bx + c$, where $a \neq 0$.

The simplest and most basic quadratic function is $f(x) = x^2$. The functional notation $f(x) = x^2$ means the same thing as $y = x^2$. These notations will be used interchangeably in this course.



Observe that for $f(x) = x^2$, if (x, y) is a point on the graph, $(-x, y)$ is also. This is because $y = x^2$ and $y = (-x)^2$ produce the same y-value. For the graph $f(x) = x^2$, the y-axis is the line of symmetry. The point (0, 0) is the minimum or lowest point of the graph and is called the vertex.

Throughout this lesson and Lesson 3, you will see how a, p, and q in the quadratic function $f(x) = a(x - p)^2 + q$, determine the size of the corresponding parabola and its placement on the coordinate plane. These changes that occur in the placement and size of a function are called **transformations**.

The Effect of 'a' in the Quadratic Function $y = ax^2$

Example 1

Use a table of values or a graphing calculator to graph the following functions on the same coordinate plane. State the similarities and differences that occur.

a) $f(x) = x^2$

b) $g(x) = 2x^2$

c) $k(x) = \frac{1}{2}x^2$

Solution

A table of values for each of the functions is listed below.

	x	-3	-2	-1	0	1	2	3
a)	$f(x)$	9	4	1	0	1	4	9
b)	$g(x)$	18	8	2	0	2	8	18
c)	$k(x)$	$\frac{9}{2}$	2	$\frac{1}{2}$	0	$\frac{1}{2}$	2	$\frac{9}{2}$

Notice that $f(x)$ is the basic quadratic function.

You have: $g(-3) = 18$ which is equal to $2 \cdot f(-3) = 2(9) = 18$

$g(-2) = 8$ which is equal to $2 \cdot f(-2) = 2(4) = 8$, and so on.

Also, $k(-3) = \frac{9}{2}$ which is equal to $\frac{1}{2} f(-3) = \frac{1}{2}(9) = \frac{9}{2}$

$k(-2) = 2$ which is equal to $\frac{1}{2} f(-2) = \frac{1}{2}(4) = 2$, and so on.

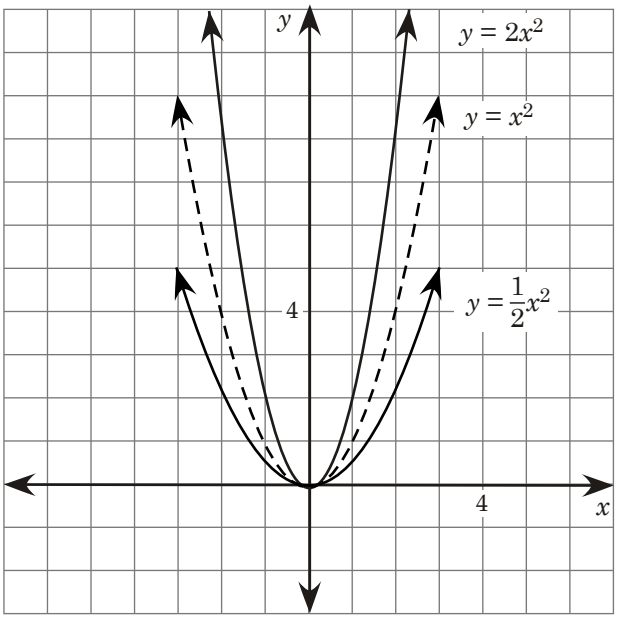
In general, if $f(x) = x^2$ and $j(x) = ax^2$, then $j(x) = a \cdot f(x)$ for all x. This means the y-values are affected by the multiplication of $f(x)$ by a. If (x, y) is a point on $f(x)$, then (x, ay) is a point on $j(x)$. Mapping notation is written as $(x, y) \rightarrow (x, ay)$.

Include this description and the following rules for 'a' on your resource sheet.

The effect of 'a' on the graph $f(x) = ax^2$:

- If $a > 1$, the graph of $f(x) = ax^2$ can be obtained from the graph of $y = x^2$ with a **vertical stretch** of the graph of $y = x^2$ by multiplying the y-values by a. The resulting graph is narrower than the graph of $y = x^2$.
- If $0 < a < 1$ (any real number between 0 and 1), the graph of $f(x) = ax^2$ can be obtained from the graph of $y = x^2$ with a **vertical compression** of the graph of $y = x^2$ by multiplying each of the y-values by a. The resulting graph is wider than the graph of $y = x^2$.

The following graph will help you to see the difference between a vertical stretch and a vertical compression. For comparison, the graph of $f(x) = x^2$ is dashed. Remember, functional notation is used interchangeably—that is, $f(x) = x^2$ means the same as $y = x^2$.



Compare some of the coordinates of $y = x^2$ and $y = 2x^2$, as shown below.

$y = x^2$	(0, 0)	(1, 1)	(2, 4)
$y = 2x^2$	(0, 0)	(1, 2)	(2, 8)

In each case, the y-values of $y = 2x^2$ are double the y-values of $y = x^2$. Therefore, the graph of $y = 2x^2$ increases more quickly than the graph of $y = x^2$. This results in a vertical stretch of the graph of $y = x^2$ to obtain the graph of $y = 2x^2$. The graph of $y = 2x^2$ is narrower than the graph of $y = x^2$.

Now, let's compare the y-values of $y = x^2$ and $y = \frac{1}{2}x^2$.

$y = x^2$	(0, 0)	(1, 1)	(2, 4)
$y = \frac{1}{2}x^2$	(0, 0)	$\left(1, \frac{1}{2}\right)$	(2, 2)

In each case, the y-values of $y = \frac{1}{2}x^2$ are half the y-values of $y = x^2$. Therefore, the graph of $y = \frac{1}{2}x^2$ increases slower than the graph of $y = x^2$. This results in a vertical compression of the graph of $y = x^2$ to obtain the graph of $y = \frac{1}{2}x^2$. The graph of $y = \frac{1}{2}x^2$ is wider than the graph of $y = x^2$.

Notice the similarities and differences amongst all three graphs:

1. The graphs open upward.
2. The graphs have the same axis of symmetry, $x = 0$.
3. The graphs share the same vertex (0, 0) and each has a minimum y-value of 0.
4. The greater the coefficient a when $a > 1$, the narrower the corresponding parabola (e.g., the graph of $j(x) = 3x^2$ will be narrower than the graph of $f(x) = 2x^2$).
5. The smaller the coefficient a when $0 < a < 1$, the wider the graph (e.g., the graph of $j(x) = \frac{1}{3}x^2$ will be wider than the graph of $f(x) = \frac{1}{2}x^2$).
6. The domain is $\{x \mid x \in \Re\}$ for all functions.
7. The range is $\{y \mid y \geq 0\}$ for all functions.
8. The zero or the x-intercept is 0.
9. The y-intercept is 0.

Reflections in the x-axis

Example 2

Graph the following functions on the same co-ordinate system using a table of values or a graphing calculator.

Note: $-x^2$ means $-(x^2)$. This is because of BEDMAS. The order of operations rule says to do the squaring (exponents) before multiplying by negative one (multiplication).

a) $F(x) = -x^2$

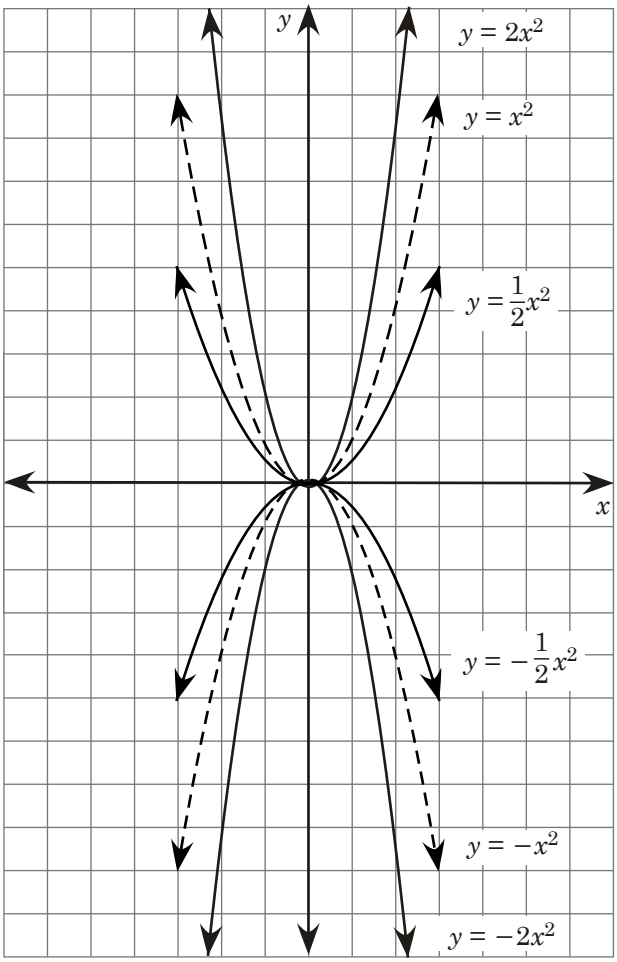
b) $m(x) = -2x^2$

c) $n(x) = -\frac{1}{2}x^2$

Solution

A table of values for each of the functions is listed below.

	$f(x) = x^2$	$(-3, 9)$	$(-2, 4)$	$(-1, 1)$	$(0, 0)$	$(1, 1)$	$(2, 4)$	$(3, 9)$
a)	$F(x) = -x^2$	$(-3, -9)$	$(-2, -4)$	$(-1, -1)$	$(0, 0)$	$(1, -1)$	$(2, -4)$	$(3, -9)$
b)	$m(x) = -2x^2$	$(-3, -18)$	$(-2, -8)$	$(-1, -2)$	$(0, 0)$	$(1, -2)$	$(2, -8)$	$(3, -18)$
c)	$n(x) = -\frac{1}{2}x^2$	$\left(-3, -\frac{9}{2}\right)$	$(-2, -2)$	$\left(-1, -\frac{1}{2}\right)$	$(0, 0)$	$\left(1, -\frac{1}{2}\right)$	$(2, -2)$	$\left(3, -\frac{9}{2}\right)$



If you compare the graph of $f(x) = x^2$ with the graph of $F(x) = -x^2$, you can note the change in the y-values. Each one has been changed to its opposite value. If (x, y) is a point on $f(x)$, then $(x, -y)$ is a point on $F(x)$. Therefore, $F(x)$ is the mirror reflection or mirror image of $f(x)$ in the x-axis. The mapping notation is $(x, y) \rightarrow (x, -y)$.

If you compare the graph of $f(x)$ with $m(x)$, you can note the changes in the y-values. Each has been changed to its opposite value and doubled. You can see that the graph of $f(x)$ has been reflected in the x-axis and vertically stretched by a factor of 2 to get $m(x)$. The graph of $m(x)$ is narrower than the graph of $f(x)$. If (x, y) is a point on $f(x)$, then $(x, -2y)$ is a point on $m(x)$.

If you compare $f(x)$ with $n(x)$, you can note the changes in the y-values. Each has been changed to its opposite value and halved. You can see that the graph of $f(x)$ has been reflected in the x-axis and vertically compressed by multiplying by $\frac{1}{2}$. The graph of $n(x)$ is wider than the graph of $f(x)$. If (x, y) is a point on $f(x)$, then $\left(x, -\frac{1}{2}y\right)$ is a point on $n(x)$.

Notice the following characteristics of graphs with negative a values, where a is the coefficient for the x^2 term.

1. The graphs open downward.
2. The graphs share the line of symmetry, $x = 0$.
3. The graphs share a common vertex (0, 0) and have a maximum y-value of 0.
4. The graphs have the same shapes as those for the corresponding positive values of a, but now they open downward.
5. The domain is $\{x \mid x \in \Re\}$ for all functions.
6. The range is $\{y \mid y \leq 0\}$ for all functions.
7. The zero or the x-intercept is 0.
8. The y-intercept is 0.

The Effect of '$-a$' on the Graph of $f(x) = -ax^2$

Any graph of the form $y = -ax^2$ can be obtained by reflecting, or flipping, the graph of $y = ax^2$ over the x-axis. If (x, y) is a point on the graph of $y = ax^2$, then $(x, -y)$ is a point on the graph of $y = -ax^2$.

Include this definition on your resource sheet.

Example 3

Describe how the graph of $g(x) = -5x^2$ can be obtained from the graph of $f(x) = x^2$. Sketch the graph of $g(x) = -5x^2$.

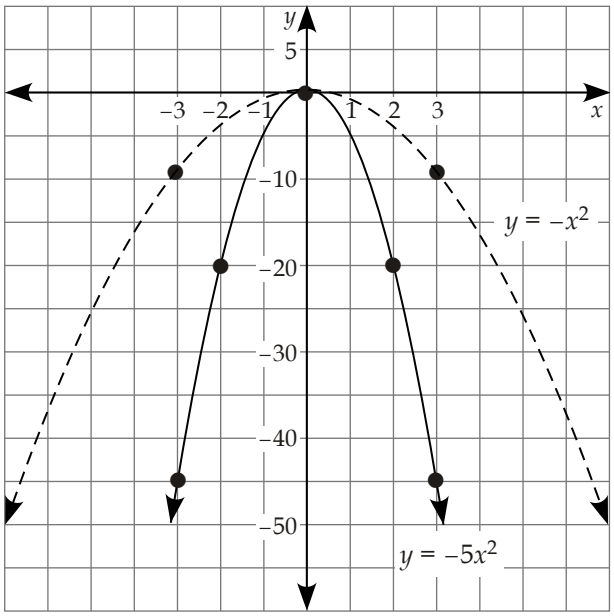
Solution

There are two methods for finding the solution. You can vertically stretch the graph of $f(x) = x^2$ by a factor of 5 and then reflect in the x-axis. Or, you can reflect the graph of $f(x) = x^2$ in the x-axis and then stretch the graph of $y = -x^2$ by a factor of 5. The result will be the same because both are multiplying operations—multiply by 5 and by (−1).

Consider the following table of values.

x	−3	−2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$5x^2$	45	20	5	0	5	20	45
$g(x) = -5x^2$	−45	−20	−5	0	−5	−20	−45

Notice that the scale on the y-axis has been changed. It is different than the scale on the x-axis. The effect of narrowing the width of the curve is thereby distorted. The graph of $y = -x^2$ is shown as a dotted line.



In general, the following statements are true about each quadratic function of the type $y = ax^2$.

1. Its axis of symmetry is the y-axis.
2. The coordinates of its vertex are (0, 0).
3. Its graph opens upward if $a > 0$, or downward if $a < 0$.
4. Its graph is narrower than the graph of $y = x^2$ if $a > 1$ or $a < -1$ and wider if $-1 < a < 1$.
5. The graph has a minimum value of 0 if $a > 0$, or a maximum value of 0 if $a < 0$.

If you do not already have this information on your resource sheet, add it now.

Example 4

Write an equation in the form $y = ax^2$ that contains the point $(-2, 8)$.

Solution

Substitute $x = -2$ and $y = 8$ into $y = ax^2$ and solve for a.

$8 = a(-2)^2$

$8 = a(4)$

$2 = a$

The equation is $y = 2x^2$.

The Effect of 'q' in the Quadratic Function $y = x^2 + q$

Now that you know the effect of a in the quadratic function $y = ax^2$, you will now examine graphs of quadratic functions of the type $y = ax^2 + q$ or $f(x) = ax^2 + q$ to determine the effect of q.

What is the effect of adding a real number q to ax^2 in the defining equation $y = ax^2$?

Example 5

Using a table of values or a graphing calculator, graph the following functions on the same coordinate system, where $a = 1$.

a) $f(x) = x^2$

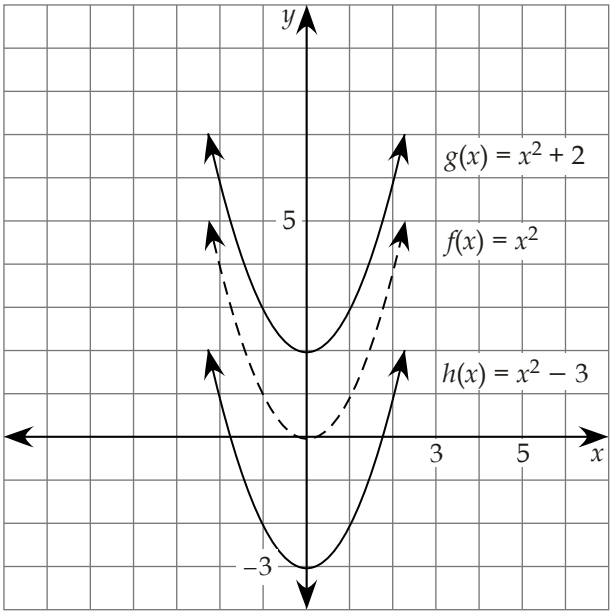
b) $g(x) = x^2 + 2$

c) $h(x) = x^2 - 3$

Solution

Your table of values should look like the following:

		−2	−1	0	1	2
a)	$f(x) = x^2$	4	1	0	1	4
b)	$g(x) = x^2 + 2$	6	3	2	3	6
c)	$h(x) = x^2 - 3$	1	−2	−3	−2	1



Notice that if the graph of $y = x^2$ is shifted up 2 units, you obtain the graph of $y = x^2 + 2$. If the graph of $y = x^2$ is shifted down 3 units, you obtain the graph of $y = x^2 - 3$

From the graph, $f(x)$ is the basic function $y = x^2$.

Observe the graphs of $g(x)$ and $h(x)$:

- The axis of symmetry is the y-axis.
- The vertices for $g(x)$ and $h(x)$ are (0, 2) and (0, −3), respectively.
- The minimum values of $g(x)$ and $h(x)$ are 2 and −3, respectively.
- The size and shape of the resulting graph is same as for $y = x^2$.

Notice that $g(-2) = f(-2) + 2$ and $h(-2) = f(-2) - 3$

In general, if $f(x) = ax^2$ and $g(x) = ax^2 + q$, then $g(x) = f(x) + q$ for all x. Also, notice that only the y-values are affected by the value of q. If (x, y) is a point on $f(x)$, then $(x, y + q)$ is a point on $g(x)$.

Include the following definitions on your resource sheet.

The graph of $f(x) = ax^2 + q$ is obtained from the graph of $y = ax^2$ by a **vertical shift**.

- If $q > 0$, the shift is up q units.
- If $q < 0$, the shift is down $-q$ units

For example, given the equation $y = 3x^2 - 2 = 3x^2 + (-2)$, the value of q is -2 and $q < 0$. The graph is shifted down $-(-2) = 2$ units.

For any q, the graph of $f(x) = ax^2 + q$ has a vertex at $(0, q)$ and the y-axis is the line of symmetry.

Example 6

Using your graphing calculator or a table of values, graph the following functions.

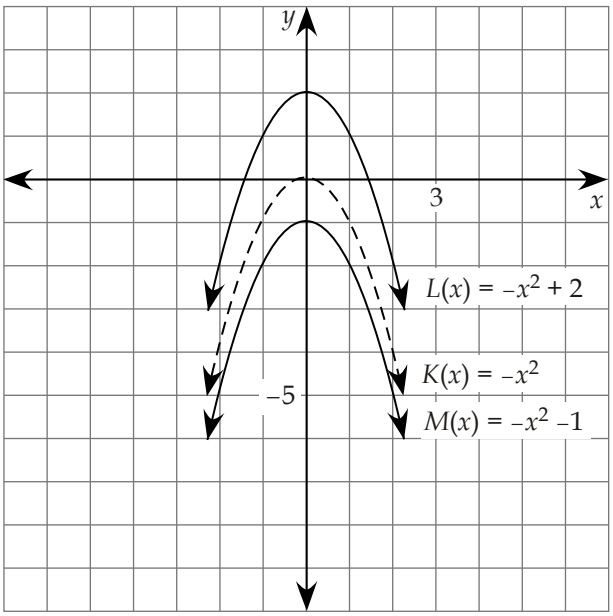
a) $K(x) = -x^2$

b) $L(x) = -x^2 + 2$

c) $M(x) = -x^2 - 1$

Solution

		-2	-1	0	1	2
a)	$K(x)$	-4	-1	0	-1	-4
b)	$L(x)$	-2	1	2	1	-2
c)	$M(x)$	-5	-2	-1	-2	-5



Notice that $L(-2) = K(-2) + 2$ and $M(-2) = K(-2) - 1$.

Notice that the pattern follows the same idea as Example 5.

The graph of L is the graph of K shifted up 2 units and the graph of M is the graph of K shifted down 1 unit.

In general, the following statements are true about each quadratic function of the type $y = ax^2 + q$.

1. The axis of symmetry is the y-axis with equation $x = 0$.
2. Coordinates of vertex: $(0, q)$.
3. $y = ax^2 + q$ is the graph of $y = ax^2$ shifted q units vertically.
   If $q > 0$, the parabola is shifted upward.
   If $q < 0$, the parabola is shifted downward.
4. If $a > 0$, the minimum value of y is q.
5. If $a < 0$, the maximum value of y is q.

If you do not already have these statements on your resource sheet, include them now.

In the following learning activity and assignment you will be expected to complete stretches, compressions, and vertical shifts without using a table of values. If you use graph paper and transform the coordinates of $y = x^2$ appropriately, you will be able to make a table of values from your graph. It is important, therefore, that you complete the following learning activity to practice your skills before you complete Assignment 3.1, which will be sent to the Distance Learning Unit for assessment.

Learning Activity 3.2

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Water is trickling out of a faucet at a speed of 3 mL per second. If Paula holds up a cup to collect the water, how much water will she collect in 3 minutes?
2. Factor: $x^2 - 9$
3. If $x = -1$ and $y = 2$, evaluate the expression: $3x^3 - xy$.
4. What is the lowest common multiple of 10 and 14?
5. At The Burger Palace, you have a choice between what type of bun and what type of burger you want. If there are three types of buns (white, whole wheat, and sesame) and three types of burgers (angus, chicken, and lean), how many different combinations of burgers are there?
6. Determine the distance between the following two points: (26, −13) and (25, −1).
7. Solve the inequality: $-6x - 7 > 3x - 9$.
8. Simplify. $(9x^2 - 7x + 5) - (2x^2 - 6x - 3)$

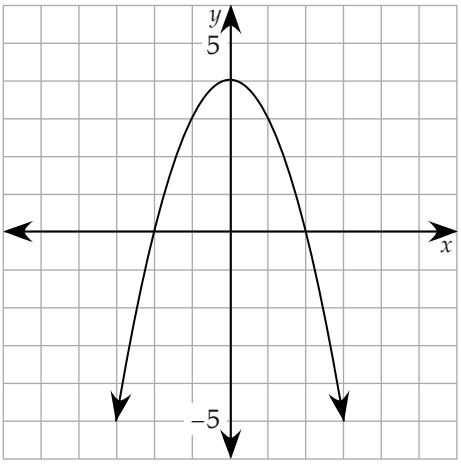
continued

Learning Activity 3.2 (continued)

Part B: Quadratic Functions: $y = ax^2$ and $y = ax^2 + q$

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Determine whether each statement is true or false. If the statement is false, rewrite it so that it is true.
   a) $x^2 + y = 1$ is a quadratic function.
   b) $t(x) = 1 - x$ is a quadratic function.
   c) $f(x) = 4x^2$ is a quadratic function.
   d) The graph of $r(x) = x^2 + 1$ is symmetric with respect to the y-axis.
   e) The graph of $y = x^2$ is a "wider" parabola than the graph of $y = \frac{1}{4}x^2$.
   f) The graph of $y = 5x^2$ is a "narrower" parabola than the graph of $y = x^2$.
   g) The graph of $y = \frac{1}{4}x^2$ is a parabola that opens downward.
   h) The graph of $y = -2x^2$ is a parabola that opens downward.
   i) The maximum value of the function pictured at the right is 4.
   j) The minimum value of the function pictured at the right is 0.
   k) The domain of the function pictured at the right is $\{y \mid y \geq 4\}$.



continued

Learning Activity 3.2 (continued)

2. Complete the following table.

	$y = \frac{1}{3}x^2$	$y = x^2$	$y = -3x^2$
Vertex			
Equation of axis of symmetry			
Domain			
Range			
x-intercepts or zeros			
Direction of opening			
Maximum y-value			
Minimum y-value			
y-intercept			

3. Describe the transformations needed to arrive at the graph of each of the following using $y = x^2$ as the reference curve.

   a) $y = 4x^2$

   b) $y = \frac{1}{2}x^2$

   c) $y = -7x^2$

4. Find the following for the graph given by $y = ax^2$:
   a) What is the direction of the opening of the graph if $a > 0$?
   b) What is the direction of the opening of the graph if $a < 0$?
   c) If $a > 0$, does $y = ax^2$ have a minimum value or a maximum value?
   d) If $a < 0$, does $y = ax^2$ have a maximum value or a minimum value?

continued

Learning Activity 3.2 (continued)

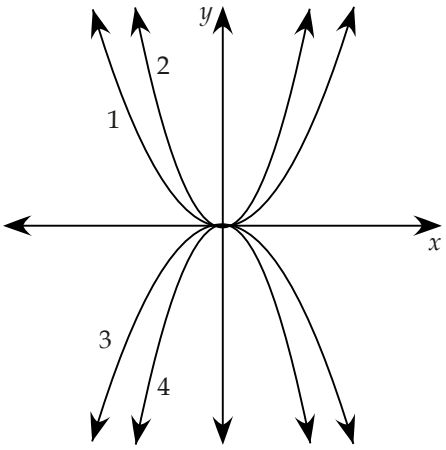
5. Match the letter corresponding to each function with the number corresponding to its graph.

   a) $y = -2x^2$

   b) $y = 1.6x^2$

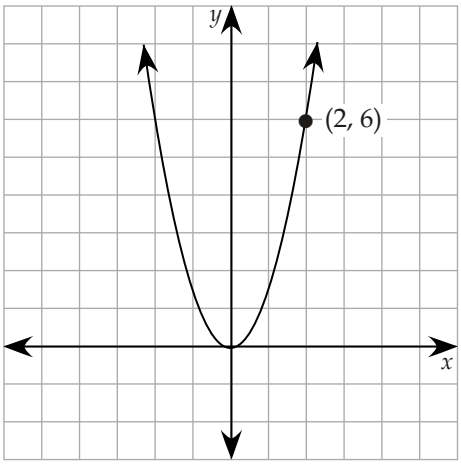
   c) $y = -2\pi x^2$

   d) $y = \sqrt{6}x^2$



6. Write an equation in the form $y = ax^2$ if $(1, -4)$ lies on the curve.

7. Write the equation of the graph. Then write the equation of its reflection in the x-axis. Note each mark represents 1 unit on the graph.



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Learning Activity 3.2 (continued)

8. Determine whether each statement is true or false. If the statement is false, rewrite it so it is true.
   a) The graph of $y = 3x^2 + 4$ is the graph of $y = 3x^2$ shifted downward 4 units.
   b) The graph of $f(x) = -x^2 - 3$ is the graph of $h(x) = -x^2$ shifted downward 3 units.
   c) The vertex of the graph of $y = 2x^2$ is the origin.
   d) The vertex of the graph of $y = \frac{1}{2}x^2 - 2$ is $(0, -2)$.
   e) An equation of the axis of symmetry of the graph of $y = x^2 - 1$ is $x = -1$.
   f) The graph of $y = x^2 - 1$ opens downward.
   g) The graph of $y = -x^2 - 2$ opens downward.
   h) The maximum value of $\{(x, y) | y = -2x^2 + 3\}$ is -2.
   i) The minimum value of is $\{(x, y) | y = 3x^2 - 4\}$ is -4.
   j) If $f(x) = 2x^2 + 3$, then $f(x) = f(-x)$, for all x.
   k) The graph of $y = x^2 + 4$ is symmetric with respect to the y-axis.
   l) The graphs of $y = x^2 + 2$ and $y = -x^2 + 2$ are mirror reflections of each other in the x-axis.
   m) The graph of $y = x^2$ is a narrower parabola than the graph of $y = \frac{1}{4}x^2$.
   n) The range of $f(x) = -x^2 + 2$ is $\{y | y \leq 2\}$.
9. A quadratic function f is defined by $f: x \rightarrow ax^2 + q$, $a \neq 0$. Describe whether the graph of f opens up or down and give its maximum or minimum value.
   a) $a > 0$ and $q > 0$
   b) $a > 0$ and $q < 0$
   c) $a < 0$ and $q > 0$
   d) $a < 0$ and $q < 0$
10. Graph $y = -3x^2 + 1$ using transformations on $y = x^2$. Then make a table of values with 5 points in it.

Lesson Summary

In this lesson, you discovered how a and q affect quadratic functions of the form $y = ax^2 + q$. You learned how a affects the basic quadratic function $y = x^2$ by a vertical stretch, vertical compression, or a reflection. You also learned how q affects the basic quadratic function $y = x^2$ by a vertical shift. These are three of the four transformations of quadratic functions that you will be learning about in this course. In the next lesson, you will be learning about the last transformation of quadratic functions: the effect of p in the quadratic function $y = a(x - p)^2$.

Assignment 3.1

Quadratic Functions and Transformations

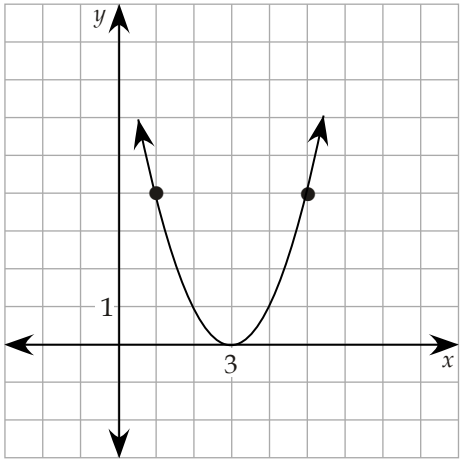
Total: 58 marks

1. Which of the following describes a quadratic function?
   (1 mark each, for a total of 4 marks)

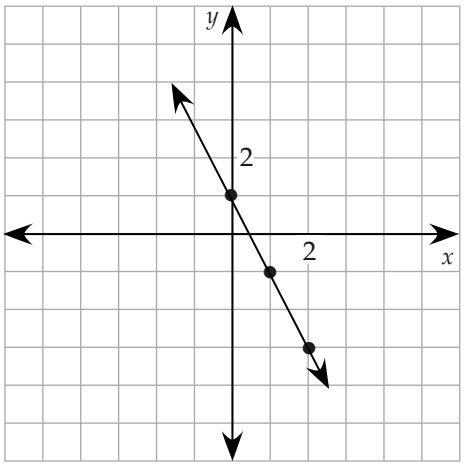
   a) $f(x) = -x^2 + 3x^2 - 1$

   b) $g(x) = x - 2$

   c)



   d)



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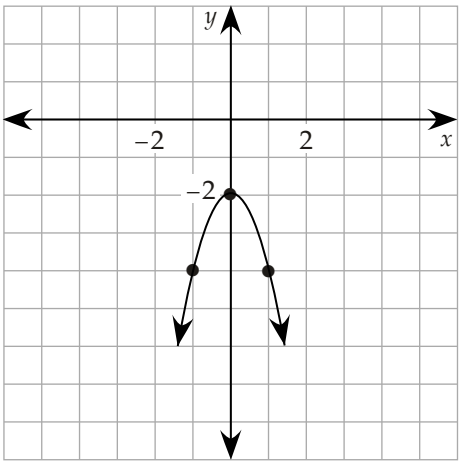
Assignment 3.1: Quadratic Functions and Transformations (continued)

2. For the following, write the domain, range, coordinates of the vertex, equation of the axis of symmetry, y-intercept, x-intercept(s), maximum value, and minimum value. *(4 marks)*



Domain: _____

Range: _____

Coordinates of the Vertex: _____

Axis of Symmetry: _____

y-intercept: _____

x-intercept(s): _____

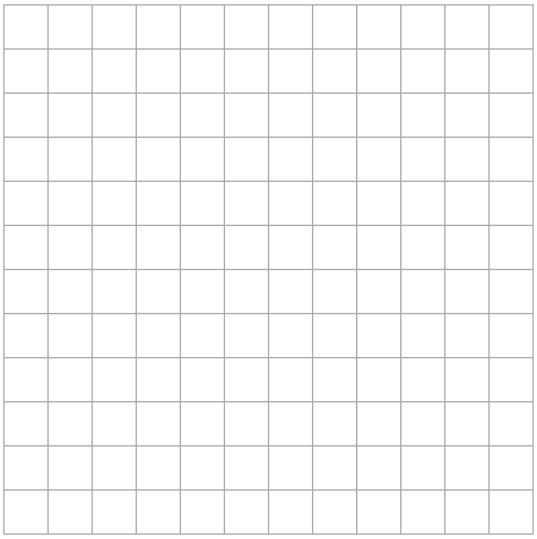
Maximum Value: _____

Minimum Value: _____

3. Sketch the graph of the following quadratic function. Fill in the table of values. Find the coordinates of the vertex, the equation of the axis of symmetry, and the maximum or minimum value. *(5 marks)*

$y = -2x^2 + 1$

x	−2	−1	0	1	2
$y = -2x^2 + 1$					



Vertex: _____

Axis of Symmetry: _____

Maximum or Minimum Value: _____

continued

Assignment 3.1: Quadratic Functions and Transformations (continued)

4. Match the equation with its graph. Use a table of values to graph each function if necessary. *(1 mark each, for a total of 4 marks)*

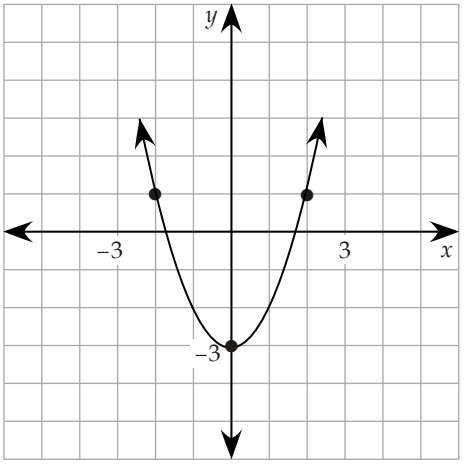
   a) $y = -x^2 + 3$

   b) $y = x^2 + 3$

   c) $y = -x^2 - 3$

   d) $y = x^2 - 3$

i)



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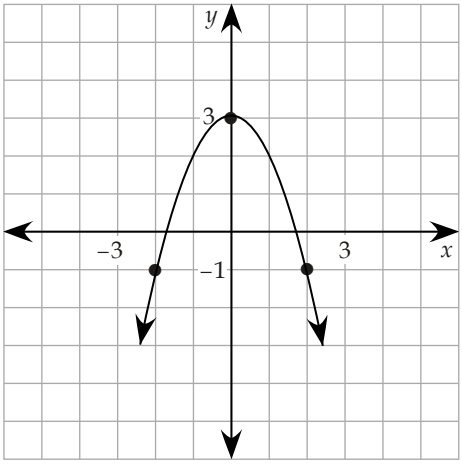

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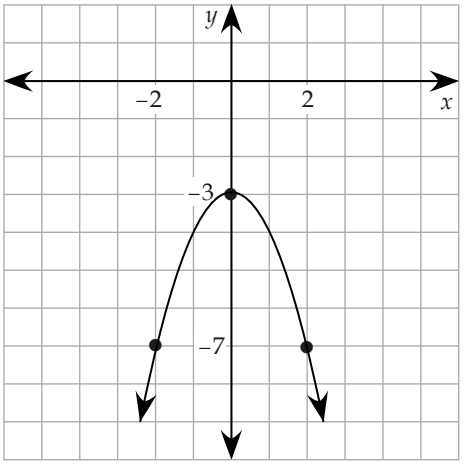

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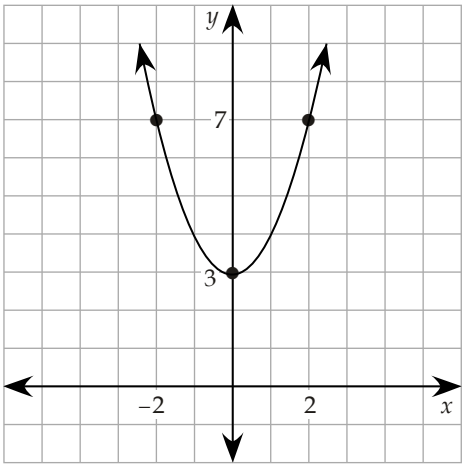

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Assignment 3.1: Quadratic Functions and Transformations (continued)

5. Write an equation in the form $y = ax^2$ if $(-2, 7)$ lies on the curve. *(2 marks)*

6. Complete the following table. *(20 marks)*

	$y = x^2 + 4$	$y = x^2$	$y = x^2 - 1$	$y = -x^2 + 1$
Vertex				
Equation of axis symmetry				
Domain				
Range				
x-intercepts				
y-intercept				
Direction of opening				
Maximum value				
Minimum value				

continued

Assignment 3.1: Quadratic Functions and Transformations (continued)

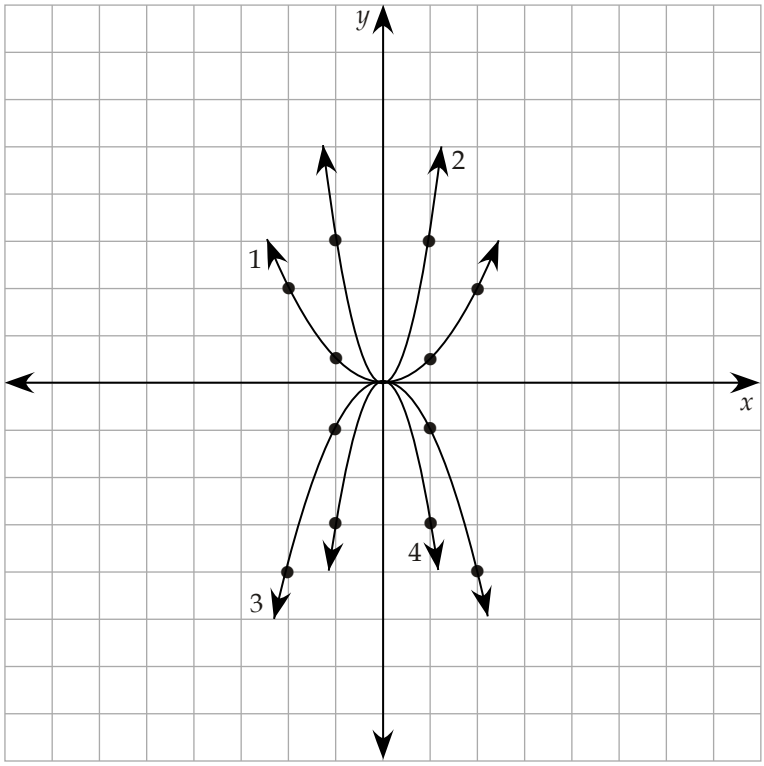
7. Match the letter corresponding to each function with the number corresponding to its graph. *(1 mark each, for a total of 4 marks)*

   a) $y = -3x^2$

   b) $y = 3x^2$

   c) $y = -x^2$

   d) $y = \frac{1}{2}x^2$



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Assignment 3.1: Quadratic Functions and Transformations (continued)

8. Describe how to arrive at the graph of each of the following if $y = x^2$ is the reference curve. *(2 marks each, for a total of 8 marks)*

   a) $f : x \rightarrow 5x^2 + 4$

   b) $f(x) = -2x^2 + 1$

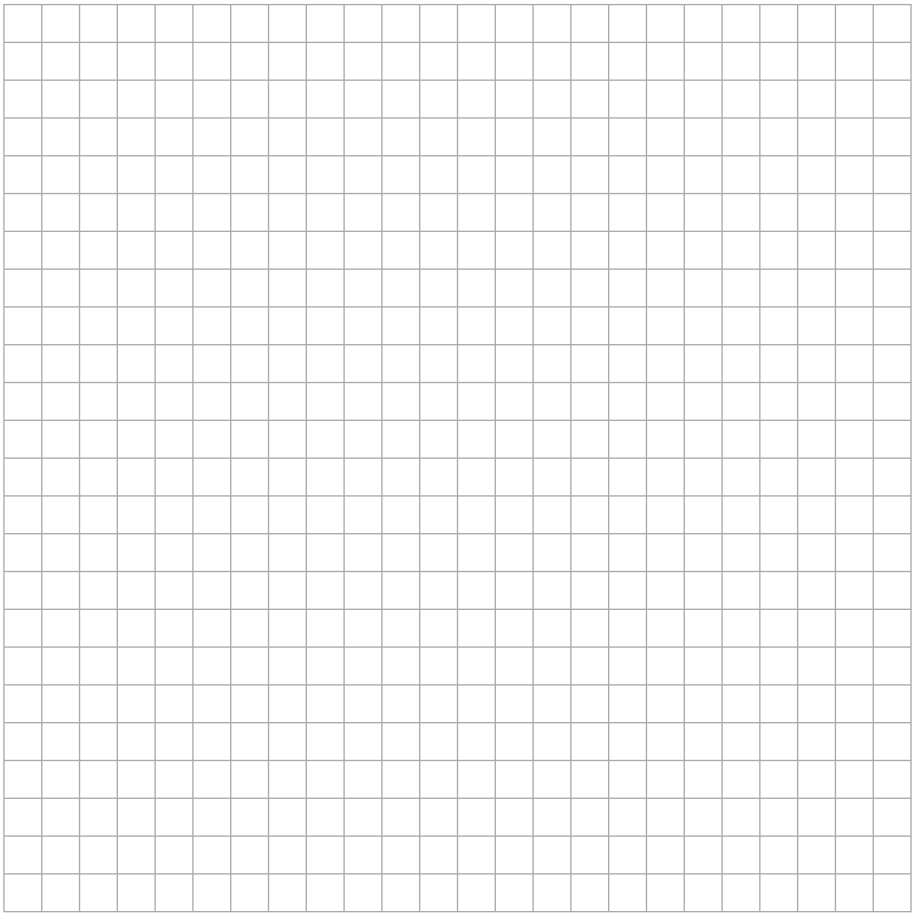
   c) $h(x) = \frac{1}{2}x^2 - 1$

   d) $-y + 3x^2 = 2$

continued

Assignment 3.1: Quadratic Functions and Transformations (continued)

9. Graph $y = -2x^2 + 3$ using transformations on $y = x^2$. Then make a table of values with 5 points in it. *(4 marks)*



x					
y					

continued

Assignment 3.1: Quadratic Functions and Transformations (continued)

10. Write an equation in the form $y = ax^2 + q$, given the vertex is at $(0, -3)$ and the parabola passes through the point $(1, -1)$. *(3 marks)*

Lesson 3: Quadratic Functions $y = a(x - p)^2$

Lesson Focus

In this lesson, you will

❑ learn how to determine characteristics of the graph $y = a(x - p)^2$

Lesson Introduction

In the last lesson, you learned the effect a and q had on the quadratic function $y = ax^2 + q$. In this lesson, you will learn the effect p has on the quadratic functions $y = (x - p)^2$ and $y = a(x - p)^2$. Once you learn this, quadratic functions will be easier to graph. You will be able to put all your knowledge of transformations of quadratic functions together to graph any quadratic function in vertex form.

Quadratic Functions

In the last lesson, you learned about quadratic functions in the form $y = ax^2 + q$. Another type of quadratic function is one defined by an equation of the form $y = a(x - p)^2$. Just as a and q had certain effects on the graph of the quadratic function, so does p.

Example 1

Use a table of values or a graphing calculator to graph the following functions on the same coordinate plane.

a) $f(x) = x^2$

b) $g(x) = (x - 6)^2$

c) $i(x) = (x + 6)^2$

Note: In this example, we let a have a fixed value of 1 to consider the effect of changes in p.

Solution

The following tables of values give some ordered pairs that satisfy the above functions.

a) $f(x) = x^2$

x	−2	−1	0	1	2
y	4	1	0	1	4

b) $g(x) = (x - 6)^2$

x	4	5	6	7	8
y	4	1	0	1	4

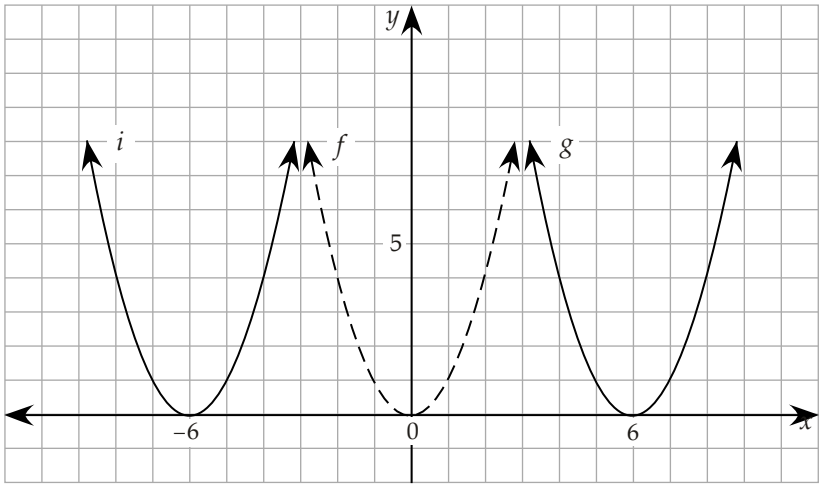
Notice each x-value in $g(x) = (x - 6)^2$ is 6 more than the corresponding x-value in $f(x)$. The y-values remain the same.

c) $i(x) = (x + 6)^2$

x	−8	−7	−6	−5	−4
y	4	1	0	1	4

Notice each x-value in $i(x) = (x + 6)^2$ is 6 less than the corresponding x-value in $f(x)$. The y-values are unchanged.

Notice that the pattern of the y-values is the same. However, because the corresponding x-values have changed, the vertex of each of the curves, $g(x)$ and $i(x)$, has been shifted horizontally. The graph of $g(x) = (x - 6)^2$ is shifted to the right 6 units and the graph of $i(x) = (x + 6)^2$ is shifted to the left 6 units. Notice also that the operation in the formula is subtract. The formula is written as $y = a(x - p)^2$, not $y = a(x + p)^2$.



Consider the following pattern:

$f(x) = x^2$

$g(x) = f(x - 6) = (x - 6)^2$

and $i(x) = f(x + 6) = (x + 6)^2$

Now consider some points on the graphs.

$g(5) = (x - 6)^2 = (5 - 6)^2 = (-1)^2 = 1$

and $g(5) = f(x - 6) = f(5 - 6) = f(-1) = (-1)^2 = 1$

Thus, $g(5) = f(1) = 1$

$i(-5) = (x + 6)^2 = (-5 + 6)^2 = 1^2 = 1$

and $i(-5) = f(-5 + 6) = f(1) = 1^2 = 1$

Thus, $i(-5) = f(1) = 1$

In general, if $f(x) = x^2$ and $g(x) = (x - p)^2$ then $g(x) = f(x - p)$ for all x.

Include the following information on your resource sheet.

The Effect of p in the Quadratic Function $f(x) = (x - p)^2$

The graph of $f(x) = (x - p)^2$ is said to be obtained from the graph of $y = x^2$ by a horizontal shift.

- If $p > 0$, the horizontal shift is p units to the right.
- If $p < 0$, the horizontal shift is $(-p)$ units to the left.

Remember that there is a subtract sign in the formula $y = a(x - p)^2$. If $p > 0$ and $p = 6$, then $y = (x - (+6))^2 = (x - 6)^2$, and the graph of $y = x^2$ would be shifted 6 units to the right. If $p < 0$ and $p = -6$, then $y = (x - (-6))^2 = (x + 6)^2$, and the graph of $y = x^2$ would be shifted 6 units to the left.

For any p, the graph of f has a vertex $(p, 0)$ and the line of symmetry $x = p$.

You can visualize the graph of $y = (x - 6)^2$ as being the graph of $y = x^2$ shifted 6 units to the right and the graph of $y = (x + 6)^2$ as being the graph of $y = x^2$ shifted 6 units to the left.

Note: $y = (x + 6)^2$ can be written as $y = (x - (-6))^2$. This format uses the definition and shows that if you have the function $y = (x + 6)^2$, then the value of p is -6 and not positive 6.

Notice the following:

- The vertex of $g(x)$ is (6, 0) and the vertex of $i(x)$ is (−6, 0).
- The equation of the axis of symmetry of $g(x)$ is $x = 6$ and of $i(x)$ is $x = -6$.
- The minimum values of $g(x)$ and $i(x)$ are 0.
- The size and shape of the $g(x)$ and $i(x)$ are the same as $f(x)$.

Example 2

Continue your investigation by giving a different values. Using a combination of shifts and stretches or compressions, graph the following functions on a coordinate plane.

a) $k(x) = 4(x-5)^2$

b) $l(x) = \frac{1}{4}(x-5)^2$

c) $m(x) = -(x-5)^2$

Solution

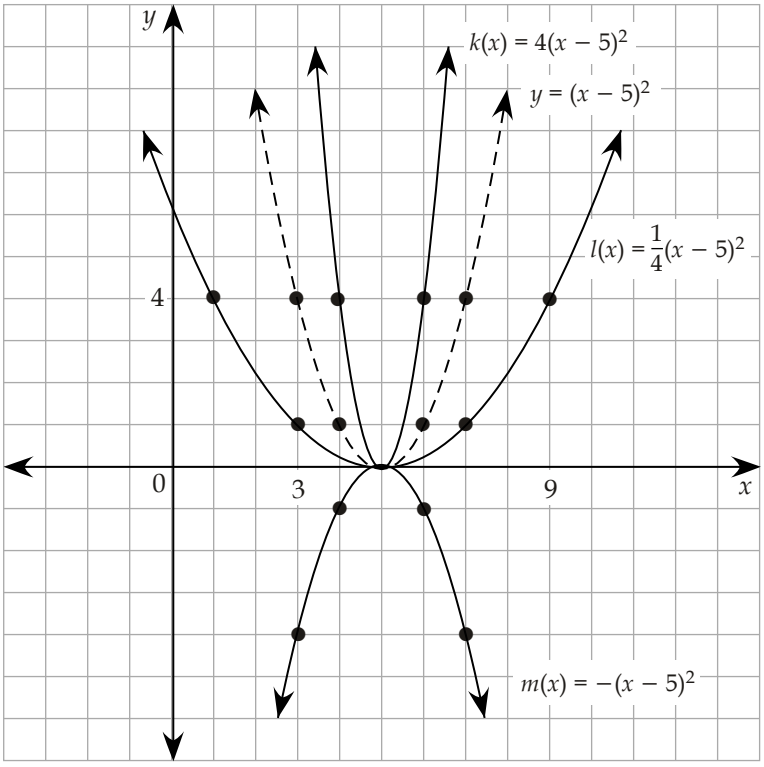
Begin with the basic graph $y = x^2$ and shift it 5 units to the right to get $y = (x-5)^2$.

a) For $k(x)$, stretch $y = (x-5)^2$ vertically by a factor of 4 to get $y = 4(x-5)^2$. Its graph will be narrower than the graph of $y = (x-5)^2$.

b) For $l(x)$, compress $y = (x-5)^2$ vertically by multiplying the y-coordinates by $\frac{1}{4}$ to get $y = \frac{1}{4}(x-5)^2$. Its graph will be wider than $y = (x-5)^2$.

c) For $m(x)$, reflect $y = (x-5)^2$ over the x-axis to get $y = -(x-5)^2$. Its graph will be the same shape as $y = (x-5)^2$.

In the figure below, the graphs of $k(x)$, $l(x)$, and $m(x)$ are contrasted with the graph of $y = (x-5)^2$ indicated by dashed marking.



Example 3

The vertex of a given parabola is at (2, 0). Another point on the graph is (−2, 8). Write its equation in the form $y = a(x - p)^2$.

Solution

Since the vertex of $y = x^2$ is shifted 2 units to the right, $p = 2$. Use the other ordered pair to substitute $y = 8$ and $x = -2$ into the equation $y = a(x - p)^2$.

$$y = a(x - p)^2$$

$$8 = a(-2 - 2)^2$$

$$8 = a(-4)^2$$

$$8 = 16a$$

$$\frac{8}{16} = a$$

$$\frac{1}{2} = a$$

The equation is $y = \frac{1}{2}(x - 2)^2$.

The following generalizations can be made for functions of the type $y = a(x - p)^2$.

1. The axis of symmetry is the line $x = p$.
2. Coordinates of the vertex are $(p, 0)$.
3. $y = a(x - p)^2$ is the graph of $y = ax^2$ shifted p units horizontally.
   If $p > 0$, the parabola is shifted to the right.
   If $p < 0$, the parabola is shifted to the left.
4. If $a > 0$, the minimum value of y is 0.
   If $a < 0$, the maximum value of y is 0.

If you do not already have these generalizations on your resource sheet, make sure you include them now.

Learning Activity 3.3

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Simplify: $(a + b)^2$
2. Evaluate: $\frac{1}{5} + \frac{2}{3}$
3. If Mark's meal comes to $16.27 with tax and he adds a tip of $1.50, how much change should he get back if he pays with a $20 bill?
4. Samuel and Cassy weigh 333 pounds together. If Samuel weighs twice as much as Cassy, how many pounds does Cassy weigh?
5. Factor: $x^2 - 3x - 28$
6. Simplify: $\left(\frac{5^5 \times 5^7}{5^2}\right)^3$
7. Rationalize the denominator: $\frac{2 - \sqrt{8}}{10\sqrt{7}}$
8. Find the slope of the line that contains the points $(2, -7)$ and $(9, -14)$.

continued

Learning Activity 3.3 (continued)

Part B: Quadratic Functions: $a(x - p)^2$

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

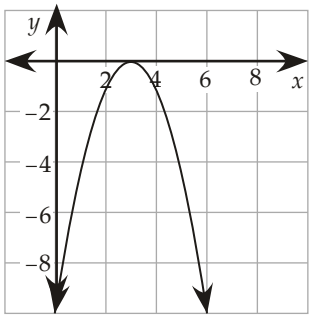
1. State whether each statement is true or false. If the statement is false, rewrite it so that it is true.
   a) The graph of $y = (x + 5)^2$ is the graph of $y = x^2$ shifted 5 units to the left.
   b) The graph of $y = (x - 7)^2$ is the graph of $y = x^2$ shifted 7 units to the left.
   c) The vertex of the graph of $y = (x - 1)^2$ is (1, 0).
   d) The vertex of the graph of $y = 2(x - 2)^2$ is (1, 0).
   e) The equation of the axis of symmetry of the graph of $y = \frac{1}{2}(x - 3)^2$ is $x = \frac{1}{2}$.
   f) The equation of the axis of symmetry of the graph of $y = -\frac{1}{2}(x + 3)^2$ is $y = -3$.
   g) The graph of $y = -\frac{1}{3}(x - 2)^2$ is a narrower parabola than the graph of $y = -3(x - 2)^2$.
   h) The graph of $y = -\frac{1}{4}(x + 4)^2$ opens downward.
   i) The quadratic function $y = \frac{1}{3}(x - 2)^2$ has 0 for its minimum value.
   j) If $g(x) = (x + 4)^2$, then $g(x - 4) = g(-x - 4)$ for all x.

continued

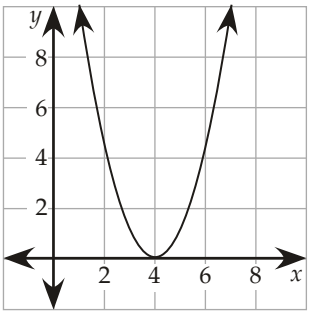
Learning Activity 3.3 (continued)

2. Match the equation with its graph.
   a) $y = (x - 4)^2$
   b) $y = x^2 + 4$
   c) $y = -(x + 2)^2$
   d) $y = -(x - 3)^2$

i)



ii)



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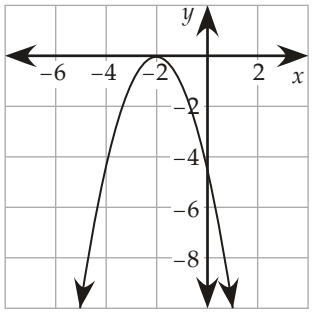

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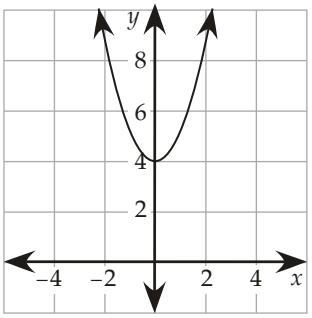


3. Without using tables of values, sketch the graphs of the following on graph paper. Provide a written explanation of the steps involved in the transformation. Then make a table of values with 5 points in it, showing how the coordinates of $y = x^2$ are transformed.
   a) $f : x \to \frac{1}{2}(x - 4)^2$
   b) $y = -2(x - 1)^2$
   c) $f(x) = 2(x + 1)^2$

continued

Learning Activity 3.3 (continued)

4. Complete the following table.

	(a)	(b)	(c)	(d)
	$y = (x + 2)^2$	$y = (x - 1)^2$	$y = 2(x - 3)^2$	$y = \frac{4}{3}(x + 6)^2$
Vertex				
Equation of axis of symmetry				
Domain				
Range				
x-intercepts				
y-intercept				
Direction of opening				
Maximum value				
Minimum value				

Lesson Summary

In this lesson, you learned about the final transformation of quadratic functions: the effect of p in the quadratic function $y = a(x - p)^2$. You should now be able to graph any quadratic function in vertex form. You will combine all of the rules for the effects of a, p, and q to graph quadratic functions in vertex form in the next lesson.

Notes

Lesson 4: Graphing Using Transformations

Lesson Focus

In this lesson, you will

- ❑ learn how to graph quadratic functions of the form $f(x) = a(x - p)^2 + q$
- ❑ learn how to write the equation of a quadratic function given characteristics of its graph
- ❑ learn how to write the equation of a quadratic function given its graph
- ❑ learn how to determine the number of x-intercepts of a quadratic function using a and q

Lesson Introduction

In the previous lessons, you explored the effects of the parameters a, p, and q on the graphs of quadratic functions. In this lesson, you will combine the effects of all of these parameters together to investigate graphs of quadratic functions of the form $f(x) = a(x - p)^2 + q$.

Transformations

The **vertex form** of a quadratic function $y = f(x)$ is $f(x) = a(x - p)^2 + q$, where a, p, and q are real numbers, and in this course they are rational numbers.

Transforming $y = x^2$ to $f(x) = a(x - p)^2 + q$.

The graph of a quadratic function in vertex form $f(x) = a(x - p)^2 + q$ can be obtained from the graph of $y = x^2$ through following a sequence of transformations, usually performed in the order of "multiplying then adding." In other words, do the stretching, compressing, and reflecting first, and then the shifting.

1. Vertical stretch/compression: Stretch by a factor of a if $a > 1$ or $a < -1$ and compress by a factor of $\frac{1}{a}$ if $-1 < a < 1$. $(x, y) \rightarrow (x, ay)$
2. Reflect in x-axis if $a < 0$. $(x, y) \rightarrow (x, -y)$
3. Horizontal shift p units: Shift right if $p > 0$ and left if $p < 0$. $(x, y) \rightarrow (x + p, y)$
4. Vertical shift q units: Shift up if $q > 0$ and down if $q < 0$. $(x, y) \rightarrow (x, y + q)$

The vertex of the graph of f is the point (p, q), and its line of symmetry is $x = p$.

The complete transformation from $y = x^2$ to $y = a(x - p)^2 + q$ is:

$(x, y) \rightarrow (x + p, ay + q)$

Note: The stretch or compression factor is always a number greater than 1. If $a > 1$, the stretch factor is a. If $0 < a < 1$, the compression factor is $\frac{1}{a}$, and $\frac{1}{a} > 1$.

If you have not updated your resource sheet to include all of the above information, now would be a good time to do so.

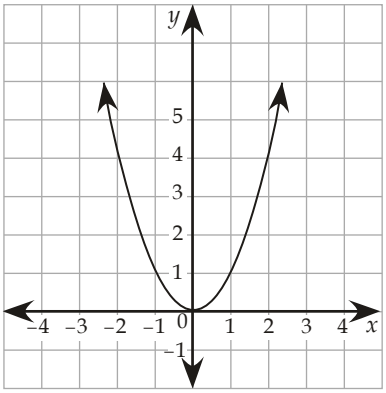
Example 1

Sketch the graph given by $y = (x + 2)^2 - 3$.

Solution

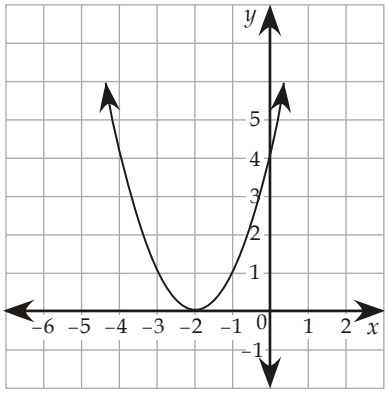
1. Sketch the graph of $y = x^2$.

   Notice the key coordinates (0, 0), (1, 1), (2, 4), and (3, 9). Others can be found by reflection over the axis of symmetry.



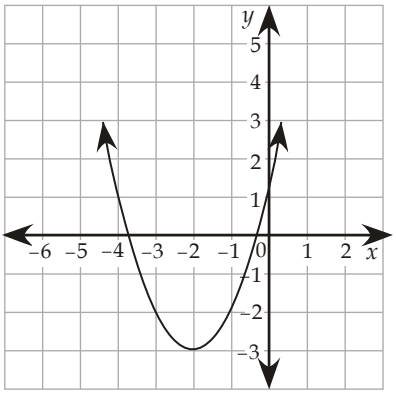
2. To obtain the graph of $y = (x + 2)^2$, apply a horizontal shift of 2 units to the left to the graph in Step 1.

   For each coordinate listed in Step 1, subtract 2 from the x-values to get $(-2, 0)$, $(-1, 1)$, $(0, 4)$, and $(1, 9)$.



3. To obtain the graph of $y = (x + 2)^2 - 3$, apply a vertical shift of 3 units downward to the graph in Step 2.

   For each of the coordinates in Step 2, subtract 3 from the y-values to get $(-2, -3)$, $(-1, -2)$, $(0, 1)$, and $(1, 6)$.



Therefore, the graph of $y = (x + 2)^2 - 3$ is obtained.

The transformation formula to transform $y = x^2$ into $y = (x + 2)^2 - 3$ is $(x, y) \rightarrow (x + (-2), y - 3)$ or $(x, y) \rightarrow (x - 2, y - 3)$.

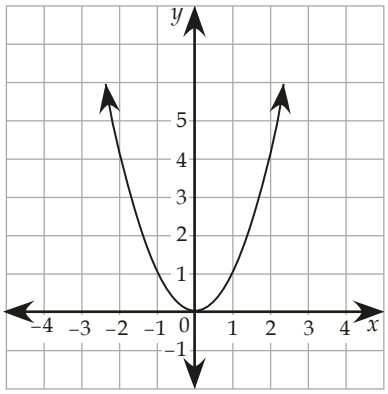
In Example 1, notice that since $a = 1$, there was no vertical stretch or compression. In the next example, $a = -1$ and a reflection will be applied.

Example 2

Sketch the graph given by $y = -(x - 1)^2 + 2$.

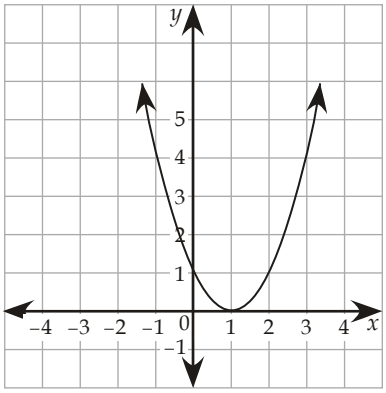
Solution

1. Sketch the graph of $y = x^2$.



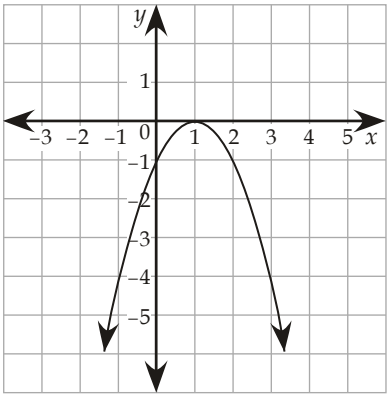
2. To obtain the graph of $y = (x - 1)^2$, apply a horizontal shift of 1 unit to the right to the graph in Step 1.

   Add 1 to each x-value: $(x, y) \rightarrow (x + 1, y)$.



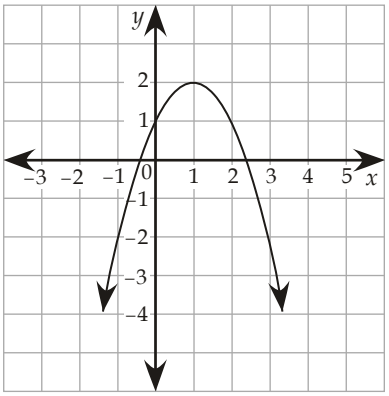
3. To obtain the graph of $y = -(x - 1)^2$, apply a reflection to the graph in Step 2 in the x-axis.

   Multiply each y-value by (-1). $(x, y) \rightarrow (x, -y)$.



4. To obtain the graph of $y = -(x - 1)^2 + 2$, apply a vertical shift of 2 units up to the graph in Step 3.

   Add 2 to each y-value: $(x, y) \rightarrow (x, y + 2)$



The complete transformation from $y = x^2$ to $y = -(x - 1)^2 + 2$ is $(x, y) \rightarrow (x + 1, -y + 2)$.

Note that the same graph results if you apply both horizontal and vertical shifts first and then the reflection. The value of a is -1.

Example 3

Write a sequence of transformations that will produce the graph of $y = -(x - 7)^2 - 3$ from $y = x^2$.

Solution

Reflect $y = x^2$ over the x-axis, shift 7 units to the right and 3 units downward.

Below is a summary of observations concerning the graph of $y = a(x - p)^2 + q$. Include these on your resource sheet if you haven't done so already.

1. The graph is a parabola.
2. The coordinates of the vertex are (p, q).
3. An equation defining the axis of symmetry is $x = p$.
4. The graph opens upward if $a > 0$ or downward if $a < 0$.
5. The graph is narrower than the graph of $y = x^2$ if $a > 1$ or $a < -1$ and wider if $-1 < a < 1$.
6. If $a > 0$, the minimum value of y is at $y = q$.
7. If $a < 0$, the maximum value of y is at $y = q$.

Example 4

A graph has the following properties:

- The vertex is at $(1, -7)$
- The graph passes through the point $(3, -23)$

Write a quadratic function in the form $y = a(x - p)^2 + q$ that represents this graph.

Solution

The vertex is $(1, -7)$, which corresponds to the point (p, q). Therefore, $p = 1$ and $q = -7$.

$$y = a(x - 1)^2 - 7$$

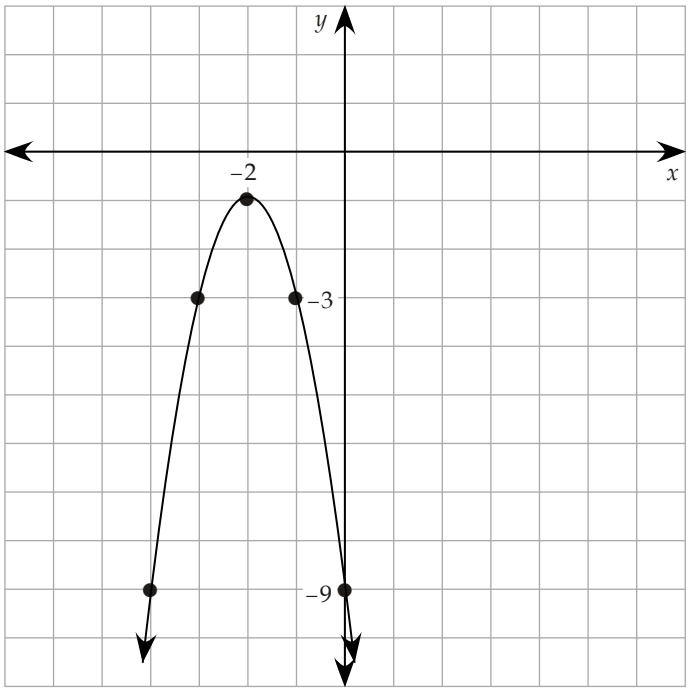
You also know the parabola passes through the point $(3, -23)$. Substitute these values into the above equation for x and y.

$$-23 = a(3 - 1)^2 - 7$$
$$-23 = a(2)^2 - 7$$
$$-23 = 4a - 7$$
$$-16 = 4a$$
$$-4 = a$$

Therefore, the quadratic function is $y = -4(x - 1)^2 - 7$.

Example 5

Write a quadratic function in the form $y = a(x - p)^2 + q$ that represents the following graph.



Solution

It is possible to find a quadratic function in vertex form for this graph using two methods. Method #1 involves finding the coordinates of the vertex and one other point. Method #2 involves comparing the graph with the graph of $y = x^2$.

Method #1: Algebraically

To use Method #1, you need to find the coordinates of the vertex and of one other point. The coordinates of the vertex are (−2, −1), and of one other point is (−1, −3).

In vertex form, the coordinates of the vertex are (p, q).

$(p, q) = (-2, -1)$

$\therefore p = -2$

$q = -1$

Substituting these values for p and q into the vertex form of the quadratic function:

$y = a(x - (-2))^2 - 1$

$y = a(x + 2)^2 - 1$

You also have another point on the curve, $(-1, -3)$. You can substitute these values into the quadratic function for x and y.

$-3 = a(-1 + 2)^2 - 1$

$-3 = a(1)^2 - 1$

$-3 = a - 1$

$-2 = a$

Therefore, the quadratic function representing this graph is $y = -2(x + 2)^2 - 1$.

Method #2: Graphically

To use Method #2, you need to compare the given graph with the graph of $y = x^2$. To do this, it may be helpful to recall the key numbers on its graph.

The vertex is located at $(-2, -1)$. Therefore, $p = -2$ and $q = -1$. In other words, this graph is shifted two units to the left and one unit down, compared to $y = x^2$.

The graph opens downward, so the value of a is negative.

The graph is narrower than the graph of $y = x^2$. Therefore, $a < -1$.

To determine the value of a, compare the vertical distances of points on this new parabola to the graph of $y = x^2$.

Compare the point $(2, 4)$ on $y = x^2$ to the point $(0, -9)$ on the parabola. Moving two units to the right of the vertex at $(-2, -1)$ shows the vertical distance from the vertex should be 4 units down from -1. But the point on the curve is $(0, -9)$, which is 8 units down from -1. This is twice as much as expected; the y-value is doubled, and the value of a must be -2. Remember, $a < 0$ because the graph opens downward.

The quadratic function in vertex form is $y = -2(x + 2)^2 - 1$.

Determining the Number of *x*-intercepts of a Quadratic Function using *a* and *q*

Now that you know how to graph quadratic functions using transformations, you can determine how many x-intercepts a quadratic function has by considering a couple of methods. One method is to consider values of a and q. The other method considers a, p, and q and a simple sketch.

Example 6

Determine the number of intercepts for each quadratic function.

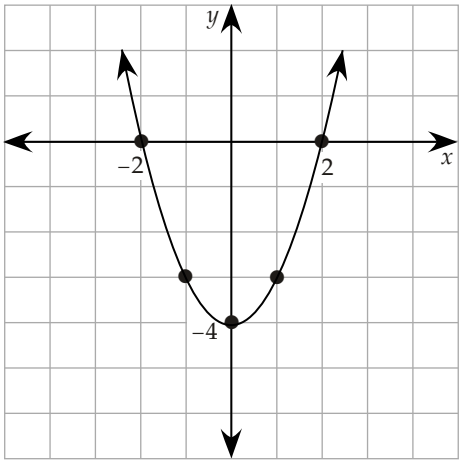
a) $y = x^2 - 4$

b) $y = 2x^2 - 5$

Solution

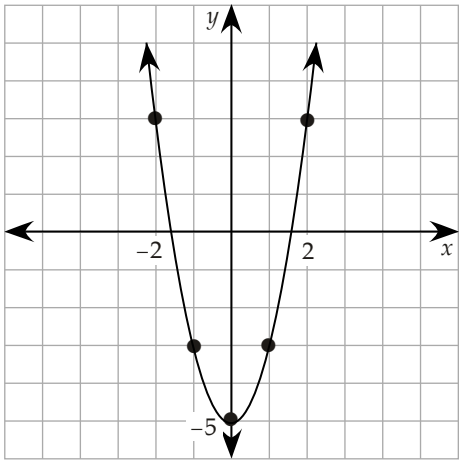
a) $y = x^2$ is shifted down 4 units to obtain the graph of $y = x^2 - 4$.

This graph has two x-intercepts, as it crosses the x-axis twice.



b) $y = x^2$ is shifted down 5 units and stretched vertically by a factor of 2 to obtain the graph of $y = 2x^2 - 5$.

This graph has two x-intercepts, as it crosses the x-axis twice.



Consider the quadratic function in vertex form, $y = a(x - p)^2 + q$. In both (a) and (b) above, the value for a was greater than zero. Also, the value for q was less than zero. Both of these graphs contained two x-intercepts. Therefore, when $a > 0$ and $q < 0$, the graph of the quadratic function contains two x-intercepts.

Example 7

Determine the number of x-intercepts for each quadratic function.

a) $y = x^2$

b) $y = 2x^2$

Solution

a) The basic quadratic graph has one x-intercept as the vertex of this parabola is on the x-axis.

b) $y = x^2$ is stretched vertically by a factor of 2 to obtain the graph of $y = 2x^2$. This graph has one x-intercept, as the vertex of this parabola is on the x-axis.

Consider the quadratic function in vertex form, $y = a(x - p)^2 + q$. In both (a) and (b) above, the value for a was greater than zero. Also, the value for q equaled zero. Both of these graphs contained one x-intercept. Therefore, when $a > 0$ and $q = 0$, the graph of the quadratic function contains one x-intercept.

Example 8

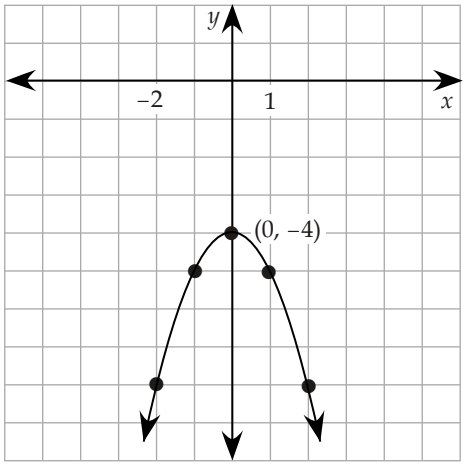
Determine the number of x-intercepts for each quadratic function.

a) $y = -x^2 - 4$

b) $y = -2x^2 - 5$

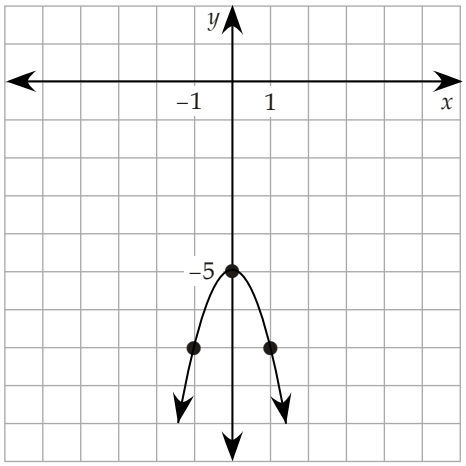
Solution

a) $y = x^2$ is reflected over the x-axis and shifted down 4 units to obtain the graph of $y = -x^2 - 4$.



This graph has no x-intercepts, as it does not cross the x-axis.

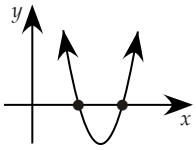
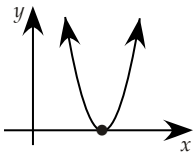
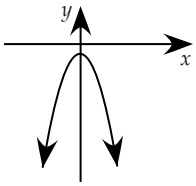
b) $y = x^2$ is stretched vertically by a factor of 2, reflected over the x-axis, and shifted down 5 units to obtain the graph of $y = -2x^2 - 5$.



This graph has no x-intercepts, as it does not cross the x-axis.

Consider the quadratic function in vertex form, $y = a(x - p)^2 + q$. In both (a) and (b) above, the value for a was less than zero. Also, the value for q was less than zero. Both of these graphs contained no x-intercepts. Therefore, when $a < 0$ and $q < 0$, the graph of the quadratic function contains no x-intercepts.

The following chart displays the information you discovered in Examples 6, 7, and 8. Note that not all possible cases of a- and q-values are represented in this chart. You may want to explore the other cases and see if you can find any patterns.

Value of a	Value of q	Graph	Number of x-intercepts
$a > 0$	$q < 0$		2 This graph crosses the x-axis twice, as the vertex is below the x-axis and the graph opens upward.
$a > 0$	$q = 0$		1 This graph crosses the x-axis once, as the vertex is on the x-axis.
$a < 0$	$q < 0$		0 This graph does not cross the x-axis, as the vertex is below the x-axis and the graph opens downward.

However, perhaps the easiest way to determine the number of x-intercepts is to quickly make a sketch of the parabola if it is given in vertex form. There is no need to memorize this chart.

Learning Activity 3.4

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. You need to make 100 cupcakes for a bake sale. However, your recipe only allows you to make cupcakes in multiples of 12. What is the least number of cupcakes you can make to have enough cupcakes for the bake sale?
2. What is the area of a park that measures 13 km by 62 km?
3. Evaluate: $\left[\left(\frac{3}{4}\right)-\left(\frac{1}{2}\right)\right]+\left[\left(\frac{1}{6}\right)+\left(\frac{5}{3}\right)\right]$
4. What is the slope of the line defined by the equation $y = 2x + 3$?
5. Does the parabola $y = -3x^2 + 4x + 2$ open up or down?
6. List the factors of 1000.
7. Write this percent as a decimal: 8.07%
8. Is 71 a prime or composite number?

Part B: Transformations of Quadratic Functions

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. State whether each statement is true or false. If the statement is false, rewrite it so it is true.
   a) The vertex of the graph of $y = (x - 4)^2 + 2$ is $(-4, 2)$.
   b) The vertex of the graph of $y = 3(x - 5)^2 - 2$ is $(5, 2)$.
   c) The vertex of the graph of $y = 2x^2 + 7$ is $(2, 7)$.
   d) The vertex of the graph of $y = -3x^2 + 4$ is $(0, 4)$.

continued

Learning Activity 3.4 (continued)

e) The vertex of the graph of $y = -2(x - 3)^2$ is $(-2, 3)$.

f) The equation of the axis of symmetry of the graph of $y = -3(x - 6)^2 + 1$ is $x = 1$.

g) The equation of the axis of symmetry of the graph of $y = 3\left(x - \frac{1}{2}\right)^2 + 4$ is $x = \frac{1}{2}$.

h) The equation of the axis of symmetry of the graph of $y = (x + 2)^2 + 1$ is $x = -2$.

i) The graph of $y = 4(x - 3)^2 - 1$ opens downward.

j) The graph of $y = 4(x + 2)^2 - 3$ opens upward.

k) The range of $y = -2(x + 3)^2 + 4$ is $\{y \mid y \leq 4\}$.

l) The maximum value of $y = 3(x - 5)^2 - 2$ is -2.

2. Match each equation with its graph.

a) $y = 3x^2$

b) $y = \frac{-1}{3}x^2$

c) $y = 3(x + 2)^2$

d) $y + 1 = 3(x + 2)^2$

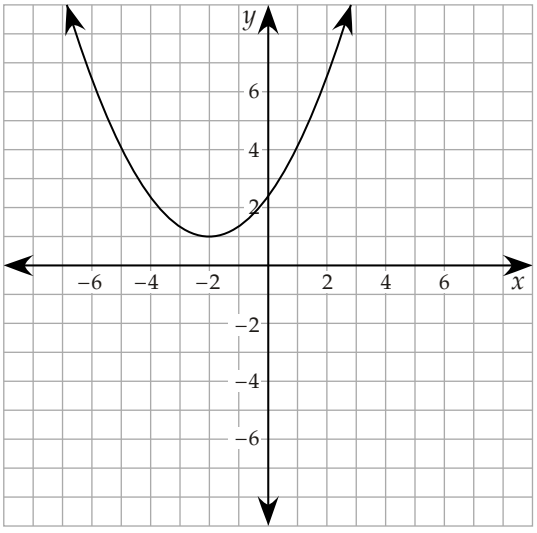
e) $y - 1 = \frac{1}{3}(x + 2)^2$

f) $y + 1 = -3(x - 2)^2$

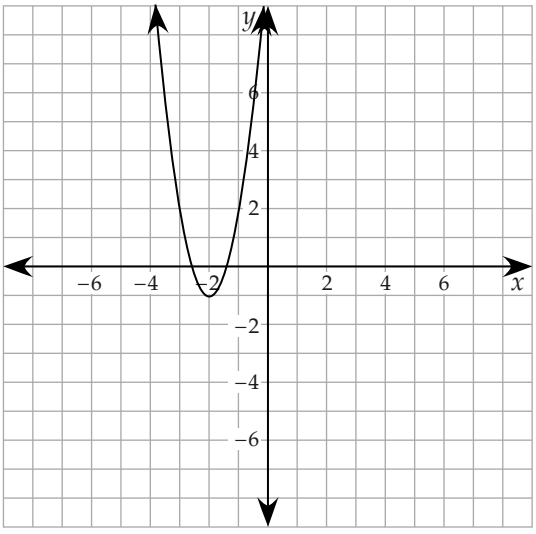
g) $y = -\frac{1}{3}(x - 2)^2$

h) $y - 1 = -\frac{1}{3}(x - 2)^2$

i)



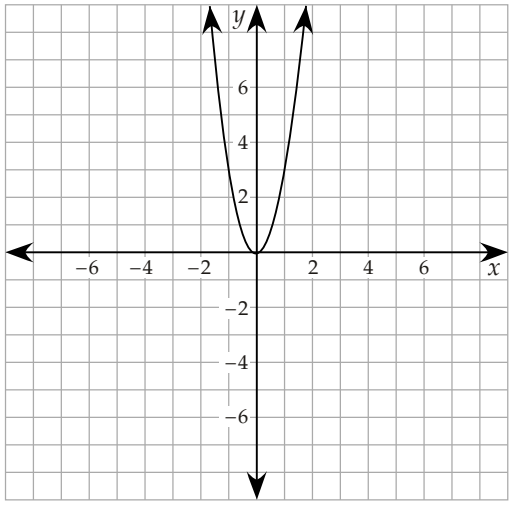
ii)



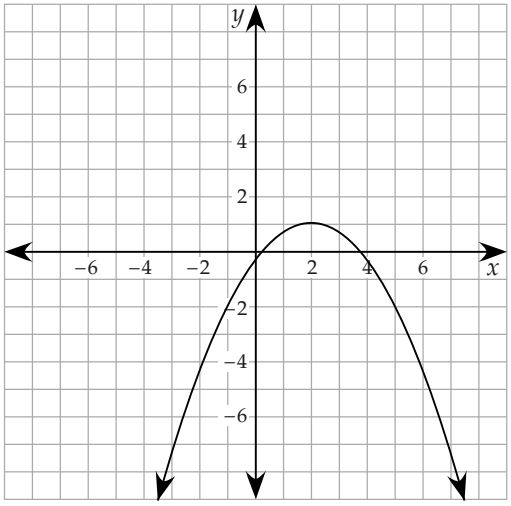
continued

Learning Activity 3.4 (continued)

iii)



iv)



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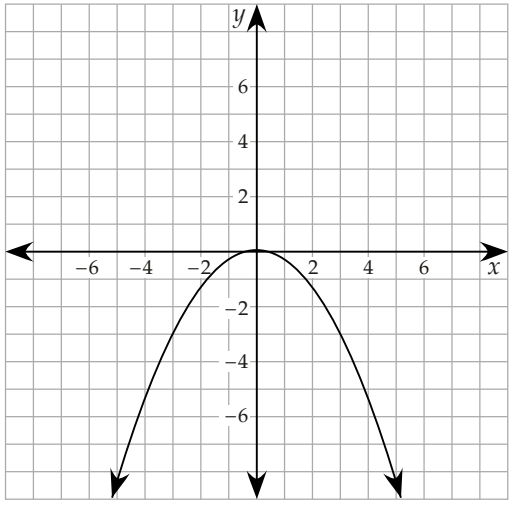

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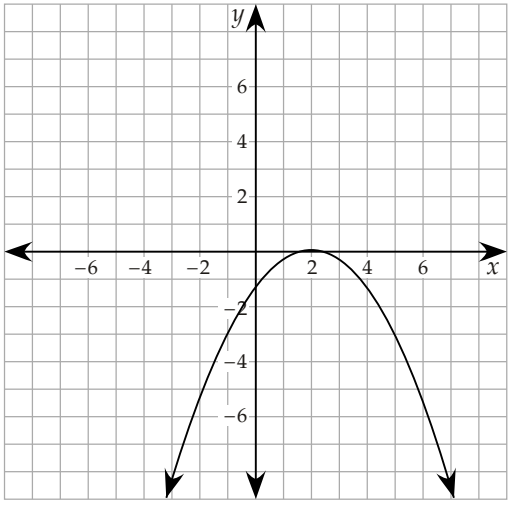
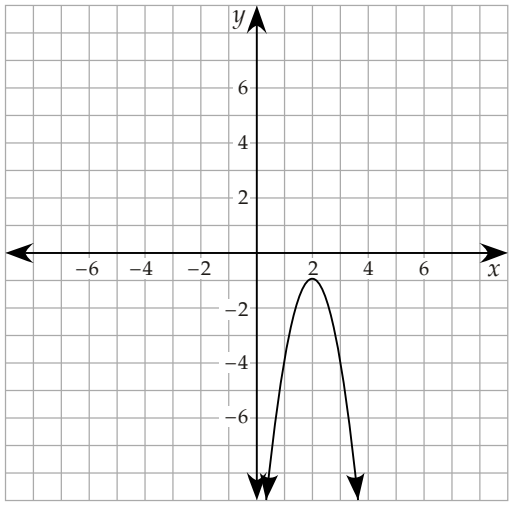
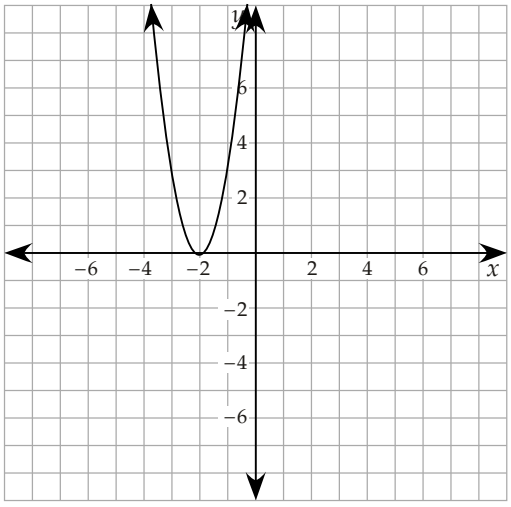


vii)



viii)



continued

Learning Activity 3.4 (continued)

3. Complete the following table.

	(a)	(b)	(c)	(d)
	$y = (x - 1)^2 + 2$	$y = -(x - 1)^2 - 2$	$y = (x + 1)^2 + 2$	$y = -(x + 1)^2 - 2$
Vertex				
Equation of axis of symmetry				
Domain				
Range				
y-intercept				
Direction of opening				
Maximum value				
Minimum value				

4. For each of the following parabolas, state:
   - the direction of opening
   - the coordinates of the vertex
   - the equation of the axis of symmetry
   - whether the given graph is narrower or wider than $y = x^2$

   a) $y = 2(x + 1)^2$

   b) $y = -\frac{1}{2}(x - 1)^2 + 6$

   c) $y = 2(x + 6)^2 - 10$

   d) $y = 6(x - 1)^2 + 8$

continued

Learning Activity 3.4 (continued)

5. Without making a table of values, sketch the graph of each of the following functions. Explain the various steps. State the vertex and the equation of the axis of symmetry. Use graph paper.

   a) $y = 2(x+1)^2$

   b) $y = -\frac{1}{2}(x-1)^2 + 6$

   c) $y = 2(x+6)^2 - 10$

   d) $y = 6(x-1)^2 + 8$

6. For each of the following quadratic functions,
   - specify whether the function has a minimum or a maximum value and what that value is.
   - specify the range.
   - find the y-intercept.

   a) $f(x) = -\left(x + \frac{1}{2}\right)^2 + 2$

   b) $y = 2(x+4)^2 - 2$

   c) $g(x) = -3(x-2)^2 - 1$

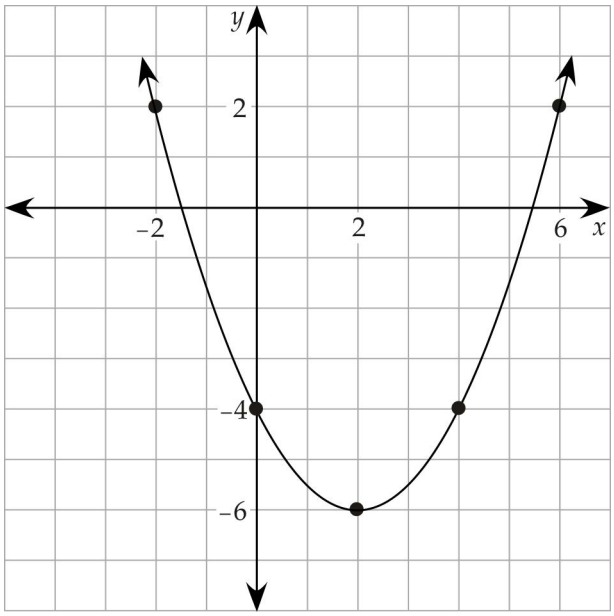
   d) $y = \frac{1}{2}\left[(x+1)^2 + 4\right]$

7. If $\left(1, \frac{3}{2}\right)$ is a point on the quadratic function represented by $y = (x-2)^2 + q$, what real number is the minimum value of the function?

continued

Learning Activity 3.4 (continued)

8. Write a quadratic function in the form $y = a(x - p)^2 + q$ that represents the following graph.



9. A graph has the following properties:
   - The vertex is at (−2, 5)
   - The graph passes through the point (2, −35)

   Write a quadratic function in the form $y = a(x - p)^2 + q$ that represents this graph.

10. Determine the number of intercepts of each quadratic function by considering the values of a and q.
    a) $y = -2(x - 3)^2 - 6$
    b) $y = 4x^2 - 1$
    c) $y = 6x^2$
    d) $y = -(x + 1)^2$

11. Redo question 10 above by considering the vertex (p, q) and the direction of opening using the value of a. Make a quick sketch.

Lesson Summary

In this lesson, you combined your knowledge of transformations of quadratic functions. You learned how to graph any quadratic function when you are given the function in vertex form. In the next lesson, you will learn how to transform quadratic functions in general form to quadratic functions in vertex form by completing the square.

Assignment 3.2

Graphing Quadratic Functions Using Transformations

Total: 65 marks

1. Match the equation with its graph. *(1 mark each, for a total of 4 marks)*

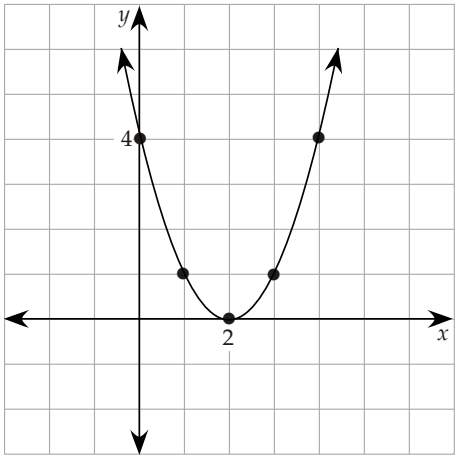
   a) $y = (x + 4)^2$ _____

   b) $y = -(x - 4)^2$ _____

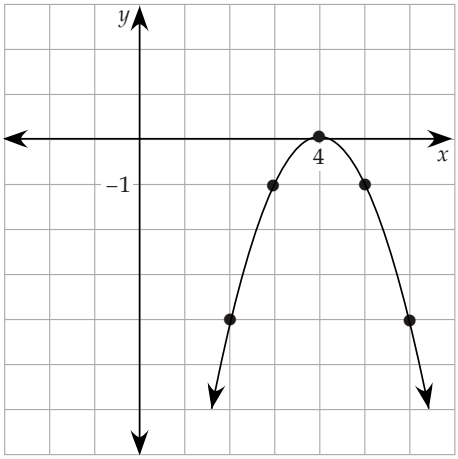
   c) $y = -(x + 8)^2$ _____

   d) $y = (x - 2)^2$ _____

i)



ii)



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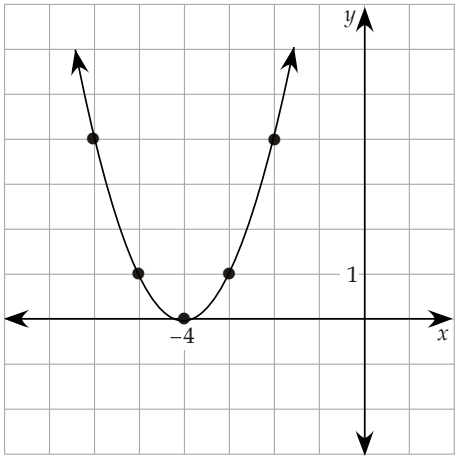

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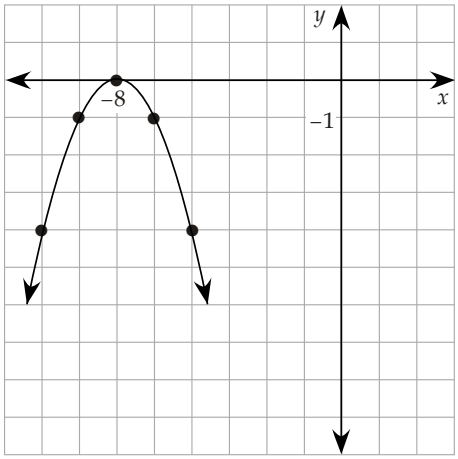
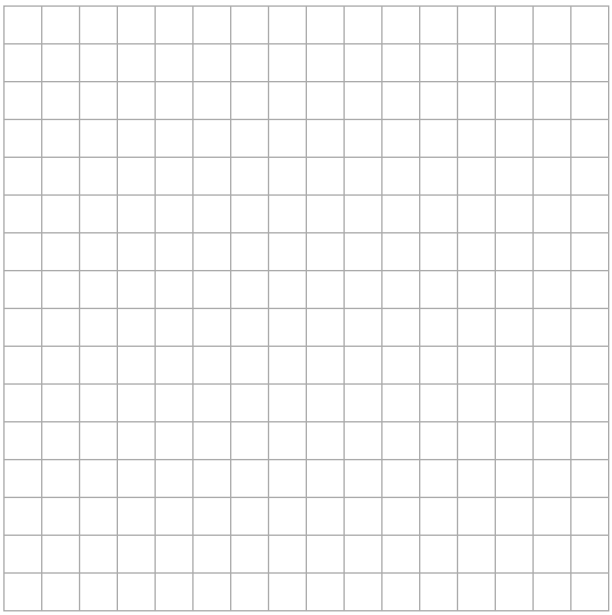


continued

Assignment 3.2: Graphing Quadratic Functions Using Transformations (continued)

2 Without using tables of values, sketch the graphs of the following on graph paper. Provide a written explanation of the steps involved in the transformation. Then, make a table of values with 5 points in it, showing how the coordinates of $y = x^2$ are transformed into points on the new graph. *(4 marks each, for a total of 8 marks)*

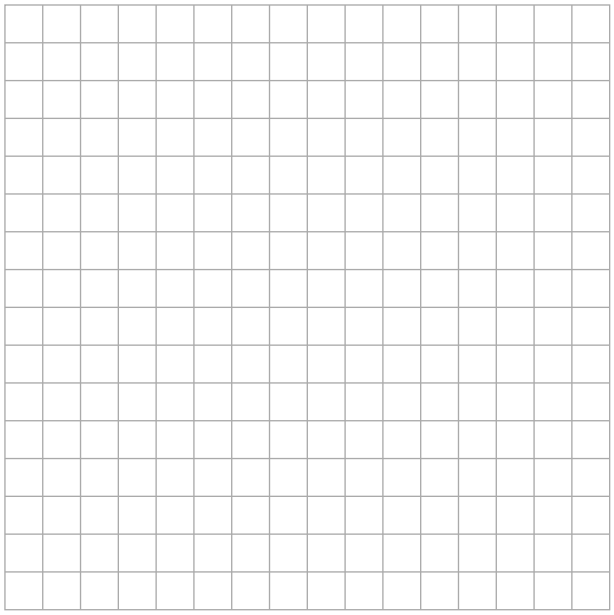
a) $f(x) = -\frac{1}{2}(x + 2)^2$



continued

Assignment 3.2: Graphing Quadratic Functions Using Transformations (continued)

b) $y = 3(x-1)^2$



continued

Assignment 3.2: Graphing Quadratic Functions Using Transformations (continued)

3. Complete the following table. *(12 marks)*

	$y = -(x + 1)^2$	$y = \frac{1}{3}(x - 5)^2$	$y = 2(x + 3)^2$
Vertex			
Equation of axis of symmetry			
Domain			
Range			
x-intercept(s)			
y-intercept			
Direction of opening			
Maximum value			
Minimum value			

4. Match each equation with its graph. *(1 mark each, for a total of 4 marks)*

a) $y = 3(x - 7)^2 + 3$

b) $y + 2 = -(x - 2)^2$

c) $y = \frac{1}{2}(x + 2)^2 - 4$

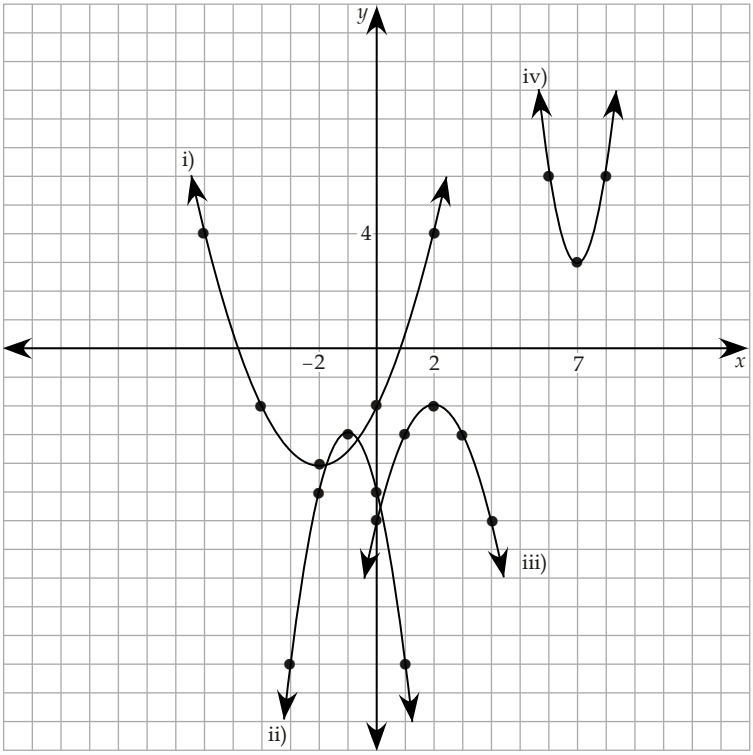
d) $y + 3 = -2(x + 1)^2$

a) _____

b) _____

c) _____

d) _____



continued

Assignment 3.2: Graphing Quadratic Functions Using Transformations (continued)

5. Complete the following table. *(12 marks)*

	$y = 2(x - 2)^2 + 1$	$y = 3(x + 1)^2 + 4$	$y = -(x + 8)^2 - 7$
Vertex			
Equation of axis of symmetry			
Domain			
Range			
y-intercept			
Direction of opening			
Maximum value			
Minimum value			

6. For each of the following parabolas, state the following: *(1 mark each, for a total of 10 marks)*
   a) The direction of opening
   b) The coordinates of the vertex
   c) The equation of the axis of symmetry
   d) Whether the given graph is narrower or wider than $y = x^2$
   e) The number of x-intercepts the graph has

   i) $y = -\frac{1}{2}(x + 5)^2 - 3$

   a) _____
   b) _____
   c) _____
   d) _____
   e) _____

   ii) $y = 2(x + 8)^2 - 9$

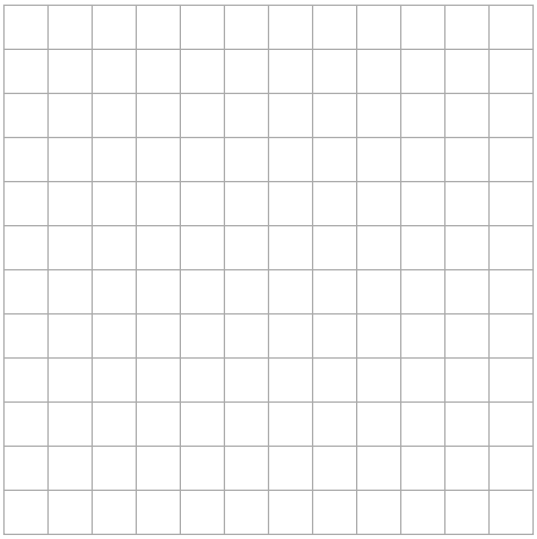
   a) _____
   b) _____
   c) _____
   d) _____
   e) _____

continued

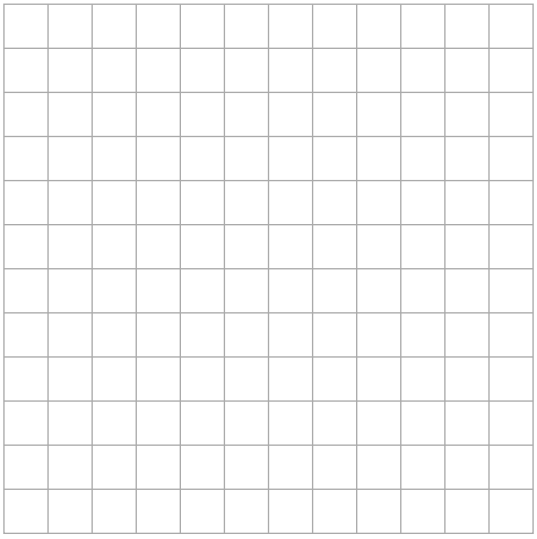
Assignment 3.2: Graphing Quadratic Functions Using Transformations (continued)

7. Without making a table of values, sketch the graph of each of the following functions. Explain the various transformations. State the vertex and the equation of the axis of symmetry. *(5 marks each, for a total of 10 marks)*

   a) $y = 3(x + 4)^2 - 2$



   b) $y = \frac{1}{2}(x - 3)^2 + 4$

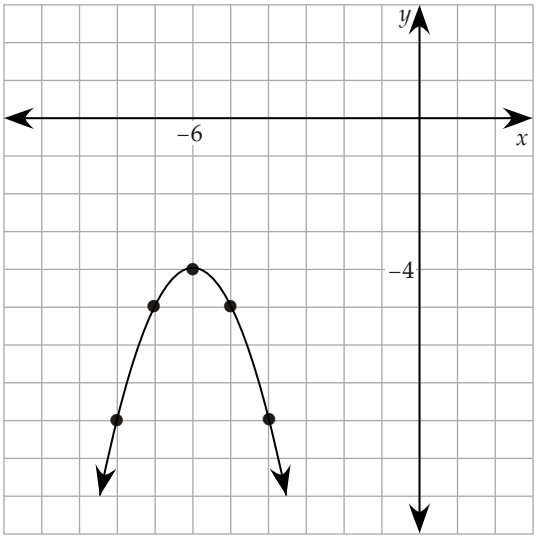


continued

Assignment 3.2: Graphing Quadratic Functions Using Transformations (continued)

8. If (3, 6) is a point on the quadratic function represented by $y = -(x + 4)^2 + q$, what real number is the maximum value of the function? *(2 marks)*

9. Write a quadratic function in the form $y = a(x - p)^2 + q$ that represents the following graph. *(3 marks)*



Notes

Lesson 5: Completing the Square

Lesson Focus

In this lesson, you will

- ❑ learn how to apply a geometric approach to completing the square using tiles
- ❑ learn how to transform $f(x) = ax^2 + bx + c$ into the form $f(x) = a(x - p)^2 + q$ by using an algebraic method of completing the square
- ❑ learn how to use quadratic functions to model situations

Lesson Introduction

You have now learned how to graph quadratic functions when they are given to you in vertex form. What happens if you are asked to graph a quadratic function in general form? This can be difficult to do without creating a table of values. Therefore, it is useful to transform quadratic functions from general form into vertex form. You can do this by learning how to complete the square. Completing the square is an important algebraic skill to master. It involves factoring and algebraic manipulation so it is a good opportunity to practise these skills. Are you up for the challenge?

Completing the Square

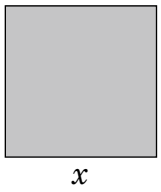
You discovered how to find the maximum or minimum value of a quadratic function when it is written in the form $f(x) = a(x - p)^2 + q$. You will now concentrate on writing any general quadratic function $f(x) = ax^2 + bx + c$ into the form $f(x) = a(x - p)^2 + q$. This process is called **completing the square**. Include this definition on your resource sheet.

In algebra, there are many times when you can complete the square to help solve a problem. Completing the square in quadratic expressions can be modelled with geometric shapes called *tiles*.

Example 1

a) Model x^2 with a square tile that is x units long on each side. Its area is $(x)(x) = x^2$.

Solution



b) Model x with a rectangular tile that is x units long and 1 unit wide. Its area is $(x)(1) = x$.

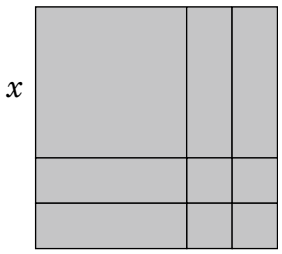
Solution

c) Model a unit with a square tile that is 1 unit long on each side. Its area is $(1)(1) = 1$.

Solution

d) Model a square that has 1 x^2-tile, 4 x-tiles, and 4 unit tiles. The area of this square is the sum of the tiles $x^2 + 4x + 4$.

Solution



Since the length of a side of this square is $x + 2$, the area can be written as $(x + 2)(x + 2) = (x + 2)^2$.

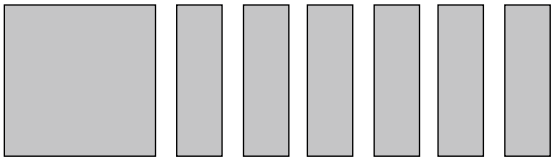
This model of $x^2 + 4x + 4$ forms a complete square. The quadratic expression is a complete square and is called a **perfect square trinomial**.

A perfect square trinomial such as $x^2 + 4x + 4$ can also be written in factored form as $(x + 2)^2$. This form is called the **square of a binomial**.

Include all of the above definitions on your resource sheet.

Example 2

a) Write a quadratic expression modelled by these tiles.



b) Rearrange all the tiles and find the minimum number of additional unit tiles needed to form a square.

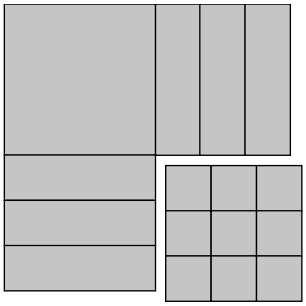
c) Write the perfect square trinomial modelled by the completed square.

Solution

a) There are 1 x^2-tile, 6 x-tiles, and 0 unit tiles. The expression is $x^2 + 6x$.

b) Take $\frac{1}{2}$ of the x-tiles and move them to the bottom side of x^2.

Now you will need 9 unit tiles to fill in the missing corner to complete the square.



c) By adding the tiles, the new square models the perfect square trinomial $x^2 + 6x + 9$.

From the above two examples a pattern emerges. In the perfect square trinomial $x^2 + 4x + 4$, the square of half the coefficient of x or $\left(\frac{4}{2}\right)^2$ is the constant term 4.

In the perfect square trinomial $x^2 + 6x + 9$, the square of half the coefficient of x or $\left(\frac{6}{2}\right)^2$ is the constant term 9.

Thus, if $x^2 + bx + c$ is a perfect square trinomial, then the constant term, c, must be $\left(\frac{b}{2}\right)^2$.

For more practice, see question 3 in Learning Activity 3.5.

You can use the technique of completing the square to transform a general quadratic function $f(x) = ax^2 + bx + c$ into the vertex form $f(x) = a(x - p)^2 + q$.

Example 3

Complete the square to find the vertex and the equation of the axis of symmetry for the quadratic function $y = x^2 + 6x$.

Solution

The number required to complete the square is the square of one-half the coefficient of the x-term.

$y = x^2 + 6x$	$\left(\frac{6}{2}\right)^2 = 3^2 = 9$
$y + 9 = x^2 + 6x + 9$	Add 9 to each side
$y + 9 = (x + 3)^2$	Factor the perfect square trinomial
$y = (x + 3)^2 - 9$	Subtract 9 from each side

The vertex is $(-3, -9)$. The vertex can also be denoted as $V(-3, -9)$

The equation of the axis of symmetry is $x = -3$.

Note: To know that you are completing the square correctly, you should be able to multiply out all the terms at any point and get back to the equation you started with. If you do not get back to the equation you started with, you know you made a mistake.

Checking to ensure $y = (x + 3)^2 - 9$ is equivalent to $y = x^2 + 6x$.

$y = (x + 3)^2 - 9$	
$y = (x + 3)(x + 3) - 9$	Expand the square
$y = x^2 + 3x + 3x + 9 - 9$	Distribute
$y = x^2 + 6x$	Combine like terms

Thus, the two forms are equivalent. Also, you could graph the quadratic function in vertex form and general form to ensure you completed the square correctly. Consider the following example.

Example 4

Express $y = x^2 - 4x + 3$ in the vertex form and find the maximum or minimum value. Verify that the quadratic function in vertex form is the same as the quadratic in general form by graphing both forms with or without a graphing calculator.

Solution

$y = x^2 - 4x + 3$	
$y - 3 = x^2 - 4x$	Isolate the x^2- and x-terms
$y - 3 + 4 = x^2 - 4x + 4$	Complete the square: $\left(\frac{4}{2}\right)^2 = 4$, and add 4 to each side
$y + 1 = (x - 2)^2$	Factor the trinomial and simplify
$y = (x - 2)^2 - 1$	Solve for y
V(2, −1)	Write the vertex

Since $a = 1$ and is positive, the parabola opens upward and the function has a minimum value of −1.

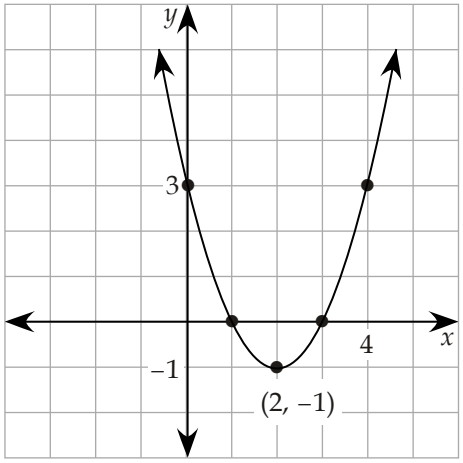
Verifying that your answer is correct by graphing both forms is shown below.

Graphing Using Graph Paper and a Table of Values

The quadratic function in general form is $y = x^2 - 4x + 3$.

The quadratic function in vertex form is $y = (x - 2)^2 - 1$.

It is possible to graph the function in vertex form by using transformations. This is the graph of $y = x^2$ shifted two units to the right and one unit down.



You can make a table of values from the graph.

x	y
0	3
1	0
2	−1
3	0
4	3

Now make a similar table of values using the general form and the same x-values obtained from the graph.

General form: $y = x^2 - 4x + 3$

Table of Values using the x-values 0, 1, 2, 3, and 4.

x	$y = x^2 - 4x + 3$
0	$y = 0^2 - 4(0) + 3 = 3$
1	$y = 1^2 - 4(1) + 3 = 0$
2	$y = 2^2 - 4(2) + 3 = -1$
3	$y = 3^2 - 4(3) + 3 = 0$
4	$y = 4^2 - 4(4) + 3 = 3$

Since the tables for each form are the same, the verification is complete.

Therefore, the quadratic function in vertex form represents the same function as the corresponding quadratic function in general form.

Example 5

Find the coordinates of the vertex and the equation of the axis of symmetry of $f(x) = 3x^2 + 12x + 16$.

Solution

Express the equation in the form $y = a(x - p)^2 + q$ by completing the square.

$f(x) = 3x^2 + 12x + 16$	
$f(x) - 16 = 3x^2 + 12x$	Remove the constant term from the right side of the equation.
$\frac{f(x) - 16}{3} = \frac{3x^2}{3} + \frac{12x}{3}$	Change the coefficient of x^2 to 1 by dividing each term by 3.
$\frac{f(x) - 16}{3} = x^2 + 4x$	Simplify.
$\frac{f(x) - 16}{3} + 4 = x^2 + 4x + 4$	Complete the square and add 4 to each side of the equation.
$\frac{f(x) - 16}{3} + 4 = (x + 2)^2$	Factor.
$\frac{f(x) - 16}{3} = (x + 2)^2 - 4$	Subtract 4 from both sides of the equation.
$f(x) - 16 = 3(x + 2)^2 - 12$	Multiply each term by 3.
$f(x) = 3(x + 2)^2 + 4$	Add 16 to both sides of the equation.
V(−2, 4)	Write the vertex.

The vertex is at (−2, 4) and the equation of the axis of symmetry is $x = -2$.

Now that you have completed a few examples of how to complete the square, it may be helpful for you to write a description of the process on your resource sheet. Feel free to include a completed example on your resource sheet if you feel the example will help you remember how to complete the square.

Although this solution looks complicated, it allows us to isolate the perfect square trinomial on the right side. Later on in this lesson, a few alternate solutions to this question are provided. Compare this solution to the solutions for Examples 11 and 12. You can use any of these methods when there is a coefficient in front of the x^2 term. Choose the one that you find easiest while still consistently letting you find the right answer.

Example 6

Express $f(x) = -2x^2 + 16x + 5$ in vertex form and find the domain and range of the function.

Solution

$$f(x) = -2x^2 + 16x + 5$$

$f(x) - 5 = -2x^2 + 16x$ Eliminate the constant term by subtracting 5 from both sides.

$\frac{f(x) - 5}{-2} = \frac{-2x^2}{-2} + \frac{16x}{-2}$ Convert the coefficient of x^2 to 1 by dividing all terms by (−2).

$\frac{f(x) - 5}{-2} = x^2 - 8x$ Simplify.

$\frac{f(x) - 5}{-2} + 16 = x^2 - 8x + 16$ Complete the square and add 16 to both sides.

$\frac{f(x) - 5}{-2} + 16 = (x - 4)^2$ Factor.

$\frac{f(x) - 5}{-2} = (x - 4)^2 - 16$ Subtract 16 from both sides of the equation.

$f(x) - 5 = -2(x - 4)^2 + 32$ Multiply each term by (−2).

$f(x) = -2(x - 4)^2 + 37$ Add 5 to both sides of the equation.

V(4, 37) Write the vertex.

Domain: $\{x \mid x \in \Re\}$

Range: $\{y \mid y \leq 37, y \in \Re\}$

Example 7

Express $y = -\frac{1}{2}x^2 - x + 4$ in vertex form. Determine whether the function has a maximum or minimum value, and determine its corresponding x-value.

Solution

$$y = -\frac{1}{2}x^2 - x + 4$$

$y - 4 = -\frac{1}{2}x^2 - x$ Subtract 4.

$\frac{y-4}{-\frac{1}{2}} = \frac{-\frac{1}{2}x^2}{-\frac{1}{2}} - \frac{x}{-\frac{1}{2}}$ Divide by $\left(-\frac{1}{2}\right)$. Be careful here.

$\frac{y-4}{-\frac{1}{2}} = x^2 + 2x$ Simplify.

$\frac{y-4}{-\frac{1}{2}} + 1 = x^2 + 2x + 1$ Complete the square and add 1 to both sides.

$\frac{y-4}{-\frac{1}{2}} + 1 = (x+1)^2$ Factor.

$\frac{y-4}{-\frac{1}{2}} = (x+1)^2 - 1$ Subtract 1 from both sides of the equation.

$y - 4 = -\frac{1}{2}(x+1)^2 + \frac{1}{2}$ Multiply by $\left(-\frac{1}{2}\right)$.

$y = -\frac{1}{2}(x+1)^2 + 4\frac{1}{2}$ Add 4.

$V\left(-1, 4\frac{1}{2}\right)$ Write the vertex.

Since the parabola opens downward, the function has a maximum value at $4\frac{1}{2}$. The corresponding x-value is -1.

Example 8

Find the values of a, p, and q that will make the statement true.

a) $x^2 - 6x = (x - p)^2 + q$

b) $3x^2 - 12x + 5 = a(x - p)^2 + q$

Solution

a) $x^2 - 6x = (x - p)^2 + q$

If you complete the square for the left side of the equation, you can determine the value for a, p, and q.

Let $y = x^2 - 6x$

$y + 9 = x^2 - 6x + 9$ Complete the square and add 9.

$y + 9 = (x - 3)^2$ Factor.

$y = (x - 3)^2 - 9$ Subtract 9.

Now compare your answer with the right side of the original equation. The value for a is 1, for p is 3, and for q is -9.

b) $3x^2 - 12x + 5 = a(x - p)^2 + q$

To find the values for a, p, and q, the expression on the left side of the equation needs to be converted into vertex form by completing the square.

Let $y = 3x^2 + 18x + 5$

$y - 5 = 3x^2 + 18x$ Remove the constant term.

$\frac{y - 5}{3} = x^2 + 6x$ Divide by 3.

$\frac{y - 5}{3} + 9 = x^2 + 6x + 9$ Complete the square and add 9.

$\frac{y - 5}{3} + 9 = (x + 3)^2$ Factor.

$\frac{y - 5}{3} = (x + 3)^2 - 9$ Subtract 9.

$y - 5 = 3(x + 3)^2 - 27$ Multiply by 3.

$y = 3(x + 3)^2 - 22$ Add 5.

Thus, $a = 3$, $p = -3$, and $q = -22$.

Common Mistakes when Completing the Square

There are many common mistakes other students make when completing the square. If you are aware of these mistakes, you are less likely to make them yourself. Consider the following examples.

Example 9

Nehemiah is trying to complete the square of the following quadratic function. His work is shown below.

$$2x^2 - 5x + 1 = y$$

$$2x^2 - 5x = y - 1$$

$$x^2 - \frac{5}{2}x = \frac{y-1}{2}$$

$$x^2 - \frac{5}{2}x + \frac{25}{4} = \frac{y-1}{2} + \frac{25}{4}$$

$$\left(x - \frac{5}{2}\right)^2 = \frac{y-1}{2} + \frac{25}{4}$$

$$2\left(x - \frac{5}{2}\right)^2 - \frac{25}{2} + 1 = y$$

$$2\left(x - \frac{5}{2}\right)^2 - \frac{27}{2} = y$$

Identify, explain, and correct Nehemiah's mistakes in completing the square.

Solution

Nehemiah completed the square correctly until he got to the fourth line. He forgot to take one-half of the coefficient of x before squaring. On the last line, he did not add correctly. He also skipped some steps, which makes checking the work more difficult.

Therefore, Nehemiah should have completed the square as follows:

$$2x^2 - 5x + 1 = y$$

$$2x^2 - 5x = y - 1$$

$$x^2 - \frac{5}{2}x = \frac{y-1}{2}$$

$$x^2 - \frac{5}{2}x + \frac{25}{16} = \frac{y-1}{2} + \frac{25}{16} \qquad \left(\frac{-5}{2} \div 2\right)^2 = \left(\frac{-5}{2} \times \frac{1}{2}\right)^2 = \left(\frac{-5}{4}\right)^2 = \frac{25}{16}$$

$$\left(x - \frac{5}{4}\right)^2 = \frac{y-1}{2} + \frac{25}{16}$$

$$\left(x - \frac{5}{4}\right)^2 - \frac{25}{16} = \frac{y-1}{2}$$

$$2\left(x - \frac{5}{4}\right)^2 - \frac{25}{16}(2) = y - 1$$

$$2\left(x - \frac{5}{4}\right)^2 - \frac{25}{8} + 1 = y$$

$$2\left(x - \frac{5}{4}\right)^2 - \frac{17}{8} = y \qquad -\frac{25}{8} + 1 = -\frac{25}{8} + \frac{8}{8} = -\frac{17}{8}$$

Don't be afraid of the fraction calculations. You could use the a b/c button or the Math:Frac program on your calculator to check your answers.

Example 10

Identify, explain, and correct the errors in the following example of completing the square.

$$y = -2x^2 = 3x - 7$$

$$y + 7 = -2x^2 - 3x$$

$$\frac{y+7}{-2} = x^2 - \frac{3}{2}x$$

$$\frac{y+7}{-2} = x^2 - \frac{3}{2}x + \frac{9}{16}$$

$$\frac{y+7}{-2} + \frac{9}{16} = \left(x - \frac{3}{4}\right)^2$$

$$\frac{y+7}{-2} = \left(x - \frac{3}{4}\right)^2 - \frac{9}{16}$$

$$y + 7 = -2\left(x - \frac{3}{4}\right)^2 - \frac{9}{16}(-2)$$

$$y = -2\left(x - \frac{3}{4}\right)^2 + \frac{9}{8} - 7$$

$$y = -2\left(x - \frac{3}{4}\right)^2 - \frac{47}{8}$$

Solution

The error in this completion of the square occurs in the third line. When dividing (-2) out each term, the -3 should turn into a positive $\frac{3}{2}$ instead of staying as negative $\frac{3}{2}$.

Therefore, the square should be completed as follows:

$$y = -2x^2 - 3x - 7$$

$$y + 7 = -2x^2 - 3x$$

$$\frac{y+7}{-2} = x^2 + \frac{3}{2}x$$

$$\frac{y+7}{-2} + \frac{9}{16} = x^2 + \frac{3}{2}x + \frac{9}{16}$$

$$\frac{y+7}{-2} + \frac{9}{16} = \left(x + \frac{3}{2}\right)^2$$

$$\frac{y+7}{-2} = \left(x + \frac{3}{2}\right)^2 - \frac{9}{16}$$

$$y + 7 = -2\left(x + \frac{3}{2}\right)^2 - \frac{9}{16}(-2)$$

$$y = -2\left(x + \frac{3}{2}\right)^2 + \frac{9}{8} - 7$$

$$y = -2\left(x + \frac{3}{2}\right)^2 - \frac{47}{8}$$

Note: A good way to ensure that you have completed the square correctly is to multiply out the vertex form of the quadratic function that you found to see if it simplifies to the general form of the quadratic function you were given in the question.

Another way to check your work is to use a different process for completing the square. Two alternate methods are shown below. You may find them easier, but they require great concentration and understanding. Compare Examples 11 and 12 with Example 5.

Example 11

Find the coordinates of the vertex and the equation of the axis of symmetry of $f(x) = 3x^2 + 12x + 16$.

Solution

Express the equation in the form $y = a(x - p)^2 + q$ by completing the square.

$f(x) = 3\left(x^2 + 4x \quad\right) + 16$	Factor the coefficient of x^2 out of the x^2- and x-terms. Leave a space and write the constant term outside the bracket.
$f(x) = 3\left(x^2 + 4x + 4 - 4\right) + 16$	Complete the square. Add and subtract 4 inside the bracket.
$f(x) = 3\left(x^2 + 4x + 4\right) + 16 + 3(-4)$	Remove (-4) from the bracket multiplied by 3.
$f(x) = 3(x + 2)^2 + 16 - 12$	Factor and simplify.
$f(x) = 3(x + 2)^2 + 4$	Simplify.
V(−2, 4)	Write the vertex.

The vertex is at $(-2, 4)$ and the equation of the axis of symmetry is $x = -2$.

Example 12

Complete the square to find the vertex for $f(x) = 3x^2 + 12x + 16$.

Solution

$f(x) = 3x^2 + 12x + 16$

$f(x) = 3(x^2 + 4x + \quad) + 16$	Factor the coefficient of x^2 out of the x^2 and x-terms. Leave one space and write the constant term outside of the brackets.
$f(x) = 3(x^2 + 4x + 4) + 16 - 4(3)$	Complete by adding 4 inside the brackets and subtracting away (4)(3) outside the brackets. You notice that 3 is both the first number and the last number on this line.
$f(x) = 3(x + 4)^2 + 16 - 12$	Factor.
$f(x) = 3(x + 4)^2 + 4$	Simplify.
V(−2, 4)	Write the vertex.

Using Quadratic Functions to Model Situations

One of the more important roles of quadratic functions is their use in modelling the path of a falling object. You can write a quadratic function that represents a situation such as the height of a basketball when thrown.

After t seconds, the height of an object is represented by the following function:

$$h(t) = -4.9t^2 + v_0t + h_0$$

where: t = time in seconds
v_0 = initial upward velocity in metres per second
h_0 = initial height (a constant)

In the equation $h(t) = -4.9t^2 + v_0t + h_0$, half of the acceleration due to the force of gravity is represented by the coefficient of the squared term. In other words, the falling object is affected by acceleration of 4.9 m/s^2.

Initially, when $t = 0$, $h(0) = -4.9(0)^2 + v_0(0) + h_0 = h_0$, a positive number. Therefore, an object has a predetermined initial height, denoted by h_0.

As the time increases, the squared term will have an increasingly larger negative value. The height will eventually reach a maximum and then become increasingly smaller as the object falls towards the ground.

Consider the following example.

Example 13

A tennis ball at an initial height of 1.5 metres is hit upwards with an initial velocity of 25 metres per second.

a) Write a quadratic function $h(t)$ for the height in metres after t seconds using the formula $h(t) = -4.9t^2 + v_0t + h_0$.

b) Use your graphing calculator, if you have one, to find the maximum height reached by the ball. Round to the nearest hundredth.

c) Complete the square algebraically to find the maximum height, rounded to the nearest hundredth.

Solution

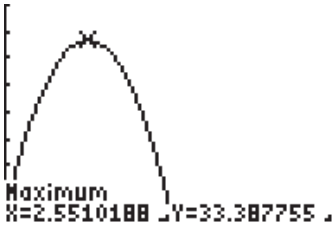
a) Substitute the given values into $h(t) = -4.9t^2 + v_0t + h_0$ where v_0 is the initial velocity in metres per second and h_0 is the initial height. This is a physics formula, which you may or may not be familiar with. You do not need to memorize this formula.

$v_0 = 25$

$h_0 = 1.5$

$h(t) = -4.9t^2 + 25t + 1.5$

b) Input $y_1 = -4.9x^2 + 25x + 1.5$



The coordinates of the vertex are V(2.55, 33.39).

Therefore, the maximum height is 33.39 m and occurs when the time is 2.55 seconds.

c) Completing the square:

$$h(t) = -4.9t^2 + 25t + 1.5$$

$$\frac{h(t) - 1.5}{-4.9} = \frac{-4.9t^2}{-4.9} + \frac{25t}{-4.9}$$

$$\frac{h(t) - 1.5}{-4.9} = t^2 - 5.102t$$

$$\frac{h(t) - 1.5}{-4.9} + \left(\frac{-5.102}{2}\right)^2 = t^2 - 5.102t + \left(\frac{-5.102}{2}\right)^2$$

$$\frac{h(t) - 1.5}{-4.9} + 6.508 = t^2 - 5.102t + 6.508$$

$$\frac{h(t) - 1.5}{-4.9} + 6.508 = (t - 2.551)^2$$

$$\frac{h(t) - 1.5}{-4.9} = (t - 2.551)^2 - 6.508$$

$$h(t) - 1.5 = -4.9(t - 2.551)^2 - 6.508(-4.9)$$

$$h(t) = -4.9(t - 2.551)^2 + 31.889 + 1.5$$

$$h(t) = -4.9(t - 2.551)^2 + 33.389$$

The vertex is (2.551, 33.389).

Thus, the maximum height is 33.39 m.

Learning Activity 3.5

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

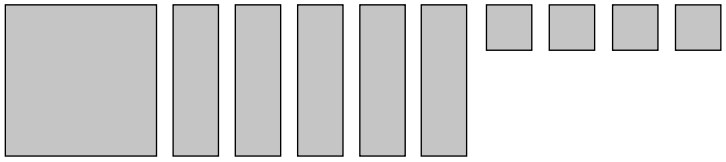
The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Factor: $x^2 - 10x + 25$
2. Your computer has a battery life of 12 hours. If you use your computer for 15 minutes in the morning to check the weather, 6 hours during school to complete school work, 25 minutes during lunch to go on your favourite social networking site, and 1.6 hours to watch a movie, how much battery life do you have left on your laptop?
3. Evaluate for $g = -2$: $5g^3 + g^2 - 6g - 13$
4. Continue the pattern: 2, 6, 18, 54, _____ , _____
5. You are at the store to buy your sister a birthday present. You have $50 to spend and the item you want to purchase is $60 on sale for 25% off. Will you have enough money to buy the item (ignoring tax)?
6. Simplify: $(x^2 - 4x + 1)(2x^2)$
7. Find the midpoint between the following two points: (2, 6) and (−7, −4)
8. What is the slope of a horizontal line?

Part B: Completing the Square

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Write a quadratic expression modelled by this set of tiles.



continued

Learning Activity 3.5 (continued)

2. Draw a tile model to represent $x^2 + 3x + 2$ in the shape of a rectangle.
3. What value of k makes the expression a perfect square trinomial? Write each expression as the square of a binomial.
   a) $x^2 + 8x + k$
   b) $x^2 - 8x + k$
   c) $x^2 + 20x + k$
   d) $x^2 - 2x + k$
   e) $x^2 - 5x + k$
   f) $x^2 + 7x + k$
4. Find the values of a, p, and q for each of the following.
   a) $x^2 - 8x = a(x - p)^2 + p$
   b) $x^2 + 5x = a(x - p)^2 + p$
   c) $\frac{1}{2}x^2 + 8x + 4 = a(x - p)^2 + p$
5. For each of the following equations, answer the questions below.
   a) $f(x) = -x^2 + 6x$
   b) $f: x \to 2x^2 + 5x - 3$
   c) $y = -4x - x^2$
      i) Express in the form $f(x) = a(x - p)^2 + q$.
      ii) Find the coordinates of the vertex and equation of the axis of symmetry.
      iii) Find the range.
      iv) Determine the maximum or minimum value.

continued

Learning Activity 3.5 (continued)

6. Given $y = ax^2 + bx + c$, the steps for expressing the function in the vertex form $y = a(x - p)^2 + q$ are written below. The coordinates of the vertex are shown. Supply the reasons for each step.

$$y = ax^2 + bx + c$$

$$y - c = ax^2 + bx$$

$$\frac{y - c}{a} = \frac{ax^2}{a} + \frac{b}{a}x$$

$$\frac{y - c}{a} = x^2 + \frac{b}{a}x$$

$$\frac{y - c}{a} + \left(\frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2$$

$$\frac{y - c}{a} + \frac{b^2}{4a^2} = \left(x + \frac{b}{2a}\right)^2$$

$$\frac{y - c}{a} = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}$$

$$y - c = a\left(x + \frac{b}{2a}\right)^2 - a \cdot \frac{b^2}{4a^2}$$

$$y = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

$$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

$$\mathrm{V}\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$$

continued

Learning Activity 3.5 (continued)

7. Dominic was asked to complete the square. His work is shown below.

$x^2 + 8x + 10 = y$

$x^2 + 8x = y - 10$

$x^2 + 8x + 64 = y - 10 + 64$

$(x + 4)^2 = y + 54$

$y = (x + 4)^2 - 54$

Identify, explain, and correct the error in his work. Verify your answer is correct by completing the square using a different method.

8. Explain why $a(x - p)^2 + q = y$ is a quadratic function.

Lesson Summary

In this lesson, you learned how to transform quadratic functions from general form into vertex form by completing the square. As you know, quadratic functions in vertex form are easier to graph by using transformations. However, as you found out in the learning activity, it is possible to determine important characteristics of quadratic functions in general form, such as the vertex, without transforming them into vertex form first. In the next lesson, you will learn more about how to determine characteristics of quadratic functions in general form.

Notes

Lesson 6: Special Features of the Quadratic Function

Lesson Focus

In this lesson, you will

- ❑ learn how to find the coordinates of the vertex of a quadratic function given its x-intercepts
- ❑ learn how to find the coordinates of the vertex of a quadratic function in general form using the formula $x = -\frac{b}{2a}$
- ❑ learn how to use interval notation to express the domain and range of relations and functions

Lesson Introduction

In the last lesson, you learned how to complete the square in order to transform quadratic functions in general form to vertex form. It is fairly easy to find the vertex of a quadratic function in vertex form $y = a(x - p)^2 + q$ by considering the values of p and q. However, it is also possible to find the vertex of a quadratic function in general form $y = ax^2 + bx + c$ by considering the values of a and b. You can also find the vertex of a quadratic function in general form by first factoring and finding the x-intercepts. Overall, the vertex can be considered to be one of the easier characteristics to find.

The Quadratic Function

The zeros of a function, f, are the values of x that make $f(x) = 0$. Zeros are another way to say x-intercepts. In order to find the x-intercepts of a quadratic function written as the product of linear functions, you use the **zero product property**.

Zero Product Property

If $ab = 0$, then either $a = 0$ or $b = 0$ must be true.

Include this property on your resource sheet.

Example 1

Find the x-coordinate of the vertex of a function whose zeros are 4 and -2. Find the axis of symmetry.

Solution

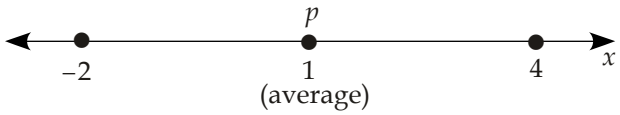
The zeros are 4 and -2.

The x-intercepts are at the points (4, 0) and $(-2, 0)$. Remember that these points are on the x-axis, and the y-coordinate of these points is always 0.

Since you know a parabola is symmetric about its axis of symmetry, the axis must lie halfway between the two x-intercepts. This average value will be the x-coordinate of the vertex.

To find the x-coordinate of the vertex, find the **average of the x-intercepts**.

$$p = \text{average of the } x\text{-intercepts} = \frac{4 + (-2)}{2} = \frac{2}{2} = 1$$



The axis of symmetry is the vertical line passing through the vertex—that is, $x = 1$.

Example 2

Find the vertex of the function $y = x^2 - 8x + 12$.

Solution

Factor $y = x^2 - 8x + 12$ to get $y = (x - 6)(x - 2)$.

Use the Zero Product Property to find the x-intercepts:

$0 = (x - 6)(x - 2)$

$x - 6 = 0$ or $x - 2 = 0$

$x = 6, x = 2$

The x-intercepts are 6 and 2.

Find the average of the x-intercepts: $\frac{6 + (+2)}{2} = \frac{8}{2} = 4$

Therefore the vertex is at $x = 4$.

Substitute $x = 4$ into the original equation:

$y = (4)^2 - 8(4) + 12$

to get $y = -4$.

Therefore, the vertex is at $(4, -4)$.

This method can be used to find the vertex of a quadratic function in general form if it factors readily. You will explore this is more depth in Module 4.

A **second method** of finding the coordinates of the vertex uses a formula developed in Learning Activity 3.5, Question 6, where you developed the generalization of completing the square to get

$$V\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$$

The x-coordinate can be derived quickly when the function is given in general form.

Example 3

Find the coordinates of the vertex and equation of axis of symmetry of $f(x) = x^2 - 6x + 4$.

Solution

This function is written in general form. Therefore, you can find the vertex by using the following steps.

1. Find the x-coordinate of the vertex using $x = -\frac{b}{2a}$.

   $a = 1 \quad b = -6$

   $$x = \frac{-(-6)}{2(1)} = \frac{6}{2} = 3$$

2. Find the y-coordinate of the vertex by substituting $x = 3$ into the original function.

   $y = x^2 - 6x + 4$

   $y = (3)^2 - 6(3) + 4$

   $y = 9 - 18 + 4$

   $y = -5$

   The vertex is at $(3, -5)$.

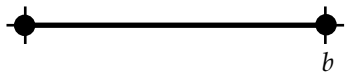
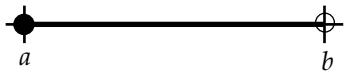
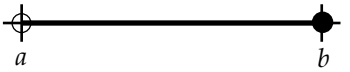
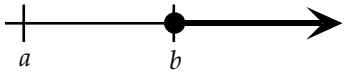
Note: If you are having trouble remembering $x = \frac{-b}{2a}$, you could complete the square to find the vertex.

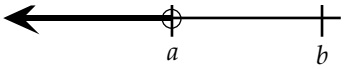
Interval Notation

In *Grade 10 Introduction to Applied and Pre-Calculus Mathematics*, you learned how to express domain and range in set notation and interval notation. Interval notation uses square brackets [ ] to indicate that all the numbers between two specified numbers and the numbers themselves are included, whereas round parentheses ( ) are used to exclude the end numbers.

The symbol ∞ is used to indicate that the interval of numbers has no upper limit; $-\infty$ means the interval has no lower limit.

The chart below provides you with the link between the set notation and interval notation.

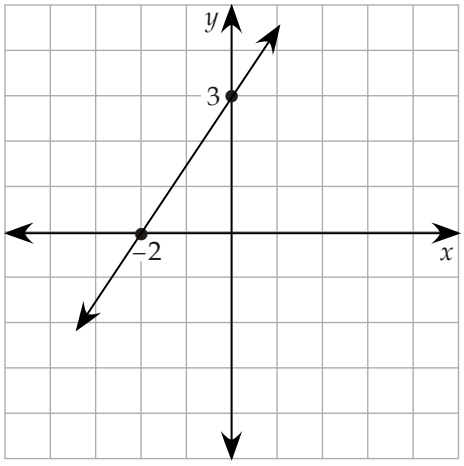
Set Notation	Corresponding Line Graph	Interval Notation
$\{x \mid a \leq x \leq b\}$		$[a, b]$ a and b are included so square brackets are used
$\{x \mid a \leq x < b\}$		$[a, b)$ the open circle indicates b is not included and so a round parenthesis is used with b
$\{x \mid a < x \leq b\}$		$(a, b]$ the open circle at a indicates a is not included and so a round parenthesis is used with a
$\{x \mid a < x < b\}$		(a, b) the open circles at a and b mean that a and b are not included—hence, the round parentheses
$\{x \mid x \geq b\}$		$[b, \infty)$ the line has a closed circle at b and goes infinitely to the right
$\{x \mid x > b\}$		(b, ∞) means b is excluded but that the line goes infinitely to the right
$\{x \mid x \leq a\}$		$(-\infty, a]$ a is included and the line continues infinitely to the left of a

Set Notation	Corresponding Line Graph	Interval Notation
$\{x \mid x < a\}$		$(-\infty, a)$ a is excluded and the line continues infinitely to the left of a
$\{a\}$		$[a, a]$ There is only one point in the set

Example 3

Find the domain and range of the following functions or relations using interval notation. Note that domain is stated as (left bound, right bound) and range is stated as (lower bound, upper bound).

a)

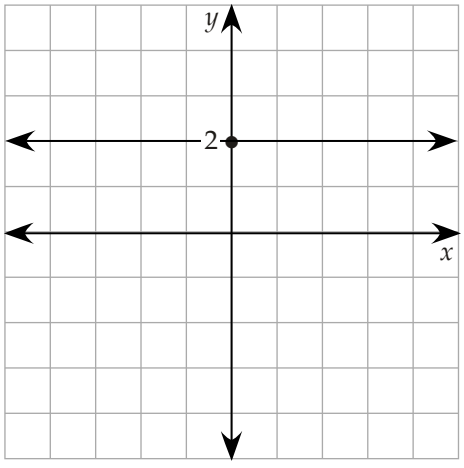


Solution

D: $(-\infty, \infty)$
$\Re$: $(-\infty, \infty)$

No restrictions are placed on the domain and range, as this line can take on all x- and y-values.

b)

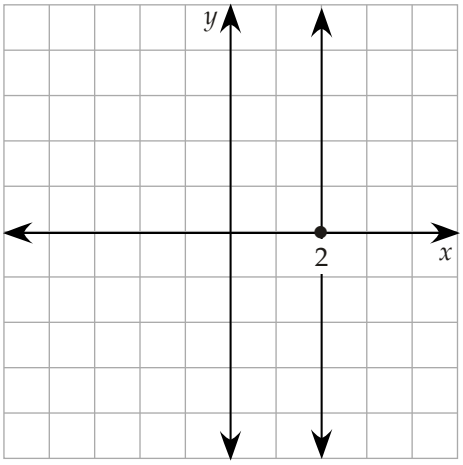


Solution

D: $(-\infty, \infty)$
$\Re$: {2}

The x-values are not restricted for this line; this set notation indicates the range is constant at 2 and is used when the interval contains a single value.

c)

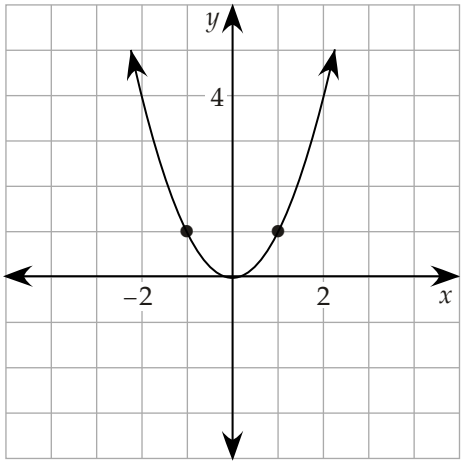


Solution

D: {2}
$\Re$: $(-\infty, \infty)$

The x-values are the same on a vertical line; y-values are not restricted.

d)

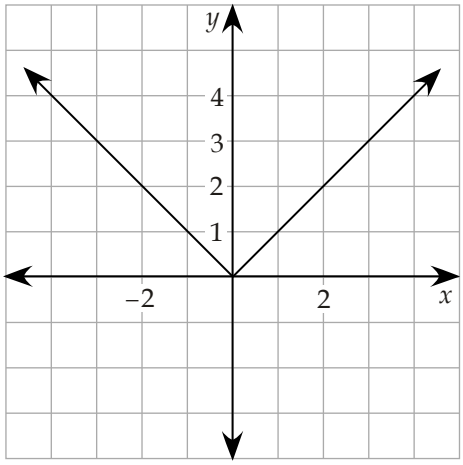


Solution

D: $(-\infty, \infty)$
$\Re$: $[0, \infty)$

The x-values are not restricted for this curve. The y-values include 0 as a lower bound and have no upper limit.

e)

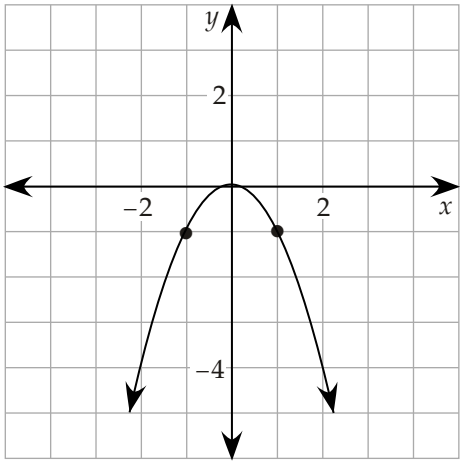


Solution

D: $(-\infty, \infty)$
$\Re$: $[0, \infty)$

Notice that the domain and range are the same as in (d), even though the function is different.

f)

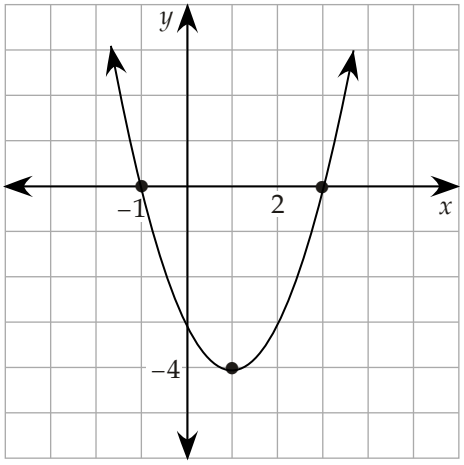


Solution

D: $(-\infty, \infty)$
$\Re$: $(-\infty, 0]$

Notice that the domain is the same as in (d); the y-values have a maximum of 0 and no lower limit.

g)

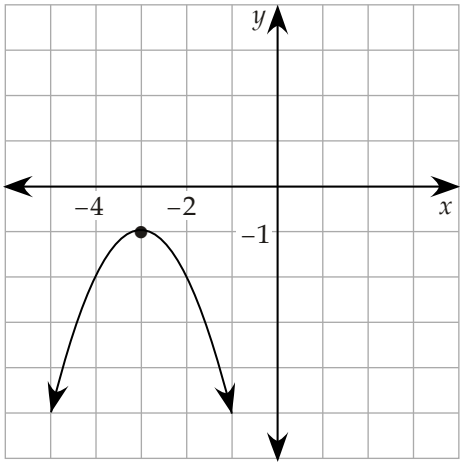


Solution

D: $(-\infty, \infty)$
$\Re$: $[-4, \infty)$

The x-values are not limited; the y-values have a minimum of -4 and increase to infinitely large numbers.

h)

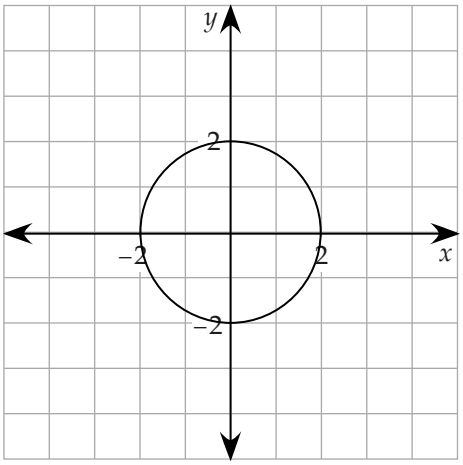


Solution

D: $(-\infty, \infty)$
$\Re$: $(-\infty, -1]$

Because the parabola opens downward, -1 is included, as well as an infinite number of y-values approaching negative infinity.

i)



Solution

D: $[-2, 2]$
$\Re$: $[-2, 2]$

Notice that it is possible to find the domain and range of relations as well as functions.

Example 4

For the function $y = x^2 + 6x + 5$, state the following:

a) the coordinates of the vertex

b) equation of axis of symmetry

c) the y-intercept

d) domain and range using interval notation

Solutions

a) The x-coordinate of the vertex is given by

$$x = \frac{-b}{2a}$$

$$a = 1,\ b = 6$$

$$x = \frac{-6}{2(1)} = -3$$

Substitute the x-value of -3 into the function to evaluate the y-coordinate of the vertex.

$$y = x^2 + 6x + 5$$

$$y = (-3)^2 + 6(-3) + 5$$

$$y = 9 - 18 + 5$$

$$y = -4$$

$$\therefore\ \mathrm{V}(-3, -4)$$

b) The axis of symmetry is the vertical line passing through the vertex:
$\therefore x = -3$.

c) The y-intercept is found by letting $x = 0$ in the function.

$y = x^2 + 6x + 5$

$y = 0^2 + 6(0) + 5$

$y = 5$

$\therefore$ the y-intercept is at the point (0, 5).

d) D: $(-\infty, \infty)$

$\Re$: $[-4, \infty)$

Make sure you complete the following learning activity, as similar questions will be on the assignment directly following this learning activity. Also, be sure to check your answers in the Module 3 Answer Key to ensure you are completing the questions correctly.

Learning Activity 3.6

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. There are 15 females and 12 males in your family. On average, each female eats 2 pieces of pizza and each male eats 3 pieces of pizza. How many pizzas will you have to buy in order for everyone to eat as much pizza as they wish if each pizza is cut into 12 pieces?
2. Multiply: $(3x + 4)(2x - 7)$
3. Evaluate for $n = 4$: $4n^2 + n - 9$
4. Calculate 30% of 42 000.
5. What is the perimeter of a square with an area of 169 units?
6. Factor completely: $4x^4 - 20x^3 + 24x^2$
7. Find the slope of the line defined by the equation $7x + 3y - 4 = 0$.
8. Multiply. $(3x^2 + 7)(8x - 4)$

continued

Learning Activity 3.6 (continued)

Part B: Special Features of Quadratic Equations

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Given the following x-intercepts and a-value,
   a) $x = -1, x = 3, a = 1$
   b) $x = -4, x = -2, a = -1$

   find the following and then sketch the graph:
   i) vertex
   ii) equation of axis of symmetry
   iii) y-intercept
   iv) direction of opening
   v) maximum value
   vi) minimum value
   vii) domain and range (state in both notations)

2. For each of the following quadratic functions
   a) $y = x^2 + 6x - 7$
   b) $y = x^2 - 4x - 60$
   c) $y = 2x^2 + 8x - 10$
   d) $y = 3x^2 + 24x + 21$

   find the following:
   i) The coordinates of the vertex using $p = -\frac{b}{2a}$ and substituting to find q
   ii) The equation of the axis of symmetry
   iii) The y-intercept
   iv) The direction of opening
   v) The maximum or minimum value
   vi) The domain and range in interval notation

Lesson Summary

In this lesson, you learned how to find the vertex of a quadratic function in general form using two methods. You also reviewed your knowledge from *Grade 10 Introduction to Applied and Pre-Calculus Mathematics*, including interval notation. You then wrote the domain and range of various functions, including quadratic functions, and relations in interval notation. In the next lesson, you will be learning about applications of quadratic functions.

Notes

Assignment 3.3

Completing the Square and Finding the Vertex

Total: 38 marks

1. What value of k makes the expression a perfect square trinomial? Write each expression as the square of a binomial. *(1 mark each, for a total of 2 marks)*

   a) $x^2 + 9x + k$

   b) $x^2 - 7x + k$

2. Find the values of a, p, and q for each of the following. *(3 marks each, for a total of 6 marks)*

   a) $x^2 - 3x = a(x - p)^2 + q$

   b) $x^2 + 10x = a(x - p)^2 + q$

(continued)

Assignment 3.3: Completing the Square and Finding the Vertex (continued)

3. For the following function, answer the questions below. *(1.5 marks each, for a total of 3 marks)*

   $f(x) = -x^2 + 7x$

   a) Express in the form $f(x) = a(x - p)^2 + q$.

   b) Find the coordinates of the vertex and equation of the axis of symmetry.

4. Lawon is trying to complete the square. His work is shown below.

$$-x^2 + 8x = y$$
$$-(x^2 - 8x) = y$$
$$-(x^2 - 8x + 16) = y + 16$$
$$-(x - 4)^2 - 16 = y$$

Identify, explain, and correct Lawon's error. *(2 marks)*

continued

Assignment 3.3: Completing the Square and Finding the Vertex (continued)

5. Create a poster with *two* quadratic functions. Your poster must include the following items. A marking rubric for the poster is provided on the following page.
   - Two quadratic functions expressed in general form *(2 marks)*
     - These quadratic functions need to have different directions of opening. Therefore, one quadratic function needs to open up and one quadratic function needs to open down.
   - Two instances of completing the square, one for each function, to express the above functions in vertex form *(4 marks)*
   - An explanation of how to tell if a quadratic function opens up or down by looking at the equation *(2 marks)*
   - The graphs of both quadratic functions on graph paper
     - On the graphs, label the vertex and mark the position of the zeros (if any). *(4 marks)*
     - State the domain and range in set notation of each function. *(4 marks)*

continued

Assignment 3.3: Completing the Square and Finding the Vertex (continued)

Module 3, Assignment 3.3: Math Poster Marking Rubric				
Category	**0**	**1**	**2**	**Score**
Quadratic Functions in General Form	No quadratic functions are included in general form.	Only one quadratic function is included in general form, or two quadratic functions are included but they have the same direction of opening.	Two quadratic functions are included in general form. One quadratic function opens up and one quadratic function opens down.	
Completing the Square for Function #1	The square is not completed correctly.	The work for completing the square is not shown, or there are a few errors in the method of completing the square.	The general form of this function is transformed into the vertex form by correctly completing the square.	
Completing the Square for Function #2	The square is not completed correctly.	The work for completing the square is not shown, or there are a few errors in the method of completing the square.	The general form of this function is transformed into the vertex form by correctly completing the square.	
Explanation of Direction of Opening	No explanation is given.	An incomplete explanation is given for the direction of opening of a parabola where one or two of the following cases are missing: a is positive, or a is negative.	An accurate explanation is given for the direction of opening of a parabola including when a is positive or negative.	
Graph of Function #1	The graph is not given or is improperly drawn and labelled.	The labels for the vertex and zeros are given, but the graph is incorrectly drawn.	An accurate graph is drawn of this function with labels for the vertex and position of the zeros (if any).	
Graph of Function #2	The graph is not given or is improperly drawn and labelled.	The labels for the vertex and zeros are given, but the graph is incorrectly drawn.	An accurate graph is drawn of this function with labels for the vertex and position of the zeros (if any).	
Graph #1 Information	The domain and range are incorrectly stated.	The domain or range is correctly stated.	The domain and range of this function are correctly stated.	
Graph #2 Information	The domain and range are incorrectly stated.	The domain or range is correctly stated.	The domain and range of this function are correctly stated.	
			TOTAL	

continued

Assignment 3.3: Completing the Square and Finding the Vertex (continued)

6. For the following quadratic function

   $y = -x^2 + 3x + 4$

   find the following: *(1 mark each, for a total of 3 marks)*

   a) The coordinates of the vertex using $p = -\frac{b}{2a}$ and substituting to find q

   b) The maximum or minimum value

   c) The domain and range in interval notation

continued

Assignment 3.3: Completing the Square and Finding the Vertex (continued)

7. For a quadratic function whose x-intercepts are -3 and -2 and whose a-value is 1, find the following:
   a) The coordinates of the vertex by finding the average of the x-intercepts. *(3 marks)*
   b) The y-intercept. *(1 mark)*
   c) The domain and range in both set notation and interval notation. *(2 marks)*

Lesson 7: Applications of Quadratic Functions

Lesson Focus

In this lesson, you will

- ❑ learn how to find a function to model problem situations that are quadratic in nature
- ❑ learn how to solve quadratic word problems

Lesson Introduction

You have learned about quadratic functions in vertex and general form, including how to graph both forms of functions. You have learned how to find important properties of quadratic functions, including their vertices. You have also learned how to complete the square. In this lesson, you will combine all of your knowledge on quadratic functions to solve problems about applications of quadratic functions.

Quadratic functions have many applications including maximizing the area, yield, height, or profit in different situations. Throughout this lesson, you will see examples of these types of maximization problems.

Applications of Quadratic Functions

The area of a rectangular figure can be modelled by a quadratic function and its graph can be used to find its maximum area.

Example 1

John has 50 m of fencing to enclose a rectangular area. He needs to know the maximum area that can be enclosed. What are the dimensions that will produce this maximum area?

Solution

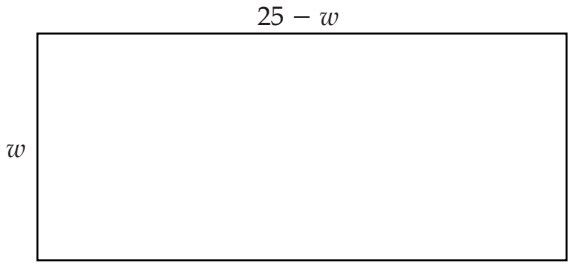
Draw a table showing the possibilities. If the perimeter is 50 m, then the length plus the width must be 25 m.

Width (m)	Length (m)	Area (m^2)
1	24	24
2	23	46
3	22	66
⋮	⋮	⋮
w	$25 - w$	$w(25 - w)$

You can generalize the pattern in the table.

Let w = width of the rectangle.

Let $25 - w$ = length



Check: The perimeter = $w + (25 - w) + w + (25 - w)$

$= 50$ m

If you represent the area by A, its equation becomes:

A = length × width

$A = (25 - w)(w)$

$A = 25w - w^2$

$A = -w^2 + 25w$

$A = -w^2 + 25w$ is a quadratic function representing the area of the rectangle.

Since the coefficient of the squared term is negative, the curve opens down. The function has a maximum value at its vertex.

By completing the square of the above function, you can find the vertex of the function. The coordinates of the vertex will be (w, A) since those are the two variables in the function equation.

$$A = -w^2 + 25w$$

$$A = -\left(w^2 - 25w \quad \right)$$

$$A + (-1)\left(\frac{25}{2}\right)^2 = -\left(w^2 - 25w + \left(\frac{25}{2}\right)^2\right)$$

$$A - \frac{625}{4} = -\left(w - \frac{25}{2}\right)^2$$

$$A = -\left(w - \frac{25}{2}\right)^2 + \frac{625}{4}$$

The vertex is $\left(\frac{25}{2}, \frac{625}{4}\right)$.

This means a width of $\frac{25}{2}$ m gives a maximum area of $\frac{625}{4}$ m^2.

Thus, the dimensions of the rectangular area must be:

$$\text{width} = \frac{25}{2} = 12.5 \text{ m}$$

$$\begin{aligned}\text{length} &= 25 - w \\ &= 25 - 12.5 \\ &= 12.5 \text{ m}\end{aligned}$$

As both the length and the width of the rectangular area have the same lengths, 12.5 m, the maximum area occurs when a square is formed from the fencing materials.

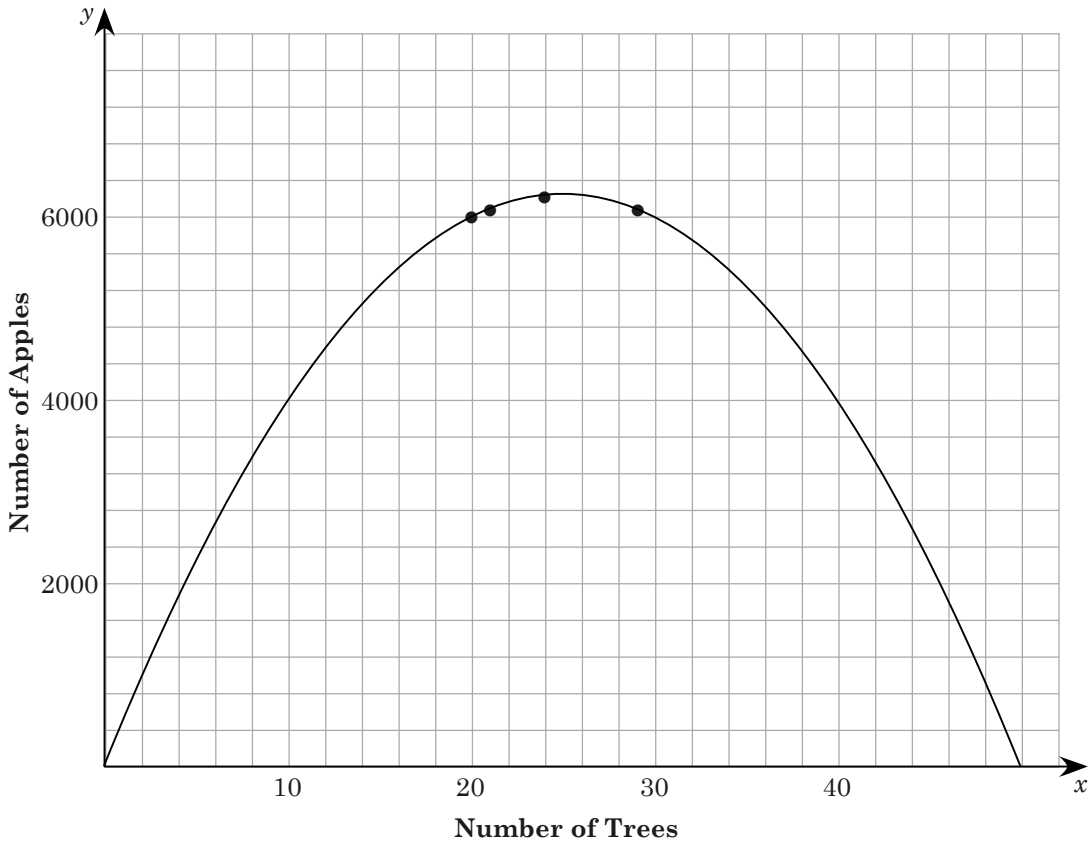
Example 2

Tom is the owner of an apple orchard. He knows that he has 20 trees on one acre of his land, and that each tree produces an average of 300 apples each season. Another acre has 21 trees and each tree produces an average of 290 apples. Another acre has 24 trees and each tree produces an average of 260 apples in a season. Another acre is examined and he finds 29 trees, and each tree produces an average of 210 apples. He proceeds to make a chart from his data:

Number of Trees per Acre (x)	Number of Apples per Tree	Total Number of Apples Produced (y)
20	300	6000
21	290	6090
24	260	6240
29	210	6090

He concludes that having more trees per acre does not necessarily equate with getting more apples per season from each acre of land. In fact, the highest yields seem to be with 24 trees per acre. However, he also concludes that due to crowding, competing for nutrients from the land, and whatever other factors may come into play, there is probably a certain number of trees per acre that will maximize the yield. He is interested in maximizing his yield and, therefore, he decides to find out the best number of trees using mathematics. Using the data from the above chart, and letting x be the number of trees and y be the total number of apples produced, he generates the points (x, y), (20, 6000), (21, 6090), (24, 6240), (29, 6090).

He plots the points as (x, y) on a graph and gets



He concludes that the complete graph may be parabolic and finds the quadratic best fit for the data to be $y = -10x^2 + 500x$. How did Tom find this function? Consider the following methods.

Method 1: Charting

A quadratic function can be found by considering the following chart.

Number of Trees (x)	Number of Apples per Tree	y = Total Yield = (# of trees)(# of apples/tree)
20	300	(20)(300) = 6000
20 + 1 = 21 (one additional tree)	290 = 300 − 10	(21)(290) = 6090
20 + 4 = 24 (four additional trees)	260 = 300 − 40	(24(260) = 6240
20 + 9 = 29 (nine additional trees)	210 = 300 − 90	(29)(210) = 6090

Can you see the pattern of the numbers that are changing (the variables) and those that remain the same (the constants)?

In the orchard with 20 trees, each tree produced 300 apples. When one more tree was added to the orchard, each tree produced 10 fewer apples. Therefore, there is a connection between the number of trees and how many apples these trees produce, namely a factor of 10 per tree. When there are 24 trees in the orchard, each tree produces 40 = 10(4) less apples than when there are 20 trees in the field. Therefore, these four additional trees contribute to each tree producing 4(10) = 40 less apples.

You can now write statements to identify the variables:

Let n = number of additional trees

Let y = total yield

You can generalize the pattern and write:

Number of trees = $20 + n$

Number of apples per tree = $300 - 10n$

You now have a quadratic function that you can simplify, find its vertex, and thus answer the farmer's question.

$$y = (20 + n)(300 - 10n)$$

$$y = 6000 - 200n + 300n - 10n^2$$

$$y = 6000 + 100n - 10n^2$$

$$y = -10n^2 + 100n + 6000$$

$$y - 6000 = -10n^2 + 100n$$

$$\frac{y - 6000}{-10} = \frac{-10n^2}{-10} + \frac{100n}{-10}$$

$$\frac{y - 6000}{-10} = n^2 - 10n$$

$$\frac{y - 6000}{-10} + 25 = n^2 - 10n + 25$$

$$\frac{y - 6000}{-10} + 25 = (n - 5)^2$$

$$y - 6000 = -10\left[(n - 5)^2 - 25\right]$$

$$y = -10(n - 5)^2 + 6250$$

V(n, y) = V(5, 6250). Therefore, the maximum yield occurs when the number of trees per acre is $20 + n = 20 + 5 = 25$ trees. This is the same answer as shown on Tom's graph.

An alternate method of finding the vertex is to use the formula:

$n = -\frac{b}{2a}$ for $y = -10n^2 + 100n + 6000$

$$n = \frac{-100}{2(-10)}$$

$$n = \frac{-100}{-20}$$

$$n = 5$$

This is the same value that you found when you used the method of completing the square. You can then find the y-coordinate of the vertex by substituting this value back into the equation.

$y = -10n^2 + 100n + 6000$

$y = -10(5)^2 + 100(5) + 6000$

$y = -250 + 500 + 6000$

$y = 6250$

Therefore, the coordinates of the vertex are V(5, 6250). This is the same point as the one found using the previous method. Thus, it is possible to use either the completing the square method or the formula method to find the vertex. You will be using both of these methods in the upcoming learning activity.

Method 2:

The farmer also uses his graph of the quadratic function to find the maximum yield:

Looking at the highest point of the curve, the vertex, he finds a maximum number of 6250 apples reached when there are 25 trees per acre.

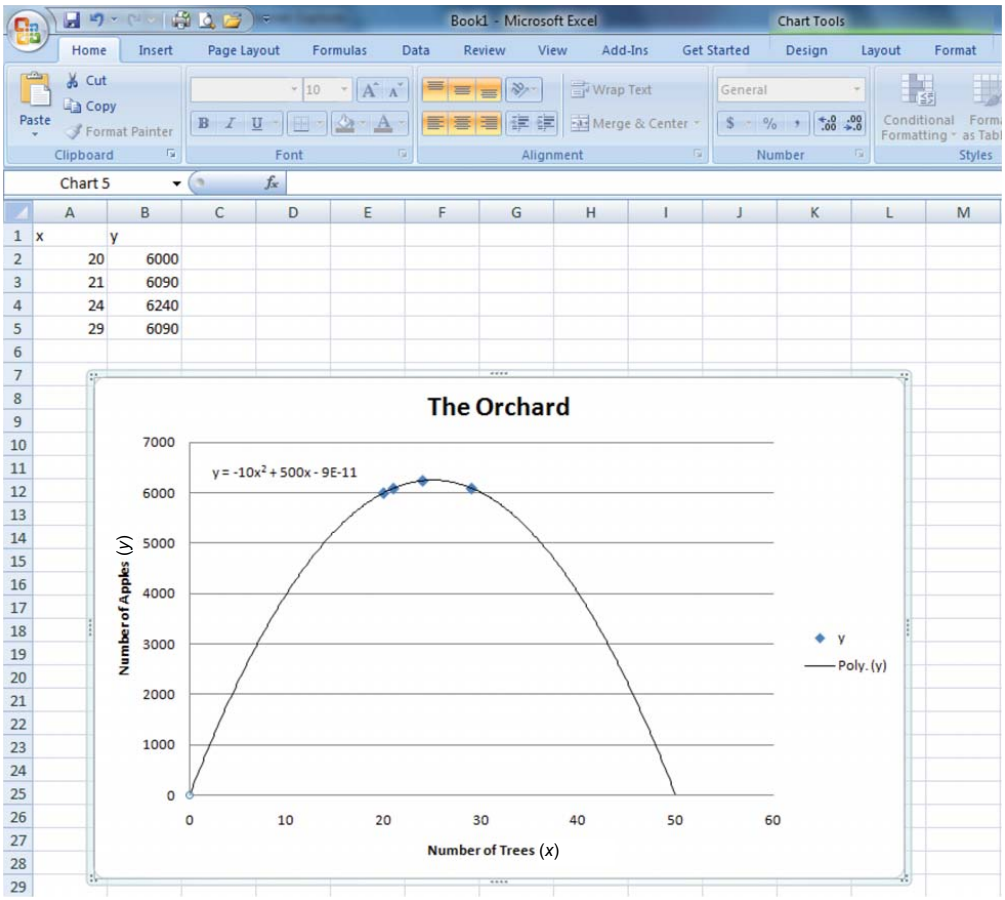
To maximize the yield, he should plant 25 trees per acre and expect to get 6250 apples.

Notice that the graph also indicates that no apples will be produced where there are no trees and also where there are 50 trees. That is when $x = 0$ and $x = 50$. These two x-values are the x-intercepts or zeros of the function.

Method 3: Computer Technology

The quadratic function can also be found using Microsoft Excel. Let x represent the number of trees and let y represent the total number of apples produced. Input the data and use a scatterplot to get the graph and trendline to calculate its equation.

The completed graph with a function defining the curve is shown below.



The function defining this curve is $y = -10x^2 + 500x - 9E\text{-}11$, where y represents the yield and x represents the number of trees. This is the same as $y = -10x^2 + 500x - 0.000\,000\,000\,09$. This is essentially the same as $y = -10x^2 + 500x$. Therefore, you have arrived at the same function using Excel as Tom did by graphing his data. Also, by extending the trendline, you can see that the x-intercepts of this graph are 0 and 50, which are the same values that you found before. Thus, it is possible to solve application problems involving quadratic functions by using a computer program such as Microsoft Excel.

Note that this is a different function than $y = -10n^2 + 100n + 6000$ because the variables t and n represent different amounts. The values for y are the same at the vertex.

Example 3

A car dealership can sell 20 cars per week at a profit of $2400 for each car. The dealership has determined that for every $300 increase in profit per car, one less car per week is sold. What is the maximum the dealership can make and how many cars would then be sold per week?

Solution

Make a chart to establish a pattern.

Profit per Car	Number of Cars Sold per Week	Total Profit = (Profit per Car) (Number of Cars Sold)
2400 (original amount)	20	(2400)(20)
2400 + 300(1) (first increase)	20 − 1 (one less car sold)	[2400 + 300(1)](20 − 1)
2400 + 300(2) (second increase)	20 − 2 (two less cars sold)	[2400 + 300(2)](20 − 2)
2400 + 300(n) (n is the number of increases in profit/car)	20 − n (or n is the number of decreases in cars sold)	[2400 + 300(n)](20 − n)

Now you can make the "let" statements to identify the variables, write the quadratic function, find the vertex, check to ensure a maximum occurs, and answer the question.

Let n = number of increases in the profit per car
(or the number of decreases in cars sold)

Let P = total weekly profit

The quadratic function is $P = (2400 + 300n)(20 - n)$.

Now you can simplify, and by completing the square or using the formula, $-\frac{b}{2a}$, you can find the vertex. The method of completing the square is shown below.

$P = (2400 + 300n)(20 - n)$

$P = 48\,000 - 2400n + 6000n - 300n^2$

$P = -300n^2 + 3600\text{n} + 48\,000$

$P = -300(n^2 - 12n) + 48\,000$

$P = -300(n^2 - 12n + 36) + 48\,000 - 36(-300)$

$P = -300(n - 6)^2 + 48\,000 + 10\,800$

$P = -300(n - 6)^2 + 58\,800$

Vertex: $V(n, P) = V(6, 58\,800)$

At the vertex, $n = 6$ and $P = 58\,800$. Since $a = -300$, the curve opens downward and there is a maximum. The number of increases in profit is 6. The number of decreases in cars is 6. The maximum profit is $58,800.

Now you can write the answer. The maximum the dealership can make is $58,800. The number of cars sold per week = 20 − 6 = 14 cars.

Example 4

If 400 children attend an afternoon movie, the price of admission is $8 per ticket. For each $1 added to the price of the admission, the attendance drops by 40 children. How many increases of $1 should the theatre add to the price in order to maximize income?

Solution

Approach 1: Using the Graphing Calculator

Choose some prices, and calculate the numbers of people expected to attend and the income.

Price of Ticket (x)	Attendees	Income for Theatre (y)
$8	400	$3200.00
$9	360	$3240.00
$10	320	$3200.00

Now enter the data for x and y into Stat Editor of your graphing calculator and calculate a quadratic regression equation.

The required quadratic regression equation that approximates the data is $y = -40x^2 + 720x$.

Enter and graph $y = -40x^2 + 720x$.

When the function is graphed and a maximum value is found, the answer is $y = 3240$ when $x = 9$, which indicates that the price of the ticket should be $9 in order to bring about a maximum income of $3240.00 for the theatre.

Approach 2: Algebraic

Let y = income for the theatre

Let n = number of $1 increases

Notice that the n variable represents the number of increases in the ticket price, not the ticket price itself.

Income = (Price per ticket) × (children in attendance)

$y = (8 + 1n)(400 - 40n)$

$y = 3200 - 320n + 400n - 40n^2$

$y = -40n^2 + 80n + 3200$

$y = -40(n^2 - 2n \quad) + 3200$

$y = -40(n^2 - 2n + 1) + 3200 - 1(-40)$

$y = -40(n - 1)^2 + 3240$

$V(1, 3240) = V(n, y)$

Since $a = -40$, the parabola opens down and has a maximum value at its vertex.

The vertex represents one increase of $1 and shows that the maximum income is $3240.00. One $1 increase brings the ticket price to $9, which is the same as is given by the graphing calculator approach.

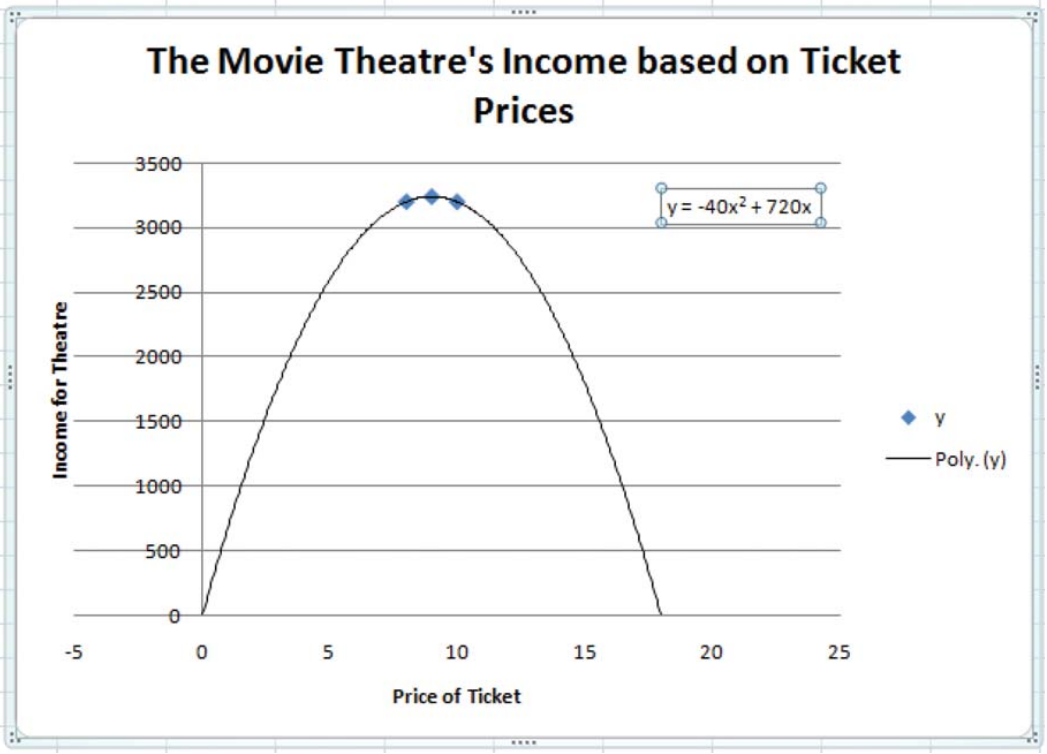
Approach 3: Microsoft Excel

Use the same steps as you used in Example 2 to find an approximation of the quadratic function to represent this information.

Let x = the price per ticket

Let y = the income for the theatre

A completed graph with a function defining the curve is shown below. Its vertex appears to be around 9 with the maximum income between 3000 and 3500.



The function defining this curve is $y = -40x^2 + 720x$. Thus, you have arrived at the same function using Microsoft Excel and by using a graphing calculator. The algebraic approach gave an exact equation for the function.

The vertex of this function, $y = -40x^2 + 720x$, occurs at $\frac{-b}{2a} = -\frac{720}{2(-40)} = 9$.

Substituting 9 into the function gives:

$y = -40(9)^2 + 720(9)$

$y = 3240$

Therefore, the vertex is at (9, 3240). The maximum income of this theatre is $3240 and occurs when the ticket price is $9.

Note: You will not be tested on how to use Microsoft Excel to create a graph or to find a curve of best fit using the graphing calculator for a graph in this course. However, it is useful to know how to do it, as using Microsoft Excel can be an easy way to check whether you are completing the questions correctly.

Learning Activity 3.7

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. You want to save up to buy a car worth $6000 in two years. How much will you have to save per month to afford this car?
2. On your first three assignments in Canadian history, you received marks of 82%, 63%, and 77%. What is your average assignment mark in Canadian history?
3. Factor: $2x(x - 4) + 3(x - 4)$
4. What is the domain of the parabola $y = 3x^2 - 5x - 12$?
5. Evaluate when $y = 5$: $\left(\frac{30}{5}\right) - \left(\frac{75}{y}\right)$
6. Estimate: $\sqrt{122}$
7. How many millimetres are in 6.3 metres?
8. What is the greatest common factor of $18x^3y^4z + 6x^2y^6z^3$?

Part B: Applications of Quadratic Functions

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

Each question can be solved by using either completing the square or by using the formula for the vertex. Practice both methods! The solution method given in the answer key is indicated in each question.

continued

Learning Activity 3.7 (continued)

1. The height, h, in metres, after the launching of a rocket at any time, t, in seconds, is defined by the following equation.

   $$h = -3t^2 + 9t + \frac{81}{4}$$

   Find the maximum height reached by the rocket and the time it takes to reach this height. Use the completing-the-square method.

2. At the local beach, the lifeguard has 620 m of marker buoys to rope off a safe swimming area. Calculate the dimensions of the rectangular swimming area to create the maximum swimming area if one side of the area is to be the beach. Use the formula method.

3. If 65 apple trees are planted in an orchard, the average yield per tree will be 1500 apples per year. For each additional tree planted in the orchard, the annual yield per tree drops by 20 apples. How many trees should be planted in order to produce a maximum yield? Use the formula method.

4. The difference between two numbers is 14. Find the two numbers so that their product is a minimum. Use the completing-the-square method.

5. Honest John's used car lot sells an average of 20 cars per week at an average price of $6,400 each. Honest John would like to increase the average price by $300; however, he knows that his sales would fall by one car if he does. If the dealer's cost per car is $4000, for what price should he sell the cars to maximize profits? Use the formula method.

6. A rectangular field is to be enclosed by a fence and divided into three smaller plots by two fences parallel to one of the sides. Find the dimensions of the largest such field if 800 m of fencing is available. Use the formula method.

7. Tom Katz's company sold tickets to a Bon Jovi concert. If 1000 or less people purchased tickets, the price was $60.00 per ticket. For every 100 tickets sold over 1000, the ticket was discounted by $3.00. What ticket price will give a maximum income? Use the completing-the-square method.

Lesson Summary

In this lesson, you applied your knowledge from the previous lessons to solve applications problems involving quadratic functions. For example, you learned how to find the maximum area, profit, and yield by finding the vertex of a quadratic function. This is the last lesson in the module. However, make sure you complete the last assignment in this module, Assignment 3.4, before you continue on to Module 4.

Notes

Assignment 3.4

Applications of Quadratic Functions

Total: 19 marks

1. A projectile is shot straight up from a height of 6 metres with an initial velocity of 80 metres per second. Its height in metres above the ground after t seconds is given by the equation $h = 6 + 80t - 5t^2$. After how many seconds does the projectile reach its maximum height, and what is this height? Use the formula method. *(4 marks)*

2. A rectangular parking lot is to be placed with one side along a straight street where no fence is required. If 1000 metres of wire are available for fencing the other three sides of the lot, what is the maximum area the lot may have, and what are the dimensions of the lot of a maximum area? Use the formula method. *(5 marks)*

continued

Assignment 3.4: Applications of Quadratic Functions (continued)

3. Find two positive real numbers whose sum is 13 and whose product is a maximum. Use the completing-the-square method. *(5 marks)*

continued

Assignment 3.4: Applications of Quadratic Functions (continued)

4. The price of a video game is $50, and 40 games are sold each day. For each $1.00 the price is raised, the store sells one fewer video game. If each video game costs $18 to make, how much should the price be set at to maximize profits? Use the completing-the-square method. *(5 marks)*

Notes

Module 3 Summary

In this module, you extended your knowledge of linear functions to learn about quadratic functions. You learned how linear functions produce a straight line when graphed, while quadratic functions produce a parabola, or a U-shape when graphed. You then learned how to find the important characteristics of quadratic functions in general form and vertex form. Completing the square, or transforming quadratic functions in general form to quadratic functions in vertex form, was another important skill you learned. When quadratic functions are in vertex form, they can be graphed using transformations. After learning all of this, you were able to solve application problems involving quadratic functions.

In the next module, you will build on your knowledge of rational expressions and quadratic functions to learn how to solve rational equations and quadratic equations.

Submitting Your Assignments

It is now time for you to submit Assignments 3.1 to 3.4 to the Distance Learning Unit so that you can receive some feedback on how you are doing in this course. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 3 assignments and organize your material in the following order:

- ❑ Module 3 Cover Sheet (found at the end of the course Introduction)
- ❑ Assignment 3.1: Quadratic Functions and Transformations
- ❑ Assignment 3.2: Graphing Quadratic Functions Using Transformations
- ❑ Assignment 3.3: Completing the Square and Finding the Vertex
- ❑ Assignment 3.4: Applications of Quadratic Functions

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Notes

Grade 11 Pre-Calculus Mathematics (30S)

Module 3
Quadratic Functions

Learning Activity Answer Keys

Module 3:
Quadratic Functions

Learning Activity 3.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Evaluate for $t = 5$: $3t^2 - 6t + 8$
2. The daily recommended intake of water is around 2 litres. If Ashton drinks 250 mL of water for breakfast, 550 mL of water for lunch, 620 mL of water during basketball practice, and 300 mL during dinner, how much water does he still need drink to reach the recommended amount?
3. Solve for x: $\frac{7}{8} + 4 = \frac{x}{8}$
4. Ms. Sparling's class has 24 students. If there are 3 girls for every boy, how many boys are in the class?
5. What is the greatest common factor of 3^4 and 3^5?
6. Express as a radical: $16^{\frac{4}{3}}$. Do not simplify.
7. Rationalize the denominator: $\frac{9\sqrt{2}}{4\sqrt{5}}$
8. Find the slope of the line perpendicular to the line $2x + y = 1$.

Answers:

1. 53 ($3(25) - 30 + 8 = 75 - 30 + 8 = 45 + 8 = 53$)
2. 280 mL (2000 mL − (250 mL + 550 mL + 620 mL + 300 mL) = 2000 mL − 1720 mL = 280 mL)
3. $39 \left(8\left(\frac{7}{8} + 4\right) = \left(\frac{x}{8}\right)(8); 7 + 32 = x; 39 = x \right)$
4. 6 (The ratio 3 girls : 1 boy implies you need 4 students in each group; $24 \div 4 = 6$, so there are 6 boys in the class.)
5. 3^4
6. $\sqrt[3]{16^4}$

7. $\frac{9\sqrt{10}}{20}$ $\left(\frac{9\sqrt{2}}{4\sqrt{5}} = \frac{9\sqrt{2}\sqrt{5}}{4(5)} = \frac{9\sqrt{10}}{20}\right)$

8. $\frac{1}{2}$ $(y = -2x + 1)$

Part B: Quadratic Functions

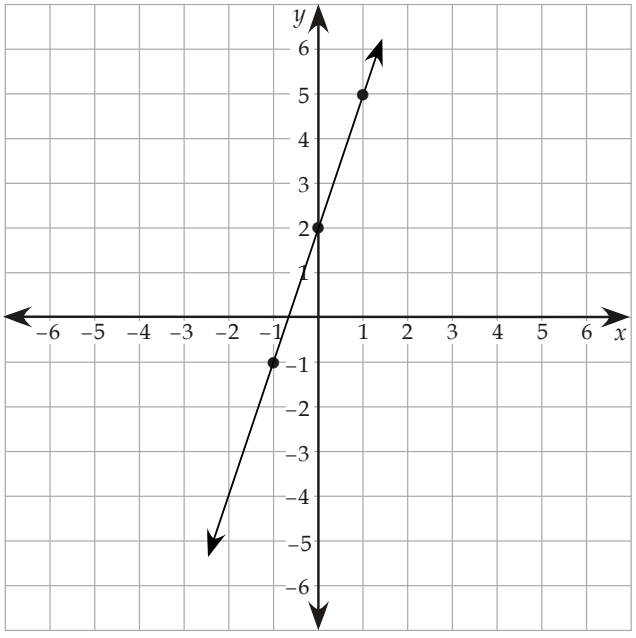
Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Draw the graph of the following functions:
   a) $y = 3x + 2$
   b) $2x - y + 1 = 0$
   c) Join these points with a smooth curve.

x	-3	-2	0	2	3
y	10	5	1	5	10

Answers:

a) $y = 3x + 2$ is the equation of a linear function with slope of $\frac{3}{1}$ and y-intercept of 2.

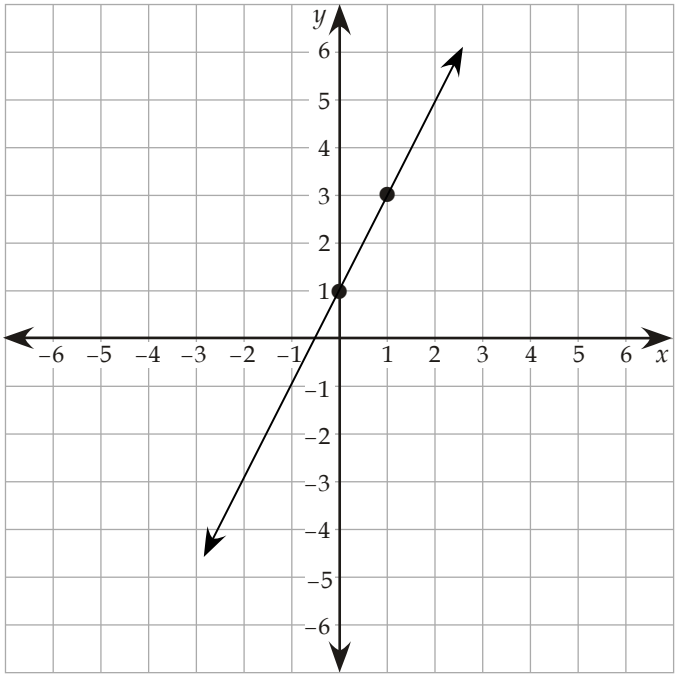


b) Change into the slope-intercept form:

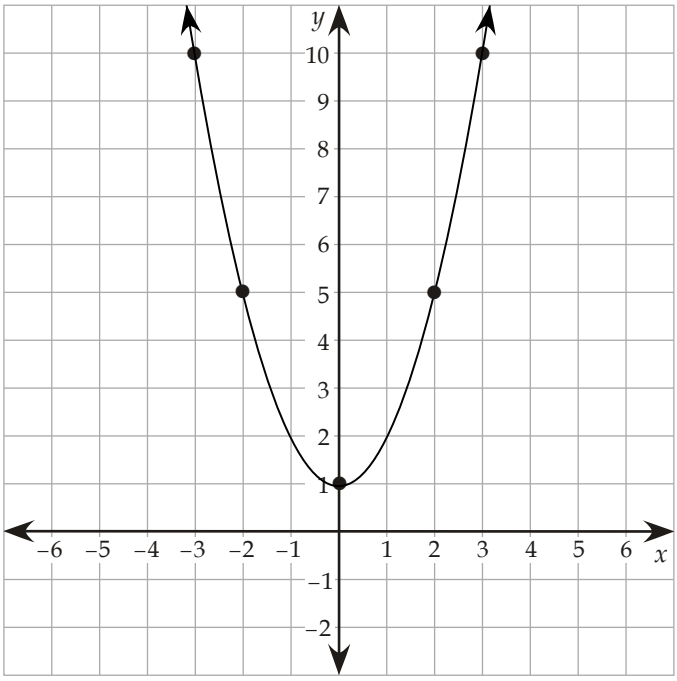
$y = mx + b$; $-y = -2x - 1$; $y = 2x + 1$

slope = $\frac{2}{1}$

y-intercept = 1



c)



2. Find the slope and intercepts for the linear functions in Question 1.

Answer:

a) $y = 3x + 2$

To find the x-intercept, let $y = 0$.

$0 = 3x + 2$

$-2 \quad -2$

$-2 = 3x$

$\frac{-2}{3} = \frac{3x}{3}$

$x = \frac{-2}{3}$

b) $2x - y + 1 = 0$

Let $y = 0$

$2x - 0 + 1 = 0$

$2x + 1 = 0$

$-1 \quad -1$

$2x = -1$

$\frac{2x}{2} = \frac{-1}{2}$

$x = \frac{-1}{2}$

c) This function does not intersect the x-axis. There are no x-intercepts.

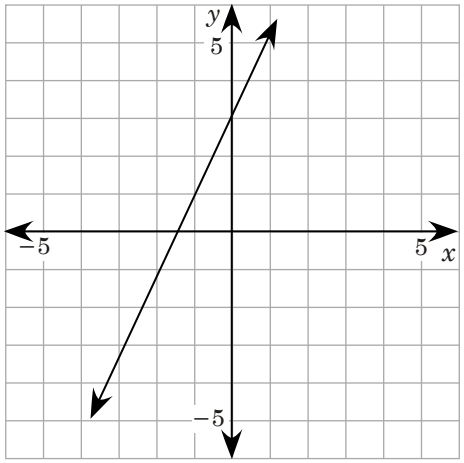
3. Which of the following describe a quadratic function?

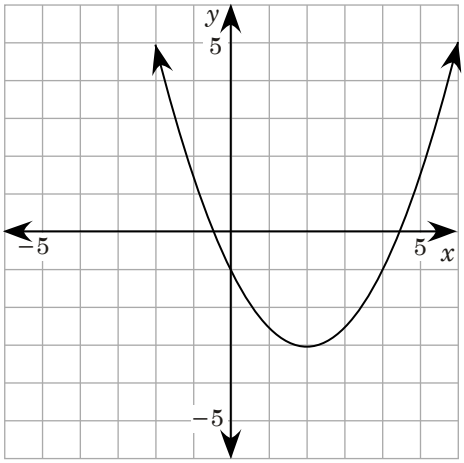
a) $f : x \rightarrow 3x^2$

b) $g : x \rightarrow 2x^3 - 5$

c) $y = 2x^2 - x + 1$

d) $f(x) = 3x - \frac{1}{x}$

e) 

f) 

Answers:

a) quadratic because it is in the form $f(x) = ax^2 + bx + c$

b) not quadratic since it has the variable x raised to the third power

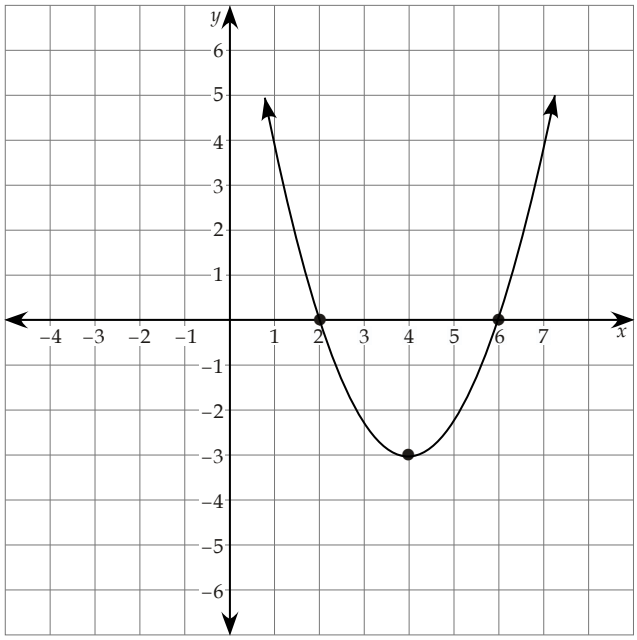
c) quadratic because it is of the form $f(x) = ax^2 + bx + c$

d) not quadratic

e) linear, not quadratic

f) quadratic since its graph is a parabola, which opens up

4. A graph of a quadratic function is shown below.



a) What is the domain of the graph?

b) What is the range of the graph?

c) What are the coordinates of the vertex?

d) What is the equation of the axis of symmetry?

e) What are the zeros of the function?

f) What are the x-intercepts?

g) Does the curve have a maximum? What is its value?

h) Does the curve have a minimum? What is its value?

Answers:

a) Domain $\{x \mid x \in \Re\}$

b) Range: $\{y \mid y \geq -3, y \in \Re\}$

c) Vertex: (4, –3). Remember this is the turning point.

d) Axis of symmetry equation: $x = 4$. This is the equation of the vertical line that passes through the vertex.

e) Zeros: 2, 6. Remember this is not an ordered pair. It represents the x-coordinates of the points where the curve intersects the x-axis.

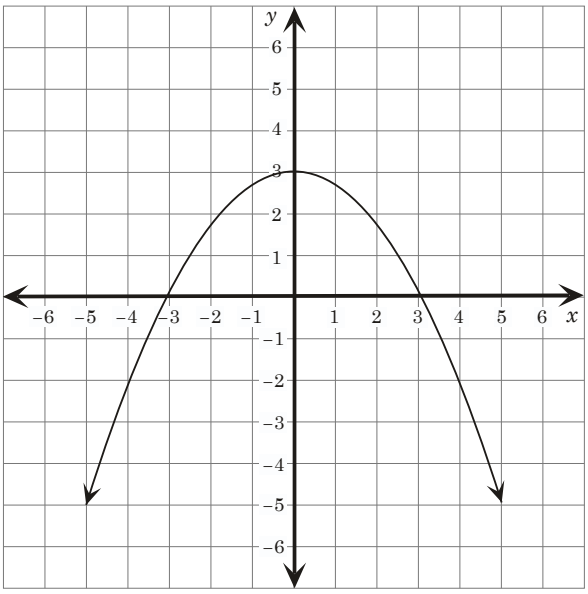
f) x-intercepts: 2, 6

g) Maximum value does not exist.

h) Minimum value is at -3, the y-coordinate of the vertex.

5. For each of the following, write the domain, range, coordinates of the vertex, equation of the axis of symmetry, y-intercept, and maximum or minimum value.

a)



Answer:

D: $\{x \mid x \in \Re\}$

R: $\{y \mid y \leq 3, y \in \Re\}$

Vertex coordinates: (0, 3)

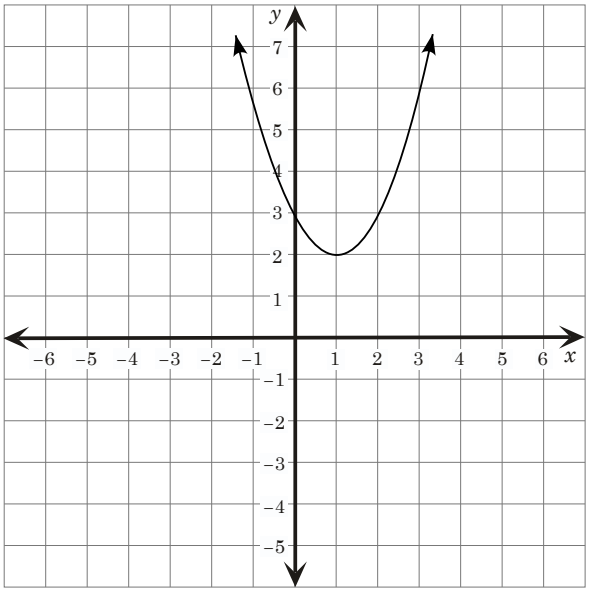
Axis of symmetry equation: $x = 0$

y-intercept: 3 is the y-coordinate of the point where the curve crosses the y-axis

Maximum value at 3

Minimum value does not exist

b)



Answer:

D: $\{x \mid x \in \Re\}$

R: $\{y \mid y \geq 2, y \in \Re\}$

Vertex: (1, 2)

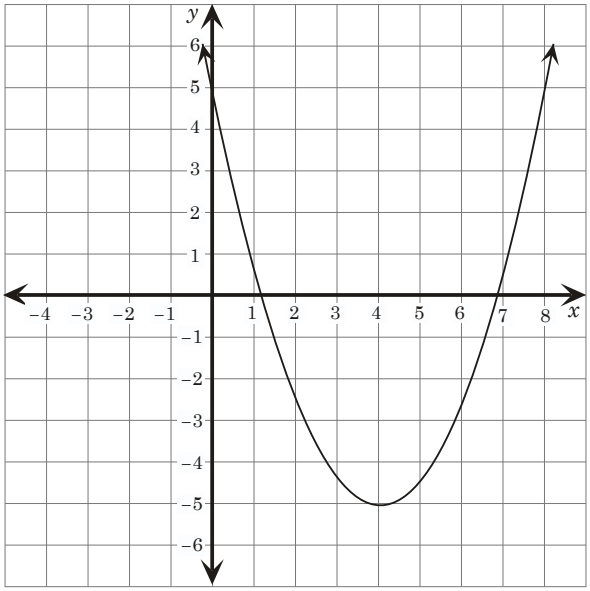
Axis of symmetry equation: $x = 1$

y-intercept: 3

Maximum value does not exist

Minimum value is 2

c)



Answer:

D: $\{x \mid x \in \Re\}$

R: $\{y \mid y \geq -5, y \in \Re\}$

Vertex: (4, −5)

Axis of symmetry equation: $x = 4$

y-intercept: 5

Maximum value: none

Minimum value: −5

6. Sketch the graphs of the following quadratic functions. Use a suitable table of values. Find the domain, range, coordinates of vertex, equation of axis of symmetry, zeros, x-intercept(s), y-intercept, and the maximum value or minimum value.

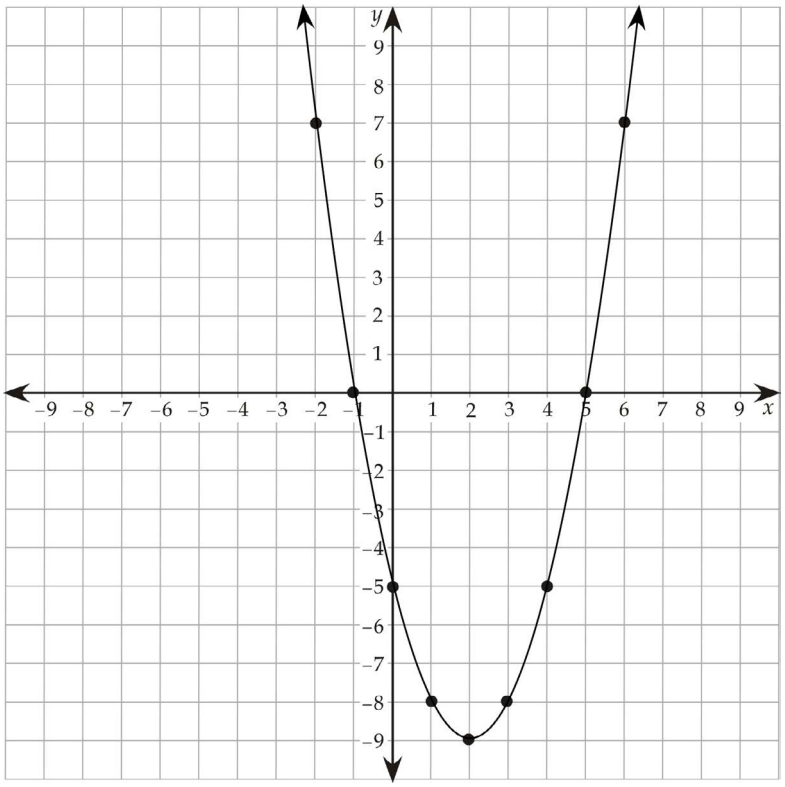
   a) $f(x) = x^2 - 4x - 5$

   b) $f : x \to -x^2 - 2x + 8$

   Answer:

   a)

x	-2	-1	0	1	2	3	4	5	6
y	7	0	-5	-8	-9	-8	-5	0	7



Domain: $\{x \mid x \in \Re\}$

Range: $\{y \mid y \geq -9, y \in \Re\}$

Vertex: $(2, -9)$

Axis of symmetry equation: $x = 2$

Zeros: -1 and 5

x-intercepts: -1 and 5

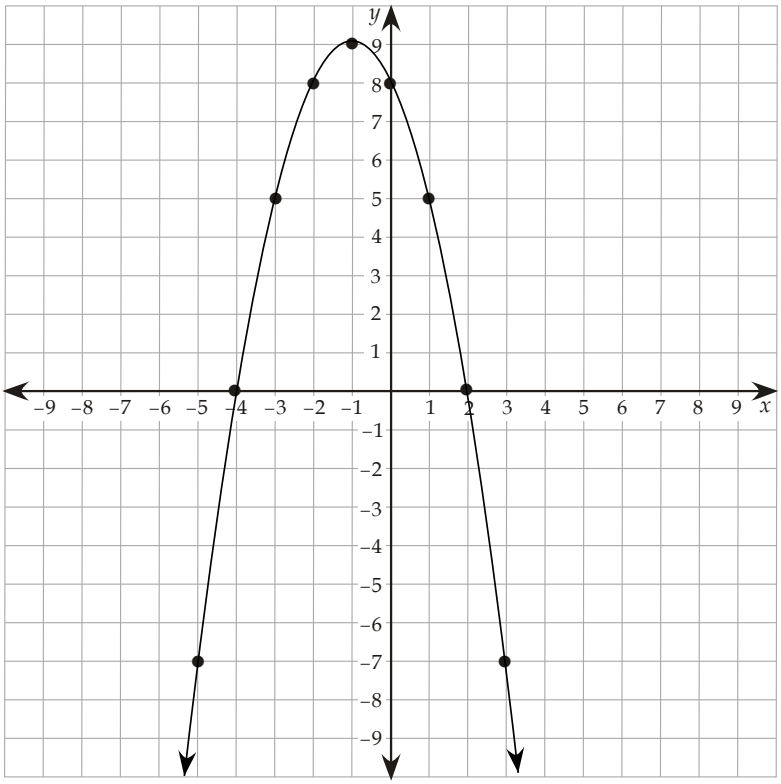
y-intercept: -5

Maximum value does not exist

Minimum value is at -9

b)

x	−5	−4	−3	−2	−1	0	1	2	3
y	−7	0	5	8	9	8	5	0	−7



Domain: $\{x \mid x \in \Re\}$

Range: $\{y \mid y \leq 9, y \in \Re\}$

Vertex: $(-1, 9)$

Axis of symmetry equation: $x = -1$

Zeros: -4 and 2

x-intercepts: -4 and 2

y-intercept: 8

Maximum value at 9

Minimum value does not exist

7. Match the equation with its graph. Use a table of values to graph each function if necessary.

   a) $y = x^2 - 2$

   b) $y = x^2 + 4x - 1$

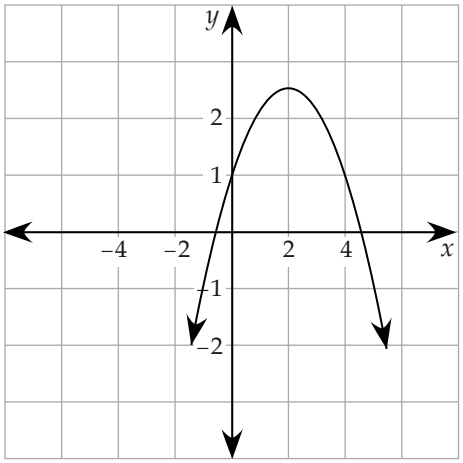
   c) $y = -x^2 - 4x - 3$

   d) $y = x^2 + 2$

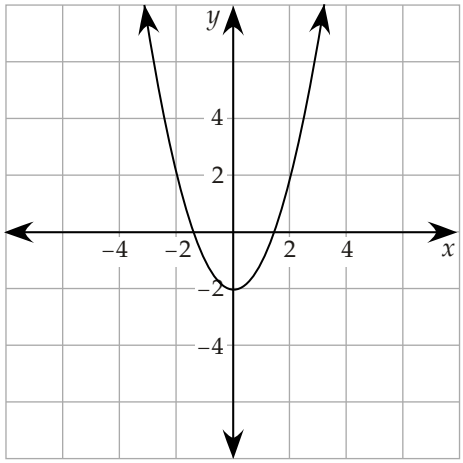
   e) $y = -x^2 + 4x + 1$

   f) $y = -x^2 + 2$

i)



ii)



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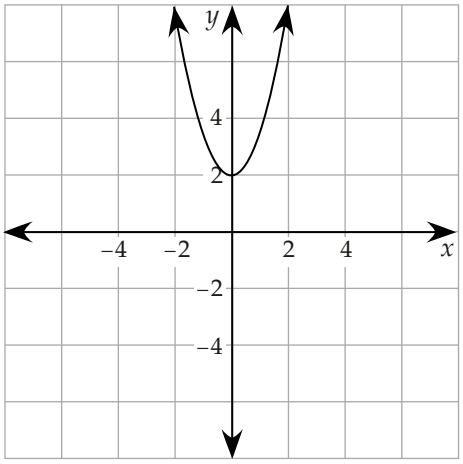

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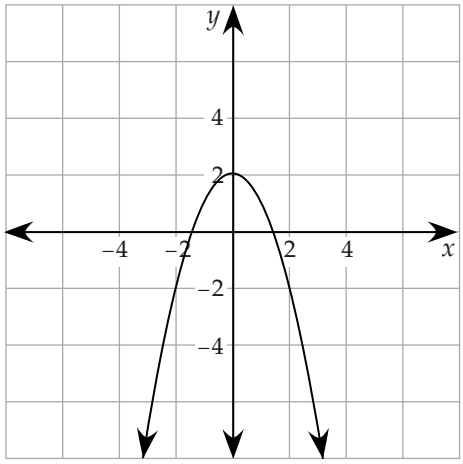
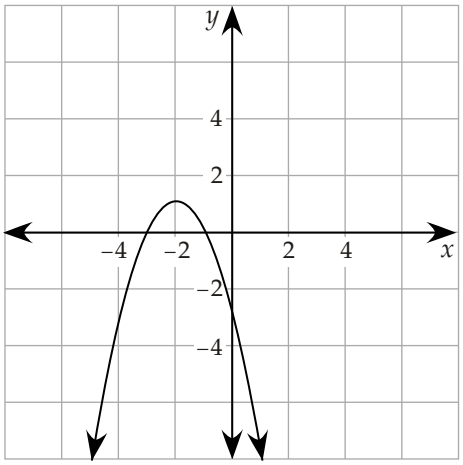
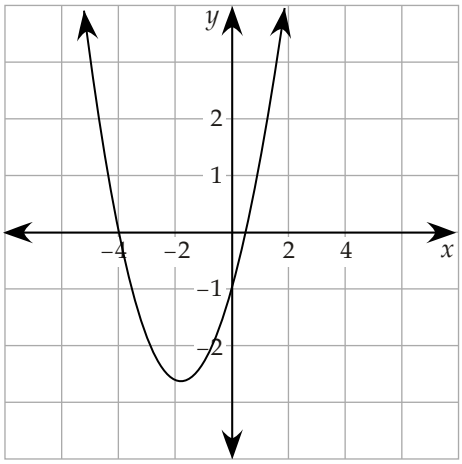


v)



vi)



Answers:

a) and ii)

b) and vi)

c) and v)

d) and iii)

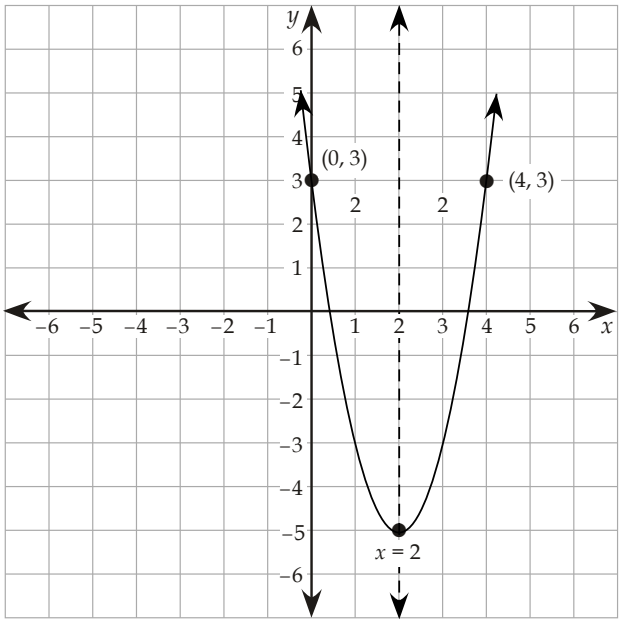
e) and i)

f) and iv)

8. a) The point (4, 3) is on the graph of $f(x) = 2x^2 - 8x + 3$. Use symmetry to find the coordinates of another point on the parabola if the axis of symmetry is $x = 2$.

   Answer:

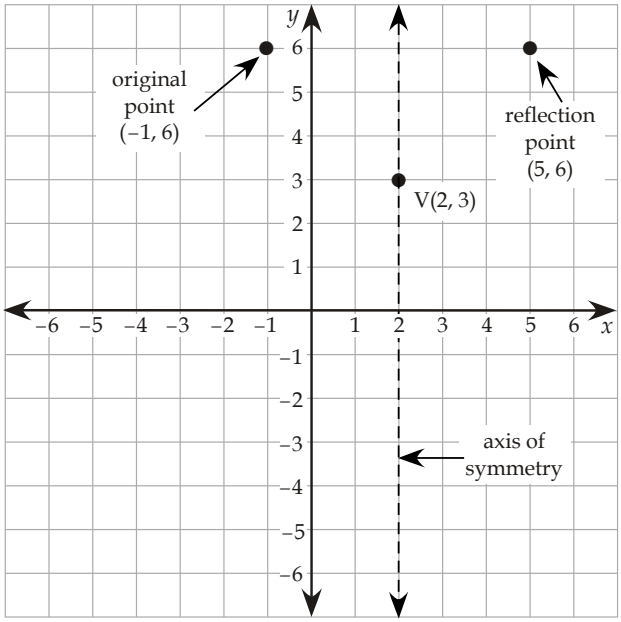
   Since the axis of symmetry for the graph of the function is the line $x = 2$, and since the point (4, 3) is 2 units to the right of it, then the reflection point must be 2 units to the left of it. Thus, (0, 3) is the required point.



b) A parabola has a vertex at (2, 3) and passes through the point P(−1, 6). Find the coordinates of a third point on the parabola.

Answer:

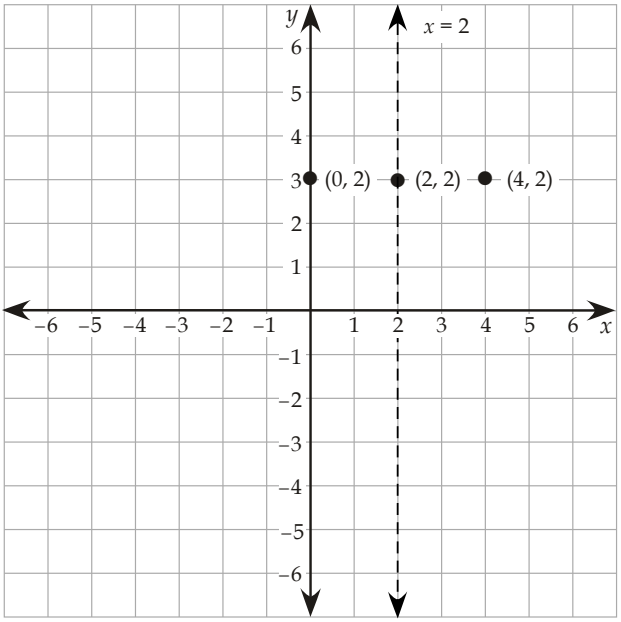
(−1, 6) is 3 units left of the axis of symmetry, so the reflection point must be 3 units to the right of the axis. Thus, (5, 6) is the required point.



c) Given certain reflection image points (0, 2) and (4, 2) on a parabola, find the equation of the line of symmetry.

Answer:

The axis of symmetry lies halfway between the reflection image points. Therefore, the point halfway between them is at (2, 2) and the equation of the axis of symmetry is $x = 2$.



Learning Activity 3.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Water is trickling out of a faucet at a speed of 3 mL per second. If Paula holds up a cup to collect the water, how much water will she collect in 3 minutes?
2. Factor: $x^2 - 9$
3. If $x = -1$ and $y = 2$, evaluate the expression: $3x^3 - xy$.
4. What is the lowest common multiple of 10 and 14?
5. At The Burger Palace, you have a choice between what type of bun and what type of burger you want. If there are three types of buns (white, whole wheat, and sesame) and three types of burgers (angus, chicken, and lean), how many different combinations of burgers are there?
6. Determine the distance between the following two points: (26, −13) and (25, −1).
7. Solve the inequality: $-6x - 7 > 3x - 9$.
8. Simplify. $(9x^2 - 7x + 5) - (2x^2 - 6x - 3)$

Answers:

1. 540 mL (There are $60 \times 3 = 180$ seconds in 3 minutes. 3 mL × 180 seconds = 540 mL.)
2. $(x - 3)(x + 3)$
3. $3(-1)^3 - (-1)(2) = -3 + 2 = -1$
4. 70
5. 9 (3 choices of bun × 3 choices of burgers = 9 choices)
6. $\sqrt{145}$ $\left(d = \sqrt{(25-26)^2 + (-1-(-13))^2} = \sqrt{(-1)^2 + (12)^2} = \sqrt{1+144} = \sqrt{145}\right)$
7. $-9x > -2$

   (Recall: When you divide by a negative, you need to flip the inequality sign.

   $x < \frac{-2}{-9}; x < \frac{2}{9}$)
8. $9x^2 - 7x + 5 - 2x^2 + 6x + 3 = 7x^2 - x + 8$

Part B: Quadratic Functions: $y = ax^2$ and $y = ax^2 + q$

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Determine whether each statement is true or false. If the statement is false, rewrite it so that it is true.

   a) $x^2 + y = 1$ is a quadratic function.

   b) $t(x) = 1 - x$ is a quadratic function.

   c) $f(x) = 4x^2$ is a quadratic function.

   d) The graph of $r(x) = x^2 + 1$ is symmetric with respect to the y-axis.

   e) The graph of $y = x^2$ is a "wider" parabola than the graph of $y = \frac{1}{4}x^2$.

   f) The graph of $y = 5x^2$ is a "narrower" parabola than the graph of $y = x^2$.

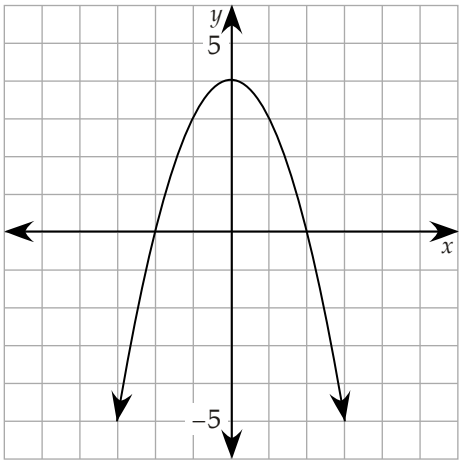
   g) The graph of $y = \frac{1}{4}x^2$ is a parabola that opens downward.

   h) The graph of $y = -2x^2$ is a parabola that opens downward.

   i) The maximum value of the function pictured at the right is 4.

   j) The minimum value of the function pictured at the right is 0.

   k) The domain of the function pictured at the right is $\{y \mid y \geq 4\}$.



   Answers:

   a) True, because it rearranges to $y = -x^2 + 1$, which is quadratic.

   b) False, because it has no squared term and so is linear. The equation $t(x) = 1 - x$ is a linear function whereas $t(x) = 1 - x^2$ is a quadratic function.

   c) True, because there is an x^2 term.

   d) True, since the axis of symmetry is the y-axis.

   e) False, since the graph of $y = x^2$ is narrower than $y = \frac{1}{4}x^2$. The graph of $y = \frac{1}{4}x^2$ is a wider parabola than the graph of $y = x^2$.

f) True, because the value of a is 5, which is greater than 1.

g) False, because when $a > 0$ the parabola opens upward. The graph $y = \frac{1}{4}x^2$ of is a parabola that opens upward or the graph of $y = -\frac{1}{4}x^2$ is a parabola that opens downward.

h) True, because when $a < 0$, the parabola opens downward.

i) True, because the function has a maximum value when the function opens downward.

j) False, because the minimum value is not defined when the parabola opens downward and continues indefinitely. The minimum value of the function pictured at the right cannot be stated.

k) False, because the range of the function is $\{y \mid y \leq 4\}$.

2. Complete the following table.

	$y = \frac{1}{3}x^2$	$y = x^2$	$y = -3x^2$
Vertex	(0, 0)	(0, 0)	(0, 0)
Equation of axis of symmetry	$x = 0$	$x = 0$	$x = 0$
Domain	$\{x \mid x \in \Re\}$	$\{x \mid x \in \Re\}$	$\{x \mid x \in \Re\}$
Range	$\{y \mid y \geq 0\}$	$\{y \mid y \geq 0\}$	$\{y \mid y \leq 0\}$
x-intercepts or zeros	0	0	0
Direction of opening	upward	upward	downward
Maximum y-value	none	none	0
Minimum y-value	0	0	cannot be determined
y-intercept	0	0	0

3. Describe the transformations needed to arrive at the graph of each of the following using $y = x^2$ as the reference curve.

   a) $y = 4x^2$

   b) $y = \frac{1}{2}x^2$

   c) $y = -7x^2$

   Answers:

   a) $y = x^2$ is stretched vertically by a factor of 4; each coordinate (x, y) on the original curve becomes $(x, 4y)$ on the new curve.

   b) $y = x^2$ is compressed vertically by multiplying each y-coordinate by $\frac{1}{2}$.

   c) $y = x^2$ is stretched vertically by a factor of 7 and reflected over the x-axis.

4. Find the following for the graph given by $y = ax^2$:

   a) What is the direction of the opening of the graph if $a > 0$?

   b) What is the direction of the opening of the graph if $a < 0$?

   c) If $a > 0$, does $y = ax^2$ have a minimum value or a maximum value?

   d) If $a < 0$, does $y = ax^2$ have a maximum value or a minimum value?

   Answers:

   a) upward

   b) downward

   c) minimum

   d) maximum

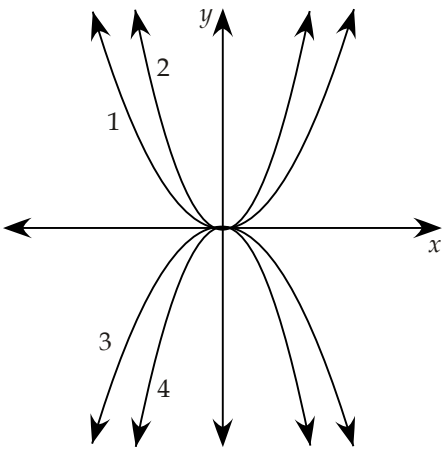
5. Match the letter corresponding to each function with the number corresponding to its graph.

a) $y = -2x^2$

b) $y = 1.6x^2$

c) $y = -2\pi x^2$

d) $y = \sqrt{6}x^2$



Answers:

a) and 3)

b) and 1)

c) and 4)

d) and 2)

6. Write an equation in the form $y = ax^2$ if $(1, -4)$ lies on the curve.

Answer:

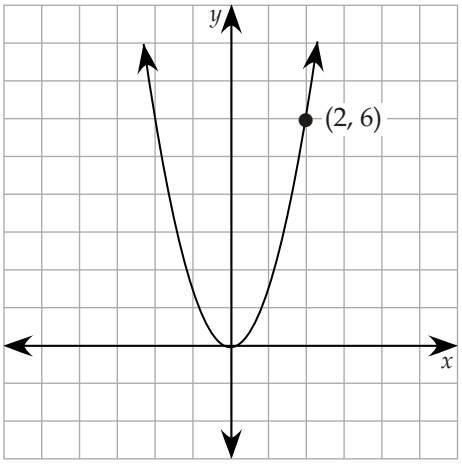
Substitute in $x = 1$, $y = -4$ into $y = ax^2$.

$-4 = a(1)^2$

$-4 = a$

Therefore, the equation is $y = -4x^2$.

7. Write the equation of the graph. Then write the equation of its reflection in the x-axis. Note each mark represents 1 unit on the graph.



Answer:

A point on the curve is (2, 6). The vertex is (0, 0). The equation is $y = ax^2$.

$\therefore$ Substitute $x = 2$, $y = 6$

$6 = a(2)^2$

$6 = 4a$

$\frac{6}{4} = a$

$\frac{3}{2} = a$

$y = \frac{3}{2}x^2$

The reflection in the x-axis of this parabola will have the same size and vertex but will open downward. Its equation is $y = -\frac{3}{2}x^2$.

8. Determine whether each statement is true or false. If the statement is false, rewrite it so it is true.
   a) The graph of $y = 3x^2 + 4$ is the graph of $y = 3x^2$ shifted downward 4 units.
   b) The graph of $f(x) = -x^2 - 3$ is the graph of $h(x) = -x^2$ shifted downward 3 units.
   c) The vertex of the graph of $y = 2x^2$ is the origin.
   d) The vertex of the graph of $y = \frac{1}{2}x^2 - 2$ is $(0, -2)$.
   e) An equation of the axis of symmetry of the graph of $y = x^2 - 1$ is $x = -1$.
   f) The graph of $y = x^2 - 1$ opens downward.
   g) The graph of $y = -x^2 - 2$ opens downward.
   h) The maximum value of $\{(x, y) | y = -2x^2 + 3\}$ is -2.
   i) The minimum value of is $\{(x, y) | y = 3x^2 - 4\}$ is -4.
   j) If $f(x) = 2x^2 + 3$, then $f(x) = f(-x)$, for all x.
   k) The graph of $y = x^2 + 4$ is symmetric with respect to the y-axis.
   l) The graphs of $y = x^2 + 2$ and $y = -x^2 + 2$ are mirror reflections of each other in the x-axis.
   m) The graph of $y = x^2$ is a narrower parabola than the graph of $y = \frac{1}{4}x^2$.
   n) The range of $f(x) = -x^2 + 2$ is $\{y | y \leq 2\}$.

Answers:

a) False. The graph of $y = 3x^2 + 4$ is the graph of $y = 3x^2$ shifted upward 4 units or the graph of $y = 3x^2 - 4$ is the graph of $y = 3x^2$ shifted downwards 4 units.

b) True

c) True

d) True

e) False. An equation of the axis of symmetry of $y = x^2 - 1$ is $x = 0$.

f) False. The graph of $y = x^2 - 1$ opens upward or the graph of $y = -x^2 - 1$ opens downward.

g) True

h) False. The maximum value of $y = -2x^2 + 3$ is 3.

i) True

j) True. $f(-x) = 2(-x)^2 + 3$

$= 2x^2 + 3$

Thus, $f(-x) = f(x)$.

k) True

l) False. The graph of $y = x^2 + 2$ and $y = -x^2 + 2$ are mirror reflections of each other in the line $y = 2$.

m) True

n) True

9. A quadratic function f is defined by $f: x \to ax^2 + q$, a $\neq 0$. Describe whether the graph of f opens up or down and give its maximum or minimum value.

a) $a > 0$ and $q > 0$

b) $a > 0$ and $q < 0$

c) $a < 0$ and $q > 0$

d) $a < 0$ and $q < 0$

Answers:

a) The graph opens upward and the minimum value, q, is positive.

b) The graph opens upward and the minimum value of q is negative.

c) The graph opens downward and the maximum value of q is positive.

d) The graph opens downward and the maximum value of q is negative.

10. Graph $y = -3x^2 + 1$ using transformations on $y = x^2$. Then make a table of values with 5 points in it.

Answer:

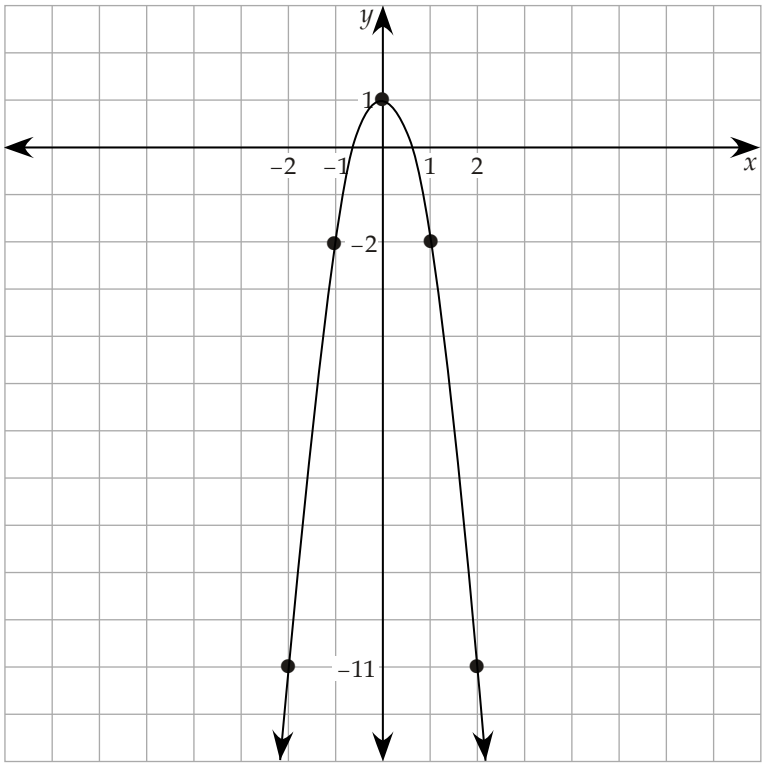
Recall the key coordinates of $y = x^2$: from the vertex at (0, 0), we move over 1 and up 1 to mark (1, 1). Then from the vertex, move over 2 and up 4 to mark (2, 4), then over 3 and up 9 from the vertex to mark (3, 9).

Now, let's transform the points (0, 0), (1, 1), (2, 4), and (3, 9) for the equation: $y = -3x^2 + 1$.

The vertex is (0, 1). The curve opens down. There is a vertical stretch of 3.

On your graph paper, starting at (0, 1), move right 1 and down 1(3) = 3 to mark the point (1, −2). Then, move right 2 from the vertex and down 4(3) = 12 to mark the point (2, −11). Then, right 3 from the vertex and down 9(3) = 27 to mark the point (3, −26).

Now reflect these points in the axis of symmetry to get (−1, −2) and (−2, −11). Now draw the graph and make a table of values from the graph.



x	2	1	0	−1	−2
y	−11	−2	1	−2	−11

Learning Activity 3.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Simplify: $(a + b)^2$
2. Evaluate: $\frac{1}{5} + \frac{2}{3}$
3. If Mark's meal comes to $16.27 with tax and he adds a tip of $1.50, how much change should he get back if he pays with a $20 bill?
4. Samuel and Cassy weigh 333 pounds together. If Samuel weighs twice as much as Cassy, how many pounds does Cassy weigh?
5. Factor: $x^2 - 3x - 28$
6. Simplify: $\left(\frac{5^5 \times 5^7}{5^2}\right)^3$
7. Rationalize the denominator: $\frac{2 - \sqrt{8}}{10\sqrt{7}}$
8. Find the slope of the line that contains the points $(2, -7)$ and $(9, -14)$.

Answers:

1. $(a + b)(a + b) = a^2 + 2ab + b^2$
2. $\frac{3}{15} + \frac{10}{15} = \frac{13}{15}$
3. $2.23 ($16.27 + $1.50 = $17.77; $20.00 − $17.77 = $2.23)
4. $111 \left(\frac{333}{3}\right)$
5. $(x - 7)(x + 4)$
6. $5^{30} \left(\left(\frac{5^5 \times 5^7}{5^2}\right)^3 = \left(\frac{5^{12}}{5^2}\right)^3 = \left(5^{10}\right)^3 = 5^{30}\right)$

7. $\frac{2\sqrt{7}-\sqrt{56}}{70}$ $\left(\frac{2-\sqrt{8}}{10\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{2\sqrt{7}-\sqrt{56}}{10 \cdot 7} = \frac{2\sqrt{7}-\sqrt{56}}{70}\right.$

$\left.\text{or } \frac{2\sqrt{7}-2\sqrt{14}}{70} \text{ or } \frac{\sqrt{7}-\sqrt{14}}{35}\right)$

8. -1 $\left(m = \frac{y_2 - y_1}{x_2 - x_1}; m = \frac{-14-(-7)}{9-2} = -\frac{7}{7} = -1\right)$

Part B: Quadratic Functions: $a(x - p)^2$

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. State whether each statement is true or false. If the statement is false, rewrite it so that it is true.
   a) The graph of $y = (x + 5)^2$ is the graph of $y = x^2$ shifted 5 units to the left.
   b) The graph of $y = (x - 7)^2$ is the graph of $y = x^2$ shifted 7 units to the left.
   c) The vertex of the graph of $y = (x - 1)^2$ is (1, 0).
   d) The vertex of the graph of $y = 2(x - 2)^2$ is (1, 0).
   e) The equation of the axis of symmetry of the graph of $y = \frac{1}{2}(x-3)^2$ is $x = \frac{1}{2}$.
   f) The equation of the axis of symmetry of the graph of $y = -\frac{1}{2}(x+3)^2$ is $y = -3$.
   g) The graph of $y = -\frac{1}{3}(x-2)^2$ is a narrower parabola than the graph of $y = -3(x - 2)^2$.
   h) The graph of $y = -\frac{1}{4}(x+4)^2$ opens downward.
   i) The quadratic function $y = \frac{1}{3}(x-2)^2$ has 0 for its minimum value.
   j) If $g(x) = (x + 4)^2$, then $g(x - 4) = g(-x - 4)$ for all x.

Answers:

a) True

b) False. The graph of $y = (x - 7)^2$ is the graph of $y = x^2$ shifted 7 units to the right or the graph of $y = (x + 7)^2$ is the graph of $y = x^2$ shifted 7 units to the left.

c) True

d) False. The vertex of the graph of $y = 2(x - 1)^2$ is (1, 0) or the vertex of the graph of $y = 2(x - 2)^2$ is (2, 0).

e) False. The equation of the axis of symmetry of the graph of $y = \frac{1}{2}(x - 3)^2$ is $x = 3$ or the equation of the axis of symmetry of the graph of $y = 3\left(x - \frac{1}{2}\right)^2$ is $x = \frac{1}{2}$.

f) False. The equation of the axis of symmetry of the graph of $y = -\frac{1}{2}(x + 3)^2$ is $x = -3$.

g) False. The graph of $y = -\frac{1}{3}(x - 2)^2$ is a wider parabola than the graph of $y = -3(x - 2)^2$ or the graph of $y = -3(x - 2)^2$ is a narrower parabola than the graph of $y = -\frac{1}{3}(x - 2)^2$.

h) True

i) True

j) True

$$g(x - 4) = \big((x - 4) + 4\big)^2$$
$$= (x)^2$$
$$= x^2$$
$$g(-x - 4) = \big((-x - 4) + 4\big)^2$$
$$= (-x)^2$$
$$= x^2$$

Thus, $g(x - 4) = g(-x - 4)$.

2. Match the equation with its graph.

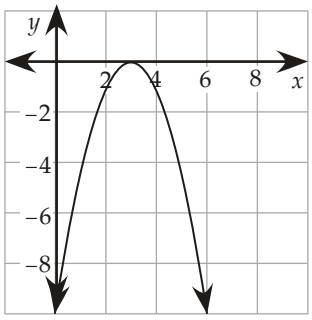
a) $y = (x - 4)^2$

b) $y = x^2 + 4$

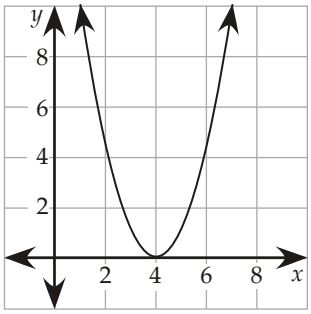
c) $y = -(x + 2)^2$

d) $y = -(x - 3)^2$

i)



ii)



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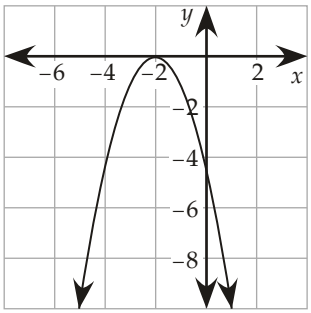
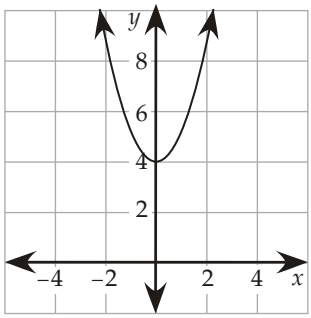


iv)



Answers:

a) ii

b) iv

c) iii

d) i

3. Without using tables of values, sketch the graphs of the following on graph paper. Provide a written explanation of the steps involved in the transformation. Then make a table of values with 5 points in it, showing how the coordinates of $y = x^2$ are transformed.

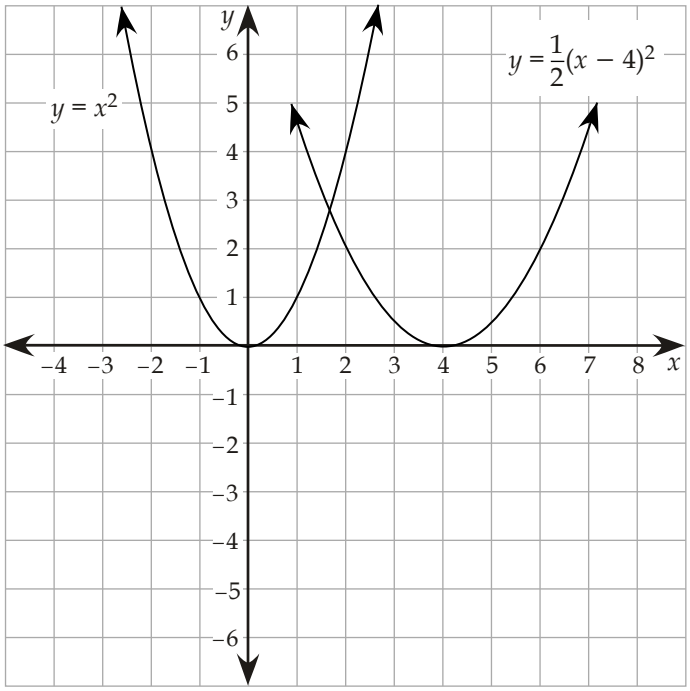
a) $f : x \to \frac{1}{2}(x - 4)^2$

b) $y = -2(x - 1)^2$

c) $f(x) = 2(x + 1)^2$

Answers:

a) $y = x^2$ compressed vertically by multiplying the y-values by $\frac{1}{2}$ and shifted horizontally 4 units to the right.

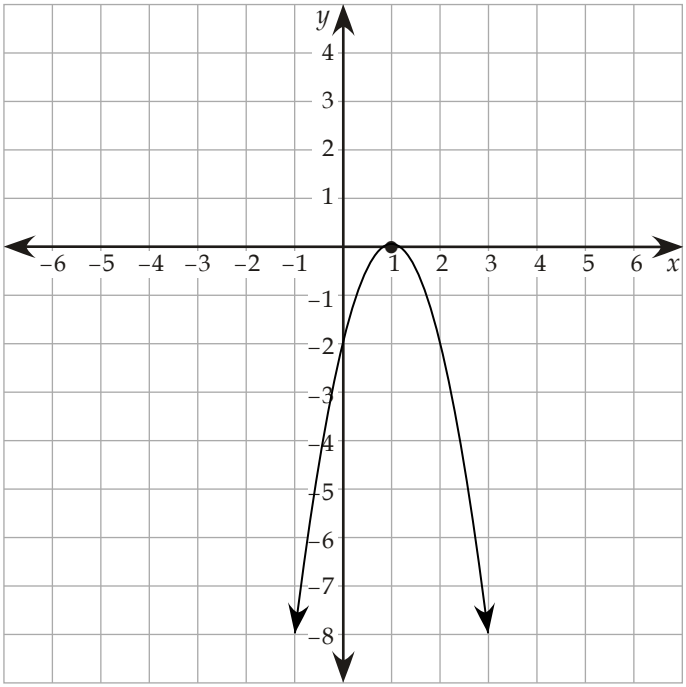


$f : x \rightarrow \frac{1}{2}(x-4)^2$

Transformation: $(x, y) \rightarrow \left(x + 4, \frac{1}{2}y\right)$

Coordinates for $y = x^2$	**Coordinates for $f(x)$**
(2, 4)	(6, 2)
(1, 1)	$\left(5, \frac{1}{2}\right)$
(0, 0)	(4, 0)
(−1, 1)	$\left(3, \frac{1}{2}\right)$
(−2, 4)	(2, 2)

b) $y = x^2$ stretched vertically by a factor of 2, reflected over the x-axis, and shifted horizontally 1 unit to the right.

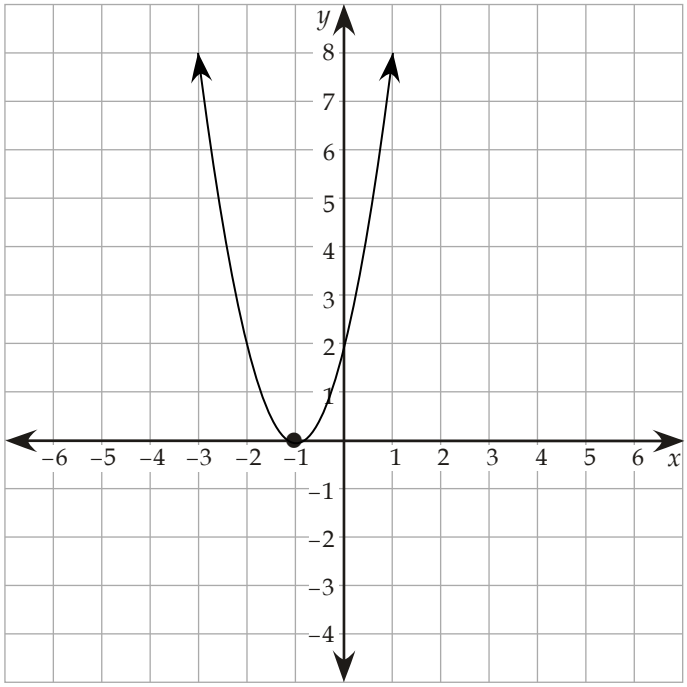


$y = -2(x - 1)^2$

Transformation: $(x, y) \rightarrow (x + 1, -2y)$

Coordinates for $y = x^2$	**Coordinates for $f(x)$**
(2, 4)	(3, −8)
(1, 1)	(2, −2)
(0, 0)	(1, 0)
(−1, 1)	(0, −2)
(−2, 4)	(−1, −8)

c) $y = x^2$ stretched vertically by a factor of 2 and shifted horizontally 1 unit to the left.



$f(x) = 2(x + 1)^2$

Transformation: $(x, y) \rightarrow (x - 1, 2y)$

Coordinates for $y = x^2$	Coordinates for $f(x)$
(2, 4)	(1, 8)
(1, 1)	(0, 2)
(0, 0)	(−1, 0)
(−1, 1)	(−2, 2)
(−2, 4)	(−3, 8)

4. Complete the following table.

Answer:

	(a)	(b)	(c)	(d)
	$y = (x + 2)^2$	$y = (x - 1)^2$	$y = 2(x - 3)^2$	$y = \frac{4}{3}(x + 6)^2$
Vertex	$(-2, 0)$	$(1, 0)$	$(3, 0)$	$(-6, 0)$
Equation of axis of symmetry	$x = -2$	$x = 1$	$x = 3$	$x = -6$
Domain	$\{x \mid x \in \Re\}$	$\{x \mid x \in \Re\}$	$\{x \mid x \in \Re\}$	$\{x \mid x \in \Re\}$
Range	$\{y \mid y \geq 0\}$	$\{y \mid y \geq 0\}$	$\{y \mid y \geq 0\}$	$\{y \mid y \geq 0\}$
x-intercepts	-2	1	3	-6
y-intercept*	4	1	18	48
Direction of opening	up	up	up	up
Maximum value	none	none	none	none
Minimum value	0	0	0	0

* **Note**:

To find the y-intercepts, let $x = 0$ in the equation and solve for y.

a) $y = (x + 2)^2 = (0 + 2)^2 = 4$

b) $y = (x - 1)^2 = (0 - 1)^2 = 1$

c) $y = 2(x - 3)^2 = 2(0 - 3)^2 = 2(9) = 18$

d) $y = \frac{4}{3}(x + 6)^2 = \frac{4}{3}(0 + 6)^2 = \frac{4}{3}(36) = 48$

Learning Activity 3.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. You need to make 100 cupcakes for a bake sale. However, your recipe only allows you to make cupcakes in multiples of 12. What is the least number of cupcakes you can make to have enough cupcakes for the bake sale?
2. What is the area of a park that measures 13 km by 62 km?
3. Evaluate: $\left[\left(\frac{3}{4}\right)-\left(\frac{1}{2}\right)\right]+\left[\left(\frac{1}{6}\right)+\left(\frac{5}{3}\right)\right]$
4. What is the slope of the line defined by the equation $y = 2x + 3$?
5. Does the parabola $y = -3x^2 + 4x + 2$ open up or down?
6. List the factors of 1000.
7. Write this percent as a decimal: 8.07%
8. Is 71 a prime or composite number?

Answers:

1. 9 dozen ($12 \times 8 = 96$; $12 \times 9 = 108$; If you make 8 dozen, you will only have 96 cupcakes. Therefore, you need to make 9 dozen to make sure you have enough cupcakes.)
2. 806 ($13 \times 62 = 10 \times 62 + 3 \times 62 = 620 + 186 = 806$)
3. $\frac{25}{12}\left(\left[\frac{9}{12}-\frac{6}{12}\right]+\left[\frac{2}{12}+\frac{20}{12}\right]=\frac{3}{12}+\frac{22}{12}=\frac{25}{12}\right)$
4. 2
5. Down (The a-value is negative.)
6. 1, 2, 4, 5, 8, 10, 20, 25, 40, 50, 100, 125, 200, 250, 500, and 1000.
7. 0.0807
8. Prime number

Part B: Transformations of Quadratic Functions

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. State whether each statement is true or false. If the statement is false, rewrite it so it is true.
   a) The vertex of the graph of $y = (x - 4)^2 + 2$ is $(-4, 2)$.
   b) The vertex of the graph of $y = 3(x - 5)^2 - 2$ is $(5, 2)$.
   c) The vertex of the graph of $y = 2x^2 + 7$ is $(2, 7)$.
   d) The vertex of the graph of $y = -3x^2 + 4$ is $(0, 4)$.
   e) The vertex of the graph of $y = -2(x - 3)^2$ is $(-2, 3)$.
   f) The equation of the axis of symmetry of the graph of $y = -3(x - 6)^2 + 1$ is $x = 1$.
   g) The equation of the axis of symmetry of the graph of $y = 3\left(x - \frac{1}{2}\right)^2 + 4$ is $x = \frac{1}{2}$.
   h) The equation of the axis of symmetry of the graph of $y = (x + 2)^2 + 1$ is $x = -2$.
   i) The graph of $y = 4(x - 3)^2 - 1$ opens downward.
   j) The graph of $y = 4(x + 2)^2 - 3$ opens upward.
   k) The range of $y = -2(x + 3)^2 + 4$ is $\{y \mid y \leq 4\}$.
   l) The maximum value of $y = 3(x - 5)^2 - 2$ is -2.

   Answers:

   a) False. The vertex of the graph of $y = (x - 4)^2 + 2$ is $(4, 2)$ or the vertex of the graph of $y = (x + 4)^2 + 2$ is $(-4, 2)$.
   b) False. The vertex of the graph of $y = 3(x - 5)^2 - 2$ is $(5, -2)$ or the vertex of the graph of $y = 3(x - 5)^2 + 2$ is $(5, 2)$.
   c) False. The vertex of the graph of $y = 2x^2 + 7$ is $(0, 7)$, or the vertex of the graph of $y = 2(x - 2)^2 + 7$ is $(2, 7)$.
   d) True.
   e) False. The vertex of the graph of $y = -2(x - 3)^2$ is $(3, 0)$, or the vertex of the graph of $y = -2(x + 2)^2 + 3$ is $(-2, 3)$.

f) False. The equation of the axis of symmetry of the graph of $y = -3(x - 6)^2 + 1$ is $x = 6$, or the equation of the axis of symmetry of the graph of $y = -3(x - 1)^2 + 1$ is $x = 1$.

g) True.

h) True.

i) False. The graph of $y = 4(x - 3)^2 - 1$ opens upward, or the graph of $y = -4(x - 3)^2 - 1$ opens downward.

j) True.

k) True.

l) False. The minimum value of $y = 3(x - 5)^2 - 2$ is -2.

2. Match each equation with its graph.

a) $y = 3x^2$

b) $y = \frac{-1}{3}x^2$

c) $y = 3(x + 2)^2$

d) $y + 1 = 3(x + 2)^2$

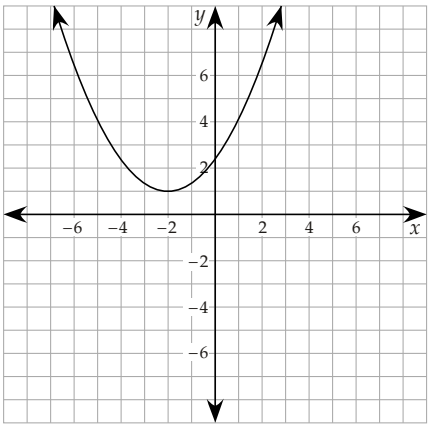
e) $y - 1 = \frac{1}{3}(x + 2)^2$

f) $y + 1 = -3(x - 2)^2$

g) $y = -\frac{1}{3}(x - 2)^2$

h) $y - 1 = -\frac{1}{3}(x - 2)^2$

i)



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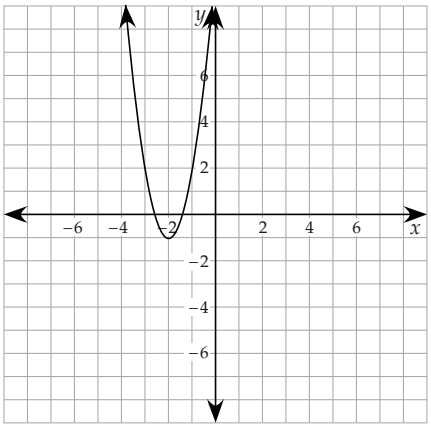

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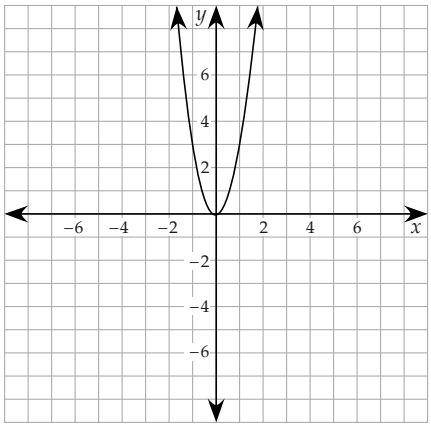

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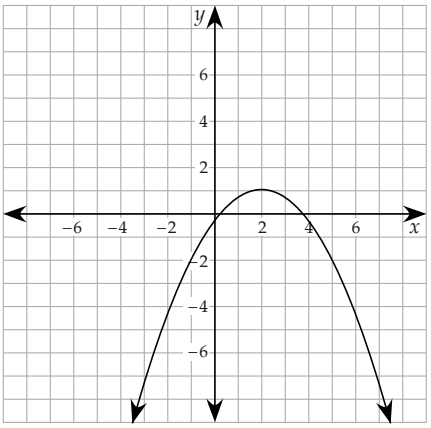
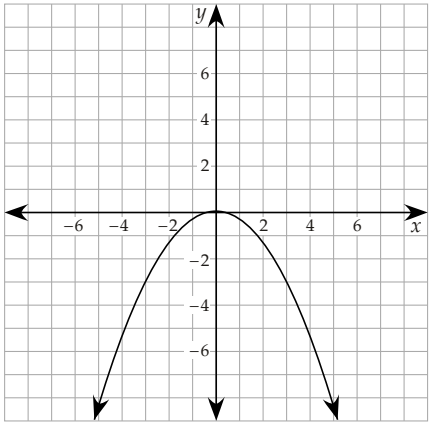
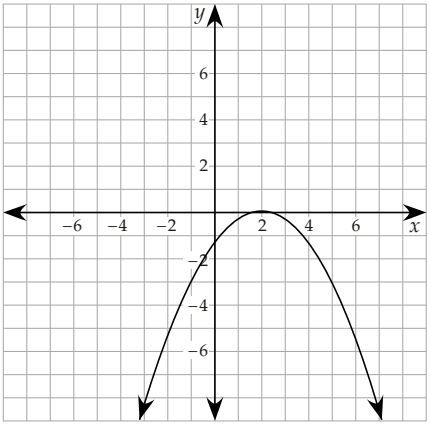


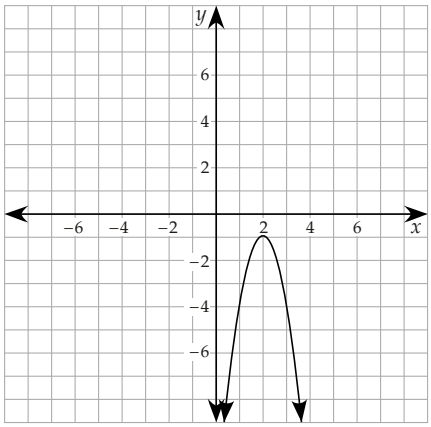
v)



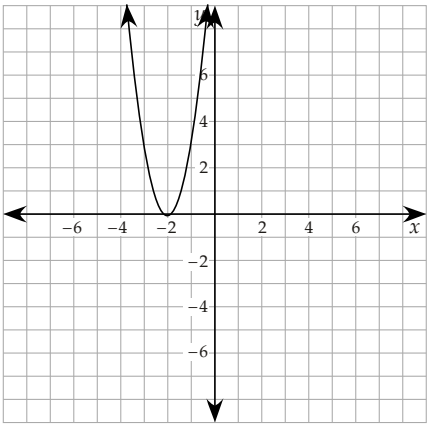
vi)



vii)



viii)



Answers:

a) and iii)
b) and v)
c) and viii)
d) and ii)
e) and i)
f) and vii)
g) and vi)
h) and iv)

3. Complete the following table.

	(a)	(b)	(c)	(d)
	$y = (x - 1)^2 + 2$	$y = -(x - 1)^2 - 2$	$y = (x + 1)^2 + 2$	$y = -(x + 1)^2 - 2$
Vertex	(1, 2)	(1, −2)	(−1, 2)	(−1, −2)
Equation of axis of symmetry	$x = 1$	$x = 1$	$x = -1$	$x = -1$
Domain	$\{x \mid x \in \Re\}$	$\{x \mid x \in \Re\}$	$\{x \mid x \in \Re\}$	$\{x \mid x \in \Re\}$
Range	$\{y \mid y \geq 2\}$	$\{y \mid y \leq -2\}$	$\{y \mid y \geq 2\}$	$\{y \mid y \leq -2\}$
y-intercept*	3	−3	3	−3
Direction of opening	up	down	up	down
Maximum value	none	−2	none	−2
Minimum value	2	none	2	none

* The y-intercepts can be found by letting $x = 0$ and solving for y.

a) $y = (x - 1)^2 + 2$
$y = (0 - 1)^2 + 2$
$y = 1 + 2$
$y = 3$

b) $y = -(x - 1)^2 - 2$
$y = -(0 - 1)^2 - 2$
$y = -1 - 2$
$y = -3$

c) $y = (x + 1)^2 + 2$
$y = (0 + 1)^2 + 2$
$y = 1 + 2$
$y = 3$

d) $y = -(x + 1)^2 - 2$
$y = -(0 + 1)^2 - 2$
$y = -1 - 2$
$y = -3$

4. For each of the following parabolas, state:
   - the direction of opening
   - the coordinates of the vertex
   - the equation of the axis of symmetry
   - whether the given graph is narrower or wider than $y = x^2$

   a) $y = 2(x + 1)^2$

   b) $y = -\frac{1}{2}(x - 1)^2 + 6$

   c) $y = 2(x + 6)^2 - 10$

   d) $y = 6(x - 1)^2 + 8$

   Answers:

   a) Direction of opening: up
   Vertex: $(-1, 0)$
   Axis of Symmetry: $x = -1$
   Narrower than $y = x^2$

   b) Direction of opening: down
   Vertex: $(1, 6)$
   Axis of Symmetry: $x = 1$
   Wider than $y = x^2$

   c) Direction of opening: up
   Vertex: $(-6, -10)$
   Axis of Symmetry: $x = -6$
   Narrower than $y = x^2$

   d) Direction of opening: up
   Vertex: $(1, 8)$
   Axis of Symmetry: $x = 1$
   Narrower than $y = x^2$

5. Without making a table of values, sketch the graph of each of the following functions. Explain the various steps. State the vertex and the equation of the axis of symmetry. Use graph paper.

   a) $y = 2(x + 1)^2$

   b) $y = -\frac{1}{2}(x - 1)^2 + 6$

   c) $y = 2(x + 6)^2 - 10$

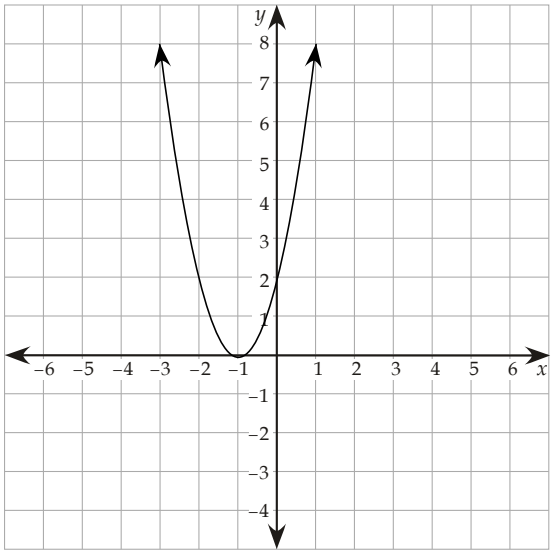
   d) $y = 6(x - 1)^2 + 8$

   Answers:

   a) $y = x^2$ is stretched vertically by a factor of 2 and shifted horizontally to the left by 1 unit.

   Vertex: (−1, 0)

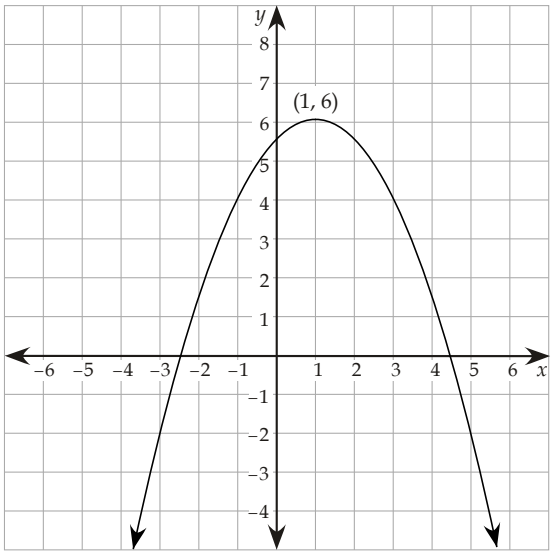
   Axis of symmetry: $x = -1$



b) $y = x^2$ is compressed vertically by a factor of 2, reflected over the x-axis, shifted horizontally to the right one unit and up 6 units.

Vertex: (1, 6)

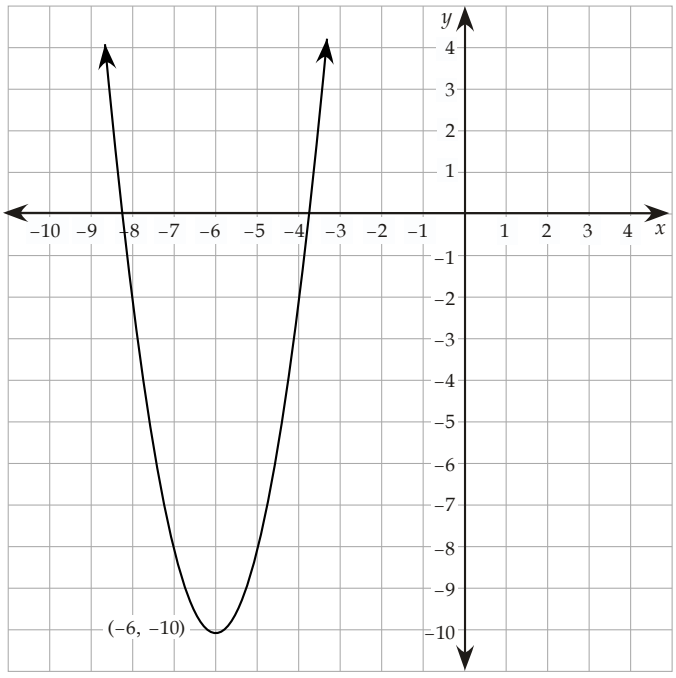
Axis of symmetry: $x = 1$



c) $y = x^2$ is stretched vertically by a factor of 2, shifted left 6 units and down 10 units.

Vertex: (−6, −10)

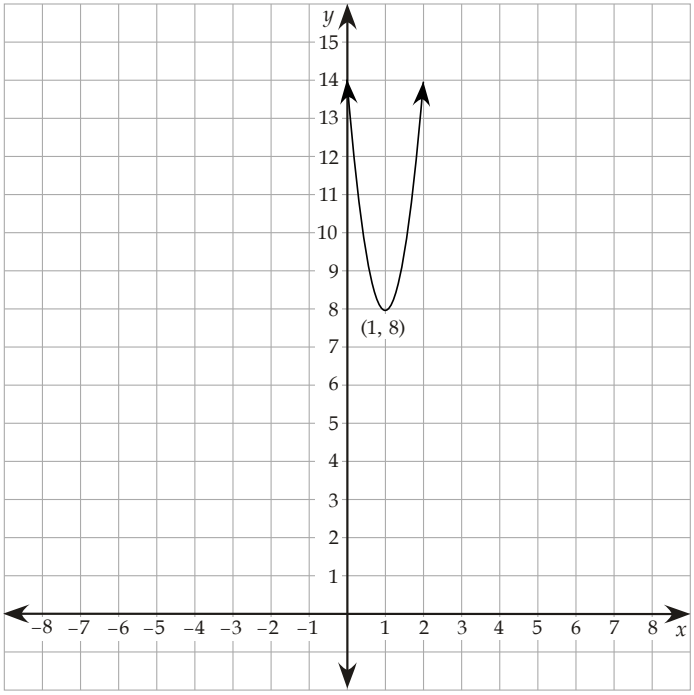
Axis of symmetry: $x = -6$



d) $y = x^2$ is stretched vertically by a factor of 6 shifted 1 unit right and 8 units up.

Vertex: (1, 8)

Axis of symmetry: $x = 1$



6. For each of the following quadratic functions,
   - specify whether the function has a minimum or a maximum value and what that value is.
   - specify the range.
   - find the y-intercept.

   a) $f(x) = -\left(x + \frac{1}{2}\right)^2 + 2$

   b) $y = 2(x + 4)^2 - 2$

   c) $g(x) = -3(x - 2)^2 - 1$

   d) $y = \frac{1}{2}\left[(x + 1)^2 + 4\right]$

Answers:

a) Maximum value at 2

Range: $\{y \mid y \leq 2\}$

$$f(x) = -\left(x + \frac{1}{2}\right)^2 + 2$$

let $x = 0$

$$f(0) = -\left(0 + \frac{1}{2}\right)^2 + 2$$

$$f(0) = -\left(\frac{1}{4}\right) + 2$$

$$f(0) = 1\frac{3}{4}$$

The y-intercept is $1\frac{3}{4}$.

b) Minimum value at -2

Range: $\{y \mid y \geq -2\}$

$y = 2(x + 4)^2 - 2$

let $x = 0$

$y = 2(0 + 4)^2 - 2$

$y = (2)(16) - 2$

$y = 30$

The y-intercept is 30.

c) Maximum value at -1

Range: $\{y \mid y \leq -1\}$

$g(x) = -3(x - 2)^2 - 1$

let $x = 0$

$g(0) = -3(0 - 2)^2 - 1$

$g(0) = -3(4) - 1$

$g(0) = -13$

The y-intercept is -13.

d) Remove the square brackets first by distributing the $\frac{1}{2}$ through the brackets to get $y = \frac{1}{2}(x+1)^2 + 2$.

Minimum value at 2

Range: $\{y \mid y \geq 2\}$

$y = \frac{1}{2}\left[(x+1)^2 + 4\right]$

let $x = 0$

$y = \frac{1}{2}\left[(0+1)^2 + 4\right]$

$y = \frac{1}{2}[1+4]$

$y = \frac{5}{2}$

The y-intercept is $\frac{5}{2}$.

7. If $\left(1, \frac{3}{2}\right)$ is a point on the quadratic function represented by $y = (x-2)^2 + q$, what real number is the minimum value of the function?

Answer:

$y = (x-2)^2 + q$

Substitute $x = 1$ and $y = \frac{3}{2}$ into the above equation.

$\frac{3}{2} = (1-2)^2 + q$

$\frac{3}{2} = 1 + q$

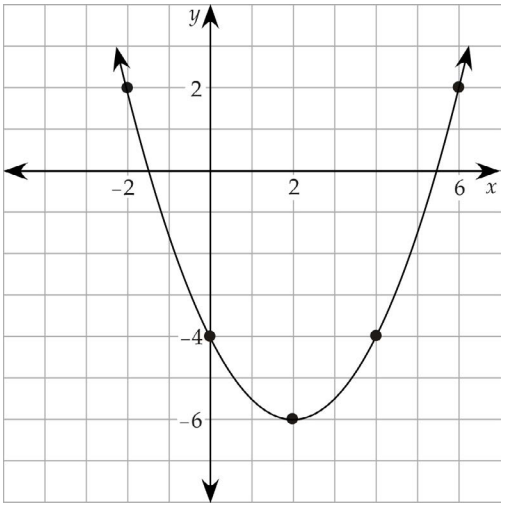
$\frac{3}{2} - \frac{2}{2} = q$

$\frac{1}{2} = q$

q represents the minimum value because the parabola opens up.

Therefore, the minimum value is $\frac{1}{2}$.

8. Write a quadratic function in the form $y = a(x - p)^2 + q$ that represents the following graph.



Answer:

Method #1: Algebraically

To use Method #1, you need to find the coordinates of the vertex and one other point. The coordinates of the vertex are (2, −6), and one other point is (4, −4).

Now, in vertex form, the coordinates of the vertex are (p, q).

$(p, q) = (2, -6)$

$\therefore \quad p = 2$

$q = -6$

Substituting these values for p and q into the vertex form of the quadratic function:

$y = a(x - 2)^2 - 6$

You also have another point on the curve, (4, −4). You can substitute these values into the quadratic function for x and y.

$-4 = a(4 - 2)^2 - 6$

$-4 = a(2)^2 - 6$

$-4 + 6 = 4a$

$2 = 4a$

$\frac{1}{2} = a$

Therefore, the quadratic function representing this graph is $y = \frac{1}{2}(x - 2)^2 - 6$.

Method #2: Graphically

To use Method #2, you need to compare this graph with the graph of $y = x^2$. The vertex is located at $(2, -6)$. Therefore, $p = 2$ and $q = -6$. In other words, this graph is shifted two units to the right and six units down.

The graph opens up, so the value of a is greater than 0.

The graph is wider than the graph of $y = x^2$. Therefore, $0 < a < 1$.

To determine the value of a, compare the vertical distances of points on this new parabola to the graph of $y = x^2$.

Compare the point $(2, 4)$ on $y = x^2$ to the point $(4, -4)$ on the translated parabola. Since the vertical distance from the vertex is half as much, the y-value is halved, and the value of $a = \frac{1}{2}$.

The quadratic function in vertex form is $y = \frac{1}{2}(x - 2)^2 - 6$.

9. A graph has the following properties:
   - The vertex is at $(-2, 5)$
   - The graph passes through the point $(2, -35)$

   Write a quadratic function in the form $y = a(x - p)^2 + q$ that represents this graph.

   Answer:

   The vertex is at $(-2, 5)$. Therefore, $p = -2$ and $q = 5$.

   $y = a(x + 2)^2 + 5$

   Another point is $(2, -35)$. Substitute these values into the equation $y = a(x + 2)^2 + 5$ to find the value of a.

   $$-35 = a(2 + 2)^2 + 5$$
   $$-35 - 5 = a(4)^2$$
   $$-40 = 16a$$
   $$\frac{-40}{16} = a$$
   $$\frac{-5}{2} = a$$

   Therefore, the quadratic function is $y = \frac{-5}{2}(x + 2)^2 + 5$.

10. Determine the number of intercepts of each quadratic function by considering the values of a and q.

    a) $y = -2(x - 3)^2 - 6$

    b) $y = 4x^2 - 1$

    c) $y = 6x^2$

    d) $y = -(x + 1)^2$

    Answers:

    a) In this quadratic function, $a = -2$ and $q = -6$. Therefore, both a and q are less than zero. This means that the corresponding graph of the quadratic function has no x-intercepts.

    b) In this quadratic function, $a = 4$ and $q = -1$. Therefore, a is greater than zero while q is less than zero. This means that the corresponding graph of the quadratic function has two x-intercepts.

    c) In this quadratic function, $a = 6$ and $q = 0$. Therefore, a is greater than zero while q equals zero. This means that the corresponding graph of the quadratic function has one x-intercept.

    d) In this quadratic function, $a = -1$ and $q = 0$. Therefore, a is less than zero while $q = 0$. There are no rules for this situation on the chart presented in this lesson. However, $q = 0$. Therefore, this graph sits on the x-axis. Also, this graph is a reflection through the x-axis of the graph $y = (x + 1)^2$. The graph of $y = (x + 1)^2$ has one x-intercept. Therefore, if you reflect this graph, which is located on the x-axis, over the x-axis, the resulting graph, $y = -(x + 1)^2$, will still have one x-intercept.

11. Redo question 10 above by considering the vertex (p, q) and the direction of opening using the value of a. Make a quick sketch.

    Answer:

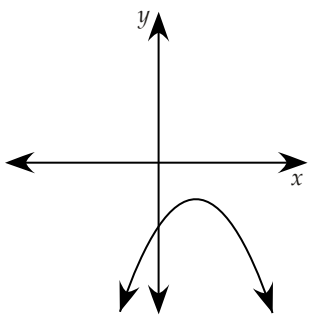
    a) $y = -2(x - 3)^2 - 6$

    Vertex $(3, -6)$ and a is negative

    Opens down

    Sketch:

    $\therefore$ no x-intercepts



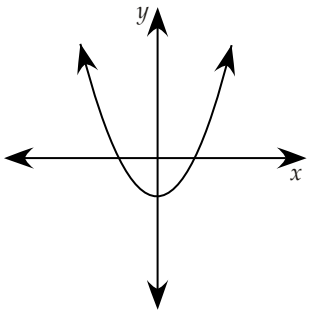
b) $y = 4x^2 - 1$

Vertex $(0, -1)$ and a is positive

Opens up

Sketch:

$\therefore$ two x-intercepts



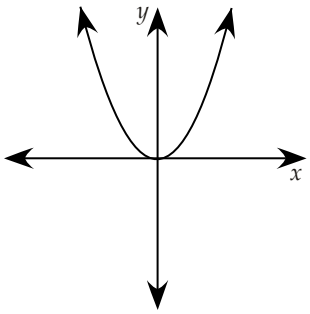
c) $y = 6x^2$

Vertex $(0, 0)$ and a is positive

Opens up

Sketch:

$\therefore$ one x-intercept



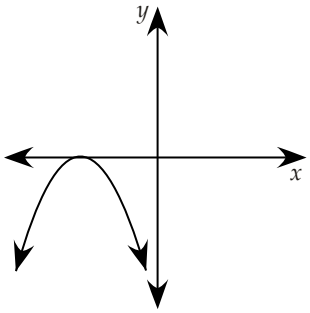
d) $y = -(x + 1)^2$

Vertex $(-1, 0)$ and a is negative

Opens down

Sketch:

$\therefore$ one x-intercept



Learning Activity 3.5

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Factor: $x^2 - 10x + 25$
2. Your computer has a battery life of 12 hours. If you use your computer for 15 minutes in the morning to check the weather, 6 hours during school to complete school work, 25 minutes during lunch to go on your favourite social networking site, and 1.6 hours to watch a movie, how much battery life do you have left on your laptop?
3. Evaluate for $g = -2$: $5g^3 + g^2 - 6g - 13$
4. Continue the pattern: 2, 6, 18, 54, _____ , _____
5. You are at the store to buy your sister a birthday present. You have $50 to spend and the item you want to purchase is $60 on sale for 25% off. Will you have enough money to buy the item (ignoring tax)?
6. Simplify: $(x^2 - 4x + 1)(2x^2)$
7. Find the midpoint between the following two points: (2, 6) and (−7, −4)
8. What is the slope of a horizontal line?

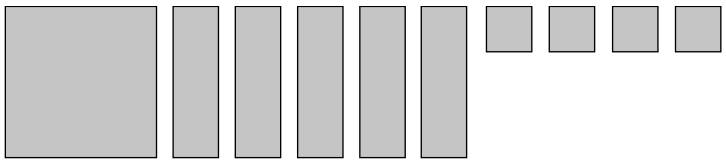
Answers:

1. $(x - 5)(x - 5)$ or $(x - 5)^2$
2. 3 hours and 44 minutes (You used your computer for 15 minutes + 6 hours + 25 minutes + 1 hour + 36 minutes = 7 hours and 76 minutes = 8 hours and 16 minutes. 12 hours − 8 hours and 16 minutes = 3 hours and 44 minutes.)
3. −37 ($5(-8) + 4 + 12 - 13 = -40 + 16 - 13 = -53 + 16 = -37$)
4. 162, 486
5. Yes (25% of $60 = $15; $60 − $15 = $45)
6. $2x^4 - 8x^3 + 2x^2$
7. $\left(-\frac{5}{2}, 1\right)$ $\left(\text{M} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{2 + (-7)}{2}, \frac{6 + (-4)}{2}\right) = \left(-\frac{5}{2}, \frac{2}{2}\right) = \left(-\frac{5}{2}, 1\right)\right)$
8. 0

Part B: Completing the Square

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Write a quadratic expression modelled by this set of tiles.



Answer:

$x^2 + 5x + 4$

2. Draw a tile model to represent $x^2 + 3x + 2$ in the shape of a rectangle.

Answer:

x^2	x	x
x	1	1

3. What value of k makes the expression a perfect square trinomial? Write each expression as the square of a binomial.

a) $x^2 + 8x + k$ b) $x^2 - 8x + k$

c) $x^2 + 20x + k$ d) $x^2 - 2x + k$

e) $x^2 - 5x + k$ f) $x^2 + 7x + k$

Answers:

a) $x^2 + 8x + k$

$$k = \left(\frac{8}{2}\right)^2 = 16$$

$$x^2 + 8x + 16 = (x + 4)^2$$

b) $x^2 - 8x + k$

$$k = \left(\frac{-8}{2}\right)^2 = 16$$

$$x^2 - 8x + 16 = (x - 4)^2$$

c) $x^2 + 20x + k$

$$k = \left(\frac{20}{2}\right)^2 = 100$$

$$x^2 + 20x + 100 = (x + 10)^2$$

d) $x^2 - 2x + k$

$$k = \left(\frac{-2}{2}\right)^2 = 1$$

$$x^2 - 2x + 1 = (x - 1)^2$$

e) $x^2 - 5x + k$

$$k = \left(\frac{-5}{2}\right)^2 = \frac{25}{4}$$

$$x^2 - 5x + \frac{25}{4} = \left(x - \frac{5}{2}\right)^2$$

f) $x^2 + 7x + k$

$$k = \left(\frac{7}{2}\right)^2 = \frac{49}{4}$$

$$x^2 + 7x + \frac{49}{4} = \left(x + \frac{7}{2}\right)^2$$

4. Find the values of a, p, and q for each of the following.

   a) $x^2 - 8x = a(x - p)^2 + p$

   b) $x^2 + 5x = a(x - p)^2 + p$

   c) $\frac{1}{2}x^2 + 8x + 4 = a(x - p)^2 + p$

   Answers:

   Each solution shows a different method for completing the square. Whichever one you choose to use is fine. All of them will be used throughout the rest of the module, but you only need to know and use one.

   a) $x^2 - 8x = a(x - p)^2 + q$

   let $y = x^2 - 8x$

   $y + 16 = x^2 - 8x + 16$

   $y + 16 = (x - 4)^2$

   $y = (x - 4)^2 - 16$

   Thus, $(x - 4)^2 - 16 = a(x - p)^2 + q$
   and $a = 1$, $p = 4$, $q = -16$

   b) $x^2 + 5x = a(x - p)^2 + q$

   let $y = x^2 + 5x$

   $y = \left(x^2 + 5x + \frac{25}{4} - \frac{25}{4}\right)$

   $y = \left(x + \frac{5}{2}\right)^2 - \frac{25}{4}$

   Thus, $\left(x + \frac{5}{2}\right)^2 - \frac{25}{4} = a(x - p)^2 + q$

   and $a = 1$, $p = -\frac{5}{2}$, $q = -\frac{25}{4}$

c) $\frac{1}{2}x^2 + 8x + 4 = a(x - p)^2 + q$

let $y = \frac{1}{2}x^2 + 8x + 4$

$y = \frac{1}{2}\left(x^2 + 16x \quad\right) + 4$

$y = \frac{1}{2}\left(x^2 + 16x + 64\right) + 4 - 64\left(\frac{1}{2}\right)$

$y = \frac{1}{2}(x + 8)^2 + 4 - 32$

$y = \frac{1}{2}(x + 8)^2 - 28$

Thus, $\frac{1}{2}(x + 8)^2 - 28 = a(x - p)^2 + q$

and $a = \frac{1}{2}$, $p = -8$, $q = -28$

5. For each of the following equations, answer the questions below.
   a) $f(x) = -x^2 + 6x$
   b) $f: x \to 2x^2 + 5x - 3$
   c) $y = -4x - x^2$
      i) Express in the form $f(x) = a(x - p)^2 + q$.
      ii) Find the coordinates of the vertex and equation of the axis of symmetry.
      iii) Find the range.
      iv) Determine the maximum or minimum value.

Answers:

a) $f(x) = -x^2 + 6x$

i) $f(x) = -x^2 + 6x$

$f(x) = -1(x^2 - 6x)$ Factor out -1 to make sure that the coefficient of x^2 is 1.

$f(x) = -1(x^2 - 6x + 9) - 9(-1)$ Complete the square.

$f(x) = -1(x - 3)^2 + 9$ Factor and simplify.

ii) V(3, 9); equation of axis of symmetry: $x = 3$

iii) Range: $\{y \mid y \leq 9\}$

iv) Maximum value of 9.

b) $f: x \rightarrow 2x^2 + 5x - 3$

i) For convenience, replace $f: x \rightarrow$ with y.

$y = 2x^2 + 5x - 3$

$y = 2\left(x^2 + \frac{5}{2}x\right) - 3$ Factor out 2 to make the coefficient of x^2 equal to 1.

$y = 2\left(x^2 + \frac{5}{2}x + \frac{25}{16}\right) - 3 - \frac{25}{16}(2)$ Complete the square.

$y = 2\left(x + \frac{5}{4}\right)^2 - 3 - \frac{25}{8}$ Simplify.

$y = 2\left(x + \frac{5}{4}\right)^2 - \frac{49}{8}$

$f: x \rightarrow 2\left(x + \frac{5}{4}\right)^2 - \frac{49}{8}$

ii) $V\left(-\frac{5}{4}, -\frac{49}{8}\right)$; equation of axis of symmetry: $x = -\frac{5}{4}$

iii) Range: $\left\{y \mid y \geq -\frac{49}{8}\right\}$

iv) Minimum value of $-\frac{49}{8}$

c) $y = -4x - x^2$

i) $y = -x^2 - 4x$ Rearrange.

$y = -\left(x^2 + 4x\right)$ Factor out (−1).

$y - 4 = -\left(x^2 + 4x + \left(\frac{4}{2}\right)^2\right)$ Complete the square and add −4 to each side.

$y = -\left(x + 2\right)^2 + 4$ Factor and add 4.

ii) V(−2, 4); equation of axis of symmetry: $x = -2$

iii) Range: $\{y \mid y \leq 4\}$

iv) Maximum value of 4

6. Given $y = ax^2 + bx + c$, the steps for expressing the function in the vertex form $y = a(x - p)^2 + q$ are written below. The coordinates of the vertex are shown. Supply the reasons for each step.

$y = ax^2 + bx + c$ Given

$y - c = ax^2 + bx$ Subtract c

$\frac{y - c}{a} = \frac{ax^2}{a} + \frac{b}{a}x$ Divide by a

$\frac{y - c}{a} = x^2 + \frac{b}{a}x$ Simplify

$\frac{y - c}{a} + \left(\frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2$ Complete the square

$\frac{y - c}{a} + \frac{b^2}{4a^2} = \left(x + \frac{b}{2a}\right)^2$ Factor

$\frac{y - c}{a} = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}$ Subtract $\frac{b^2}{4a^2}$

$y - c = a\left(x + \frac{b}{2a}\right)^2 - a \cdot \frac{b^2}{4a^2}$ Multiply by a

$y = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$ Add c and simplify

$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$ Simplify

$V\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$ Write the vertex

You can use $x = -\frac{b}{2a}$ for finding the x-coordinate at the vertex if the given equation is in the form $y = ax^2 + bx + c$. You can substitute that value into the quadratic equation to find the corresponding y-coordinate.

You can write this formula on your resource sheet.

7. Dominic was asked to complete the square. His work is shown below.

$$x^2 + 8x + 10 = y$$
$$x^2 + 8x = y - 10$$
$$x^2 + 8x + 64 = y - 10 + 64$$
$$(x + 4)^2 = y + 54$$
$$y = (x + 4)^2 - 54$$

Identify, explain, and correct the error in his work. Verify your answer is correct by completing the square using a different method.

Answer:

When completing the square, Dominic forgot to divide eight by two before squaring. Therefore, Dominic added 64 to both sides when he should have added 16 to both sides. The correct completion of the square is:

$$x^2 + 8x = y - 10$$
$$x^2 + 8x + 16 = y - 10 + 16$$
$$(x + 4)^2 = y + 6$$
$$y = (x + 4)^2 - 6$$

Verification:

$$x^2 + 8x + 10 = y$$
$$(x^2 + 8x \quad) + 10 = y$$
$$(x^2 + 8x + 16) + 10 - 16 = y$$
$$(x + 4)^2 - 6 = y$$

This is the same answer as the one obtained above.

8. Explain why $a(x - p)^2 + q = y$ is a quadratic function.

   Answer:

   A quadratic function is a function in the form $ax^2 + bx + c = y$.

   If you multiply out the function $a(x - p)^2 + q = y$, you get:

   $a(x^2 - 2px + p^2) + q = y$

   $ax^2 - 2apx + (ap^2 + q) = y$

   Therefore, when expanded, the quadratic function in vertex form is now in the form $ax^2 + bx + c = y$, where $b = -2ap$ and $c = ap^2 + q$.

Learning Activity 3.6

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. There are 15 females and 12 males in your family. On average, each female eats 2 pieces of pizza and each male eats 3 pieces of pizza. How many pizzas will you have to buy in order for everyone to eat as much pizza as they wish if each pizza is cut into 12 pieces?
2. Multiply: $(3x + 4)(2x - 7)$
3. Evaluate for $n = 4$: $4n^2 + n - 9$
4. Calculate 30% of 42 000.
5. What is the perimeter of a square with an area of 169 units?
6. Factor completely: $4x^4 - 20x^3 + 24x^2$
7. Find the slope of the line defined by the equation $7x + 3y - 4 = 0$.
8. Multiply. $(3x^2 + 7)(8x - 4)$

Answers:

1. 6 pizzas (15 × 2 = 30 pieces of pizza; 12 × 3 = 36 pieces of pizza; 30 + 36 = 66 pieces of pizza; 6 × 12 = 72, so you need 6 pizzas but will have some left)
2. $6x^2 - 13x - 28$
3. 59 (4(16) + 4 − 9 = 64 + 4 − 9 = 59)
4. 12 600 (10% of 42,000 = 4200; 30% of 42,000 = 4200 × 3 = 12 600)
5. 52 units (side length = $\sqrt{169} = 13$; perimeter = 13 × 4 = 52 units)
6. $4x^2(x - 2)(x - 3)$ [$4x^4 - 20x^3 + 24x^2 = 4x^2(x^2 - 5x + 6)$, which factors further to $4x^2(x - 2)(x - 3)$]
7. $-\frac{7}{3}$ $\left(3y = -7x + 4;\ y = -\frac{7}{3}x + \frac{4}{3}\right)$
8. $24x^3 - 12x^2 + 56x - 28$

Part B: Special Features of Quadratic Equations

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Given the following x-intercepts and a-value,

   a) $x = -1$, $x = 3$, $a = 1$

   b) $x = -4$, $x = -2$, $a = -1$

   find the following and then sketch the graph:

   i) vertex

   ii) equation of axis of symmetry

   iii) y-intercept

   iv) direction of opening

   v) maximum value

   vi) minimum value

   vii) domain and range (state in both notations)

   Answers:

   a) i) The x-intercepts are -1 and 3.

   The x-coordinate of the vertex is the average of the x-intercepts:

   $$\frac{-1+3}{2} = \frac{2}{2} = 1$$

   The equation for the function is $y = a(x - 1)^2 + q$.

   The value for a is given as 1. One point on the curve is at an x-intercept—say, $x = 3$. Using the point (3, 0), solve for q.

   $0 = 1(3 - 1)^2 + q$

   $0 = 4 + q$

   $-4 = q$

   The function is $y = (x - 1)^2 - 4$.

   The vertex is V(1, -4).

   ii) Axis of symmetry equation: $x = 1$

iii) y-intercept

Let $x = 0$ to find y-intercept for $y = (x - 1)^2 - 4$

$$y = (0 - 1)^2 - 4$$
$$y = 1 - 4$$
$$y = -3$$

iv) Opens upward because a is positive

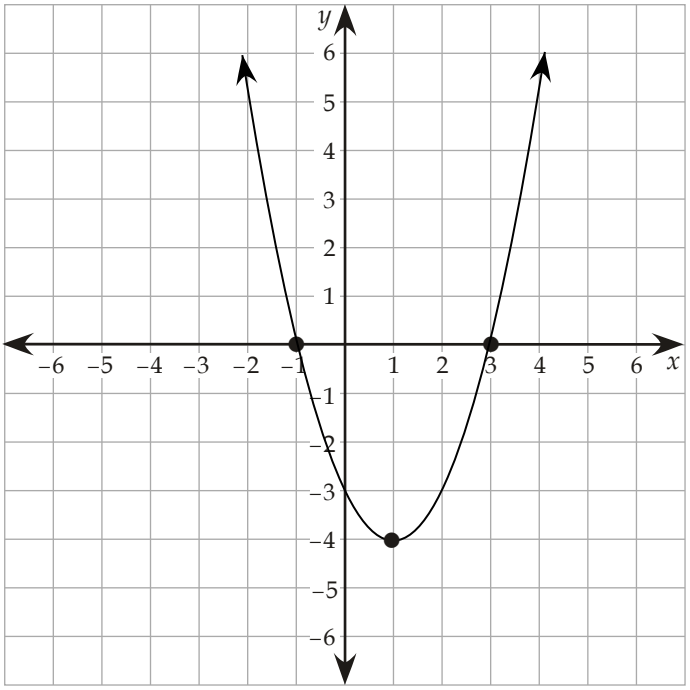
v) Maximum value is not defined

vi) Minimum value corresponding to the y-coordinate of the vertex $= -4$

vii)

Set Notation	Interval Notation
D: $\{x \mid x \in \Re\}$	$(-\infty, \infty)$
R: $\{y \mid y \geq -4, y \in \Re\}$	$[-4, \infty)$

Graph



b) i) The x-intercepts are -4 and -2.

x-coordinate at vertex is the average of the x-intercepts:

$$\frac{-4+(-2)}{2} = \frac{-6}{2} = -3$$

The equation for the function is $y = a(x + 3)^2 + q$.

Using the x-intercept to get a point on the curve, solve for q.

x-intercept = -4

point $(-4, 0)$ and $a = -1$

$y = a(x + 3)^2 + q$

$0 = -1(-4 + 3)^2 + q$

$0 = -1 + q$

$1 = q$

The function is $y = -(x + 3)^2 + 1$.

The vertex is V$(-3, 1)$.

ii) Axis of symmetry equation: $x = -3$

iii) y-intercept

Let $x = 0$ for $y = -(x + 3)^2 + 1$

$y = -(0 + 3)^2 + 1$

$y = -9 + 1$

$y = -8$

iv) Opens downward, since a is negative

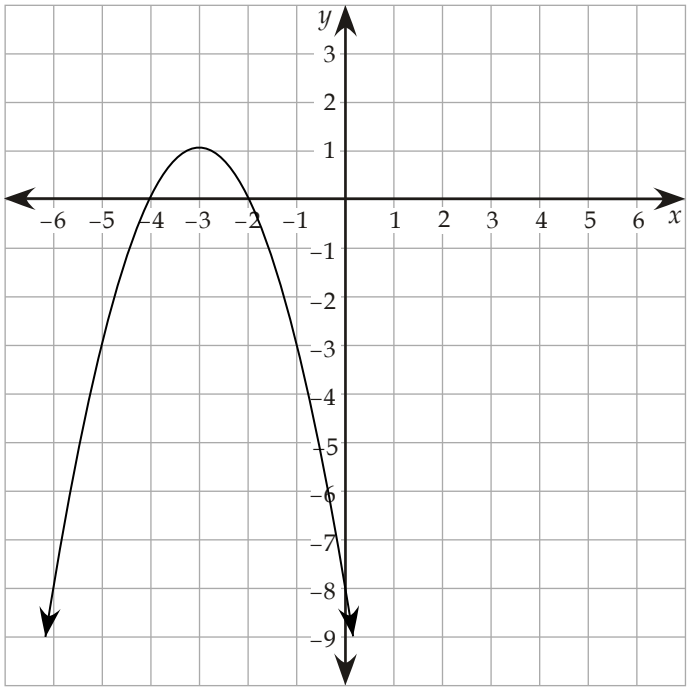
v) Maximum value is 1

vi) No minimum value

vii)

Set Notation	Interval Notation
D: $\{x \mid x \in \Re\}$	$(-\infty, \infty)$
R: $\{y \mid y \leq 1, y \in \Re\}$	$(-\infty, 1]$

Graph



2. For each of the following quadratic functions
   a) $y = x^2 + 6x - 7$
   b) $y = x^2 - 4x - 60$
   c) $y = 2x^2 + 8x - 10$
   d) $y = 3x^2 + 24x + 21$

   find the following:

   i) The coordinates of the vertex using $p = -\frac{b}{2a}$ and substituting to find q
   ii) The equation of the axis of symmetry
   iii) The y-intercept
   iv) The direction of opening
   v) The maximum or minimum value
   vi) The domain and range in interval notation

Answers:

a) $y = x^2 + 6x - 7$

i) x-coordinate of vertex = $\frac{-b}{2a}$

$a = 1 \qquad b = 6$

$p = \frac{-6}{2(1)} = -3$

y-coordinate of the vertex:

$q = (-3)^2 + 6(-3) - 7$

$q = 9 - 18 - 7$

$q = -16$

V(−3, −16)

ii) Axis of symmetry equation: $x = -3$

iii) y-intercept:

substitute $x = 0$ into:

$y = x^2 + 6x - 7$

$y = (0)^2 + 6(0) - 7$

$y = -7$

iv) Because $a > 0$, the parabola opens upward.

v) Since the parabola opens upward, the function has a minimum value of −16.

vi) D: $(-\infty, \infty)$

R: $[-16, \infty)$

b) $y = x^2 - 4x - 60$

i) x-coordinate of vertex = $\frac{-b}{2a}$

$a = 1 \qquad b = -4$

$p = \frac{-(-4)}{2(1)} = \frac{4}{2} = 2$

y-coordinate of the vertex:

$q = (2)^2 - 4(2) - 60$

$q = 4 - 8 - 60$

$q = -64$

V(2, −64)

ii) Axis of symmetry equation: $x = 2$

iii) y-intercept:

substitute $x = 0$ into:

$y = x^2 - 4x - 60$

$y = (0)^2 - 4(0) - 60$

$y = -60$

iv) Opens upward, since $a > 0$

v) Since the parabola opens upward, the function has a minimum value at −64.

vi) D: $(-\infty, \infty)$

R: $[-64, \infty)$

c) $y = 2x^2 + 8x - 10$

i) x-coordinate of vertex $= -\frac{b}{2a}$

$a = 2 \qquad b = 8$

$p = \frac{-8}{2(2)} = -2$

y-coordinate of the vertex:

$q = 2(-2)^2 + 8(-2) - 10$

$q = 2(4) - 16 - 10$

$q = -18$

V$(-2, -18)$

ii) Axis of symmetry equation: $x = -2$

iii) y-intercept:

substitute $x = 0$ into:

$y = 2x^2 + 8x - 10$

$y = 2(0)^2 + 8(0) - 10$

$y = -10$

iv) Opens upward because $a > 0$

v) Because the parabola opens upward, the minimum value is -18.

vi) D: $(-\infty, \infty)$

R: $[-18, \infty)$

d) $y = 3x^2 + 24x + 21$

i) x-coordinate of vertex = $\frac{-b}{2a}$

$a = 3 \qquad b = 24$

$p = \frac{-24}{2(3)} = -4$

y-coordinate of the vertex:

$q = 3(-4)^2 + 24(-4) + 21$

$q = 48 - 96 + 21$

$q = -27$

V$(-4, -27)$

ii) Axis of symmetry equation: $x = -4$

iii) y-intercept:

substitute $x = 0$ into

$y = 3x^2 + 24x + 21$

$y = 3(0)^2 + 24(0) + 21$

$y = 21$

iv) Opens upward because $a > 0$

v) Because the parabola opens upward, the minimum value is -27.

vi) D: $(-\infty, \infty)$

R: $[-27, \infty)$

Learning Activity 3.7

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. You want to save up to buy a car worth $6000 in two years. How much will you have to save per month to afford this car?
2. On your first three assignments in Canadian history, you received marks of 82%, 63%, and 77%. What is your average assignment mark in Canadian history?
3. Factor: $2x(x - 4) + 3(x - 4)$
4. What is the domain of the parabola $y = 3x^2 - 5x - 12$?
5. Evaluate when $y = 5$: $\left(\frac{30}{5}\right) - \left(\frac{75}{y}\right)$
6. Estimate: $\sqrt{122}$
7. How many millimetres are in 6.3 metres?
8. What is the greatest common factor of $18x^3y^4z + 6x^2y^6z^3$?

Answers:

1. $250 (Two years = 24 months; $\frac{6000}{24} = \frac{3000}{12} = \frac{1500}{6} = \$250$
2. 74% $\left(82 + 63 + 77 = 222; \frac{222}{3} = 74\right)$
3. $(2x + 3)(x - 4)$
4. Domain: $\{x \mid x \in \Re\}$
5. -9 $\left(\frac{30}{5} - \frac{75}{5} = 6 - 15 = -9\right)$
6. $\approx$11 (122 is between 121 and 144; therefore, $\sqrt{122}$ is between 11 and 12, but closer to 11)
7. 6300 millimetres (1000 millimetres in 1 metre; 6.3 × 1000 = 6300 millimetres)
8. $6x^2y^4z$.

Part B: Applications of Quadratic Functions

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

Each question can be solved by using either completing the square or by using the formula for the vertex. Practice both methods! The solution method given in the answer key is indicated in each question.

1. The height, h, in metres, after the launching of a rocket at any time, t, in seconds, is defined by the following equation.

   $$h = -3t^2 + 9t + \frac{81}{4}$$

   Find the maximum height reached by the rocket and the time it takes to reach this height. Use the completing-the-square method.

   Answer:

   $$h(t) = -3t^2 + 9t + \frac{81}{4}$$

   $$h(t) - \frac{81}{4} + (-3)\left(\frac{-3}{2}\right)^2 = -3\left(t^2 - 3t + \left(\frac{-3}{2}\right)^2\right)$$

   $$h(t) - \frac{81}{4} - \frac{27}{4} = -3\left(t - \frac{3}{2}\right)^2$$

   $$h(t) = -3\left(t - \frac{3}{2}\right)^2 + \frac{108}{4}$$

   $$h(t) = -3\left(t - \frac{3}{2}\right)^2 + 27$$

   The curve opens downward.

   The maximum height of 27 m is reached in 1.5 seconds.

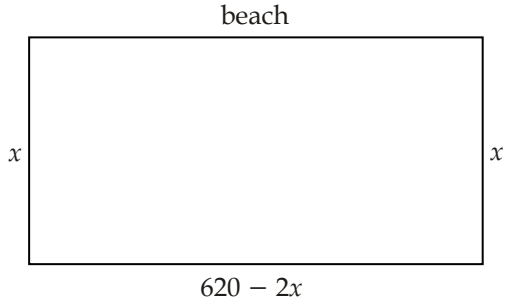
2. At the local beach, the lifeguard has 620 m of marker buoys to rope off a safe swimming area. Calculate the dimensions of the rectangular swimming area to create the maximum swimming area if one side of the area is to be the beach. Use the formula method.

   Answer:

   Let x = width of the swimming area in metres

   $620 - 2x$ = length of the swimming area in metres

   Let A(x) represent the area of the rectangle.



   Remember, you only need three sides roped off.

   $\therefore$ Area: $A(x) = x(620 - 2x)$

   $= 620x - 2x^2$

   The maximum area occurs at the vertex of the quadratic function because the parabola opens down. You need only find the x-coordinate of the vertex.

$$x = \frac{-b}{2a} = -\frac{620}{2(-2)} = 155 \text{ m}$$

   When the width is 155 m, the length is $620 - 2(155) = 310$ m.

   The dimensions of the swimming area are 155 m × 310 m.

3. If 65 apple trees are planted in an orchard, the average yield per tree will be 1500 apples per year. For each additional tree planted in the orchard, the annual yield per tree drops by 20 apples. How many trees should be planted in order to produce a maximum yield? Use the formula method.

Answer:

	Number of Trees	**Yield per Tree**	**Annual Yield**
originally	65	1500	65(1500)
one additional tree	65 + 1	1500 − 20(1)	(65 + 1)[1500 − 20(1)]
two additional trees	65 + 2	1500 − 20(2)	(65 + 2)[1500 − 20(2)]
n additional trees	65 + n	1500 − 20(n)	(65 + n)[1500 − 20(n)]

Maximum yield is required.

The equation will have a vertex at (n, Y) where:

n = number of additional trees

Y = annual yield

$Y = (65 + n)(1500 - 20n)$

$Y = 97\,500 - 1300n + 1500n - 20n^2$

$Y = -20n^2 + 200n + 97\,500$

The curve opens downward. Thus, we have a maximum at the vertex.

Using the formula method to find the vertex:

$$x = \frac{-b}{2a}$$

$$x = \frac{-200}{2(-20)}$$

$$x = 5$$

Thus, 5 additional trees should be planted. There will be 70 trees altogether in the orchard.

4. The difference between two numbers is 14. Find the two numbers so that their product is a minimum. Use the completing-the-square method.

   Answer:

   Let x = larger of the two numbers

   $x - 14$ = smaller of the two numbers

   Let P(x) = the product of the two numbers

   $$P(x) = x(x - 14)$$

   $$= x^2 - 14x$$

   $$P(x) + 49 = x^2 - 14x + 49$$

   $$P(x) = (x - 7)^2 - 49$$

   Since a = 1, the parabola opens up and a minimum occurs at x = 7. If x = 7, then $x - 14 = -7$.

   The two numbers are 7 and -7, giving a minimum product of -49.

5. Honest John's used car lot sells an average of 20 cars per week at an average price of $6,400 each. Honest John would like to increase the average price by $300; however, he knows that his sales would fall by one car if he does. If the dealer's cost per car is $4000, for what price should he sell the cars to maximize profits? Use the formula method.

   Answer:

Number of Cars Sold	Net Profit per Car	Total Net Profit
20	2400	20(2400)
20 − 1	2400 + 3000(1)	(20 − 1)[2400 + 300(1)]
20 − 2	2400 + 3000(2)	(20 − 2)[2400 + 300(2)]
20 − n	2400 + 3000(n)	(20 − n)[2400 + 300(n)]

   Let P = total net profit

   Let n = number of increases in average price per car (or number of decreases in cars sold)

   $P = (20 - n)(2400 + 300\text{n})$

   $P = 48\,000 + 6000n - 2400n - 300n^2$

   $P = -300n^2 + 3600n + 48\,000$

   The curve opens downward. The maximum occurs at the vertex.

At the vertex, $n = \dfrac{-b}{2a} = \dfrac{-3600}{2(-300)} = 6.$

Honest John should sell his cars at a price, including cost, of:

Price = 2400 + 300(6) + 4000

= 2400 + 1800 + 4000

= $8200

6. A rectangular field is to be enclosed by a fence and divided into three smaller plots by two fences parallel to one of the sides. Find the dimensions of the largest such field if 800 m of fencing is available. Use the formula method.

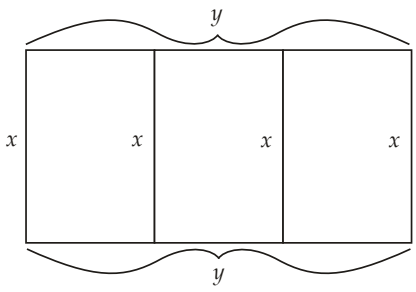
Answer:

x = width

y = length

Total lengths = $2y$

Total widths = $4x$



$2y + 4x = 800$

$y = 400 - 2x$ Divide by two and solve for y.

Area = A

A = length × width

$A = (400 - 2x)x$

$A = -2x^2 + 400x$

Using $x = \dfrac{-b}{2a} = \dfrac{-400}{2(-2)} = 100$ m.

$y = 400 - 2x = 400 - 2(100) = 200$ m

The curve opens down; there is a maximum at the vertex.

The largest area occurs when the dimensions of the field are 100 m × 200 m.

7. Tom Katz's company sold tickets to a Bon Jovi concert. If 1000 or less people purchased tickets, the price was \$60.00 per ticket. For every 100 tickets sold over 1000, the ticket was discounted by \$3.00. What ticket price will give a maximum income? Use the completing-the-square method.

Answer:

Number of Tickets Purchased	Price per Ticket	Income
1000	60	1000(60)
1000 + 100(1)	60 − 3(1)	[1000 + 100(1)][60 − 3(1)]
1000 + 100(2)	60 − 3(2)	[1000 + 100(2)][60 − 3(2)]
1000 + 100(n)	60 − 3(n)	[1000 + 100(n)][60 − 3(n)]

Let n = number of groups of 100 tickets sold (or number of decreases in price of ticket)

Let I = income

$I = (1000 + 100n)(60 - 3n)$

$I = 60\,000 - 3000n + 6000n - 300n^2$

$I = -300n^2 + 3000n + 60\,000$

$I = -300(n^2 - 10n + 25) + 60\,000 - 25(-300)$

$I = -300(n - 5)^2 + 60\,000 + 7500$

$I = -300(n - 5)^2 + 67\,500$

V(5, 67 500) = V(n, I)

The curve opens downward; the maximum occurs at the vertex.

The ticket price for maximum income must be:

$60 - 3(5) = 60 - 15 = \$45.$

Notes

Grade 11 Pre-Calculus Mathematics (30S)

Module 3
Quadratic Functions

Learning Activity Answer Keys

Module 3:
Quadratic Functions

Learning Activity 3.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Evaluate for $t = 5$: $3t^2 - 6t + 8$
2. The daily recommended intake of water is around 2 litres. If Ashton drinks 250 mL of water for breakfast, 550 mL of water for lunch, 620 mL of water during basketball practice, and 300 mL during dinner, how much water does he still need drink to reach the recommended amount?
3. Solve for x: $\frac{7}{8} + 4 = \frac{x}{8}$
4. Ms. Sparling's class has 24 students. If there are 3 girls for every boy, how many boys are in the class?
5. What is the greatest common factor of 3^4 and 3^5?
6. Express as a radical: $16^{\frac{4}{3}}$. Do not simplify.
7. Rationalize the denominator: $\frac{9\sqrt{2}}{4\sqrt{5}}$
8. Find the slope of the line perpendicular to the line $2x + y = 1$.

Answers:

1. 53 ($3(25) - 30 + 8 = 75 - 30 + 8 = 45 + 8 = 53$)
2. 280 mL (2000 mL − (250 mL + 550 mL + 620 mL + 300 mL) = 2000 mL − 1720 mL = 280 mL)
3. $39 \left(8\left(\frac{7}{8} + 4\right) = \left(\frac{x}{8}\right)(8); 7 + 32 = x; 39 = x \right)$
4. 6 (The ratio 3 girls : 1 boy implies you need 4 students in each group; $24 \div 4 = 6$, so there are 6 boys in the class.)
5. 3^4
6. $\sqrt[3]{16^4}$

7. $\frac{9\sqrt{10}}{20}$ $\left(\frac{9\sqrt{2}}{4\sqrt{5}} = \frac{9\sqrt{2}\sqrt{5}}{4(5)} = \frac{9\sqrt{10}}{20}\right)$

8. $\frac{1}{2}$ $(y = -2x + 1)$

Part B: Quadratic Functions

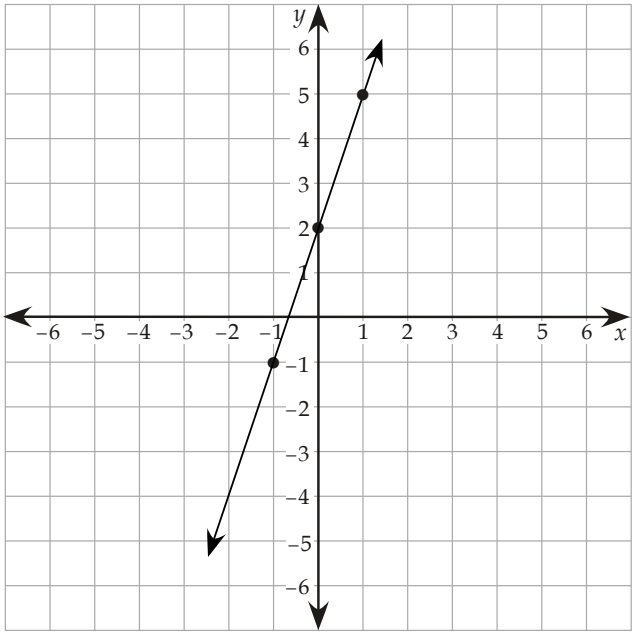
Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Draw the graph of the following functions:
   a) $y = 3x + 2$
   b) $2x - y + 1 = 0$
   c) Join these points with a smooth curve.

x	-3	-2	0	2	3
y	10	5	1	5	10

Answers:

a) $y = 3x + 2$ is the equation of a linear function with slope of $\frac{3}{1}$ and y-intercept of 2.

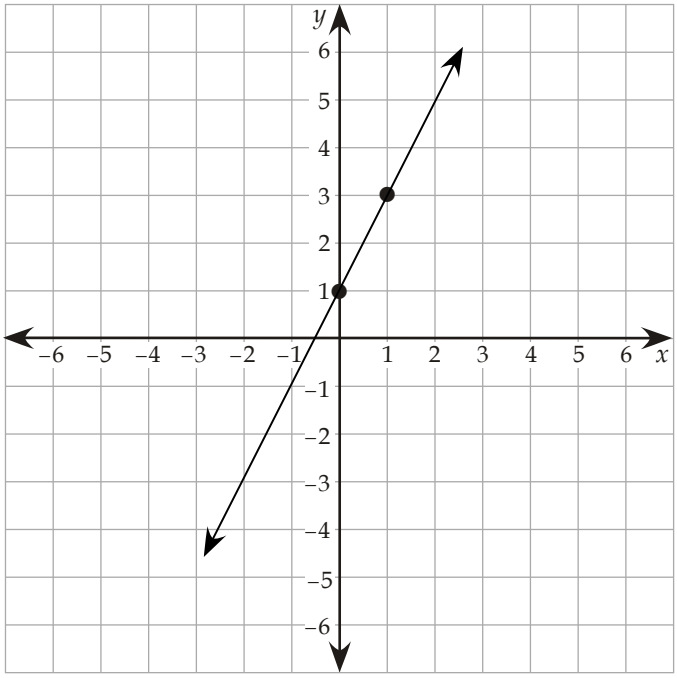


b) Change into the slope-intercept form:

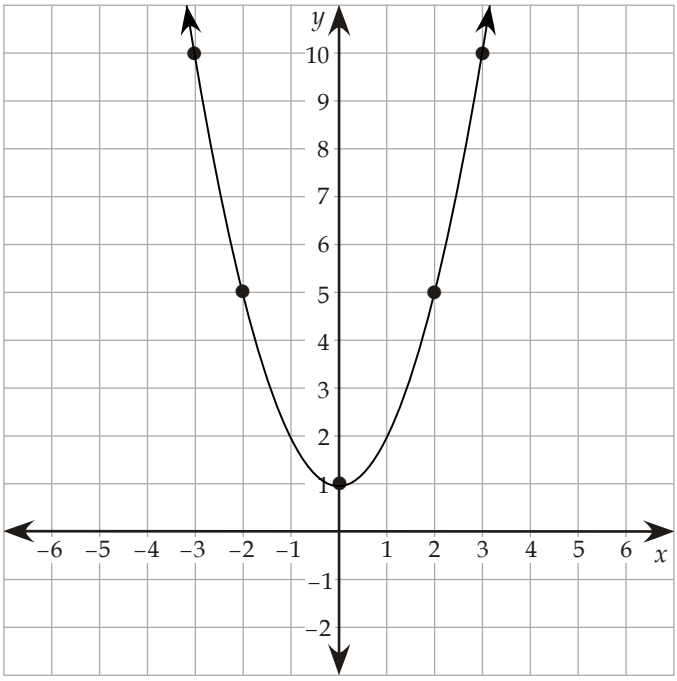
$y = mx + b$; $-y = -2x - 1$; $y = 2x + 1$

slope = $\frac{2}{1}$

y-intercept = 1



c)



2. Find the slope and intercepts for the linear functions in Question 1.

Answer:

a) $y = 3x + 2$

To find the x-intercept, let $y = 0$.

$0 = 3x + 2$

$-2 \quad -2$

$-2 = 3x$

$\frac{-2}{3} = \frac{3x}{3}$

$x = \frac{-2}{3}$

b) $2x - y + 1 = 0$

Let $y = 0$

$2x - 0 + 1 = 0$

$2x + 1 = 0$

$-1 \quad -1$

$2x = -1$

$\frac{2x}{2} = \frac{-1}{2}$

$x = \frac{-1}{2}$

c) This function does not intersect the x-axis. There are no x-intercepts.

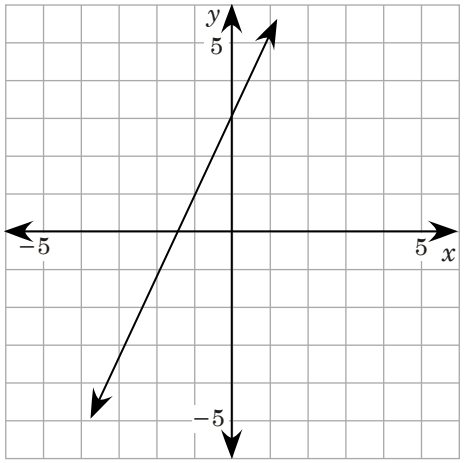
3. Which of the following describe a quadratic function?

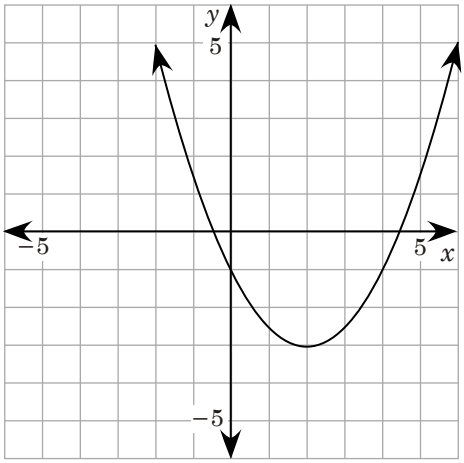
a) $f : x \rightarrow 3x^2$

b) $g : x \rightarrow 2x^3 - 5$

c) $y = 2x^2 - x + 1$

d) $f(x) = 3x - \frac{1}{x}$

e) 

f) 

Answers:

a) quadratic because it is in the form $f(x) = ax^2 + bx + c$

b) not quadratic since it has the variable x raised to the third power

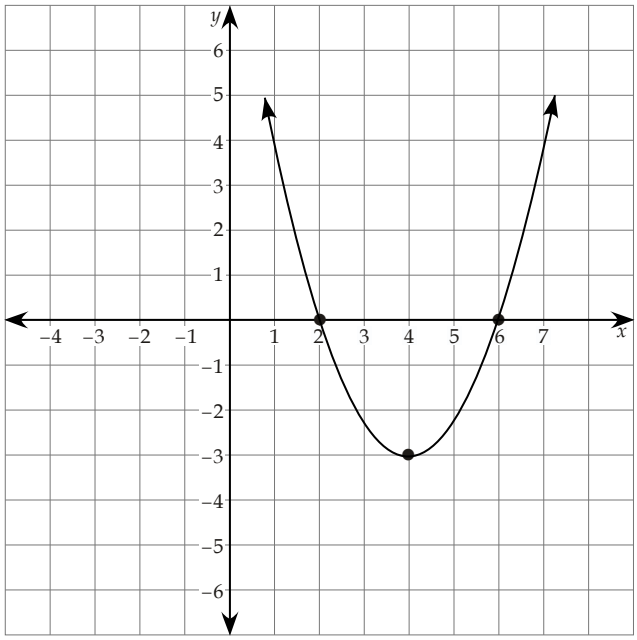
c) quadratic because it is of the form $f(x) = ax^2 + bx + c$

d) not quadratic

e) linear, not quadratic

f) quadratic since its graph is a parabola, which opens up

4. A graph of a quadratic function is shown below.



a) What is the domain of the graph?

b) What is the range of the graph?

c) What are the coordinates of the vertex?

d) What is the equation of the axis of symmetry?

e) What are the zeros of the function?

f) What are the x-intercepts?

g) Does the curve have a maximum? What is its value?

h) Does the curve have a minimum? What is its value?

Answers:

a) Domain $\{x \mid x \in \Re\}$

b) Range: $\{y \mid y \geq -3, y \in \Re\}$

c) Vertex: (4, –3). Remember this is the turning point.

d) Axis of symmetry equation: $x = 4$. This is the equation of the vertical line that passes through the vertex.

e) Zeros: 2, 6. Remember this is not an ordered pair. It represents the x-coordinates of the points where the curve intersects the x-axis.

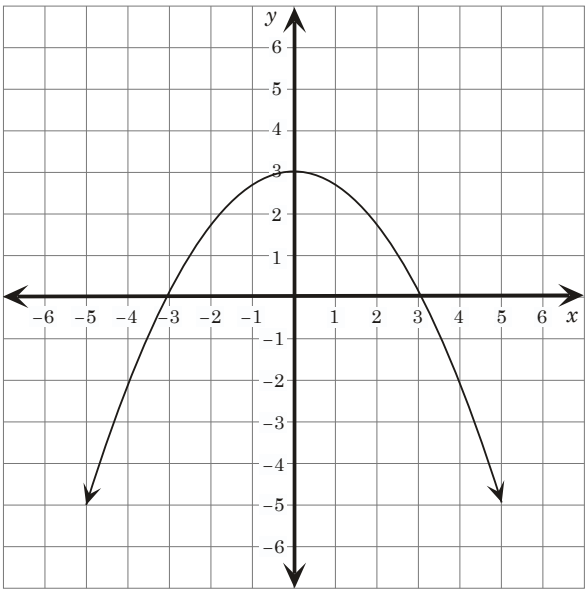
f) x-intercepts: 2, 6

g) Maximum value does not exist.

h) Minimum value is at -3, the y-coordinate of the vertex.

5. For each of the following, write the domain, range, coordinates of the vertex, equation of the axis of symmetry, y-intercept, and maximum or minimum value.

a)



Answer:

D: $\{x \mid x \in \Re\}$

R: $\{y \mid y \leq 3, y \in \Re\}$

Vertex coordinates: (0, 3)

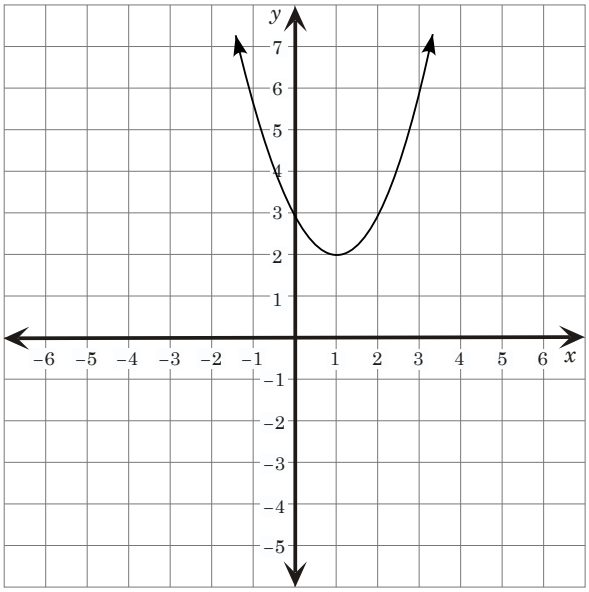
Axis of symmetry equation: $x = 0$

y-intercept: 3 is the y-coordinate of the point where the curve crosses the y-axis

Maximum value at 3

Minimum value does not exist

b)



Answer:

D: $\{x \mid x \in \Re\}$

R: $\{y \mid y \geq 2, y \in \Re\}$

Vertex: (1, 2)

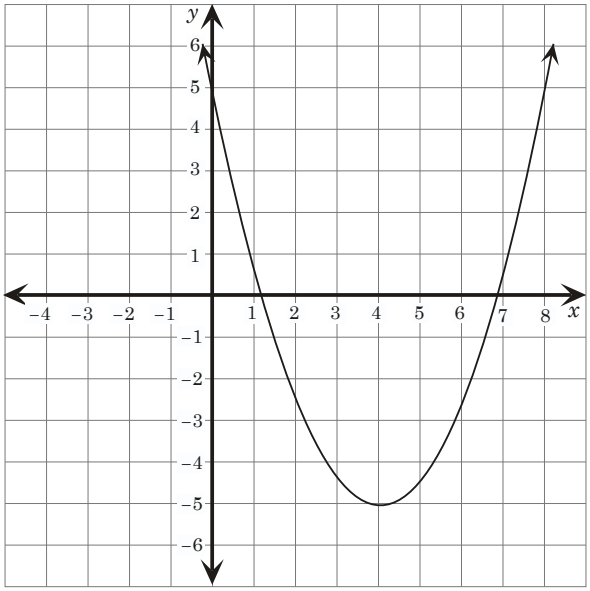
Axis of symmetry equation: $x = 1$

y-intercept: 3

Maximum value does not exist

Minimum value is 2

c)



Answer:

D: $\{x \mid x \in \Re\}$

R: $\{y \mid y \geq -5, y \in \Re\}$

Vertex: (4, −5)

Axis of symmetry equation: $x = 4$

y-intercept: 5

Maximum value: none

Minimum value: −5

6. Sketch the graphs of the following quadratic functions. Use a suitable table of values. Find the domain, range, coordinates of vertex, equation of axis of symmetry, zeros, x-intercept(s), y-intercept, and the maximum value or minimum value.

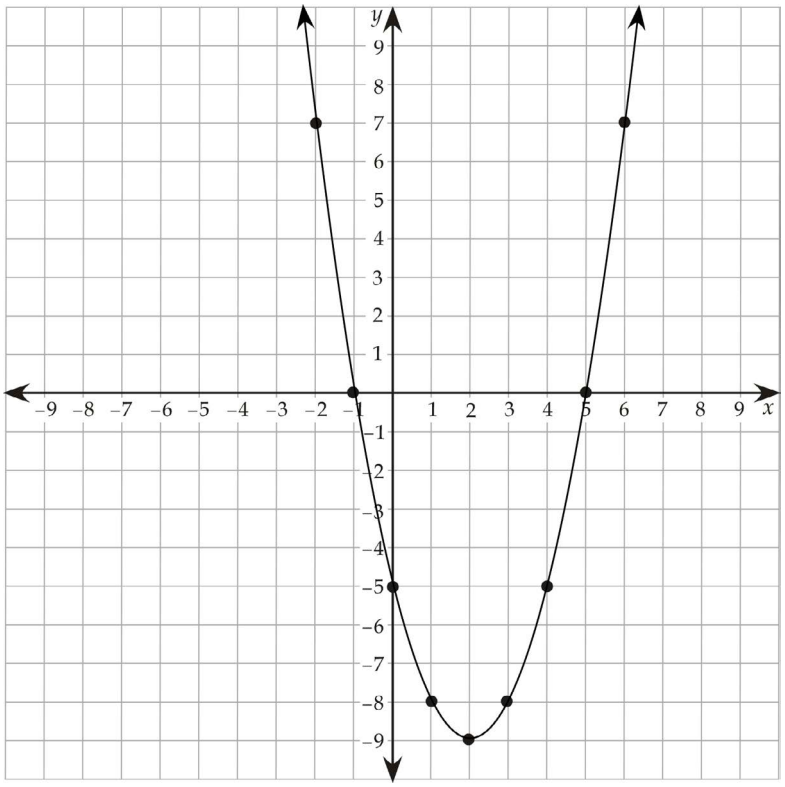
   a) $f(x) = x^2 - 4x - 5$

   b) $f : x \to -x^2 - 2x + 8$

   Answer:

   a)

x	-2	-1	0	1	2	3	4	5	6
y	7	0	-5	-8	-9	-8	-5	0	7



Domain: $\{x \mid x \in \Re\}$

Range: $\{y \mid y \geq -9, y \in \Re\}$

Vertex: $(2, -9)$

Axis of symmetry equation: $x = 2$

Zeros: -1 and 5

x-intercepts: -1 and 5

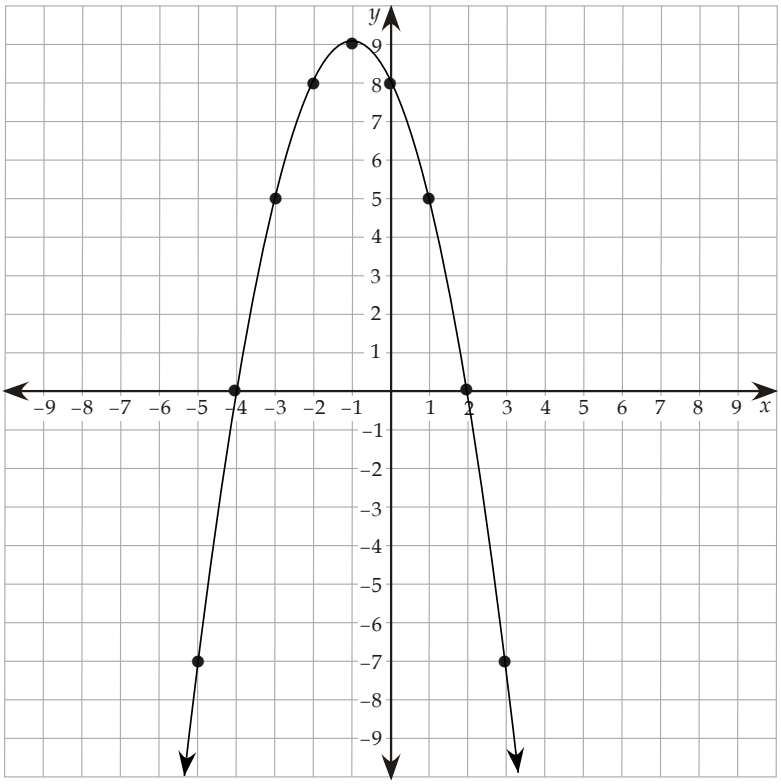
y-intercept: -5

Maximum value does not exist

Minimum value is at -9

b)

x	−5	−4	−3	−2	−1	0	1	2	3
y	−7	0	5	8	9	8	5	0	−7



Domain: $\{x \mid x \in \Re\}$

Range: $\{y \mid y \leq 9, y \in \Re\}$

Vertex: $(-1, 9)$

Axis of symmetry equation: $x = -1$

Zeros: -4 and 2

x-intercepts: -4 and 2

y-intercept: 8

Maximum value at 9

Minimum value does not exist

7. Match the equation with its graph. Use a table of values to graph each function if necessary.

   a) $y = x^2 - 2$

   b) $y = x^2 + 4x - 1$

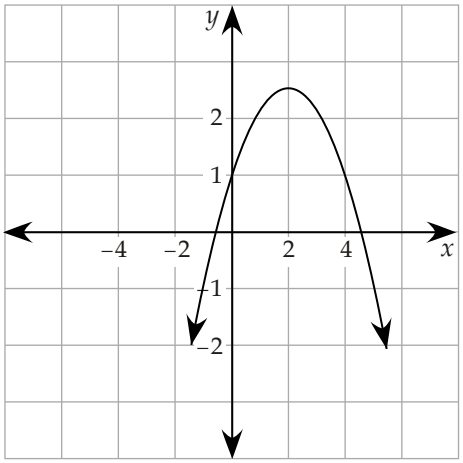
   c) $y = -x^2 - 4x - 3$

   d) $y = x^2 + 2$

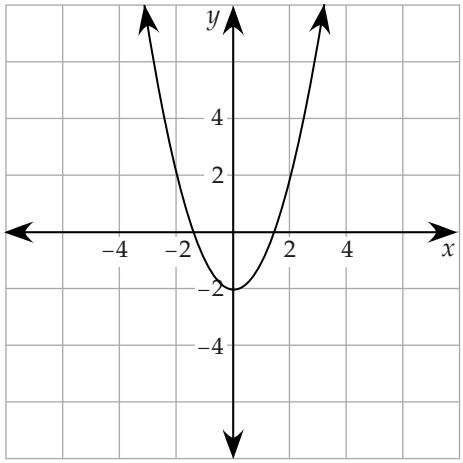
   e) $y = -x^2 + 4x + 1$

   f) $y = -x^2 + 2$

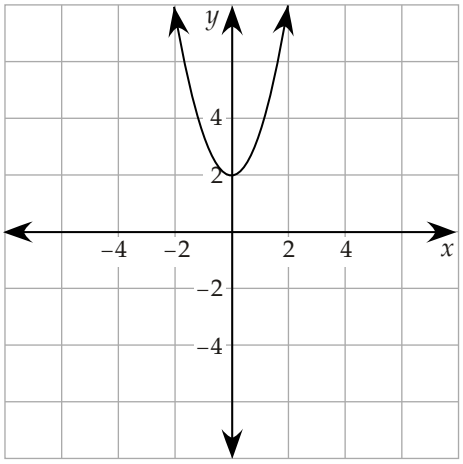
i)



ii)



iii)



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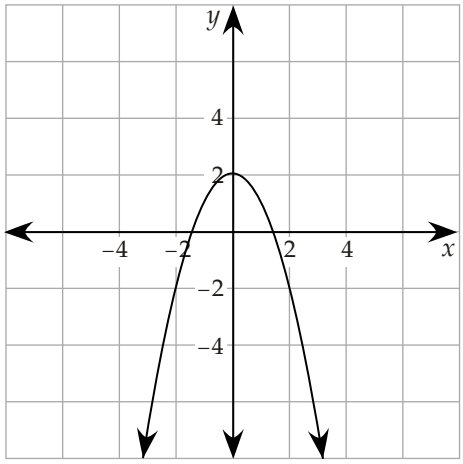
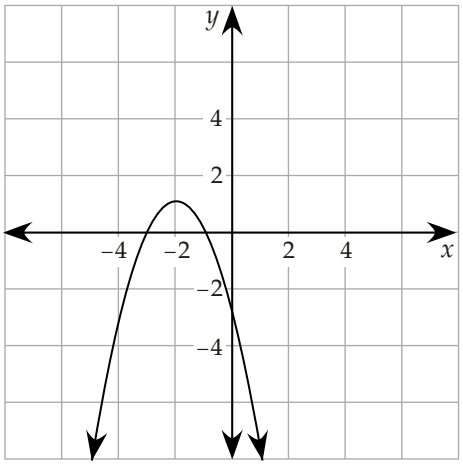


v)



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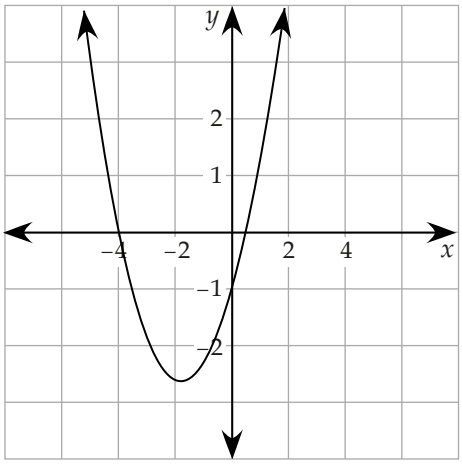


Answers:

a) and ii)

b) and vi)

c) and v)

d) and iii)

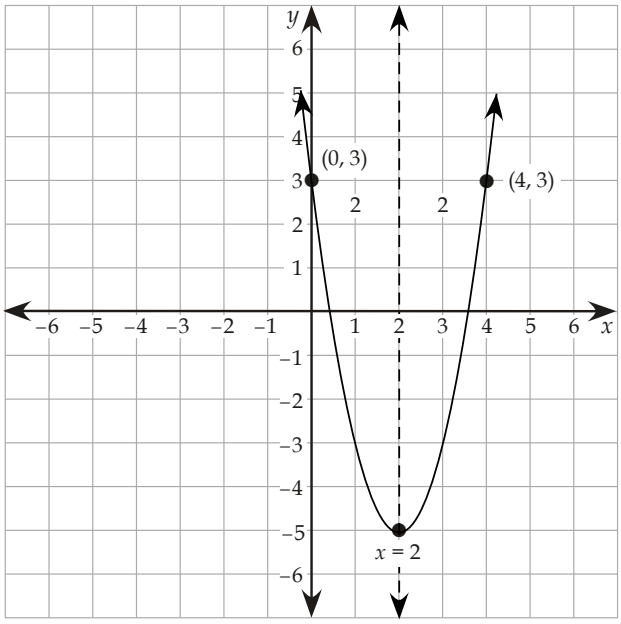
e) and i)

f) and iv)

8. a) The point (4, 3) is on the graph of $f(x) = 2x^2 - 8x + 3$. Use symmetry to find the coordinates of another point on the parabola if the axis of symmetry is $x = 2$.

   Answer:

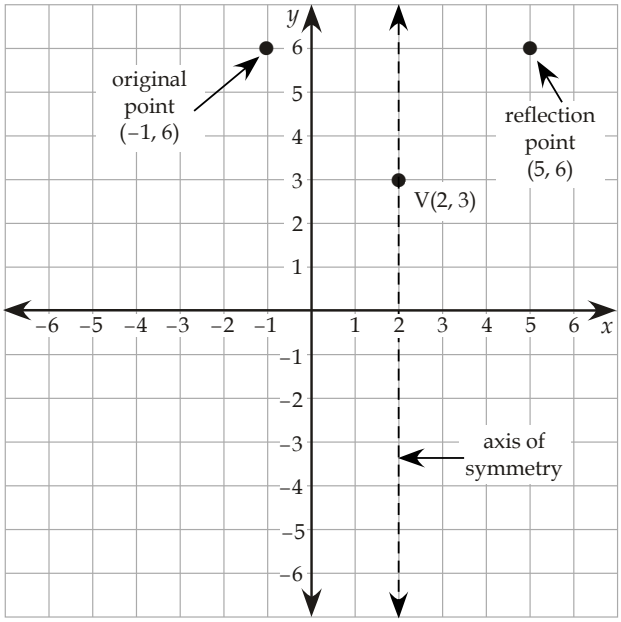
   Since the axis of symmetry for the graph of the function is the line $x = 2$, and since the point (4, 3) is 2 units to the right of it, then the reflection point must be 2 units to the left of it. Thus, (0, 3) is the required point.



b) A parabola has a vertex at (2, 3) and passes through the point P(−1, 6). Find the coordinates of a third point on the parabola.

Answer:

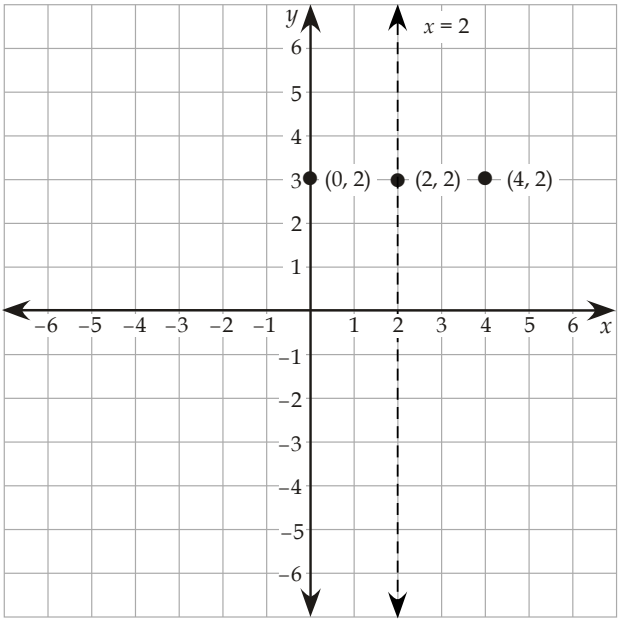
(−1, 6) is 3 units left of the axis of symmetry, so the reflection point must be 3 units to the right of the axis. Thus, (5, 6) is the required point.



c) Given certain reflection image points (0, 2) and (4, 2) on a parabola, find the equation of the line of symmetry.

Answer:

The axis of symmetry lies halfway between the reflection image points. Therefore, the point halfway between them is at (2, 2) and the equation of the axis of symmetry is $x = 2$.



Learning Activity 3.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Water is trickling out of a faucet at a speed of 3 mL per second. If Paula holds up a cup to collect the water, how much water will she collect in 3 minutes?
2. Factor: $x^2 - 9$
3. If $x = -1$ and $y = 2$, evaluate the expression: $3x^3 - xy$.
4. What is the lowest common multiple of 10 and 14?
5. At The Burger Palace, you have a choice between what type of bun and what type of burger you want. If there are three types of buns (white, whole wheat, and sesame) and three types of burgers (angus, chicken, and lean), how many different combinations of burgers are there?
6. Determine the distance between the following two points: (26, −13) and (25, −1).
7. Solve the inequality: $-6x - 7 > 3x - 9$.
8. Simplify. $(9x^2 - 7x + 5) - (2x^2 - 6x - 3)$

Answers:

1. 540 mL (There are $60 \times 3 = 180$ seconds in 3 minutes. 3 mL × 180 seconds = 540 mL.)
2. $(x - 3)(x + 3)$
3. $3(-1)^3 - (-1)(2) = -3 + 2 = -1$
4. 70
5. 9 (3 choices of bun × 3 choices of burgers = 9 choices)
6. $\sqrt{145}$ $\left(d = \sqrt{(25-26)^2 + (-1-(-13))^2} = \sqrt{(-1)^2 + (12)^2} = \sqrt{1+144} = \sqrt{145}\right)$
7. $-9x > -2$

   (Recall: When you divide by a negative, you need to flip the inequality sign.

   $x < \frac{-2}{-9}; x < \frac{2}{9}$)
8. $9x^2 - 7x + 5 - 2x^2 + 6x + 3 = 7x^2 - x + 8$

Part B: Quadratic Functions: $y = ax^2$ and $y = ax^2 + q$

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Determine whether each statement is true or false. If the statement is false, rewrite it so that it is true.

   a) $x^2 + y = 1$ is a quadratic function.

   b) $t(x) = 1 - x$ is a quadratic function.

   c) $f(x) = 4x^2$ is a quadratic function.

   d) The graph of $r(x) = x^2 + 1$ is symmetric with respect to the y-axis.

   e) The graph of $y = x^2$ is a "wider" parabola than the graph of $y = \frac{1}{4}x^2$.

   f) The graph of $y = 5x^2$ is a "narrower" parabola than the graph of $y = x^2$.

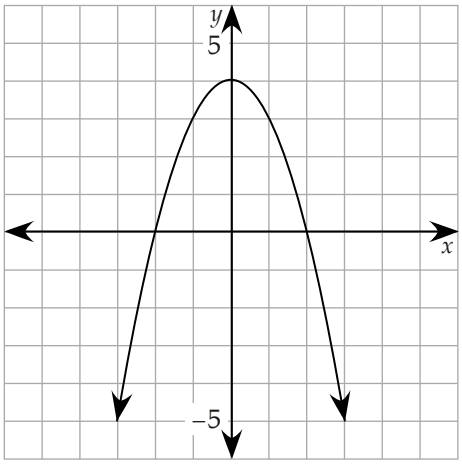
   g) The graph of $y = \frac{1}{4}x^2$ is a parabola that opens downward.

   h) The graph of $y = -2x^2$ is a parabola that opens downward.

   i) The maximum value of the function pictured at the right is 4.

   j) The minimum value of the function pictured at the right is 0.

   k) The domain of the function pictured at the right is $\{y \mid y \geq 4\}$.



   Answers:

   a) True, because it rearranges to $y = -x^2 + 1$, which is quadratic.

   b) False, because it has no squared term and so is linear. The equation $t(x) = 1 - x$ is a linear function whereas $t(x) = 1 - x^2$ is a quadratic function.

   c) True, because there is an x^2 term.

   d) True, since the axis of symmetry is the y-axis.

   e) False, since the graph of $y = x^2$ is narrower than $y = \frac{1}{4}x^2$. The graph of $y = \frac{1}{4}x^2$ is a wider parabola than the graph of $y = x^2$.

f) True, because the value of a is 5, which is greater than 1.

g) False, because when $a > 0$ the parabola opens upward. The graph $y = \frac{1}{4}x^2$ of is a parabola that opens upward or the graph of $y = -\frac{1}{4}x^2$ is a parabola that opens downward.

h) True, because when $a < 0$, the parabola opens downward.

i) True, because the function has a maximum value when the function opens downward.

j) False, because the minimum value is not defined when the parabola opens downward and continues indefinitely. The minimum value of the function pictured at the right cannot be stated.

k) False, because the range of the function is $\{y \mid y \leq 4\}$.

2. Complete the following table.

	$y = \frac{1}{3}x^2$	$y = x^2$	$y = -3x^2$
Vertex	(0, 0)	(0, 0)	(0, 0)
Equation of axis of symmetry	$x = 0$	$x = 0$	$x = 0$
Domain	$\{x \mid x \in \Re\}$	$\{x \mid x \in \Re\}$	$\{x \mid x \in \Re\}$
Range	$\{y \mid y \geq 0\}$	$\{y \mid y \geq 0\}$	$\{y \mid y \leq 0\}$
x-intercepts or zeros	0	0	0
Direction of opening	upward	upward	downward
Maximum y-value	none	none	0
Minimum y-value	0	0	cannot be determined
y-intercept	0	0	0

3. Describe the transformations needed to arrive at the graph of each of the following using $y = x^2$ as the reference curve.

   a) $y = 4x^2$

   b) $y = \frac{1}{2}x^2$

   c) $y = -7x^2$

   Answers:

   a) $y = x^2$ is stretched vertically by a factor of 4; each coordinate (x, y) on the original curve becomes $(x, 4y)$ on the new curve.

   b) $y = x^2$ is compressed vertically by multiplying each y-coordinate by $\frac{1}{2}$.

   c) $y = x^2$ is stretched vertically by a factor of 7 and reflected over the x-axis.

4. Find the following for the graph given by $y = ax^2$:

   a) What is the direction of the opening of the graph if $a > 0$?

   b) What is the direction of the opening of the graph if $a < 0$?

   c) If $a > 0$, does $y = ax^2$ have a minimum value or a maximum value?

   d) If $a < 0$, does $y = ax^2$ have a maximum value or a minimum value?

   Answers:

   a) upward

   b) downward

   c) minimum

   d) maximum

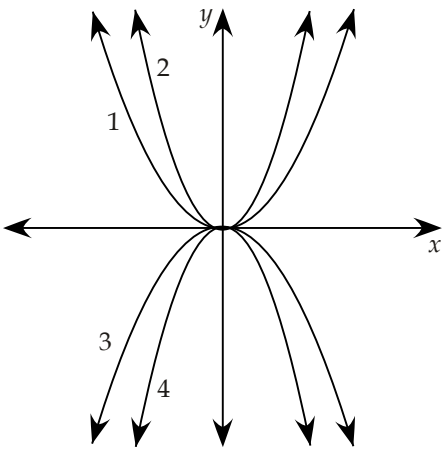
5. Match the letter corresponding to each function with the number corresponding to its graph.

a) $y = -2x^2$

b) $y = 1.6x^2$

c) $y = -2\pi x^2$

d) $y = \sqrt{6}x^2$



Answers:

a) and 3)

b) and 1)

c) and 4)

d) and 2)

6. Write an equation in the form $y = ax^2$ if $(1, -4)$ lies on the curve.

Answer:

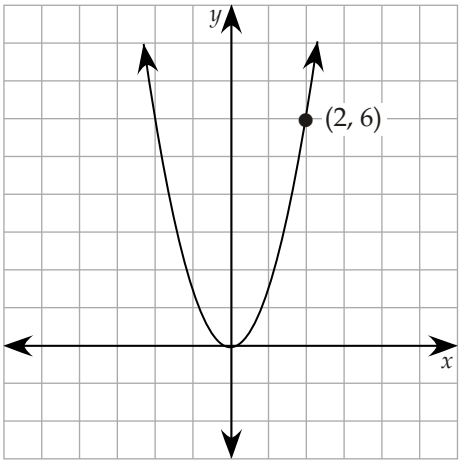
Substitute in $x = 1$, $y = -4$ into $y = ax^2$.

$-4 = a(1)^2$

$-4 = a$

Therefore, the equation is $y = -4x^2$.

7. Write the equation of the graph. Then write the equation of its reflection in the x-axis. Note each mark represents 1 unit on the graph.



Answer:

A point on the curve is (2, 6). The vertex is (0, 0). The equation is $y = ax^2$.

$\therefore$ Substitute $x = 2$, $y = 6$

$6 = a(2)^2$

$6 = 4a$

$\frac{6}{4} = a$

$\frac{3}{2} = a$

$y = \frac{3}{2}x^2$

The reflection in the x-axis of this parabola will have the same size and vertex but will open downward. Its equation is $y = -\frac{3}{2}x^2$.

8. Determine whether each statement is true or false. If the statement is false, rewrite it so it is true.
   a) The graph of $y = 3x^2 + 4$ is the graph of $y = 3x^2$ shifted downward 4 units.
   b) The graph of $f(x) = -x^2 - 3$ is the graph of $h(x) = -x^2$ shifted downward 3 units.
   c) The vertex of the graph of $y = 2x^2$ is the origin.
   d) The vertex of the graph of $y = \frac{1}{2}x^2 - 2$ is $(0, -2)$.
   e) An equation of the axis of symmetry of the graph of $y = x^2 - 1$ is $x = -1$.
   f) The graph of $y = x^2 - 1$ opens downward.
   g) The graph of $y = -x^2 - 2$ opens downward.
   h) The maximum value of $\{(x, y) | y = -2x^2 + 3\}$ is -2.
   i) The minimum value of is $\{(x, y) | y = 3x^2 - 4\}$ is -4.
   j) If $f(x) = 2x^2 + 3$, then $f(x) = f(-x)$, for all x.
   k) The graph of $y = x^2 + 4$ is symmetric with respect to the y-axis.
   l) The graphs of $y = x^2 + 2$ and $y = -x^2 + 2$ are mirror reflections of each other in the x-axis.
   m) The graph of $y = x^2$ is a narrower parabola than the graph of $y = \frac{1}{4}x^2$.
   n) The range of $f(x) = -x^2 + 2$ is $\{y | y \leq 2\}$.

Answers:

a) False. The graph of $y = 3x^2 + 4$ is the graph of $y = 3x^2$ shifted upward 4 units or the graph of $y = 3x^2 - 4$ is the graph of $y = 3x^2$ shifted downwards 4 units.

b) True

c) True

d) True

e) False. An equation of the axis of symmetry of $y = x^2 - 1$ is $x = 0$.

f) False. The graph of $y = x^2 - 1$ opens upward or the graph of $y = -x^2 - 1$ opens downward.

g) True

h) False. The maximum value of $y = -2x^2 + 3$ is 3.

i) True

j) True. $f(-x) = 2(-x)^2 + 3$

$= 2x^2 + 3$

Thus, $f(-x) = f(x)$.

k) True

l) False. The graph of $y = x^2 + 2$ and $y = -x^2 + 2$ are mirror reflections of each other in the line $y = 2$.

m) True

n) True

9. A quadratic function f is defined by $f: x \to ax^2 + q$, a $\neq 0$. Describe whether the graph of f opens up or down and give its maximum or minimum value.

a) $a > 0$ and $q > 0$

b) $a > 0$ and $q < 0$

c) $a < 0$ and $q > 0$

d) $a < 0$ and $q < 0$

Answers:

a) The graph opens upward and the minimum value, q, is positive.

b) The graph opens upward and the minimum value of q is negative.

c) The graph opens downward and the maximum value of q is positive.

d) The graph opens downward and the maximum value of q is negative.

10. Graph $y = -3x^2 + 1$ using transformations on $y = x^2$. Then make a table of values with 5 points in it.

Answer:

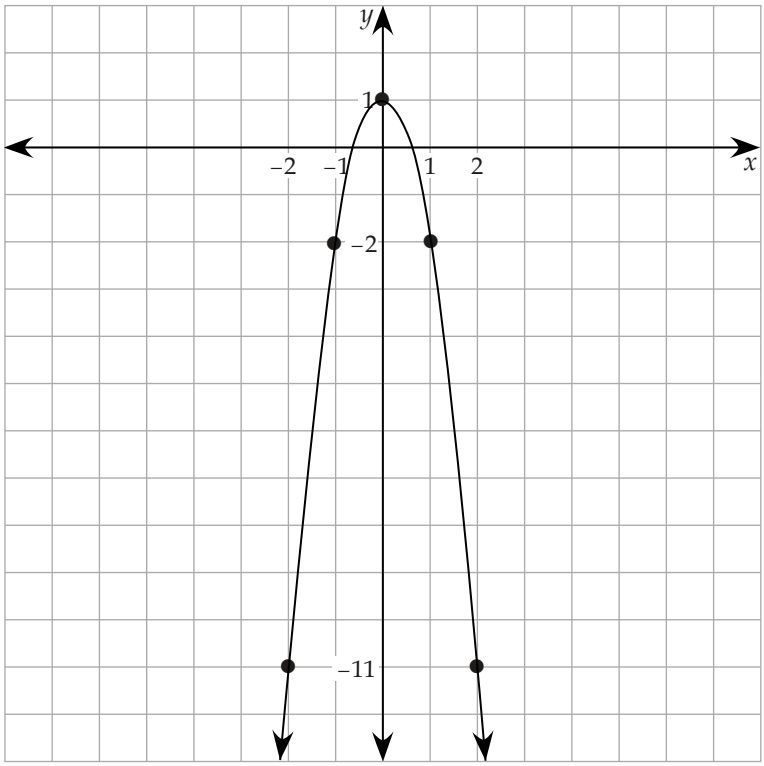
Recall the key coordinates of $y = x^2$: from the vertex at (0, 0), we move over 1 and up 1 to mark (1, 1). Then from the vertex, move over 2 and up 4 to mark (2, 4), then over 3 and up 9 from the vertex to mark (3, 9).

Now, let's transform the points (0, 0), (1, 1), (2, 4), and (3, 9) for the equation: $y = -3x^2 + 1$.

The vertex is (0, 1). The curve opens down. There is a vertical stretch of 3.

On your graph paper, starting at (0, 1), move right 1 and down 1(3) = 3 to mark the point (1, −2). Then, move right 2 from the vertex and down 4(3) = 12 to mark the point (2, −11). Then, right 3 from the vertex and down 9(3) = 27 to mark the point (3, −26).

Now reflect these points in the axis of symmetry to get (−1, −2) and (−2, −11). Now draw the graph and make a table of values from the graph.



x	2	1	0	−1	−2
y	−11	−2	1	−2	−11

Learning Activity 3.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Simplify: $(a + b)^2$
2. Evaluate: $\frac{1}{5} + \frac{2}{3}$
3. If Mark's meal comes to $16.27 with tax and he adds a tip of $1.50, how much change should he get back if he pays with a $20 bill?
4. Samuel and Cassy weigh 333 pounds together. If Samuel weighs twice as much as Cassy, how many pounds does Cassy weigh?
5. Factor: $x^2 - 3x - 28$
6. Simplify: $\left(\frac{5^5 \times 5^7}{5^2}\right)^3$
7. Rationalize the denominator: $\frac{2 - \sqrt{8}}{10\sqrt{7}}$
8. Find the slope of the line that contains the points $(2, -7)$ and $(9, -14)$.

Answers:

1. $(a + b)(a + b) = a^2 + 2ab + b^2$
2. $\frac{3}{15} + \frac{10}{15} = \frac{13}{15}$
3. $2.23 ($16.27 + $1.50 = $17.77; $20.00 − $17.77 = $2.23)
4. $111 \left(\frac{333}{3}\right)$
5. $(x - 7)(x + 4)$
6. $5^{30} \left(\left(\frac{5^5 \times 5^7}{5^2}\right)^3 = \left(\frac{5^{12}}{5^2}\right)^3 = \left(5^{10}\right)^3 = 5^{30}\right)$

7. $\frac{2\sqrt{7}-\sqrt{56}}{70}$ $\left(\frac{2-\sqrt{8}}{10\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{2\sqrt{7}-\sqrt{56}}{10 \cdot 7} = \frac{2\sqrt{7}-\sqrt{56}}{70}\right.$

$\left.\text{or } \frac{2\sqrt{7}-2\sqrt{14}}{70} \text{ or } \frac{\sqrt{7}-\sqrt{14}}{35}\right)$

8. -1 $\left(m = \frac{y_2 - y_1}{x_2 - x_1}; m = \frac{-14-(-7)}{9-2} = -\frac{7}{7} = -1\right)$

Part B: Quadratic Functions: $a(x - p)^2$

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. State whether each statement is true or false. If the statement is false, rewrite it so that it is true.
   a) The graph of $y = (x + 5)^2$ is the graph of $y = x^2$ shifted 5 units to the left.
   b) The graph of $y = (x - 7)^2$ is the graph of $y = x^2$ shifted 7 units to the left.
   c) The vertex of the graph of $y = (x - 1)^2$ is (1, 0).
   d) The vertex of the graph of $y = 2(x - 2)^2$ is (1, 0).
   e) The equation of the axis of symmetry of the graph of $y = \frac{1}{2}(x-3)^2$ is $x = \frac{1}{2}$.
   f) The equation of the axis of symmetry of the graph of $y = -\frac{1}{2}(x+3)^2$ is $y = -3$.
   g) The graph of $y = -\frac{1}{3}(x-2)^2$ is a narrower parabola than the graph of $y = -3(x - 2)^2$.
   h) The graph of $y = -\frac{1}{4}(x+4)^2$ opens downward.
   i) The quadratic function $y = \frac{1}{3}(x-2)^2$ has 0 for its minimum value.
   j) If $g(x) = (x + 4)^2$, then $g(x - 4) = g(-x - 4)$ for all x.

Answers:

a) True

b) False. The graph of $y = (x - 7)^2$ is the graph of $y = x^2$ shifted 7 units to the right or the graph of $y = (x + 7)^2$ is the graph of $y = x^2$ shifted 7 units to the left.

c) True

d) False. The vertex of the graph of $y = 2(x - 1)^2$ is (1, 0) or the vertex of the graph of $y = 2(x - 2)^2$ is (2, 0).

e) False. The equation of the axis of symmetry of the graph of $y = \frac{1}{2}(x - 3)^2$ is $x = 3$ or the equation of the axis of symmetry of the graph of $y = 3\left(x - \frac{1}{2}\right)^2$ is $x = \frac{1}{2}$.

f) False. The equation of the axis of symmetry of the graph of $y = -\frac{1}{2}(x + 3)^2$ is $x = -3$.

g) False. The graph of $y = -\frac{1}{3}(x - 2)^2$ is a wider parabola than the graph of $y = -3(x - 2)^2$ or the graph of $y = -3(x - 2)^2$ is a narrower parabola than the graph of $y = -\frac{1}{3}(x - 2)^2$.

h) True

i) True

j) True

$$g(x - 4) = \big((x - 4) + 4\big)^2$$
$$= (x)^2$$
$$= x^2$$
$$g(-x - 4) = \big((-x - 4) + 4\big)^2$$
$$= (-x)^2$$
$$= x^2$$

Thus, $g(x - 4) = g(-x - 4)$.

2. Match the equation with its graph.

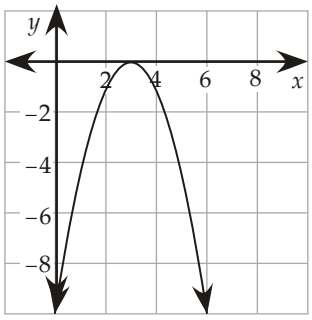
a) $y = (x - 4)^2$

b) $y = x^2 + 4$

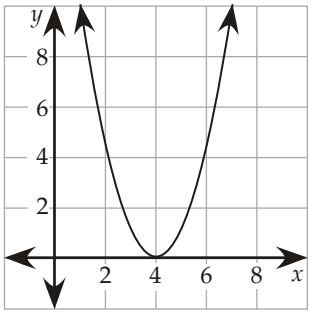
c) $y = -(x + 2)^2$

d) $y = -(x - 3)^2$

i)



ii)



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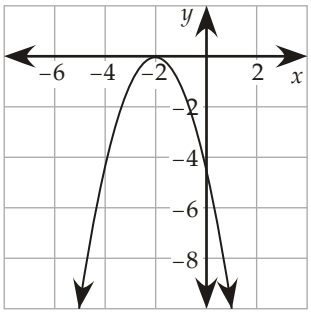

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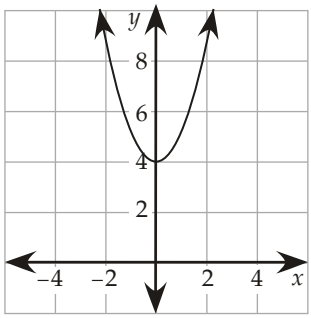


Answers:

a) ii

b) iv

c) iii

d) i

3. Without using tables of values, sketch the graphs of the following on graph paper. Provide a written explanation of the steps involved in the transformation. Then make a table of values with 5 points in it, showing how the coordinates of $y = x^2$ are transformed.

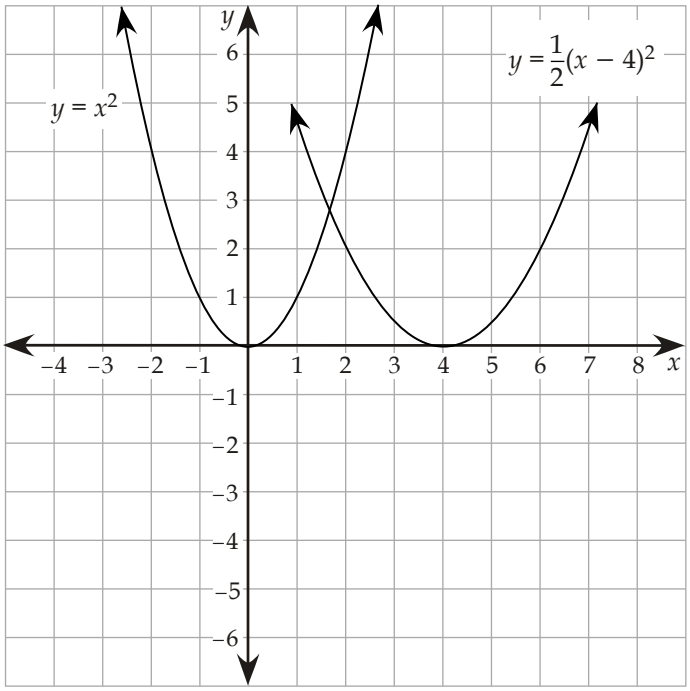
a) $f : x \to \frac{1}{2}(x - 4)^2$

b) $y = -2(x - 1)^2$

c) $f(x) = 2(x + 1)^2$

Answers:

a) $y = x^2$ compressed vertically by multiplying the y-values by $\frac{1}{2}$ and shifted horizontally 4 units to the right.

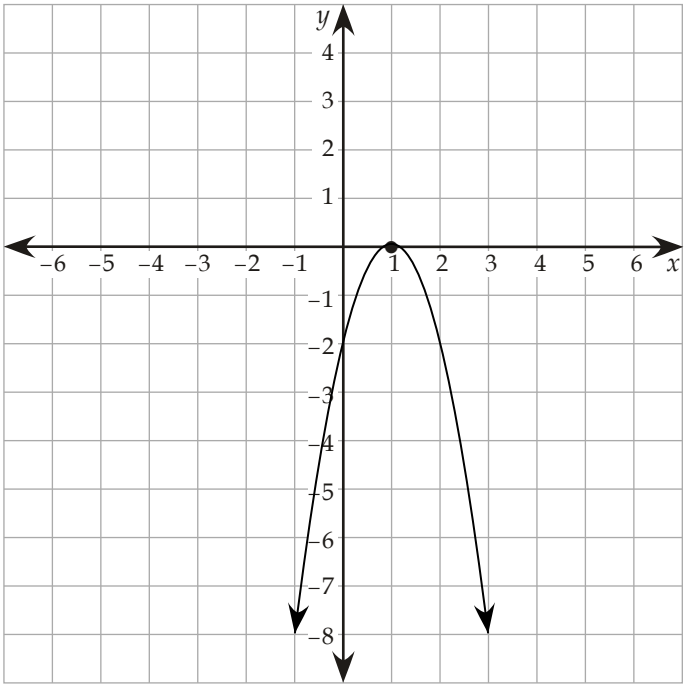


$f : x \rightarrow \frac{1}{2}(x-4)^2$

Transformation: $(x, y) \rightarrow \left(x + 4, \frac{1}{2}y\right)$

Coordinates for $y = x^2$	**Coordinates for $f(x)$**
(2, 4)	(6, 2)
(1, 1)	$\left(5, \frac{1}{2}\right)$
(0, 0)	(4, 0)
(−1, 1)	$\left(3, \frac{1}{2}\right)$
(−2, 4)	(2, 2)

b) $y = x^2$ stretched vertically by a factor of 2, reflected over the x-axis, and shifted horizontally 1 unit to the right.

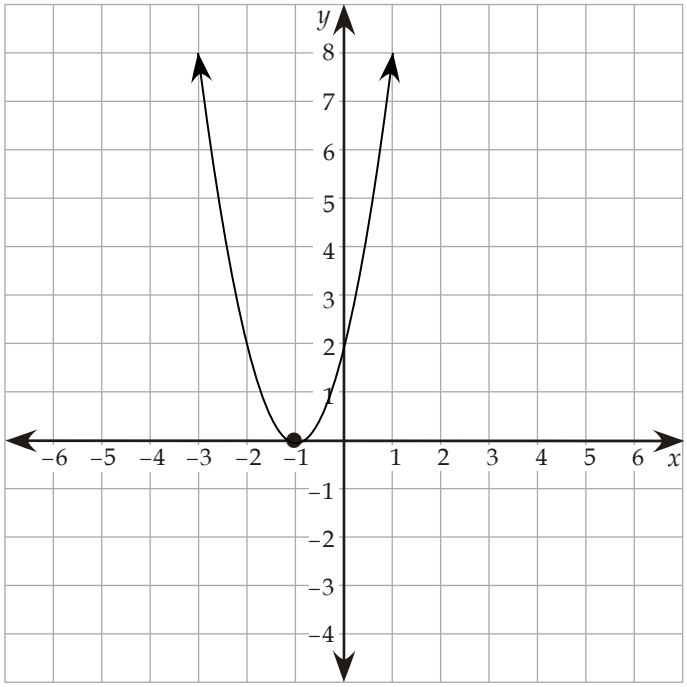


$y = -2(x - 1)^2$

Transformation: $(x, y) \rightarrow (x + 1, -2y)$

Coordinates for $y = x^2$	**Coordinates for $f(x)$**
(2, 4)	(3, −8)
(1, 1)	(2, −2)
(0, 0)	(1, 0)
(−1, 1)	(0, −2)
(−2, 4)	(−1, −8)

c) $y = x^2$ stretched vertically by a factor of 2 and shifted horizontally 1 unit to the left.



$f(x) = 2(x + 1)^2$

Transformation: $(x, y) \rightarrow (x - 1, 2y)$

Coordinates for $y = x^2$	Coordinates for $f(x)$
(2, 4)	(1, 8)
(1, 1)	(0, 2)
(0, 0)	(−1, 0)
(−1, 1)	(−2, 2)
(−2, 4)	(−3, 8)

4. Complete the following table.

Answer:

	(a)	(b)	(c)	(d)
	$y = (x + 2)^2$	$y = (x - 1)^2$	$y = 2(x - 3)^2$	$y = \frac{4}{3}(x + 6)^2$
Vertex	$(-2, 0)$	$(1, 0)$	$(3, 0)$	$(-6, 0)$
Equation of axis of symmetry	$x = -2$	$x = 1$	$x = 3$	$x = -6$
Domain	$\{x \mid x \in \Re\}$	$\{x \mid x \in \Re\}$	$\{x \mid x \in \Re\}$	$\{x \mid x \in \Re\}$
Range	$\{y \mid y \geq 0\}$	$\{y \mid y \geq 0\}$	$\{y \mid y \geq 0\}$	$\{y \mid y \geq 0\}$
x-intercepts	-2	1	3	-6
y-intercept*	4	1	18	48
Direction of opening	up	up	up	up
Maximum value	none	none	none	none
Minimum value	0	0	0	0

* **Note**:

To find the y-intercepts, let $x = 0$ in the equation and solve for y.

a) $y = (x + 2)^2 = (0 + 2)^2 = 4$

b) $y = (x - 1)^2 = (0 - 1)^2 = 1$

c) $y = 2(x - 3)^2 = 2(0 - 3)^2 = 2(9) = 18$

d) $y = \frac{4}{3}(x + 6)^2 = \frac{4}{3}(0 + 6)^2 = \frac{4}{3}(36) = 48$

Learning Activity 3.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. You need to make 100 cupcakes for a bake sale. However, your recipe only allows you to make cupcakes in multiples of 12. What is the least number of cupcakes you can make to have enough cupcakes for the bake sale?
2. What is the area of a park that measures 13 km by 62 km?
3. Evaluate: $\left[\left(\frac{3}{4}\right)-\left(\frac{1}{2}\right)\right]+\left[\left(\frac{1}{6}\right)+\left(\frac{5}{3}\right)\right]$
4. What is the slope of the line defined by the equation $y = 2x + 3$?
5. Does the parabola $y = -3x^2 + 4x + 2$ open up or down?
6. List the factors of 1000.
7. Write this percent as a decimal: 8.07%
8. Is 71 a prime or composite number?

Answers:

1. 9 dozen ($12 \times 8 = 96$; $12 \times 9 = 108$; If you make 8 dozen, you will only have 96 cupcakes. Therefore, you need to make 9 dozen to make sure you have enough cupcakes.)
2. 806 ($13 \times 62 = 10 \times 62 + 3 \times 62 = 620 + 186 = 806$)
3. $\frac{25}{12}\left(\left[\frac{9}{12}-\frac{6}{12}\right]+\left[\frac{2}{12}+\frac{20}{12}\right]=\frac{3}{12}+\frac{22}{12}=\frac{25}{12}\right)$
4. 2
5. Down (The a-value is negative.)
6. 1, 2, 4, 5, 8, 10, 20, 25, 40, 50, 100, 125, 200, 250, 500, and 1000.
7. 0.0807
8. Prime number

Part B: Transformations of Quadratic Functions

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. State whether each statement is true or false. If the statement is false, rewrite it so it is true.
   a) The vertex of the graph of $y = (x - 4)^2 + 2$ is $(-4, 2)$.
   b) The vertex of the graph of $y = 3(x - 5)^2 - 2$ is $(5, 2)$.
   c) The vertex of the graph of $y = 2x^2 + 7$ is $(2, 7)$.
   d) The vertex of the graph of $y = -3x^2 + 4$ is $(0, 4)$.
   e) The vertex of the graph of $y = -2(x - 3)^2$ is $(-2, 3)$.
   f) The equation of the axis of symmetry of the graph of $y = -3(x - 6)^2 + 1$ is $x = 1$.
   g) The equation of the axis of symmetry of the graph of $y = 3\left(x - \frac{1}{2}\right)^2 + 4$ is $x = \frac{1}{2}$.
   h) The equation of the axis of symmetry of the graph of $y = (x + 2)^2 + 1$ is $x = -2$.
   i) The graph of $y = 4(x - 3)^2 - 1$ opens downward.
   j) The graph of $y = 4(x + 2)^2 - 3$ opens upward.
   k) The range of $y = -2(x + 3)^2 + 4$ is $\{y \mid y \leq 4\}$.
   l) The maximum value of $y = 3(x - 5)^2 - 2$ is -2.

   Answers:

   a) False. The vertex of the graph of $y = (x - 4)^2 + 2$ is $(4, 2)$ or the vertex of the graph of $y = (x + 4)^2 + 2$ is $(-4, 2)$.
   b) False. The vertex of the graph of $y = 3(x - 5)^2 - 2$ is $(5, -2)$ or the vertex of the graph of $y = 3(x - 5)^2 + 2$ is $(5, 2)$.
   c) False. The vertex of the graph of $y = 2x^2 + 7$ is $(0, 7)$, or the vertex of the graph of $y = 2(x - 2)^2 + 7$ is $(2, 7)$.
   d) True.
   e) False. The vertex of the graph of $y = -2(x - 3)^2$ is $(3, 0)$, or the vertex of the graph of $y = -2(x + 2)^2 + 3$ is $(-2, 3)$.

f) False. The equation of the axis of symmetry of the graph of $y = -3(x - 6)^2 + 1$ is $x = 6$, or the equation of the axis of symmetry of the graph of $y = -3(x - 1)^2 + 1$ is $x = 1$.

g) True.

h) True.

i) False. The graph of $y = 4(x - 3)^2 - 1$ opens upward, or the graph of $y = -4(x - 3)^2 - 1$ opens downward.

j) True.

k) True.

l) False. The minimum value of $y = 3(x - 5)^2 - 2$ is -2.

2. Match each equation with its graph.

a) $y = 3x^2$

b) $y = \frac{-1}{3}x^2$

c) $y = 3(x + 2)^2$

d) $y + 1 = 3(x + 2)^2$

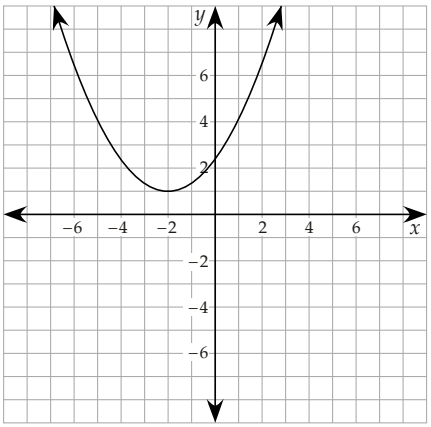
e) $y - 1 = \frac{1}{3}(x + 2)^2$

f) $y + 1 = -3(x - 2)^2$

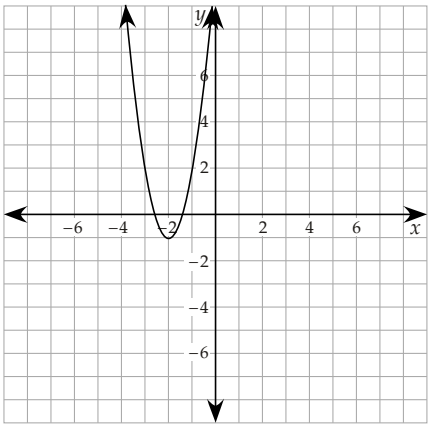
g) $y = -\frac{1}{3}(x - 2)^2$

h) $y - 1 = -\frac{1}{3}(x - 2)^2$

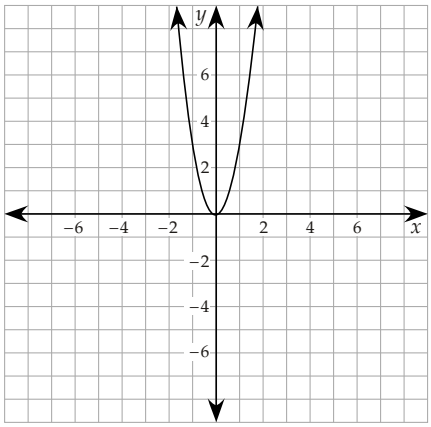
i)



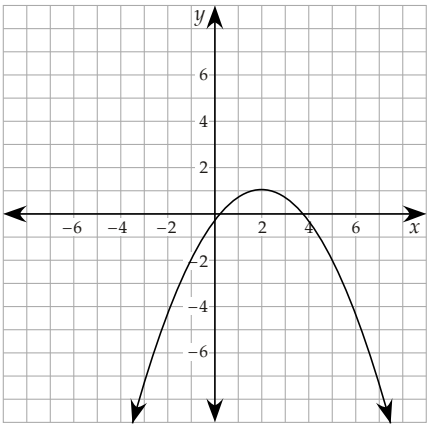
ii)



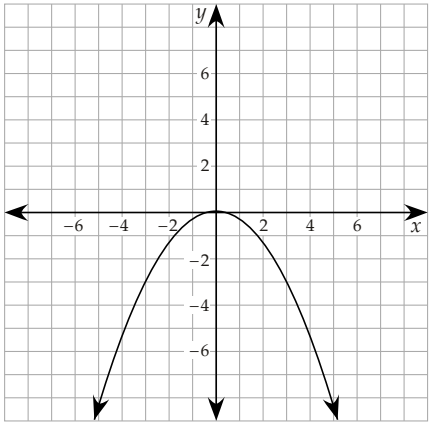
iii)



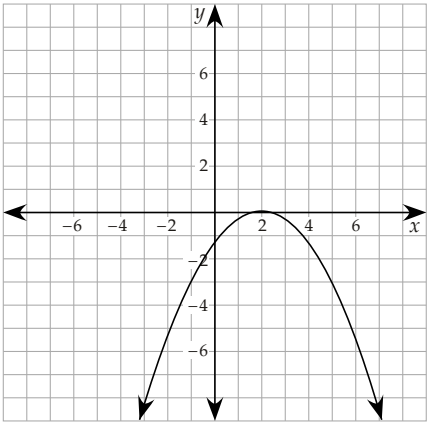
iv)



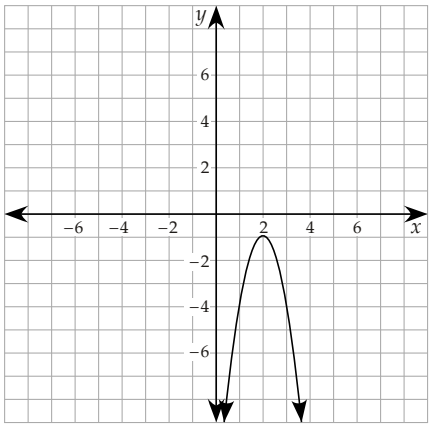
v)



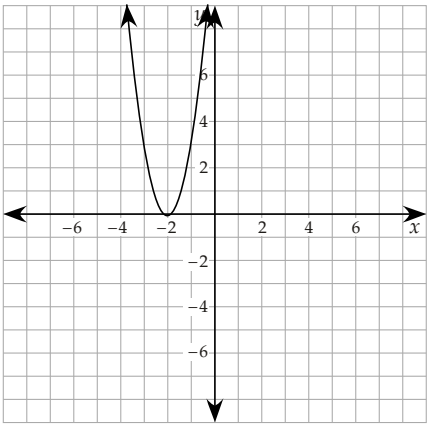
vi)



vii)



viii)



Answers:

a) and iii)
b) and v)
c) and viii)
d) and ii)
e) and i)
f) and vii)
g) and vi)
h) and iv)

3. Complete the following table.

	(a)	(b)	(c)	(d)
	$y = (x - 1)^2 + 2$	$y = -(x - 1)^2 - 2$	$y = (x + 1)^2 + 2$	$y = -(x + 1)^2 - 2$
Vertex	(1, 2)	(1, −2)	(−1, 2)	(−1, −2)
Equation of axis of symmetry	$x = 1$	$x = 1$	$x = -1$	$x = -1$
Domain	$\{x \mid x \in \Re\}$	$\{x \mid x \in \Re\}$	$\{x \mid x \in \Re\}$	$\{x \mid x \in \Re\}$
Range	$\{y \mid y \geq 2\}$	$\{y \mid y \leq -2\}$	$\{y \mid y \geq 2\}$	$\{y \mid y \leq -2\}$
y-intercept*	3	−3	3	−3
Direction of opening	up	down	up	down
Maximum value	none	−2	none	−2
Minimum value	2	none	2	none

* The y-intercepts can be found by letting $x = 0$ and solving for y.

a) $y = (x - 1)^2 + 2$
$y = (0 - 1)^2 + 2$
$y = 1 + 2$
$y = 3$

b) $y = -(x - 1)^2 - 2$
$y = -(0 - 1)^2 - 2$
$y = -1 - 2$
$y = -3$

c) $y = (x + 1)^2 + 2$
$y = (0 + 1)^2 + 2$
$y = 1 + 2$
$y = 3$

d) $y = -(x + 1)^2 - 2$
$y = -(0 + 1)^2 - 2$
$y = -1 - 2$
$y = -3$

4. For each of the following parabolas, state:
   - the direction of opening
   - the coordinates of the vertex
   - the equation of the axis of symmetry
   - whether the given graph is narrower or wider than $y = x^2$

   a) $y = 2(x + 1)^2$

   b) $y = -\frac{1}{2}(x - 1)^2 + 6$

   c) $y = 2(x + 6)^2 - 10$

   d) $y = 6(x - 1)^2 + 8$

   Answers:

   a) Direction of opening: up
   Vertex: $(-1, 0)$
   Axis of Symmetry: $x = -1$
   Narrower than $y = x^2$

   b) Direction of opening: down
   Vertex: $(1, 6)$
   Axis of Symmetry: $x = 1$
   Wider than $y = x^2$

   c) Direction of opening: up
   Vertex: $(-6, -10)$
   Axis of Symmetry: $x = -6$
   Narrower than $y = x^2$

   d) Direction of opening: up
   Vertex: $(1, 8)$
   Axis of Symmetry: $x = 1$
   Narrower than $y = x^2$

5. Without making a table of values, sketch the graph of each of the following functions. Explain the various steps. State the vertex and the equation of the axis of symmetry. Use graph paper.

   a) $y = 2(x + 1)^2$

   b) $y = -\frac{1}{2}(x - 1)^2 + 6$

   c) $y = 2(x + 6)^2 - 10$

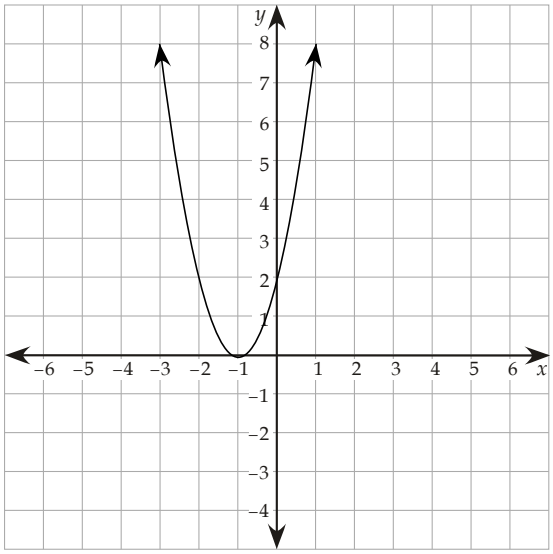
   d) $y = 6(x - 1)^2 + 8$

   Answers:

   a) $y = x^2$ is stretched vertically by a factor of 2 and shifted horizontally to the left by 1 unit.

   Vertex: (−1, 0)

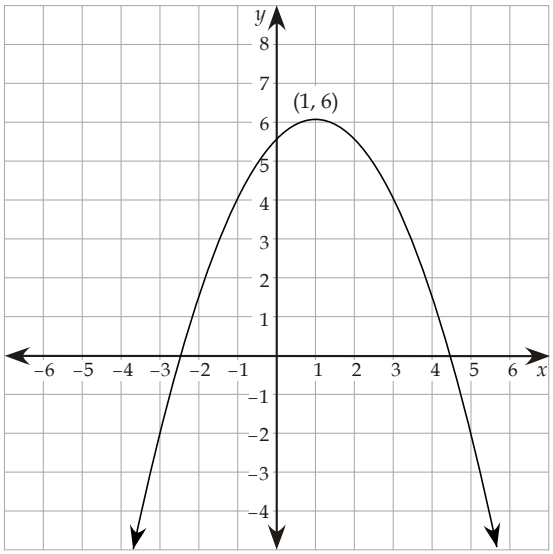
   Axis of symmetry: $x = -1$



b) $y = x^2$ is compressed vertically by a factor of 2, reflected over the x-axis, shifted horizontally to the right one unit and up 6 units.

Vertex: (1, 6)

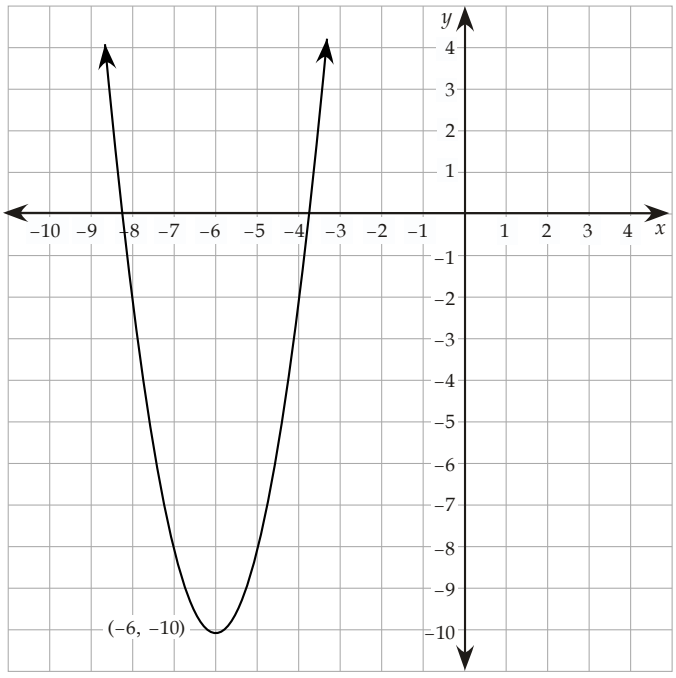
Axis of symmetry: $x = 1$



c) $y = x^2$ is stretched vertically by a factor of 2, shifted left 6 units and down 10 units.

Vertex: (−6, −10)

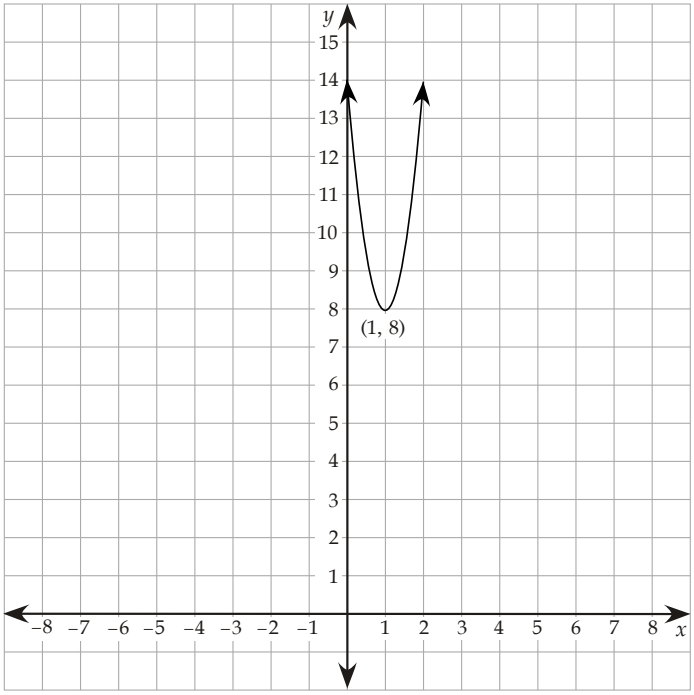
Axis of symmetry: $x = -6$



d) $y = x^2$ is stretched vertically by a factor of 6 shifted 1 unit right and 8 units up.

Vertex: (1, 8)

Axis of symmetry: $x = 1$



6. For each of the following quadratic functions,
   - specify whether the function has a minimum or a maximum value and what that value is.
   - specify the range.
   - find the y-intercept.

   a) $f(x) = -\left(x + \frac{1}{2}\right)^2 + 2$

   b) $y = 2(x + 4)^2 - 2$

   c) $g(x) = -3(x - 2)^2 - 1$

   d) $y = \frac{1}{2}\left[(x + 1)^2 + 4\right]$

Answers:

a) Maximum value at 2

Range: $\{y \mid y \leq 2\}$

$$f(x) = -\left(x + \frac{1}{2}\right)^2 + 2$$

let $x = 0$

$$f(0) = -\left(0 + \frac{1}{2}\right)^2 + 2$$

$$f(0) = -\left(\frac{1}{4}\right) + 2$$

$$f(0) = 1\frac{3}{4}$$

The y-intercept is $1\frac{3}{4}$.

b) Minimum value at -2

Range: $\{y \mid y \geq -2\}$

$y = 2(x + 4)^2 - 2$

let $x = 0$

$y = 2(0 + 4)^2 - 2$

$y = (2)(16) - 2$

$y = 30$

The y-intercept is 30.

c) Maximum value at -1

Range: $\{y \mid y \leq -1\}$

$g(x) = -3(x - 2)^2 - 1$

let $x = 0$

$g(0) = -3(0 - 2)^2 - 1$

$g(0) = -3(4) - 1$

$g(0) = -13$

The y-intercept is -13.

d) Remove the square brackets first by distributing the $\frac{1}{2}$ through the brackets to get $y = \frac{1}{2}(x+1)^2 + 2$.

Minimum value at 2

Range: $\{y \mid y \geq 2\}$

$y = \frac{1}{2}\left[(x+1)^2 + 4\right]$

let $x = 0$

$y = \frac{1}{2}\left[(0+1)^2 + 4\right]$

$y = \frac{1}{2}[1+4]$

$y = \frac{5}{2}$

The y-intercept is $\frac{5}{2}$.

7. If $\left(1, \frac{3}{2}\right)$ is a point on the quadratic function represented by $y = (x-2)^2 + q$, what real number is the minimum value of the function?

Answer:

$y = (x-2)^2 + q$

Substitute $x = 1$ and $y = \frac{3}{2}$ into the above equation.

$\frac{3}{2} = (1-2)^2 + q$

$\frac{3}{2} = 1 + q$

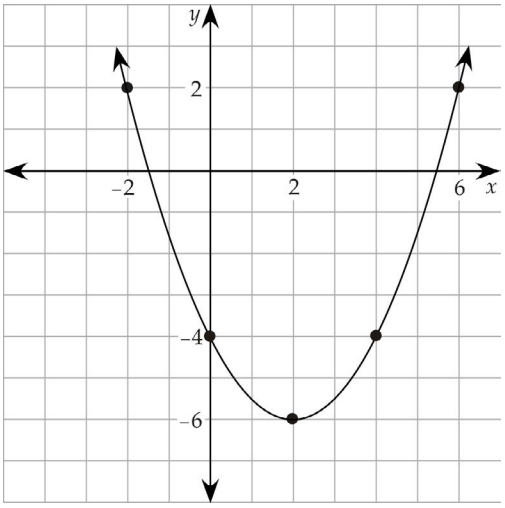
$\frac{3}{2} - \frac{2}{2} = q$

$\frac{1}{2} = q$

q represents the minimum value because the parabola opens up.

Therefore, the minimum value is $\frac{1}{2}$.

8. Write a quadratic function in the form $y = a(x - p)^2 + q$ that represents the following graph.



Answer:

Method #1: Algebraically

To use Method #1, you need to find the coordinates of the vertex and one other point. The coordinates of the vertex are (2, −6), and one other point is (4, −4).

Now, in vertex form, the coordinates of the vertex are (p, q).

$(p, q) = (2, -6)$

$\therefore \quad p = 2$

$q = -6$

Substituting these values for p and q into the vertex form of the quadratic function:

$y = a(x - 2)^2 - 6$

You also have another point on the curve, (4, −4). You can substitute these values into the quadratic function for x and y.

$-4 = a(4 - 2)^2 - 6$

$-4 = a(2)^2 - 6$

$-4 + 6 = 4a$

$2 = 4a$

$\frac{1}{2} = a$

Therefore, the quadratic function representing this graph is $y = \frac{1}{2}(x - 2)^2 - 6$.

Method #2: Graphically

To use Method #2, you need to compare this graph with the graph of $y = x^2$. The vertex is located at $(2, -6)$. Therefore, $p = 2$ and $q = -6$. In other words, this graph is shifted two units to the right and six units down.

The graph opens up, so the value of a is greater than 0.

The graph is wider than the graph of $y = x^2$. Therefore, $0 < a < 1$.

To determine the value of a, compare the vertical distances of points on this new parabola to the graph of $y = x^2$.

Compare the point $(2, 4)$ on $y = x^2$ to the point $(4, -4)$ on the translated parabola. Since the vertical distance from the vertex is half as much, the y-value is halved, and the value of $a = \frac{1}{2}$.

The quadratic function in vertex form is $y = \frac{1}{2}(x - 2)^2 - 6$.

9. A graph has the following properties:
   - The vertex is at $(-2, 5)$
   - The graph passes through the point $(2, -35)$

   Write a quadratic function in the form $y = a(x - p)^2 + q$ that represents this graph.

   Answer:

   The vertex is at $(-2, 5)$. Therefore, $p = -2$ and $q = 5$.

   $y = a(x + 2)^2 + 5$

   Another point is $(2, -35)$. Substitute these values into the equation $y = a(x + 2)^2 + 5$ to find the value of a.

   $$-35 = a(2 + 2)^2 + 5$$
   $$-35 - 5 = a(4)^2$$
   $$-40 = 16a$$
   $$\frac{-40}{16} = a$$
   $$\frac{-5}{2} = a$$

   Therefore, the quadratic function is $y = \frac{-5}{2}(x + 2)^2 + 5$.

10. Determine the number of intercepts of each quadratic function by considering the values of a and q.

    a) $y = -2(x - 3)^2 - 6$

    b) $y = 4x^2 - 1$

    c) $y = 6x^2$

    d) $y = -(x + 1)^2$

    Answers:

    a) In this quadratic function, $a = -2$ and $q = -6$. Therefore, both a and q are less than zero. This means that the corresponding graph of the quadratic function has no x-intercepts.

    b) In this quadratic function, $a = 4$ and $q = -1$. Therefore, a is greater than zero while q is less than zero. This means that the corresponding graph of the quadratic function has two x-intercepts.

    c) In this quadratic function, $a = 6$ and $q = 0$. Therefore, a is greater than zero while q equals zero. This means that the corresponding graph of the quadratic function has one x-intercept.

    d) In this quadratic function, $a = -1$ and $q = 0$. Therefore, a is less than zero while $q = 0$. There are no rules for this situation on the chart presented in this lesson. However, $q = 0$. Therefore, this graph sits on the x-axis. Also, this graph is a reflection through the x-axis of the graph $y = (x + 1)^2$. The graph of $y = (x + 1)^2$ has one x-intercept. Therefore, if you reflect this graph, which is located on the x-axis, over the x-axis, the resulting graph, $y = -(x + 1)^2$, will still have one x-intercept.

11. Redo question 10 above by considering the vertex (p, q) and the direction of opening using the value of a. Make a quick sketch.

    Answer:

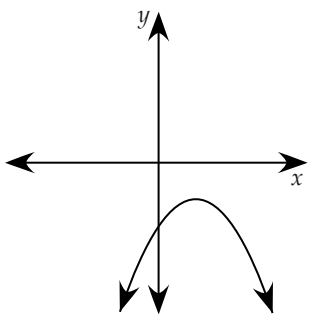
    a) $y = -2(x - 3)^2 - 6$

    Vertex $(3, -6)$ and a is negative

    Opens down

    Sketch:

    $\therefore$ no x-intercepts



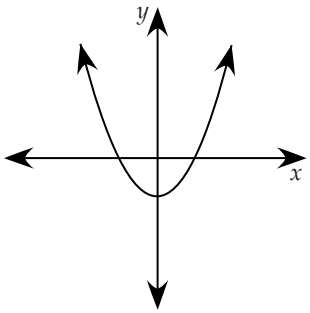
b) $y = 4x^2 - 1$

Vertex $(0, -1)$ and a is positive

Opens up

Sketch:

$\therefore$ two x-intercepts



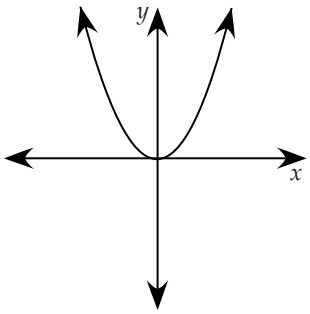
c) $y = 6x^2$

Vertex $(0, 0)$ and a is positive

Opens up

Sketch:

$\therefore$ one x-intercept



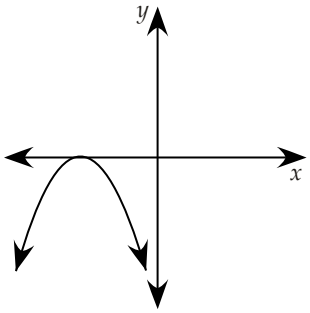
d) $y = -(x + 1)^2$

Vertex $(-1, 0)$ and a is negative

Opens down

Sketch:

$\therefore$ one x-intercept



Learning Activity 3.5

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Factor: $x^2 - 10x + 25$
2. Your computer has a battery life of 12 hours. If you use your computer for 15 minutes in the morning to check the weather, 6 hours during school to complete school work, 25 minutes during lunch to go on your favourite social networking site, and 1.6 hours to watch a movie, how much battery life do you have left on your laptop?
3. Evaluate for $g = -2$: $5g^3 + g^2 - 6g - 13$
4. Continue the pattern: 2, 6, 18, 54, _____ , _____
5. You are at the store to buy your sister a birthday present. You have $50 to spend and the item you want to purchase is $60 on sale for 25% off. Will you have enough money to buy the item (ignoring tax)?
6. Simplify: $(x^2 - 4x + 1)(2x^2)$
7. Find the midpoint between the following two points: (2, 6) and (−7, −4)
8. What is the slope of a horizontal line?

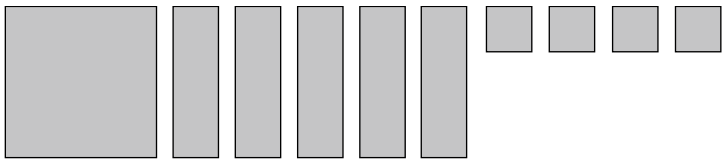
Answers:

1. $(x - 5)(x - 5)$ or $(x - 5)^2$
2. 3 hours and 44 minutes (You used your computer for 15 minutes + 6 hours + 25 minutes + 1 hour + 36 minutes = 7 hours and 76 minutes = 8 hours and 16 minutes. 12 hours − 8 hours and 16 minutes = 3 hours and 44 minutes.)
3. −37 ($5(-8) + 4 + 12 - 13 = -40 + 16 - 13 = -53 + 16 = -37$)
4. 162, 486
5. Yes (25% of $60 = $15; $60 − $15 = $45)
6. $2x^4 - 8x^3 + 2x^2$
7. $\left(-\frac{5}{2}, 1\right)$ $\left(\text{M} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{2 + (-7)}{2}, \frac{6 + (-4)}{2}\right) = \left(-\frac{5}{2}, \frac{2}{2}\right) = \left(-\frac{5}{2}, 1\right)\right)$
8. 0

Part B: Completing the Square

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Write a quadratic expression modelled by this set of tiles.



Answer:

$x^2 + 5x + 4$

2. Draw a tile model to represent $x^2 + 3x + 2$ in the shape of a rectangle.

Answer:

x^2	x	x
x	1	1

3. What value of k makes the expression a perfect square trinomial? Write each expression as the square of a binomial.

a) $x^2 + 8x + k$ b) $x^2 - 8x + k$

c) $x^2 + 20x + k$ d) $x^2 - 2x + k$

e) $x^2 - 5x + k$ f) $x^2 + 7x + k$

Answers:

a) $x^2 + 8x + k$

$$k = \left(\frac{8}{2}\right)^2 = 16$$

$$x^2 + 8x + 16 = (x + 4)^2$$

b) $x^2 - 8x + k$

$$k = \left(\frac{-8}{2}\right)^2 = 16$$

$$x^2 - 8x + 16 = (x - 4)^2$$

c) $x^2 + 20x + k$

$$k = \left(\frac{20}{2}\right)^2 = 100$$

$$x^2 + 20x + 100 = (x + 10)^2$$

d) $x^2 - 2x + k$

$$k = \left(\frac{-2}{2}\right)^2 = 1$$

$$x^2 - 2x + 1 = (x - 1)^2$$

e) $x^2 - 5x + k$

$$k = \left(\frac{-5}{2}\right)^2 = \frac{25}{4}$$

$$x^2 - 5x + \frac{25}{4} = \left(x - \frac{5}{2}\right)^2$$

f) $x^2 + 7x + k$

$$k = \left(\frac{7}{2}\right)^2 = \frac{49}{4}$$

$$x^2 + 7x + \frac{49}{4} = \left(x + \frac{7}{2}\right)^2$$

4. Find the values of a, p, and q for each of the following.

   a) $x^2 - 8x = a(x - p)^2 + p$

   b) $x^2 + 5x = a(x - p)^2 + p$

   c) $\frac{1}{2}x^2 + 8x + 4 = a(x - p)^2 + p$

   Answers:

   Each solution shows a different method for completing the square. Whichever one you choose to use is fine. All of them will be used throughout the rest of the module, but you only need to know and use one.

   a) $x^2 - 8x = a(x - p)^2 + q$

   let $y = x^2 - 8x$

   $y + 16 = x^2 - 8x + 16$

   $y + 16 = (x - 4)^2$

   $y = (x - 4)^2 - 16$

   Thus, $(x - 4)^2 - 16 = a(x - p)^2 + q$
   and $a = 1$, $p = 4$, $q = -16$

   b) $x^2 + 5x = a(x - p)^2 + q$

   let $y = x^2 + 5x$

   $y = \left(x^2 + 5x + \frac{25}{4} - \frac{25}{4}\right)$

   $y = \left(x + \frac{5}{2}\right)^2 - \frac{25}{4}$

   Thus, $\left(x + \frac{5}{2}\right)^2 - \frac{25}{4} = a(x - p)^2 + q$

   and $a = 1$, $p = -\frac{5}{2}$, $q = -\frac{25}{4}$

c) $\frac{1}{2}x^2 + 8x + 4 = a(x - p)^2 + q$

let $y = \frac{1}{2}x^2 + 8x + 4$

$y = \frac{1}{2}\left(x^2 + 16x \quad\right) + 4$

$y = \frac{1}{2}\left(x^2 + 16x + 64\right) + 4 - 64\left(\frac{1}{2}\right)$

$y = \frac{1}{2}(x + 8)^2 + 4 - 32$

$y = \frac{1}{2}(x + 8)^2 - 28$

Thus, $\frac{1}{2}(x + 8)^2 - 28 = a(x - p)^2 + q$

and $a = \frac{1}{2}$, $p = -8$, $q = -28$

5. For each of the following equations, answer the questions below.
   a) $f(x) = -x^2 + 6x$
   b) $f: x \to 2x^2 + 5x - 3$
   c) $y = -4x - x^2$
      i) Express in the form $f(x) = a(x - p)^2 + q$.
      ii) Find the coordinates of the vertex and equation of the axis of symmetry.
      iii) Find the range.
      iv) Determine the maximum or minimum value.

Answers:

a) $f(x) = -x^2 + 6x$

i) $f(x) = -x^2 + 6x$

$f(x) = -1(x^2 - 6x)$ Factor out -1 to make sure that the coefficient of x^2 is 1.

$f(x) = -1(x^2 - 6x + 9) - 9(-1)$ Complete the square.

$f(x) = -1(x - 3)^2 + 9$ Factor and simplify.

ii) V(3, 9); equation of axis of symmetry: $x = 3$

iii) Range: $\{y \mid y \leq 9\}$

iv) Maximum value of 9.

b) $f: x \rightarrow 2x^2 + 5x - 3$

i) For convenience, replace $f: x \rightarrow$ with y.

$y = 2x^2 + 5x - 3$

$y = 2\left(x^2 + \frac{5}{2}x\right) - 3$ Factor out 2 to make the coefficient of x^2 equal to 1.

$y = 2\left(x^2 + \frac{5}{2}x + \frac{25}{16}\right) - 3 - \frac{25}{16}(2)$ Complete the square.

$y = 2\left(x + \frac{5}{4}\right)^2 - 3 - \frac{25}{8}$ Simplify.

$y = 2\left(x + \frac{5}{4}\right)^2 - \frac{49}{8}$

$f: x \rightarrow 2\left(x + \frac{5}{4}\right)^2 - \frac{49}{8}$

ii) $V\left(-\frac{5}{4}, -\frac{49}{8}\right)$; equation of axis of symmetry: $x = -\frac{5}{4}$

iii) Range: $\left\{y \mid y \geq -\frac{49}{8}\right\}$

iv) Minimum value of $-\frac{49}{8}$

c) $y = -4x - x^2$

i) $y = -x^2 - 4x$ Rearrange.

$y = -\left(x^2 + 4x\right)$ Factor out (−1).

$y - 4 = -\left(x^2 + 4x + \left(\frac{4}{2}\right)^2\right)$ Complete the square and add −4 to each side.

$y = -\left(x + 2\right)^2 + 4$ Factor and add 4.

ii) V(−2, 4); equation of axis of symmetry: $x = -2$

iii) Range: $\{y \mid y \leq 4\}$

iv) Maximum value of 4

6. Given $y = ax^2 + bx + c$, the steps for expressing the function in the vertex form $y = a(x - p)^2 + q$ are written below. The coordinates of the vertex are shown. Supply the reasons for each step.

$y = ax^2 + bx + c$ Given

$y - c = ax^2 + bx$ Subtract c

$\frac{y - c}{a} = \frac{ax^2}{a} + \frac{b}{a}x$ Divide by a

$\frac{y - c}{a} = x^2 + \frac{b}{a}x$ Simplify

$\frac{y - c}{a} + \left(\frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2$ Complete the square

$\frac{y - c}{a} + \frac{b^2}{4a^2} = \left(x + \frac{b}{2a}\right)^2$ Factor

$\frac{y - c}{a} = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}$ Subtract $\frac{b^2}{4a^2}$

$y - c = a\left(x + \frac{b}{2a}\right)^2 - a \cdot \frac{b^2}{4a^2}$ Multiply by a

$y = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$ Add c and simplify

$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$ Simplify

$V\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$ Write the vertex

You can use $x = -\frac{b}{2a}$ for finding the x-coordinate at the vertex if the given equation is in the form $y = ax^2 + bx + c$. You can substitute that value into the quadratic equation to find the corresponding y-coordinate.

You can write this formula on your resource sheet.

7. Dominic was asked to complete the square. His work is shown below.

$$x^2 + 8x + 10 = y$$
$$x^2 + 8x = y - 10$$
$$x^2 + 8x + 64 = y - 10 + 64$$
$$(x + 4)^2 = y + 54$$
$$y = (x + 4)^2 - 54$$

Identify, explain, and correct the error in his work. Verify your answer is correct by completing the square using a different method.

Answer:

When completing the square, Dominic forgot to divide eight by two before squaring. Therefore, Dominic added 64 to both sides when he should have added 16 to both sides. The correct completion of the square is:

$$x^2 + 8x = y - 10$$
$$x^2 + 8x + 16 = y - 10 + 16$$
$$(x + 4)^2 = y + 6$$
$$y = (x + 4)^2 - 6$$

Verification:

$$x^2 + 8x + 10 = y$$
$$(x^2 + 8x \quad) + 10 = y$$
$$(x^2 + 8x + 16) + 10 - 16 = y$$
$$(x + 4)^2 - 6 = y$$

This is the same answer as the one obtained above.

8. Explain why $a(x - p)^2 + q = y$ is a quadratic function.

   Answer:

   A quadratic function is a function in the form $ax^2 + bx + c = y$.

   If you multiply out the function $a(x - p)^2 + q = y$, you get:

   $a(x^2 - 2px + p^2) + q = y$

   $ax^2 - 2apx + (ap^2 + q) = y$

   Therefore, when expanded, the quadratic function in vertex form is now in the form $ax^2 + bx + c = y$, where $b = -2ap$ and $c = ap^2 + q$.

Learning Activity 3.6

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. There are 15 females and 12 males in your family. On average, each female eats 2 pieces of pizza and each male eats 3 pieces of pizza. How many pizzas will you have to buy in order for everyone to eat as much pizza as they wish if each pizza is cut into 12 pieces?
2. Multiply: $(3x + 4)(2x - 7)$
3. Evaluate for $n = 4$: $4n^2 + n - 9$
4. Calculate 30% of 42 000.
5. What is the perimeter of a square with an area of 169 units?
6. Factor completely: $4x^4 - 20x^3 + 24x^2$
7. Find the slope of the line defined by the equation $7x + 3y - 4 = 0$.
8. Multiply. $(3x^2 + 7)(8x - 4)$

Answers:

1. 6 pizzas (15 × 2 = 30 pieces of pizza; 12 × 3 = 36 pieces of pizza; 30 + 36 = 66 pieces of pizza; 6 × 12 = 72, so you need 6 pizzas but will have some left)
2. $6x^2 - 13x - 28$
3. 59 (4(16) + 4 − 9 = 64 + 4 − 9 = 59)
4. 12 600 (10% of 42,000 = 4200; 30% of 42,000 = 4200 × 3 = 12 600)
5. 52 units (side length = $\sqrt{169} = 13$; perimeter = 13 × 4 = 52 units)
6. $4x^2(x - 2)(x - 3)$ [$4x^4 - 20x^3 + 24x^2 = 4x^2(x^2 - 5x + 6)$, which factors further to $4x^2(x - 2)(x - 3)$]
7. $-\frac{7}{3}$ $\left(3y = -7x + 4;\ y = -\frac{7}{3}x + \frac{4}{3}\right)$
8. $24x^3 - 12x^2 + 56x - 28$

Part B: Special Features of Quadratic Equations

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Given the following x-intercepts and a-value,

   a) $x = -1$, $x = 3$, $a = 1$

   b) $x = -4$, $x = -2$, $a = -1$

   find the following and then sketch the graph:

   i) vertex

   ii) equation of axis of symmetry

   iii) y-intercept

   iv) direction of opening

   v) maximum value

   vi) minimum value

   vii) domain and range (state in both notations)

   Answers:

   a) i) The x-intercepts are -1 and 3.

   The x-coordinate of the vertex is the average of the x-intercepts:

   $$\frac{-1+3}{2} = \frac{2}{2} = 1$$

   The equation for the function is $y = a(x - 1)^2 + q$.

   The value for a is given as 1. One point on the curve is at an x-intercept—say, $x = 3$. Using the point (3, 0), solve for q.

   $0 = 1(3 - 1)^2 + q$

   $0 = 4 + q$

   $-4 = q$

   The function is $y = (x - 1)^2 - 4$.

   The vertex is V(1, -4).

   ii) Axis of symmetry equation: $x = 1$

iii) y-intercept

Let $x = 0$ to find y-intercept for $y = (x - 1)^2 - 4$

$$y = (0 - 1)^2 - 4$$
$$y = 1 - 4$$
$$y = -3$$

iv) Opens upward because a is positive

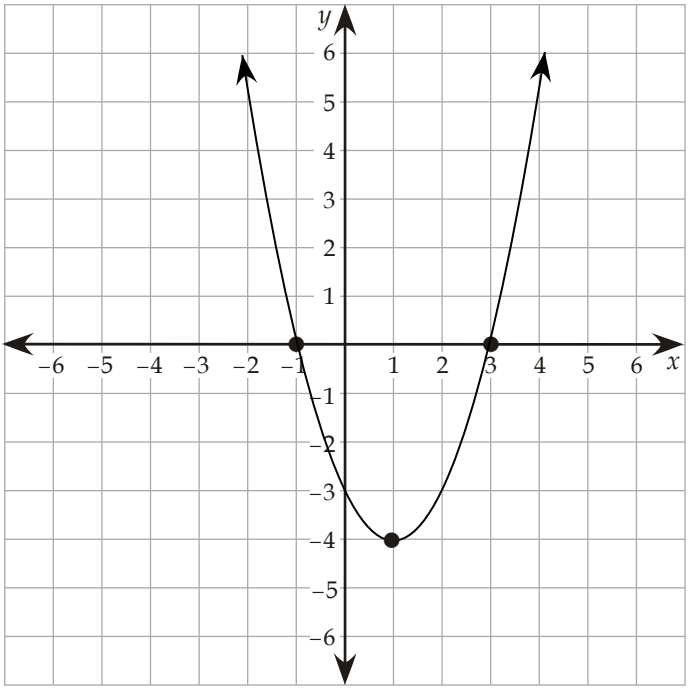
v) Maximum value is not defined

vi) Minimum value corresponding to the y-coordinate of the vertex $= -4$

vii)

Set Notation	Interval Notation
D: $\{x \mid x \in \Re\}$	$(-\infty, \infty)$
R: $\{y \mid y \geq -4, y \in \Re\}$	$[-4, \infty)$

Graph



b) i) The x-intercepts are -4 and -2.

x-coordinate at vertex is the average of the x-intercepts:

$$\frac{-4+(-2)}{2} = \frac{-6}{2} = -3$$

The equation for the function is $y = a(x + 3)^2 + q$.

Using the x-intercept to get a point on the curve, solve for q.

x-intercept = -4

point $(-4, 0)$ and $a = -1$

$y = a(x + 3)^2 + q$

$0 = -1(-4 + 3)^2 + q$

$0 = -1 + q$

$1 = q$

The function is $y = -(x + 3)^2 + 1$.

The vertex is V$(-3, 1)$.

ii) Axis of symmetry equation: $x = -3$

iii) y-intercept

Let $x = 0$ for $y = -(x + 3)^2 + 1$

$y = -(0 + 3)^2 + 1$

$y = -9 + 1$

$y = -8$

iv) Opens downward, since a is negative

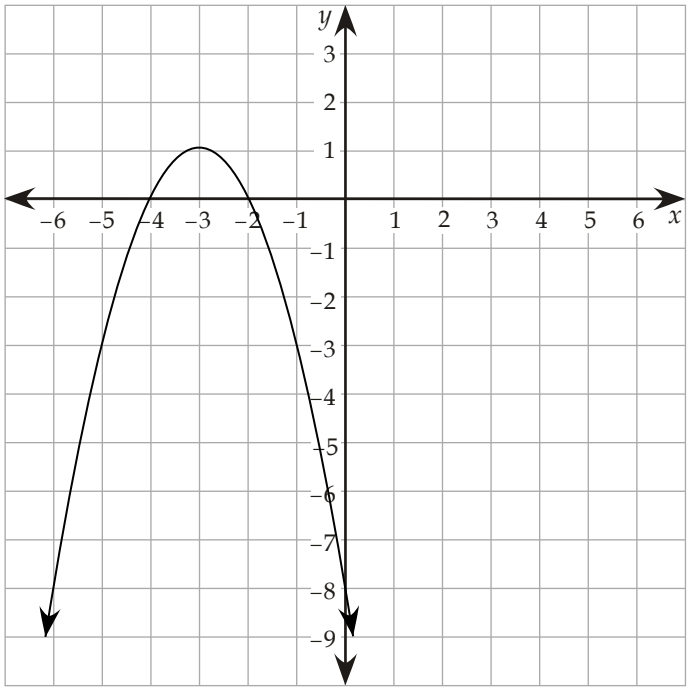
v) Maximum value is 1

vi) No minimum value

vii)

Set Notation	Interval Notation
D: $\{x \mid x \in \Re\}$	$(-\infty, \infty)$
R: $\{y \mid y \leq 1, y \in \Re\}$	$(-\infty, 1]$

Graph



2. For each of the following quadratic functions
   a) $y = x^2 + 6x - 7$
   b) $y = x^2 - 4x - 60$
   c) $y = 2x^2 + 8x - 10$
   d) $y = 3x^2 + 24x + 21$

   find the following:

   i) The coordinates of the vertex using $p = -\frac{b}{2a}$ and substituting to find q
   ii) The equation of the axis of symmetry
   iii) The y-intercept
   iv) The direction of opening
   v) The maximum or minimum value
   vi) The domain and range in interval notation

Answers:

a) $y = x^2 + 6x - 7$

i) x-coordinate of vertex = $\frac{-b}{2a}$

$a = 1 \qquad b = 6$

$p = \frac{-6}{2(1)} = -3$

y-coordinate of the vertex:

$q = (-3)^2 + 6(-3) - 7$

$q = 9 - 18 - 7$

$q = -16$

V(−3, −16)

ii) Axis of symmetry equation: $x = -3$

iii) y-intercept:

substitute $x = 0$ into:

$y = x^2 + 6x - 7$

$y = (0)^2 + 6(0) - 7$

$y = -7$

iv) Because $a > 0$, the parabola opens upward.

v) Since the parabola opens upward, the function has a minimum value of −16.

vi) D: $(-\infty, \infty)$

R: $[-16, \infty)$

b) $y = x^2 - 4x - 60$

i) x-coordinate of vertex = $\frac{-b}{2a}$

$a = 1 \qquad b = -4$

$p = \frac{-(-4)}{2(1)} = \frac{4}{2} = 2$

y-coordinate of the vertex:

$q = (2)^2 - 4(2) - 60$

$q = 4 - 8 - 60$

$q = -64$

V(2, −64)

ii) Axis of symmetry equation: $x = 2$

iii) y-intercept:

substitute $x = 0$ into:

$y = x^2 - 4x - 60$

$y = (0)^2 - 4(0) - 60$

$y = -60$

iv) Opens upward, since $a > 0$

v) Since the parabola opens upward, the function has a minimum value at −64.

vi) D: $(-\infty, \infty)$

R: $[-64, \infty)$

c) $y = 2x^2 + 8x - 10$

i) x-coordinate of vertex $= -\frac{b}{2a}$

$a = 2 \qquad b = 8$

$p = \frac{-8}{2(2)} = -2$

y-coordinate of the vertex:

$q = 2(-2)^2 + 8(-2) - 10$

$q = 2(4) - 16 - 10$

$q = -18$

V$(-2, -18)$

ii) Axis of symmetry equation: $x = -2$

iii) y-intercept:

substitute $x = 0$ into:

$y = 2x^2 + 8x - 10$

$y = 2(0)^2 + 8(0) - 10$

$y = -10$

iv) Opens upward because $a > 0$

v) Because the parabola opens upward, the minimum value is -18.

vi) D: $(-\infty, \infty)$

R: $[-18, \infty)$

d) $y = 3x^2 + 24x + 21$

i) x-coordinate of vertex = $\frac{-b}{2a}$

$a = 3 \qquad b = 24$

$p = \frac{-24}{2(3)} = -4$

y-coordinate of the vertex:

$q = 3(-4)^2 + 24(-4) + 21$

$q = 48 - 96 + 21$

$q = -27$

V$(-4, -27)$

ii) Axis of symmetry equation: $x = -4$

iii) y-intercept:

substitute $x = 0$ into

$y = 3x^2 + 24x + 21$

$y = 3(0)^2 + 24(0) + 21$

$y = 21$

iv) Opens upward because $a > 0$

v) Because the parabola opens upward, the minimum value is -27.

vi) D: $(-\infty, \infty)$

R: $[-27, \infty)$

Learning Activity 3.7

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. You want to save up to buy a car worth $6000 in two years. How much will you have to save per month to afford this car?
2. On your first three assignments in Canadian history, you received marks of 82%, 63%, and 77%. What is your average assignment mark in Canadian history?
3. Factor: $2x(x - 4) + 3(x - 4)$
4. What is the domain of the parabola $y = 3x^2 - 5x - 12$?
5. Evaluate when $y = 5$: $\left(\frac{30}{5}\right) - \left(\frac{75}{y}\right)$
6. Estimate: $\sqrt{122}$
7. How many millimetres are in 6.3 metres?
8. What is the greatest common factor of $18x^3y^4z + 6x^2y^6z^3$?

Answers:

1. $250 (Two years = 24 months; $\frac{6000}{24} = \frac{3000}{12} = \frac{1500}{6} = \$250$
2. 74% $\left(82 + 63 + 77 = 222; \frac{222}{3} = 74\right)$
3. $(2x + 3)(x - 4)$
4. Domain: $\{x \mid x \in \Re\}$
5. -9 $\left(\frac{30}{5} - \frac{75}{5} = 6 - 15 = -9\right)$
6. $\approx$11 (122 is between 121 and 144; therefore, $\sqrt{122}$ is between 11 and 12, but closer to 11)
7. 6300 millimetres (1000 millimetres in 1 metre; 6.3 × 1000 = 6300 millimetres)
8. $6x^2y^4z$.

Part B: Applications of Quadratic Functions

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

Each question can be solved by using either completing the square or by using the formula for the vertex. Practice both methods! The solution method given in the answer key is indicated in each question.

1. The height, h, in metres, after the launching of a rocket at any time, t, in seconds, is defined by the following equation.

   $$h = -3t^2 + 9t + \frac{81}{4}$$

   Find the maximum height reached by the rocket and the time it takes to reach this height. Use the completing-the-square method.

   Answer:

   $$h(t) = -3t^2 + 9t + \frac{81}{4}$$

   $$h(t) - \frac{81}{4} + (-3)\left(\frac{-3}{2}\right)^2 = -3\left(t^2 - 3t + \left(\frac{-3}{2}\right)^2\right)$$

   $$h(t) - \frac{81}{4} - \frac{27}{4} = -3\left(t - \frac{3}{2}\right)^2$$

   $$h(t) = -3\left(t - \frac{3}{2}\right)^2 + \frac{108}{4}$$

   $$h(t) = -3\left(t - \frac{3}{2}\right)^2 + 27$$

   The curve opens downward.

   The maximum height of 27 m is reached in 1.5 seconds.

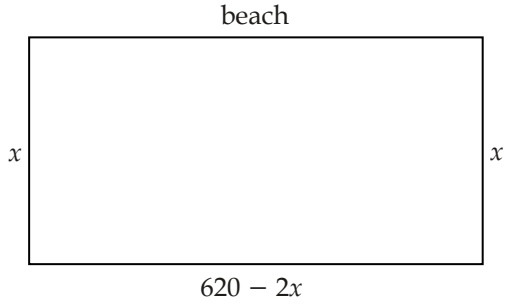
2. At the local beach, the lifeguard has 620 m of marker buoys to rope off a safe swimming area. Calculate the dimensions of the rectangular swimming area to create the maximum swimming area if one side of the area is to be the beach. Use the formula method.

   Answer:

   Let x = width of the swimming area in metres

   $620 - 2x$ = length of the swimming area in metres

   Let A(x) represent the area of the rectangle.



   Remember, you only need three sides roped off.

   $\therefore$ Area: $A(x) = x(620 - 2x)$

   $= 620x - 2x^2$

   The maximum area occurs at the vertex of the quadratic function because the parabola opens down. You need only find the x-coordinate of the vertex.

$$x = \frac{-b}{2a} = -\frac{620}{2(-2)} = 155 \text{ m}$$

   When the width is 155 m, the length is $620 - 2(155) = 310$ m.

   The dimensions of the swimming area are 155 m × 310 m.

3. If 65 apple trees are planted in an orchard, the average yield per tree will be 1500 apples per year. For each additional tree planted in the orchard, the annual yield per tree drops by 20 apples. How many trees should be planted in order to produce a maximum yield? Use the formula method.

Answer:

	Number of Trees	**Yield per Tree**	**Annual Yield**
originally	65	1500	65(1500)
one additional tree	65 + 1	1500 − 20(1)	(65 + 1)[1500 − 20(1)]
two additional trees	65 + 2	1500 − 20(2)	(65 + 2)[1500 − 20(2)]
n additional trees	65 + n	1500 − 20(n)	(65 + n)[1500 − 20(n)]

Maximum yield is required.

The equation will have a vertex at (n, Y) where:

n = number of additional trees

Y = annual yield

$Y = (65 + n)(1500 - 20n)$

$Y = 97\,500 - 1300n + 1500n - 20n^2$

$Y = -20n^2 + 200n + 97\,500$

The curve opens downward. Thus, we have a maximum at the vertex.

Using the formula method to find the vertex:

$$x = \frac{-b}{2a}$$

$$x = \frac{-200}{2(-20)}$$

$$x = 5$$

Thus, 5 additional trees should be planted. There will be 70 trees altogether in the orchard.

4. The difference between two numbers is 14. Find the two numbers so that their product is a minimum. Use the completing-the-square method.

   Answer:

   Let x = larger of the two numbers

   $x - 14$ = smaller of the two numbers

   Let P(x) = the product of the two numbers

   $$P(x) = x(x - 14)$$

   $$= x^2 - 14x$$

   $$P(x) + 49 = x^2 - 14x + 49$$

   $$P(x) = (x - 7)^2 - 49$$

   Since a = 1, the parabola opens up and a minimum occurs at x = 7. If x = 7, then $x - 14 = -7$.

   The two numbers are 7 and -7, giving a minimum product of -49.

5. Honest John's used car lot sells an average of 20 cars per week at an average price of $6,400 each. Honest John would like to increase the average price by $300; however, he knows that his sales would fall by one car if he does. If the dealer's cost per car is $4000, for what price should he sell the cars to maximize profits? Use the formula method.

   Answer:

Number of Cars Sold	Net Profit per Car	Total Net Profit
20	2400	20(2400)
20 − 1	2400 + 3000(1)	(20 − 1)[2400 + 300(1)]
20 − 2	2400 + 3000(2)	(20 − 2)[2400 + 300(2)]
20 − n	2400 + 3000(n)	(20 − n)[2400 + 300(n)]

   Let P = total net profit

   Let n = number of increases in average price per car (or number of decreases in cars sold)

   $P = (20 - n)(2400 + 300\text{n})$

   $P = 48\,000 + 6000n - 2400n - 300n^2$

   $P = -300n^2 + 3600n + 48\,000$

   The curve opens downward. The maximum occurs at the vertex.

At the vertex, $n = \dfrac{-b}{2a} = \dfrac{-3600}{2(-300)} = 6.$

Honest John should sell his cars at a price, including cost, of:

Price = 2400 + 300(6) + 4000

= 2400 + 1800 + 4000

= $8200

6. A rectangular field is to be enclosed by a fence and divided into three smaller plots by two fences parallel to one of the sides. Find the dimensions of the largest such field if 800 m of fencing is available. Use the formula method.

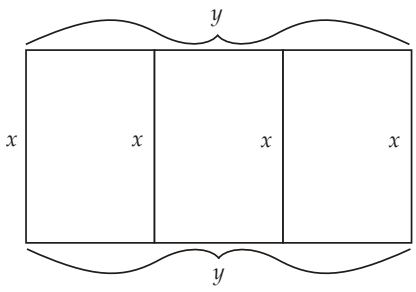
Answer:

x = width

y = length

Total lengths = $2y$

Total widths = $4x$



$2y + 4x = 800$

$y = 400 - 2x$ Divide by two and solve for y.

Area = A

A = length × width

$A = (400 - 2x)x$

$A = -2x^2 + 400x$

Using $x = \dfrac{-b}{2a} = \dfrac{-400}{2(-2)} = 100$ m.

$y = 400 - 2x = 400 - 2(100) = 200$ m

The curve opens down; there is a maximum at the vertex.

The largest area occurs when the dimensions of the field are 100 m × 200 m.

7. Tom Katz's company sold tickets to a Bon Jovi concert. If 1000 or less people purchased tickets, the price was \$60.00 per ticket. For every 100 tickets sold over 1000, the ticket was discounted by \$3.00. What ticket price will give a maximum income? Use the completing-the-square method.

Answer:

Number of Tickets Purchased	Price per Ticket	Income
1000	60	1000(60)
1000 + 100(1)	60 − 3(1)	[1000 + 100(1)][60 − 3(1)]
1000 + 100(2)	60 − 3(2)	[1000 + 100(2)][60 − 3(2)]
1000 + 100(n)	60 − 3(n)	[1000 + 100(n)][60 − 3(n)]

Let n = number of groups of 100 tickets sold (or number of decreases in price of ticket)

Let I = income

$I = (1000 + 100n)(60 - 3n)$

$I = 60\,000 - 3000n + 6000n - 300n^2$

$I = -300n^2 + 3000n + 60\,000$

$I = -300(n^2 - 10n + 25) + 60\,000 - 25(-300)$

$I = -300(n - 5)^2 + 60\,000 + 7500$

$I = -300(n - 5)^2 + 67\,500$

V(5, 67 500) = V(n, I)

The curve opens downward; the maximum occurs at the vertex.

The ticket price for maximum income must be:

$60 - 3(5) = 60 - 15 = \$45.$

Notes

Grade 11 Pre-Calculus Mathematics (30S)

Module 4
Solving Quadratic and Rational Equations

Module 4:
Solving Quadratic and Rational Equations

Introduction

In Module 3, you learned about quadratic *functions*. In this module, you will learn how to solve quadratic *equations*. There are many different methods used to solve quadratic equations including factoring, taking square roots, using the quadratic formula, and even completing the square. In this module, you will learn when it is best to use each method. You will learn how to derive the quadratic formula and important properties of the discriminant (a part of the quadratic formula). The last part of this module introduces rational equations and presents some problems where, in order to solve them, you will be required to use rational equations.

In Module 2, you learned how to add, subtract, multiply, and divide rational expressions. This knowledge will be helpful when you learn how to solve rational equations in this module. As mentioned in Module 2, it is possible to solve application problems involving rational equations, such as motion problems.

Assignments in Module 4

When you have completed the assignments for Module 4, submit your completed assignments to the Distance Learning Unit either by mail or electronically through the learning management system (LMS). The staff will forward your work to your tutor/marker.

Lesson	Assignment Number	Assignment Title
2	Assignment 4.1	Solving Quadratic Equations by Graphing and Taking Square Roots
4	Assignment 4.2	Solving Quadratic Equations
6	Assignment 4.3	The Discriminant and Solving Rational Equations

Resource Sheet

When you write your midterm examination, you are encouraged to take a Midterm Examination Resource Sheet with you into the examination. This sheet will be one letter-sized page, 8½ " by 11", with both sides in your handwriting or typewritten. You will submit it with your examination, but you do not receive any marks for it.

Many students have found that preparing a resource sheet is an excellent way to review. It provides you with a summary of the important facts of each module. You should complete a resource sheet for each module to help with your studying and reviewing. Lesson summaries and module summaries are included for you to use as a guide.

You may use the list of instructions provided below to help you with preparing your resource sheet for the material in Module 4. On this sheet, you should record math terms and definitions, formulas, sample questions, or a list of places where you often make mistakes. You should also identify special areas that require extra attention or review by writing the page numbers.

After you have completed each module's resource sheet, you may summarize the sheets from Modules 1, 2, 3, and 4 to prepare your Midterm Examination Resource Sheet. The midterm examination for this course is based on Modules 1 to 4.

Resource Sheet for Module 4

As you go through the lessons of this module, you may want to consider the following suggestions regarding the creation of a resource sheet.

1. List all the important math terms, and define them if necessary.
2. List all the formulas and perhaps a sample problem that shows how the formula is used.
3. If necessary, write the solutions to some problems, showing in detail how you did the calculations.
4. Copy any questions that represent the key points of the lesson, and perhaps include the solutions as well.
5. Identify the problems you found most difficult, and copy the page numbers onto the resource sheet so that you can review them before writing the examination. You may also copy the problems and the solutions onto your resource sheet, and later write them onto your Midterm Examination Resource Sheet.
6. Write any comments, ideas, shortcuts, or other reminders that may be helpful during an examination.

Writing Your Midterm Examination

You will write the midterm examination when you have completed Module 4 of this course. The midterm examination is based on Modules 1 to 4, and is worth 20 percent of your final mark in the course. To do well on the midterm examination, you should review all the work you complete in Modules 1 to 4, including all the learning activities and assignments. You will write the midterm examination under supervision.

Notes

Lesson 1: Solving Quadratic Equations Graphically

Lesson Focus

In this lesson, you will

- ❑ learn how to solve quadratic equations graphically
- ❑ how to recognize the connections between the zeros of a quadratic function, the x-intercepts of the graph of a quadratic function, and the roots of the quadratic equation

Lesson Introduction

In Module 3, you learned how to graph quadratic functions. This will be extremely useful in this lesson when you are asked to solve quadratic equations by graphing. As you worked through Module 3, you learned that a quadratic function has infinitely many points on its graph. When the function takes on the specific value of 0, you have a quadratic equation. The graph of the related quadratic function can be used to solve the quadratic equation. You will learn how to solve quadratic equations in this way in this lesson.

What is a Quadratic Equation?

Quadratic equations and quadratic functions are not the same. Include the following definitions on your resource sheet to help you distinguish between the two.

Note: You should have a definition for a quadratic function on your resource sheet for Module 3.

A **quadratic equation** is an equation that can be written in the general form $ax^2 + bx + c = 0$ where a, b, and c are real numbers, and $a \neq 0$. Note that a, b, and c could be irrational. However, this course will not include any quadratic equations with irrational coefficients.

Notice that a quadratic equation contains only one variable.

The number a is called the **leading coefficient**.

The solutions of a quadratic equation are called its **roots**. The roots of a quadratic equation are also the zeros of the corresponding function or the x-intercepts of the corresponding graph.

Solving Quadratic Equations

In Module 3, you learned how to find the x-intercepts of a quadratic function. You also learned that the x-intercepts of a quadratic function are the same as the zeros of a quadratic function.

Solving Quadratic Equations by Graphing

The x-intercepts and the zeros of a quadratic function are also the same as the *roots* of a quadratic equation. Thus, when you are asked to solve a quadratic equation, this is the same as you being asked to find the x-intercepts or the zeros of the corresponding quadratic function. Consider the following example.

Example 1

a) Find the zeros of the quadratic function by graphing:
$f(x) = x^2 + 8x + 7$

b) Solve the quadratic equation, $-3 = -(x - 2)^2 - 2$, for its roots, by graphing its corresponding function and then finding the x-intercepts from the graph.

Solution

This example asks the same question in three different ways. You are to find the zeros of a function, the roots of a quadratic equation, and the x-intercepts of a graph.

a) The function is $f(x) = x^2 + 8x + 7$. Its zeros occur at the points where the function value or y-value is zero. Those points always occur where the function intersects the x-axis. Thus, you need to find the x-intercepts. You can graph this function by using a calculator or by using algebra. Both methods are shown below.

Method 1: Algebraic Method for Graphing

In Module 3, you learned several methods for graphing a quadratic function in general form, such as changing to vertex form by completing the square, or using the vertex formula, $p = -\frac{b}{2a}$, to find the vertex. Once you find the vertex, you can apply the appropriate transformations to graph the function. The completing-the-square method is shown below.

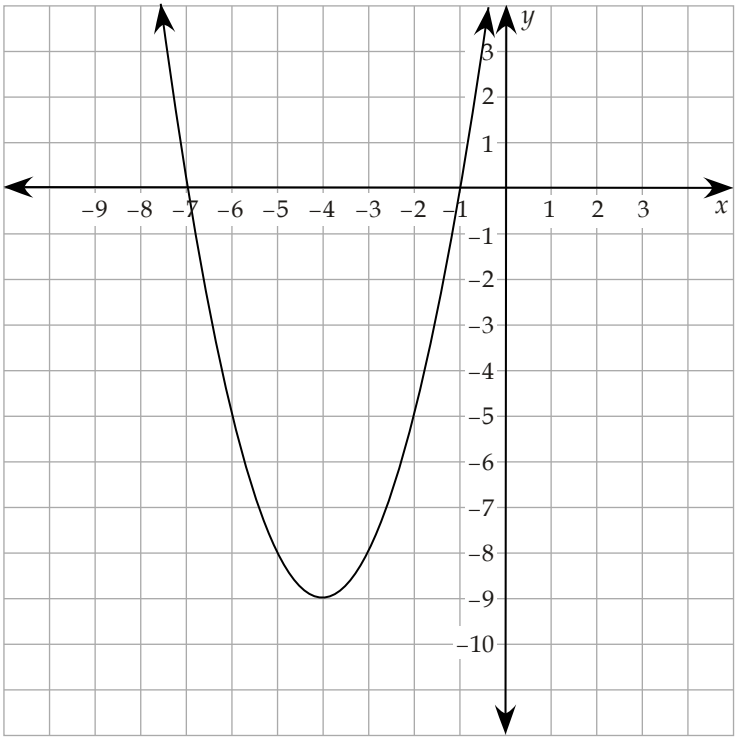
$$\begin{aligned} f(x) &= x^2 + 8x + 7 \\ &= \left(x^2 + 8x + 16\right) + 7 - 16 \\ &= (x + 4)^2 - 9 \end{aligned}$$

The vertex is at (−4, −9). The parabola opens up and is standard size.

You can use your knowledge of the key points on the basic parabola $y = x^2$, and the laws of transformations to plot the points of $f(x) = x^2 + 8x + 7$. (If you need to review these techniques, refer back to Module 3.) The formula for the transformation is $(x, y) \rightarrow (x - 4, y - 9)$.

	Transforming (−1, 1) and (1, 1)	**Transforming (−2, 4) and (2, 4)**	**Transforming (−3, 9) and (3, 9)**
Transformation for the key coordinates for $y = x^2$	Move 1 unit left or right from the vertex and up 1 unit	Move 2 units left or right from the vertex and 4 units up	Move 3 units left or right from the vertex and 9 units up
for $f(x) = x^2 + 8x + 7$ vertex (−4, −9)	(−4 ± 1, −9 + 1) giving the points: (–3, –8) and (–5, –8)	(−4 ± 2, −9 + 4) giving the points: (−2, −5) and (−6, −5)	(−4 ± 3, −9 + 9) giving the points: (−1, 0) and (−7, 0)

The graph is:



The zeros are at the points where the graph intersects the x-axis. These points are (−7, 0) and (−1, 0). The following connections should be made:

- The zeros of the function are −7 and −1.
- The x-intercepts are −7 and −1.
- The solutions of the equation, $x^2 + 8x + 7 = 0$, are −7 and −1.
- The roots of the equation, $x^2 + 8x + 7 = 0$, are −7 and −1.

Note that this graphing method works when the equation is "nice," when the zeros are integer values. However, if the parabola did not cross the x-axis at integral points, then estimating the exact value of the x-intercepts from the graph would be difficult. The graphing method does, however, give you a picture of what is meant by "solving a quadratic equation."

Method 2: Using the TI-83 Graphing Calculator

Refer to the manual that came with your calculator for more specific instructions regarding how to graph equations and find the zeros. The use of a graphing calculator is not required for this course.

b) Start by rearranging the equation into vertex form.

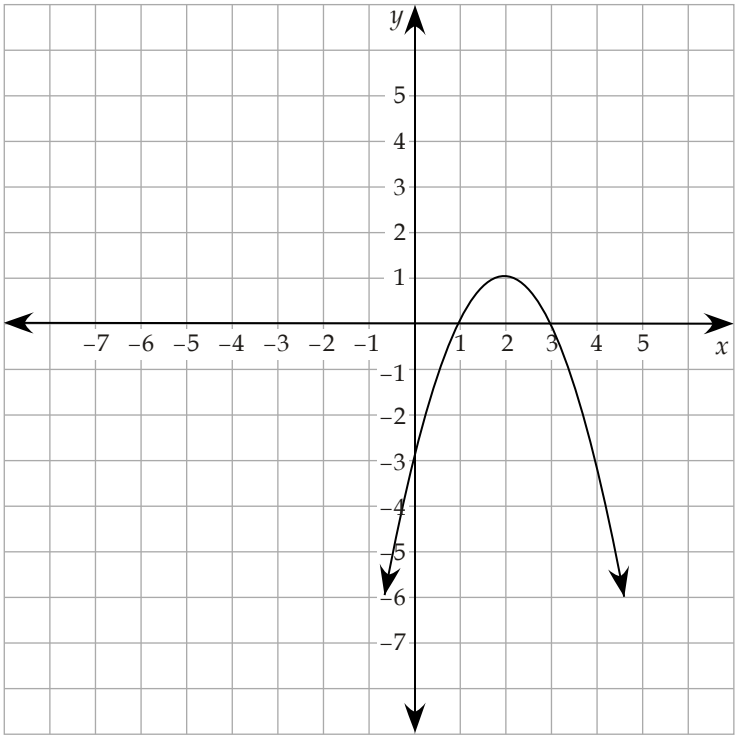
$$-3 = -(x-2)^2 - 2$$

$$0 = -(x-2)^2 - 2 + 3$$

$$0 = -(x-2)^2 + 1$$

Write the corresponding function as $y = -(x-2)^2 + 1$.

Then draw the graph of the function with vertex (2, 1), opening down, and standard size. Its graph is shown below.



The x-intercepts can be read from the graph. The x-intercepts are 1 and 3. Thus, the roots are 1 and 3.

To ensure the validity of these answers, you can substitute the values into the original equation. If the solutions can be verified, then 1 and 3 are the roots.

Verify: $-3 = -(x - 2)^2 - 2$

for $x = 1$	for $x = 3$
LHS $= -3$	LHS $= -3$
RHS $= -(x - 2)^2 - 2$	RHS $= -(x - 2)^2 - 2$
$= -(1 - 2)^2 - 2$	$= -(3 - 2)^2 - 2$
$= -(-1)^2 - 2$	$= -1 - 2$
$= -3$	$= -3$
LHS = RHS	LHS = RHS

Since both solutions are verified, the roots are $x = 1$, $x = 3$.

The following connections should be made for the function $y = -(x - 2)^2 + 1$:

- The curve crosses the x-axis at the points (1, 0) and (3, 0).
- The zeros are 1 and 3.
- The x-intercepts of the graph are 1 and 3.
- The roots of the equation $0 = -(x - 2)^2 + 1$ are 1 and 3.
- The solutions of the equation $0 = -(x - 2)^2 + 1$ are 1 and 3.

If you were not able to read the zeros from the graph, you could use the graphing calculator to solve for the zeros, or you could use an algebraic solving method, as will be shown in the next few lessons.

Note: If $ax^2 + bx + c = 0$ is the given equation, then $ax^2 + bx + c = y$ is the corresponding or related function. Similarly, if $ax^2 + bx + c = y$ is the given function, then $ax^2 + bx + c = 0$ is the corresponding or related equation.

Example 2

Consider the quadratic function $y = x^2 + 2x - 24$.

a) Graph the quadratic function $y = x^2 + 2x - 24$ by completing the square to find the vertex form and then find the x-intercepts from the graph.
b) Check your answer using a graphing calculator, if you have one. Having a graphing calculator is not a necessity for this course.
c) Where does the graph intersect the x-axis?
d) What are the zeros of the function?
e) What are the roots of the equation?
f) Check each root is actually a solution to $x^2 + 2x - 24 = 0$. (This is a good idea since you are reading your answers from the graph rather than solving for them.)

Solution

a) $y = x^2 + 2x - 24$

$y + 24 = x^2 + 2x$

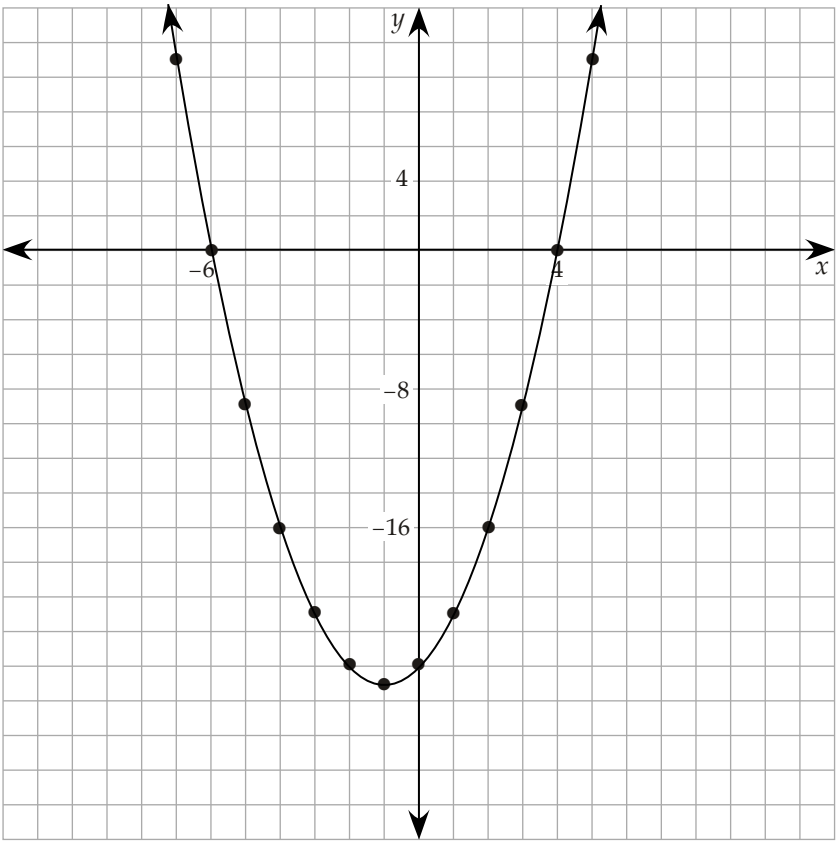
$y + 24 + 1 = x^2 + 2x + 1$

$y + 25 = (x + 1)^2$

$y = (x + 1)^2 - 25$

V(−1, −25)

In the graph shown below, notice that the y-scale is different from the x-scale.



The x-intercepts are −6 and 4.

b) Check your answer using a graphing calculator if you have one. The calculator should confirm that the zeros are −6 and 4.

c) The graph intersects the x-axis at $(-6, 0)$ and $(4, 0)$.

d) The zeros are -6 and 4.

e) The roots of the equation are -6 and 4, they are the same as the zeros of the function.

f) Check by substituting into $x^2 + 2x - 24 = 0$.

For $x = -6$

$\text{LHS} = x^2 + 2x - 24 = 0$

$= (-6)^2 + 2(-6) - 24$

$= 36 - 12 - 24$

$= 0$

$\text{RHS} = 0$

$\therefore$ LHS = RHS

For $x = 4$

$\text{LHS} = x^2 + 2x - 24$

$= (4)^2 + 2(4) - 24$

$= 16 + 8 - 24$

$= 0$

$\text{RHS} = 0$

$\therefore$ LHS = RHS

$\therefore$ Both -6 and 4 are roots of the equation.

Learning Activity 4.1

Complete the following, and check your answers in the learning activity keys found at the end of this module.

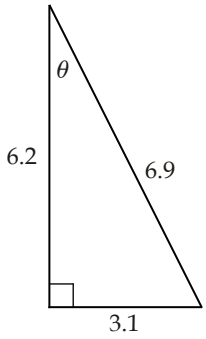
Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Brooklyn can type 55 words per minute. If her essay took her an hour to write, how many words are in her essay?
2. What is the greatest common factor of $3x^5y^3$ and $6x^2y^8$?
3. Find the vertex of the following parabola: $y = -2x^2 + 8x - 1$
4. Solve for x: $6x + 5 = 7$
5. How many centimetres are in one kilometre?

continued

Use the following right-angled triangle to answer the next three questions. Do not simplify your answers.



6. Identify the sine ratio for θ.
7. Identify the cosine ratio for θ.
8. Identify the tangent ratio for θ.

Part B: Solving Quadratic Equations by Graphing

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Rearrange each of the following into the general form of a quadratic equation with the leading coefficient greater than 0.
   a) $-2x^2 + 10 = -6$
   b) $3x(x + 2) = 8$
   c) $5x^2 - 7 + 3x = 7$
   d) $(x + 3)^2 = -4$
   e) $(x + 3)^2 - 4 = 5$
   f) $\dfrac{x^2 - 4}{3} = x$
   g) $(2x - 3)(x + 2) = 0$

continued

Learning Activity 4.1 (continued)

2. Use the graph of the related function to solve each equation.
   a) $0 = (x - 2)^2 - 4$
   b) $0 = x^2 - 6x + 5$
   c) $x^2 + 6x + 1 = -7$
3. a) Graph the quadratic function $y = x^2 - 2x - 8$ by completing the square, and then find the x-intercepts from the graph.
   b) Check your answer using a graphing calculator if you have one.
   c) Where does the graph intersect the x-axis?
   d) What are the zeros of the function?
   e) What are the roots of the equation $x^2 - 2x - 8 = 0$?
   f) Verify each root.
4. Solve each of the following equations by graphing.
   a) $x^2 + 2x - 8 = 0$
   b) $x^2 + 4x + 3 = 0$
   c) $x^2 + 8x = -15$
   d) $9 - x^2 = 0$
   e) $2x^2 - 12x + 10 = 0$

continued

5. Kalia has been asked to solve the quadratic equation $-x^2 + 5x - 4 = 0$ by graphing. Her work for completing the square and graphing the corresponding quadratic function is shown below.

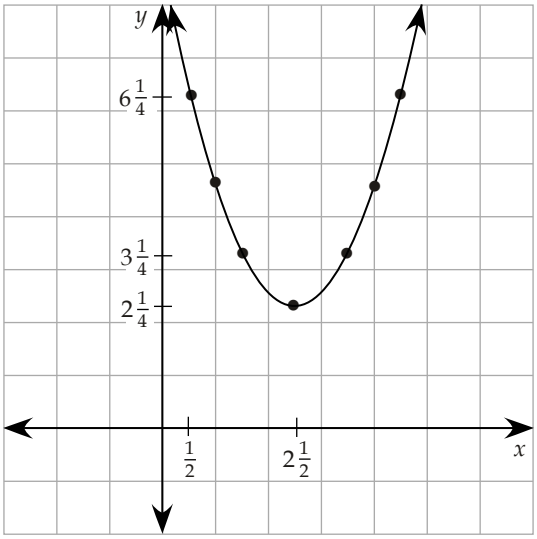
$$y = -x^2 + 5x - 4$$

$$y + 4 = -\left(x^2 - 5x\right)$$

$$y + 4 - \frac{25}{4} = -\left(x^2 - 5x + \frac{25}{4}\right)$$

$$y + \frac{16}{4} - \frac{25}{4} = -\left(x - \frac{5}{2}\right)^2$$

$$y = -\left(x - \frac{5}{2}\right)^2 + \frac{9}{4}$$



From this graph, Kalia concludes there are no x-intercepts.

Identify, explain, and correct Kalia's mistake.

Lesson Summary

In this lesson, you learned how to solve quadratic equations by graphing. To do this, you used the knowledge you gained in Module 3 about completing the square and graphing quadratic functions using transformations. You also learned about the connection between the zeros, the x-intercepts of a graph of a quadratic function, the roots of a quadratic equation, and the solution to a quadratic equation.

In the next lesson, you will learn another algebraic way to solve specific types of quadratic equations by taking square roots.

Notes

Lesson 2: Solving Quadratic Equations by Determining Square Roots

Lesson Focus

In this lesson, you will

- ❑ learn how to solve quadratic equations by taking square roots

Lesson Introduction

In the last lesson, you learned how to solve quadratic equations by graphing. However, sometimes finding roots by graphing a quadratic function can be difficult because they are not integer values. Sometimes it is easier to solve quadratic equations algebraically by determining square roots. It is possible to solve quadratic equations in this way if there is no x-term in the quadratic equation and when you complete the square, as seen in Lesson 4. Thus, you will be able to solve quadratic equations by taking square roots in special circumstances.

Solving Quadratic Equations

Before you learn how to solve quadratic equations by taking square roots, it will be helpful for you to remember how to simplify square roots. In *Grade 10 Introduction to Applied and Pre-Calculus Mathematics*, you learned how to simplify radicals. A **radical** is another name for a **root** (such as a square root or cube root), and the symbol $\sqrt{\ }$ may also be called the **radical symbol**.

You also learned that simplifying a radical expression, such as $\sqrt{32}$, does *not* mean you find the decimal approximation. Instead, you are writing it as another expression with the same exact value, with no perfect square factors of the number under the radical sign.

Step 1: 32 can be written as the product of 16 and 2. To simplify $\sqrt{32}$, write it as $\sqrt{(16)(2)}$ or $\left(\sqrt{16}\right)\left(\sqrt{2}\right)$.

Step 2: The expression $\left(\sqrt{16}\right)\left(\sqrt{2}\right)$ has a perfect square factor of 16, whose root is 4. Rewrite the $\sqrt{16}$ as 4. The final product is an equivalent, simplified expression for $\sqrt{32} = 4\sqrt{2}$.

Example 1

Simplify the following radical expressions.

a) $\sqrt{1244}$

b) $\sqrt{180}$

c) $\sqrt{75}$

Solution

a) 1224 can be written as the product of 36 and 34, where 36 is the largest perfect square factor of 1224.

$$\sqrt{1224} = \left(\sqrt{36}\right)\left(\sqrt{34}\right)$$

36 is a perfect square whose root is 6.

Thus, $6\sqrt{34}$ is the simplified form of $\sqrt{1224}$, as $\sqrt{34}$ cannot be simplified any further. It does not contain any perfect square factors.

b) 180 can be written as the product of 36 and 5.

$$\sqrt{180} = \left(\sqrt{36}\right)\left(\sqrt{5}\right)$$

Again, 36 is a perfect square whose root is 6.

Thus, $6\sqrt{5}$ is the simplified form of $\sqrt{180}$.

c) 75 can be written as the product of 25 and 3.

$$\sqrt{75} = \left(\sqrt{25}\right)\left(\sqrt{3}\right)$$

25 is a perfect square whose root is 5.

Thus, the simplified form of $\sqrt{75}$ is $5\sqrt{3}$.

Solving Quadratic Equations by Taking Square Roots

Now that you have reviewed how to simplify radical expressions, it is possible to learn how to solve quadratic equations by taking square roots. However, you can only solve quadratic equations of this type by taking square roots when they are in the form $ax^2 + c = 0$. In other words, there is no x-term. For equations of this type, in the general form of the quadratic equation, the value of b is zero.

Example 2

Find the value of a, b, and c for the equation $3x^2 - 42 = 0$.

Solution

If you compare the equation with the general form, $ax^2 + bx + c = 0$, then $a = 3$, $b = 0$, and $c = -42$.

Example 3

Solve the equation: $2x^2 - 50 = 0$.

Solution

$2x^2 - 50 = 0$

$2x^2 = 50$	Adding 50 to each side
$x^2 = 25$	Dividing each side by 2
$x = \pm\sqrt{25}$	Taking the square root of each side
$x = \pm 5$	

Note that when you are finding the square root of a variable (x), you need to include both the positive and the negative answers. Therefore, there are two solutions for this equation. There is more discussion on this topic of square roots in Module 5.

You could check your work for Example 3 by graphing the corresponding function, $y = 2x^2 - 50$, and then finding the zeros of the function at the points $(5, 0)$ and $(-5, 0)$.

Example 4

Solve $x^2 = 0$.

Solution

If $x^2 = 0$, then $x = 0$. There is only one solution for this equation.

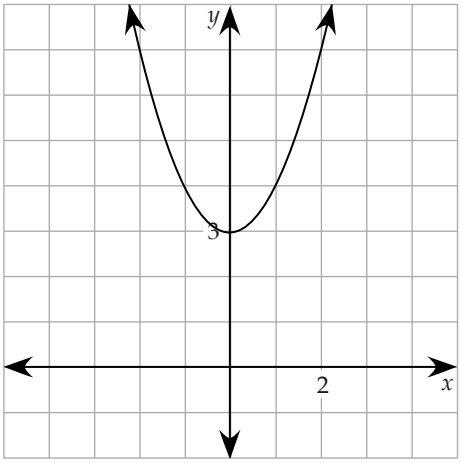
Example 5

Solve $x^2 + 3 = 0$.

Solution

When you arrange the equation and write $x^2 = -3$, you can take the square root of both sides to solve for x. However, you will not get a real number answer. There is no real number solution. No real number squared will give an answer of -3.

If you graph the related function, $y = x^2 + 3$, you get a parabola with vertex at (0, 3), opening up. The parabola does not intersect the x-axis. The function has no zeros. The equation has no real roots. The graph has no x-intercepts.



The following learning activity will allow you to practice the skills you learned in this lesson. After completing this learning activity, make sure you check your answers in the Module 4 Learning Activity Answer Keys to ensure you are completing the questions correctly. This learning activity will help prepare you for the upcoming assignment.

Learning Activity 4.2

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Cameron is packing his suitcase for Mexico. If his suitcase measures 110 cm by 40 cm by 60 cm, what is the volume of his suitcase?
2. You and your friend are going out for dinner and you want to split the bill evenly. If the bill comes to $76.36 and you want to tip the waiter $4, how much should you give the waiter?
3. Factor: $14m - 28$
4. Does the parabola $y = 3x^2 + 2x - 1$ open up or down?
5. Continue the pattern: A, E, F, H, I, K, L, M, _____ , _____
6. Find the midpoint between the two points $(9, -24)$ and $(4, 12)$.
7. Solve the inequality, $13x - 2 > x$.
8. Find the slope of the line parallel to the line $8x = 3y - 2$.

Part B: Solving Quadratic Equations by Taking Square Roots

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Find the roots of the following equations.
   a) $x^2 = 16$
   b) $3x^2 - 18 = 0$
   c) $x^2 - 6 = -5$
   d) $6x^2 + 3 = 2x^2 + 7$

continued

e) $x^2 = 6x^2 - 1$

f) $4x^2 - 9 = -3$

g) $x^2 - 20 = 0$

h) $24 - 2x^2 = 0$

i) $(x - 2)^2 = 5$

j) $\frac{2}{3}x^2 + 6 = 18$

2. A square, ABCD, with sides of length 6 cm, is circumscribed by a circle. The four points, A, B, C, and D, are on the circumference of the circle. Find the area of a circle. Leave π in your answer. Remember if π is not approximated, your answer is exact. If π is approximated, your answer is approximate.

3. Jordon is asked to solve the following quadratic equation by taking square roots. His work is shown below.

$x^2 - 40 = 0$

$x^2 = 40$

$x = \pm\sqrt{40}$

$x = \pm 4\sqrt{10}$

Identify, explain, and correct Jordon's error.

Lesson Summary

In this lesson, you learned another method of solving quadratic equations. First, you reviewed how to simplify radical expressions. Then, you learned how to solve quadratic equations of the form $ax^2 + c = 0$ by taking square roots. In the next lesson, you will learn an additional way of solving quadratic equations by factoring.

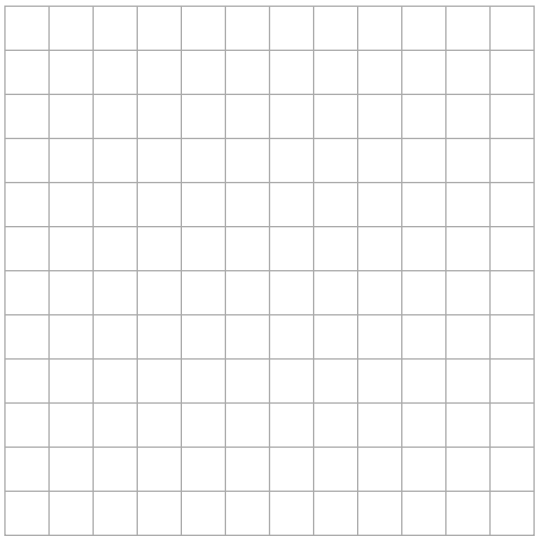
Assignment 4.1

Solving Quadratic Equations by Graphing and Taking Square Roots

Total: 33 marks

1. Use the graph of the related function to find the roots of each equation. *(3 marks each, for a total of 6 marks)*

   a) $0 = (x - 1)^2 - 4$



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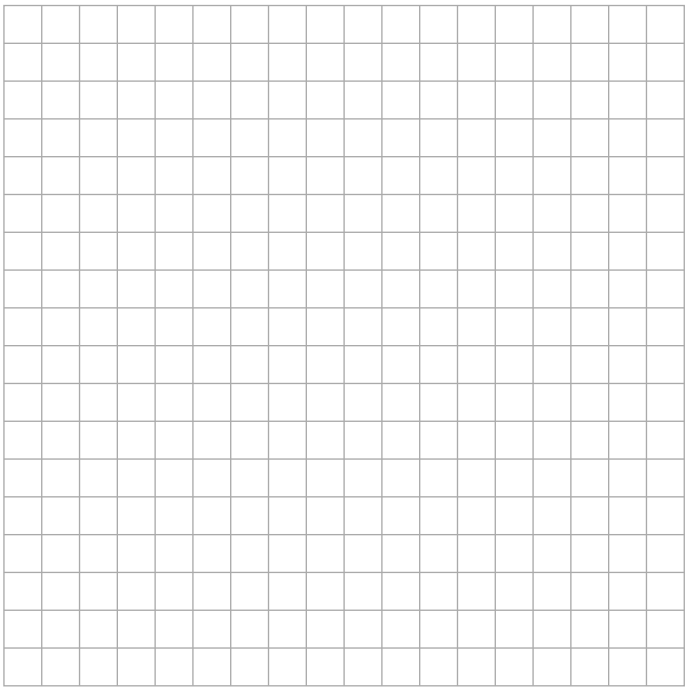
Assignment 4.1: Solving Quadratic Equations by Graphing and Taking Square Roots (continued)

b) $-x^2 + 6x + 7 = 0$



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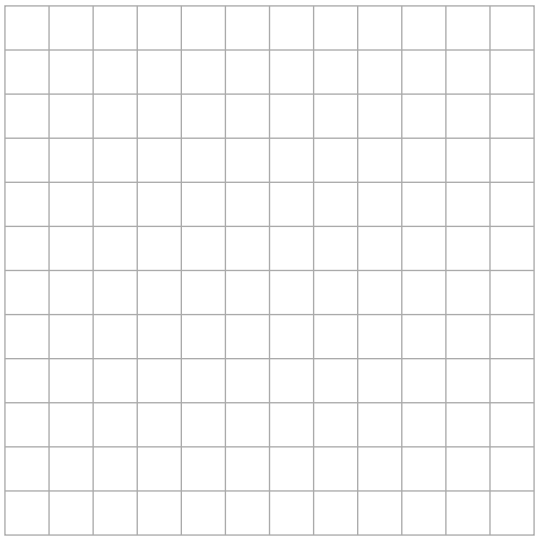
Assignment 4.1: Solving Quadratic Equations by Graphing and Taking Square Roots (continued)

2. a) Graph the quadratic function $y = x^2 + x - 12$. Find the vertex by completing the square. *(3 marks)*



b) Verify the x-intercepts of the function are 3 and -4 by substitution. *(2 marks)*

c) What are the zeros of the function? *(1 mark)*

d) What are the roots of the equation $0 = x^2 + x - 12$? *(1 mark)*

continued

Assignment 4.1: Solving Quadratic Equations by Graphing and Taking Square Roots (continued)

3. Identify, explain, and correct the mistake Julie made when solving the following quadratic equation by taking square roots. Her work is shown below. *(2 marks)*

$4x^2 = 8 + 2x^2$

$2x^2 = 8$

$x^2 = 4$

$x = \sqrt{4}$

$x = 2$

4. Solve the following equations by taking square roots. Write your answers as exact values in simplified form.

   a) $x^2 = 76$ *(1 mark)*

   b) $7x^2 - 21 = 0$ *(2 marks)*

continued

Assignment 4.1: Solving Quadratic Equations by Graphing and Taking Square Roots (continued)

c) $\frac{x^2}{2} - 3 = 5x^2 - 8$ *(4 marks)*

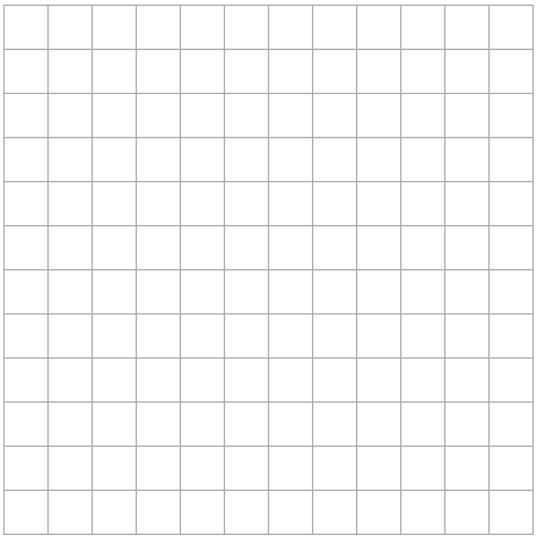
d) $(x + 2)^2 = 4$ *(3 marks)*

continued

Assignment 4.1: Solving Quadratic Equations by Graphing and Taking Square Roots (continued)

5. Solve the following equations by using the graphing method or the square root method. Explain why you chose the graphing or square root method for each equation. *(4 marks each, for a total of 8 marks)*

   a) $x^2 - 3 = 2x$



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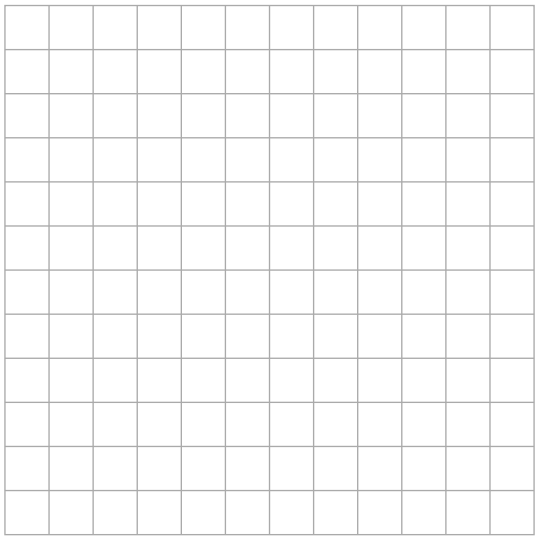
Assignment 4.1: Solving Quadratic Equations by Graphing and Taking Square Roots (continued)

b) $x^2 + 4 = 4x^2$



Notes

Lesson 3: Solving Quadratic Equations by Factoring

Lesson Focus

In this lesson, you will

- ❑ learn how to solve quadratic equations by factoring
- ❑ learn how to apply solving quadratic equations to word problems
- ❑ learn how to write the factored form of a quadratic function

Lesson Introduction

You learned how to solve quadratic equations by graphing in Lesson 1. In some cases, only approximate roots can be determined. However, it is also possible to find the roots of some quadratics by factoring. This method is based on the Zero Product Property. You have encountered this property earlier in this course.

In Lesson 2, you examined quadratic equations of the type $ax^2 + c = 0$, where $b = 0$. You developed a method for solving those types by using inverse operations.

Solving Quadratic Equations

The Zero Product Property states that if $ab = 0$, at least one of a and b has to equal zero. This is also true if a and b are factors of polynomials. Therefore, if $(x - 3)(x + 4) = 0$, either $(x - 3)$ has to equal zero, or $(x + 4)$ has to equal zero. It is thus possible to solve quadratic equations by factoring them, and equating each factor to zero.

Solving Quadratic Equations by Factoring

When using the Zero Product Property to solve quadratic equations, make sure the equation is rearranged into general form, having one side equal to zero. Consider the following examples.

Example 1

Solve $4x^2 - 9 = 0$ by factoring.

Solution

$(2x - 3)(2x + 3) = 0$

$2x - 3 = 0$ or $2x + 3 = 0$

$2x = 3$ $\qquad$ $2x = -3$

$x = \frac{3}{2}$ $\qquad$ $x = \frac{-3}{2}$

You can also use factoring to solve some equations of the type $ax^2 + bx = 0$ or $ax^2 + bx + c = 0$.

Example 2

Solve equations of the form $ax^2 + bx = 0$ by factoring.

a) $3x^2 = 7x$

Solution

$3x^2 - 7x = 0$	Set up equation with one side equal to zero.
$x(3x - 7) = 0$	Factor out the common factor of x.
$x = 0$ or $3x - 7 = 0$	Use the Zero Product Property.
$3x = 7$	Solve for x.
$x = \frac{7}{3}$	

b) $2x^2 - 16x = 0$

Solution

$2x(x - 8) = 0$	Factor out common factor of $2x$.
$2x = 0$ or $x - 8 = 0$	
$x = 0$ or $x = 8$	

Example 3

Solve equations of the form $ax^2 + bx + c = 0$ by factoring.

a) $3x^2 + 4x = 4$

Solution

$3x^2 + 4x - 4 = 0$ Rearrange the equation.

$(3x - 2)(x + 2) = 0$ Factor.

$3x - 2 = 0$ or $x + 2 = 0$

$x = \frac{2}{3}$ $x = -2$

b) $x^2 + 9 = 6x$

Solution

$x^2 - 6x + 9 = 0$

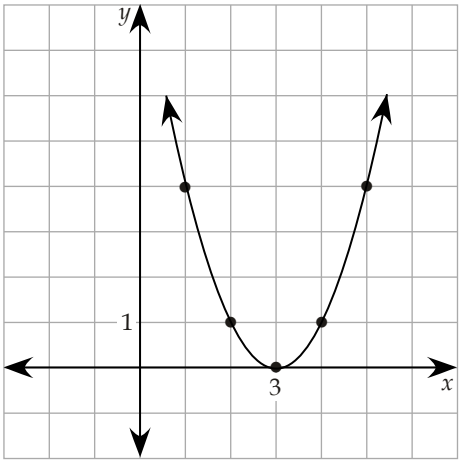
$(x - 3)(x - 3) = 0$

$x - 3 = 0$ or $x - 3 = 0$

$x = 3$ $x = 3$

The answer is 3.

When the same number repeats as a root or a zero, it is called a **double root** or **double zero**. Include this definition on your resource sheet.



In the graph of $x^2 - 6x + 9 = y$, notice that the vertex is on the x-axis. Also, the root is the same number as the x-coordinate of the vertex. This is because double roots occur when the vertex of a parabola touches the x-axis.

c) $2x^2 + 4x - 16 = 0$

Solution

$2(x^2 + 2x - 8) = 0$ Always factor the common factor out first.

$\frac{2(x^2 + 2x - 8)}{2} = \frac{0}{2}$ Divide by 2.

$(x + 4)(x - 2) = 0$ Factor the trinomial.

$x + 4 = 0$ or $x - 2 = 0$

$x = -4$ $x = 2$

d) $x^3 + 5x^2 - 6x = 0$

Solution

This is not a quadratic equation. However, you can still solve it by factoring and see that this method can be extended to equations that are not quadratics.

$x(x^2 + 5x - 6) = 0$ Factor out the common by factoring x.

$x(x + 6)(x - 1) = 0$ Factor the remaining trinomial.

$x = 0$ or $x + 6 = 0$ or $x - 1 = 0$ Use ZPP.

$x = 0$ or $x = -6$ or $x = 1$

Note: There are three roots for this cubic equation.

Solving Quadratic Word Problems

Example 4

The surface area of a rectangular prism is 184 cm^2. Its length is 12 cm more than its width. Its height is 4 cm. Find the dimensions of the prism.

($SA = 2lw + 2lh + 2wh$)

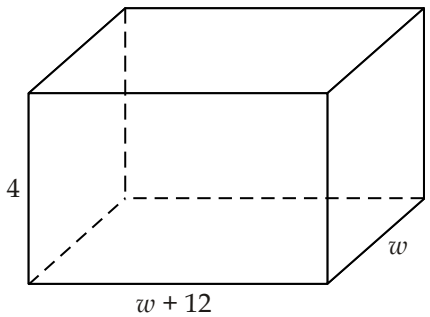
Solution

Let w = width of the prism in cm

$l = w + 12$

Given $h = 4$ cm

$SA = 184$ cm^2



$$SA = 2lw + 2lh + 2wh$$

$$184 = 2(w + 12)(w) + 2(w + 12)(4) + 2(w)(4)$$

$$184 = 2\left[w^2 + 12w + 4w + 48 + 4w\right]$$

$$184 = 2\left[w^2 + 20w + 48\right]$$

$$92 = w^2 + 20w + 48$$

$$0 = w^2 + 20w - 44$$

$$0 = (w + 22)(w - 2)$$

$w + 22 = 0$ $\qquad$ $w - 2 = 0$

$w = -22$ $\qquad$ $w = 2$

Since width must be positive, the only solution is $w = 2$. The dimensions of the prism are:

width = 2 cm

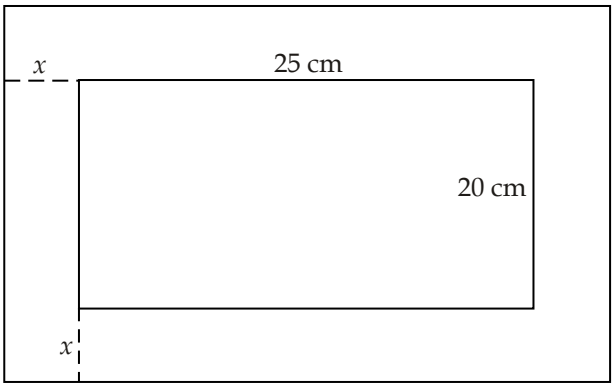
length = 2 + 12 = 14 cm

height = 4 cm

Example 5

A picture with dimensions 25 cm by 20 cm needs to be framed. The frame is to have uniform width and the picture and its frame have an area 150% larger than the area of the picture. Find the dimensions of the picture and its frame.

Solution



Let x = the width of the frame.

Area of picture = 25 × 20 = 500 cm^2.

Area of picture and frame = 500 × 150% = 750 cm^2.

Length of picture and frame = $25 + 2x$

Width of picture and frame = $20 + 2x$

Area = length × width

$$750 = (25 + 2x)(20 + 2x)$$

$$750 = 500 + 50x + 40x + 4x^2$$

$$0 = 4x^2 + 90x - 250$$

$$0 = 2x^2 + 45x - 125$$

$$0 = 2x^2 + 50x - 5x - 125$$

$$0 = 2x(x + 25) - 5(x + 25)$$

$$0 = (2x - 5)(x + 25)$$

$$2x - 5 = 0 \qquad x + 25 = 0$$

$$x = \frac{5}{2} \qquad x = -25$$

Since x must be positive, the only solution is $x = \frac{5}{2}$.

The dimensions of the picture and frame are:

$$\text{length} = 25 + 2x = 25 + 2\left(\frac{5}{2}\right) = 30 \text{ cm}$$

$$\text{width} = 20 + 2x = 20 + 2\left(\frac{5}{2}\right) = 25 \text{ cm}$$

Factored Form of a Quadratic Function

The factored form of a quadratic function includes the leading coefficient, a, and the x-intercepts of the function, x_1 and x_2, and is written:

$$y = a(x - x_1)(x - x_2)$$

Example 6

Write the quadratic function $y = 3x^2 - 3x - 60$ in factored form.

Solution

Factored form requires the values for the leading coefficient and the x-intercepts.

The function is $y = 3x^2 - 3x - 60$.

The leading coefficient, a, is 3.

To find the x-intercepts, you factor the quadratic equation:

$$0 = 3x^2 - 3x - 60$$
$$0 = 3(x^2 - x - 20)$$
$$0 = 3(x - 5)(x + 4)$$
$$x - 5 = 0 \qquad x + 4 = 0$$
$$x = 5 \qquad x = -4$$

The required factored form of the quadratic function is:

$$y = 3(x - 5)(x - (-4))$$
$$y = 3(x - 5)(x + 4)$$

Example 7

The zeros of a function are -6 and 2. The function passes through the point $(-4, 6)$. Write the function in factored form.

Solution

The factored form of a quadratic function is:

$$y = a(x - x_1)(x - x_2)$$

You are given the zeros, or x-intercepts, -6 and 2. The equation becomes:

$$y = a(x - (-6))(x - 2)$$
$$y = a(x + 6)(x - 2)$$

You know the function passes through the point $(-4, 6)$. These values can be substituted into the equation to solve for a.

$$y = a(x+6)(x-2)$$
$$6 = a(-4+6)(-4-2)$$
$$6 = a(2)(-6)$$
$$6 = a(-12)$$
$$-\frac{1}{2} = a$$

The function in factored form is $y = -\frac{1}{2}(x+6)(x-2)$.

Learning Activity 4.3

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Your mom bought 500 g of deli meat. If you put an average of 125 g of meat on each sandwich you make, how many days will the deli meat last for (assuming you only eat one sandwich a day)?
2. You are playing soccer on a co-ed team. You have 20 players on your team and 3 of them are girls. You need at least 20% of your players to be female according to the league. Do you have enough female players?
3. Simplify: $(x + 5)^2$
4. Give a simplified quadratic function that has zeros of -1 and -5 and $a = 1$.
5. Simplify: $2^4 \times 3^2$
6. Express as a power: $\sqrt{15}$
7. Are these fractions equivalent, $\frac{\sqrt{2}}{\sqrt{3}}$ and $\frac{\sqrt{4}}{3}$?
8. The x-intercepts of a parabola are -4 and 2, and the y-intercept is 10. Write the quadratic function in factored form.

continued

Learning Activity 4.3 (continued)

Part B: Solving Quadratic Equations by Factoring

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Solve these equations by factoring.
   a) $x^2 - x - 12 = 0$
   b) $x^2 - x - 20 = 0$
   c) $-x^2 - 2x + 3 = 0$
   d) $x^2 + 9x + 18 = 0$
   e) $2x^2 + 3x - 2 = 0$

2. Rearrange each of the following equations and solve by factoring, if possible.
   a) $10x^2 = 7x + 12$
   b) $5x^2 + 21x = 54$
   c) $3x(x - 2) - x(x + 1) + 5 = 0$
   d) $x^2 + \frac{9}{2}x - 2\frac{1}{2} = 0$
   e) $x^2 + 9 = 0$
   f) $16x^2 - 64 = 0$
   g) $x^2 + 12x - 28 = 0$

3. One leg of a right triangle is 7 m longer than the other leg. The hypotenuse is 17 m long. Find the length of each leg.

4. Two square checkerboards together have an area of 1300 cm^2. One has sides that are 10 cm longer than the other. Find the length of the sides of each checkerboard.

5. Find 3 consecutive odd integers such that the product of the second and the third is 63.

6. A rectangular piece of cardboard is 5 cm longer than it is wide. A 3 cm-by-3 cm square is cut out of each corner, and the four flaps are folded up to form a box with a volume of 450 cm^3. What were the length and width of the original piece of cardboard? (Volume = length × width × height)

continued

Learning Activity 4.3 (continued)

7. Farrah was asked to solve a quadratic equation by factoring. Her work is shown below.

   $x^2 - 10x + 24 = 0$

   $(x - 6)(x - 4) = 0$

   Therefore, $x = -6$ or $x = -4$.

   Identify, explain, and correct the errors Farrah made when trying to solve this quadratic equation.

8. Write the quadratic function $y = -4(x - 3)(x + 1)$ in vertex form using the average of the x-intercepts.

Lesson Summary

In this lesson, you used some of the knowledge you gained in Module 3 to learn how to solve quadratic equations by factoring. You learned that you needed to arrange the equation into general form, having one side equal to zero, before you could factor the quadratic equation. Then, you equated each factor of the quadratic equation to zero to solve the equation. This method only works if you are able to factor the quadratic equation. In the next lesson, you will be learning two new methods of solving quadratic equations. These methods are useful when it may be difficult to graph the quadratic equation and the equation is not easily factorable.

Lesson 4: Solving Quadratic Equations by Completing the Square and by Using the Quadratic Formula

Lesson Focus

In this lesson, you will

- ❑ learn how to solve a quadratic equation by completing the square
- ❑ learn how solve a quadratic equation by using the quadratic formula
- ❑ learn how to solve equations having a quadratic pattern

Lesson Introduction

In previous lessons, you have learned how to solve quadratic equations by factoring, taking square roots and graphing. Some quadratic equations are easier to solve by factoring than by graphing. However, not all quadratic equations are factorable. One technique used to solve quadratic equations that are not easily factorable is called completing the square. This method follows a pattern similar to the method you used to complete the square in Module 3. Another method used to solve quadratic equations that are not easily factorable is the quadratic formula.

Solving Quadratic Equations

In Module 3, you learned how to complete the square. When completing the square to solve a quadratic equation, you must preserve the equality. Therefore, when you add a constant to one side of an equation, you need to add the same constant to the other side of the equation as well.

Solving Quadratic Equations by Completing the Square

There are two cases regarding the solution to $ax^2 + bx + c = 0$ by completing the square.

Case 1: Quadratic equations having a leading coefficient of 1

Example 1

Solve by completing the square: $x^2 - 4x + 2 = 0$.

Solution

$x^2 - 4x + 2 = 0$	Subtract 2 from each side to isolate the variables.
$x^2 - 4x = -2$	
$x^2 - 4x + (-2)^2 = -2 + (-2)^2$	Add the square of one-half the coefficient of x to each side.
$x^2 - 4x + 4 = -2 + 4$	Simplify.
$(x - 2)^2 = -2 + 4$	Factor.
$(x - 2)^2 = 2$	Add.
$x - 2 = \pm\sqrt{2}$	Take the square root of each side.
$x = 2 \pm \sqrt{2}$	Add 2 to each side.

Remember:

- $2 + \sqrt{2}$ and $2 - \sqrt{2}$ are the roots of the quadratic equation
- $2 + \sqrt{2}$ and $2 - \sqrt{2}$ are the zeros of the function
- $2 + \sqrt{2}$ and $2 - \sqrt{2}$ are the x-intercepts of the graph
- $\left(2 + \sqrt{2}, 0\right)$ and $\left(2 - \sqrt{2}, 0\right)$ are the points where the graph crosses the x-axis

Quadratic equations having a leading coefficient that is not 1

If the leading coefficient of the quadratic is not 1, you divide both sides of the equation by the coefficient before you complete the square.

Example 2

Find the roots of the equation by completing the square: $2x^2 - 6x - 7 = 0$

Solution

$$2x^2 - 6x - 7 = 0$$

$$x^2 - 3x = \frac{7}{2}$$ Add 7 to each side and divide by 2.

$$x^2 - 3x + \left(\frac{-3}{2}\right)^2 = \frac{7}{2} + \left(\frac{-3}{2}\right)^2$$ Add the square of $\frac{1}{2}(-3)$ to each side.

$$\left(x - \frac{3}{2}\right)^2 = \frac{7}{2} + \frac{9}{4}$$ Factor and simplify.

$$\left(x - \frac{3}{2}\right)^2 = \frac{23}{4}$$ Add the fractions.

$$x - \frac{3}{2} = \pm\frac{\sqrt{23}}{2}$$ Take the square root.

$$x = \frac{3}{2} \pm \frac{\sqrt{23}}{2}$$ Add $\frac{3}{2}$ to both sides.

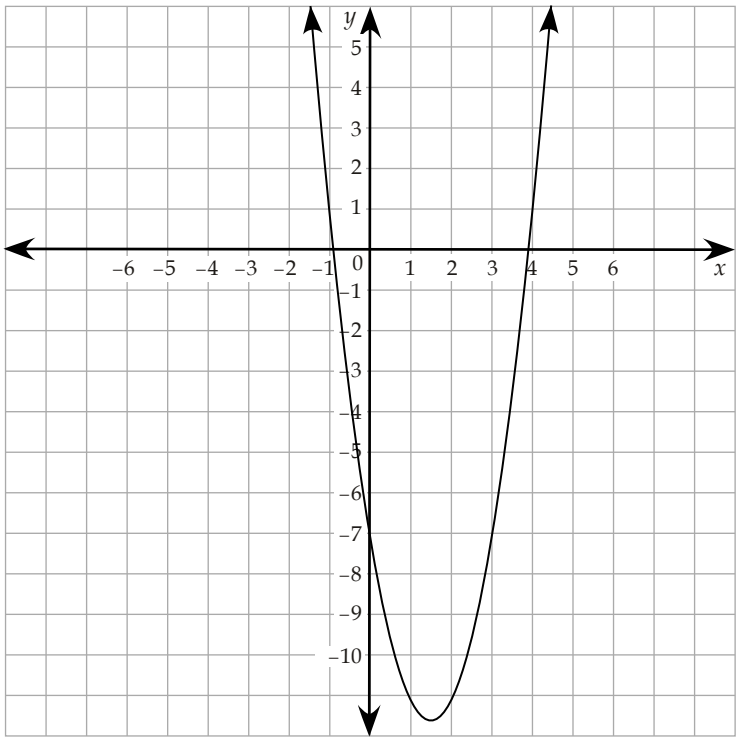
$$x = \frac{3 \pm \sqrt{23}}{2}$$ Add the fractions.

These are the exact roots of the equation. Using substitution to check solutions that have fractions and square roots is quite difficult. Instead, a graphing check using the graphing calculator will give the approximate placement of the intersection of the curve with the x-axis.
For example:

$$\frac{3}{2} + \frac{\sqrt{23}}{2} \approx 3.9$$

$$\frac{3}{2} - \frac{\sqrt{23}}{2} \approx -0.9$$

V(1.5, −11.5)



Using the graphing calculator, the graph of $y = 2x^2 - 6x - 7$ has approximate x-intercepts of 3.9 and -0.9.

Example 3

A basketball follows a path $h(d) = -0.05d^2 + 1.5d + 4$ when thrown. The basketball's height is represented by h while the basketball's horizontal distance is represented by d, where $d \geq 0$. If no one catches the basketball and it falls to the ground, its height will be zero.

a) Write a quadratic equation to describe the basketball's distance when its height is zero.

b) Solve the quadratic equation you created in (a) by completing the square to find out how far the basketball travels when thrown if no one catches it and it falls to the ground.

Solution

a) The basketball follows a path represented by the equation $h(d) = -0.05d^2 + 1.5d + 4$. When $h = 0$, the equation becomes $-0.05d^2 + 1.5d + 4 = 0$.

b)

$$-0.05d^2 + 1.5d + 4 = 0$$

Dividing both sides of the equation by -0.05.

$$d^2 - 30d - 80 = 0$$

Adding 80 to both sides

$$d^2 - 30d = 80$$

Completing the square by adding $\left(\frac{-30}{2}\right)^2$ to each side.

$$d^2 - 30d + 225 = 80 + 225$$

$$(d - 15)^2 = 305$$

Factoring and simplifying.

$$d - 15 = \pm\sqrt{305}$$

Taking the square root of each side.

$$d = 15 \pm \sqrt{305}$$

Add 15 to both sides.

Therefore, the distance the basketball travels is either $15 + \sqrt{305}$ or $15 - \sqrt{305}$.

$$15 + \sqrt{305} \approx 32.464$$

$$15 - \sqrt{305} \approx -2.464$$

Since distance travelled cannot be negative, as defined in the function by $d \geq 0$, the only solution is $15 + \sqrt{305}$.

The solution, $d = 15 - \sqrt{305} \approx -2.464$ is called an **extraneous root**. An **extraneous root** is defined as a number obtained when solving an equation that does not satisfy the initial restrictions on the variable. The basketball cannot travel a negative distance, and thus $d = 15 - \sqrt{305} \approx -2.464$ cannot be the correct answer. Therefore, $d = 15 + \sqrt{305} \approx 32.464$.

Include this definition on your resource sheet.

Solving Quadratic Equations by Using the Quadratic Formula

To derive a formula for solving any quadratic equation, you can complete the square for the general quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$.

$ax^2 + bx + c = 0$	General form.
$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$	Divide by a.
$x^2 + \frac{b}{a}x = -\frac{c}{a}$	Isolate terms with x onto one side.
$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$	Complete the square by adding $\left(\frac{1}{2} \cdot \frac{b}{a}\right)^2$ to each side.
$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$	Factor and simplify.
$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$	Add the fractions.
$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$	Take the square root of each side.
$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$	Subtract $\frac{b}{2a}$ from each side.
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Add the fractions.

The quadratic formula is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and can be used to solve equations of the form $ax^2 + bx + c = 0$ where $a \neq 0$.

Include this formula on your resource sheet. This formula is used extensively throughout mathematics courses. It would be extremely beneficial if you memorized this formula so you do not have to keep referring back to your resource sheet.

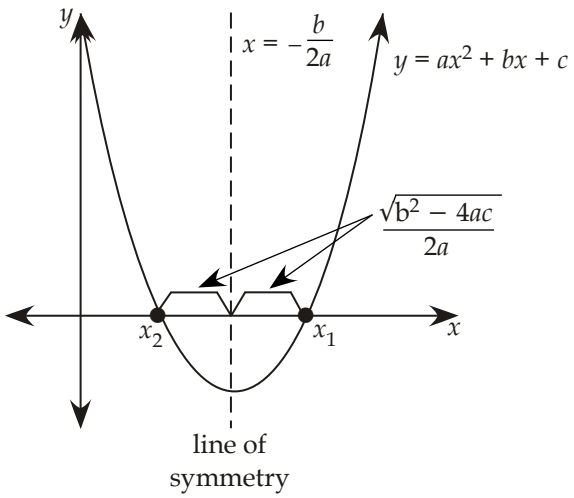
Note: You do not have to memorize the derivation of the formula, but notice how similar this work is to Module 3, Learning Activity 3.5, Question 6. Being able to algebraically manipulate polynomial equations and expressions will be useful to you in future mathematics courses.

When solving an equation using the quadratic formula, it is important to remember that $\left(-b \pm \sqrt{b^2 - 4ac}\right)$ is all divided by $2a$.

However, if you think of the quadratic formula as two fractions instead of one, you can find an easy way to find the vertex of a quadratic function in the $y = ax^2 + bx + c$ form.

$$x_1 = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x_2 = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$



As you can see, $x = -\frac{b}{2a}$ is the line of symmetry of the parabola and $-\frac{b}{2a}$ is the x-value of its vertex, as you learned in Module 3.

You can use the quadratic formula to solve any quadratic equation. When using the quadratic formula, you must set up the equation in the general form of $ax^2 + bx + c = 0$ where $a \neq 0$, so that a, b, and c can be correctly identified.

Example 4

Using the quadratic formula, solve $2x^2 + 5x - 3 = 0$.

Solution

$a = 2, b = 5, c = -3$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-5 \pm \sqrt{5^2 - 4(2)(-3)}}{2(2)}$$

$$x = \frac{-5 \pm \sqrt{25 + 24}}{4}$$

$$x = \frac{-5 \pm \sqrt{49}}{4}$$

$$x = \frac{-5 \pm 7}{4}$$

$$x = \frac{-5 + 7}{4} \quad \text{or} \quad x = \frac{-5 - 7}{4}$$

$$x = \frac{2}{4} \qquad\qquad x = \frac{-12}{4}$$

$$x = \frac{1}{2} \qquad\qquad x = -3$$

Note: This equation could also have been solved by factoring because the radicand, 49, is a perfect square.

Example 5

Solve $x^2 + 3x - 9 = 0$.

Solution

Note: This equation does not factor. Therefore, either the quadratic formula or the method of completing the square would have to be used.

$a = 1, b = 3, c = -9$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(-9)}}{2(1)}$$

$$x = \frac{-3 \pm \sqrt{9 + 36}}{2}$$

$$x = \frac{-3 \pm \sqrt{45}}{2}$$

$$x = \frac{-3 \pm 3\sqrt{5}}{2}$$

$$x = \frac{-3 + 3\sqrt{5}}{2} \text{ or } x = \frac{-3 - 3\sqrt{5}}{2}$$

Equations having a Quadratic Pattern

Some equations do not appear to be quadratic at first glance. However, a closer look reveals that they actually have a quadratic form or pattern.

Example 6

Solve: $2x^4 - 5x^2 + 2 = 0$

Solution

Notice that the equation can be written in quadratic form:

$2(x^2)^2 - 5(x^2)^1 + 2 = 0$

Temporarily replace x^2 by another letter p to obtain a quadratic equation:

$2p^2 - 5p + 2 = 0$

Factoring, you get

$$(2p-1)(p-2)=0$$

$$p=\frac{1}{2} \text{ or } p=2$$

Replacing p by x^2, you get

$$x^2=\frac{1}{2} \qquad \text{or} \qquad x^2=2$$

$$x=\pm\sqrt{\frac{1}{2}} \qquad\qquad x=\pm\sqrt{2}$$

$$x=\pm\frac{1}{\sqrt{2}} \qquad\qquad x=\pm\sqrt{2}$$

The answers are $x=\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \sqrt{2}, -\sqrt{2}.$

If you rationalize the denominators, the answers are $x=\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}, \sqrt{2}, -\sqrt{2}.$

Remember, you should always check your answers in the original equation, in case you have made an error.

Check for the equation $2x^4-5x^2+2=0$.

$$x=\sqrt{2}$$

$$\text{LHS}=2\left(\sqrt{2}\right)^4-5\left(\sqrt{2}\right)^2+2$$

$$=8-10+2$$

$$=0$$

$$\text{RHS}=0$$

$$\therefore \text{LHS}=\text{RHS}$$

$$\therefore x=\sqrt{2} \text{ is a solution}$$

$x=-\sqrt{2}$ also works because the powers involved in this equation are even. A negative value to an even power is always positive, as squaring eliminates the negative sign.

$$x = \frac{\sqrt{2}}{2}$$

$$\text{LHS} = 2\left(\frac{\sqrt{2}}{2}\right)^4 - 5\left(\frac{\sqrt{2}}{2}\right)^2 + 2$$

$$= 2\left(\frac{4}{16}\right) - 5\left(\frac{2}{4}\right) + 2$$

$$= \frac{1}{2} - \frac{10}{4} + 2$$

$$= \frac{2}{4} - \frac{10}{4} + \frac{8}{4}$$

$$= 0$$

$$\text{RHS} = 0$$

$$\therefore \text{LHS} = \text{RHS}$$

$\therefore x = \frac{\sqrt{2}}{2}$ is a solution

$x = \frac{-\sqrt{2}}{2}$ also works because the powers involved in this equation are even.

Example 7

Solve: $5\left(\frac{2}{x-3}\right)^2 + 8\left(\frac{2}{x-3}\right) + 3 = 0$

Solution

Note that $x = 3$ is a non-permissible value for x. If $x = 3$, the denominator will have a value of zero, and division by zero is not possible.

This equation displays a quadratic pattern. If $p = \frac{2}{x-3}$, you can rewrite the equation and factor to solve it.

$$5p^2 + 8p + 3 = 0$$

$$(5p + 3)(p + 1) = 0$$

$p = \frac{-3}{5}$ or $p = -1$

Replace p by $\frac{2}{x-3}$ to solve for x.

$$\frac{2}{x-3} = \frac{-3}{5} \quad \text{or} \quad \frac{2}{x-3} = -1$$

$-3(x-3) = 10$	$-1(x-3) = 2$	Cross multiply.
$-3x + 9 = 10$	$-x + 3 = 2$	
$-3x = 1$	$-x = -1$	
$x = -\frac{1}{3}$	$x = 1$	

Check to see that roots satisfy the original equation to ensure you have no errors in your work and that your solution doesn't contain a non-permissible value for x.

Check solutions in the original equation $5\left(\frac{2}{x-3}\right)^2 + 8\left(\frac{2}{x-3}\right) + 3 = 0$.

Check: $x = -\frac{1}{3}$

$$\text{LHS} = 5\left(\frac{2}{x-3}\right)^2 + 8\left(\frac{2}{x-3}\right) + 3$$

$$= 5\left(\frac{2}{-\frac{1}{3}-\frac{9}{3}}\right)^2 + 8\left(\frac{2}{-\frac{1}{3}-\frac{9}{3}}\right) + 3$$

$$= 5\left(\frac{2}{\frac{-10}{3}}\right)^2 + 8\left(\frac{2}{\frac{-10}{3}}\right) + 3$$

Note: $\frac{2}{\frac{-10}{3}} = 2 \div \left(\frac{-10}{3}\right) = 2 \times \frac{-3}{10} = \frac{-3}{5}$

$$= 5\left(\frac{-3}{5}\right)^2 + 8\left(\frac{-3}{5}\right) + 3$$

$$= 5\left(\frac{9}{25}\right) + \left(\frac{-24}{5}\right) + 3$$

$$= \frac{45}{25} + \left(\frac{-120}{25}\right) + \frac{75}{25}$$

$$= 0$$

RHS = 0

$\therefore$ LHS = RHS

$\therefore x = -\frac{1}{3}$ is a solution

Check: $x = 1$

$$\text{LHS} = 5\left(\frac{2}{x-3}\right)^2 + 8\left(\frac{2}{x-3}\right) + 3$$

$$= 5\left(\frac{2}{1-3}\right)^2 + 8\left(\frac{2}{1-3}\right) + 3$$

$$= 5\left(\frac{2}{-2}\right)^2 + 8\left(\frac{2}{-2}\right) + 3$$

$$= 5(-1)^2 + 8(-1) + 3$$

$$= 5 - 8 + 3$$

$$= 0$$

RHS = 0

$\therefore$ LHS = RHS

$\therefore x = 1$ is a solution

Learning Activity 4.4

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Evaluate: $\sqrt{115 + \sqrt{27 + \sqrt{76 + \sqrt{25}}}}$
2. Britney is making cookies. The recipe calls for $\frac{3}{8}$ of a cup of butter. If Britney is tripling the recipe, how much butter does she need? Write your answer as a mixed fraction.
3. Joanne makes $400 in a month. If Joanne needs to pay $66 for her phone bill, $40 for gas to drive her parents' car, and $120 for a haircut, will she have enough to buy the $150 jeans she wants?
4. Factor: $\frac{1}{3}x^2 - \frac{121}{3}$
5. Find the vertex of the following parabola: $y = 3x^2 + 6x - 5$
6. Rationalize the denominator: $\frac{6}{\sqrt{2}}$
7. Find the slope of the line perpendicular to the line, $7x + 3y - 4 = 0$.
8. The circumference of a circle is 92π centimetres. What is the radius of the circle?

continued

Part B: Solving Quadratic Equations by Completing the Square and Using the Quadratic Formula

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. For each quadratic equation below, state the values of a, b, and c for the general equation format.
   a) $x^2 - 2x - 5 = 0$
   b) $3x^2 - 2x + 5 = 0$
   c) $5x^2 - 3x = 8$
   d) $2(x^2 - 2x) - 1 = 0$
   e) $5x^2 = 9x$
   f) $4 - 2x^2 = 9x$

2. Solve these equations by factoring or by using the quadratic formula. Be sure to state the formula before substituting values into it, as this will help you memorize it.
   a) $\frac{x^2}{2} + x - \frac{15}{2} = 0$
   b) $2w^2 - 3w + 1 = 0$
   c) $7w^2 - 3w = 0$
   d) $1 = 25x^2$
   e) $x^2 - 0.1x - 0.06 = 0$
   f) $-x^2 - 7x - 1 = 0$

3. Use the completing-the-square method to find the roots of each equation below.
   a) $3x^2 - 6x - 15 = 0$
   b) $2x^2 - 4x - 1 = 0$
   c) $9x^2 - 18x - 7 = 0$
   d) $x^2 - x - 3 = 0$

4. Find the zeros of the function using the quadratic formula.
   a) $f(x) = 5x^2 - x - 3$
   b) $y = 2x^2 + 6x - 1$

continued

Learning Activity 4.4 (continued)

5. Find the roots of the quadratic equation $6x^2 + 5x - 6 = 0$
   a) using the quadratic formula
   b) by factoring
   Which method do you prefer and why?
6. Find the zeros of the quadratic function $f(x) = x^2 + 8x + 15$ by
   a) factoring
   b) using the quadratic formula
   c) graphing
   d) completing the square.
   Which method do you prefer and why?
7. Refer to the Solving Quadratic Equations chart found at the end of this learning activity. For each of the five methods of solving quadratic equations, describe when each method could be best used. Fill in your answers on the chart.
8. Write a quadratic equation in general form that satisfies each of the following descriptions. Explain your answers.
   a) two distinct real roots
   b) one distinct real root, or two equal roots
   c) no real roots
9. Is the quadratic formula always the most efficient way to solve quadratic equations? Explain with examples.
10. A rocket is launched and follows a path described by the quadratic function $h(d) = -0.02d^2 + 0.4d + 1$. The height is represented by h while the horizontal distance is represented by d and both are measured in metres. How far away from its original position does this rocket land?

continued

Learning Activity 4.4 (continued)

11. David is trying to solve a quadratic equation by using the quadratic formula. His work is shown below.

$10x^2 + 2x + 1 = 0$

$a = 10, b = 2, c = 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(10)}}{2(10)}$$

$$x = \frac{-2 \pm \sqrt{4 - 40}}{20}$$

$$x = \frac{-2 \pm \sqrt{-36}}{20}$$

$$x = \frac{-2 \pm 6}{20}$$

$$x = \frac{-2 + 6}{20} = \frac{4}{20} = \frac{1}{5}$$

$$x = \frac{-2 - 6}{20} = \frac{-8}{20} = \frac{-2}{5}$$

Identify, explain, and correct David's error.

12. Solve the equation:

$$(x^2 - 5)^2 + 3(x^2 - 5) + 2 = 0$$

Lesson Summary

In this lesson, you learned two additional ways of solving quadratic equations—completing the square and the quadratic formula. You now have five different techniques that you can use to solve a quadratic equation. In the learning activity, you also explained when each technique is most useful and you learned to solve equations with a quadratic pattern. In the next lesson, you will be learning more about the quadratic formula and how it can be used to determine the number of roots a quadratic equation has, without solving the equation.

Solving Quadratic Equations

Graphing

Use when:

Square Roots

Use when:

Factoring

Use when:

Completing the Square

Use when:

Quadratic Formula

Use when:

Assignment 4.2

Solving Quadratic Equations

Total: 49 marks

1. Solve these equations by factoring. *(2 marks each, for a total of 12 marks)*

   a) $x^2 + 7x + 12 = 0$

   b) $4x^2 + 4x - 3 = 0$

   c) $\frac{x^2}{3} - \frac{4}{3}x = 7$

continued

Assignment 4.2: Solving Quadratic Equations (continued)

d) $x^2 - 30 = -x$

e) $13x^2 - 14x = 8 - 2x^2$

f) $2x^2 + 8 - 20x = -10x^2 + 8x$

continued

Assignment 4.2: Solving Quadratic Equations (continued)

2. Two positive numbers differ by 4 and the sum of their squares is 136. Find the numbers. *(3 marks)*

3. A concrete pathway x metres wide is being built around the outside of a 40 m × 30 m garden. The area of the pathway is 984 m^2. Find the width of the pathway. *(5 marks)*

continued

Assignment 4.2: Solving Quadratic Equations (continued)

4. Solve the following quadratic equations by completing the square. *(2 marks each, for a total of 6 marks)*

   a) $x^2 - 14x - 44 = 0$

   b) $x^2 - 14x = 95$

   c) $10x^2 - 4x = 32$

continued

Assignment 4.2: Solving Quadratic Equations (continued)

5. Solve the following quadratic equations by using the quadratic formula. *(2 marks each, for a total of 6 marks)*

   a) $2x^2 - 12x - 14 = 0$

   b) $8x^2 - 11x + 3 = 0$

   c) $3x^2 + x - 11 = 0$

continued

Assignment 4.2: Solving Quadratic Equations (continued)

6. Solve the following quadratic equations using a different method for each one. Explain why you chose each particular method for each equation. *(2 marks each, for a total of 8 marks)*

   a) $(x + 3)^2 = 49$

   b) $2x^2 - 5x - 3 = 0$

   c) $6x^2 - 28x + 15 = 0$

   d) $x^2 + 6x - 11 = 0$

continued

Assignment 4.2: Solving Quadratic Equations (continued)

7. Find the roots of the quadratic equation $3x^2 - 5x - 1 = 0$ to one decimal place. Use your calculator. *(3 marks)*

8. Solve: $2x^4 - 3x^2 + 1 = 0$ *(2 marks)*

continued

Assignment 4.2: Solving Quadratic Equations (continued)

9. A football is kicked and its path is described by the following quadratic function: $h(d) = -5d^2 + 20d + 35$. The football's height is represented by h while the football's horizontal distance travelled is represented by d. Both variables are measured in metres and both must have zero or positive values. If nothing interacts with the football after it is kicked, how far does the football travel before it hits the ground? Round your answer(s) to three decimal places. *(4 marks)*

Lesson 5: The Discriminant

Lesson Focus

In this lesson, you will

- ❑ learn how to use the discriminant of the quadratic formula to describe the nature of the roots of a quadratic equation
- ❑ learn how to apply solving linear inequalities in one variable to problems involving the nature of the roots of a quadratic equation

Lesson Introduction

In the first four lessons of this module, you learned how to solve quadratic equations using five different techniques. In Lesson 4, you learned how to use the quadratic formula to solve quadratic equations. The *discriminant* is an aspect of the quadratic formula that can allow you to determine how many roots a quadratic equation has without solving for the roots. In this lesson, you will consider the value of the discriminant of a given quadratic equation, the roots of the quadratic equation, the number of the roots, the type of the roots, and the graph of the corresponding quadratic function.

The Discriminant

You have found any quadratic equation of the form $ax^2 + bx + c = 0$ can be solved using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression under the square root sign, $b^2 - 4ac$, is called the **discriminant** and it enables you to observe the nature of the roots without actually finding the roots. The **nature of the roots** describes how many roots a quadratic equation has and whether they are rational, irrational, or non-existent.

Include the above definition and any other information you deem to be important on your resource sheet.

The Roots of a Quadratic Equation

There are four situations that can occur when you are finding the roots of a quadratic equation. The roots can be real and rational, real and irrational, real and equal, or none. You can determine which situation will occur by examining the value of the discriminant.

Case 1: When $b^2 - 4ac > 0$ and is a perfect square

Sample Equation: $x^2 - 6x + 5 = 0$

Value of Discriminant: $b^2 - 4ac$

$a = 1, b = -6, c = 5$

$b^2 - 4ac = (-6)^2 - 4(1)(5) = 36 - 20 = 16$

Note that the value of $b^2 - 4ac$ is positive and a perfect square.

Roots of Quadratic: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$r_1 = \frac{-(-6) + \sqrt{16}}{2} = \frac{6 + 4}{2} = \frac{10}{2} = 5$$

$$r_2 = \frac{-(-6) - \sqrt{16}}{2} = \frac{6 - 4}{2} = \frac{2}{2} = 1$$

Nature of Roots: r_1 and r_2 are referred to as two *real, rational roots.*

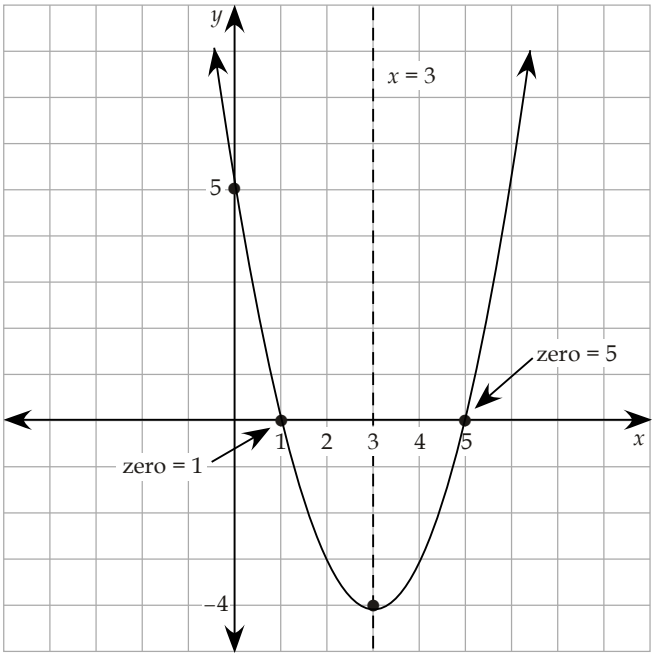
Graph of the Quadratic Function:

$y = x^2 - 6x + 5$

$y = (x^2 - 6x + 9) + 5 - 9$

$y = (x - 3)^2 - 4$

Vertex $(3, -4)$



Description of Zeros: The zeros are referred to as two real, rational zeros at 1 and 5.

Note: When the discriminant is a perfect square, the quadratic equation can also be solved more easily by factoring:

$$x^2 - 6x + 5 = 0$$
$$(x - 5)(x - 1) = 0$$
$$x = 5, 1$$

Case 2: When $b^2 - 4ac > 0$ and is not a perfect square

Sample Equation: $x^2 - 6x + 7 = 0$

Value of Discriminant: $b^2 - 4ac$

$a = 1, b = -6, c = 7$

$b^2 - 4ac = (-6)^2 - 4(1)(7) = 36 - 28 = 8$

Note: The value of $b^2 - 4ac$ is not a perfect square, although it is greater than zero.

Roots of Quadratic:
$$r_1 = \frac{-(-6) + \sqrt{8}}{2(1)} = \frac{6 + 2\sqrt{2}}{2} = 3 + \sqrt{2}$$
$$r_2 = \frac{-(-6) - \sqrt{8}}{2(1)} = \frac{6 - 2\sqrt{2}}{2} = 3 - \sqrt{2}$$

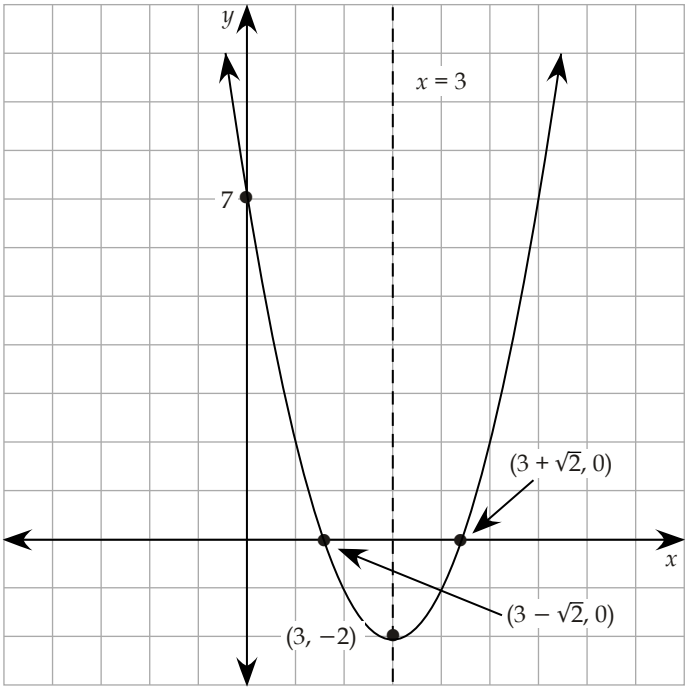
Nature of Roots: r_1 and r_2 are referred to as two *real, irrational roots*.

Graph of the Quadratic Function: $y = x^2 - 6x + 7$

$y = (x^2 - 6x + 9) + 7 - 9$

$y = (x - 3)^2 - 2$

Vertex $(3, -2)$



Description of Zeros: Two real, irrational zeros at $3 + \sqrt{2}$ and $3 - \sqrt{2}$.

Case 3: When $b^2 - 4ac = 0$

Sample Equation: $x^2 - 6x + 9 = 0$

Value of Discriminant: $b^2 - 4ac$

$a = 1, b = -6, c = 9$

$b^2 - 4ac = (-6)^2 - 4(1)(9) = 36 - 36 = 0$

Note: The the value of $b^2 - 4ac$ is zero.

Roots of Quadratic: $r_1 = \dfrac{-(-6) + \sqrt{0}}{2(1)} = \dfrac{6 + 0}{2} = 3$

$r_2 = \dfrac{-(-6) - \sqrt{0}}{2(1)} = \dfrac{6 - 0}{2} = 3$

Nature of Roots: r_1 and r_2 are the same real number.

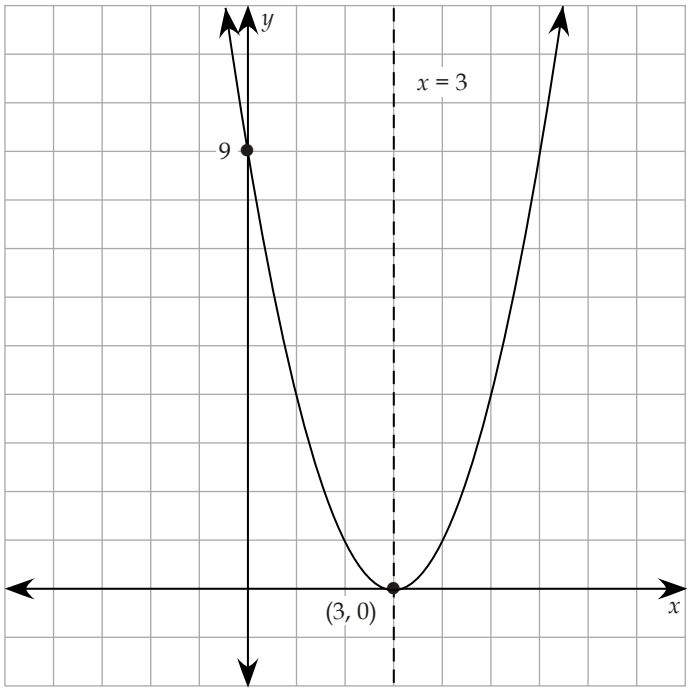
r_1 and r_2 are referred to as *real and equal roots*.

Graph of the Quadratic Function:

$y = x^2 - 6x + 9$

$y = (x - 3)^2$

Vertex (3, 0)



Description of Zeros: There is one zero. The graph only touches the x-axis at its vertex.

Case 4: When $b^2 - 4ac < 0$

Sample Equation: $x^2 - 6x + 14 = 0$

Value of Discriminant: $b^2 - 4ac$

$a = 1, b = -6, c = 14$

$b^2 - 4ac = (-6)^2 - 4(1)(14) = 36 - 56 = -20$

Note: The value of the discriminant is less than zero.

Roots of Quadratic:

$$r_1 = \frac{-(-6) + \sqrt{-20}}{2} = \frac{6 + \sqrt{-20}}{2} = \text{no real number answer}$$

$$r_2 = \frac{-(-6) - \sqrt{-20}}{2} = \frac{6 - \sqrt{-20}}{2} = \text{no real number answer}$$

The square root of a negative number is not a real number.

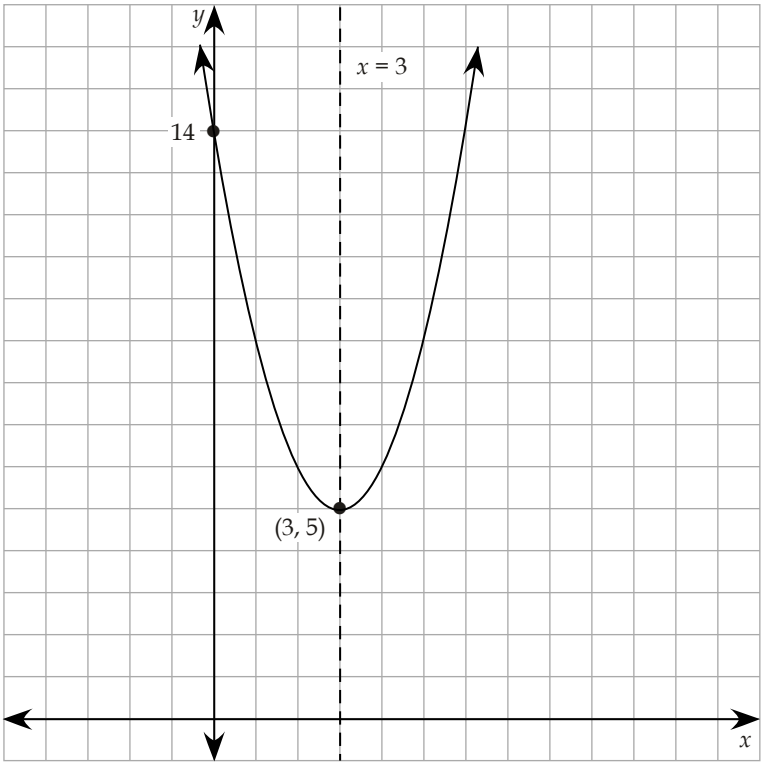
Nature of Roots: There are no real roots.

Graph of the Quadratic Function: $y = x^2 - 6x + 14$

$y = (x^2 - 6x + 9) + 14 - 9$

$y = (x - 3)^2 + 5$

V(3, 5), opens up, standard size



Description of Zeros: There are no zeros since the graph does not cross or touch the x-axis.

The following table displays the information you discovered in this lesson about the nature of the roots of a quadratic equation. Include this information on your resource sheet.

Nature of the Roots for the Quadratic Equation: $ax^2 + bx + c = 0$ where a, b, and c are rational numbers, and $a \neq 0$	
Value of the Discriminant $b^2 - 4ac$	**Nature of the Roots**
$b^2 - 4ac > 0$ and is a perfect square	2 real, rational, unequal roots
$b^2 - 4ac > 0$ and is not a perfect square	2 real, irrational, unequal roots
$b^2 - 4ac = 0$	one real root (same as: 2 real, equal roots, or one double, real root)
$b^2 - 4ac < 0$	no real roots

Note that a, b, and c are designated as being rational numbers. However, if they are irrational numbers, the summary chart is not valid in some cases. The following example shows when a false conclusion may occur.

Example 1

Find the nature of the roots of the quadratic equation and check your answer by finding the roots of the equation.

$$\sqrt{2}x^2 + \sqrt{17}x - 4\sqrt{2} = 0$$

Solution

$$a = \sqrt{2},\ b = \sqrt{17},\ c = -4\sqrt{2}$$

Notice these coefficients are irrational.

The discriminant is:

$$\begin{aligned} b^2 = 4ac &= \left(\sqrt{17}\right)^2 - 4\left(\sqrt{2}\right)\left(-4\sqrt{2}\right) \\ &= 17 + 16(2) \\ &= 17 + 32 \\ &= 49 \end{aligned}$$

The discriminant is a perfect square. The usual conclusion would be there are two rational roots.

Now, you can find those roots to see if they are rational.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-\sqrt{17} \pm \sqrt{49}}{2\sqrt{2}} \\ &= \frac{-\sqrt{17} \pm 7}{2\sqrt{2}} \end{aligned}$$

However, these roots are irrational numbers. The conclusion is false.

False conclusions may occur if the coefficients of the quadratic equation are not rational. Note that further study in this topic will be covered at higher levels of math. In this course, all the coefficients of a given quadratic equation are rational.

Linear Inequalities in One Variable

Because questions involving the nature of the roots sometimes involve inequalities, it will be helpful for you to review how to solve linear inequalities. You learned how to do this in *Grade 9 Mathematics*.

Solving linear inequalities is equivalent to solving equations except for one important rule.

When **multiplying** or **dividing** each side of the inequality by a negative number, **reverse** the direction of the inequality sign. Add this rule to your resource sheet.

The reason for this rule is shown in the following example.

Example 2

Is the following statement true or false? If the statement is false, rewrite the statement to make it true.

If $4 < 6$, then $(-1)4 < (-1)(6)$.

Solution

False.

The first part of the statement, $4 < 6$, is true, but if you simplify the second part of the statement, you get $-4 < -6$, which is a false statement. Thus, the correct statement should have the inequality sign reversed.

If $4 < 6$, then $(-1)4 > (-1)(6)$.

The following example gives you practice in solving a linear inequality in one variable.

Example 3

Solve each of the following. Write your answer as an inequality and as an interval using interval notation.

a) $3 - 2x > -5$

b) $3(4 - x) \leq x - (3 - 5x)$

Solution

a) $3 - 2x > -5$

$-2x > -5 - 3$

$-2x > -8$

$\frac{-2x}{-2} < \frac{-8}{-2}$ Division by a negative number reverses the direction of the inequality sign.

$x < 4$

In interval notation, the solution is $(-\infty, 4)$.

b) $3(4 - x) \leq x - (3 - 5x)$

$12 - 3x \leq x - 3 + 5x$

$12 - 3x \leq 6x - 3$

$12 + 3 \leq 6x + 3x$

$15 \leq 9x$

$\frac{15}{9} \leq x$

$\frac{5}{3} \leq x$

$x \geq \frac{5}{3}$

The solution can be shown in interval notation as $\left[\frac{5}{3}, \infty\right)$.

Example 4

For what values of m will $x^2 + 8x - m = 0$ have real and unequal roots?

Solution

$x^2 + 8x - m = 0$

$a = 1, b = 8, c = -m$

For real and unequal roots, $b^2 - 4ac > 0$.

$$8^2 - 4(1)(-m) > 0$$
$$64 + 4m > 0$$
$$4m > -64$$
$$m > -16$$

The value of m must be greater than -16 for the given equation to have real and unequal roots.

Example 5

Write a quadratic equation that has two different, real solutions.

Solution

A quadratic equation in general form has two real number solutions when $b^2 - 4ac > 0$. Many solutions are possible.

One way to solve this problem is to choose any values of a and c, then find a value of b that satisfies this inequality.

If $a = 3, c = 11$

$$b^2 - 4ac > 0$$
$$b^2 - 4(3)(11) > 0$$
$$b^2 - 132 > 0$$
$$b^2 > 132$$

A value of b could be 12 because $12^2 > 132$. The equation is $3x^2 + 12x + 11 = 0$.

Note that Example 5 is a quadratic inequality that was solved by giving an example that satisfies the conditions of the problem.

Learning Activity 4.5

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Janique is taking piano lessons and her teacher tells her she needs to practice piano for at least 7 hours a week. If Janique practices for 45 minutes each day from Monday to Friday, 1.25 hours on Saturday, and 1.75 hours on Sunday, will Janique have practiced enough?
2. If you sleep for 9 hours a day, what fraction of the day are you awake (in lowest terms)?
3. Evaluate for $j = 3$: $5j - 2j^2 + 7$
4. Evaluate: $2^{(2^3)}$
5. What is the area of a triangle with a base of 4 cm and a height of 8 cm?
6. Factor completely: $36x^2 - 1$
7. Two angles in an isosceles triangle measure 16° and 148°. What is the measure of the third angle?
8. Express as a radical: $8^{\frac{3}{2}}$

continued

Learning Activity 4.5 (continued)

Part B: The Discriminant

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Determine whether each statement is true or false. If the statement is false, rewrite it so it is true.
   a) The discriminant of $2x^2 + 5x + 6 = 0$ is 23.
   b) For each quadratic equation $ax^2 + bx + c = 0$, if $b^2 - 4ac > 0$, then the equation has two real roots.
   c) For each quadratic equation $ax^2 + bx + c = 0$, if $b^2 - 4ac < 0$, then the equation has no real number solutions.
   d) $\dfrac{-6 + \sqrt{32}}{10} = \dfrac{-3 + 8\sqrt{2}}{5}$
   e) The equation $x^2 + 2x + 1 = 0$ has two distinct real roots.
2. If the discriminant of a quadratic equation has the given value, state the nature of the roots.
   a) -15
   b) 25
   c) -9
   d) 0
   e) 50
3. How many times would the graph of $y = ax^2 + bx + c$ (with a, b, and c as rational numbers and $a \neq 0$) intersect the x-axis if the value of the discriminant is
   a) negative?
   b) zero?
   c) positive?

continued

Learning Activity 4.5 (continued)

4. Determine the nature of the roots by calculating the discriminant for each equation.
   a) $x^2 - 8x + 16 = 0$
   b) $x^2 + 2x + 7 = 0$
   c) $p^2 - 16 = 0$
   d) $2x^2 + x = 5$

5. Find value(s) of k so that each equation has real and equal roots.
   a) $kx^2 - 6x + 2 = 0$
   b) $x^2 + (k - 8)x + 9 = 0$

6. Sketch the graph of a quadratic function whose discriminant is positive and whose vertex is in the second quadrant.

7. Find the value of k for each of the following situations.
   a) $3x^2 - 2x + k = 0$ has two real roots
   b) $x^2 + kx + 3k = 0$ has a double root
   c) $kx^2 + 8x = 4$ has no real roots

8. For each of the following quadratic equations, solve by first using the discriminant to determine if its value is a perfect square. If it is a perfect square, use the factoring method to solve. If it is not a perfect square, use the quadratic formula.
   a) $12x^2 - 19x - 21 = 0$
   b) $18x^2 - 35x + 12 = 0$
   c) $60x^2 - 59x - 20 = 0$

9. Complete the following Quadratic Equation Puzzle with your learning partner. Cut out all the separate pieces and arrange them into rows on the Quadratic Equation Puzzle worksheet so that each row corresponds to a quadratic equation. In each row, the quadratic equation must be in general form and vertex form, and you must include the value of the discriminant, the number of roots, the types of roots, and the actual roots themselves.

Notes

Quadratic Equation Puzzle Worksheet

General Form of Quadratic Equation	Vertex Form of Quadratic Function	Value of the Discriminant	Number of Roots	Type of Roots	Roots

Quadratic Equation Puzzle Pieces

$y = 25\left(x - \frac{1}{5}\right)^2$	none	1	$25x^2 - 10x + 1 = 0$	Real and Irrational	-20
$3x^2 + 8x - 35 = 0$	No Real Roots	484	0	$\frac{1}{5}$	$-5, \frac{7}{3}$
Real and Rational	2	$y = 9\left(x + \frac{1}{3}\right)^2 - 5$	2	$x^2 - 4x + 4 = 0$	Real and Equal
$\frac{-1+\sqrt{5}}{3}, \frac{-1-\sqrt{5}}{3}$	$9x^2 + 6x - 4 = 0$	0	$y = 3\left(x + \frac{4}{3}\right)^2 - \frac{121}{3}$	Real and Equal	0
2	$y = 2\left(x - \frac{1}{2}\right)^2 + \frac{5}{2}$	1	$y = (x - 2)^2$	$2x^2 - 2x + 3 = 0$	180

Lesson Summary

In this lesson, you learned how the discriminant of the quadratic formula can be used to determine the nature of the roots of the quadratic equation. The roots of a quadratic equation can either be real and irrational, real and rational, real and equal, or none.

You learned how to simplify, multiply, divide, add, and subtract rational expressions in Module 2. In the next lesson, your focus will switch to dealing with solving rational equations.

Notes

Lesson 6: Solving Rational Equations

Lesson Focus

In this lesson, you will

- ❑ learn how to solve rational equations
- ❑ learn how to determine the non-permissible values for rational equations
- ❑ learn how to determine if you have an extraneous solution
- ❑ learn how to solve rational equation application problems

Lesson Introduction

In Module 2, you learned how to add, subtract, multiply, divide, and simplify rational expressions. In this lesson, you will learn how to solve rational equations. Solving rational equations is different than performing operations or simplifying rational expressions. Therefore, in this lesson, you will learn new techniques in order to solve rational equations. In order to not confuse rational expressions with rational equations, examples throughout this lesson outline their differences.

Rational Equations

An equation involving rational expressions is a **rational equation**. The algebraic method of solution is based on the **least common multiple**. The least common multiple (LCM) of two or more polynomials is the smallest polynomial that is a multiple of each of the original polynomials.

The following steps display how to solve rational equations. These are not the same steps you used to simplify rational expressions. Therefore, make sure you copy these steps down on your resource sheet. To solve rational equations, follow these steps:

1. Factor all denominators and numerators. Reduce each rational expression, if possible.
2. Find the LCM of the denominators. Call it the LCD (lowest common denominator).

3. Remove the denominators of the rational expressions by multiplying each rational expression in the equation by the least common denominator. Make sure you cancel to reduce the fractions.
4. Solve the resulting equation.
5. Check your solution(s) that none of these numbers are non-permissible values of the original denominators.

Note: Be careful not to confuse a rational **expression** and a rational **equation**.

Expression

Simplify.

$\frac{x-2}{x-1}+\frac{2}{x}$

LCD $= (x-1)(x)$

$$\frac{x-2}{x-1}+\frac{2}{x}=\frac{x-2}{x-1}\cdot\frac{x}{x}+\frac{2}{x}\cdot\frac{x-1}{x-1}$$
$$=\frac{(x-2)x+2(x-1)}{(x-1)x}$$
$$=\frac{x^2-2x+2x-2}{(x-1)x}$$
$$=\frac{x^2-2}{(x-1)x},\ x\neq 0,1$$

Equation

Solve for x.

$\frac{x-2}{x-1}+\frac{2}{x}=\frac{7}{6}$

LCD $= (6x)(x-1)$

$$\frac{x-2}{x-1}(6x)(x-1)+\frac{2}{x}(6x)(x-1)=\frac{7}{6}(6x)(x-1)$$
$$(x-2)6x+12(x-1)=7x(x-1)$$
$$6x^2-12x+12x-12=7x^2-7x$$
$$0=x^2-7x+12$$
$$0=(x-3)(x-4)$$

$x-3=0$ or $x-4=0$

$x=3$ or $x=4,\ x\neq 0,1$

When simplifying a rational expression, the denominator remains unless it cancels with a common factor in the numerator. The non-permissible values are determined from the original expression.

In solving a rational equation, the denominator is cancelled by multiplying every term in the equation by the least common denominator. This process clears the rational equation of the denominators and produces an equivalent equation with restrictions. These non-permissible values are shown beside the solution.

Include the above explanation on your resource sheet to help you distinguish between simplifying a rational expression and solving a rational equation.

Example 1

a) Simplify.

$$\frac{3}{x-1}-\frac{2}{x+1}$$

b) Solve for x.

$$\frac{3}{x-1}-\frac{2}{x+1}=1$$

Solution

a) LCD = $(x - 1)(x + 1)$

$$\frac{3}{x-1}-\frac{2}{x+1}$$

$$=\frac{3}{x-1}\cdot\frac{x+1}{x+1}-\frac{2}{x+1}\cdot\frac{x-1}{x-1}$$

$$=\frac{3(x+1)-2(x-1)}{(x-1)(x+1)}$$

$$=\frac{3x+3-2x+2}{(x-1)(x+1)}$$

$$=\frac{x+5}{(x-1)(x+1)},\ x\neq 1,-1$$

Note: The denominator is still present in the answer.

b) $\dfrac{3}{x-1} - \dfrac{2}{x+1} = 1$

$\text{LCD} = (x-1)(x+1)$

$$\frac{3}{x-1}(x-1)(x+1) - \frac{2}{x+1}(x-1)(x+1) = 1(x-1)(x+1)$$

Each term is multiplied by the LCD.

$$3(x+1) - 2(x-1) = (x-1)(x+1)$$

Reduce each term by cancelling to clear the equation of its denominators.

$$3x + 3 - 2x + 2 = x^2 + x - x - 1$$

$$x + 5 = x^2 - 1$$

$$0 = x^2 - x - 6$$

$$0 = (x-3)(x+2)$$

$$x - 3 = 0 \text{ or } x + 2 = 0$$

$$x = 3 \text{ or } \quad x = -2$$

Both of these answers are acceptable since the restrictions are $x \neq 1$ and $x \neq -1$.

Example 2

Solve for x.

a) $\dfrac{2}{x^2-1}-1=\dfrac{1}{x-1}$

b) $\dfrac{1}{x}+\dfrac{2}{x+1}=\dfrac{3}{2x}$

Solution

a) $\dfrac{2}{x^2-1}-1=\dfrac{1}{x-1}$

$\dfrac{2}{(x-1)(x+1)}-1=\dfrac{1}{x-1}$ Factor denominator.

$\text{LCD}=(x-1)(x+1)$

$\dfrac{2}{(x-1)(x+1)}(x-1)(x+1)-1(x-1)(x+1)$ Multiply by LCD.

$=\dfrac{1}{x-1}(x-1)(x+1)$

$2-1(x-1)(x+1)=1(x+1)$ Cancel to remove the denominators.

$2-\left(x^2-1\right)=x+1$ Simplify.

$2-x^2+1=x+1$

$0=x^2+x-2$

$0=(x+2)(x-1)$

$x+2=0$ or $x-1=0$

$x=-2$ or $x=1$

From the original equation, the non-permissible values are $x=-1$ and $x=1$. Therefore, $x=1$ is an extraneous solution and $x=-2$ is the only solution.

b) $\frac{1}{x}+\frac{2}{x+1}=\frac{3}{2x}$

LCD $=(2x)(x+1)$

$$\frac{1}{x}(2x)(x+1)+\frac{2}{x+1}(2x)(x+1)=\frac{3}{2x}(2x)(x+1)$$

$$(2)(x+1)+(2)(2x)=3(x+1)$$

$$2x+2+4x=3x+3$$

$$3x=1$$

$$x=\frac{1}{3}$$

From the original equation, the restrictions are $x \neq 0$, $x \neq -1$.

Therefore, there are no extraneous roots and $x=\frac{1}{3}$ is the only solution.

Example 3

Solve: $\frac{2}{x}+\frac{1}{2}=\frac{3}{x}$

Solution

Method 1: Algebraic

$$\frac{2}{x}+\frac{1}{2}=\frac{3}{x}$$

LCD $=2x$

$$\frac{2}{x}(2x)+\frac{1}{2}(2x)=\frac{3}{x}(2x)$$

$$4+x=6$$

$x=2$, $x \neq 0$

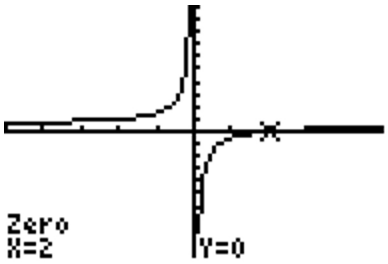
Method 2: Graphing Calculator

You can also use your graphing calculator to solve for x. First, rearrange the equation to equal zero as:

$$\frac{2}{x} + \frac{1}{2} - \frac{3}{x} = 0 \text{ or } \frac{1}{2} - \frac{1}{x} = 0$$

Now graph $y = \frac{1}{2} - \frac{1}{x}$ and find the zeros.

The x-intercepts of the graph are the solutions to the equation when it is set equal to zero.



The only zero is at $x = 2$.

Notice also that the graph of the equation approaches but does not cross the y-axis, where $x = 0$. This is because the non-permissible value for this function is $x = 0$.

Applications of Rational Equations

It is possible to solve motion problems by solving rational equations. Consider the following example.

Example 4

Sophie can canoe 5 km in the same time as it takes Skyler to cover 30 km in his motor boat. If Skyler's boat travels 40 km/h faster than Sophie's canoe, how fast is Sophie canoeing? Round to the nearest tenth.

Solution

Use the formula $d = rt$, where d = distance, t = time, and r = rate or speed

Let s = Sophie's speed and let Skyler's speed = $s + 40$

Rearrange the distance formula to get $t = \frac{d}{r}$. The time is the constant in this equation. Therefore, you can set up equations for Sophie's time and Skyler's time and then equate them.

Sophie: $\frac{d}{r} = \frac{5}{s} = t$

Skyler: $\frac{d}{r} = \frac{30}{s+40} = t$

$$\frac{5}{s} = \frac{30}{s+40}$$

LCD = $(s)(s+40)$

$$\frac{5}{s}(s)(s+40) = \frac{30}{s+40}(s)(s+40)$$

$$5(s+40) = 30s$$

$$5s + 200 = 30s$$

$$200 = 25s$$

$$s = 8,\ s \neq 0,\ s \neq -40$$

Therefore, Sophie is canoeing at 8 km/h.

It is also possible to solve application problems using a table to set up the information. Consider the following example.

Example 5

Mark can ride his bicycle 2 km/h faster than Tamara. They begin at the same spot and at the same time, travelling in opposite directions. After travelling for the same length of time, they stop and realize that Mark has travelled 27 km and Tamara has travelled 21 km. How fast was Mark travelling?

Solution

To get the entries for the "Time" column, fill in the rate and distance columns first and then find the ratio of distance to rate.

	Rate	**Time**	**Distance**
Mark	$r + 2$	$\frac{\text{distance}}{\text{rate}} = \frac{27}{r+2}$	27
Tamara	r	$\frac{\text{distance}}{\text{rate}} = \frac{21}{r}$	21

Mark's time of travel = Tamara's time of travel.

$$\frac{27}{r+2} = \frac{21}{r}$$

LCD = $r(r + 2)$

$$\frac{27}{(r+2)} r(r+2) = \frac{21}{r} r(r+2)$$

$$27r = 21(r+2)$$

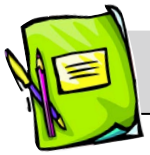
$$27r = 21r + 42$$

$$6r = 42$$

$$r = 7,\ r \neq 0, -2$$

Mark's rate of travel was $r + 2 = 7 + 2 = 9$ km/h.

This is the last learning activity in this module. Learning activities are designed to help prepare you for the assignments and exams. Make sure you complete this learning activity, as there is an assignment directly following it that contains similar questions. Also, you will be preparing for your midterm examination at the end of this module. All the learning activities you have completed will help prepare you for this examination.



Learning Activity 4.6

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Multiply: $(3x^2)(4xy^4)$
2. Your dog can run at a speed of 10 m/s. If you are standing half a kilometre away from your dog, how long will it take for your dog to run to you?
3. What are the x-intercepts of the function $y = x^2 + 3x + 2$?

continued

4. At a restaurant, Megan orders a meal advertised as containing 500 calories. She also orders a chocolate milkshake, which contains 490 calories. If Megan only eats 75% of her meal but drinks her entire milkshake, how many calories did Megan consume?
5. Calculate: $\frac{1}{5} + \frac{5}{6} + \frac{2}{15}$
6. Multiply: $(x - y)(3x^2 - 7x + 2)$
7. State the independent and dependent variables for the following situation:

   When the temperature changes, the height of the mercury in a thermometer either rises or falls.
8. Express as an entire radical: $4x^{\frac{3}{2}}$

Part B: Solving Rational Equations

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Determine whether each statement is true or false. If the statement is false, rewrite it so it is true.

   a) If $x \neq 2$, then $(x - 2)\frac{4x}{x - 2}$ is equivalent to $4x$.

   b) If $x \neq 3$, then $2(x - 3)\left(\frac{6}{x - 3} + \frac{x}{2}\right)$ is equivalent to $6 + x$.

   c) If $x \neq 0$, then the equations $x - \frac{1}{x} = 0$ and $x^2 - 1 = 0$ are equivalent.

   d) If $x \neq -3$ and $x \neq -4$, then $\frac{x - 1}{x + 3} = \frac{x - 2}{x + 4}$ is equivalent to

   $(x - 1)(x + 4) = (x + 3)(x - 2)$.

continued

Learning Activity 4.6 (continued)

2. Solve each of the following rational equations. Make sure you state the non-permissible values and check for extraneous roots.

   a) $x = \dfrac{-2}{x-3}$

   b) $\dfrac{2x-9}{x-7} + \dfrac{x}{2} = \dfrac{5}{x-7}$

   c) $x - 4 = \dfrac{-1}{x}$

   d) $\dfrac{3x^2}{3x+1} - 2 = \dfrac{2x+1}{3x+1}$

   e) $\dfrac{2x}{x-3} + \dfrac{1}{2x+3} + \dfrac{3x+9}{2x^2-3x-9} = 0$

   f) $\dfrac{x^2+12}{x-3} = \dfrac{7x}{x-3}$

3. The distance from Winnipeg to Minneapolis is about 600 km. You drove from Winnipeg to Minneapolis 10 km/h faster than from Minneapolis to Winnipeg. If the total driving time for the round trip was 22 hours, how fast did you drive from Winnipeg to Minneapolis?

4. House painters, Gwen and Megan, want to know how long it will take them if they work together to paint a house. Gwen, working alone, can paint a house in 13 hours. Megan, working alone, can paint a house in 11 hours. How long will it take them to paint a house if they work together? Round your answer to the nearest hundredth.

5. Jeff is trying to fill his above-ground pool with water. He knows that one of his hoses can fill the pool in 32 hours and another in 26 hours. If Jeff uses both hoses to fill the pool, how long will it take to fill the pool with water? Round your answer to the nearest tenth.

Lesson Summary

In this lesson, you learned how to solve rational equations. You also learned how to solve application problems involving rational equations. Make sure you complete Assignment 4.3, which is the last assignment in this module and the last assignment before your midterm examination.

Assignment 4.3

The Discriminant and Solving Rational Equations

Total: 47 marks

1. If the discriminant of a quadratic equation is the given value, state the nature of the roots. *(1 mark each, for a total of 4 marks)*

   a) 25

   b) -4

   c) 27

   d) 0

2. Determine the nature of the roots of the following equations. *(2 marks each, for a total of 10 marks)*

   a) $\frac{x^2}{2} + 4x + 4 = 0$

continued

Assignment 4.3: The Discriminant and Solving Rational Equations (continued)

b) $\frac{x-1}{2} - x^2 - 3 = 0$

c) $2(x^2 - 3) = 4x$

d) $6x^2 - x + 2 = 0$

e) $4x^2 - 12x + 9 = 0$

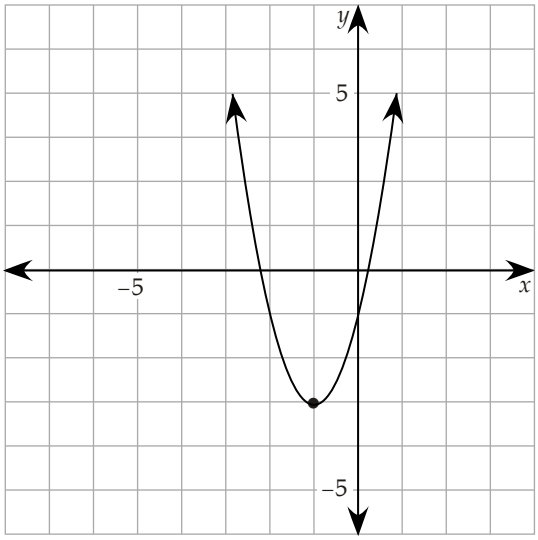
continued

Assignment 4.3: The Discriminant and Solving Rational Equations (continued)

3. For what values of k will the equation $2x^2 + 4x + (2 - k - k^2)$ have exactly one root? *(4 marks)*

4. Given the graph of $y = ax^2 + bx + c$, state whether the discriminant is zero, positive, or negative. *(1 mark each, for a total of 3 marks)*

   a)



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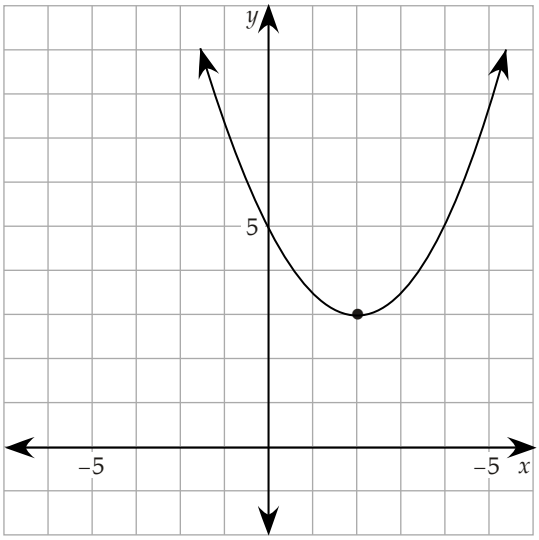
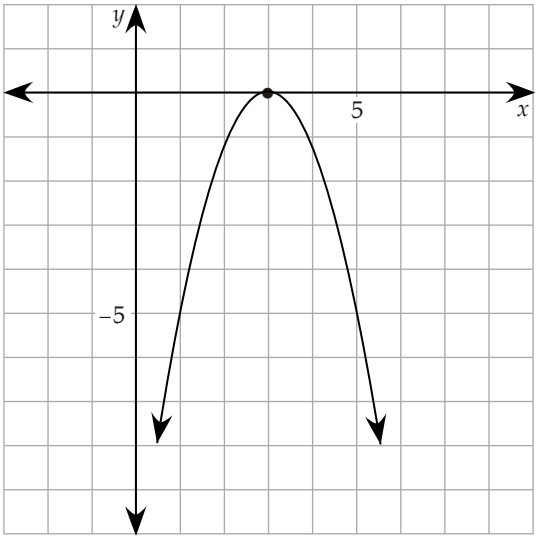
Assignment 4.3: The Discriminant and Solving Rational Equations (continued)

b)



c)



continued

Assignment 4.3: The Discriminant and Solving Rational Equations (continued)

5. Solve for x. State the non-permissible values of each equation. *(3 marks each, for a total of 15 marks)*

   a) $\dfrac{x}{x-1} - \dfrac{2}{x} = \dfrac{1}{x-1}$

   b) $\dfrac{1}{x-2} = \dfrac{x}{10-5x}$

continued

Assignment 4.3: The Discriminant and Solving Rational Equations (continued)

c) $\frac{x}{x-14} = -\frac{2}{x+1}$

d) $\frac{x+1}{x+3} - \frac{2x+1}{x-4} = \frac{-6x-39}{x^2-x-12}$

continued

Assignment 4.3: The Discriminant and Solving Rational Equations (continued)

e) $\dfrac{x}{x^2 + x - 42} = \dfrac{2x + 3}{x + 7} + \dfrac{x}{x - 6}$

continued

Assignment 4.3: The Discriminant and Solving Rational Equations (continued)

6. You and your family are going on a road trip to visit your recently married cousin in British Columbia. The total round trip distance is 2100 km. On the way to B.C., you encounter no setbacks and you are able to drive on average at a constant speed. On the way home from B.C., you encounter a snowstorm and your mom has to slow down to avoid getting into an accident. Thus, your average speed decreases by 10 km/h. If the total driving time was 42 hours, what was your average speed on the way to B.C.? Recall: Distance = speed × time. Round to the nearest tenth. *(6 marks)*

continued

Assignment 4.3: The Discriminant and Solving Rational Equations (continued)

7. To increase her endurance, a speed skater decides to include running as part of her workout. During her workout, she will run for 10 kilometres at a constant speed. After running, she will speed skate for 5 kilometres at 12 km/h faster than she can run. If her entire workout lasts for 1 hour, how fast can this athlete run? Round to the nearest hundredth. *(5 marks)*

Notes

Module 4 Summary

Congratulations! You are halfway done this course! In this module, you extended your knowledge of quadratic functions to learn how to solve quadratic equations using different methods. It is possible to solve quadratic equations by graphing, taking square roots, factoring, using the quadratic formula, and completing the square. These methods are useful in different situations, which you summarized in the organizational chart in Learning Activity 4.4. You also extended your knowledge from Module 2 on rational *expressions* to learn how to solve rational *equations*. Then, you learned how to solve applications problems involving motion by analyzing rational equations.

Before you continue on to Module 5, it is time to take your midterm exam. The following information will tell you how to apply and prepare for your midterm examination. It is also a good idea to complete the midterm practice examination, as it will let you know what areas you are struggling with, and perhaps need to study further.

In Module 5, you are going to be studying radicals. You had a brief look at radicals in *Grade 10 Introduction to Applied and Pre-Calculus Mathematics*. You also reviewed some of your knowledge on radicals in Lesson 2 in this module. In Module 5, you will learn how to perform operations on radicals and also solve radical equations.

Submitting Your Assignments

It is now time for you to submit Assignments 4.1 to 4.3 to the Distance Learning Unit so that you can receive some feedback on how you are doing in this course. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 4 assignments and organize your material in the following order:

- ❑ Module 4 Cover Sheet (found at the end of the course Introduction)
- ❑ Assignment 4.1: Solving Quadratic Equations by Graphing and Taking Square Roots
- ❑ Assignment 4.2: Solving Quadratic Equations
- ❑ Assignment 4.3: The Discriminant and Solving Rational Equations

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Midterm Examination

Congratulations, you have finished Module 4 in the course. The midterm examination is out of 100 marks and worth 20% of your final mark. In order to do well on this examination, you should review all of your learning activities and assignments from Modules 1 to 4.

You will complete this examination while being supervised by a proctor. You should already have made arrangements to have the examination sent to the proctor from the Distance Learning Unit. If you have not yet made arrangements to write it, then do so now. The instructions for doing so are provided in the Introduction to this module.

You will need to bring the following items to the examination: pens/pencils (2 or 3 of each), blank paper, a ruler, a scientific or graphing calculator, and your Examination Resource Sheet. A maximum of 2.5 hours is available to complete your midterm examination. When you have completed it, the proctor will then forward it for assessment. Good luck!

At this point you will also have to combine your resource sheets from Modules 1 to 4 onto one 8½" × 11" paper (you may use both sides). Be sure you have all the formulas, definitions, and strategies that you think you will need. This paper can be brought into the examination with you. We suggest that you divide your paper into two quadrants on each side so that each quadrant contains information from one module.

Examination Review

You are now ready to begin preparing for your midterm examination. Please review the content, learning activities, and assignments from Modules 1 to 4.

The midterm practice examination is also an excellent study aid for reviewing Modules 1 to 4.

You will learn what types of questions will appear on the examination and what material will be assessed. Remember, your mark on the midterm examination determines 20% of your final mark in this course and you will have 2.5 hours to complete the examination.

Midterm Practice Examination and Answer Key

To help you succeed in your examination, a practice examination can be found in the learning management system (LMS). The midterm practice examination is very similar to the actual examination that you will be writing. The answer key is also included so that, when you have finished writing the practice examination, you can check your answers. This will give you the confidence that you need to do well on your examination. If you do not have access to the Internet, contact the Distance Learning Unit at 1-800-465-9915 to get a copy of the practice examination and the answer key.

To get the most out of your midterm practice examination, follow these steps:

1. Study for the midterm practice examination as if it were an actual examination.
2. Review those learning activities and assignments from Modules 1 to 4 that you found the most challenging. Reread those lessons carefully and learn the concepts.
3. Contact your learning partner and your tutor/marker if you need help.
4. Review your lessons from Modules 1 to 4, including all of your notes, learning activities, and assignments.
5. Use your module resource sheets to make a draft of your Midterm Examination Resource Sheet. You can use both sides of an 8½" by 11" piece of paper.
6. Bring the following to the midterm practice examination: pens/pencils (2 or 3 of each), blank paper, a ruler, a scientific or graphing calculator, and your Midterm Examination Resource Sheet.
7. Write your midterm practice examination as if it were an actual examination. In other words, write the entire examination in one sitting, and don't check your answers until you have completed the entire examination. Remember that the time allowed for writing the midterm examination is 2.5 hours.
8. Once you have completed the entire practice examination, check your answers against the answer key. Review the questions that you got wrong. For each of those questions, you will need to go back into the course and learn the things that you have missed.
9. Go over your resource sheet. Was anything missing or is there anything that you didn't need to have on it? Make adjustments to your Midterm Examination Resource Sheet. Once you are happy with it, make a photocopy that you can keep.

Notes

Grade 11 Pre-Calculus Mathematics (30S)

Module 4
Solving Quadratic and Rational Equations

Learning Activity Answer Keys

Module 4:
Solving Quadratic and Rational Equations

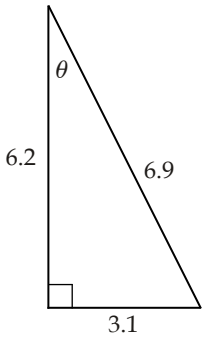
Learning Activity 4.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Brooklyn can type 55 words per minute. If her essay took her an hour to write, how many words are in her essay?
2. What is the greatest common factor of $3x^5y^3$ and $6x^2y^8$?
3. Find the vertex of the following parabola: $y = -2x^2 + 8x - 1$
4. Solve for x: $6x + 5 = 7$
5. How many centimetres are in one kilometre?

Use the following right-angled triangle to answer the next three questions. Do not simplify your answers.



6. Identify the sine ratio for θ.
7. Identify the cosine ratio for θ.
8. Identify the tangent ratio for θ.

Answers:

1. 3300 (55 × 60; There are 60 minutes in an hour, therefore, Brooklyn's essay contains 3300 words.)
2. $3x^2y^3$
3. (2, 7) $\left(x = -\frac{8}{-4} = 2;\ y = -2(4) + 8(2) - 1 = -8 + 16 - 1 = 7\right)$
4. $\frac{1}{3}$ $\left(6x = 2,\ x = \frac{2}{6} \text{ or } \frac{1}{3}\right)$
5. 100 000 cm in 1 km (There are 100 cm in 1 m and 1000 m in 1 km. Therefore, 100 × 1000 = 100 000 cm in 1 km.)
6. $\frac{3.1}{6.9}$
7. $\frac{6.2}{6.9}$
8. $\frac{3.1}{6.2}$

Part B: Solving Quadratic Equations by Graphing

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Rearrange each of the following into the general form of a quadratic equation with the leading coefficient greater than 0.
   a) $-2x^2 + 10 = -6$
   b) $3x(x + 2) = 8$
   c) $5x^2 - 7 + 3x = 7$
   d) $(x + 3)^2 = -4$
   e) $(x + 3)^2 - 4 = 5$
   f) $\frac{x^2 - 4}{3} = x$
   g) $(2x - 3)(x + 2) = 0$

Answers:

a) $-2x^2 + 10 = -6$

$-2x^2 + 16 = 0$ Add 6 to each side.

$2x^2 - 16 = 0$ Multiply by (-1).

b) $3x(x + 2) = 8$

$3x^2 + 6x = 8$ Distribute $3x$ into the bracket.

$3x^2 + 6x - 8 = 0$ Subtract 8 from each side.

c) $5x^2 - 7 + 3x = 7$

$5x^2 + 3x - 14 = 0$ Subtract 7 from each side and rearrange.

d) $(x + 3)^2 = -4$

$x^2 + 6x + 9 = -4$ Expand $(x + 3)^2$.

$x^2 + 6x + 13 = 0$ Add +4 to each side.

e) $(x + 3)^2 - 4 = 5$

$x^2 + 6x + 9 - 4 = 5$ Expand $(x + 3)^2$.

$x^2 + 6x + 5 = 5$ Simplify.

$x^2 + 6x = 0$ Subtract 5 from each side.

f) $\dfrac{x^2 - 4}{3} = x$

$x^2 - 4 = 3x$ Multiply each side by 3.

$x^2 - 3x - 4 = 0$ Subtract $3x$ from each side.

g) $(2x - 3)(x + 2) = 0$

$2x^2 + 4x - 3x - 6 = 0$ Multiply.

$2x^2 + x - 6 = 0$ Simplify.

2. Use the graph of the related function to solve each equation.

   a) $0 = (x - 2)^2 - 4$

   b) $0 = x^2 - 6x + 5$

   c) $x^2 + 6x + 1 = -7$

Answers:

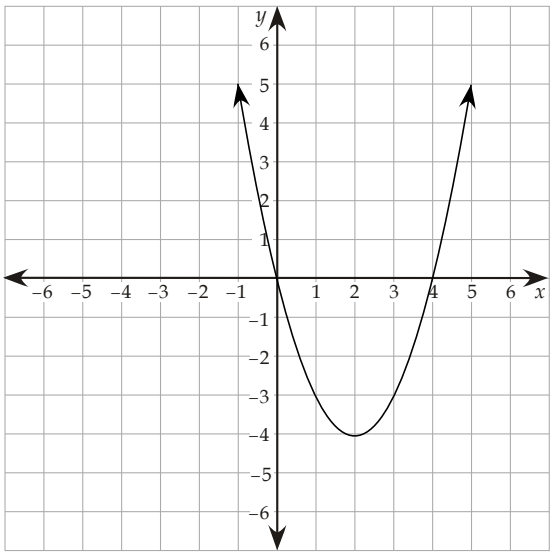
Method 1: Using Graph Paper

a) $0 = (x - 2)^2 - 4$

Graph the function:
$y = (x - 2)^2 - 4$

Vertex at (2, −4), opens up, standard size

The zeros are 4 and 0. The solutions of the equation are 4 and 0.



b) $0 = x^2 - 6x + 5$

The function is $y = x^2 - 6x + 5$.

First, complete the square to find the vertex and then graph the parabola.

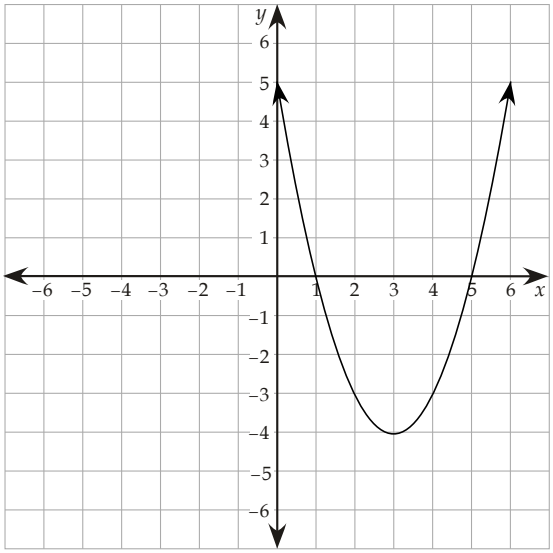
$y = x^2 - 6x + 5$

$y = (x^2 - 6x + 9) + 5 - 9$

$y = (x - 3)^2 - 4$

$\therefore$ Vertex is (3, −4); the curve opens up and is standard size.

The zeros are 1 and 5 and the solutions are 1 and 5.



c) $x^2 + 6x + 1 = 7$

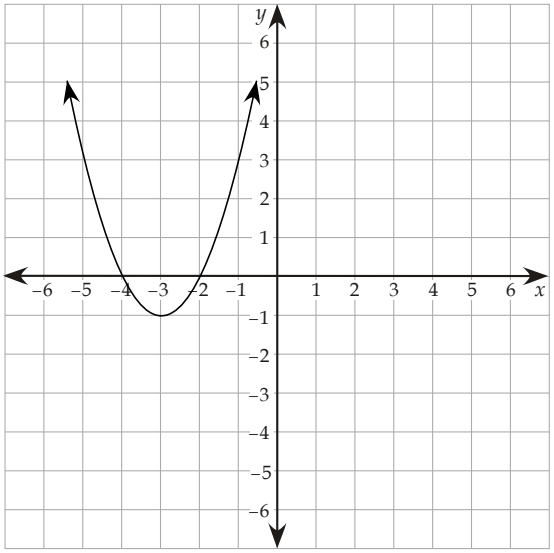
$x^2 + 6x + 8 = 0$

The function is $y = x^2 + 6x + 8$.

Upon completing the square, you get $y = (x + 3)^2 - 1$.

The vertex is $(-3, -1)$; the curve opens up and is normal size.

The zeros are -4 and -2.
The solutions are -4 and -2.

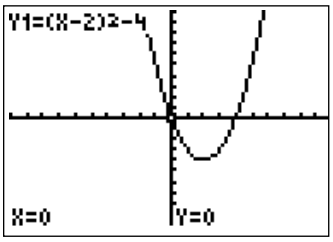


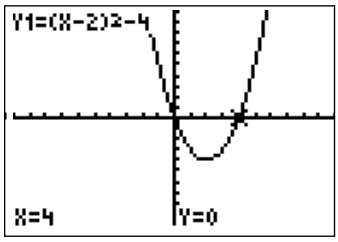
Method 2: Using a graphing calculator

a) ENTER $Y_1 = (x - 2)^2 - 4$

Set Window to standard size: XMIN −10 XMAX +10
YMIN −10 YMAX +10

Use TRACE and the arrow keys to approximate the x-values of the points where the graph crosses the x-axis.





Answers: $x = 0$ and 4

b) Press Y=

ENTER $Y_1 = x^2 - 6x + 5$

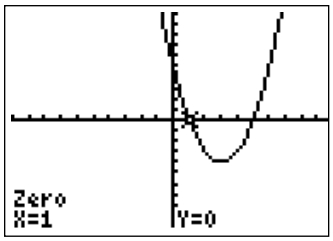
Set Window to standard size: XMIN −10 XMAX +10
YMIN −10 YMAX +10

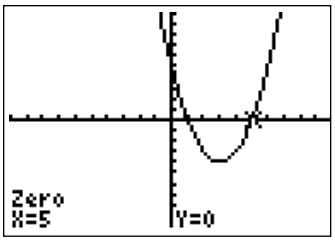
Use TRACE to approximate the x-values of the point where the parabola intersects the x-axis.

Answers: $x = 1$ and $x = 5$

Using 2nd (CALC) ZERO will give a more accurate answer.

Set left and right bounds and guess. Press ENTER after each.

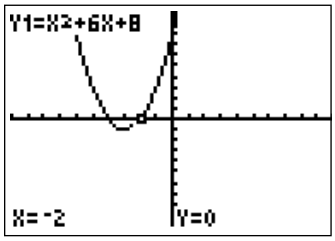


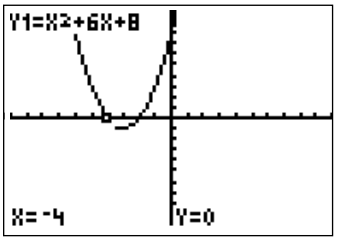


c) Press Y=

ENTER $Y_1 = x^2 + 6x + 8$

Use TRACE to approximate the zeros.





Answers: $x = -2$ or $x = -4$

3. a) Graph the quadratic function $y = x^2 - 2x - 8$ by completing the square, and then find the x-intercepts from the graph.
   b) Check your answer using a graphing calculator if you have one.
   c) Where does the graph intersect the x-axis?
   d) What are the zeros of the function?
   e) What are the roots of the equation $x^2 - 2x - 8 = 0$?
   f) Verify each root.

Answers:

a) $y = x^2 - 2x - 8$

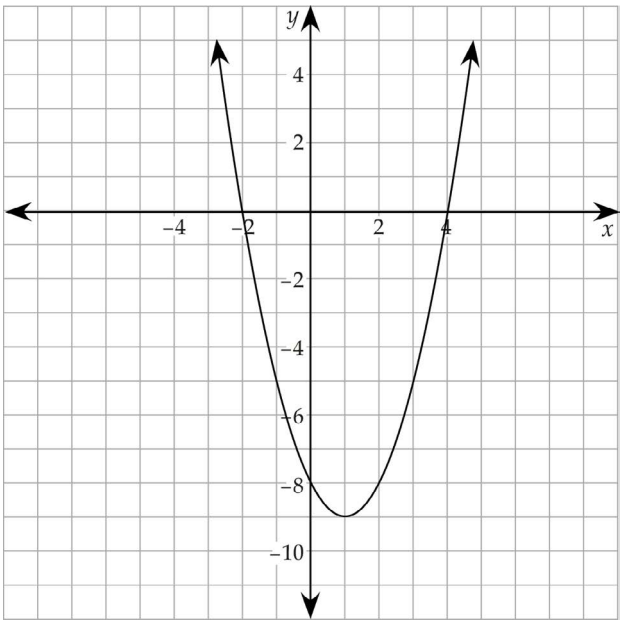
$y + 8 = x^2 - 2x$

$y + 8 + 1 = x^2 - 2x + 1$

$y + 9 = (x - 1)^2$

$y = (x - 1)^2 - 9$

V(1, −9)



The x-intercepts are −2 and 4.

b) Check your answer using a graphing calculator, finding the zeros of the function to be −2 and 4.

c) $y = x^2 - 2x + 8$

The points can be read from the graph.

The graph intersects the x-axis at the points (−2, 0) and (4, 0).

d) The zeros are 4, −2.

e) The roots of the equation are 4, −2 since they are the same as the zeros of the function.

f) Check by substituting into $x^2 - 2x - 8 = 0$.

For $x = 4$

$\text{LHS} = x^2 - 2x - 8$

$= 4^2 - 2(4) - 8$

$= 16 - 8 - 8$

$= 0$

$\text{RHS} = 0$

$\text{LHS} = \text{RHS}$

For $x = -2$

$\text{LHS} = x^2 - 2x - 8$

$= (-2)^2 - 2(-2) - 8$

$= 4 + 4 - 8$

$= 0$

$\text{RHS} = 0$

$\text{LHS} = \text{RHS}$

Thus, both roots are verified.

4. Solve each of the following equations by graphing.

a) $x^2 + 2x - 8 = 0$

b) $x^2 + 4x + 3 = 0$

c) $x^2 + 8x = -15$

d) $9 - x^2 = 0$

e) $2x^2 - 12x + 10 = 0$

Answers:

a) $x^2 + 2x - 8 = 0$

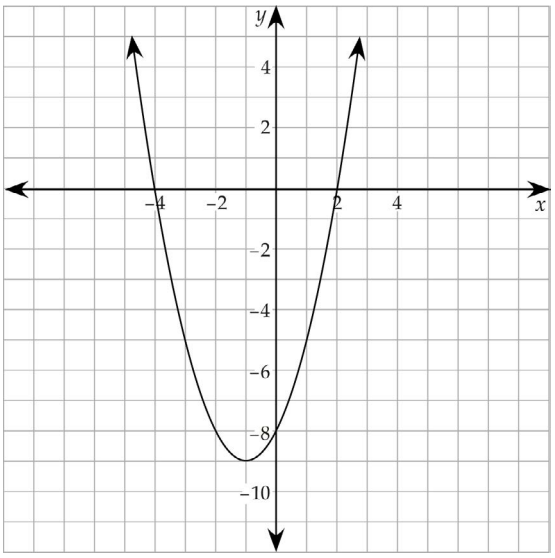
Let $y = x^2 + 2x - 8$

$y = (x^2 + 2x \quad) - 8$

$y = (x^2 + 2x + 1) - 8 - 1$

$y = (x + 1)^2 - 9$

V(−1, −9)



The solutions are $x = -4, 2$.

b) $x^2 + 4x + 3 = 0$

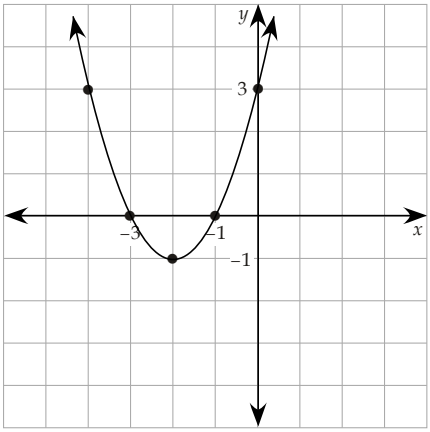
Let $y = x^2 + 4x + 3$

$y = (x^2 + 4x \quad) + 3$

$y = (x^2 + 4x + 4) + 3 - 4$

$y = (x + 2)^2 - 1$

V(−2, −1)



The solutions are $x = -3, -1$.

c) $x^2 + 8x = -15$

$x^2 + 8x + 15 = 0$

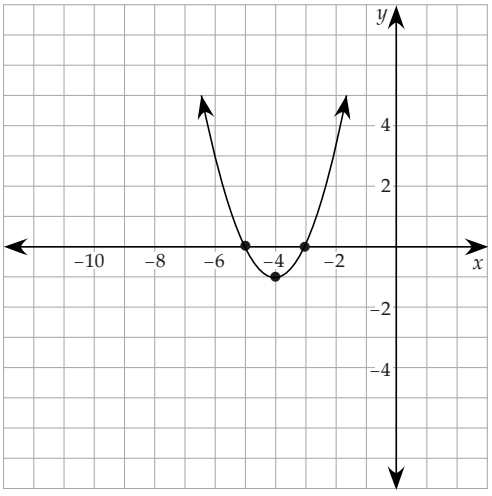
Let $y = x^2 + 8x + 15$

$y = (x^2 + 8x \quad) + 15$

$y = (x^2 + 8x + 16) + 15 - 16$

$y = (x + 4)^2 - 1$

V(−4, −1)

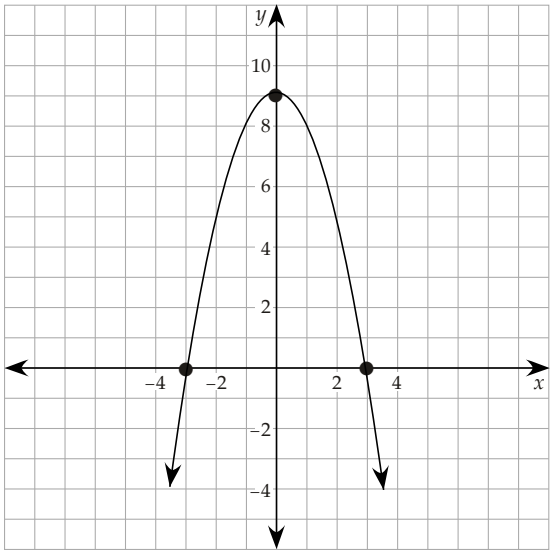


The solutions are $x = -5, -3$.

d) $9 - x^2 = 0$

Let $y = -x^2 + 9$

V(0, 9)



The solutions are $x = \pm 3$.

e) $2x^2 - 12x + 10 = 0$

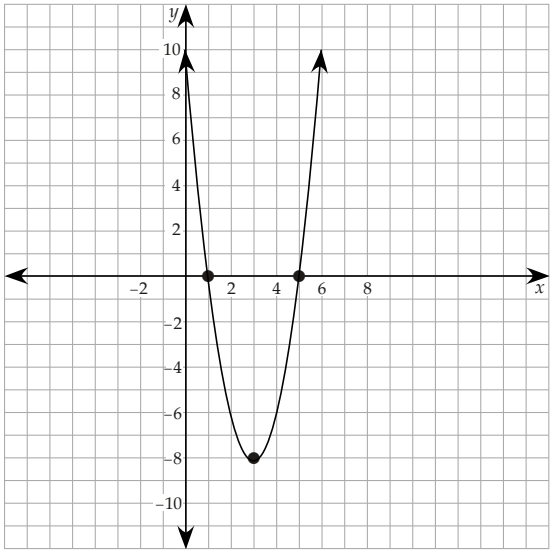
Let $y = 2x^2 - 12x + 10$

$y = 2(x^2 - 6x \quad) + 10$

$y = 2(x^2 - 6x + 9) + 10 - 9(2)$

$y = 2(x - 3)^2 - 8$

V(3, −8)



The solutions are $x = 1, 5$.

5. Kalia has been asked to solve the quadratic equation $-x^2 + 5x - 4 = 0$ by graphing. Her work for completing the square and graphing the corresponding quadratic function is shown below.

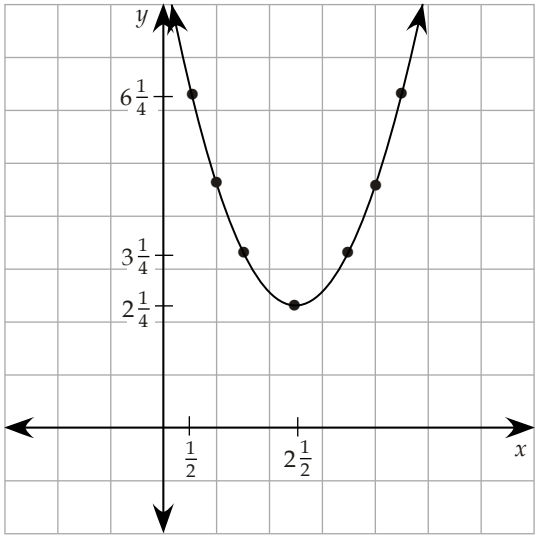
$$y = -x^2 + 5x - 4$$

$$y + 4 = -\left(x^2 - 5x\right)$$

$$y + 4 - \frac{25}{4} = -\left(x^2 - 5x + \frac{25}{4}\right)$$

$$y + \frac{16}{4} - \frac{25}{4} = -\left(x - \frac{5}{2}\right)^2$$

$$y = -\left(x - \frac{5}{2}\right)^2 + \frac{9}{4}$$

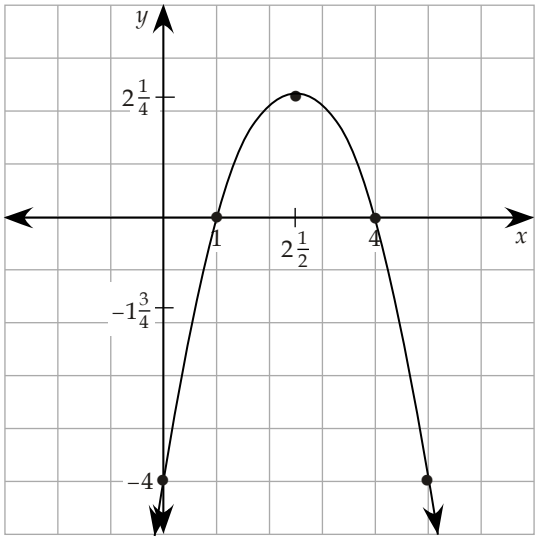


From this graph, Kalia concludes there are no x-intercepts.

Identify, explain, and correct Kalia's mistake.

Answer:

Kalia graphed the quadratic function incorrectly. Because there is a negative sign before the x^2 term, this parabola should open down. The correct graph is shown below.



From this graph, the x-intercepts are 1 and 4. Therefore, the solutions to the quadratic equation $-x^2 + 5x - 4 = 0$ are 1 and 4.

Learning Activity 4.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Cameron is packing his suitcase for Mexico. If his suitcase measures 110 cm by 40 cm by 60 cm, what is the volume of his suitcase?
2. You and your friend are going out for dinner and you want to split the bill evenly. If the bill comes to $76.36 and you want to tip the waiter $4, how much should you give the waiter?
3. Factor: $14m - 28$
4. Does the parabola $y = 3x^2 + 2x - 1$ open up or down?
5. Continue the pattern: A, E, F, H, I, K, L, M, ____ , ____
6. Find the midpoint between the two points $(9, -24)$ and $(4, 12)$.
7. Solve the inequality, $13x - 2 > x$.
8. Find the slope of the line parallel to the line $8x = 3y - 2$.

Answers:

1. 264 000 cm^3 ($110 \times 40 \times 60 = 110 \times 2400 = 264\,000\ cm^3$)
2. $\$42.18 \left(\frac{\$76.36}{2} + \$4 = \$38.18 + \$4 = \$42.18 \right)$
3. $14(m - 2)$
4. Up
5. N, T (The pattern is letters made from only straight lines.)
6. $\left(\frac{13}{2}, -6\right) \left(\text{M} = \left(\frac{9+4}{2}, \frac{-24+12}{2} \right) = \left(\frac{13}{2}, -\frac{12}{2} \right) = \left(\frac{13}{2}, -6 \right) \right)$
7. $x > \frac{1}{6} (12x > 2)$
8. $\frac{8}{3} \left(3y = 8x + 2; \frac{3y}{3} = \frac{8}{3}x + \frac{2}{3}; y = \frac{8}{3}x + \frac{2}{3} \right)$

Part B: Solving Quadratic Equations by Taking Square Roots

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Find the roots of the following equations.

   Recall: Finding the roots of an equation is identical to solving an equation.

   a) $x^2 = 16$

   Answer:

   $x = \pm\sqrt{16}$

   $x = \pm 4$

   b) $3x^2 - 18 = 0$

   Answer:

   $3x^2 = 18$

   $x^2 = 6$

   $x = \pm\sqrt{6}$

   c) $x^2 - 6 = -5$

   Answer:

   $x^2 = -5 + 6$

   $x^2 = 1$

   $x = \pm\sqrt{1}$

   $x = \pm 1$

   d) $6x^2 + 3 = 2x^2 + 7$

   Answer:

   $4x^2 = 4$

   $x^2 = 1$

   $x = \pm\sqrt{1}$

   $x = \pm 1$

   e) $x^2 = 6x^2 - 1$

   Answer:

   $1 = 5x^2$

   $x^2 = \frac{1}{5}$

   $x = \pm\sqrt{\frac{1}{5}}$

   $x = \pm\frac{1}{\sqrt{5}}$ or

   $x = \pm\frac{\sqrt{5}}{5}$

   f) $4x^2 - 9 = -3$

   Answer:

   $4x^2 = 6$

   $x^2 = \frac{3}{2}$

   $x = \pm\sqrt{\frac{3}{2}}$

   $x = \pm\frac{\sqrt{3}}{\sqrt{2}}$ or

   $x = \pm\frac{\sqrt{6}}{2}$

Note: For (e), you can rationalize the denominator or not. Either method of answering is acceptable. Remember, to rationalize a monomial radical denominator, multiply top and bottom of the fraction by the radical:

$$\frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{25}} = \frac{\sqrt{5}}{5}$$

g) $x^2 - 20 = 0$

Answer:

$x^2 = 20$

$x = \pm\sqrt{20}$

$x = \pm\sqrt{4}\sqrt{5}$

$x = \pm 2\sqrt{5}$

h) $24 - 2x^2 = 0$

Answer:

$2x^2 = 24$

$x^2 = 12$

$x = \pm\sqrt{12}$

$x = \pm\sqrt{4}\sqrt{3}$

$x = \pm 2\sqrt{3}$

i) $(x - 2)^2 = 5$

Answer:

Note: This question does not exactly fit the pattern. You have a binomial squared. However, you can still easily take the square root of both sides to eliminate the square and then add 2 to both sides.

$\sqrt{(x-2)^2} = \pm\sqrt{5}$

$x - 2 = \pm\sqrt{5}$

$x = 2 \pm \sqrt{5}$

Therefore, $x = 2 - \sqrt{5}$ or $x = 2 + \sqrt{5}$.

j) $\frac{2}{3}x^2 + 6 = 18$

Answer:

$3\left(\frac{2}{3}x^2 + 6\right) = 3(18)$ Multiply both sides of the equation by 3.

$2x^2 + 18 = 54$ Simplify.

$x^2 + 9 = 27$ Divide both sides of the equation by 2.

$x^2 = 18$

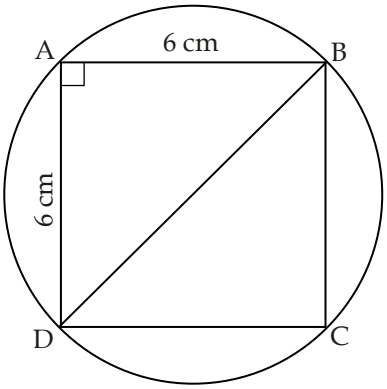
$x = \pm\sqrt{18}$

$x = \pm\sqrt{9}\sqrt{2}$

$x = \pm 3\sqrt{2}$

2. A square, ABCD, with sides of length 6 cm, is circumscribed by a circle. The four points, A, B, C, and D, are on the circumference of the circle. Find the area of a circle. Leave π in your answer. Remember if π is not approximated, your answer is exact. If π is approximated, your answer is approximate.

Answer:



Because ABCD is a square, $\angle A$ is 90° and the diagonal of the square will pass through the centre of the circle.

$$BD^2 = AD^2 + AB^2$$
$$BD^2 = 6^2 + 6^2$$
$$BD = \sqrt{36 + 36}$$
$$BD = \sqrt{36 \cdot 2}$$
$$BD = 6\sqrt{2}$$
$$\text{Radius} = \frac{1}{2}\left(6\sqrt{2}\right)$$
$$= 3\sqrt{2}$$
$$\text{Area of circle} = \pi r^2$$
$$= \pi \left(3\sqrt{2}\right)^2$$
$$= 18\pi \text{ cm}^2$$

3. Jordon is asked to solve the following quadratic equation by taking square roots. His work is shown below.

$$x^2 - 40 = 0$$
$$x^2 = 40$$
$$x = \pm\sqrt{40}$$
$$x = \pm 4\sqrt{10}$$

Identify, explain, and correct Jordon's error.

Answer:

Jordon's work is complete up until the last step. However, $\sqrt{40}$ should be simplified as follows:

40 can be written as the product of 4 and 10.

$$\sqrt{40} = \left(\sqrt{4}\right)\left(\sqrt{10}\right) = 2\sqrt{10}$$

Therefore, $x = \pm 2\sqrt{10}$ is the correct answer.

Learning Activity 4.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Your mom bought 500 g of deli meat. If you put an average of 125 g of meat on each sandwich you make, how many days will the deli meat last for (assuming you only eat one sandwich a day)?
2. You are playing soccer on a co-ed team. You have 20 players on your team and 3 of them are girls. You need at least 20% of your players to be female according to the league. Do you have enough female players?
3. Simplify: $(x + 5)^2$
4. Give a simplified quadratic function that has zeros of -1 and -5 and $a = 1$.
5. Simplify: $2^4 \times 3^2$
6. Express as a power: $\sqrt{15}$
7. Are these fractions equivalent, $\frac{\sqrt{2}}{\sqrt{3}}$ and $\frac{\sqrt{4}}{3}$?
8. The x-intercepts of a parabola are -5 and 2 and the y-intercept is 10. Write the quadratic function in factored form.

Answers:

1. 4 days $\left(\frac{500}{125} = 4\right)$
2. No $\left(\frac{3}{20} = \frac{15}{100} = 15\%\right)$
3. $x^2 + 10x + 25$ $[(x + 5)^2 = (x + 5)(x + 5) = x^2 + 5x + 5x + 25]$
4. $y = 1(x + 1)(x + 5) = x^2 + 6x + 5$
5. 144 (16×9)
6. $15^{\frac{1}{2}}$
7. These fractions are not equivalent. $\left(\frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3}, \text{ not } \frac{\sqrt{4}}{3}\right)$
8. $y = -(x + 5)(x - 2)$ (The factored form is $y = a(x + 5)(x - 2)$. Since the y-intercept is 10, $10 = a(0 + 5)(0 - 2)$ and the value of a is -1.)

Part B: Solving Quadratic Equations by Factoring

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Solve these equations by factoring.

   a) $x^2 - x - 12 = 0$

   Answer:

   $x^2 - x - 12 = 0$

   $(x-4)(x+3) = 0$

   $x - 4 = 0 \quad \text{or} \quad x + 3 = 0$

   $x = 4 \quad \text{or} \quad x = -3$

   b) $x^2 - x - 20 = 0$

   Answer:

   $x^2 - x - 20 = 0$

   $(x-5)(x+4) = 0$

   $x - 5 = 0 \quad \text{or} \quad x + 4 = 0$

   $x = 5 \quad \text{or} \quad x = -4$

   c) $-x^2 - 2x + 3 = 0$

   Answer:

   $-x^2 - 2x + 3 = 0$

   $x^2 + 2x - 3 = 0$ Multiply each side by (-1).

   $(x+3)(x-1) = 0$

   $x + 3 = 0 \quad \text{or} \quad x - 1 = 0$

   $x = -3 \quad \text{or} \quad x = 1$

d) $x^2 + 9x + 18 = 0$

Answer:

$x^2 + 9x + 18 = 0$

$(x + 6)(x + 3) = 0$

$x + 6 = 0$ or $x + 3 = 0$

$x = -6$ or $x = -3$

e) $2x^2 + 3x - 2 = 0$

Answer:

$2x^2 + 3x - 2 = 0$

$(2x - 1)(x + 2) = 0$

$2x - 1 = 0$ or $x + 2 = 0$

$x = \frac{1}{2}$ or $x = -2$

2. Rearrange each of the following equations and solve by factoring, if possible.

a) $10x^2 = 7x + 12$

Answer:

$10x^2 - 7x - 12 = 0$

$(5x + 4)(2x - 3) = 0$

$5x + 4 = 0$ or $2x - 3 = 0$

$x = -\frac{4}{5}$ or $x = \frac{3}{2}$

b) $5x^2 + 21x = 54$

Answer:

$5x^2 + 21x - 54 = 0$

$(5x - 9)(x + 6) = 0$

$5x - 9 = 0$ or $x + 6 = 0$

$x = \frac{9}{5}$ or $x = -6$

c) $3x(x-2)-x(x+1)+5=0$

Answer:

$3x(x-2)-x(x+1)+5=0$

$3x^2-6x-x^2-x+5=0$

$2x^2-7x+5=0$

$(2x-5)(x-1)=0$

$2x-5=0 \quad \text{or} \quad x-1=0$

$x=\frac{5}{2} \quad \text{or} \quad x=1$

d) $x^2+\frac{9}{2}x-2\frac{1}{2}=0$

Answer:

$x^2+\frac{9}{2}x-2\frac{1}{2}=0$

$x^2+\frac{9}{2}x-\frac{5}{2}=0$

$2x^2+9x-5=0$ Multiply by 2.

$(2x-1)(x+5)=0$

$2x-1=0 \quad \text{or} \quad x+5=0$

$x=\frac{1}{2} \quad \text{or} \quad x=-5$

e) $x^2+9=0$

Answer:

$x^2+9=0$

Does not factor.

Solve by using inverse operations.

$x^2=-9$

$x=\pm\sqrt{-9}$

$x=$ no real answer

f) $16x^2 - 64 = 0$

Answer:

$$16x^2 - 64 = 0$$

$$16\left(x^2 - 4\right) = 0$$

$$16(x-2)(x+2) = 0$$

$$(x-2)(x+2) = 0 \qquad \text{Divide by 16.}$$

$$x - 2 = 0 \quad \text{or} \quad x + 2 = 0$$

$$x = 2 \quad \text{or} \quad x = -2$$

g) $x^2 + 12x - 28 = 0$

Answer:

$$x^2 + 12x - 28 = 0$$

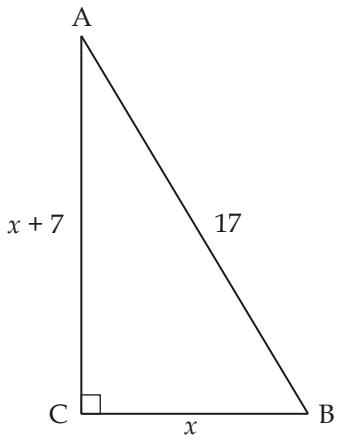
$$(x+14)(x-2) = 0$$

$$x + 14 = 0 \quad \text{or} \quad x - 2 = 0$$

$$x = -14 \quad \text{or} \quad x = 2$$

3. One leg of a right triangle is 7 m longer than the other leg. The hypotenuse is 17 m long. Find the length of each leg.

Answer:



Let x = shorter leg of the right triangle

$x + 7$ = larger leg of the right triangle

Because the triangle has a right angle,

$$a^2 + b^2 = c^2$$

$$x^2 + (x+7)^2 = 17^2$$

$$x^2 + x^2 + 14x + 49 = 289$$

$$2x^2 + 14x - 240 = 0$$

$$2\left(x^2 + 7x - 120\right) = 0$$

$$x^2 + 7x - 120 = 0$$

$$(x+15)(x-8) = 0$$

$x + 15 = 0$ or $x - 8 = 0$

$x = -15$ or $x = 8$

$x = -15$ is not possible as a length

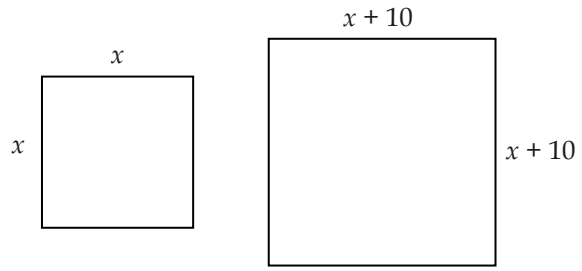
$\therefore x = 8$ m is the shorter side

$x + 7 = 15$ m is the longer side

Note: When solving word problems, writing a statement at the beginning of your solution that identifies what the variables represent is a necessary part of your solution.

4. Two square checkerboards together have an area of 1300 cm^2. One has sides that are 10 cm longer than the other. Find the length of the sides of each checkerboard.

   Answer:



Let x = length of side of smaller square

Let $x + 10$ = length of side of larger square

Sum of areas = 1300 cm^2

$$(x \cdot x) + (x + 10)(x + 10) = 1300$$

$$x^2 + x^2 + 20x + 100 = 1300$$

$$2x^2 + 20x - 1200 = 0$$

$$2\left(x^2 + 10x - 600\right) = 0$$

$$x^2 + 10x - 600 = 0$$

$$(x - 20)(x + 30) = 0$$

$x = 20$ or $x = -30$

Since x represents a length, the negative value of -30 must be ignored.

The only solution can be $x = 20$ and $x + 10 = 30$.

Thus, the length of each side of the smaller checkerboard is 20 cm and of the bigger checkerboard 30 cm.

5. Find 3 consecutive odd integers such that the product of the second and the third is 63.

Answer:

Let x = first odd integer

$x + 2$ = second odd integer

$x + 4$ = third odd integer

$(x + 2)(x + 4) = 63$

$x^2 + 6x + 8 = 63$

$x^2 + 6x - 55 = 0$

$(x + 11)(x - 5) = 0$

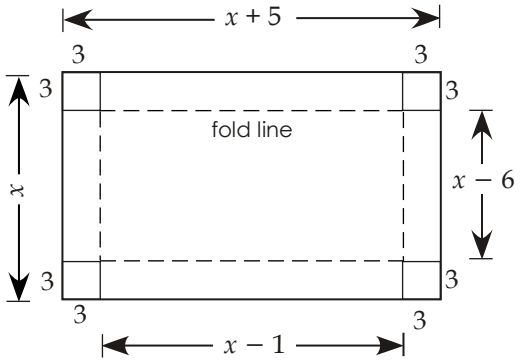
$x = -11$ or $x = 5$

If $x = -11$ is first integer
$x + 2 = -9$ is second integer
$x + 4 = -7$ is third integer

If $x = 5$ is first integer
$x + 2 = 7$ is second integer
$x + 4 = 9$ is third integer

6. A rectangular piece of cardboard is 5 cm longer than it is wide. A 3 cm-by-3 cm square is cut out of each corner, and the four flaps are folded up to form a box with a volume of 450 cm^3. What were the length and width of the original piece of cardboard? (Volume = length × width × height)

Answer:



Let x = width of rectangle

$x + 5$ = length of rectangle

3 = height of box = side of square removed from each corner

$x + 5 - 6$ or

$x - 1$ = length of box

$x - 6$ = width of box

$450 = (x - 1)(x - 6)(3)$ Formula for volume.

$150 = (x - 1)(x - 6)$ Divide by 3.

$150 = x^2 - 7x + 6$

$0 = x^2 - 7x - 144$

$0 = (x - 16)(x + 9)$

$x = 16$ or $x = -9$

Since the width cannot be negative, our solution is $x = 16$.

The dimensions of the original piece of cardboard are 16 cm wide and 21 cm long.

7. Farrah was asked to solve a quadratic equation by factoring. Her work is shown below.

   $x^2 - 10x + 24 = 0$

   $(x - 6)(x - 4) = 0$

   Therefore, $x = -6$ or $x = -4$.

   Identify, explain, and correct the errors Farrah made when trying to solve this quadratic equation.

   Answer:

   Farrah did not equate each factor of the quadratic equation to zero. Instead, she just took the terms inside the brackets as solutions. The correct solutions are:

   $x - 6 = 0$ or $x - 4 = 0$

   $x = 6$ $x = 4$

8. Write the quadratic function $y = -4(x - 3)(x + 1)$ in vertex form using the average of the x-intercepts.

   Answer:

   For $y = -4(x - 3)(x + 1)$, the x-intercepts are $x = 3$ and $x = -1$. The x-coordinate of the vertex is $\frac{3 + (-1)}{2} = 1$. The y-coordinate of the vertex is $x = -4(1 - 3)(1 + 1) = -4(-2)(2) = 16$. The leading coefficient, a, is -4. The function in vertex form is $y = -4(x - 1)^2 + 16$.

Learning Activity 4.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Evaluate: $\sqrt{115 + \sqrt{27 + \sqrt{76 + \sqrt{25}}}}$
2. Britney is making cookies. The recipe calls for $\frac{3}{8}$ of a cup of butter. If Britney is tripling the recipe, how much butter does she need? Write your answer as a mixed fraction.
3. Joanne makes $400 in a month. If Joanne needs to pay $66 for her phone bill, $40 for gas to drive her parents' car, and $120 for a haircut, will she have enough to buy the $150 jeans she wants?
4. Factor: $\frac{1}{3}x^2 - \frac{121}{3}$
5. Find the vertex of the following parabola: $y = 3x^2 + 6x - 5$
6. Rationalize the denominator: $\frac{6}{\sqrt{2}}$
7. Find the slope of the line perpendicular to the line, $7x + 3y - 4 = 0$.
8. The circumference of a circle is 92π centimetres. What is the radius of the circle?

Answers:

1. $11 \left(\sqrt{115 + \sqrt{27 + \sqrt{76 + \sqrt{25}}}}\,; \sqrt{115 + \sqrt{27 + \sqrt{76 + 5}}}\,; \sqrt{115 + \sqrt{27 + \sqrt{81}}}\,; \right.$
   $\left. \sqrt{115 + \sqrt{27 + 9}}\,; \sqrt{115 + \sqrt{36}}\,; \sqrt{115 + 6}\,; \sqrt{121}\,; = 11 \right)$
2. $1\frac{1}{8}$ cups of butter $\left(\frac{3}{8} \times 3 = \frac{9}{8} \right)$
3. Yes ($400 − ($66 + $40 + $120) = $400 − $226 = $174)
4. $\frac{1}{3}(x - 11)(x + 11)\left[\frac{1}{3}x^2 - \frac{121}{3} = \frac{1}{3}\left(x^2 - 121\right) = \frac{1}{3}(x - 11)(x + 11) \right]$

5 $(-1, -8)$ $\left(x = -\frac{6}{2(3)} = -\frac{6}{6} = -1;\ y = 3(-1)^2 + 6(-1) - 5 = 3 - 6 - 5 = -8\right)$

6. $3\sqrt{2}$ $\left(\frac{6}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}\right)$

7. $\frac{3}{7}$ $\left(3y = -7x + 4;\ y = -\frac{7}{3}x + \frac{4}{3}\right)$

   (Recall: Perpendicular lines have slopes that are negative reciprocals.)

8. 46 cm $\left(C = 2\pi r;\ C = 92\pi;\ \therefore 2\pi r = 92\pi;\right.$

   $\left. r = \frac{92\pi}{2\pi};\ r = \frac{92}{2} = 46 \text{ cm}\right)$

Part B: Solving Quadratic Equations by Completing the Square and Using the Quadratic Formula

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. For each quadratic equation below, state the values of a, b, and c for the general equation format.
   a) $x^2 - 2x - 5 = 0$
   b) $3x^2 - 2x + 5 = 0$
   c) $5x^2 - 3x = 8$
   d) $2(x^2 - 2x) - 1 = 0$
   e) $5x^2 = 9x$
   f) $4 - 2x^2 = 9x$

   Answers:

   a) $x^2 - 2x - 5 = 0$
   $a = 1, b = -2, c = -5$

   b) $3x^2 - 2x + 5 = 0$
   $a = 3, b = -2, c = 5$

   c) $5x^2 - 3x - 8 = 0$
   $a = 5, b = -3, c = -8$

   d) $2(x^2 - 2x) - 1 = 0$
   $2x^2 - 4x - 1 = 0$
   $a = 2, b = -4, c = -1$

   e) $5x^2 = 9x$
   $5x^2 - 9x = 0$
   $a = 5, b = -9, c = 0$

   f) $4 - 2x^2 = 9x$
   $-2x^2 - 9x + 4 = 0$
   $2x^2 + 9x - 4 = 0$
   $a = 2, b = 9, c = -4$

2. Solve these equations by factoring or by using the quadratic formula. Be sure to state the formula before substituting values into it, as this will help you memorize it

a) $\frac{x^2}{2} + x - \frac{15}{2} = 0$

b) $2w^2 - 3w + 1 = 0$

c) $7w^2 - 3w = 0$

d) $1 = 25x^2$

e) $x^2 - 0.1x - 0.06 = 0$

f) $-x^2 - 7x - 1 = 0$

Answers:

a) $\frac{x^2}{2} + x - \frac{15}{2} = 0$

$\frac{1}{2}\left(x^2 + 2x - 15\right) = 0$

$\frac{1}{2}(x + 5)(x - 3) = 0$

$x + 5 = 0$ or $x - 3 = 0$

$x = -5$ or $x = 3$

b) $2w^2 - 3w + 1 = 0$

$(2w - 1)(w - 1) = 0$

$2w - 1 = 0$ or $w - 1 = 0$

$w = \frac{1}{2}$ or $w = 1$

c) $7w^2 - 3w = 0$

$w(7w - 3) = 0$

$w = 0$ or $7w - 3 = 0$

$w = 0$ or $w = \frac{3}{7}$

d) $1 = 25x^2$

$0 = 25x^2 - 1$

$0 = (5x - 1)(5x + 1)$

$5x - 1 = 0$ or $5x + 1 = 0$

$x = \frac{1}{5}$ or $x = -\frac{1}{5}$

e) $x^2 - 0.1x - 0.06 = 0$

$(x - 0.3)(x + 0.2) = 0$

$x - 0.3 = 0$ or $x + 0.2 = 0$

$x = 0.3$ or $x = -0.2$

f) $-x^2 - 7x - 1 = 0$

$x^2 + 7x + 1 = 0$

$a = 1, b = 7, c = 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-7 \pm \sqrt{49 - 4}}{2}$$

$$x = \frac{-7 \pm \sqrt{45}}{2}$$

$$x = \frac{-7 \pm 3\sqrt{5}}{2}$$

3. Use the completing-the-square method to find the roots of each equation below.
   a) $3x^2 - 6x - 15 = 0$
   b) $2x^2 - 4x - 1 = 0$
   c) $9x^2 - 18x - 7 = 0$
   d) $x^2 - x - 3 = 0$

Answers:

a) $3x^2 - 6x - 15 = 0$

$3\left(x^2 - 2x \quad\right) = 15$

$3\left(x^2 - 2x + 1\right) = 15 + 3(1)$

$3(x-1)^2 = 18$

$(x-1)^2 = 6$

$x - 1 = \pm\sqrt{6}$

$x = 1 \pm \sqrt{6}$

b) $2x^2 - 4x - 1 = 0$

$2\left(x^2 - 2x \quad\right) = 1$

$2\left(x^2 - 2x + 1\right) = 1 + 2(1)$

$2(x-1)^2 = 3$

$(x-1)^2 = \frac{3}{2}$

$x - 1 = \pm\sqrt{\frac{3}{2}}$

$x = 1 \pm \frac{\sqrt{3}}{\sqrt{2}}$

$x = 1 \pm \frac{\sqrt{6}}{2}$

$x = \frac{2 \pm \sqrt{6}}{2}$

c) $9x^2 - 18x - 7 = 0$

$9\left(x^2 - 2x \quad\right) = 7$

$9\left(x^2 - 2x + 1\right) = 7 + 9(1)$

$9(x-1)^2 = 16$

$(x-1)^2 = \frac{16}{9}$

$x - 1 = \pm\frac{4}{3}$

$x = 1 \pm \frac{4}{3}$

$x = 1 + \frac{4}{3}, 1 - \frac{4}{3}$

$x = \frac{7}{3}, -\frac{1}{3}$

d) $x^2 - x - 3 = 0$

$x^2 - x + \frac{1}{4} = 3 + \frac{1}{4}$

$\left(x - \frac{1}{2}\right)^2 = \frac{13}{4}$

$x - \frac{1}{2} = \pm\sqrt{\frac{13}{4}}$

$x = \frac{1}{2} \pm \frac{\sqrt{13}}{2}$

$x = \frac{1 \pm \sqrt{13}}{2}$

4. Find the zeros of the function using the quadratic formula.

   a) $f(x) = 5x^2 - x - 3$

   b) $y = 2x^2 + 6x - 1$

Answers:

a) $f(x) = 5x^2 - x - 3$

$0 = 5x^2 - x - 3$

$a = 5,\ b = -1,\ c = -3$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{+1 \pm \sqrt{(-1)^2 - 4(5)(-3)}}{2(5)}$$

$$x = \frac{1 \pm \sqrt{1 + 60}}{10}$$

$$x = \frac{1 \pm \sqrt{61}}{10}$$

$$x = \frac{1 + \sqrt{61}}{10} \text{ or } \frac{1 - \sqrt{61}}{10}$$

b) $y = 2x^2 + 6x - 1$

$0 = 2x^2 + 6x - 1$

$a = 2,\ b = 6,\ c = -1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-6 \pm \sqrt{(6)^2 - 4(2)(-1)}}{2(2)}$$

$$x = \frac{-6 \pm \sqrt{36 + 8}}{4}$$

$$x = \frac{-6 \pm \sqrt{44}}{4}$$

$$x = \frac{-6 \pm 2\sqrt{11}}{4}$$

$$x = \frac{2\left(-3 \pm \sqrt{11}\right)}{2(2)}$$

$$x = \frac{-3 \pm \sqrt{11}}{2}$$

$$x = \frac{-3 + \sqrt{11}}{2} \text{ or } \frac{-3 - \sqrt{11}}{2}$$

5. Find the roots of the quadratic equation $6x^2 + 5x - 6 = 0$

   a) using the quadratic formula

   b) by factoring

   Which method do you prefer and why?

   Answers:

   a) $6x^2 + 5x - 6 = 0$

   $a = 6, b = 5, c = -6$

   $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

   $$x = \frac{-5 \pm \sqrt{(5)^2 - 4(6)(-6)}}{2(6)}$$

   $$x = \frac{-5 \pm \sqrt{169}}{12}$$

   $$x = \frac{-5 \pm 13}{12} = \frac{-5 + 13}{12}, \frac{-5 - 13}{12} = \frac{8}{12}, -\frac{18}{12}$$

   $$x = \frac{2}{3} \text{ or } -\frac{3}{2}$$

   b) $6x^2 + 5x - 6 = 0$

   $(2x + 3)(3x - 2) = 0$

   $2x + 3 = 0$ $\qquad$ $3x - 2 = 0$

   $2x = -3$ $\qquad$ $3x = 2$

   $x = \frac{-3}{2}$ $\qquad$ $x = \frac{2}{3}$

You may prefer either method. Perhaps you find factoring easier than plugging values into a formula. Or, you may like the certainty of using a formula. Both methods are important for future mathematics courses so you should practise both.

6. Find the zeros of the quadratic function $f(x) = x^2 + 8x + 15$ by
   a) factoring
   b) using the quadratic formula
   c) graphing
   d) completing the square.

   Which method do you prefer and why?

   Answers:

   a) $0 = x^2 + 8x + 15$
   $0 = (x + 3)(x + 5)$
   $x = -3, -5$

   b) $a = 1, b = 8, c = 15$

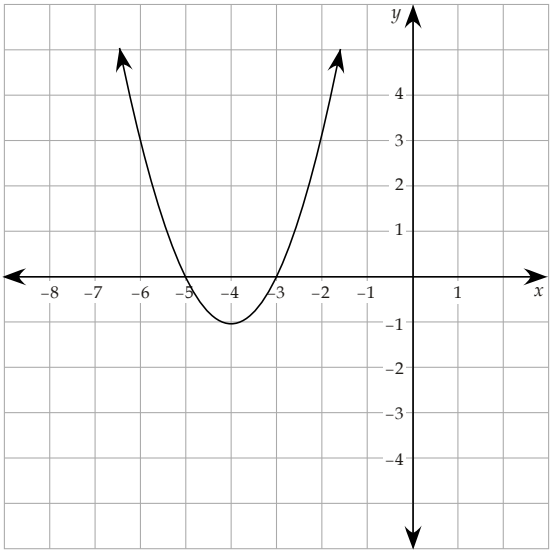
$$x = \frac{-8 \pm \sqrt{64 - 60}}{2}$$

$$x = \frac{-8 \pm \sqrt{4}}{2}$$

$$x = \frac{-8 \pm 2}{2} = \frac{-8 + 2}{2}, \frac{-8 - 2}{2}$$

   $x = -3$ or -5

   c) $y = x^2 + 8x + 15$
   $y = (x^2 + 8x + 16) + 15 - 16$
   $y = (x + 4)^2 - 1$
   Vertex: $(-4, -1)$, opens up, standard size
   Zeros: $x = -3, -5$



d)
$$0 = x^2 + 8x + 15$$
$$0 = \left(x^2 + 8x + 16\right) + 15 - 16$$
$$0 = (x + 4)^2 - 1$$
$$1 = (x + 4)^2$$
$$x + 4 = \pm 1$$
$$x = \pm 1 - 4$$
$$x = 1 - 4, -1 - 4$$
$$x = -3, -5$$

You may prefer any of the above methods for a variety of reasons. It is a good idea to practise them all—check your answers if you try more than one method. Some questions may be easier to solve using one method while others are easier to solve using a different method.

7. Refer to the Solving Quadratic Equations chart found at the end of this learning activity. For each of the five methods of solving quadratic equations, describe when each method could be best used. Fill in your answers on the chart.

Answer:

See the Solving Quadratic Equations chart at the end of the answers for Learning Activity 4.4 for possible descriptions. Discuss your answers with your learning partner or your tutor/marker if you are unsure.

8. Write a quadratic equation in general form that satisfies each of the following descriptions. Explain your answers.
   a) two distinct real roots
   b) one distinct real root, or two equal roots
   c) no real roots

Answers:

Answers will vary—possible answers are given. Check your answers with your learning partner or tutor/marker.

a) If you pick the roots to be $x = 1$ and $x = 5$, then $(x - 1)$ and $(x - 5)$ are factors of the corresponding quadratic equation. Thus, the corresponding quadratic equation could be $(x - 1)(x - 5) = 0$ or $x^2 - 6x + 5 = 0$.

b) If you pick the two equal roots to be $x = 1$ and $x = 1$, then $(x - 1)$ and $(x - 1)$ are factors of the corresponding quadratic equation. Thus, the quadratic equation is $(x - 1)(x - 1) = 0$ or $x^2 - 2x + 1 = 0$.

c) A quadratic equation that has no real roots has a corresponding quadratic function that does not intersect the x-axis. For example, $y = (x - 2)^2 + 1$ has its vertex at (2, 1) and opens up. In general form, the equation is

$$0 = (x - 2)^2 + 1$$
$$0 = x^2 - 4x + 4 + 1$$
$$0 = x^2 - 4x + 5$$

Also, when you use the quadratic formula, you get the following result:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{+4 \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)}$$
$$x = \frac{4 \pm \sqrt{-4}}{2}$$

x = not a real number. Thus, there are no roots.

9. Is the quadratic formula always the most efficient way to solve quadratic equations? Explain with examples.

Answer:

No. In Question 6, the quadratic equation could easily be solved by factoring. Therefore, the quadratic formula could sometimes make an easy question unnecessarily hard.

10. A rocket is launched and follows a path described by the quadratic function $h(d) = -0.02d^2 + 0.4d + 1$. The height is represented by h while the horizontal distance is represented by d, and both are measured in metres. How far away from its original position does this rocket land?

Answer:

When the rocket lands, its height is zero. Therefore, the quadratic equation is $-0.02d^2 + 0.4d + 1 = 0$. To find the distance at this time, you need to solve for the roots of this quadratic equation. The completing-the-square method is shown below.

$-0.02d^2 + 0.4d + 1 = 0$

$d^2 - 20d - 50 = 0$	Dividing by -0.02.
$d^2 - 20d = 50$	Add 50.
$d^2 - 20d + 100 = 50 + 100$	Completing the square.
$(d - 10)^2 = 150$	Factoring and simplifying.
$d - 10 = \pm\sqrt{150}$	Taking square roots.
$d = 10 \pm 5\sqrt{6}$	

Two possible answers are:

$d = 10 + 5\sqrt{6} \approx 22.247$

$d = 10 - 5\sqrt{6} \approx -2.247$

$d = 10 - 5\sqrt{6}$ is a root of the equation but not a solution to the problem. The rocket cannot travel a negative distance. Thus, $10 - 5\sqrt{6}$ is an extraneous root. The only solution is $d = 10 + 5\sqrt{6}$. Therefore, the rocket has travelled $\left(10 + 5\sqrt{6}\right)$ m.

11. David is trying to solve a quadratic equation by using the quadratic formula. His work is shown below.

$10x^2 + 2x + 1 = 0$

$a = 10, b = 2, c = 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(10)}}{2(10)}$$

$$x = \frac{-2 \pm \sqrt{4 - 40}}{20}$$

$$x = \frac{-2 \pm \sqrt{-36}}{20}$$

$$x = \frac{-2 \pm 6}{20}$$

$$x = \frac{-2 + 6}{20} = \frac{4}{20} = \frac{1}{5}$$

$$x = \frac{-2 - 6}{20} = \frac{-8}{20} = \frac{-2}{5}$$

Identify, explain, and correct David's error.

Answer:

David took the square root of a negative number (−36). It is not possible to find a real number answer for the square root of a negative number. Therefore, there is no solution to this quadratic equation in real numbers like David thought there was.

12. Solve the equation:

$(x^2 - 5)^2 + 3(x^2 - 5) + 2 = 0$

Answer:

Notice that there is a quadratic pattern in the equation. Temporarily, let $a = x^2 - 5$ and substitute into the original equation.

$\therefore a^2 + 3a + 2 = 0$

$(a + 2)(a + 1) = 0$

$a = -2$ or $a = -1$

Now solve for x after replacing a with $x^2 - 5$.

$x^2 - 5 = -2$ or $x^2 - 5 = -1$

$x^2 = 3$ $x^2 - 4 = 0$

$x = \pm\sqrt{3}$ $(x - 2)(x + 2) = 0$

$x = 2$ or $x = -2$

Solutions:

$x = \{-2, -\sqrt{3}, \sqrt{3}, 2\}$

Solving Quadratic Equations

Graphing

Use when:

The quadratic equation is in vertex form with integer values for a, p, and q that can easily fit on a graph.

Square Roots

Use when:

There is no x-term, when $b = 0$ in the general form of the quadratic equation, or when the equation is presented as a binomial squared.

Factoring

Use when:

The equation, in general form, can be factored easily.

Completing the Square

Use when:

The x^2 term has a coefficient of 1, and the coefficient of the x-term is even.

Quadratic Formula

Use when:

The equation cannot be factored easily or at all, or when completing the square would involve fractions.

Learning Activity 4.5

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Janique is taking piano lessons and her teacher tells her she needs to practice piano for at least 7 hours a week. If Janique practices for 45 minutes each day from Monday to Friday, 1.25 hours on Saturday, and 1.75 hours on Sunday, will Janique have practiced enough?
2. If you sleep for 9 hours a day, what fraction of the day are you awake (in lowest terms)?
3. Evaluate for $j = 3$: $5j - 2j^2 + 7$
4. Evaluate: $2^{(2^3)}$
5. What is the area of a triangle with a base of 4 cm and a height of 8 cm?
6. Factor completely: $36x^2 - 1$
7. Two angles in an isosceles triangle measure 16° and 148°. What is the measure of the third angle?
8. Express as a radical: $8^{\frac{3}{2}}$

Answers:

1. No [Janique will still need to practice for 0.25 hours, or 15 minutes. (45 minutes = 0.75 hours; Total time = $0.75 \times 5 + 1.25 + 1.75 = 3.75 + 1.25 + 1.75 = 6.75$ hours)]
2. $\frac{5}{8}$ $\left(24 - 9 = 15; \frac{15}{24} = \frac{5}{8}\right)$
3. 4 [$5(3) - 2(9) + 7 = 15 - 18 + 7 = 4$]
4. 256 $\left(2^{2^3} = 2^8 = 256\right)$

5. $16\ \text{cm}^2$ $\left(\text{Area} = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 4 \times 8 = 16\ \text{cm}^2\right)$

6. $(6x - 1)(6x + 1)$

7. 16°

   (**Method 1:** The sum of the angles in a triangle must be 180°. Therefore, the missing angle is 180° − 16° − 148° = 16°.

   Method 2: An isosceles triangle will have two angles that are the same and all three angles add up to 180°. Therefore, the third angle must measure 16°, since you cannot have two angles of 148° in a triangle.)

8. $\sqrt{8^3}$ or $\left(\sqrt{8}\right)^3$

Part B: The Discriminant

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Determine whether each statement is true or false. If the statement is false, rewrite it so it is true.
   a) The discriminant of $2x^2 + 5x + 6 = 0$ is 23.
   b) For each quadratic equation $ax^2 + bx + c = 0$, if $b^2 - 4ac > 0$, then the equation has two real roots.
   c) For each quadratic equation $ax^2 + bx + c = 0$, if $b^2 - 4ac < 0$, then the equation has no real number solutions.
   d) $\frac{-6 + \sqrt{32}}{10} = \frac{-3 + 8\sqrt{2}}{5}$
   e) The equation $x^2 + 2x + 1 = 0$ has two distinct real roots.

   Answers:

   a) False. The discriminant of $2x^2 + 5x + 6 = 0$ is $b^2 - 4ac = 5^2 - 4(2)(6) = 25 - 48 = -23$.
   b) True
   c) True
   d) False. $\frac{-6 + \sqrt{32}}{10} = \frac{-6 + 4\sqrt{2}}{10} = \frac{-3 + 2\sqrt{2}}{5}$
   e) False. The equation $x^2 + 2x + 1 = 0$ has one real root since $b^2 - 4ac = 2^2 - 4(1)(1) = 0$.

2. If the discriminant of a quadratic equation has the given value, state the nature of the roots.

   Answer:

	Discriminant	Nature of Roots:
a)	-15	no real roots
b)	25	2 real, rational roots
c)	-9	no real roots
d)	0	1 real root
e)	50	2 real, irrational roots

3. How many times would the graph of $y = ax^2 + bx + c$ (with a, b, and c as rational numbers and $a \neq 0$) intersect the x-axis if the value of the discriminant is

   a) negative?

   Answer: Never

   b) zero?

   Answer: Once

   c) positive?

   Answer: Twice

4. Determine the nature of the roots by calculating the discriminant for each equation.

   a) $x^2 - 8x + 16 = 0$

   Answer:

   $x^2 - 8x + 16 = 0$

   $a = 1, b = -8, c = 16$

   $b^2 - 4ac = (-8)^2 - 4(1)(16)$

   $= 64 - 64$

   $= 0$

   Because $b^2 - 4ac = 0$, there is one real root.

b) $x^2 + 2x + 7 = 0$

Answer:

$x^2 + 2x + 7 = 0$

$a = 1, b = 2, c = 7$

$b^2 - 4ac = (2)^2 - 4(1)(7)$

$= 4 - 28$

$= -24$

Because $b^2 - 4ac < 0$, there are no real roots.

c) $p^2 - 16 = 0$

Answer:

$p^2 - 16 = 0$

$a = 1, b = 0, c = -16$

$b^2 - 4ac = (0)^2 - 4(1)(-16)$

$= 64$

Because $b^2 - 4ac > 0$ and equal to a perfect square, there are two real, rational roots.

d) $2x^2 + x = 5$

Answer:

$2x^2 + x = 5$

$2x^2 + x - 5 = 0$

$a = 2, b = 1, c = -5$

$b^2 - 4ac = (1)^2 - 4(2)(-5)$

$= 1 + 40$

$= 41$

Because $b^2 - 4ac > 0$ and not equal to a perfect square, there are two real, irrational roots.

5. Find value(s) of k so that each equation has real and equal roots.

a) $kx^2 - 6x + 2 = 0$

Answer:

$kx^2 - 6x + 2 = 0$

If the roots are equal, $b^2 - 4ac = 0$.

$a = k$, $b = -6$, $c = 2$

$$(-6)^2 - 4(k)(2) = 0$$
$$36 - 8k = 0$$
$$-8k = -36$$
$$k = \frac{-36}{-8}$$
$$k = \frac{9}{2}$$

b) $x^2 + (k - 8)x + 9 = 0$

Answer:

$x^2 + (k - 8)x + 9 = 0$

If the roots are equal, $b^2 - 4ac = 0$.

$a = 1$, $b = k - 8$, $c = 9$

Method 1:

$$(k - 8)^2 - 4(1)(9) = 0$$
$$k^2 - 16k + 64 - 36 = 0$$
$$k^2 - 16k + 28 = 0$$
$$(k - 2)(k - 14) = 0$$

$k = 2$ or $k = 14$

Method 2:

$$(k - 8)^2 - 4(1)(9) = 0$$
$$(k - 8)^2 - 36 = 0$$
$$(k - 8)^2 = 36$$
$$k - 8 = \pm 6$$

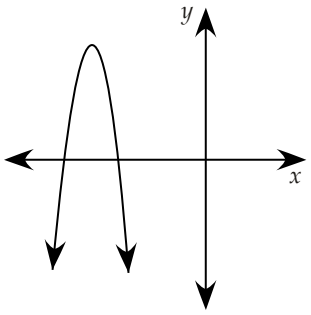
$k = 8 - 6$ or $k = 8 + 6$

$k = 2$ or $k = 14$

6. Sketch the graph of a quadratic function whose discriminant is positive and whose vertex is in the second quadrant.

   Answer:

   The graph must intersect the x-axis twice because a positive discriminant tells us that the quadratic function has two real roots. It must open down.



7. Find the value of k for each of the following situations.

   a) $3x^2 - 2x + k = 0$ has two real roots

   Answer:

   $3x^2 - 2x + k = 0$

   If the equation has two real roots, $b^2 - 4ac > 0$.

   $a = 3$, $b = -2$, $c = k$

$$b^2 - 4ac > 0$$

$$(-2)^2 - 4(3)k > 0$$

$$4 - 12k > 0$$

$$-12k > -4$$

$$\frac{-12k}{-12} < \frac{-4}{-12}$$

$$k < \frac{1}{3}$$

   Remember that when dividing an inequality by a negative number, the inequality sign reverses direction.

b) $x^2 + kx + 3k = 0$ has a double root

Answer:

$x^2 + kx + 3k = 0$

If the equation has a double root, then $b^2 - 4ac$ must be 0.

$$b^2 - 4ac = 0$$
$$(k)^2 - 4(1)(3k) = 0$$
$$k^2 - 12k = 0$$
$$k(k - 12) = 0$$

$k = 0, k = 12$

c) $kx^2 + 8x = 4$ has no real roots

Answer:

$$kx^2 + 8x = 4$$
$$kx^2 + 8x - 4 = 0$$

If the equation has no real roots, then $b^2 - 4ac < 0$.

$a = k, b = 8, c = -4$

$$b^2 - 4ac < 0$$
$$8^2 - 4(k)(-4) < 0$$
$$64 + 16k < 0$$
$$16k < -64$$
$$k < -4$$

8. For each of the following quadratic equations, solve by first using the discriminant to determine if its value is a perfect square. If it is a perfect square, use the factoring method to solve. If it is not a perfect square, use the quadratic formula.

a) $12x^2 - 19x - 21 = 0$

Answer:

$12x^2 - 19x - 21 = 0$

$a = 12,\ b = -19,\ c = -21$

$b^2 - 4ac = (-19)^2 - 4(12)(-21)$

$= 1369$

$= 37^2$

The discriminant is a perfect square. The equation can be factored.

$$12x^2 - 19x - 21 = 0$$

$$12x^2 - 28x + 9x - 21 = 0$$

$$4x(3x - 7) + 3(3x - 7) = 0$$

$$(4x + 3)(3x - 7) = 0$$

$$x = -\frac{3}{4} \text{ and } x = \frac{7}{3}$$

b) $18x^2 - 35x + 12 = 0$

Answer:

$18x^2 - 35x + 12 = 0$

$a = 18,\ b = -35,\ c = 12$

$b^2 - 4ac = (-35)^2 - 4(18)(12)$

$= 361$

$= 19^2$

The discriminant is a perfect square. The equation can be factored.

$$18x^2 - 35x + 12 = 0$$

$$18x^2 - 27x - 8x + 12 = 0$$

$$9x(2x - 3) - 4(2x - 3) = 0$$

$$(9x - 4)(2x - 3) = 0$$

$$x = \frac{4}{9}, \frac{3}{2}$$

c) $60x^2 - 59x - 20 = 0$

Answer:

$60x^2 - 59x - 20 = 0$

$a = 60,\ b = -59,\ c = -20$

$b^2 - 4ac = (-59)^2 - 4(60)(-20)$

$= 8281$

$= 91^2$

The discriminant is a perfect square. The equation can be factored.

$$60x^2 - 59x - 20 = 0$$

$$60x^2 + 16x - 75x - 20 = 0$$

$$4x(15x + 4) - 5(15x + 4) = 0$$

$$(4x - 5)(15x + 4) = 0$$

$x = \frac{5}{4}$ and $x = -\frac{4}{15}$

9. Complete the following Quadratic Equation Puzzle with your learning partner. Cut out all the separate pieces and arrange them into rows on the Quadratic Equation Puzzle worksheet so that each row corresponds to a quadratic equation. In each row, the quadratic equation must be in general form and vertex form, and you must include the value of the discriminant, the number of roots, the types of roots, and the actual roots themselves.

Quadratic Equation Puzzle Key

General Form of Quadratic Equation	Vertex Form of Quadratic Function	Value of the Discriminant	Number of Roots	Type of Roots	Roots
$9x^2 + 6x - 4 = 0$	$y = 9\left(x + \frac{1}{3}\right)^2 - 5$	180	2	Real and Irrational	$\frac{-1+\sqrt{5}}{3}, \frac{-1-\sqrt{5}}{3}$
$3x^2 + 8x - 35 = 0$	$y = 3\left(x + \frac{4}{3}\right)^2 - \frac{121}{3}$	484	2	Real and Rational	$-5, \frac{7}{3}$
$2x^2 - 2x + 3 = 0$	$y = 2\left(x - \frac{1}{2}\right)^2 + \frac{5}{2}$	-20	0	None	No Real Roots
$x^2 - 4x + 4 = 0$	$y = (x - 2)^2$	0	1	Real and Equal	2
$25x^2 - 10x + 1 = 0$	$y = 25\left(x - \frac{1}{5}\right)^2$	0	1	Real and Equal	$\frac{1}{5}$

Learning Activity 4.6

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Multiply: $(3x^2)(4xy^4)$
2. Your dog can run at a speed of 10 m/s. If you are standing half a kilometre away from your dog, how long will it take for your dog to run to you?
3. What are the x-intercepts of the function $y = x^2 + 3x + 2$?
4. At a restaurant, Megan orders a meal advertised as containing 500 calories. She also orders a chocolate milkshake, which contains 490 calories. If Megan only eats 75% of her meal but drinks her entire milkshake, how many calories did Megan consume?
5. Calculate: $\frac{1}{5} + \frac{5}{6} + \frac{2}{15}$
6. Multiply: $(x - y)(3x^2 - 7x + 2)$
7. State the independent and dependent variables for the following situation:

   When the temperature changes, the height of the mercury in a thermometer either rises or falls.

8. Express as an entire radical: $4x^{\frac{3}{2}}$

Answers:

1. $12x^3y^4$
2. 50 seconds (half a kilometre = 500 metres; $\frac{500}{10} = 50$)
3. -2 and -1 (Factoring: $(x + 2)(x + 1)$)
4. 865 (75% of 500 = 375; 490 + 375 = 865 calories)
5. $\frac{7}{6}$ $\left(\frac{6}{30} + \frac{25}{30} + \frac{4}{30} = \frac{35}{30} = \frac{7}{6}\right)$

6. $3x^3 - 7x^2 + 2x - 3x^2y + 7xy - 2y$

7. Independent Variable: The temperature

   Dependent Variable: The height of the mercury in the thermometer

8. $\sqrt{16x^3}$ $\left(4x^{\frac{3}{2}} = 4\sqrt{x^3} = \sqrt{4^2x^3} = \sqrt{16x^3}\right)$

Part B: Solving Rational Equations

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Determine whether each statement is true or false. If the statement is false, rewrite it so it is true.

   a) If $x \neq 2$, then $(x-2)\dfrac{4x}{x-2}$ is equivalent to $4x$.

   b) If $x \neq 3$, then $2(x-3)\left(\dfrac{6}{x-3} + \dfrac{x}{2}\right)$ is equivalent to $6 + x$.

   c) If $x \neq 0$, then the equations $x - \dfrac{1}{x} = 0$ and $x^2 - 1 = 0$ are equivalent.

   d) If $x \neq -3$ and $x \neq -4$, then $\dfrac{x-1}{x+3} = \dfrac{x-2}{x+4}$ is equivalent to $(x-1)(x+4) = (x+3)(x-2)$.

   Answers:

   a) True

   b) False. If $x \neq 3$, then $2(x-3)\left(\dfrac{6}{x-3} + \dfrac{x}{2}\right)$ is equivalent to $12 + x^2 - 3x$.

   c) True

   d) True

2. Solve each of the following rational equations. Make sure you state the non-permissible values and check for extraneous roots.

a) $x = -\dfrac{2}{x-3}$

Answer:

LCD = $x - 3$ and $x \neq 3$

$$x(x-3) = \frac{-2}{x-3}(x-3)$$

$$x^2 - 3x = -2$$

$$x^2 - 3x + 2 = 0$$

$$(x-2)(x-1) = 0$$

$$x = 2, 1$$

Check:

Are any of the solutions the non-permissible value? No.

$\therefore$ The solutions are $x = 2$ or $x = 1$, $x \neq 3$.

b) $\dfrac{2x-9}{x-7} + \dfrac{x}{2} = \dfrac{5}{x-7}$

Answer:

LCD = $2(x - 7)$ and $x \neq 7$.

$$\frac{2x-9}{x-7}(2)(x-7) + \frac{x}{2}(2)(x-7) = \frac{5}{x-7}(2)(x-7)$$

$$2(2x-9) + x(x-7) = 5(2)$$

$$4x - 18 + x^2 - 7x = 10$$

$$x^2 - 3x - 18 - 10 = 0$$

$$x^2 - 3x - 28 = 0$$

$$(x-7)(x+4) = 0$$

$$x = 7, -4$$

Check:

It is not possible for x to be 7.

$\therefore$ $x = -4$ is the only solution and $x \neq 7$.

c) $x - 4 = -\dfrac{1}{x}$

Answer:

LCD = x, $x \neq 0$

$$x(x-4) = x\left(\frac{-1}{x}\right)$$

$$x^2 - 4x = -1$$

$x^2 - 4x + 1 = 0$ Not factorable

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16-4}}{2}$$

$$= \frac{4 \pm \sqrt{12}}{2}$$

$$= \frac{4 \pm \sqrt{4}\sqrt{3}}{2}$$

$$= \frac{4 \pm 2\sqrt{3}}{2}$$

$$= \frac{2\left(2 \pm \sqrt{3}\right)}{2}$$

$$= 2 \pm \sqrt{3}$$

Check:

Non-permissible values are okay.

$\therefore$ Solutions are $x = 2 \pm \sqrt{3}$ and $x \neq 0$.

d) $\dfrac{3x^2}{3x+1} - 2 = \dfrac{2x+1}{3x+1}$

Answer:

LCD $= 3x+1$, $x \neq -\dfrac{1}{3}$

$$\frac{3x^2}{3x+1} - 2 = \frac{2x+1}{3x+1}$$

$$\left(\frac{3x^2}{3x+1}\right)(3x+1) - 2(3x+1) = \frac{(2x+1)}{(3x+1)}(3x+1)$$

$$3x^2 - 6x - 2 = 2x + 1$$

$$3x^2 - 8x - 3 = 0$$

$$(3x+1)(x-3) = 0$$

$3x+1=0$ $\qquad$ $x-3=0$

$x = -\dfrac{1}{3}$ $\qquad$ $x = 3$

Check:

It is not permissible for x to be $-\dfrac{1}{3}$.

$\therefore x = 3$ is the only solution and $x \neq -\dfrac{1}{3}$.

e) $\dfrac{2x}{x-3} + \dfrac{1}{2x+3} + \dfrac{3x+9}{2x^2-3x-9} = 0$

Answer:

$$\frac{2x}{x-3} + \frac{1}{2x+3} + \frac{3x+9}{(2x+3)(x-3)} = 0$$

$\therefore$ LCD $= (2x+3)(x-3)$ and $x \neq 3, -\dfrac{3}{2}$

$$\frac{2x}{(x-3)}(x-3)(2x+3) + \frac{1}{(2x+3)}(2x+3)(x-3) +$$

$$\frac{(3x+9)}{(2x+3)(x-3)}(2x+3)(x-3) = 0(2x+3)(x-3)$$

$$2x(2x+3)+(x-3)+(3x+9)=0$$

$$4x^2+6x+x-3+3x+9=0$$

$$4x^2+10x+6=0$$

$$2x^2+5x+3=0$$

$$(2x+3)(x+1)=0$$

$2x+3=0$ $\qquad$ $x+1=0$

$x=\dfrac{-3}{2}$ $\qquad$ $x=-1$

Check:

It is not permissible for x to be $\dfrac{-3}{2}$.

$\therefore$ The only solution is $x=-1$ and $x\neq 3, -\dfrac{3}{2}$.

f) $\dfrac{x^2+12}{x-3}=\dfrac{7x}{x-3}$

Answer:

LCD = $x-3, x\neq 3$

$$\frac{x^2+12}{x-3}(x-3)=\frac{7x}{x-3}(x-3)$$

$$x^2+12=7x$$

$$x^2-7x+12=0$$

$$(x-4)(x-3)=0$$

$$x=4, 3$$

Check:

It is not permissible for x to be 3.

$\therefore$ The only solution is $x = 4$ and $x \neq 3$.

3. The distance from Winnipeg to Minneapolis is about 600 km. You drove from Winnipeg to Minneapolis 10 km/h faster than from Minneapolis to Winnipeg. If the total driving time for the round trip was 22 hours, how fast did you drive from Winnipeg to Minneapolis?

Answer:

Fill in the rate and distance, and then find the ratio for time = $\frac{\text{distance}}{\text{rate}}$ for each.

	Rate ×	Time =	Distance
Winnipeg to Minneapolis	$r + 10$	$\frac{600}{r + 10}$	600
Minneapolis to Winnipeg	r	$\frac{600}{r}$	600

Driving from Winnipeg to Minneapolis + Driving from Minneapolis to Winnipeg = 22 hours.

$$\frac{600}{r + 10} + \frac{600}{r} = 22$$

LCD = $r(r + 10)$

$$\frac{600}{r + 10} r(r + 10) + \frac{600}{r} r(r + 10) = 22r(r + 10)$$

$$600r + 600(r + 10) = 22r(r + 10)$$

$$600r + 600r + 6000 = 22r^2 + 220r$$

$$0 = 22r^2 - 980r - 6000$$

$$0 = 11r^2 - 490r - 3000$$

$$0 = 11r^2 - 550r + 60r - 3000$$

$$0 = 11r(r - 50) + 60(r - 50)$$

$$0 = (11r + 60)(r - 50)$$

$$11r + 60 = 0 \text{ or } r - 50 = 0$$

$$r = -\frac{60}{11} \text{ or } r = 50$$

The only acceptable answer is $r = 50$, since rate cannot be negative. Therefore, you drove $r + 10 = 50 + 10 = 60$ km/h from Winnipeg to Minneapolis. Note that this answer is not very reasonable unless you were stopping often for coffee and the lineup at the border was long.

4. House painters, Gwen and Megan, want to know how long it will take them if they work together to paint a house. Gwen, working alone, can paint a house in 13 hours. Megan, working alone, can paint a house in 11 hours. How long will it take them to paint a house if they work together? Round your answer to the nearest hundredth.

   Answer:

   Consider what portion of the house each person paints in one hour.

   Gwen paints a house in 13 hours. Her rate is $\frac{1}{13}$ house per hour.

   Megan's rate is $\frac{1}{11}$ house per hour.

   Let t = time, in hours, for Megan and Gwen to paint the house.

   $$\frac{1}{13}t + \frac{1}{11}t = 1$$

   The equation is equal to one because that represents the painting of one house.

   Now solve the equation.

   LCD = (11)(13) = 143

   $$\frac{t}{13} + \frac{t}{11} = 1$$

   $\frac{t}{13}(11)(13) + \frac{t}{11}(11)(13) = 1(11)(13)$ Multiply by the LCD to clear denominators.

   $$11t + 13t = 143$$

   $$24t = 143$$

   $$t = \frac{143}{24}$$

   $$t = 5.96$$

   Thus, Gwen and Megan working together can paint a house in 5.96 hours.

5. Jeff is trying to fill his above-ground pool with water. He knows that one of his hoses can fill the pool in 32 hours and another in 26 hours. If Jeff uses both hoses to fill the pool, how long will it take to fill the pool with water? Round your answer to the nearest tenth.

   Answer:

   Hose 1: rate per hour is $\frac{1}{32}$ of the pool per hour

   Hose 2: rate per hour is $\frac{1}{26}$ of the pool per hour

   Let t = the number of hours to fill the pool when both hoses are used.

   Let 1 represent the pool filled completely.

   $$\frac{1}{32}t + \frac{1}{26}t = 1$$

   To solve the equation, first find the LCD.

   $32 = 2^5$

   $26 = 2 \times 13$

   $\text{LCD} = 2^5 \times 13 = 32 \times 13 = 416$

   Now, multiply each term by the LCD and solve for t.

   $$\frac{t}{32} + \frac{t}{26} = 1$$

   $$\frac{t}{32}(416) + \frac{t}{26}(416) = 1(416)$$

   $$13t + 16t = 416$$

   $$29t = 416$$

   $$t = \frac{416}{29}$$

   $$t = 14.3$$

   Both hoses working together can fill the pool in 14.3 hours.

Notes

Grade 11 Pre-Calculus Mathematics (30S)

Module 4
Solving Quadratic and Rational Equations

Learning Activity Answer Keys

Module 4:
Solving Quadratic and Rational Equations

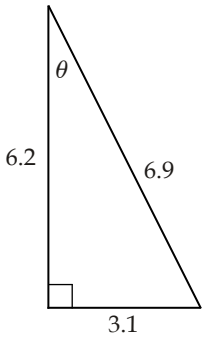
Learning Activity 4.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Brooklyn can type 55 words per minute. If her essay took her an hour to write, how many words are in her essay?
2. What is the greatest common factor of $3x^5y^3$ and $6x^2y^8$?
3. Find the vertex of the following parabola: $y = -2x^2 + 8x - 1$
4. Solve for x: $6x + 5 = 7$
5. How many centimetres are in one kilometre?

Use the following right-angled triangle to answer the next three questions. Do not simplify your answers.



6. Identify the sine ratio for θ.
7. Identify the cosine ratio for θ.
8. Identify the tangent ratio for θ.

Answers:

1. 3300 (55 × 60; There are 60 minutes in an hour, therefore, Brooklyn's essay contains 3300 words.)
2. $3x^2y^3$
3. (2, 7) $\left(x = -\frac{8}{-4} = 2;\ y = -2(4) + 8(2) - 1 = -8 + 16 - 1 = 7\right)$
4. $\frac{1}{3}$ $\left(6x = 2,\ x = \frac{2}{6} \text{ or } \frac{1}{3}\right)$
5. 100 000 cm in 1 km (There are 100 cm in 1 m and 1000 m in 1 km. Therefore, 100 × 1000 = 100 000 cm in 1 km.)
6. $\frac{3.1}{6.9}$
7. $\frac{6.2}{6.9}$
8. $\frac{3.1}{6.2}$

Part B: Solving Quadratic Equations by Graphing

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Rearrange each of the following into the general form of a quadratic equation with the leading coefficient greater than 0.
   a) $-2x^2 + 10 = -6$
   b) $3x(x + 2) = 8$
   c) $5x^2 - 7 + 3x = 7$
   d) $(x + 3)^2 = -4$
   e) $(x + 3)^2 - 4 = 5$
   f) $\frac{x^2 - 4}{3} = x$
   g) $(2x - 3)(x + 2) = 0$

Answers:

a) $-2x^2 + 10 = -6$

$-2x^2 + 16 = 0$ Add 6 to each side.

$2x^2 - 16 = 0$ Multiply by (-1).

b) $3x(x + 2) = 8$

$3x^2 + 6x = 8$ Distribute $3x$ into the bracket.

$3x^2 + 6x - 8 = 0$ Subtract 8 from each side.

c) $5x^2 - 7 + 3x = 7$

$5x^2 + 3x - 14 = 0$ Subtract 7 from each side and rearrange.

d) $(x + 3)^2 = -4$

$x^2 + 6x + 9 = -4$ Expand $(x + 3)^2$.

$x^2 + 6x + 13 = 0$ Add +4 to each side.

e) $(x + 3)^2 - 4 = 5$

$x^2 + 6x + 9 - 4 = 5$ Expand $(x + 3)^2$.

$x^2 + 6x + 5 = 5$ Simplify.

$x^2 + 6x = 0$ Subtract 5 from each side.

f) $\dfrac{x^2 - 4}{3} = x$

$x^2 - 4 = 3x$ Multiply each side by 3.

$x^2 - 3x - 4 = 0$ Subtract $3x$ from each side.

g) $(2x - 3)(x + 2) = 0$

$2x^2 + 4x - 3x - 6 = 0$ Multiply.

$2x^2 + x - 6 = 0$ Simplify.

2. Use the graph of the related function to solve each equation.

   a) $0 = (x - 2)^2 - 4$

   b) $0 = x^2 - 6x + 5$

   c) $x^2 + 6x + 1 = -7$

Answers:

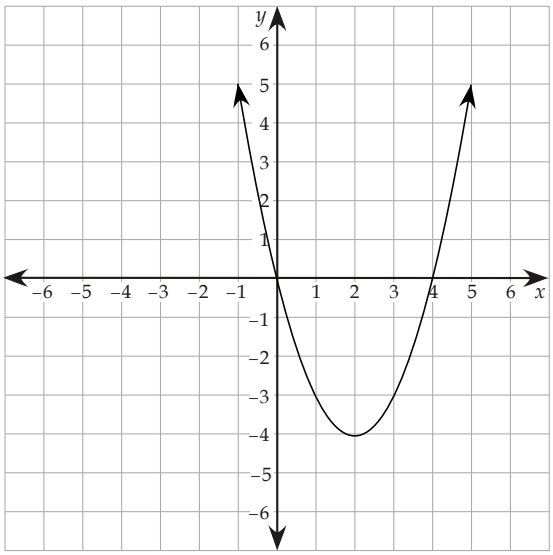
Method 1: Using Graph Paper

a) $0 = (x - 2)^2 - 4$

Graph the function:
$y = (x - 2)^2 - 4$

Vertex at (2, −4), opens up, standard size

The zeros are 4 and 0. The solutions of the equation are 4 and 0.



b) $0 = x^2 - 6x + 5$

The function is $y = x^2 - 6x + 5$.

First, complete the square to find the vertex and then graph the parabola.

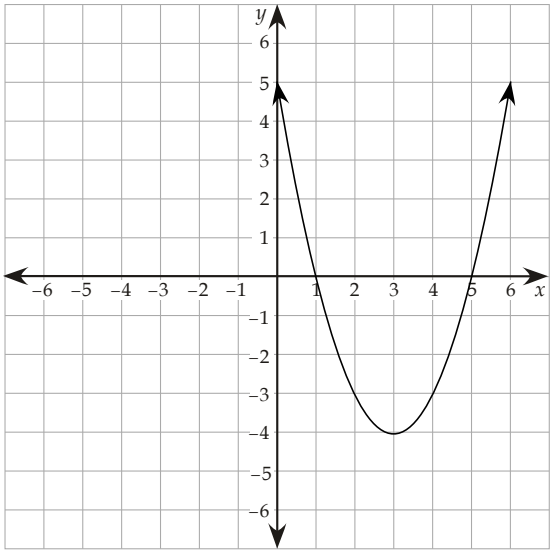
$y = x^2 - 6x + 5$

$y = (x^2 - 6x + 9) + 5 - 9$

$y = (x - 3)^2 - 4$

$\therefore$ Vertex is (3, −4); the curve opens up and is standard size.

The zeros are 1 and 5 and the solutions are 1 and 5.



c) $x^2 + 6x + 1 = 7$

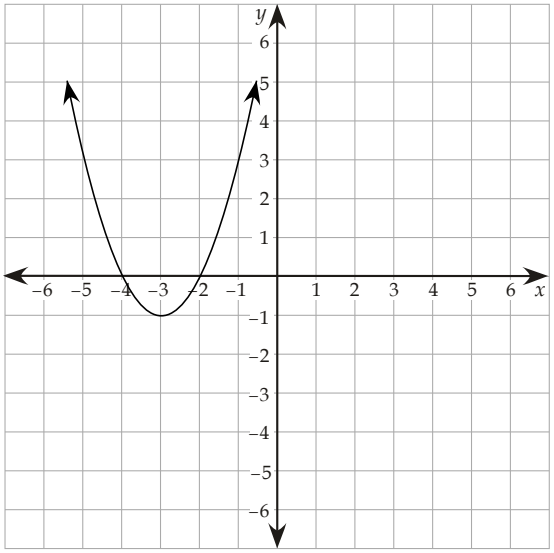
$x^2 + 6x + 8 = 0$

The function is $y = x^2 + 6x + 8$.

Upon completing the square, you get $y = (x + 3)^2 - 1$.

The vertex is $(-3, -1)$; the curve opens up and is normal size.

The zeros are -4 and -2.
The solutions are -4 and -2.

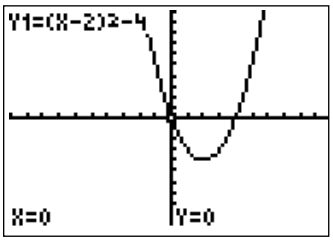


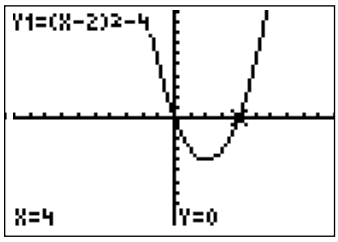
Method 2: Using a graphing calculator

a) ENTER $Y_1 = (x - 2)^2 - 4$

Set Window to standard size: XMIN −10 XMAX +10
YMIN −10 YMAX +10

Use TRACE and the arrow keys to approximate the x-values of the points where the graph crosses the x-axis.





Answers: $x = 0$ and 4

b) Press Y=

ENTER $Y_1 = x^2 - 6x + 5$

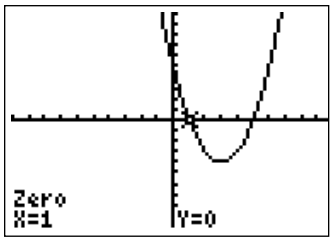
Set Window to standard size: XMIN −10 XMAX +10
YMIN −10 YMAX +10

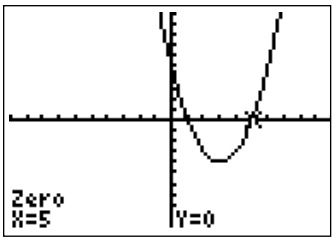
Use TRACE to approximate the x-values of the point where the parabola intersects the x-axis.

Answers: $x = 1$ and $x = 5$

Using 2nd (CALC) ZERO will give a more accurate answer.

Set left and right bounds and guess. Press ENTER after each.

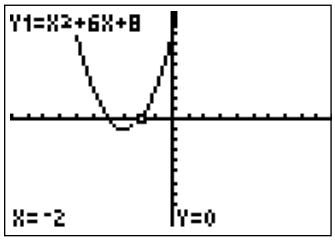


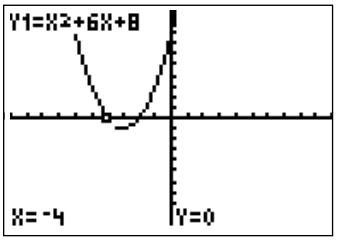


c) Press Y=

ENTER $Y_1 = x^2 + 6x + 8$

Use TRACE to approximate the zeros.





Answers: $x = -2$ or $x = -4$

3. a) Graph the quadratic function $y = x^2 - 2x - 8$ by completing the square, and then find the x-intercepts from the graph.
   b) Check your answer using a graphing calculator if you have one.
   c) Where does the graph intersect the x-axis?
   d) What are the zeros of the function?
   e) What are the roots of the equation $x^2 - 2x - 8 = 0$?
   f) Verify each root.

Answers:

a) $y = x^2 - 2x - 8$

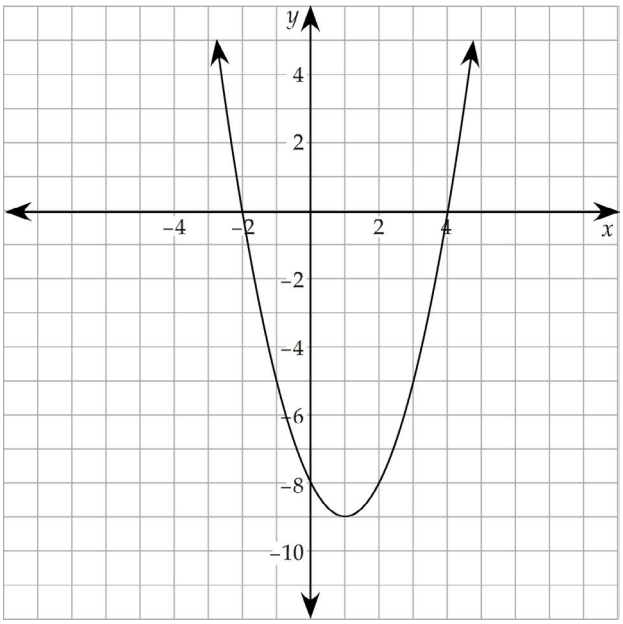
$y + 8 = x^2 - 2x$

$y + 8 + 1 = x^2 - 2x + 1$

$y + 9 = (x - 1)^2$

$y = (x - 1)^2 - 9$

V(1, −9)



The x-intercepts are −2 and 4.

b) Check your answer using a graphing calculator, finding the zeros of the function to be −2 and 4.

c) $y = x^2 - 2x + 8$

The points can be read from the graph.

The graph intersects the x-axis at the points (−2, 0) and (4, 0).

d) The zeros are 4, −2.

e) The roots of the equation are 4, −2 since they are the same as the zeros of the function.

f) Check by substituting into $x^2 - 2x - 8 = 0$.

For $x = 4$

$\text{LHS} = x^2 - 2x - 8$

$= 4^2 - 2(4) - 8$

$= 16 - 8 - 8$

$= 0$

$\text{RHS} = 0$

$\text{LHS} = \text{RHS}$

For $x = -2$

$\text{LHS} = x^2 - 2x - 8$

$= (-2)^2 - 2(-2) - 8$

$= 4 + 4 - 8$

$= 0$

$\text{RHS} = 0$

$\text{LHS} = \text{RHS}$

Thus, both roots are verified.

4. Solve each of the following equations by graphing.

a) $x^2 + 2x - 8 = 0$

b) $x^2 + 4x + 3 = 0$

c) $x^2 + 8x = -15$

d) $9 - x^2 = 0$

e) $2x^2 - 12x + 10 = 0$

Answers:

a) $x^2 + 2x - 8 = 0$

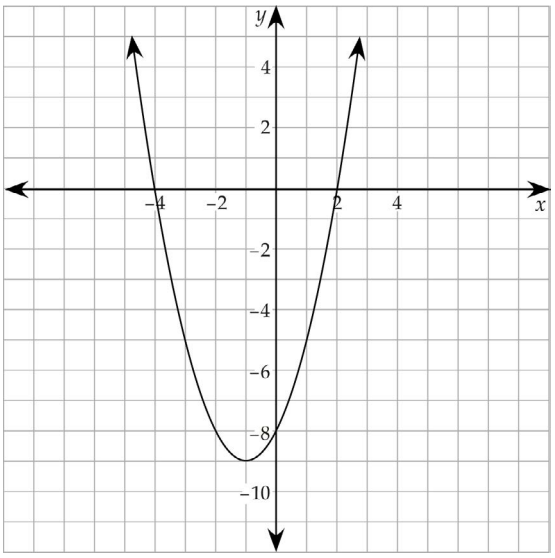
Let $y = x^2 + 2x - 8$

$y = (x^2 + 2x \quad) - 8$

$y = (x^2 + 2x + 1) - 8 - 1$

$y = (x + 1)^2 - 9$

V(−1, −9)



The solutions are $x = -4, 2$.

b) $x^2 + 4x + 3 = 0$

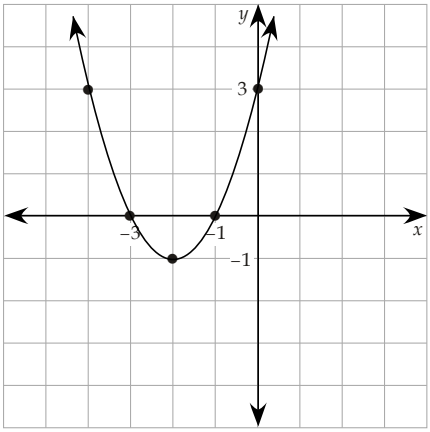
Let $y = x^2 + 4x + 3$

$y = (x^2 + 4x \quad) + 3$

$y = (x^2 + 4x + 4) + 3 - 4$

$y = (x + 2)^2 - 1$

V(−2, −1)



The solutions are $x = -3, -1$.

c) $x^2 + 8x = -15$

$x^2 + 8x + 15 = 0$

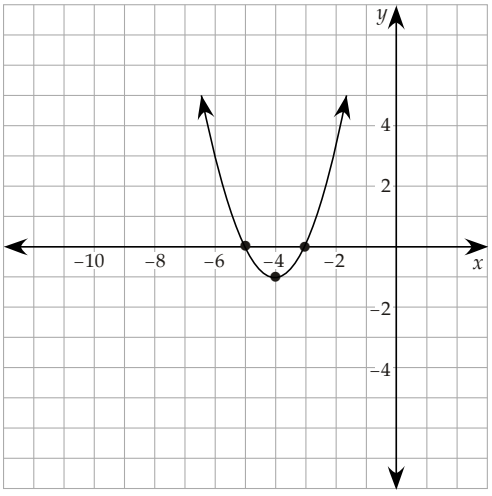
Let $y = x^2 + 8x + 15$

$y = (x^2 + 8x \quad) + 15$

$y = (x^2 + 8x + 16) + 15 - 16$

$y = (x + 4)^2 - 1$

V(−4, −1)

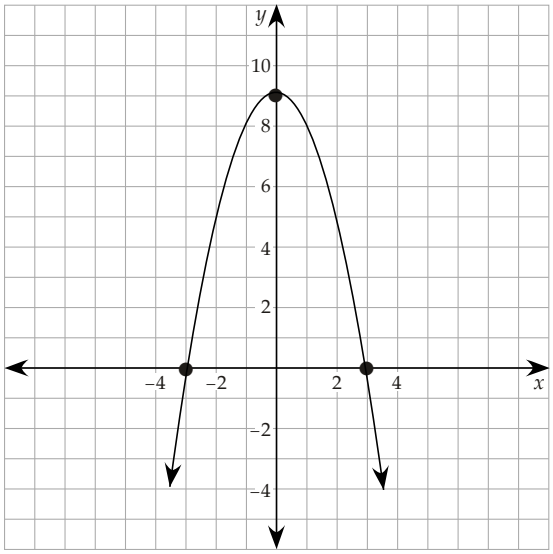


The solutions are $x = -5, -3$.

d) $9 - x^2 = 0$

Let $y = -x^2 + 9$

V(0, 9)



The solutions are $x = \pm 3$.

e) $2x^2 - 12x + 10 = 0$

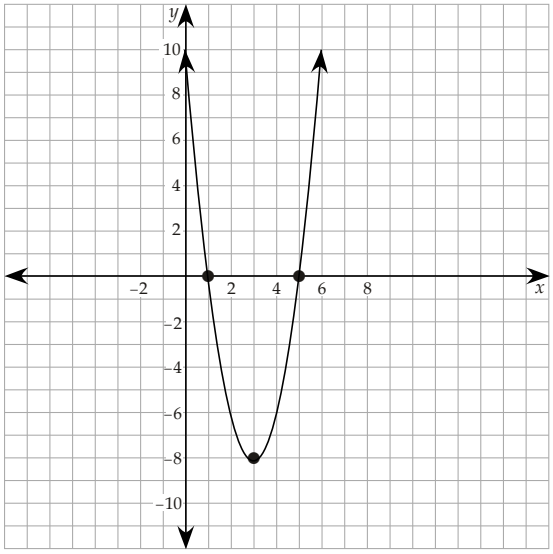
Let $y = 2x^2 - 12x + 10$

$y = 2(x^2 - 6x \quad) + 10$

$y = 2(x^2 - 6x + 9) + 10 - 9(2)$

$y = 2(x - 3)^2 - 8$

V(3, −8)



The solutions are $x = 1, 5$.

5. Kalia has been asked to solve the quadratic equation $-x^2 + 5x - 4 = 0$ by graphing. Her work for completing the square and graphing the corresponding quadratic function is shown below.

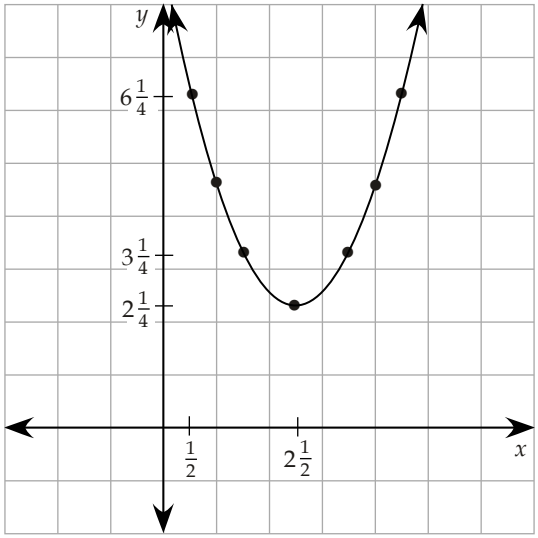
$$y = -x^2 + 5x - 4$$

$$y + 4 = -\left(x^2 - 5x\right)$$

$$y + 4 - \frac{25}{4} = -\left(x^2 - 5x + \frac{25}{4}\right)$$

$$y + \frac{16}{4} - \frac{25}{4} = -\left(x - \frac{5}{2}\right)^2$$

$$y = -\left(x - \frac{5}{2}\right)^2 + \frac{9}{4}$$

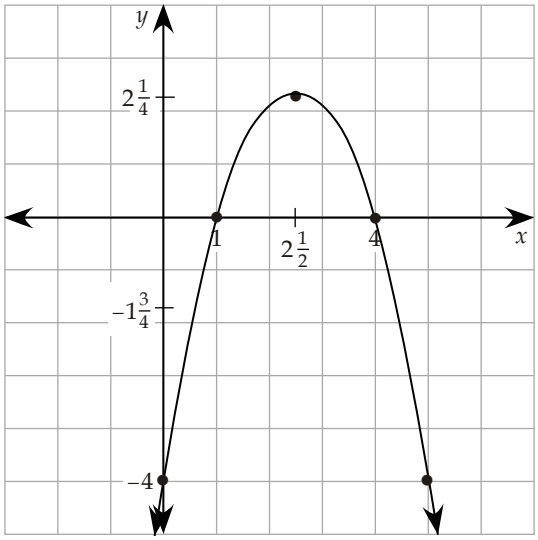


From this graph, Kalia concludes there are no x-intercepts.

Identify, explain, and correct Kalia's mistake.

Answer:

Kalia graphed the quadratic function incorrectly. Because there is a negative sign before the x^2 term, this parabola should open down. The correct graph is shown below.



From this graph, the x-intercepts are 1 and 4. Therefore, the solutions to the quadratic equation $-x^2 + 5x - 4 = 0$ are 1 and 4.

Learning Activity 4.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Cameron is packing his suitcase for Mexico. If his suitcase measures 110 cm by 40 cm by 60 cm, what is the volume of his suitcase?
2. You and your friend are going out for dinner and you want to split the bill evenly. If the bill comes to $76.36 and you want to tip the waiter $4, how much should you give the waiter?
3. Factor: $14m - 28$
4. Does the parabola $y = 3x^2 + 2x - 1$ open up or down?
5. Continue the pattern: A, E, F, H, I, K, L, M, ____ , ____
6. Find the midpoint between the two points $(9, -24)$ and $(4, 12)$.
7. Solve the inequality, $13x - 2 > x$.
8. Find the slope of the line parallel to the line $8x = 3y - 2$.

Answers:

1. 264 000 cm^3 ($110 \times 40 \times 60 = 110 \times 2400 = 264\,000\ cm^3$)
2. $\$42.18 \left(\frac{\$76.36}{2} + \$4 = \$38.18 + \$4 = \$42.18 \right)$
3. $14(m - 2)$
4. Up
5. N, T (The pattern is letters made from only straight lines.)
6. $\left(\frac{13}{2}, -6\right) \left(\text{M} = \left(\frac{9+4}{2}, \frac{-24+12}{2} \right) = \left(\frac{13}{2}, -\frac{12}{2} \right) = \left(\frac{13}{2}, -6 \right) \right)$
7. $x > \frac{1}{6} (12x > 2)$
8. $\frac{8}{3} \left(3y = 8x + 2; \frac{3y}{3} = \frac{8}{3}x + \frac{2}{3}; y = \frac{8}{3}x + \frac{2}{3} \right)$

Part B: Solving Quadratic Equations by Taking Square Roots

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Find the roots of the following equations.

   Recall: Finding the roots of an equation is identical to solving an equation.

   a) $x^2 = 16$

   Answer:

   $x = \pm\sqrt{16}$

   $x = \pm 4$

   b) $3x^2 - 18 = 0$

   Answer:

   $3x^2 = 18$

   $x^2 = 6$

   $x = \pm\sqrt{6}$

   c) $x^2 - 6 = -5$

   Answer:

   $x^2 = -5 + 6$

   $x^2 = 1$

   $x = \pm\sqrt{1}$

   $x = \pm 1$

   d) $6x^2 + 3 = 2x^2 + 7$

   Answer:

   $4x^2 = 4$

   $x^2 = 1$

   $x = \pm\sqrt{1}$

   $x = \pm 1$

   e) $x^2 = 6x^2 - 1$

   Answer:

   $1 = 5x^2$

   $x^2 = \frac{1}{5}$

   $x = \pm\sqrt{\frac{1}{5}}$

   $x = \pm\frac{1}{\sqrt{5}}$ or

   $x = \pm\frac{\sqrt{5}}{5}$

   f) $4x^2 - 9 = -3$

   Answer:

   $4x^2 = 6$

   $x^2 = \frac{3}{2}$

   $x = \pm\sqrt{\frac{3}{2}}$

   $x = \pm\frac{\sqrt{3}}{\sqrt{2}}$ or

   $x = \pm\frac{\sqrt{6}}{2}$

Note: For (e), you can rationalize the denominator or not. Either method of answering is acceptable. Remember, to rationalize a monomial radical denominator, multiply top and bottom of the fraction by the radical:

$$\frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{25}} = \frac{\sqrt{5}}{5}$$

g) $x^2 - 20 = 0$

Answer:

$x^2 = 20$

$x = \pm\sqrt{20}$

$x = \pm\sqrt{4}\sqrt{5}$

$x = \pm 2\sqrt{5}$

h) $24 - 2x^2 = 0$

Answer:

$2x^2 = 24$

$x^2 = 12$

$x = \pm\sqrt{12}$

$x = \pm\sqrt{4}\sqrt{3}$

$x = \pm 2\sqrt{3}$

i) $(x - 2)^2 = 5$

Answer:

Note: This question does not exactly fit the pattern. You have a binomial squared. However, you can still easily take the square root of both sides to eliminate the square and then add 2 to both sides.

$\sqrt{(x-2)^2} = \pm\sqrt{5}$

$x - 2 = \pm\sqrt{5}$

$x = 2 \pm \sqrt{5}$

Therefore, $x = 2 - \sqrt{5}$ or $x = 2 + \sqrt{5}$.

j) $\frac{2}{3}x^2 + 6 = 18$

Answer:

$3\left(\frac{2}{3}x^2 + 6\right) = 3(18)$ Multiply both sides of the equation by 3.

$2x^2 + 18 = 54$ Simplify.

$x^2 + 9 = 27$ Divide both sides of the equation by 2.

$x^2 = 18$

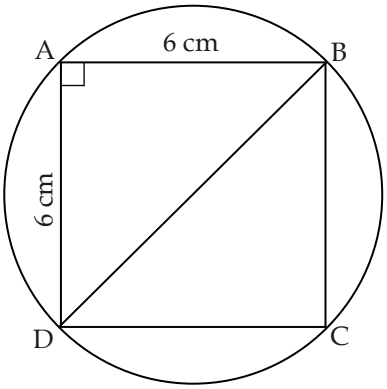
$x = \pm\sqrt{18}$

$x = \pm\sqrt{9}\sqrt{2}$

$x = \pm 3\sqrt{2}$

2. A square, ABCD, with sides of length 6 cm, is circumscribed by a circle. The four points, A, B, C, and D, are on the circumference of the circle. Find the area of a circle. Leave π in your answer. Remember if π is not approximated, your answer is exact. If π is approximated, your answer is approximate.

Answer:



Because ABCD is a square, $\angle A$ is 90° and the diagonal of the square will pass through the centre of the circle.

$$BD^2 = AD^2 + AB^2$$
$$BD^2 = 6^2 + 6^2$$
$$BD = \sqrt{36 + 36}$$
$$BD = \sqrt{36 \cdot 2}$$
$$BD = 6\sqrt{2}$$
$$\text{Radius} = \frac{1}{2}\left(6\sqrt{2}\right)$$
$$= 3\sqrt{2}$$
$$\text{Area of circle} = \pi r^2$$
$$= \pi \left(3\sqrt{2}\right)^2$$
$$= 18\pi \text{ cm}^2$$

3. Jordon is asked to solve the following quadratic equation by taking square roots. His work is shown below.

$$x^2 - 40 = 0$$
$$x^2 = 40$$
$$x = \pm\sqrt{40}$$
$$x = \pm 4\sqrt{10}$$

Identify, explain, and correct Jordon's error.

Answer:

Jordon's work is complete up until the last step. However, $\sqrt{40}$ should be simplified as follows:

40 can be written as the product of 4 and 10.

$$\sqrt{40} = \left(\sqrt{4}\right)\left(\sqrt{10}\right) = 2\sqrt{10}$$

Therefore, $x = \pm 2\sqrt{10}$ is the correct answer.

Learning Activity 4.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Your mom bought 500 g of deli meat. If you put an average of 125 g of meat on each sandwich you make, how many days will the deli meat last for (assuming you only eat one sandwich a day)?
2. You are playing soccer on a co-ed team. You have 20 players on your team and 3 of them are girls. You need at least 20% of your players to be female according to the league. Do you have enough female players?
3. Simplify: $(x + 5)^2$
4. Give a simplified quadratic function that has zeros of -1 and -5 and $a = 1$.
5. Simplify: $2^4 \times 3^2$
6. Express as a power: $\sqrt{15}$
7. Are these fractions equivalent, $\frac{\sqrt{2}}{\sqrt{3}}$ and $\frac{\sqrt{4}}{3}$?
8. The x-intercepts of a parabola are -5 and 2 and the y-intercept is 10. Write the quadratic function in factored form.

Answers:

1. 4 days $\left(\frac{500}{125} = 4\right)$
2. No $\left(\frac{3}{20} = \frac{15}{100} = 15\%\right)$
3. $x^2 + 10x + 25$ $[(x + 5)^2 = (x + 5)(x + 5) = x^2 + 5x + 5x + 25]$
4. $y = 1(x + 1)(x + 5) = x^2 + 6x + 5$
5. 144 (16×9)
6. $15^{\frac{1}{2}}$
7. These fractions are not equivalent. $\left(\frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3}, \text{ not } \frac{\sqrt{4}}{3}\right)$
8. $y = -(x + 5)(x - 2)$ (The factored form is $y = a(x + 5)(x - 2)$. Since the y-intercept is 10, $10 = a(0 + 5)(0 - 2)$ and the value of a is -1.)

Part B: Solving Quadratic Equations by Factoring

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Solve these equations by factoring.

   a) $x^2 - x - 12 = 0$

   Answer:

   $x^2 - x - 12 = 0$

   $(x-4)(x+3) = 0$

   $x - 4 = 0 \quad \text{or} \quad x + 3 = 0$

   $x = 4 \quad \text{or} \quad x = -3$

   b) $x^2 - x - 20 = 0$

   Answer:

   $x^2 - x - 20 = 0$

   $(x-5)(x+4) = 0$

   $x - 5 = 0 \quad \text{or} \quad x + 4 = 0$

   $x = 5 \quad \text{or} \quad x = -4$

   c) $-x^2 - 2x + 3 = 0$

   Answer:

   $-x^2 - 2x + 3 = 0$

   $x^2 + 2x - 3 = 0$ Multiply each side by (-1).

   $(x+3)(x-1) = 0$

   $x + 3 = 0 \quad \text{or} \quad x - 1 = 0$

   $x = -3 \quad \text{or} \quad x = 1$

d) $x^2 + 9x + 18 = 0$

Answer:

$x^2 + 9x + 18 = 0$

$(x + 6)(x + 3) = 0$

$x + 6 = 0$ or $x + 3 = 0$

$x = -6$ or $x = -3$

e) $2x^2 + 3x - 2 = 0$

Answer:

$2x^2 + 3x - 2 = 0$

$(2x - 1)(x + 2) = 0$

$2x - 1 = 0$ or $x + 2 = 0$

$x = \frac{1}{2}$ or $x = -2$

2. Rearrange each of the following equations and solve by factoring, if possible.

a) $10x^2 = 7x + 12$

Answer:

$10x^2 - 7x - 12 = 0$

$(5x + 4)(2x - 3) = 0$

$5x + 4 = 0$ or $2x - 3 = 0$

$x = -\frac{4}{5}$ or $x = \frac{3}{2}$

b) $5x^2 + 21x = 54$

Answer:

$5x^2 + 21x - 54 = 0$

$(5x - 9)(x + 6) = 0$

$5x - 9 = 0$ or $x + 6 = 0$

$x = \frac{9}{5}$ or $x = -6$

c) $3x(x-2)-x(x+1)+5=0$

Answer:

$3x(x-2)-x(x+1)+5=0$

$3x^2-6x-x^2-x+5=0$

$2x^2-7x+5=0$

$(2x-5)(x-1)=0$

$2x-5=0 \quad \text{or} \quad x-1=0$

$x=\frac{5}{2} \quad \text{or} \quad x=1$

d) $x^2+\frac{9}{2}x-2\frac{1}{2}=0$

Answer:

$x^2+\frac{9}{2}x-2\frac{1}{2}=0$

$x^2+\frac{9}{2}x-\frac{5}{2}=0$

$2x^2+9x-5=0$ Multiply by 2.

$(2x-1)(x+5)=0$

$2x-1=0 \quad \text{or} \quad x+5=0$

$x=\frac{1}{2} \quad \text{or} \quad x=-5$

e) $x^2+9=0$

Answer:

$x^2+9=0$

Does not factor.

Solve by using inverse operations.

$x^2=-9$

$x=\pm\sqrt{-9}$

$x=$ no real answer

f) $16x^2 - 64 = 0$

Answer:

$$16x^2 - 64 = 0$$

$$16\left(x^2 - 4\right) = 0$$

$$16(x-2)(x+2) = 0$$

$$(x-2)(x+2) = 0 \qquad \text{Divide by 16.}$$

$$x - 2 = 0 \quad \text{or} \quad x + 2 = 0$$

$$x = 2 \quad \text{or} \quad x = -2$$

g) $x^2 + 12x - 28 = 0$

Answer:

$$x^2 + 12x - 28 = 0$$

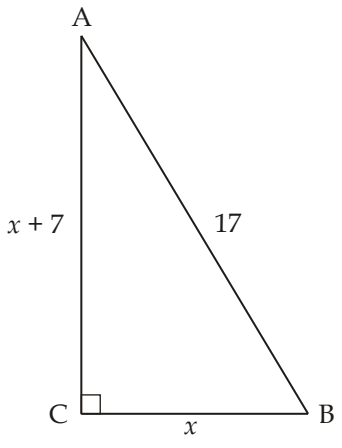
$$(x+14)(x-2) = 0$$

$$x + 14 = 0 \quad \text{or} \quad x - 2 = 0$$

$$x = -14 \quad \text{or} \quad x = 2$$

3. One leg of a right triangle is 7 m longer than the other leg. The hypotenuse is 17 m long. Find the length of each leg.

Answer:



Let x = shorter leg of the right triangle

$x + 7$ = larger leg of the right triangle

Because the triangle has a right angle,

$$a^2 + b^2 = c^2$$

$$x^2 + (x+7)^2 = 17^2$$

$$x^2 + x^2 + 14x + 49 = 289$$

$$2x^2 + 14x - 240 = 0$$

$$2\left(x^2 + 7x - 120\right) = 0$$

$$x^2 + 7x - 120 = 0$$

$$(x+15)(x-8) = 0$$

$x + 15 = 0$ or $x - 8 = 0$

$x = -15$ or $x = 8$

$x = -15$ is not possible as a length

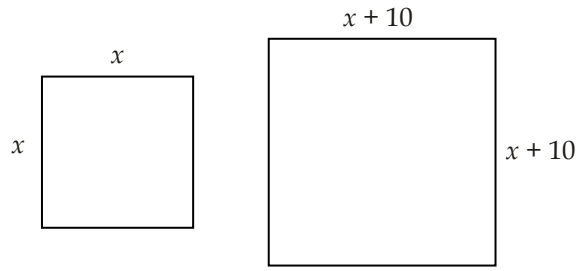
$\therefore x = 8$ m is the shorter side

$x + 7 = 15$ m is the longer side

Note: When solving word problems, writing a statement at the beginning of your solution that identifies what the variables represent is a necessary part of your solution.

4. Two square checkerboards together have an area of 1300 cm^2. One has sides that are 10 cm longer than the other. Find the length of the sides of each checkerboard.

   Answer:



Let x = length of side of smaller square

Let $x + 10$ = length of side of larger square

Sum of areas = 1300 cm^2

$$(x \cdot x) + (x + 10)(x + 10) = 1300$$

$$x^2 + x^2 + 20x + 100 = 1300$$

$$2x^2 + 20x - 1200 = 0$$

$$2\left(x^2 + 10x - 600\right) = 0$$

$$x^2 + 10x - 600 = 0$$

$$(x - 20)(x + 30) = 0$$

$x = 20$ or $x = -30$

Since x represents a length, the negative value of -30 must be ignored.

The only solution can be $x = 20$ and $x + 10 = 30$.

Thus, the length of each side of the smaller checkerboard is 20 cm and of the bigger checkerboard 30 cm.

5. Find 3 consecutive odd integers such that the product of the second and the third is 63.

Answer:

Let x = first odd integer

$x + 2$ = second odd integer

$x + 4$ = third odd integer

$(x + 2)(x + 4) = 63$

$x^2 + 6x + 8 = 63$

$x^2 + 6x - 55 = 0$

$(x + 11)(x - 5) = 0$

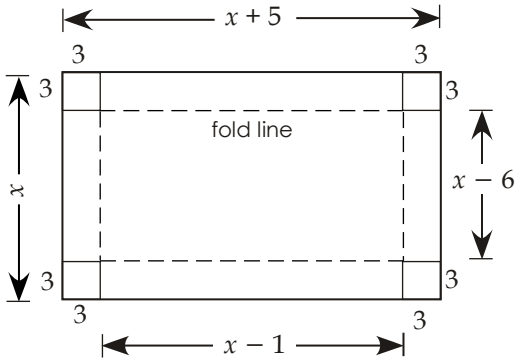
$x = -11$ or $x = 5$

If $x = -11$ is first integer
$x + 2 = -9$ is second integer
$x + 4 = -7$ is third integer

If $x = 5$ is first integer
$x + 2 = 7$ is second integer
$x + 4 = 9$ is third integer

6. A rectangular piece of cardboard is 5 cm longer than it is wide. A 3 cm-by-3 cm square is cut out of each corner, and the four flaps are folded up to form a box with a volume of 450 cm^3. What were the length and width of the original piece of cardboard? (Volume = length × width × height)

Answer:



Let x = width of rectangle

$x + 5$ = length of rectangle

3 = height of box = side of square removed from each corner

$x + 5 - 6$ or

$x - 1$ = length of box

$x - 6$ = width of box

$450 = (x - 1)(x - 6)(3)$ Formula for volume.

$150 = (x - 1)(x - 6)$ Divide by 3.

$150 = x^2 - 7x + 6$

$0 = x^2 - 7x - 144$

$0 = (x - 16)(x + 9)$

$x = 16$ or $x = -9$

Since the width cannot be negative, our solution is $x = 16$.

The dimensions of the original piece of cardboard are 16 cm wide and 21 cm long.

7. Farrah was asked to solve a quadratic equation by factoring. Her work is shown below.

   $x^2 - 10x + 24 = 0$

   $(x - 6)(x - 4) = 0$

   Therefore, $x = -6$ or $x = -4$.

   Identify, explain, and correct the errors Farrah made when trying to solve this quadratic equation.

   Answer:

   Farrah did not equate each factor of the quadratic equation to zero. Instead, she just took the terms inside the brackets as solutions. The correct solutions are:

   $x - 6 = 0$ or $x - 4 = 0$

   $x = 6$ $x = 4$

8. Write the quadratic function $y = -4(x - 3)(x + 1)$ in vertex form using the average of the x-intercepts.

   Answer:

   For $y = -4(x - 3)(x + 1)$, the x-intercepts are $x = 3$ and $x = -1$. The x-coordinate of the vertex is $\frac{3 + (-1)}{2} = 1$. The y-coordinate of the vertex is $x = -4(1 - 3)(1 + 1) = -4(-2)(2) = 16$. The leading coefficient, a, is -4. The function in vertex form is $y = -4(x - 1)^2 + 16$.

Learning Activity 4.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Evaluate: $\sqrt{115 + \sqrt{27 + \sqrt{76 + \sqrt{25}}}}$
2. Britney is making cookies. The recipe calls for $\frac{3}{8}$ of a cup of butter. If Britney is tripling the recipe, how much butter does she need? Write your answer as a mixed fraction.
3. Joanne makes $400 in a month. If Joanne needs to pay $66 for her phone bill, $40 for gas to drive her parents' car, and $120 for a haircut, will she have enough to buy the $150 jeans she wants?
4. Factor: $\frac{1}{3}x^2 - \frac{121}{3}$
5. Find the vertex of the following parabola: $y = 3x^2 + 6x - 5$
6. Rationalize the denominator: $\frac{6}{\sqrt{2}}$
7. Find the slope of the line perpendicular to the line, $7x + 3y - 4 = 0$.
8. The circumference of a circle is 92π centimetres. What is the radius of the circle?

Answers:

1. $11 \left(\sqrt{115 + \sqrt{27 + \sqrt{76 + \sqrt{25}}}}\,; \sqrt{115 + \sqrt{27 + \sqrt{76 + 5}}}\,; \sqrt{115 + \sqrt{27 + \sqrt{81}}}\,; \right.$
   $\left. \sqrt{115 + \sqrt{27 + 9}}\,; \sqrt{115 + \sqrt{36}}\,; \sqrt{115 + 6}\,; \sqrt{121}\,; = 11 \right)$
2. $1\frac{1}{8}$ cups of butter $\left(\frac{3}{8} \times 3 = \frac{9}{8} \right)$
3. Yes ($400 − ($66 + $40 + $120) = $400 − $226 = $174)
4. $\frac{1}{3}(x - 11)(x + 11)\left[\frac{1}{3}x^2 - \frac{121}{3} = \frac{1}{3}\left(x^2 - 121\right) = \frac{1}{3}(x - 11)(x + 11) \right]$

5 $(-1, -8)$ $\left(x = -\frac{6}{2(3)} = -\frac{6}{6} = -1;\ y = 3(-1)^2 + 6(-1) - 5 = 3 - 6 - 5 = -8\right)$

6. $3\sqrt{2}$ $\left(\frac{6}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}\right)$

7. $\frac{3}{7}$ $\left(3y = -7x + 4;\ y = -\frac{7}{3}x + \frac{4}{3}\right)$

   (Recall: Perpendicular lines have slopes that are negative reciprocals.)

8. 46 cm $\left(C = 2\pi r;\ C = 92\pi;\ \therefore 2\pi r = 92\pi;\right.$

   $\left. r = \frac{92\pi}{2\pi};\ r = \frac{92}{2} = 46 \text{ cm}\right)$

Part B: Solving Quadratic Equations by Completing the Square and Using the Quadratic Formula

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. For each quadratic equation below, state the values of a, b, and c for the general equation format.
   a) $x^2 - 2x - 5 = 0$
   b) $3x^2 - 2x + 5 = 0$
   c) $5x^2 - 3x = 8$
   d) $2(x^2 - 2x) - 1 = 0$
   e) $5x^2 = 9x$
   f) $4 - 2x^2 = 9x$

   Answers:

   a) $x^2 - 2x - 5 = 0$
   $a = 1, b = -2, c = -5$

   b) $3x^2 - 2x + 5 = 0$
   $a = 3, b = -2, c = 5$

   c) $5x^2 - 3x - 8 = 0$
   $a = 5, b = -3, c = -8$

   d) $2(x^2 - 2x) - 1 = 0$
   $2x^2 - 4x - 1 = 0$
   $a = 2, b = -4, c = -1$

   e) $5x^2 = 9x$
   $5x^2 - 9x = 0$
   $a = 5, b = -9, c = 0$

   f) $4 - 2x^2 = 9x$
   $-2x^2 - 9x + 4 = 0$
   $2x^2 + 9x - 4 = 0$
   $a = 2, b = 9, c = -4$

2. Solve these equations by factoring or by using the quadratic formula. Be sure to state the formula before substituting values into it, as this will help you memorize it

a) $\frac{x^2}{2} + x - \frac{15}{2} = 0$

b) $2w^2 - 3w + 1 = 0$

c) $7w^2 - 3w = 0$

d) $1 = 25x^2$

e) $x^2 - 0.1x - 0.06 = 0$

f) $-x^2 - 7x - 1 = 0$

Answers:

a) $\frac{x^2}{2} + x - \frac{15}{2} = 0$

$\frac{1}{2}\left(x^2 + 2x - 15\right) = 0$

$\frac{1}{2}(x + 5)(x - 3) = 0$

$x + 5 = 0$ or $x - 3 = 0$

$x = -5$ or $x = 3$

b) $2w^2 - 3w + 1 = 0$

$(2w - 1)(w - 1) = 0$

$2w - 1 = 0$ or $w - 1 = 0$

$w = \frac{1}{2}$ or $w = 1$

c) $7w^2 - 3w = 0$

$w(7w - 3) = 0$

$w = 0$ or $7w - 3 = 0$

$w = 0$ or $w = \frac{3}{7}$

d) $1 = 25x^2$

$0 = 25x^2 - 1$

$0 = (5x - 1)(5x + 1)$

$5x - 1 = 0$ or $5x + 1 = 0$

$x = \frac{1}{5}$ or $x = -\frac{1}{5}$

e) $x^2 - 0.1x - 0.06 = 0$

$(x - 0.3)(x + 0.2) = 0$

$x - 0.3 = 0$ or $x + 0.2 = 0$

$x = 0.3$ or $x = -0.2$

f) $-x^2 - 7x - 1 = 0$

$x^2 + 7x + 1 = 0$

$a = 1, b = 7, c = 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-7 \pm \sqrt{49 - 4}}{2}$$

$$x = \frac{-7 \pm \sqrt{45}}{2}$$

$$x = \frac{-7 \pm 3\sqrt{5}}{2}$$

3. Use the completing-the-square method to find the roots of each equation below.
   a) $3x^2 - 6x - 15 = 0$
   b) $2x^2 - 4x - 1 = 0$
   c) $9x^2 - 18x - 7 = 0$
   d) $x^2 - x - 3 = 0$

Answers:

a) $3x^2 - 6x - 15 = 0$

$3\left(x^2 - 2x \quad\right) = 15$

$3\left(x^2 - 2x + 1\right) = 15 + 3(1)$

$3(x-1)^2 = 18$

$(x-1)^2 = 6$

$x - 1 = \pm\sqrt{6}$

$x = 1 \pm \sqrt{6}$

b) $2x^2 - 4x - 1 = 0$

$2\left(x^2 - 2x \quad\right) = 1$

$2\left(x^2 - 2x + 1\right) = 1 + 2(1)$

$2(x-1)^2 = 3$

$(x-1)^2 = \frac{3}{2}$

$x - 1 = \pm\sqrt{\frac{3}{2}}$

$x = 1 \pm \frac{\sqrt{3}}{\sqrt{2}}$

$x = 1 \pm \frac{\sqrt{6}}{2}$

$x = \frac{2 \pm \sqrt{6}}{2}$

c) $9x^2 - 18x - 7 = 0$

$9\left(x^2 - 2x \quad\right) = 7$

$9\left(x^2 - 2x + 1\right) = 7 + 9(1)$

$9(x-1)^2 = 16$

$(x-1)^2 = \frac{16}{9}$

$x - 1 = \pm\frac{4}{3}$

$x = 1 \pm \frac{4}{3}$

$x = 1 + \frac{4}{3}, 1 - \frac{4}{3}$

$x = \frac{7}{3}, -\frac{1}{3}$

d) $x^2 - x - 3 = 0$

$x^2 - x + \frac{1}{4} = 3 + \frac{1}{4}$

$\left(x - \frac{1}{2}\right)^2 = \frac{13}{4}$

$x - \frac{1}{2} = \pm\sqrt{\frac{13}{4}}$

$x = \frac{1}{2} \pm \frac{\sqrt{13}}{2}$

$x = \frac{1 \pm \sqrt{13}}{2}$

4. Find the zeros of the function using the quadratic formula.

   a) $f(x) = 5x^2 - x - 3$

   b) $y = 2x^2 + 6x - 1$

Answers:

a) $f(x) = 5x^2 - x - 3$

$0 = 5x^2 - x - 3$

$a = 5,\ b = -1,\ c = -3$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{+1 \pm \sqrt{(-1)^2 - 4(5)(-3)}}{2(5)}$$

$$x = \frac{1 \pm \sqrt{1 + 60}}{10}$$

$$x = \frac{1 \pm \sqrt{61}}{10}$$

$$x = \frac{1 + \sqrt{61}}{10} \text{ or } \frac{1 - \sqrt{61}}{10}$$

b) $y = 2x^2 + 6x - 1$

$0 = 2x^2 + 6x - 1$

$a = 2,\ b = 6,\ c = -1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-6 \pm \sqrt{(6)^2 - 4(2)(-1)}}{2(2)}$$

$$x = \frac{-6 \pm \sqrt{36 + 8}}{4}$$

$$x = \frac{-6 \pm \sqrt{44}}{4}$$

$$x = \frac{-6 \pm 2\sqrt{11}}{4}$$

$$x = \frac{2\left(-3 \pm \sqrt{11}\right)}{2(2)}$$

$$x = \frac{-3 \pm \sqrt{11}}{2}$$

$$x = \frac{-3 + \sqrt{11}}{2} \text{ or } \frac{-3 - \sqrt{11}}{2}$$

5. Find the roots of the quadratic equation $6x^2 + 5x - 6 = 0$

   a) using the quadratic formula

   b) by factoring

   Which method do you prefer and why?

   Answers:

   a) $6x^2 + 5x - 6 = 0$

   $a = 6, b = 5, c = -6$

   $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

   $$x = \frac{-5 \pm \sqrt{(5)^2 - 4(6)(-6)}}{2(6)}$$

   $$x = \frac{-5 \pm \sqrt{169}}{12}$$

   $$x = \frac{-5 \pm 13}{12} = \frac{-5 + 13}{12}, \frac{-5 - 13}{12} = \frac{8}{12}, -\frac{18}{12}$$

   $$x = \frac{2}{3} \text{ or } -\frac{3}{2}$$

   b) $6x^2 + 5x - 6 = 0$

   $(2x + 3)(3x - 2) = 0$

   $2x + 3 = 0$ $\qquad$ $3x - 2 = 0$

   $2x = -3$ $\qquad$ $3x = 2$

   $x = \frac{-3}{2}$ $\qquad$ $x = \frac{2}{3}$

You may prefer either method. Perhaps you find factoring easier than plugging values into a formula. Or, you may like the certainty of using a formula. Both methods are important for future mathematics courses so you should practise both.

6. Find the zeros of the quadratic function $f(x) = x^2 + 8x + 15$ by
   a) factoring
   b) using the quadratic formula
   c) graphing
   d) completing the square.

   Which method do you prefer and why?

   Answers:

   a) $0 = x^2 + 8x + 15$
   $0 = (x + 3)(x + 5)$
   $x = -3, -5$

   b) $a = 1, b = 8, c = 15$

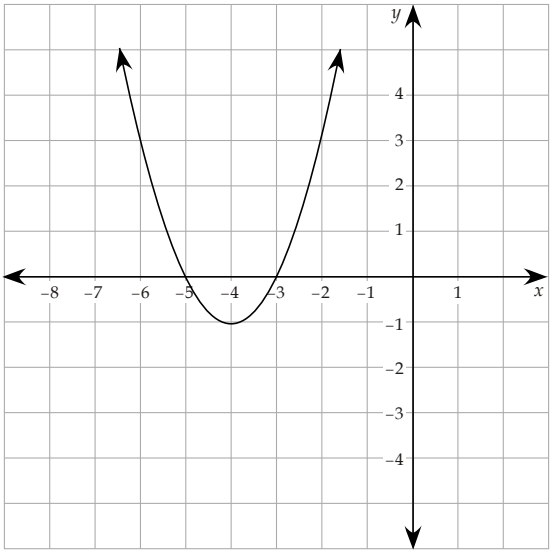
$$x = \frac{-8 \pm \sqrt{64 - 60}}{2}$$

$$x = \frac{-8 \pm \sqrt{4}}{2}$$

$$x = \frac{-8 \pm 2}{2} = \frac{-8 + 2}{2}, \frac{-8 - 2}{2}$$

   $x = -3$ or -5

   c) $y = x^2 + 8x + 15$
   $y = (x^2 + 8x + 16) + 15 - 16$
   $y = (x + 4)^2 - 1$
   Vertex: $(-4, -1)$, opens up, standard size
   Zeros: $x = -3, -5$



d)
$$0 = x^2 + 8x + 15$$
$$0 = \left(x^2 + 8x + 16\right) + 15 - 16$$
$$0 = (x + 4)^2 - 1$$
$$1 = (x + 4)^2$$
$$x + 4 = \pm 1$$
$$x = \pm 1 - 4$$
$$x = 1 - 4, -1 - 4$$
$$x = -3, -5$$

You may prefer any of the above methods for a variety of reasons. It is a good idea to practise them all—check your answers if you try more than one method. Some questions may be easier to solve using one method while others are easier to solve using a different method.

7. Refer to the Solving Quadratic Equations chart found at the end of this learning activity. For each of the five methods of solving quadratic equations, describe when each method could be best used. Fill in your answers on the chart.

Answer:

See the Solving Quadratic Equations chart at the end of the answers for Learning Activity 4.4 for possible descriptions. Discuss your answers with your learning partner or your tutor/marker if you are unsure.

8. Write a quadratic equation in general form that satisfies each of the following descriptions. Explain your answers.
   a) two distinct real roots
   b) one distinct real root, or two equal roots
   c) no real roots

Answers:

Answers will vary—possible answers are given. Check your answers with your learning partner or tutor/marker.

a) If you pick the roots to be $x = 1$ and $x = 5$, then $(x - 1)$ and $(x - 5)$ are factors of the corresponding quadratic equation. Thus, the corresponding quadratic equation could be $(x - 1)(x - 5) = 0$ or $x^2 - 6x + 5 = 0$.

b) If you pick the two equal roots to be $x = 1$ and $x = 1$, then $(x - 1)$ and $(x - 1)$ are factors of the corresponding quadratic equation. Thus, the quadratic equation is $(x - 1)(x - 1) = 0$ or $x^2 - 2x + 1 = 0$.

c) A quadratic equation that has no real roots has a corresponding quadratic function that does not intersect the x-axis. For example, $y = (x - 2)^2 + 1$ has its vertex at (2, 1) and opens up. In general form, the equation is

$$0 = (x - 2)^2 + 1$$
$$0 = x^2 - 4x + 4 + 1$$
$$0 = x^2 - 4x + 5$$

Also, when you use the quadratic formula, you get the following result:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{+4 \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)}$$
$$x = \frac{4 \pm \sqrt{-4}}{2}$$

x = not a real number. Thus, there are no roots.

9. Is the quadratic formula always the most efficient way to solve quadratic equations? Explain with examples.

Answer:

No. In Question 6, the quadratic equation could easily be solved by factoring. Therefore, the quadratic formula could sometimes make an easy question unnecessarily hard.

10. A rocket is launched and follows a path described by the quadratic function $h(d) = -0.02d^2 + 0.4d + 1$. The height is represented by h while the horizontal distance is represented by d, and both are measured in metres. How far away from its original position does this rocket land?

Answer:

When the rocket lands, its height is zero. Therefore, the quadratic equation is $-0.02d^2 + 0.4d + 1 = 0$. To find the distance at this time, you need to solve for the roots of this quadratic equation. The completing-the-square method is shown below.

$-0.02d^2 + 0.4d + 1 = 0$

$d^2 - 20d - 50 = 0$	Dividing by -0.02.
$d^2 - 20d = 50$	Add 50.
$d^2 - 20d + 100 = 50 + 100$	Completing the square.
$(d - 10)^2 = 150$	Factoring and simplifying.
$d - 10 = \pm\sqrt{150}$	Taking square roots.
$d = 10 \pm 5\sqrt{6}$	

Two possible answers are:

$d = 10 + 5\sqrt{6} \approx 22.247$

$d = 10 - 5\sqrt{6} \approx -2.247$

$d = 10 - 5\sqrt{6}$ is a root of the equation but not a solution to the problem. The rocket cannot travel a negative distance. Thus, $10 - 5\sqrt{6}$ is an extraneous root. The only solution is $d = 10 + 5\sqrt{6}$. Therefore, the rocket has travelled $\left(10 + 5\sqrt{6}\right)$ m.

11. David is trying to solve a quadratic equation by using the quadratic formula. His work is shown below.

$10x^2 + 2x + 1 = 0$

$a = 10, b = 2, c = 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(10)}}{2(10)}$$

$$x = \frac{-2 \pm \sqrt{4 - 40}}{20}$$

$$x = \frac{-2 \pm \sqrt{-36}}{20}$$

$$x = \frac{-2 \pm 6}{20}$$

$$x = \frac{-2 + 6}{20} = \frac{4}{20} = \frac{1}{5}$$

$$x = \frac{-2 - 6}{20} = \frac{-8}{20} = \frac{-2}{5}$$

Identify, explain, and correct David's error.

Answer:

David took the square root of a negative number (−36). It is not possible to find a real number answer for the square root of a negative number. Therefore, there is no solution to this quadratic equation in real numbers like David thought there was.

12. Solve the equation:

$(x^2 - 5)^2 + 3(x^2 - 5) + 2 = 0$

Answer:

Notice that there is a quadratic pattern in the equation. Temporarily, let $a = x^2 - 5$ and substitute into the original equation.

$\therefore a^2 + 3a + 2 = 0$

$(a + 2)(a + 1) = 0$

$a = -2$ or $a = -1$

Now solve for x after replacing a with $x^2 - 5$.

$x^2 - 5 = -2$ or $x^2 - 5 = -1$

$x^2 = 3$ $x^2 - 4 = 0$

$x = \pm\sqrt{3}$ $(x - 2)(x + 2) = 0$

$x = 2$ or $x = -2$

Solutions:

$x = \{-2, -\sqrt{3}, \sqrt{3}, 2\}$

Solving Quadratic Equations

Graphing

Use when:

The quadratic equation is in vertex form with integer values for a, p, and q that can easily fit on a graph.

Square Roots

Use when:

There is no x-term, when $b = 0$ in the general form of the quadratic equation, or when the equation is presented as a binomial squared.

Factoring

Use when:

The equation, in general form, can be factored easily.

Completing the Square

Use when:

The x^2 term has a coefficient of 1, and the coefficient of the x-term is even.

Quadratic Formula

Use when:

The equation cannot be factored easily or at all, or when completing the square would involve fractions.

Learning Activity 4.5

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Janique is taking piano lessons and her teacher tells her she needs to practice piano for at least 7 hours a week. If Janique practices for 45 minutes each day from Monday to Friday, 1.25 hours on Saturday, and 1.75 hours on Sunday, will Janique have practiced enough?
2. If you sleep for 9 hours a day, what fraction of the day are you awake (in lowest terms)?
3. Evaluate for $j = 3$: $5j - 2j^2 + 7$
4. Evaluate: $2^{(2^3)}$
5. What is the area of a triangle with a base of 4 cm and a height of 8 cm?
6. Factor completely: $36x^2 - 1$
7. Two angles in an isosceles triangle measure 16° and 148°. What is the measure of the third angle?
8. Express as a radical: $8^{\frac{3}{2}}$

Answers:

1. No [Janique will still need to practice for 0.25 hours, or 15 minutes. (45 minutes = 0.75 hours; Total time = $0.75 \times 5 + 1.25 + 1.75 = 3.75 + 1.25 + 1.75 = 6.75$ hours)]
2. $\frac{5}{8}$ $\left(24 - 9 = 15; \frac{15}{24} = \frac{5}{8}\right)$
3. 4 [$5(3) - 2(9) + 7 = 15 - 18 + 7 = 4$]
4. 256 $\left(2^{2^3} = 2^8 = 256\right)$

5. $16\ \text{cm}^2$ $\left(\text{Area} = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 4 \times 8 = 16\ \text{cm}^2\right)$

6. $(6x - 1)(6x + 1)$

7. 16°

   (**Method 1:** The sum of the angles in a triangle must be 180°. Therefore, the missing angle is 180° − 16° − 148° = 16°.

   Method 2: An isosceles triangle will have two angles that are the same and all three angles add up to 180°. Therefore, the third angle must measure 16°, since you cannot have two angles of 148° in a triangle.)

8. $\sqrt{8^3}$ or $\left(\sqrt{8}\right)^3$

Part B: The Discriminant

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Determine whether each statement is true or false. If the statement is false, rewrite it so it is true.
   a) The discriminant of $2x^2 + 5x + 6 = 0$ is 23.
   b) For each quadratic equation $ax^2 + bx + c = 0$, if $b^2 - 4ac > 0$, then the equation has two real roots.
   c) For each quadratic equation $ax^2 + bx + c = 0$, if $b^2 - 4ac < 0$, then the equation has no real number solutions.
   d) $\frac{-6 + \sqrt{32}}{10} = \frac{-3 + 8\sqrt{2}}{5}$
   e) The equation $x^2 + 2x + 1 = 0$ has two distinct real roots.

   Answers:

   a) False. The discriminant of $2x^2 + 5x + 6 = 0$ is $b^2 - 4ac = 5^2 - 4(2)(6) = 25 - 48 = -23$.
   b) True
   c) True
   d) False. $\frac{-6 + \sqrt{32}}{10} = \frac{-6 + 4\sqrt{2}}{10} = \frac{-3 + 2\sqrt{2}}{5}$
   e) False. The equation $x^2 + 2x + 1 = 0$ has one real root since $b^2 - 4ac = 2^2 - 4(1)(1) = 0$.

2. If the discriminant of a quadratic equation has the given value, state the nature of the roots.

   Answer:

	Discriminant	Nature of Roots:
a)	-15	no real roots
b)	25	2 real, rational roots
c)	-9	no real roots
d)	0	1 real root
e)	50	2 real, irrational roots

3. How many times would the graph of $y = ax^2 + bx + c$ (with a, b, and c as rational numbers and $a \neq 0$) intersect the x-axis if the value of the discriminant is

   a) negative?

   Answer: Never

   b) zero?

   Answer: Once

   c) positive?

   Answer: Twice

4. Determine the nature of the roots by calculating the discriminant for each equation.

   a) $x^2 - 8x + 16 = 0$

   Answer:

   $x^2 - 8x + 16 = 0$

   $a = 1, b = -8, c = 16$

   $b^2 - 4ac = (-8)^2 - 4(1)(16)$

   $= 64 - 64$

   $= 0$

   Because $b^2 - 4ac = 0$, there is one real root.

b) $x^2 + 2x + 7 = 0$

Answer:

$x^2 + 2x + 7 = 0$

$a = 1, b = 2, c = 7$

$b^2 - 4ac = (2)^2 - 4(1)(7)$

$= 4 - 28$

$= -24$

Because $b^2 - 4ac < 0$, there are no real roots.

c) $p^2 - 16 = 0$

Answer:

$p^2 - 16 = 0$

$a = 1, b = 0, c = -16$

$b^2 - 4ac = (0)^2 - 4(1)(-16)$

$= 64$

Because $b^2 - 4ac > 0$ and equal to a perfect square, there are two real, rational roots.

d) $2x^2 + x = 5$

Answer:

$2x^2 + x = 5$

$2x^2 + x - 5 = 0$

$a = 2, b = 1, c = -5$

$b^2 - 4ac = (1)^2 - 4(2)(-5)$

$= 1 + 40$

$= 41$

Because $b^2 - 4ac > 0$ and not equal to a perfect square, there are two real, irrational roots.

5. Find value(s) of k so that each equation has real and equal roots.

a) $kx^2 - 6x + 2 = 0$

Answer:

$kx^2 - 6x + 2 = 0$

If the roots are equal, $b^2 - 4ac = 0$.

$a = k$, $b = -6$, $c = 2$

$$(-6)^2 - 4(k)(2) = 0$$
$$36 - 8k = 0$$
$$-8k = -36$$
$$k = \frac{-36}{-8}$$
$$k = \frac{9}{2}$$

b) $x^2 + (k - 8)x + 9 = 0$

Answer:

$x^2 + (k - 8)x + 9 = 0$

If the roots are equal, $b^2 - 4ac = 0$.

$a = 1$, $b = k - 8$, $c = 9$

Method 1:

$$(k - 8)^2 - 4(1)(9) = 0$$
$$k^2 - 16k + 64 - 36 = 0$$
$$k^2 - 16k + 28 = 0$$
$$(k - 2)(k - 14) = 0$$

$k = 2$ or $k = 14$

Method 2:

$$(k - 8)^2 - 4(1)(9) = 0$$
$$(k - 8)^2 - 36 = 0$$
$$(k - 8)^2 = 36$$
$$k - 8 = \pm 6$$

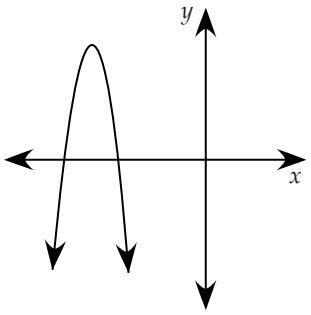
$k = 8 - 6$ or $k = 8 + 6$

$k = 2$ or $k = 14$

6. Sketch the graph of a quadratic function whose discriminant is positive and whose vertex is in the second quadrant.

   Answer:

   The graph must intersect the x-axis twice because a positive discriminant tells us that the quadratic function has two real roots. It must open down.



7. Find the value of k for each of the following situations.

   a) $3x^2 - 2x + k = 0$ has two real roots

   Answer:

   $3x^2 - 2x + k = 0$

   If the equation has two real roots, $b^2 - 4ac > 0$.

   $a = 3$, $b = -2$, $c = k$

$$b^2 - 4ac > 0$$

$$(-2)^2 - 4(3)k > 0$$

$$4 - 12k > 0$$

$$-12k > -4$$

$$\frac{-12k}{-12} < \frac{-4}{-12}$$

$$k < \frac{1}{3}$$

   Remember that when dividing an inequality by a negative number, the inequality sign reverses direction.

b) $x^2 + kx + 3k = 0$ has a double root

Answer:

$x^2 + kx + 3k = 0$

If the equation has a double root, then $b^2 - 4ac$ must be 0.

$$b^2 - 4ac = 0$$
$$(k)^2 - 4(1)(3k) = 0$$
$$k^2 - 12k = 0$$
$$k(k - 12) = 0$$

$k = 0, k = 12$

c) $kx^2 + 8x = 4$ has no real roots

Answer:

$$kx^2 + 8x = 4$$
$$kx^2 + 8x - 4 = 0$$

If the equation has no real roots, then $b^2 - 4ac < 0$.

$a = k, b = 8, c = -4$

$$b^2 - 4ac < 0$$
$$8^2 - 4(k)(-4) < 0$$
$$64 + 16k < 0$$
$$16k < -64$$
$$k < -4$$

8. For each of the following quadratic equations, solve by first using the discriminant to determine if its value is a perfect square. If it is a perfect square, use the factoring method to solve. If it is not a perfect square, use the quadratic formula.

a) $12x^2 - 19x - 21 = 0$

Answer:

$12x^2 - 19x - 21 = 0$

$a = 12,\ b = -19,\ c = -21$

$b^2 - 4ac = (-19)^2 - 4(12)(-21)$

$= 1369$

$= 37^2$

The discriminant is a perfect square. The equation can be factored.

$$12x^2 - 19x - 21 = 0$$

$$12x^2 - 28x + 9x - 21 = 0$$

$$4x(3x - 7) + 3(3x - 7) = 0$$

$$(4x + 3)(3x - 7) = 0$$

$$x = -\frac{3}{4} \text{ and } x = \frac{7}{3}$$

b) $18x^2 - 35x + 12 = 0$

Answer:

$18x^2 - 35x + 12 = 0$

$a = 18,\ b = -35,\ c = 12$

$b^2 - 4ac = (-35)^2 - 4(18)(12)$

$= 361$

$= 19^2$

The discriminant is a perfect square. The equation can be factored.

$$18x^2 - 35x + 12 = 0$$

$$18x^2 - 27x - 8x + 12 = 0$$

$$9x(2x - 3) - 4(2x - 3) = 0$$

$$(9x - 4)(2x - 3) = 0$$

$$x = \frac{4}{9}, \frac{3}{2}$$

c) $60x^2 - 59x - 20 = 0$

Answer:

$60x^2 - 59x - 20 = 0$

$a = 60,\ b = -59,\ c = -20$

$b^2 - 4ac = (-59)^2 - 4(60)(-20)$

$= 8281$

$= 91^2$

The discriminant is a perfect square. The equation can be factored.

$$60x^2 - 59x - 20 = 0$$

$$60x^2 + 16x - 75x - 20 = 0$$

$$4x(15x + 4) - 5(15x + 4) = 0$$

$$(4x - 5)(15x + 4) = 0$$

$x = \frac{5}{4}$ and $x = -\frac{4}{15}$

9. Complete the following Quadratic Equation Puzzle with your learning partner. Cut out all the separate pieces and arrange them into rows on the Quadratic Equation Puzzle worksheet so that each row corresponds to a quadratic equation. In each row, the quadratic equation must be in general form and vertex form, and you must include the value of the discriminant, the number of roots, the types of roots, and the actual roots themselves.

Quadratic Equation Puzzle Key

General Form of Quadratic Equation	Vertex Form of Quadratic Function	Value of the Discriminant	Number of Roots	Type of Roots	Roots
$9x^2 + 6x - 4 = 0$	$y = 9\left(x + \frac{1}{3}\right)^2 - 5$	180	2	Real and Irrational	$\frac{-1+\sqrt{5}}{3}, \frac{-1-\sqrt{5}}{3}$
$3x^2 + 8x - 35 = 0$	$y = 3\left(x + \frac{4}{3}\right)^2 - \frac{121}{3}$	484	2	Real and Rational	$-5, \frac{7}{3}$
$2x^2 - 2x + 3 = 0$	$y = 2\left(x - \frac{1}{2}\right)^2 + \frac{5}{2}$	-20	0	None	No Real Roots
$x^2 - 4x + 4 = 0$	$y = (x - 2)^2$	0	1	Real and Equal	2
$25x^2 - 10x + 1 = 0$	$y = 25\left(x - \frac{1}{5}\right)^2$	0	1	Real and Equal	$\frac{1}{5}$

Learning Activity 4.6

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Multiply: $(3x^2)(4xy^4)$
2. Your dog can run at a speed of 10 m/s. If you are standing half a kilometre away from your dog, how long will it take for your dog to run to you?
3. What are the x-intercepts of the function $y = x^2 + 3x + 2$?
4. At a restaurant, Megan orders a meal advertised as containing 500 calories. She also orders a chocolate milkshake, which contains 490 calories. If Megan only eats 75% of her meal but drinks her entire milkshake, how many calories did Megan consume?
5. Calculate: $\frac{1}{5} + \frac{5}{6} + \frac{2}{15}$
6. Multiply: $(x - y)(3x^2 - 7x + 2)$
7. State the independent and dependent variables for the following situation:

   When the temperature changes, the height of the mercury in a thermometer either rises or falls.

8. Express as an entire radical: $4x^{\frac{3}{2}}$

Answers:

1. $12x^3y^4$
2. 50 seconds (half a kilometre = 500 metres; $\frac{500}{10} = 50$)
3. -2 and -1 (Factoring: $(x + 2)(x + 1)$)
4. 865 (75% of 500 = 375; 490 + 375 = 865 calories)
5. $\frac{7}{6}$ $\left(\frac{6}{30} + \frac{25}{30} + \frac{4}{30} = \frac{35}{30} = \frac{7}{6}\right)$

6. $3x^3 - 7x^2 + 2x - 3x^2y + 7xy - 2y$

7. Independent Variable: The temperature

   Dependent Variable: The height of the mercury in the thermometer

8. $\sqrt{16x^3}$ $\left(4x^{\frac{3}{2}} = 4\sqrt{x^3} = \sqrt{4^2x^3} = \sqrt{16x^3}\right)$

Part B: Solving Rational Equations

Remember, these questions are similar to the ones that will be on your assignments and examinations. So, if you were able to answer them correctly, you are likely to do well on your assignments and examinations. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Determine whether each statement is true or false. If the statement is false, rewrite it so it is true.

   a) If $x \neq 2$, then $(x-2)\dfrac{4x}{x-2}$ is equivalent to $4x$.

   b) If $x \neq 3$, then $2(x-3)\left(\dfrac{6}{x-3} + \dfrac{x}{2}\right)$ is equivalent to $6 + x$.

   c) If $x \neq 0$, then the equations $x - \dfrac{1}{x} = 0$ and $x^2 - 1 = 0$ are equivalent.

   d) If $x \neq -3$ and $x \neq -4$, then $\dfrac{x-1}{x+3} = \dfrac{x-2}{x+4}$ is equivalent to $(x-1)(x+4) = (x+3)(x-2)$.

   Answers:

   a) True

   b) False. If $x \neq 3$, then $2(x-3)\left(\dfrac{6}{x-3} + \dfrac{x}{2}\right)$ is equivalent to $12 + x^2 - 3x$.

   c) True

   d) True

2. Solve each of the following rational equations. Make sure you state the non-permissible values and check for extraneous roots.

a) $x = -\dfrac{2}{x-3}$

Answer:

LCD = $x - 3$ and $x \neq 3$

$$x(x-3) = \frac{-2}{x-3}(x-3)$$

$$x^2 - 3x = -2$$

$$x^2 - 3x + 2 = 0$$

$$(x-2)(x-1) = 0$$

$$x = 2, 1$$

Check:

Are any of the solutions the non-permissible value? No.

$\therefore$ The solutions are $x = 2$ or $x = 1$, $x \neq 3$.

b) $\dfrac{2x-9}{x-7} + \dfrac{x}{2} = \dfrac{5}{x-7}$

Answer:

LCD = $2(x - 7)$ and $x \neq 7$.

$$\frac{2x-9}{x-7}(2)(x-7) + \frac{x}{2}(2)(x-7) = \frac{5}{x-7}(2)(x-7)$$

$$2(2x-9) + x(x-7) = 5(2)$$

$$4x - 18 + x^2 - 7x = 10$$

$$x^2 - 3x - 18 - 10 = 0$$

$$x^2 - 3x - 28 = 0$$

$$(x-7)(x+4) = 0$$

$$x = 7, -4$$

Check:

It is not possible for x to be 7.

$\therefore$ $x = -4$ is the only solution and $x \neq 7$.

c) $x - 4 = -\dfrac{1}{x}$

Answer:

LCD = x, $x \neq 0$

$$x(x-4) = x\left(\frac{-1}{x}\right)$$

$$x^2 - 4x = -1$$

$x^2 - 4x + 1 = 0$ Not factorable

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16-4}}{2}$$

$$= \frac{4 \pm \sqrt{12}}{2}$$

$$= \frac{4 \pm \sqrt{4}\sqrt{3}}{2}$$

$$= \frac{4 \pm 2\sqrt{3}}{2}$$

$$= \frac{2\left(2 \pm \sqrt{3}\right)}{2}$$

$$= 2 \pm \sqrt{3}$$

Check:

Non-permissible values are okay.

$\therefore$ Solutions are $x = 2 \pm \sqrt{3}$ and $x \neq 0$.

d) $\dfrac{3x^2}{3x+1} - 2 = \dfrac{2x+1}{3x+1}$

Answer:

LCD $= 3x+1$, $x \neq -\dfrac{1}{3}$

$$\frac{3x^2}{3x+1} - 2 = \frac{2x+1}{3x+1}$$

$$\left(\frac{3x^2}{3x+1}\right)(3x+1) - 2(3x+1) = \frac{(2x+1)}{(3x+1)}(3x+1)$$

$$3x^2 - 6x - 2 = 2x + 1$$

$$3x^2 - 8x - 3 = 0$$

$$(3x+1)(x-3) = 0$$

$3x+1=0$ $\qquad$ $x-3=0$

$x = -\dfrac{1}{3}$ $\qquad$ $x = 3$

Check:

It is not permissible for x to be $-\dfrac{1}{3}$.

$\therefore x = 3$ is the only solution and $x \neq -\dfrac{1}{3}$.

e) $\dfrac{2x}{x-3} + \dfrac{1}{2x+3} + \dfrac{3x+9}{2x^2-3x-9} = 0$

Answer:

$$\frac{2x}{x-3} + \frac{1}{2x+3} + \frac{3x+9}{(2x+3)(x-3)} = 0$$

$\therefore$ LCD $= (2x+3)(x-3)$ and $x \neq 3, -\dfrac{3}{2}$

$$\frac{2x}{(x-3)}(x-3)(2x+3) + \frac{1}{(2x+3)}(2x+3)(x-3) +$$

$$\frac{(3x+9)}{(2x+3)(x-3)}(2x+3)(x-3) = 0(2x+3)(x-3)$$

$$2x(2x+3)+(x-3)+(3x+9)=0$$

$$4x^2+6x+x-3+3x+9=0$$

$$4x^2+10x+6=0$$

$$2x^2+5x+3=0$$

$$(2x+3)(x+1)=0$$

$2x+3=0$ $\qquad$ $x+1=0$

$x=\dfrac{-3}{2}$ $\qquad$ $x=-1$

Check:

It is not permissible for x to be $\dfrac{-3}{2}$.

$\therefore$ The only solution is $x=-1$ and $x\neq 3, -\dfrac{3}{2}$.

f) $\dfrac{x^2+12}{x-3}=\dfrac{7x}{x-3}$

Answer:

LCD = $x-3, x\neq 3$

$$\frac{x^2+12}{x-3}(x-3)=\frac{7x}{x-3}(x-3)$$

$$x^2+12=7x$$

$$x^2-7x+12=0$$

$$(x-4)(x-3)=0$$

$$x=4, 3$$

Check:

It is not permissible for x to be 3.

$\therefore$ The only solution is $x = 4$ and $x \neq 3$.

3. The distance from Winnipeg to Minneapolis is about 600 km. You drove from Winnipeg to Minneapolis 10 km/h faster than from Minneapolis to Winnipeg. If the total driving time for the round trip was 22 hours, how fast did you drive from Winnipeg to Minneapolis?

Answer:

Fill in the rate and distance, and then find the ratio for time = $\frac{\text{distance}}{\text{rate}}$ for each.

	Rate ×	Time =	Distance
Winnipeg to Minneapolis	$r + 10$	$\frac{600}{r + 10}$	600
Minneapolis to Winnipeg	r	$\frac{600}{r}$	600

Driving from Winnipeg to Minneapolis + Driving from Minneapolis to Winnipeg = 22 hours.

$$\frac{600}{r + 10} + \frac{600}{r} = 22$$

LCD = $r(r + 10)$

$$\frac{600}{r + 10} r(r + 10) + \frac{600}{r} r(r + 10) = 22r(r + 10)$$

$$600r + 600(r + 10) = 22r(r + 10)$$

$$600r + 600r + 6000 = 22r^2 + 220r$$

$$0 = 22r^2 - 980r - 6000$$

$$0 = 11r^2 - 490r - 3000$$

$$0 = 11r^2 - 550r + 60r - 3000$$

$$0 = 11r(r - 50) + 60(r - 50)$$

$$0 = (11r + 60)(r - 50)$$

$$11r + 60 = 0 \text{ or } r - 50 = 0$$

$$r = -\frac{60}{11} \text{ or } r = 50$$

The only acceptable answer is $r = 50$, since rate cannot be negative. Therefore, you drove $r + 10 = 50 + 10 = 60$ km/h from Winnipeg to Minneapolis. Note that this answer is not very reasonable unless you were stopping often for coffee and the lineup at the border was long.

4. House painters, Gwen and Megan, want to know how long it will take them if they work together to paint a house. Gwen, working alone, can paint a house in 13 hours. Megan, working alone, can paint a house in 11 hours. How long will it take them to paint a house if they work together? Round your answer to the nearest hundredth.

   Answer:

   Consider what portion of the house each person paints in one hour.

   Gwen paints a house in 13 hours. Her rate is $\frac{1}{13}$ house per hour.

   Megan's rate is $\frac{1}{11}$ house per hour.

   Let t = time, in hours, for Megan and Gwen to paint the house.

   $$\frac{1}{13}t + \frac{1}{11}t = 1$$

   The equation is equal to one because that represents the painting of one house.

   Now solve the equation.

   LCD = (11)(13) = 143

   $$\frac{t}{13} + \frac{t}{11} = 1$$

   $\frac{t}{13}(11)(13) + \frac{t}{11}(11)(13) = 1(11)(13)$ Multiply by the LCD to clear denominators.

   $$11t + 13t = 143$$

   $$24t = 143$$

   $$t = \frac{143}{24}$$

   $$t = 5.96$$

   Thus, Gwen and Megan working together can paint a house in 5.96 hours.

5. Jeff is trying to fill his above-ground pool with water. He knows that one of his hoses can fill the pool in 32 hours and another in 26 hours. If Jeff uses both hoses to fill the pool, how long will it take to fill the pool with water? Round your answer to the nearest tenth.

   Answer:

   Hose 1: rate per hour is $\frac{1}{32}$ of the pool per hour

   Hose 2: rate per hour is $\frac{1}{26}$ of the pool per hour

   Let t = the number of hours to fill the pool when both hoses are used.

   Let 1 represent the pool filled completely.

   $$\frac{1}{32}t + \frac{1}{26}t = 1$$

   To solve the equation, first find the LCD.

   $32 = 2^5$

   $26 = 2 \times 13$

   $\text{LCD} = 2^5 \times 13 = 32 \times 13 = 416$

   Now, multiply each term by the LCD and solve for t.

   $$\frac{t}{32} + \frac{t}{26} = 1$$

   $$\frac{t}{32}(416) + \frac{t}{26}(416) = 1(416)$$

   $$13t + 16t = 416$$

   $$29t = 416$$

   $$t = \frac{416}{29}$$

   $$t = 14.3$$

   Both hoses working together can fill the pool in 14.3 hours.

Notes

Grade 11 Pre-Calculus Mathematics (30S)

Module 5
Radicals

Module 5: Radicals

Introduction

In Module 4, you learned how to solve quadratic equations by taking square roots. You also reviewed your knowledge of simplifying square roots from *Grade 10 Introduction to Applied and Pre-Calculus Mathematics*. Another name for the math notation for square roots is *radicals*. In this module, you will build on your knowledge of simplifying radical expressions. You will also learn how to add, subtract, multiply, and divide radical expressions.

Radical expressions have applications in trigonometry, electricity, and driving. They also appear in formulas designed to find the volume or surface area of 3-D objects. It is possible to solve these types of applications problems by solving radical equations. Once you learn how to perform operations on radical expressions, you will learn how to solve radical equations.

Assignments in Module 5

When you have completed the assignments for Module 5, submit your completed assignments to the Distance Learning Unit either by mail or electronically through the learning management system (LMS). The staff will forward your work to your tutor/marker.

Lesson	Assignment Number	Assignment Title
2	Assignment 5.1	Simplifying Radical Expressions
4	Assignment 5.2	Multiplying and Dividing Radical Expressions
5	Assignment 5.3	Solving Radical Equations

Final Examination Information

The breakdown of the final examination will be 20% for the first half of the course and 80% for the second half.

Resource Sheet

When you write your final examination, you are encouraged to take a Final Examination Resource Sheet with you into the examination. This sheet will be one letter-sized page, 8½ " by 11", with both sides in your handwriting or typewritten. You will submit it with your examination, but you do not receive any marks for it.

Many students have found that preparing a resource sheet is an excellent way to review. It provides you with a summary of the important facts of each module. You should complete a resource sheet for each module to help with your studying and reviewing. Lesson summaries and module summaries are included for you to use as a guide.

You may use the list of instructions that follows to help you with preparing your resource sheet for the material in Module 5. On this sheet, you should record math terms and definitions, formulas, sample questions, or a list of places where you often make mistakes. You should also identify special areas that require extra attention or review by writing the page numbers.

After you have completed each module's resource sheet, you may summarize the sheets from Modules 1 to 8 to prepare your Final Examination Resource Sheet. The final examination for this course is based on Modules 1 to 8.

Resource Sheet for Module 5

As you go through the lessons of this module, you may want to consider the following suggestions regarding the creation of a resource sheet.

1. List all the important math terms, and define them if necessary.
2. List all the formulas and perhaps a sample problem that shows how the formula is used.
3. If necessary, write the solutions to some problems, showing in detail how you did the calculations.
4. Copy any questions that represent the key points of the lesson, and perhaps include the solutions as well.
5. Identify the problems you found most difficult, and copy the page numbers onto the resource sheet so that you can review them before writing the examination. You may also copy the problems and the solutions onto your resource sheet, and later write them onto your Final Examination Resource Sheet.
6. Write any comments, ideas, shortcuts, or other reminders that may be helpful during an examination.

Lesson 1: Simplifying Radical Expressions

Lesson Focus

In this lesson, you will

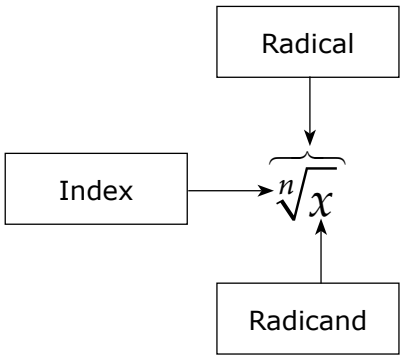
- ❑ learn how to express an entire radical as a mixed radical
- ❑ learn how to express a mixed radical as an entire radical
- ❑ learn how to compare and order a set of radical expressions
- ❑ learn how to use the rules and conventions of square roots
- ❑ learn how to identify the values of a variable for which a radical expression is defined

Lesson Introduction

In Module 4, you learned how to simplify radical expressions involving numerical radicands. In this lesson, you will expand on that knowledge and learn how to simplify radical expressions involving variable and numerical radicands. This is a crucial step to learn as it will be helpful to know when you are performing operations on radical expressions in later lessons.

Simplifying Radical Expressions

Before you learn how to simplify radical expressions, you should become familiar with the parts of a radical. Consider the following diagram. This diagram will allow you to see the difference between the index and the radicand of a radical.



Radical: Any expression written using the $\sqrt[n]{\ }$ symbol

Radicand: Everything under the radical symbol.

Radical Symbol: $\sqrt{\ }$

Index: A natural number ≥ 2, used to indicate the type of root, square root, cube root, etc.

Expressing Entire Radicals as a Mixed Radical

In Module 4, you reviewed how to simplify radical expressions. To do this, you used the following properties of radicals with $n = 2$—in other words, using a square root.

The generalized form of the rules are given for n of any value. Also note that in these rules, the radicand is assumed to be positive or zero. Negative radicands are discussed later in this lesson.

First Property of Radicals (Inverse Operations Property)

$\sqrt[n]{a^n} = a$ where $a \geq 0$

n is a positive integer ≥ 2

When the power and the index are the same, the answer is the base. Note, if $a < 0$ and n is even, then the rule can be written more generally as, $\sqrt{a^2} = |a|$. See Example 10.

Product Property of Radicals

$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ where $a \geq 0$

$b \geq 0$

n is a positive integer ≥ 2

The root of a product is the product of the roots.

Quotient Property of Radicals

$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ where $a \geq 0$

$b > 0$ (**Note:** $b \neq 0$ since it is the denominator.)

n is a positive integer ≥ 2

The root of a quotient is the quotient of the roots.

Example 1

Simplify the following radical expressions.

a) $\sqrt{50}$

b) $\sqrt{\frac{81}{36}}$

c) $\sqrt{1800}$

d) $\sqrt[3]{16}$

Solution

a) You need to look for perfect square factors of the radicand. For example, 25 is a perfect square factor of 50. Using the Product Property of Radicals, $\sqrt{50}$ can be written as $\sqrt{25} \cdot \sqrt{2}$.

$$\sqrt{50} = \sqrt{25} \cdot \sqrt{2}$$

Using the First Property of Radicals, $\sqrt{25}$ can be written as $\sqrt{(5)^2}$, which simplifies to 5 because $\sqrt{(5)^2} = \sqrt[2]{5^2} = 5$. The explanation is given this way to prepare you for roots of variables.

$$\sqrt{25} \cdot \sqrt{2} = \sqrt{(5)^2} \cdot \sqrt{2} = 5\sqrt{2}$$

Thus, $5\sqrt{2}$ is called a mixed radical because it is the product of a rational number and a radical.

b) Using the Quotient Property of Radicals, $\sqrt{\frac{81}{36}}$ can be written as $\frac{\sqrt{81}}{\sqrt{36}}$.

Using the First Property of Radicals, $\sqrt{81}$ can be written as $\sqrt{9^2}$, which simplifies to 9. Also, $\sqrt{36}$ can be written as $\sqrt{6^2}$, which simplifies to 6.

$$\frac{\sqrt{81}}{\sqrt{36}} = \frac{\sqrt{9^2}}{\sqrt{6^2}} = \frac{9}{6} = \frac{3}{2}$$

c) Simplify in several steps, especially if the number is large.

$$\begin{aligned} \sqrt{1800} &= \sqrt{100}\sqrt{18} \\ &= 10\sqrt{18} \\ &= 10\sqrt{9}\sqrt{2} \\ &= 10(3)\sqrt{2} \\ &= 30\sqrt{2} \end{aligned}$$

or

$$\begin{aligned} \sqrt{1800} &= \sqrt{900}\sqrt{2} \\ &= 30\sqrt{2} \end{aligned}$$

Notice that an entire square root can be simplified to a mixed radical if the original radicand has a *perfect square as one of its factors*.

d) $\sqrt[3]{16} = \sqrt[3]{8}\sqrt[3]{2}$

$$= 2\sqrt[3]{2}$$

Example 2

Simplify, given $x \geq 0$ and $y \geq 0$.

a) $\sqrt{x^2}$

b) $\sqrt{x^6}$

c) $\sqrt{x^7 y^4}$

d) $\sqrt{8x^2 y^4}$

e) $\sqrt{25x^6 y^2}$

f) $\sqrt{50x^3 y^{10}}$

g) $\sqrt[3]{128x^{14}}$

Solution

a) Using the First Property of Radicals, $\sqrt{x^2}$ can be written as $\sqrt[2]{x^2}$.

$\sqrt{x^2} = \sqrt[2]{x^2} = x$, as long as $x \geq 0$.

b) $\sqrt{x^6}$

You can write the radicand, x^6, as $(x^3)^2$, using the following rule of exponents:

Power of a Power Law: When a power is raised to a power, multiply the exponents.

$(x^3)^2 = (x)^{(3)(2)} = x^6$

The radical becomes $\sqrt{\left(x^3\right)^2}$.

$$\sqrt{x^6} = \sqrt[2]{\left(x^3\right)^2}$$

Using the first rule of radicals, $\sqrt[2]{\left(x^3\right)^2} = x^3$, $x \geq 0$.

$$\therefore \sqrt{x^6} = \sqrt{\left(x^3\right)^2} = x^3$$

c) $\sqrt{x^7y^4}$

Using the Product Property of Radicals, $\sqrt{x^7y^4}$ can be written as $\sqrt{x^7}\sqrt{y^4}$.

Consider $\sqrt{x^7}$.

x^7 can be written as $(x^3)^2(x)$.

The radical then becomes $\sqrt{(x^3)^2(x)}$.

Consider $\sqrt{y^4}$.

y^4 can be written as $(y^2)^2$.

The radical then becomes. $\sqrt{(y^2)^2}$.

Note: You want to group exponents into powers with an exponent of 2 (perfect squares). This is because the index of the radical is "2."

Therefore, $\sqrt{x^7y^4} = \sqrt{x^7}\sqrt{y^4} = \sqrt{(x^3)^2(x)}\sqrt{(y^2)^2}$.

Using the product property again:

$$\sqrt{(x^3)^2(x)}\sqrt{(y^2)^2} = \sqrt{(x^3)^2}\sqrt{x}\sqrt{(y^2)^2}$$

Using the First Property of Radicals, $\sqrt{(x^3)^2} = x^3$ and $\sqrt{(y^2)^2} = y^2$.

$$\sqrt{(x^3)^2}\sqrt{x}\sqrt{(y^2)^2} = x^3\sqrt{x}y^2$$

When writing radical expressions, the variables that are not part of the radicand are written before the radical symbol to avoid confusion.

$$x^3\sqrt{x}y^2 = x^3y^2\sqrt{x}$$

The simplified form of $\sqrt{x^7y^4}$ is $x^3y^2\sqrt{x}$.

If you were to redo this example without the lines of explanation, you might write the following steps to show your work.

$$\begin{aligned}\sqrt{x^7y^4} &= \sqrt{x^6}\cdot\sqrt{x}\cdot\sqrt{y^4}\\ &= \sqrt{(x^3)^2}\sqrt{x}\sqrt{(y^2)^2}\\ &= x^3y^2\sqrt{x}\end{aligned}$$

d) $\sqrt{8x^2y^4}$

Look for perfect square factors of the radicand. Using the Product Property of Radicals, $\sqrt{8x^2y^4}$ can be written as $\sqrt{4}\sqrt{2}\sqrt{x^2}\sqrt{\left(y^2\right)^2}$.

Using the First Property of Radicals, $\sqrt{4} = 2$, $\sqrt{x^2} = x$, and $\sqrt{\left(y^2\right)^2} = y^2$.

$$\sqrt{4}\sqrt{2}\sqrt{x^2}\sqrt{\left(y^2\right)^2} = 2\sqrt{2}xy^2 = 2xy^2\sqrt{2}$$

The simplified form of $\sqrt{8x^2y^4}$ is $2xy^2\sqrt{2}$.

e) $\sqrt{25x^6y^2}$

Using the Product Property of Radicals, $\sqrt{25x^6y^2}$ can be written as $\sqrt{25}\sqrt{\left(x^3\right)^2}\sqrt{y^2}$.

Using the First Property of Radicals, $\sqrt{25} = 5$, $\sqrt{\left(x^3\right)^2} = x^3$, and $\sqrt{y^2} = y$.

$$\sqrt{25}\sqrt{\left(x^3\right)^2}\sqrt{y^2} = 5x^3y$$

The simplified form of $\sqrt{25x^6y^2}$ is $5x^3y$.

f) $\sqrt{50x^3y^{10}}$

Using the Product Property of Radicals, $\sqrt{50x^3y^{10}}$ can be written as $\sqrt{25}\sqrt{2}\sqrt{x^2}\sqrt{x}\sqrt{\left(y^5\right)^2}$.

Using the First Property of Radicals, $\sqrt{25} = 5$, $\sqrt{x^2} = x$, and $\sqrt{\left(y^5\right)^2} = y^5$.

$$\sqrt{25}\sqrt{2}\sqrt{x^2}\sqrt{x}\sqrt{\left(y^5\right)^2} = 5\sqrt{2}x\sqrt{x}y^5 = 5xy^5\sqrt{2}\sqrt{x}$$

Using the Product Property of Radicals in reverse, $\sqrt{2}\sqrt{x}$ can be written as $\sqrt{2x}$.

$$5xy^5\sqrt{2}\sqrt{x} = 5xy^5\sqrt{2x}$$

The simplified form of $\sqrt{50x^3y^{10}}$ is $5xy^5\sqrt{2x}$.

Note: As you can see, for the variables, when the index is 2, the exponent of the answer is one-half of the exponent of the question. If there is a remainder when dividing the exponent by the index, the variable with the remainder exponent remains inside the radical sign.

g) Look for perfect cube factors of the radicand. For example, 64 is a perfect cube factor of 128 and x^{12} is a perfect cube factor of x^{14}.

$$\begin{aligned}\sqrt[3]{128x^{14}} &= \sqrt[3]{64}\,\sqrt[3]{2}\,\sqrt[3]{x^{12}}\,\sqrt[3]{x^2} \\ &= (4)\left(\sqrt[3]{2}\right)\left(x^4\right)\left(\sqrt[3]{x^2}\right) \\ &= 4x^4\sqrt[3]{2x^2}\end{aligned}$$

Example 3

Simplify $\sqrt{x^6}$ and $\sqrt{x^7}$, $x \geq 0$.

Solution

$\sqrt{x^6} = x^3$, since $\dfrac{6}{2} = 3$ or $\sqrt{x^6} = \sqrt{\left(x^3\right)^2} = x^3$.

While $\sqrt{x^7} = x^3\sqrt{x}$, since $\dfrac{7}{2} = 3$, with remainder 1.

Or, $\sqrt{x^7} = \sqrt{\left(x^3\right)^2 x} = x^3\sqrt{x}$.

Note: Students find it helpful to memorize the perfect squares, perfect cubes, perfect quads, etc., up to 100. For example, the perfect squares are 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, and the perfect cubes are 1, 8, 27, 64.

There are other ways to simplify radicals using prime factorization or rational exponents.

Example 4

Simplify the following expressions. Assume that $x \geq 0$ and $y \geq 0$.

a) $\sqrt{x^3y^2}$

b) $\sqrt[3]{x^4}$

Solution

a) It is possible to simplify this expression by using the First Property of Radicals. However, it is also possible to simplify this expression using the method of prime factorization. To use this method, you need to factor the radicand completely.

$x^3y^2 = (x)(x)(x)(y)(y)$

Thus, $\sqrt{x^3y^2}$ becomes $\sqrt{(x)(x)(x)(y)(y)}$.

Because the index is 2, you need to combine pairs (groups of 2) of identical factors to make perfect squares.

$$\sqrt{(x)(x)(x)(y)(y)} = \sqrt{(x^2)(x)(y^2)} = xy\sqrt{x}$$

b) It is possible to simplify this expression by using the First Property of Radicals or prime factorization. However, a third method is to simplify this expression by using rational exponents.

Recall: $\sqrt[n]{x^m} = x^{\frac{m}{n}}$

Thus, $\sqrt[3]{x^4} = x^{\frac{4}{3}}$

Now, you need to change the exponent of x from an improper fraction into a mixed number.

Recall: An **improper fraction** is a fraction that is larger than one. In other words, the numerator is greater than the denominator.

$$x^{\frac{4}{3}} = x^{\frac{3}{3}}x^{\frac{1}{3}} = xx^{\frac{1}{3}}$$

Once the exponent is fully simplified, determine if there are any fractional exponents. If there are fractional exponents, convert that part of the expression back into a radical.

In this example, $\frac{1}{3}$ is a fractional exponent. Therefore, $x^{\frac{1}{3}}$ needs to be converted into a radical.

$$xx^{\frac{1}{3}} = x\sqrt[3]{x}$$

You have now learned three different methods of simplifying radical expressions. You can choose to use any method you like to simplify radical expressions. If you feel it would be helpful, you can also include steps on how to simplify radical expressions on your resource sheet.

Note: A radical is in simplest form when after prime factorization of the radicand all exponents in the radicand are less than the index. Thus, if the index is 2, no perfect square factors should be in the radicand.

Expressing Mixed Radicals as Entire Radicals

If you are given a mixed radical, it is possible to rewrite it as an entire radical. This is the reverse process of expressing an entire radical as a mixed radical. Consider the following example.

Example 5

Express the following mixed radicals as entire radicals, given $x \geq 0$ and $y \geq 0$.

a) $4\sqrt{3}$

b) $x\sqrt{x}$

c) $x^2y\sqrt{xy}$

d) $x\sqrt[3]{x^2}$

Solution

a) Expressing mixed radicals as entire radicals is essentially the reverse process of expressing entire radicals as mixed radicals. Therefore, you need to express the values outside the radical as radicals themselves.

4 can also be written as $\sqrt{4^2} = \sqrt{16}$

Thus, $4\sqrt{3} = \sqrt{4^2}\sqrt{3} = \sqrt{16}\sqrt{3}$.

Reversing the Product Property of Radicals:

$$\sqrt{16}\sqrt{3} = \sqrt{(16)(3)} = \sqrt{48}$$

b) x can be written as $\sqrt{x^2}$.

$$x\sqrt{x} = \sqrt{x^2}\sqrt{x} = \sqrt{(x^2)(x)} = \sqrt{x^3}$$

c) x^2 can be written as $\sqrt{x^4}$ and y can be written as $\sqrt{y^2}$.

$$x^2y\sqrt{xy} = \sqrt{x^4}\sqrt{y^2}\sqrt{xy} = \sqrt{(x^4)(y^2)(x)(y)} = \sqrt{x^5y^3}$$

d) This time, the index is 3 and x can also be written as $\sqrt[3]{x^3}$.

$$x\sqrt[3]{x^2} = \sqrt[3]{x^3}\sqrt[3]{x^2} = \sqrt[3]{(x^3)(x^2)} = \sqrt[3]{x^5}$$

Comparing and Ordering Radical Expressions

Now that you are able to simplify radical expressions and also express simplified radical expression as entire radicals, you will be able to compare and order radical expressions. Consider the following example.

Example 6

Order the following radical expressions from least to greatest, given $x \geq 0$.

a) $2\sqrt{3}$, 8, $\sqrt{36}$

b) $3\sqrt{x}$, $\sqrt{x}$, $5\sqrt{x}$

c) $\sqrt[3]{2}$, $\sqrt{5}$, $\sqrt[4]{15}$

Solution

a) If each radical has the same index, or can be represented as having the same index, the radical expressions can be ordered as follows.

$2\sqrt{3} = \sqrt{(4)(3)} = \sqrt{12}$

$8 = \sqrt{64}$

$\sqrt{36} = \sqrt{36}$

Since $\sqrt{12} < \sqrt{36} < \sqrt{64}$, the answer is $2\sqrt{3} < \sqrt{36} < 8$.

b) If each radical expression is a multiple of some radical, you can simply compare the coefficients.

All of the radical expressions are a multiple of $\sqrt{x}$.

Since $1 < 3 < 5$, the answer is $\sqrt{x} < 3\sqrt{x} < 5\sqrt{x}$.

c) If it is not possible to express each radical expression as having the same index and the radical expressions are not multiples of some radical, unless you use fractional exponents, you will need to use estimation or your calculator to order the radical expressions.

$\sqrt[3]{2} \approx 1.2599$ or estimate that it is less than 2.

$\sqrt{5} \approx 2.2361$ or estimate that it is more than 2.

$\sqrt[4]{15} \approx 1.9680$ or estimate that it is close to $\sqrt[4]{16}$, which equals 2.

Thus, $\sqrt[3]{2} < \sqrt[4]{15} < \sqrt{5}$.

Properties of Square Roots

You should now be familiar with square roots and simplifying radical expressions. However, there are some rules that need to be understood to reduce the confusion when dealing with square roots.

From previous math courses, you know that $(-x)^2 = (-x)(-x) = x^2$. Consider the following table:

	$(-x)^2$	x^2
$x = -2$	$(-(-2))^2 = (2)^2 = 4$	$(-2)^2 = 4$
$x = -1$	$(-(-1))^2 = (1)^2 = 1$	$(-1)^2 = 1$
$x = 0$	$(-0)^2 = 0$	$0^2 = 0$
$x = 1$	$(-1)^2 = 1$	$(1)^2 = 1$
$x = 2$	$(-2)^2 = 4$	$(2)^2 = 4$
$x = 3$	$(-3)^2 = 9$	$(3)^2 = 9$

As you can see, both the $(-x)^2$ and x^2 columns are the same. This is because of the fact that when you multiply two negative numbers together, the answer is positive. Thus, $(-x)^2 = (-x)(-x) = x^2$.

When you are solving for x in an equation like $x^2 = 100$, you need to consider both the negative and positive square roots of 100. It is possible for x to equal both positive 10 and negative 10.

$(10)^2 = 100$ and $(-10)^2 = 100$

Therefore, when you have an equation in the form $x^2 = a$, where $a > 0$, you perform the operation of taking the square root on both sides and you have two solutions. Like the questions in Module 4 involving solving quadratic equations by taking square roots, you need to consider the positive and negative square roots. Thus:

$$x^2 = a$$

$$\sqrt{x^2} = \pm\sqrt{a} \qquad \text{Take the square root of both sides.}$$

$$x = \pm\sqrt{a} \qquad \text{Simplify.}$$

Example 7

Solve for x when $a = 25$ in the equation $x^2 = a$.

Solution

$$x^2 = a$$
$$a = 25$$
$$x^2 = 25$$
$$\sqrt{x^2} = \pm\sqrt{25}$$
$$x = \pm 5$$

When you introduce the radical symbol $\left(\sqrt{\ }\right)$ as part of your solution, there are two answers, 5 and (−5).

The general rule is that if $x^2 = a$, then $x = \pm\sqrt{a}$.

Instead, if you are given an equation in the form $x = \sqrt{a^2}$, then you have only the principal square root. The **principal square root** is the positive square root. The general rule is $\sqrt{a^2} = |a|$. In other words, the square root of a^2 is always a positive number. When the radical symbol is given as part of the question, there is only one solution depending on the sign of the radical given to you.

Consider the following example.

Example 8

Simplify.

a) $-\sqrt{25}$

b) $-\sqrt{16}$

Solution

a) When you are given an expression in the form $-\sqrt{x}$ and asked to simplify, this expression is not asking for the principal square root. The principal square root is the positive square root. Therefore, $\sqrt{25} = 5$. However, there is a negative sign in front of the square root. Thus, $-\sqrt{25} = -5$.

b) $-\sqrt{16} = -\left(\sqrt{16}\right) = -(4) = -4$

Example 9

Simplify.

a) $\sqrt{36}$

b) $-\sqrt{36}$

c) $\pm\sqrt{36}$

d) $\sqrt{-36}$

Solution

a) When you are asked to simplify an expression in the form $\sqrt{x}$, you are being asked for the principal square root (the positive square root).

$$\sqrt{36} = 6$$

b) In this expression, you are asked for the negative square root of 36.

$$-\sqrt{36} = -6$$

c) In this expression, you are asked for both the positive and negative square roots of 36.

$$\pm\sqrt{36} = \pm 6$$

d) It is not possible to simplify this expression any further, as there is no solution to the square root of a negative number in the real number system.

Example 10

Simplify.

a) $\sqrt{(-2)^2}$

b) $\sqrt{3^2}$

Solution

a) $\sqrt{(-2)^2} = \sqrt{4} = 2$. Compare with $\sqrt{x^2} = |x|$. In this example, $x = -2$.

Therefore, $\sqrt{(-2)^2} = |-2| = 2$.

b) $\sqrt{3^2} = \sqrt{9} = 3$. Compare with $\sqrt{x^2} = |x|$. In this example, $x = 3$.

Therefore, $\sqrt{3^2} = |3| = 3$.

If the negative square root is written, the notation would be in the form $x = -\sqrt{a^2}$ and the answer could be simplified to $x = -|a|$.

Summary of Rules for Dealing with Square Roots:

1. $(-x)^2 = x^2$ for $x \in \Re$
2. If a is any number where $a > 0$, and $x^2 = a$, then $x = \pm\sqrt{a}$. There are two answers.
3. If a is any number where $a = 0$, and $x^2 = 0$, then $x = \sqrt{0} = 0$.
4. If a is any number where $a < 0$, and $x^2 = a$, then $x = \pm\sqrt{\text{negative number}}$ and x has no real number answer.
5. If you are given a radical equation, $\sqrt{x} = a$, then you have the principal square root of x and the value of "a" must be positive or zero. There is only one answer.
6. If you are given a radical equation, $-\sqrt{x} = a$, then you do not have the principal square root and the value of "a" must be negative. There is only one answer.
7. If you are given a ≥ 0, then $\sqrt{a^2} = a$.
8. If you are given $a < 0$, then $\sqrt{a^2} = -a$. **Be careful here.** For example, $\sqrt{(-6)^2} = -(-6) = 6$. The answer is a positive number.
9. Rules 7 and 8 can be summarized by writing $\sqrt{a^2} = |a|$ for $a \in \Re$. Notice this rule refers to the principal square root because the sign in front of it is positive.

This notation may seem confusing at first; however, these rules were put in place to avoid confusion and to ensure the same answer is found by anyone solving these types of problems. You may remember these rules by knowing the following pattern. If you introduce the square root in your solution, then use $\pm$. If you are given the radical in the question, then use the sign that is given. You may find it helpful to add these rules to your resource sheet.

Identifying the Values of a Variable for Which a Radical Expression is Defined

It is not possible to take the square root of a negative number in the real number system. However, it is possible to find the cube root of a negative number. Consider the following table.

	Is the Radical Expression Defined for Positive Values?		**Is the Radical Expression Defined for Negative Values?**	
Radical Expression	Example: $x = 2$		Example: $x = -2$	
$\sqrt{x}$	$\sqrt{2} \approx 1.41$	Yes	$\sqrt{-2}$ = undefined	No
$\sqrt[3]{x}$	$\sqrt[3]{2} \approx 1.26$	Yes	$\sqrt[3]{-2} \approx -1.26$	Yes
$\sqrt[4]{x}$	$\sqrt[4]{2} \approx 1.19$	Yes	$\sqrt[4]{-2}$ = undefined	No
$\sqrt[5]{x}$	$\sqrt[5]{2} \approx 1.15$	Yes	$\sqrt[5]{-2} \approx -1.15$	Yes
$\sqrt[6]{x}$	$\sqrt[6]{2} \approx 1.12$	Yes	$\sqrt[6]{-2}$ = undefined	No

From this table, you can see that radicals are always defined when the radicand is greater than zero. However, when the radicand is less than zero, the radical is only defined if the index is odd.

Example 11

Identify the values of the variable for which the radical is defined in the set of real numbers.

a) $\sqrt{5-x}$

b) $\sqrt{x^2}$

c) $\sqrt{y^5}$

d) $\sqrt[3]{z}$

e) $\sqrt[6]{x^5}$

Solution

a) As a square root is not defined when the radicand is negative, this expression is defined when $5 - x \geq 0$, or $5 \geq x$, or $x \leq 5$.

b) $\sqrt{x^2}$ is defined for all values of x in the real numbers. This is because any value squared will become positive or zero. Also, the square root of any positive number or zero is defined. Therefore, this expression is defined for $x \in \Re$.

c) $\sqrt{y^5}$ has an index of 2. Therefore, as the index is even, this radical expression is only defined when $y^5 \geq 0$ and when $y \geq 0$.

d) $\sqrt[3]{z}$ has an index of 3. Therefore, as the index is odd, this radical expression is defined for all values of $z \in \Re$.

e) $\sqrt[6]{x^5}$ has index of 6. Therefore, as the index is even, this radical expression is only defined when $x^5 \geq 0$ and $x \geq 0$.

The following learning activity will allow you to practice the mathematical skills that you have just learned. Make sure you complete this learning activity because similar questions will appear on your next assignment and the final examination.

Learning Activity 5.1

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Jenelle's doctor told her to eat an average of 2000 calories a day. If Jenelle ate 520 calories for breakfast, 630 calories for lunch, and had two snacks with 125 calories each, how many calories does Jenelle have left for supper?
2. Multiply: $(x + 3)(x + 4)$
3. Evaluate: $\sqrt{256}$

continued

Learning Activity 5.1 (continued)

4. Solve for x: $(2x + 4)(x - 2) = 0$
5. Patrick sleeps for 8 hours a day, attends school for 6 hours a day, and plays sports for 2 hours a day. What fraction of the day (in lowest terms) does Patrick have for free time?
6. Order these values from lowest to highest: $\sqrt[3]{8}$, $2^{\frac{2}{4}}$, $2^{\frac{9}{3}}$, $2^{-\frac{1}{2}}$
7. Are these fractions equivalent? $\frac{1}{\sqrt{2}}$ and $\frac{\sqrt{2}}{2}$
8. Find the slope of the line perpendicular to the line, $3y - 2x + 7 = 0$.

Part B: Simplifying Radical Expressions

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Simplify each radical.

   a) $\sqrt{80}$
   b) $\sqrt{18}$
   c) $\sqrt{200}$
   d) $\sqrt{250}$
   e) $\sqrt{1224}$
   f) $\sqrt{288}$
   g) $\sqrt{512}$
   h) $-\sqrt{8}$
   i) $-\sqrt{800}$

2. Simplify each radical, given $x \geq 0$, $y \geq 0$, and $z \geq 0$.

   a) $\sqrt{4x^4y^6}$
   b) $\sqrt{8x^{10}y^{12}}$
   c) $\sqrt{75x^8y^5}$
   d) $\sqrt{x^4yz^2}$
   e) $\sqrt{5x^3y^7}$

continued

3. Write each mixed radical as an entire radical.

   a) $3\sqrt{15}$

   b) $3x\sqrt{5x}$

   c) $5x^2y\sqrt{3xy}$

4. Identify the greater number. (Do not use a calculator.)

   a) $\sqrt{125}$ or $3\sqrt{20}$?

   b) $16\sqrt{2}$ or $\sqrt{768}$?

5. Order the following radical expressions from least to greatest. (Do not use a calculator.)

   a) $5\sqrt{3}$, $\sqrt{49}$, $6\sqrt{5}$

   b) $3\sqrt{2}$, $6\sqrt{2}$, $\sqrt{32}$

6. Solve for x.

   a) $x^2 = 25$

   b) $x^2 = 16$

   c) $x^2 = 48$

   d) $x^2 = 0$

   e) $x^2 = -9$

7. By coverting the radical to a rational exponent, prove the First Property of Radicals:

   $\sqrt[n]{a^n} = a$ when n is a positive integer greater than 1.

8. Identify the values of the variable(s) for which the radical represents a real number.

   a) $\sqrt{8 - 2x}$

   b) $\sqrt{x^3}$

   c) $\sqrt[5]{z}$

   d) $\sqrt[4]{x^7}$

   e) $\sqrt{64a^3b^6}$

   f) $\sqrt{-432xy^2}$

Lesson Summary

In this lesson, you expanded on your knowledge of simplifying radicals with numerical radicands. You learned how to simplify radicals with variable radicands, and how to identify the values of a variable for which a radical expression is defined. As well, conventions regarding square roots were discussed. In the next lesson, you will learn how to add and subtract radical expressions.

Notes

Lesson 2: Adding and Subtracting Radical Expressions

Lesson Focus

In this lesson, you will

❑ learn how to add and subtract radical expressions

Lesson Introduction

In Lesson 1, you learned how to simplify radical expressions. This technique will prove to be extremely useful in this lesson. In previous courses, you learned how to add and subtract fractions and polynomial expressions. In this lesson, you will learn how to add and subtract radical expressions using similar techniques to the ones you would use to add and subtract fractions or polynomials.

Adding and Subtracting Radical Expressions

Until now, you have been working with expressions consisting of polynomials. Some of the concepts applied to fractions or polynomials can be used when combining radicals.

Comparing the Combining of Polynomials and Radicals

To add or subtract radicals, you add or subtract the coefficients of like radicals, just as you add or subtract the coefficients of like terms with polynomials and add or subtract fractions with common denominators. Like radicals are radicals with the same radicand and the same index.

Polynomial

$2x + 3y + 5x + y$

$= (2x + 5x) + (3y + y)$ Combine like terms.

$= 7x + 4y$

Fraction

$$\frac{1}{3}+\frac{2}{5}$$

$$=\frac{5}{15}+\frac{6}{15}$$ Combine fractions with "like" denominators.

$$=\frac{11}{15}$$

Radical

$$2\sqrt{5}+3\sqrt{2}+5\sqrt{5}+\sqrt{2}$$

$$=\left(2\sqrt{5}+5\sqrt{5}\right)+\left(3\sqrt{2}+\sqrt{2}\right)$$ Combine like radicals.

$$=7\sqrt{5}+4\sqrt{2}$$

Example 1

Simplify each of the following:

a) $3\sqrt{5}+4\sqrt{7}-\sqrt{5}+6\sqrt{7}$

b) $3\sqrt{2}+7+3\sqrt{2}+3$

c) $\left(3\sqrt{2}+\sqrt{3}-1\right)-\left(3\sqrt{3}-2\sqrt{2}-1\right)$

Solution

a) $3\sqrt{5}+4\sqrt{7}-\sqrt{5}+6\sqrt{7}$

$=2\sqrt{5}+10\sqrt{7}$

b) $3\sqrt{2}+7+3\sqrt{2}+3$

$=6\sqrt{2}+10$

c) $\left(3\sqrt{2}+\sqrt{3}-1\right)-\left(3\sqrt{3}-2\sqrt{2}-1\right)$

$=3\sqrt{2}+\sqrt{3}-1-3\sqrt{3}+2\sqrt{2}+1$

$=5\sqrt{2}-2\sqrt{3}$

Example 2

Simplify $5\sqrt{3} + \sqrt{12} - \sqrt{3}$ completely.

Solution

Your first answer to this question may be $4\sqrt{3} + \sqrt{12}$. That answer is not completely simplified. There is a simpler form. As with fractions, when you write an equivalent fraction with a common denominator, *sometimes* you can write an equivalent mixed radical with a common radicand. The term, $\sqrt{12}$, is equal to $\sqrt{4}\sqrt{3}$, which can be written as $2\sqrt{3}$, so

$$5\sqrt{3} + \sqrt{12} - \sqrt{3}$$
$$= 4\sqrt{3} + \sqrt{12}$$
$$= 4\sqrt{3} + 2\sqrt{3}$$
$$= 6\sqrt{3}$$

Example 3

Simplify.

a) $\sqrt{12} + 3\sqrt{3} + \sqrt{75}$

b) $\sqrt{50} + \sqrt{75} + \sqrt{300}$

c) $\sqrt{20} + \sqrt{150} - \sqrt{125}$

Solutions

a) $\sqrt{12} + 3\sqrt{3} + \sqrt{75}$

$= \sqrt{4}\sqrt{3} + 3\sqrt{3} + \sqrt{25}\sqrt{3}$ Factor radicands.

$= 2\sqrt{3} + 3\sqrt{3} + 5\sqrt{3}$ Write square root of the perfect squares.

$= 10\sqrt{3}$ Combine like terms.

b) $\sqrt{50} + \sqrt{75} + \sqrt{300}$

$= \sqrt{25}\sqrt{2} + \sqrt{25}\sqrt{3} + \sqrt{100}\sqrt{3}$

$= 5\sqrt{2} + 5\sqrt{3} + 10\sqrt{3}$

$= 5\sqrt{2} + 15\sqrt{3}$

Note: Unlike fractions, you cannot always make the terms have a common radical.

c) $\sqrt{20} + \sqrt{150} - \sqrt{125}$

$= \sqrt{4}\sqrt{5} + \sqrt{25}\sqrt{6} - \sqrt{25}\sqrt{5}$

$= 2\sqrt{5} + 5\sqrt{6} - 5\sqrt{5}$

$= -3\sqrt{5} + 5\sqrt{6}$

Adding and Subtracting Radical Expressions with Variable Radicands

Just as it is possible to add and subtract radical expressions with numerical radicands (such as in examples 1 through 3), it is also possible to add and subtract radical expression with variable radicands. Consider the following example.

Example 4

Simplify, given $x \geq 0$.

a) $6\sqrt{x} - 4\sqrt{x} + 5\sqrt{x}$

b) $\sqrt[3]{27x^2} - \sqrt[3]{8x^2} + \sqrt[3]{64x^2}$

c) $5x\sqrt{24x^3} - 7\sqrt{54x^5} + x^2\sqrt{6x} - x^2\sqrt{96x}$

Solution

a) Each of these terms has $\sqrt{x}$ in common. Therefore, they are all like terms and you can simply add and subtract the coefficients as indicated.

$6\sqrt{x} - 4\sqrt{x} + 5\sqrt{x}$

$= 7\sqrt{x}$

b) Before you can combine like terms, you need to simplify each radical completely.

$\sqrt[3]{27x^2} - \sqrt[3]{8x^2} + \sqrt[3]{64x^2}$

$= 3\sqrt[3]{x^2} - 2\sqrt[3]{x^2} + 4\sqrt[3]{x^2}$

$= 5\sqrt[3]{x^2}$

c) Again, before you can combine like terms, you need to simplify each radical.

$$5x\sqrt{24x^3} - 7\sqrt{54x^5} + x^2\sqrt{6x} - x^3\sqrt{96x}$$

$$= 5x\sqrt{(4)(6)\left(x^2\right)(x)} - 7\sqrt{(9)(6)\left(x^4\right)(x)} + x^2\sqrt{6x} - x^3\sqrt{(16)(6)(x)}$$

$$= 5x(2)(x)\sqrt{6x} - 7(3)\left(x^2\right)\sqrt{6x} + x^2\sqrt{6x} - x^3(4)\sqrt{6x}$$

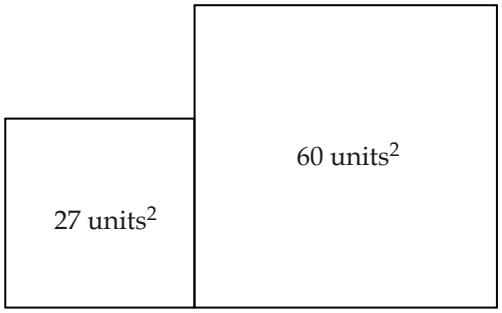
$$= 10x^2\sqrt{6x} - 21x^2\sqrt{6x} + x^2\sqrt{6x} - 4x^3\sqrt{6x}$$

$$= -10x^2\sqrt{6x} - 4x^3\sqrt{6x}$$

Note: The terms are "like terms" only if all variable factors are the same.

Example 5

A square with an area of 27 units^2 is placed next to a square with an area of 60 units^2. Determine the exact value of the perimeter of the combined figure in simplest form.



Solution

The area of a square = length × width.

However, the length and the width are the same in a square.

Therefore, the area of a square = length × length = (length)^2.

Let the length of the smaller square equal x.

$$x^2 = 27$$

$$x = \pm\sqrt{27} = \pm 3\sqrt{3}$$

Since length is never negative, x can only be $3\sqrt{3}$.

Let the length of the larger square equal y.

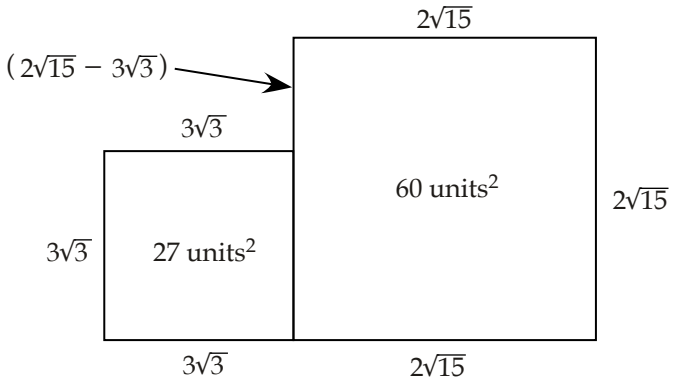
$$y^2 = 60$$

$$y = \pm\sqrt{60} = \pm 2\sqrt{15}$$

Since $y > 0$, $y = 2\sqrt{15}$.

Since you are asked for the exact value, you must leave the numbers written as radicals rather than as approximate values using a calculator.

When you are solving for a variable, you need to consider both the positive and negative square roots and also the restrictions on its value.



Now, to determine the perimeter of the larger shape, you have to sum all of the side lengths.

Perimeter:

$3\sqrt{3} + 3\sqrt{3} + 3\sqrt{3} + 2\sqrt{15} + 2\sqrt{15} + 2\sqrt{15} + \left(2\sqrt{15} - 3\sqrt{3}\right)$

$= 9\sqrt{3} + 6\sqrt{15} + 2\sqrt{15} - 3\sqrt{3}$

$= \left(6\sqrt{3} + 8\sqrt{15}\right)$ units

Learning Activity 5.2

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Alyssa is twice as old as her brother today. When she is 25 years old, her brother will be 20 years old. How many years old is Alyssa today?
2. How much would a sweater cost if it were 25% off the original price of $60 and you had to pay an additional 12% sales tax on the discounted price?
3. Simplify: $\left(\sqrt[3]{2x+10}\right)^0$
4. Simplify: $2x(3x-5)-4(x+2)$
5. Write 625 as a power with a base of 5.
6. List the factors of 68.
7. The equation of a parabola is, $y = 3x^2 - 5x + 8$. What is the y-intercept?
8. Find the slope of the line that contains the points $(7, -1)$ and $(-8, 15)$.

Part B: Adding and Subtracting Radical Expressions

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Simplify each radical expression as much as possible. Given: $x \geq 0$.

a) $2\sqrt{3}+4\sqrt{3}-\sqrt{3}$

b) $\sqrt{96}-\sqrt{54}-\sqrt{24}$

c) $3\sqrt{75}-2\sqrt{48}+\sqrt{12}$

d) $\sqrt{98}-\sqrt{50}+\sqrt{72}$

e) $5\sqrt{x}-3\sqrt{x}+6\sqrt{x}$

f) $6\sqrt{5}+3\sqrt{2}-4\sqrt{5}+\sqrt{2}$

g) $\sqrt{x^5}+3x^2\sqrt{x}-7x\sqrt{4x^3}$

h) $2x\sqrt[3]{8x^4}-4x^2\sqrt[3]{x}+\sqrt[3]{125x^7}$

continued

Learning Activity 5.2 (continued)

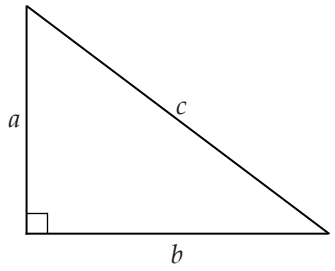
2. Find the values of the missing numerical coefficients A and B.

   a) $3\sqrt{2} + 2\sqrt{3} + A\sqrt{12} - B\sqrt{8} = \sqrt{75} - \sqrt{32}$

   b) $\sqrt{24} + A\sqrt{2} + B\sqrt{150} = 3\sqrt{2}$

3. Find the width of a rectangle if the length is $5\sqrt{3}$ cm and the perimeter is $\left(16\sqrt{3} - 4\sqrt{2}\right)$ cm.

4. Given the lengths of two sides of the right triangle, find the length of the third side.



   a) $a = 3$ $b = 4$

   b) $a = 3$ $b = 5$

   c) $a = 15$ $c = 17$

5. Why is it important to simplify radical expressions before adding or subtracting? Provide an example to support your explanation.

6. The perimeter of a square is $8\sqrt{2x}$ units. The area of a square is 56 units^2. Determine the value of x, if $x > 0$.

Lesson Summary

In this lesson, you learned how to add and subtract radical expressions. Sometimes, it was necessary to simplify a radical expression before you were able to add or subtract it. The skills you learned in Lesson 1 dealing with simplifying radical expressions were used throughout this lesson. Simplifying radical expressions also comes in handy when you multiply radical expressions, which is the focus of Lesson 3.

Before you continue on to the next lesson, complete the following assignment. Assignment 5.1 contains questions similar to those found in Learning Activity 5.1 and Learning Activity 5.2. If you completed most of those questions correctly, you should do well on Assignment 5.1.

Notes

Assignment 5.1

Simplifying Radical Expressions

Total: 42 marks

1. Simplify each radical. Given: $x \geq 0$ and $y \geq 0$.

   a) $\sqrt{75}$ *(1 mark)*

   b) $-\sqrt{400}$ *(1 mark)*

   c) $\sqrt{4800}$ *(2 marks)*

   d) $-\sqrt{648}$ *(2 marks)*

continued

Assignment 5.1: Simplifying Radical Expressions (continued)

e) $\sqrt{x^3y^2}$ *(1 mark)*

f) $-\sqrt{x^6y^7}$ *(2 marks)*

g) $\sqrt{4x^{10}y^{17}}$ *(2 marks)*

h) $-\sqrt{8x^7y^3}$ *(2 marks)*

continued

Assignment 5.1: Simplifying Radical Expressions (continued)

2. Write each mixed radical as an entire radical. Given: $x \geq 0$. *(3 marks)*

   a) $4\sqrt{7}$

   b) $x\sqrt{3x}$

   c) $3x^2y^3\sqrt{2xy}$

3. Order the following radical expressions from least to greatest. Given: $x \geq 1$. Do not use a calculator. *(4 marks)*

   a) $5\sqrt{2}$, $6\sqrt{3}$, $2\sqrt{5}$

   b) $3\sqrt{2x^3}$, $x\sqrt{8x}$, $\sqrt{32x^3}$

continued

Assignment 5.1: Simplifying Radical Expressions (continued)

4. Identify the values of the variables for which each radical expression is defined. *(7 marks)*

   a) $\sqrt{6-3x}$ *(1 mark)*

   b) $\sqrt{x^5}$ *(1 mark)*

   c) $\sqrt[3]{x^2}$ *(1 mark)*

   d) $\sqrt[4]{48r^6s^3}$ *(2 marks)*

   e) $\sqrt[3]{128x^2y^5}$ *(2 marks)*

continued

Assignment 5.1: Simplifying Radical Expressions (continued)

5. Simplify each radical expression as much as possible. Given: $x \geq 0$. *(2 marks each, for a total of 8 marks)*

   a) $4\sqrt{5} + 6\sqrt{5} - 2\sqrt{125}$

   b) $\sqrt{48} - 2\sqrt{80} - 4\sqrt{192}$

   c) $2x\sqrt{x} + 3\sqrt{x^3} - \sqrt{16x^3}$

   d) $\sqrt{x^5} - x\sqrt{4x^3} + 6x^2\sqrt{x}$

continued

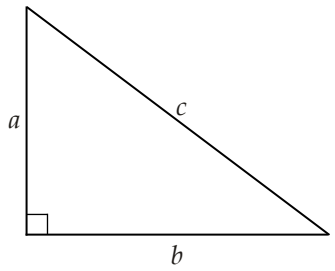
Assignment 5.1: Simplifying Radical Expressions (continued)

6. Find the width of a rectangle if the length is $\sqrt{2}$ and the perimeter is $4\sqrt{3} - 2\sqrt{2}$.
   (3 marks)

continued

Assignment 5.1: Simplifying Radical Expressions (continued)

7. Given the lengths of two sides of the right triangle, find the length of the third side. Recall: $a^2 + b^2 = c^2$. *(2 marks each, for a total of 4 marks)*



a) $a = \sqrt{2}, b = \sqrt{3}$

b) $a = \sqrt{5}, c = \sqrt{10}$

Notes

Lesson 3: Multiplying Radical Expressions

Lesson Focus

In this lesson, you will

❑ learn how to multiply radical expressions

Lesson Introduction

You have now learned how to simplify radical expressions, as well as how to add and subtract radical expressions. Both of these techniques will be useful as you learn how to multiply radical expressions in this lesson. Multiplying radical expressions is similar to multiplying polynomial expressions or multiplying fractions. As you are already familiar with multiplying polynomial expressions and fractions, the technique for multiplying radical expressions should be an easy extension of your skills.

Multiplying Radical Expressions

Radical multiplication can be compared to multiplication you already know. Compare the following expressions.

Polynomial	**Fraction**	**Radical**
$(3x)(2y) = 6xy$	$\left(\frac{2}{3}\right)\left(\frac{5}{7}\right) = \frac{10}{21}$	$\left(3\sqrt{5}\right)\left(2\sqrt{7}\right) = 6\sqrt{35}$

To multiply polynomials: The coefficients are multiplied together and the variables are multiplied together.

To multiply fractions: The numerators are multiplied together and the denominators are multiplied together.

To multiply radicals: The numbers outside each radical sign are multiplied together and the numbers inside each radical sign are multiplied together.

Include these instructions on how to multiply radical expressions on your resource sheet.

Use the **Distributive Law** to simplify these expressions:

	Polynomial	**Radical**
a)	$2x(3x + 4y)$	$2\sqrt{2}\left(3\sqrt{2} + 4\sqrt{5}\right)$
	$= 6x^2 + 8xy$	$= 6\sqrt{4} + 8\sqrt{10}$
		$= 12 + 8\sqrt{10}$
b)	$(2x + 1)(3x + 2)$	$\left(2\sqrt{3} + 1\right)\left(3\sqrt{3} + 2\right)$
	$= 2x(3x + 2) + 1(3x + 2)$	$= 2\sqrt{3}\left(3\sqrt{3} + 2\right) + 1\left(3\sqrt{3} + 2\right)$
	$= 6x^2 + 4x + 3x + 2$	$= 6\sqrt{9} + 4\sqrt{3} + 3\sqrt{3} + 2$
	$= 6x^2 + 7x + 2$	$= 6(3) + 7\sqrt{3} + 2$
		$= 18 + 7\sqrt{3} + 2$
		$= 20 + 7\sqrt{3}$

Example 1

Simplify these radical expressions. Given: $x \geq 0$.

a) $\left(\sqrt{5}\right)\left(\sqrt{10}\right)$

b) $\left(7\sqrt{3}\right)\left(5\sqrt{2}\right)$

c) $\left(2\sqrt{3}\right)\left(3\sqrt{3}\right)$

d) $\left(\sqrt{5}\right)\left(\sqrt{5}\right)$

e) $\sqrt{2}\left(\sqrt{3} - \sqrt{10}\right)$

f) $2\sqrt{3}\left(4\sqrt{3} - 3\sqrt{7}\right)$

g) $9\sqrt{2x}\left(\sqrt{2x^5} + 2\sqrt{x}\right)$

h) $\left(2 + \sqrt{3}\right)\left(2 - \sqrt{3}\right)$

i) $\left(\sqrt{x} + \sqrt{5}\right)\left(3\sqrt{x} - 2\sqrt{5}\right)$

Solutions

a) $\left(\sqrt{5}\right)\left(\sqrt{10}\right)$

$= \sqrt{50}$

$= \sqrt{25}\sqrt{2}$

$= 5\sqrt{2}$

b) $\left(7\sqrt{3}\right)\left(5\sqrt{2}\right) = 35\sqrt{6}$

c) $\left(2\sqrt{3}\right)\left(3\sqrt{3}\right) = 6\sqrt{9}$

$= 6(3)$

$= 18$

d) $\left(\sqrt{5}\right)\left(\sqrt{5}\right) = \sqrt{25}$

$= 5$

e) $\sqrt{2}\left(\sqrt{3} - \sqrt{10}\right) = \sqrt{6} - \sqrt{20}$

$= \sqrt{6} - \sqrt{4}\sqrt{5}$

$= \sqrt{6} - 2\sqrt{5}$

f) $2\sqrt{3}\left(4\sqrt{3} - 3\sqrt{7}\right) = 8\sqrt{9} - 6\sqrt{21}$

$= 8(3) - 6\sqrt{21}$

$= 24 - 6\sqrt{21}$

g) $9\sqrt{2x}\left(\sqrt{2x^5} + 2\sqrt{x}\right) = 9\sqrt{4x^6} + 18\sqrt{2x^2}$

$= 9 \cdot 2 \cdot x^3 + 18 \cdot x\sqrt{2}$

$= 18x^3 + 18x\sqrt{2}$

h) $\left(2 + \sqrt{3}\right)\left(2 - \sqrt{3}\right) = 2\left(2 - \sqrt{3}\right) + \sqrt{3}\left(2 - \sqrt{3}\right)$

$= 4 - 2\sqrt{3} + 2\sqrt{3} - \sqrt{9}$

$= 4 - \sqrt{9}$

$= 4 - 3$

$= 1$

Note: When multiplying two radicals, the answer may be a rational number! The answer contains no radical term because the polynomials are the factors for a difference of squares called conjugates.
$(a + b)(a - b) = a^2 - b^2$

i) $\left(\sqrt{x} + \sqrt{5}\right)\left(3\sqrt{x} - 2\sqrt{5}\right) = \sqrt{x}\left(3\sqrt{x} - 2\sqrt{5}\right) + \sqrt{5}\left(3\sqrt{x} - 2\sqrt{5}\right)$

$= 3\sqrt{x^2} - 2\sqrt{5x} + 3\sqrt{5x} - 2\sqrt{25}$

$= 3x + \sqrt{5x} - 2(5)$

$= 3x + \sqrt{5x} - 10$

Example 2

Simplify the following radical expression. Given: $x \geq 0$ and $y \geq 0$.

$$\left(\sqrt{xy} - 3\sqrt{y}\right)\left(\sqrt{x} + 5\sqrt{xy}\right) - \left(2\sqrt{y} - \sqrt{x}\right)^2$$

Solution

$$\left(\sqrt{xy} - 3\sqrt{y}\right)\left(\sqrt{x} + 5\sqrt{xy}\right) - \left(2\sqrt{y} - \sqrt{x}\right)^2$$

$$= \left(\sqrt{xy} - 3\sqrt{y}\right)\left(\sqrt{x} + 5\sqrt{xy}\right) - \left(2\sqrt{y} - \sqrt{x}\right)\left(2\sqrt{y} - \sqrt{x}\right)$$

$$= \sqrt{xy}\left(\sqrt{x} + 5\sqrt{xy}\right) - 3\sqrt{y}\left(\sqrt{x} + 5\sqrt{xy}\right) - \left[2\sqrt{y}\left(2\sqrt{y} - \sqrt{x}\right) - \sqrt{x}\left(2\sqrt{y} - \sqrt{x}\right)\right]$$

$$= \sqrt{x^2y} + 5\sqrt{x^2y^2} - 3\sqrt{xy} - 15\sqrt{xy^2} - \left[4\sqrt{y^2} - 2\sqrt{xy} - 2\sqrt{xy} + \sqrt{x^2}\right]$$

$$= x\sqrt{y} + 5xy - 3\sqrt{xy} - 15y\sqrt{x} - 4y + 2\sqrt{xy} + 2\sqrt{xy} - x$$

$$= x\sqrt{y} + 5xy + \sqrt{xy} - 15y\sqrt{x} - 4y - x$$

As you can see, simplifying radical expressions can become quite complicated. The following learning activity will allow you to practice multiplying radical expressions. Make sure you complete this learning activity and check your answers in the Module 5 Answer Key to ensure you are completing questions correctly. Similar questions to those found in the following learning activity will appear in your next assignment.

Learning Activity 5.3

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. School starts at 8:45 a.m. Hunter spends twenty minutes eating breakfast, fifteen minutes taking a shower, thirty-five minutes watching TV, and ten minutes biking to school. If Hunter arrives just on time for school, what time did he wake up?
2. Simplify: $\left(x + \sqrt{3}\right)^2$
3. Express 0.72 as a fraction in lowest terms.
4. Find the x-intercepts of the following parabola: $y = x^2 + 9x + 14$
5. If three angles in a parallelogram measure 286° in total, what is the measure of the fourth angle in the parallelogram?
6. Determine the distance between the following two points: (9, 24) and (4, 12)
7. Express as a power: $\sqrt[4]{5^8}$
8. Write this percent as a decimal: 3.2%

continued

Learning Activity 5.3 (continued)

Part B: Multiplying Radical Expressions

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Simplify each radical expression. Given: $x \geq 0$, $y \geq 0$.

   a) $\left(\sqrt{5}\right)\left(\sqrt{3}\right)$

   b) $\left(\sqrt{8}\right)\left(\sqrt{2}\right)$

   c) $\left(4\sqrt{10}\right)\left(3\sqrt{5}\right)$

   d) $\left(2\sqrt{5}\right)\left(-3\sqrt{5}\right)$

   e) $\left(-3\sqrt{3}\right)\left(-4\sqrt{6}\right)$

   f) $\left(2\sqrt{x}\right)\left(3\sqrt{xy^2}\right)$

   g) $\sqrt{3}\left(\sqrt{2}-\sqrt{5}\right)$

   h) $-\sqrt{2}\left(3\sqrt{2}+3\sqrt{5}\right)$

   i) $\sqrt{6}\left(3\sqrt{2}+5\sqrt{8}\right)$

   j) $\left(2\sqrt{5}+3\right)\left(2\sqrt{5}-3\right)$

   k) $\left(6\sqrt{5}-3\sqrt{10}\right)\left(4\sqrt{5}+\sqrt{10}\right)$

   l) $\left(\sqrt{2}-\sqrt{3}\right)^2$

   m) $\left(4\sqrt{7}+3\sqrt{2}\right)^2$

   n) $\left(\sqrt{x}+\sqrt{y^3}\right)\left(\sqrt{xy}\right)$

   o) $\left(2\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}-3\sqrt{y}\right)$

2. Find the exact values of the area of a square if its perimeter is $\left(4\sqrt{2}+8\right)$ cm.

3. Identify the greater number:

   a) $2\sqrt{80}$ or $\left(5\sqrt{8}\right)\left(2\sqrt{3}\right)$?

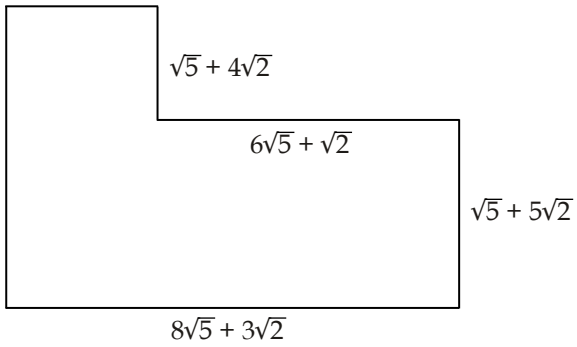
   b) $\left(3\sqrt{2}\right)\left(\sqrt{75}\right)$ or $\left(5\sqrt{2}\right)\left(3\sqrt{2}\right)$?

4. Every hour, a spider climbs up a distance in metres of $\left(2\sqrt{3}+5\sqrt{2}\right)$ and then is pushed down a distance of $\left(\sqrt{3}-\sqrt{2}\right)$. If the spider begins to climb at 12 noon, what is his distance above his starting point at 4:00 p.m. sharp?

continued

Learning Activity 5.3 (continued)

5. Find the area of the figure drawn below.



6. Determine whether $3 - \sqrt{2}$ is a root of the equation $x^2 - 6x + 7 = 0$.

7. The operation $\square$ is defined as follows: $a \square b = a^2 + ab$.

   Evaluate each of the following:

   a) $\left(2\sqrt{3}\right) \square \left(3 + \sqrt{3}\right)$

   b) $\left(2 + \sqrt{3}\right) \square \left(2 - \sqrt{3}\right)$

   c) $\left(\sqrt{2} - 5\right) \square \left(3\sqrt{5}\right)$

8. Use playing cards to create four unique radical expression multiplication problems and simplify. Take out all the face cards from your deck of cards before creating your radical expression problems. Deal out at least four cards to create each problem. Black suits (spades and clubs) represent natural numbers, while red suits (diamonds and hearts) represent radicands. If you deal out a 3 of diamonds, a 2 of clubs, a 6 of clubs, and an 8 of hearts, you will have the following expression:

   3 of diamonds = $\sqrt{3}$

   2 of clubs = 2

   6 of clubs = 6

   6 of hearts = $\sqrt{6}$

   Expression: $\sqrt{3} \cdot 2 \cdot 6 \cdot \sqrt{6}$

continued

Learning Activity 5.3 (continued)

Simplifying the Expression: $\sqrt{3} \cdot 2 \cdot 6 \cdot \sqrt{6}$

$$= 12\sqrt{18}$$

$$= 12\sqrt{(9)(2)}$$

$$= 12(3)\sqrt{2}$$

$$= 36\sqrt{2}$$

You can try a variety of random cards and then test yourself using the following examples. Check your work against the solutions given in the answer key.

a) 5C, 10S, 3H, 5D

b) 4S, 3S, 2H, 5D

c) 8H, 3C, AceD, 10S

d) 7D, 2H, 8D, 2C

Lesson Summary

In this lesson, you learned how to multiply radical expressions. Having learned this technique, you will be able to learn how to divide radical expressions in the next lesson. Both of these skills will be useful when you are asked to solve radical equations in Lesson 5.

Lesson 4: Dividing Radical Expressions

Lesson Focus

In this lesson, you will

- ❑ learn how to divide radical expressions
- ❑ learn how to identify the conjugate of a binomial expression
- ❑ learn how to rationalize the denominator of a radical expression

Lesson Introduction

You have now learned how to simplify radical expressions, as well as how to add, subtract, and multiply radical expressions. Dividing radical expressions is an extension of these skills. Dividing radial expressions is similar to dividing polynomials when you have a monomial divisor and it is similar to simplifying fractions. Radical expressions need to be *rationalized* in order to eliminate the radical from the denominator. This allows for rational division rather than using an irrational number as a divisor.

Dividing Radical Expressions

Monomial division is still a model for some radical division. Compare these expressions:

Polynomial	**Radical**
$\frac{6x^2y}{3x} = 2xy$	$\frac{6\sqrt{35}}{3\sqrt{7}} = 2\sqrt{5}$

As in multiplication of radicals, the factors outside the radical sign are divided separately from the division of the factors inside the radical sign.

To divide radical expressions, use the following **Quotient Property of Radicals**.

Note: This was also discussed in Lesson 1.

Quotient Property of Radicals

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

where $a \geq 0$

$b > 0$

n is a positive integer ≥ 2

When $n = 2$, the property becomes:

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

where $a \geq 0$ and $b > 0$

Note that the property can be used both ways:

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}} \text{ or } \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Note that b must be other than zero, since b is a divisor.

The following examples take advantage of this property to simplify the expressions.

Example 1

Simplify each quotient using the **Quotient Property of Radicals**. Given: $x > 0$ and $y > 0$.

a) $\dfrac{\sqrt{8}}{\sqrt{18}}$

b) $\dfrac{\sqrt{10}}{5}$

c) $\dfrac{\sqrt{225}}{\sqrt{3}}$

d) $\dfrac{\sqrt{24} + 2\sqrt{60} - \sqrt{3}}{\sqrt{3}}$

e) $\dfrac{\sqrt{100}}{2}$

f) $\dfrac{\sqrt{xy}}{\sqrt{y}}$

g) $\dfrac{\sqrt{8x^3y^5}}{\sqrt{2xy^3}}$

Solution

a) $\dfrac{\sqrt{8}}{\sqrt{18}} = \sqrt{\dfrac{8}{18}} = \sqrt{\dfrac{4}{9}} = \dfrac{\sqrt{4}}{\sqrt{9}} = \dfrac{2}{3}$

b) $\dfrac{\sqrt{10}}{\sqrt{5}} = \sqrt{\dfrac{10}{5}} = \sqrt{2}$

c) $\dfrac{\sqrt{225}}{\sqrt{3}} = \sqrt{\dfrac{225}{3}} = \sqrt{75} = \sqrt{25}\sqrt{3} = 5\sqrt{3}$

d) $\dfrac{\sqrt{24} + 2\sqrt{60} - \sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{24}}{\sqrt{3}} + \dfrac{2\sqrt{60}}{\sqrt{3}} - \dfrac{\sqrt{3}}{\sqrt{3}}$

$= \sqrt{\dfrac{24}{3}} + 2\sqrt{\dfrac{60}{3}} - \sqrt{\dfrac{3}{3}}$

$= \sqrt{8} + 2\sqrt{20} - \sqrt{1}$

$= \sqrt{4}\sqrt{2} + 2\sqrt{4}\sqrt{5} - 1$

$= 2\sqrt{2} + 4\sqrt{5} - 1$

e) $\dfrac{\sqrt{100}}{2} = \dfrac{10}{2} = 5$

f) $\dfrac{\sqrt{xy}}{\sqrt{y}} = \sqrt{\dfrac{xy}{y}} = \sqrt{x}$

g) $\dfrac{\sqrt{8x^3y^5}}{\sqrt{2xy^3}} = \sqrt{\dfrac{8x^3y^5}{2xy^3}} = \sqrt{4x^2y^2} = 2xy$

Question: If you were not allowed to use a calculator, which of the following quotients would be easier to write as a decimal numeral, rounded to four decimal places?

$$\frac{1}{1.4142} \text{ or } \frac{1.4142}{2}$$

Answer

The latter can be simplified very quickly using long division (or "short" division since the digits can be divided by 2):

$1.4142 \div 2 = 0.7071$

Performing the long division, $1.4142\overline{)1}$, takes a bit more time.

$1.4142\overline{)1.00000000}$ is equivalent to $14142\overline{)10000.00000}$

Now do the long division:

$$
\begin{array}{r}
0.70711 \\
14142\overline{)10000.00000} \\
\underline{98994} \\
100600 \\
\underline{98994} \\
16060 \\
\underline{14142} \\
19180 \\
\underline{14142} \\
5038
\end{array}
$$

Answer: $\dfrac{1}{1.4142} = 0.7071$

Both questions produce the same result to four decimal places.

The reason these answers are close to the same value is that $1.4142 \approx \sqrt{2}$ and you were asked to choose between two equivalent fractions:

$$\frac{1}{\sqrt{2}} \text{ and } \frac{\sqrt{2}}{2}$$

Verify that these are equivalent fractions:

$$
\begin{aligned}
\text{LHS} &= \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(1) \\
&= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) \\
&= \frac{\sqrt{2}}{\sqrt{4}} \\
&= \frac{\sqrt{2}}{2} \\
&= \text{RHS}
\end{aligned}
$$

$\therefore \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

Thus, $\dfrac{1}{1.4142} \approx \dfrac{1.4142}{2}$, but $\dfrac{1.4142}{2}$ is much easier to calculate.

Rationalizing the Denominator of a Radical Expression

Before the existence of calculators, mathematicians found the approximation for $\frac{\sqrt{2}}{2}$ easier to perform than $\frac{1}{\sqrt{2}}$. Changing $\frac{1}{\sqrt{2}}$ into $\frac{\sqrt{2}}{2}$ is called **rationalizing the denominator**.

A rationalized denominator has no radicals (which are irrational numbers) in the denominator. Even now, a radical denominator is often rationalized due to convention and because it provides an easier form to compare answers.

To rationalize the denominator of a radical expression when the denominator is a monomial, use the fact that $\sqrt{x} \cdot \sqrt{x} = \sqrt{x^2} = x$, provided $x \geq 0$. Then, multiply both the numerator and the denominator by the radical value found in the denominator. Consider the following example.

Example 2

Rationalize the denominator of each fraction.

a) $\frac{2}{\sqrt{3}}$

b) $\frac{2\sqrt{5}}{\sqrt{3}}$

c) $\frac{3\sqrt{5}}{2\sqrt{3}}$

d) $\frac{-4\sqrt{6}}{3\sqrt{2}}$

Solution

a) $\sqrt{3}$ is the radical value found in the denominator. Therefore, you need to multiply both the numerator and the denominator of the radical expression by $\sqrt{3}$ to rationalize the denominator. The expression $\frac{\sqrt{3}}{\sqrt{3}}$ is equivalent to 1. You are not changing the value of the original expression when you multiply it by 1.

$$\begin{aligned}\frac{2}{\sqrt{3}} &= \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} \\ &= \frac{2\sqrt{3}}{\sqrt{9}} \\ &= \frac{2\sqrt{3}}{3}\end{aligned}$$

b) $\frac{2\sqrt{5}}{\sqrt{3}} = \frac{2\sqrt{5}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{15}}{\sqrt{9}} = \frac{2\sqrt{15}}{3}$

c) $\frac{3\sqrt{5}}{2\sqrt{3}} = \frac{3\sqrt{5}}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{15}}{2(3)} = \frac{3\sqrt{15}}{6} = \frac{\sqrt{15}}{2}$

d) $\frac{-4\sqrt{6}}{3\sqrt{2}} = \frac{-4\sqrt{6}}{3\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{-4\sqrt{12}}{3(2)} = \frac{-4\sqrt{4}\sqrt{3}}{6} = \frac{-4(2)\sqrt{3}}{6} = \frac{-8\sqrt{3}}{6} = \frac{-4\sqrt{3}}{3}$

Now that you have seen a few examples on how to rationalize a monomial denominator, include steps for rationalizing the denominator on your resource sheet.

Example 3

Rationalize each denominator. State the values for which the radical expression is defined.

a) $\frac{2\sqrt{3}}{\sqrt{x}}$

b) $\frac{3\sqrt{y}}{2\sqrt{xy}}$

Solution

a) $\frac{2\sqrt{3}}{\sqrt{x}} = \frac{2\sqrt{3}}{\sqrt{x}}\left(\frac{\sqrt{x}}{\sqrt{x}}\right) = \frac{2\sqrt{3x}}{\sqrt{x^2}} = \frac{2\sqrt{3x}}{x}$

Since the radicand must be positive, this radical expression is only defined when $x > 0$. Since the denominator of a radical expression cannot equal zero, x cannot be zero.

b) $\frac{3\sqrt{y}}{2\sqrt{xy}} = \frac{3\sqrt{y}}{2\sqrt{xy}} \cdot \frac{\sqrt{xy}}{\sqrt{xy}} = \frac{3\sqrt{xy^2}}{2\sqrt{x^2y^2}} = \frac{3y\sqrt{x}}{2xy} = \frac{3\sqrt{x}}{2x}$

This radical expression is defined only when $x > 0$ and $y > 0$, as both of these variables appear in the denominator and cannot equal zero.

Conjugate of a Binomial

If the denominator of a radical expression is a binomial, a different process is required to rationalize the denominator. If you try to rationalize $\frac{2}{3+\sqrt{2}}$ by multiplying by $\frac{\sqrt{2}}{\sqrt{2}}$, the method fails.

For example,

$$\frac{2}{3+\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{3\sqrt{2}+\sqrt{4}} = \frac{2\sqrt{2}}{3\sqrt{2}+2}$$

You still have a radical in the denominator. However, what does work is to multiply by the **conjugate** of $3+\sqrt{2}$.

The **conjugate** of $a + b$ is $a - b$ and the conjugate of $a - b$ is $a + b$. Therefore the conjugate of $3+\sqrt{2}$ is $3-\sqrt{2}$. The conjugate of any binomial involves changing the sign between the two terms. There is only one sign change.

Include the above definition on your resource sheet.

Example 4

Find the conjugates of the following radical expressions.

a) $3-\sqrt{5}$

b) $\sqrt{3}-1$

c) $2+\sqrt{2}$

d) $-\sqrt{5}-\sqrt{7}$

Solution

a) To find the conjugate of a radical expression, you simply change the sign between the two terms in the expression. Thus, the conjugate of $3-\sqrt{5}$ is $3+\sqrt{5}$.

b) The conjugate of $\sqrt{3}-1$ is $\sqrt{3}+1$.

c) The conjugate of $2+\sqrt{2}$ is $2-\sqrt{2}$.

d) The conjugate of $-\sqrt{5}-\sqrt{7}$ is $-\sqrt{5}+\sqrt{7}$.

To rationalize the denominator of a radical expression when the denominator is a binomial, you simply multiply both the numerator and the denominator by the conjugate of the denominator. Include this explanation on your resource sheet. Again, you are multiplying the original radical expression by one and its value does not change.

For example,

$$\frac{2}{3+\sqrt{2}}\left(\frac{3-\sqrt{2}}{3-\sqrt{2}}\right) = \frac{2\left(3-\sqrt{2}\right)}{\left(3+\sqrt{2}\right)\left(3-\sqrt{2}\right)}$$

$$= \frac{6-2\sqrt{2}}{3(3)-3\sqrt{2}+3\sqrt{2}-\sqrt{4}}$$

$$= \frac{6-2\sqrt{2}}{9-2}$$

$$= \frac{6-2\sqrt{2}}{7}$$

Notice how multiplying the denominator by its conjugate eliminates the radical in the denominator. This is because you are creating a situation involving a difference of squares. When you multiply two binomials that are conjugates, the result is to square each term and subtract. Thus, this eliminates all square roots.

Example 5

Rationalize the denominator.

a) $\dfrac{2}{5-\sqrt{2}}$

b) $\dfrac{5}{\sqrt{2}+1}$

c) $\dfrac{3+\sqrt{2}}{2-\sqrt{2}}$

d) $\dfrac{3\sqrt{2}-\sqrt{5}}{2\sqrt{2}+\sqrt{5}}$

Solutions

a) $$\frac{2}{5-\sqrt{2}} = \frac{2}{5-\sqrt{2}} \times \frac{5+\sqrt{2}}{5+\sqrt{2}}$$
$$= \frac{10+2\sqrt{2}}{25+5\sqrt{2}-5\sqrt{2}-\sqrt{4}}$$
$$= \frac{10+2\sqrt{2}}{25-\sqrt{4}}$$
$$= \frac{10+2\sqrt{2}}{25-2}$$
$$= \frac{10+2\sqrt{2}}{23}$$

b) $$\frac{5}{\sqrt{2}+1} = \frac{5}{\sqrt{2}+1} \cdot \frac{\sqrt{2}-1}{\sqrt{2}-1}$$
$$= \frac{5\sqrt{2}-5}{\sqrt{4}-\sqrt{2}+\sqrt{2}-1}$$
$$= \frac{5\sqrt{2}-5}{\sqrt{4}-1}$$
$$= \frac{5\sqrt{2}-5}{2-1}$$
$$= 5\sqrt{2}-5$$

c) $$\frac{3+\sqrt{2}}{2-\sqrt{2}} = \frac{3+\sqrt{2}}{2-\sqrt{2}} \cdot \frac{2+\sqrt{2}}{2+\sqrt{2}}$$
$$= \frac{6+3\sqrt{2}+2\sqrt{2}+\sqrt{4}}{4+2\sqrt{2}-2\sqrt{2}-\sqrt{4}}$$
$$= \frac{6+3\sqrt{2}+2\sqrt{2}+\sqrt{4}}{4-\sqrt{4}}$$
$$= \frac{6+5\sqrt{2}+2}{4-2}$$
$$= \frac{8+5\sqrt{2}}{2}$$

d) $$\frac{3\sqrt{2}-\sqrt{5}}{2\sqrt{2}+\sqrt{5}}$$
$$= \frac{3\sqrt{2}-\sqrt{5}}{2\sqrt{2}+\sqrt{5}} \cdot \frac{2\sqrt{2}-\sqrt{5}}{2\sqrt{2}-\sqrt{5}}$$
$$= \frac{6\sqrt{4}-3\sqrt{10}-2\sqrt{10}+\sqrt{25}}{4\sqrt{4}-2\sqrt{10}+2\sqrt{10}-\sqrt{25}}$$
$$= \frac{6\sqrt{4}-3\sqrt{10}-2\sqrt{10}+\sqrt{25}}{4\sqrt{4}-5}$$
$$= \frac{6(2)-5\sqrt{10}+5}{4(2)-5}$$
$$= \frac{12-5\sqrt{10}+5}{8-5}$$
$$= \frac{17-5\sqrt{10}}{3}$$

Example 6

Simplify. State the values for which the radical expression is defined.

a) $\dfrac{3-\sqrt{x}}{2+\sqrt{x}}$

b) $\dfrac{\sqrt{3}-\sqrt{y}}{2\sqrt{y}+\sqrt{5}}$

Solution

a) To simplify radical expressions means to rationalize the denominator.

$$\frac{3-\sqrt{x}}{2+\sqrt{x}} = \frac{3-\sqrt{x}}{2+\sqrt{x}} \times \frac{2-\sqrt{x}}{2-\sqrt{x}}$$

$$= \frac{6-3\sqrt{x}-2\sqrt{x}+\sqrt{x^2}}{4-2\sqrt{x}+2\sqrt{x}-\sqrt{x^2}}$$

$$= \frac{6-3\sqrt{x}-2\sqrt{x}+\sqrt{x^2}}{4-x}$$

$$= \frac{6+x-5\sqrt{x}}{4-x}$$

This radical expression is only defined when $x \geq 0$ (you cannot take the square root of a negative number) and when $x \neq 4$ (an expression is not defined when its denominator equals zero).

b) $$\frac{\sqrt{3}-\sqrt{y}}{2\sqrt{y}+\sqrt{5}} = \frac{\sqrt{3}-\sqrt{y}}{2\sqrt{y}+\sqrt{5}} \times \frac{2\sqrt{y}-\sqrt{5}}{2\sqrt{y}-\sqrt{5}}$$

$$= \frac{2\sqrt{3y}-\sqrt{15}-2\sqrt{y^2}+\sqrt{5y}}{4\sqrt{y^2}-2\sqrt{5y}+2\sqrt{5y}-\sqrt{25}}$$

$$= \frac{2\sqrt{3y}-\sqrt{15}-2\sqrt{y^2}+\sqrt{5y}}{4\sqrt{y^2}-\sqrt{25}}$$

$$= \frac{2\sqrt{3y}-\sqrt{15}-2y+\sqrt{5y}}{4y-5}$$

Since you cannot take the square root of a negative number, this radical expression is only defined when $y \geq 0$. When $y = \dfrac{5}{4}$, the denominator equals zero. Thus, the radical expression is defined when $y \geq 0$ and $y \neq \dfrac{5}{4}$.

Learning Activity 5.4

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Marissa loves to eat cookie dough so she bought a tub of cookie dough that was supposed to make 270 cookies. If Marissa and her brother each eat a scoop of cookie dough that is equivalent to three cookies every day, how long will it take them to eat the entire tub of cookie dough?
2. Write 32 as a power with a base of 2.
3. Bill went on a biking trip where he biked at 20 km/h for two hours and then 30 km/h for three hours. How far did Bill bike?
4. Factor: $2x(x^2 + 1) - 3(x^2 + 1)$
5. Simplify: $16^{\frac{1}{2}}$
6. Simplify: $-32m^{12} \div 8m^4$
7. Rationalize the denominator: $\dfrac{20}{\sqrt{3}+1}$
8. Determine the length of the hypotenuse of a right-angled triangle when the other two sides measure 7 cm and 9 cm respectively.

continued

Learning Activity 5.4 (continued)

Part B: Dividing Radical Expressions

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Simplify. Present your answer with no radicals in the denominators. If the expression contains variables, state for which values of the variables the radical expression is defined.

   a) $\dfrac{\sqrt{15}}{\sqrt{3}}$

   b) $\dfrac{\sqrt{2}}{2\sqrt{3}}$

   c) $\dfrac{10\sqrt{2}}{\sqrt{5}}$

   d) $\dfrac{\sqrt{2}+5}{\sqrt{3}}$

   e) $\dfrac{8\sqrt{10}-7\sqrt{5}}{2\sqrt{5}}$

   f) $\dfrac{7}{3-\sqrt{2}}$

   g) $\dfrac{\sqrt{5}}{9\sqrt{3}-4}$

   h) $\dfrac{\sqrt{7}-\sqrt{3}}{\sqrt{7}+\sqrt{3}}$

   i) $\dfrac{4-3\sqrt{3}}{6+\sqrt{2}}$

   j) $\dfrac{2\sqrt{3}-\sqrt{2}}{2\sqrt{3}+\sqrt{2}}$

   k) $\dfrac{\sqrt{x}}{\sqrt{x}-y}$

   l) $\dfrac{\sqrt{x}+1}{\sqrt{x}}$

   m) $\dfrac{\sqrt{x^2y^7}}{\sqrt{8y^6}}$

   n) $\dfrac{3-\sqrt{x}}{\sqrt{2x}-\sqrt{3}}$

2. The area of a rectangle is $\left(4+3\sqrt{3}\right)\text{m}^2$. If the length is $\left(\sqrt{3}+2\right)\text{m}$, find the
   a) width
   b) perimeter
   c) length of its diagonal

continued

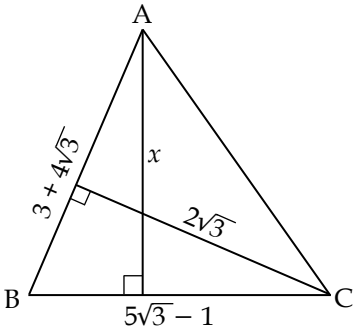
Learning Activity 5.4 (continued)

3. Observing that both these numbers are positive, is it possible that they are equal?

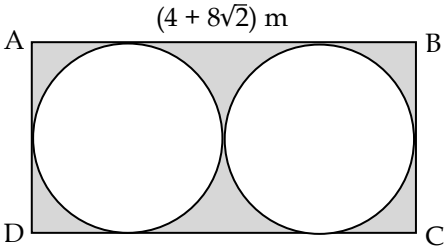
$$\frac{\sqrt{2+\sqrt{3}}}{2};\ \frac{\sqrt{2}+\sqrt{6}}{4}$$

(**Hint:** Square both numbers and compare the results.)

4. Find the length of the altitude from A, if the length of the altitude from C is $2\sqrt{3}$.



5. Find the area of the shaded region if two identical circles are cut out of rectangle ABCD, as illustrated below. The length of $AB = \left(4+8\sqrt{2}\right)$ m. Leave your answer as an exact value in terms of π.



6. What is the value of A if $\frac{2+\sqrt{3}}{A} = 5-\sqrt{3}$?

7. Simplify: $\frac{2}{\sqrt{3}-1} + \left(2+\sqrt{3}\right)\left(5-\sqrt{3}\right) - \left(2+\sqrt{3}\right)^2$

8. Write these numbers in descending order.

2, $\frac{3}{\sqrt{5}}$, and $\sqrt{3}$

Lesson Summary

In this lesson, you learned how to divide radical expressions. You also learned how to find the conjugate of a binomial, as well as how to rationalize the denominator of a fraction. These skills will be useful when you learn how to solve radical equations in the next lesson.

Assignment 5.2

Multiplying and Dividing Radical Expressions

Total: 50 marks

1. Simplify each of the following. All answers must have rationalized denominators. Assume all variables are positive. *(29 marks)*

   a) $\left(3\sqrt{2}\right)\left(\sqrt{18}\right)$ *(1 mark)*

   b) $\left(-2\sqrt{3}\right)^2$ *(1 mark)*

   c) $5\sqrt{6}\left(\sqrt{2}-4\sqrt{12}\right)$ *(2 marks)*

   d) $\left(2\sqrt{5}-3\sqrt{6}\right)^2$ *(2 marks)*

continued

Assignment 5.2: Multiplying and Dividing Radical Expressions (continued)

e) $\left(2\sqrt{6}-5\sqrt{2}\right)\left(2\sqrt{6}+5\sqrt{2}\right)$ *(2 marks)*

f) $\left(\sqrt{x^2y}+\sqrt{4x}\right)\left(\sqrt{xy}\right)$ *(2 marks)*

g) $\left(\sqrt{32xy}+\sqrt{y^3}\right)\left(\sqrt{16x^3}-\sqrt{y}\right)$ *(3 marks)*

h) $\dfrac{14\sqrt{75}}{7\sqrt{3}}$ *(2 marks)*

continued

Assignment 5.2: Multiplying and Dividing Radical Expressions (continued)

i) $\dfrac{3\sqrt{2}+4\sqrt{3}}{\sqrt{6}}$ *(4 marks)*

j) $\dfrac{\sqrt{45xy}}{\sqrt{5xy}}$ *(2 marks)*

k) $\dfrac{4+\sqrt{3}}{7-\sqrt{3}}$ *(3 marks)*

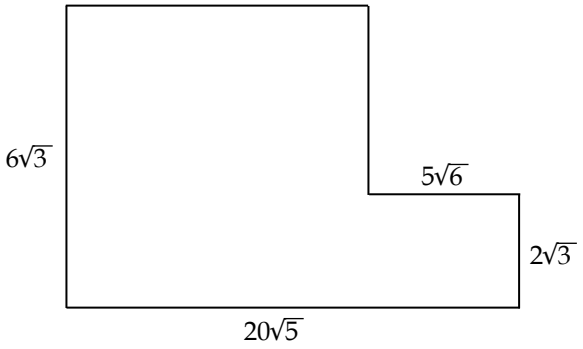
l) $\dfrac{2+\sqrt{x}}{\sqrt{y}}$ *(2 marks)*

continued

Assignment 5.2: Multiplying and Dividing Radical Expressions (continued)

m) $\dfrac{\sqrt{2x}+\sqrt{3xy}}{\sqrt{x}-\sqrt{y}}$ *(3 marks)*

2. Find the area of the following shape. *(4 marks)*



continued

Assignment 5.2: Multiplying and Dividing Radical Expressions (continued)

3. Determine whether $3 + \sqrt{5}$ is a root of the equation $x^2 - 6x + 4 = 0$. *(3 marks)*

4. Simplify the following expressions. All answers must have rationalized denominators. Determine the values of the variables for which the radical expressions are defined. *(7 marks)*

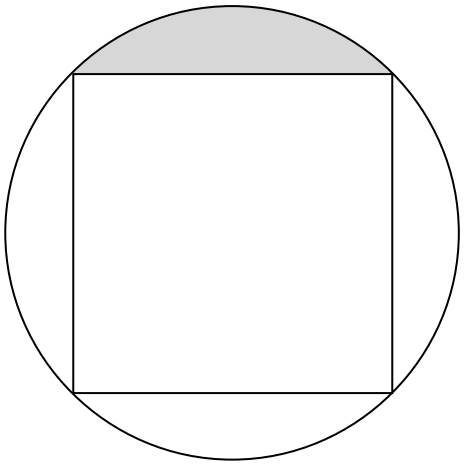
   a) $\dfrac{2 - \sqrt{y}}{3\sqrt{x}}$ *(3 marks)*

   b) $\dfrac{\sqrt{y} + 1}{5 - \sqrt{x}}$ *(4 marks)*

continued

Assignment 5.2: Multiplying and Dividing Radical Expressions (continued)

5. For this question, leave your answers in terms of π. A square is inscribed within a circle. If the area of the square is 80 cm^2, what is the exact value of the area of
   a) the circle? (**Hint:** You may want to draw a diagonal in the diagram.) *(5 marks)*
   b) the shaded region? *(2 marks)*



Lesson 5: Solving Radical Equations

Lesson Focus

In this lesson, you will

- ❑ learn how to solve radical equations
- ❑ learn how to determine if the roots of a radical equation are extraneous
- ❑ learn how to find the restrictions on the variable(s) in a radical equation
- ❑ learn how to apply solving radical equations to a word problem

Lesson Introduction

You have learned how to simplify, add, subtract, multiply, and divide radical expressions. These are all necessary skills that you can use to solve radical equations. Radical equations are equations that contain radicals or rational exponents. In this lesson, not only will you learn how to solve these types of equations, you will learn how to apply these equations and their solutions to real-life situations.

Solving Radical Equations

To solve radical equations, you try to eliminate the radicals by writing an equivalent linear or quadratic equation. You already have experience solving those.

The following property plays a key role in the simplification process.

If $a = b$, then $a^n = b^n$.

It means that both sides of an equation can be raised to the same power and the resulting expressions will still be equal.

Example 1

Solve: $\sqrt{x} = 4$

Solution

$\sqrt{x} = 4$

$\left(\sqrt{x}\right)^2 = 4^2$ Square each side to remove the square root sign.

$x = 16$

Check:

LHS $= \sqrt{x} = \sqrt{16} = 4$

RHS $= 4$

$\therefore$ LHS = RHS

Thus, $x = 16$ is the solution.

Notice that you are given the radical and it is the principal square root of x only. Notice that it is equal to a positive number. Thus, 16 is a valid solution.

Extraneous Roots of a Radical Equation

Solving a radical equation is the same thing as finding the roots of a radical equation, just as solving a quadratic equation is the same thing as finding the roots of a quadratic equation. When you are solving radical equations, you must always check your solution not only for errors, but also for **extraneous roots**. Whenever you use the method "Square Both Sides," you may be introducing a root into your solution that does not satisfy the original equation. These roots are called **extraneous roots**. Consider the following example.

Example 2

Solve: $x^{\frac{3}{2}} = -8$

Solution

$x^{\frac{3}{2}}$ can be rewritten as $\sqrt{x^3}$

Thus, you need to solve a radical equation.

$$\sqrt{x^3} = -8$$

$$\left(\sqrt{x^3}\right)^2 = (-8)^2 \qquad \text{Square both sides.}$$

$$x^3 = 64 \qquad \text{Simplify.}$$

$$\sqrt[3]{x^3} = \sqrt[3]{64} \qquad \text{Cube root both sides.}$$

$$x = 4 \qquad \text{Simplify.}$$

Check by substituting into the original equation.

$$x^{\frac{3}{2}} = -8$$

$$\text{LHS} = 4^{\frac{3}{2}}$$

$$= \left(\sqrt{4}\right)^3$$

$$= 2^3$$

$$= 8$$

$$\text{RHS} = -8$$

$$\text{LHS} \neq \text{RHS}$$

Thus, $x = 4$ is an extraneous solution and there is no other solution. So, there is no solution to this equation.

Raising both sides of an equation to the nth power may introduce extraneous or false solutions if n is even. So when you use the procedure, it is critical that you check each solution in the original equation. For instance, squaring both sides of the equation, $\sqrt{x} = -1$ produces $x = 1$. However, the root is extraneous because when you check the solution and substitute $x = 1$ into the original equation, this value does not work. The original equation has no real number solution. You would also know this is a false equation because $\sqrt{x}$ means the principal square root, which is positive and cannot equal a negative number.

Example 3

Solve: $\sqrt[3]{x} - 4 = 0$

Solution

Before raising both sides of an equation to the nth power, you must isolate the radical expression on one side of the equation to make solving easier.

$\sqrt[3]{x} = 4$ Isolate the radical.

$\left(\sqrt[3]{x}\right)^3 = 4^3$ Cube each side. The inverse operation of cube rooting is cubing.

$x = 64$

Check by substituting $x = 64$ into the original equation, $\sqrt[3]{x} - 4 = 0$.

$$\begin{aligned} \text{LHS} &= \sqrt[3]{64} - 4 \\ &= 4 - 4 \\ &= 0 \\ \text{RHS} &= 0 \\ \therefore \text{LHS} &= \text{RHS} \end{aligned}$$

Note that cubing each side of the equation does not introduce extraneous roots into the solution. Thus, the checking process is only for errors in your work.

The solution is $x = 64$.

Restrictions on Variables in a Radical Equation

Just as there are restrictions on variables in radical expressions, there are also restrictions on variables in radical equations. These restrictions appear when you have the possibility of a negative radicand when the index is even. Since in the real number system, it is not possible to find a square root of a negative number, negative radicand values are not allowed.

Example 4

Solve: $x + 2\sqrt{x-3} = 6$. Determine the restrictions on the variable in this radical equation.

Solution

In order for this radical equation to have a solution, $x - 3$ has to be greater than or equal to zero.

$$x - 3 \geq 0$$
$$x \geq 3$$

Thus, the restrictions on the variable in this radical equation are that x is any real number greater than or equal to three.

$$x + 2\sqrt{x-3} = 6$$

$$2\sqrt{x-3} = 6 - x \qquad \text{Isolate the radical.}$$

$$\left(2\sqrt{x-3}\right)^2 = (6-x)^2 \qquad \text{Square both sides.}$$

$$4(x-3) = 36 - 12x + x^2 \qquad \text{Be sure you square the binomial correctly.}$$

$$4x - 12 = 36 - 12x + x^2$$

$$0 = x^2 - 16x + 48 \qquad \text{Rearrange.}$$

$$0 = (x-4)(x-12) \qquad \text{Factor.}$$

$x = 4$ or $x = 12$

As both $x = 4$ and $x = 12$ are greater than 3, they are possible solutions to this radical equation. However, it is still necessary to check for extraneous roots of the original equation.

Check by substituting into the original equation $x + 2\sqrt{x-3} = 6$.

if $x = 4$

$$\text{LHS} = 4 + 2\sqrt{4-3}$$
$$= 4 + (2)(1)$$
$$= 6$$
$$\text{RHS} = 6$$
$$\text{LHS} = \text{RHS}$$

if $x = 12$

$$\text{LHS} = 12 + 2\sqrt{12-3}$$
$$= 12 + 2\left(\sqrt{9}\right)$$
$$= 12 + 2(3)$$
$$= 12 + 6$$
$$= 18$$
$$\text{RHS} = 6$$
$$\text{LHS} \neq \text{RHS}$$

Thus, $x = 12$ is an extraneous root. The only solution is $x = 4$.

When you squared both sides of the equation $\left(2\sqrt{x-3}\right)^2 = (6-x)^2$, you introduced the possibility of a second equation, $2\sqrt{x-3} = -(6-x)$, into the solution. As a result of squaring, extraneous roots may be introduced. A check of the roots is always necessary when you are solving radical equations.

Notice that the restrictions on the variable are usually found before solving.

If a radical equation has more than one term with a variable in the radicand, you may have to isolate and square more than once, as shown in the following example.

Example 5

Solve: $\sqrt{2x+3} - \sqrt{x+1} = 1$. Determine the restrictions on the variable x.

Solution

In order for this radical equation to have a solution, $2x + 3$ and $x + 1$ both have to be greater than or equal to zero.

$$2x + 3 \geq 0 \qquad x + 1 \geq 0$$
$$2x \geq -3 \qquad x \geq -1$$
$$x \geq -\frac{3}{2}$$

In order for this entire radical equation to be defined, x has to be greater than or equal to both $-\frac{3}{2}$ and -1. Since $-\frac{3}{2}$ is smaller than -1, this is the same as saying x has to be greater than or equal to -1. Thus, the restrictions on the variable x in this radical equation are that x has to be greater than or equal to negative one.

$\sqrt{2x+3} - \sqrt{x+1} = 1$

$\sqrt{2x+} = 1 + \sqrt{x+1}$ Rearrange to isolate one of the radicals.

$\left(\sqrt{2x+3}\right)^2 = \left(1+\sqrt{x+1}\right)^2$ Square each side.

$2x + 3 = 1 + 2\sqrt{x+1} + (x+1)$ Simplify.

$2x + 3 - 1 - x - 1 = 2\sqrt{x+1}$ Isolate the remaining radical.

$x + 1 = 2\sqrt{x+1}$ Simplify.

$(x+1)^2 = \left(2\sqrt{x+1}\right)^2$ Square each side again.

$x^2 + 2x + 1 = 4(x+1)$

$x^2 + 2x + 1 = 4x + 4$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = 3$ or $x = -1$

Both solutions satisfy the restriction $x \geq -1$.

Check for extraneous roots by substituting into the original equation $\sqrt{2x+3} - \sqrt{x+1} = 1$.

if $x = 3$	if $x = -1$
LHS $= \sqrt{2(3)+3} - \sqrt{3+1}$	LHS $= \sqrt{2(-1)+3} - \sqrt{-1+1}$
$= \sqrt{9} - \sqrt{4}$	$= \sqrt{1} - \sqrt{0}$
$= 3 - 2$	$= 1 - 0$
$= 1$	$= 1$
RHS $= 1$	RHS $= 1$
LHS = RHS	LHS = RHS

$\therefore$ Both solutions are valid.

Answer: $x = 3$ or $x = -1$

Example 6

Solve: $\sqrt{2\sqrt{x}} = 6$

Solution

The restriction on the variable is $x \geq 0$.

In order to solve this problem, you need to square both sides twice.

$\left(\sqrt{2\sqrt{x}}\right)^2 = 6^2$ Squaring both sides once.

$2\sqrt{x} = 36$ Simplifying.

$\sqrt{x} = 18$ Divide by 2.

$\left(\sqrt{x}\right)^2 = 18^2$ Square both sides.

$x = 324$ Simplify.

Check by substituting into the original equation $\sqrt{2\sqrt{x}} = 6$.

$\text{LHS} = \sqrt{2\sqrt{324}}$

$= \sqrt{2 \cdot 18}$

$= \sqrt{36}$

$= 6$

$\text{RHS} = 6$

$\text{LHS} = \text{RHS}$ The solution is $x = 324$.

Applying Solving Radical Equations to a Word Problem

Example 7

The radius, r, of a cylinder is defined by the equation $r = \sqrt{\frac{V}{\pi h}}$, where V represents the volume of the cylinder and h represents the height of the cylinder. If the radius of a cylinder is 5 cm and the volume of a cylinder is 400 cm^3, what is the height of the cylinder? Leave your answer in terms of π.

Solution

You have the following information and formula:

$r = 5$ cm

$V = 400 \text{ cm}^3$

$r = \sqrt{\dfrac{V}{\pi h}}$

By substituting the information into the given formula, you can solve for the height h.

$$r = \sqrt{\frac{V}{\pi h}}$$

$$5 = \sqrt{\frac{400}{\pi h}}$$

$$5^2 = \left(\sqrt{\frac{400}{\pi h}}\right)^2 \qquad \text{Square both sides.}$$

$$25 = \frac{400}{\pi h} \qquad \text{Simplify.}$$

$$25\pi h = 400 \qquad \text{Multiply.}$$

$$h = \frac{400}{25\pi} \qquad \text{Solve for } h.$$

$$h = \frac{16}{\pi}$$

The formula for r has the restrictions $V \geq 0$ and $h > 0$. Checking $h = \dfrac{16}{\pi}$ for being extraneous, you get

$$\text{RHS} = \sqrt{\frac{V}{\pi h}} = \sqrt{\frac{400}{\pi \cdot \frac{16}{\pi}}} = \sqrt{\frac{400}{16}} = \sqrt{25} = 5$$

$\text{LHS} = r = 5$

$\therefore \text{LHS} = \text{RHS}$

Thus, the height of the cylinder is $\dfrac{16}{\pi}$ cm.

Make sure you complete the following learning activity and check your answers in the Module 5 Answer Key. Completing this learning activity will help prepare you for the following assignment. If you get most of the questions right in Learning Activity 5.5, you are likely to do well in the final assignment in this module, Assignment 5.3.

Learning Activity 5.5

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Maddox has 130 quarters, 72 dimes, and 26 nickels. How much money does he have in total?
2. Evaluate: $\frac{3}{5} \times \frac{10}{7}$
3. If today is Friday and your math exam is in 17 days, what day of the week is your math exam on?
4. Evaluate: $52 - (-3 - 4(-26 - (-28)))$.
5. Simplify: $\frac{3x^5y^2z^8}{9x^7y^3z^2}$
6. Estimate: $\sqrt{140}$
7. Trey and Tanisha are discussing their solutions to the linear inequality $-3x > 6$. Trey believes the answer is $x > -2$, while Tanisha believes the answer is $x < -2$. Who is right?
8. Simplify: $(x - 2)^3$

continued

Learning Activity 5.5 (continued)

Part B: Solving Radical Equations

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. State whether each statement is true or false. If the statement is false, rewrite it so that it is true.
   a) For each real number x, $\sqrt{x}$ is a real number only when $x \geq 0$.
   b) For each real number x, $\sqrt{x-3}$ is a real number only when $x \geq 0$.
   c) For each real number x, $\sqrt{x^2} = x$.
   d) The real number solution set of $\sqrt{x^2} = -3$ is $\{\sqrt{3}, -\sqrt{3}\}$.
   e) The equation $(x-1)^2 = \left(\sqrt{2x-1}\right)^2$ is equivalent to the sentence $(x-1) = \sqrt{2x-1}$ or $(x-1) = -\sqrt{2x-1}$.
   f) For each real number x, if $x^2 = 16$, then $x = \sqrt{16}$.
   g) Squaring both sides of $\sqrt{x} + 2 = 3x$ yields $x + 4 = 9x^2$.
2. Simplify each of the following expressions.
   a) $\left(\sqrt{2x-1}\right)^2$
   b) $\left(5 + \sqrt{x}\right)^2$
   c) $\left(2 + \sqrt{x-5}\right)^2$
3. Find the solutions for each of the following equations. Check your solutions for extraneous roots.
   a) $x + 2 = \sqrt{2x+7}$
   b) $x = 2 - \sqrt{2x-5}$
   c) $\sqrt{2x+3} - \sqrt{x+1} = 1$
   d) $\sqrt{x^2-3} + 1 = 0$
   e) $x = \sqrt{3x-2} + 2$
   f) $\sqrt{\sqrt{x^2+6x}} = 2$
   g) $\sqrt{3x+2} = 3\sqrt{x} - \sqrt{2}$
   h) $\sqrt{1-x} + \sqrt{x} = \sqrt{x+1}$

continued

Learning Activity 5.5 (continued)

4. Solve each of the following equations.

   a) $\sqrt[5]{x} = 4$

   b) $\sqrt[3]{x} = 0.5$

   c) $x^{\frac{2}{3}} = 81$

   d) $2x^{0.25} = 3$

5. Solve each of the following equations. Determine any restrictions on the variables.

   a) $\sqrt{x-3} = \sqrt{x} + 3$

   b) $\sqrt{x+2} = 4 - \sqrt{x+2}$

6. Consider a cylindrical drink can. The radius of the can is 3 cm. The height of the can is 12 cm.

   a) Find the volume of the pop can. Given: $r = \sqrt{\frac{V}{\pi h}}$. Round your answer to two decimal places. Check your answer for extraneous roots.

   b) If the volume of a typical pop can is 355 mL, how does your answer compare to that value?

7. Kinetic energy is calculated using the formula $E_k = \frac{1}{2}mv^2$, where E_k represents the kinetic energy of an object measured in joules, m represents the mass of an object measured in kilograms and v represents the positive speed of an object. Speed is measured in metres per second. Using the formula, calculate the speed of a baseball that has a mass of 0.145 kg and 32.1 J (joules) of kinetic energy. Round your answer to two decimal places.

Lesson Summary

In this lesson, you compiled your knowledge of radical expressions to learn how to solve radical equations. Just as solving quadratic equations was the same as finding the roots of a quadratic equation, you discovered how solving radical equations is the same as finding the roots of a radical equation. Sometimes the method of "squaring both sides" introduces roots that are extraneous. In this lesson, you learned how to identify both the extraneous roots of a radical equation and the restrictions on the variable(s) in a radical equation.

Make sure you complete Assignment 5.3, which is the last assignment in this module.

Notes

Assignment 5.3

Solving Radical Equations

Total: 53 marks

1. Find the solutions for each of the following equations. *(27 marks)*

   a) $\sqrt{3+x}-5=-2$ *(3 marks)*

   b) $\sqrt{2x-1}+2=5$ *(3 marks)*

continued

Assignment 5.3: Solving Radical Equations (continued)

c) $x = 2\sqrt{2x - 3}$ *(4 marks)*

d) $\sqrt{3x + 7} = x + 3$ *(4 marks)*

continued

Assignment 5.3: Solving Radical Equations (continued)

e) $\sqrt{\frac{x+1}{3}} + 5x = 3x - 1$ *(5 marks)*

continued

Assignment 5.3: Solving Radical Equations (continued)

f) $\sqrt{\sqrt{2x+4}} = 4$ *(4 marks)*

g) $\sqrt{x+9} = \sqrt{3} + \sqrt{x}$ *(4 marks)*

continued

Assignment 5.3: Solving Radical Equations (continued)

2. Solve each of the following equations. Determine any restrictions on the variable(s). *(15 marks)*

   a) $4 = \sqrt[3]{8x}$ *(3 marks)*

   b) $\sqrt{21 - x} = 2$ *(4 marks)*

continued

Assignment 5.3: Solving Radical Equations (continued)

c) $2\sqrt{x+8} = 3\sqrt{3x+1}$ *(4 marks)*

d) $x^{\frac{3}{2}} = 27$ *(4 marks)*

continued

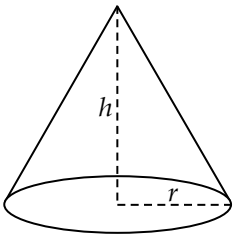
Assignment 5.3: Solving Radical Equations (continued)

3. The formula relating current (I) in amperes, power (p) in watts, and resistance (r) in ohms is $I = \sqrt{\frac{p}{r}}$. If the current of a toaster is 8 amperes and the power is 1500 watts, what is the resistance of the toaster measured in ohms? Leave your answer as a fraction in lowest terms. Check your answer for extraneous roots. *(5 marks)*

continued

Assignment 5.3: Solving Radical Equations (continued)

4. The lateral surface area of a right circular cone is given by the formula $A = \pi r\sqrt{r^2 + h^2}$, where A represents the surface area, r represents the radius of the cone, and h represents the height of the cone. Find the height of a right circular cone if the lateral surface area is 60 cm^2 and the radius of the cone is 4 cm. Round your answer to two decimal places. *(6 marks)*



Module 5 Summary

In this module, you built upon the knowledge of radicals you had from *Grade 10 Introduction to Applied and Pre-Calculus Mathematics*. In Grade 10, you learned how to simplify radical expressions containing numerical radicands. In this module, you learned how to simplify, add, subtract, multiply, and divide radical expressions containing a mix of numerical and variable radicands. You learned these skills in order to be able to solve radical equations.

As you saw, radical equations can appear in many different situations that involve the use of formulas, such as area and volume, current and resistance, and kinetic energy. If you plan to take either chemistry or physics, equations like these will appear fairly often. Now that you know how to solve these types of equations, they will be familiar when you encounter them again.

Submitting Your Assignments

It is now time for you to submit Assignments 5.1 to 5.3 to the Distance Learning Unit so that you can receive some feedback on how you are doing in this course. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 5 assignments and organize your material in the following order:

- ❑ Module 5 Cover Sheet (found at the end of the course Introduction)
- ❑ Assignment 5.1: Simplifying Radical Expressions
- ❑ Assignment 5.2: Multiplying and Dividing Radical Expressions
- ❑ Assignment 5.3: Solving Radical Equations

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Notes

Grade 11 Pre-Calculus Mathematics (30S)

Module 5
Radicals

Learning Activity Answer Keys

Module 5:
Radicals

Learning Activity 5.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Jenelle's doctor told her to eat an average of 2000 calories a day. If Jenelle ate 520 calories for breakfast, 630 calories for lunch, and had two snacks with 125 calories each, how many calories does Jenelle have left for supper?
2. Multiply: $(x + 3)(x + 4)$
3. Evaluate: $\sqrt{256}$
4. Solve for x: $(2x + 4)(x - 2) = 0$
5. Patrick sleeps for 8 hours a day, attends school for 6 hours a day, and plays sports for 2 hours a day. What fraction of the day (in lowest terms) does Patrick have for free time?
6. Order these values from lowest to highest: $\sqrt[3]{8}$, $2^{\frac{2}{4}}$, $2^{\frac{9}{3}}$, $2^{-\frac{1}{2}}$
7. Are these fractions equivalent? $\frac{1}{\sqrt{2}}$ and $\frac{\sqrt{2}}{2}$
8. Find the slope of the line perpendicular to the line, $3y - 2x + 7 = 0$.

Answers:

1. 600 $(2000 - (520 + 630 + 125 + 125) = 2000 - 1400 = 600)$
2. $x^2 + 7x + 12$ $(x^2 + 4x + 3x + 12)$
3. 16 $\left(\sqrt{2 \times 128} = \sqrt{2 \times 2 \times 64} = \sqrt{4} \times \sqrt{64} = 2 \times 8\right)$
4. -2 and 2
5. $\frac{1}{3}$ $\left(24 - (8 + 6 + 2) = 24 - 16 = 8; \frac{8}{24} = \frac{1}{3}\right)$

6. $\frac{1}{\sqrt{2}} < \sqrt{2} < 2 < 8$ or $2^{-\frac{1}{2}} < 2^{\frac{2}{4}} < \sqrt[3]{8} < 2^{\frac{9}{3}}$

$\left(\sqrt[3]{8} = 2;\ 2^{\frac{2}{4}} = 2^{\frac{1}{2}} = \sqrt{2};\ 2^{\frac{9}{3}} = 2^3 = 8;\ 2^{-\frac{1}{2}} = \frac{1}{2^{\frac{1}{2}}} = \frac{1}{\sqrt{2}}\right)$

7. Yes $\left(\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{\sqrt{2}}{2}\right)$

8. The slope of the perpendicular line is $-\frac{3}{2}$.

$\left(3y = 2x - 7;\ y = \frac{2}{3}x - \frac{7}{3};\ \text{the slope of this line is } \frac{2}{3}\right)$

Part B: Simplifying Radical Expressions

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Simplify each radical.

a) $\sqrt{80}$

b) $\sqrt{18}$

c) $\sqrt{200}$

d) $\sqrt{250}$

e) $\sqrt{1224}$

f) $\sqrt{288}$

g) $\sqrt{512}$

h) $-\sqrt{8}$

i) $-\sqrt{800}$

Answers:

a) $\sqrt{16}\sqrt{5} = 4\sqrt{5}$

b) $\sqrt{9}\sqrt{2} = 3\sqrt{2}$

c) $\sqrt{100}\sqrt{2} = 10\sqrt{2}$

d) $\sqrt{25}\sqrt{10} = 5\sqrt{10}$

e) $\sqrt{36}\sqrt{34} = 6\sqrt{34}$

f) $\sqrt{144}\sqrt{2} = 12\sqrt{2}$

g) $\sqrt{256}\sqrt{2} = 16\sqrt{2}$

h) $-\sqrt{4}\sqrt{2} = -2\sqrt{2}$

i) $-\sqrt{400}\sqrt{2} = -20\sqrt{2}$

2. Simplify each radical, given $x \geq 0$, $y \geq 0$, and $z \geq 0$.

   a) $\sqrt{4x^4y^6}$

   b) $\sqrt{8x^{10}y^{12}}$

   c) $\sqrt{75x^8y^5}$

   d) $\sqrt{x^4yz^2}$

   e) $\sqrt{5x^3y^7}$

   Answers:

   a) $\sqrt{4}\sqrt{x^4}\sqrt{y^6} = 2x^2y^3$

   b) $\sqrt{4}\sqrt{2}\sqrt{x^{10}}\sqrt{y^{12}} = 2x^5y^6\sqrt{2}$

   c) $\sqrt{25}\sqrt{3}\sqrt{x^8}\sqrt{y^4}\sqrt{y} = 5x^4y^2\sqrt{3y}$

   d) $\sqrt{x^4}\sqrt{y}\sqrt{z^2} = x^2z\sqrt{y}$

   e) $\sqrt{5}\sqrt{x^2}\sqrt{x}\sqrt{y^6}\sqrt{y} = xy^3\sqrt{5xy}$

3. Write each mixed radical as an entire radical.

   a) $3\sqrt{15}$

   b) $3x\sqrt{5x}$

   c) $5x^2y\sqrt{3xy}$

   Answers:

   a) $\sqrt{9}\sqrt{15} = \sqrt{135}$

   b) $\sqrt{9}\sqrt{x^2}\sqrt{5x} = \sqrt{45x^3}$

   c) $\sqrt{25}\sqrt{x^4}\sqrt{y^2}\sqrt{3xy} = \sqrt{75x^5y^3}$

4. Identify the greater number. (Do not use a calculator.)

   a) $\sqrt{125}$ or $3\sqrt{20}$?

   Answer:

   $\sqrt{125} = \sqrt{25}\sqrt{5} = 5\sqrt{5}$

   $3\sqrt{20} = 3\sqrt{4}\sqrt{5} = 3(2)\sqrt{5} = 6\sqrt{5}$

   As $6\sqrt{5} > 5\sqrt{5}$, then $3\sqrt{20} > \sqrt{125}$.

   b) $16\sqrt{2}$ or $\sqrt{768}$?

   Answer:

   $\sqrt{768} = \sqrt{256}\sqrt{3} = 16\sqrt{3}$

   As $\sqrt{3} > \sqrt{2}$, then $16\sqrt{3} > 16\sqrt{2}$ and $\sqrt{768} > 16\sqrt{2}$.

5. Order the following radical expressions from least to greatest. (Do not use a calculator.)

   a) $5\sqrt{3}, \sqrt{49}, 6\sqrt{5}$

   Answer:

   $5\sqrt{3} = \sqrt{25}\sqrt{3} = \sqrt{75}$

   $\sqrt{49} = \sqrt{49}$

   $6\sqrt{5} = \sqrt{36}\sqrt{5} = \sqrt{180}$

   Thus, $\sqrt{49} < \sqrt{75} < \sqrt{180}$ and $\sqrt{49} < 5\sqrt{3} < 6\sqrt{5}$.

   b) $3\sqrt{2}, 6\sqrt{2}, \sqrt{32}$

   Answer:

   $\sqrt{32} = \sqrt{16}\sqrt{2} = 4\sqrt{2}$

   Thus, $3\sqrt{2} < 4\sqrt{2} < 6\sqrt{2}$ and $3\sqrt{2} < \sqrt{32} < 6\sqrt{2}$.

6. Solve for x.

   a) $x^2 = 25$
   b) $x^2 = 16$
   c) $x^2 = 48$
   d) $x^2 = 0$
   e) $x^2 = -9$

   Answers:

   a) $x = \pm 5$
   b) $x = \pm 4$
   c) $x = \pm\sqrt{48} = \pm\sqrt{16}\sqrt{3} = \pm 4\sqrt{3}$
   d) $x = \sqrt{0} = 0$
   e) $x = \pm\sqrt{-9}$ = no real number answer

7. By converting the radical to a rational exponent, prove the First Property of Radicals:

   $\sqrt[n]{a^n} = a$ when n is a positive integer greater than 1.

   Answer:

   $\sqrt[n]{a^n}$ can be written using a rational exponent as $a^{\frac{n}{n}}$.

   Since n is a positive integer greater than one, and $a^{\frac{n}{n}} = a^1 = a$,

   thus, $\sqrt[n]{a^n} = a$ when $n > 1$.

8. Identify the values of the variable(s) for which the radical represents a real number.

a) $\sqrt{8-2x}$

Answer:

A square root is only defined when the radicand is greater than or equal to zero. You will need to solve a simple inequality.

$8 - 2x \geq 0$

$8 \geq 2x$

$4 \geq x$

Thus, this radical expression is defined when $x \leq 4$.

b) $\sqrt{x^3}$

Answer:

$\sqrt{x^3}$ has an index of 2. Therefore, as the index is even, this radical expression is only defined when $x^3 \geq 0$ and, thus, $x \geq 0$.

c) $\sqrt[5]{z}$

Answer:

$\sqrt[5]{z}$ has an index of 5. Therefore, as the index is odd, this radical expression is defined for $z \in \Re$.

d) $\sqrt[4]{x^7}$

Answer:

$\sqrt[4]{x^7}$ has an index of 4. Therefore, as the index is even, this radical expression is only defined when $x^7 \geq 0$ and, thus, $x \geq 0$.

e) $\sqrt{64a^3b^6}$

Answer:

$\sqrt{64a^3b^6}$ has an index of 2. Therefore, as the index is even, this radical expression is only defined when $64a^3b^6 \geq 0$. Since the exponent on a is odd, then a must be 0 or positive. However, b can be any value, $b \in \Re$, as b has an even exponent. Thus, this expression is defined when $a \geq 0$ and $b \in \Re$.

f) $\sqrt{-432xy^2}$

Answer:

To answer this question, you have to consider the radicand to be positive. Since the coefficient, -432, is negative, then one of the variables has to be negative. Since y^2 will never be negative, x has to be negative. Also, x or y could be zero. Thus, $x \leq 0$ and $y \in \Re$.

Learning Activity 5.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Alyssa is twice as old as her brother today. When she is 25 years old, her brother will be 20 years old. How many years old is Alyssa today?
2. How much would a sweater cost if it were 25% off the original price of $60 and you had to pay an additional 12% sales tax on the discounted price?
3. Simplify: $\left(\sqrt[3]{2x + 10}\right)^0$
4. Simplify: $2x(3x - 5) - 4(x + 2)$
5. Write 625 as a power with a base of 5.
6. List the factors of 68.
7. The equation of a parabola is, $y = 3x^2 - 5x + 8$. What is the y-intercept?
8. Find the slope of the line that contains the points $(7, -1)$ and $(-8, 15)$.

Answers:

1. Alyssa is 10. (Since Alyssa is 5 years older, she is twice as old when her brother is 5 and she is 10.)
2. $50.40 in total ($60 × 25% = $15; the sweater costs $60 − $15 = $45 without tax; 10% of $45 = $4.50; 1% of $45 = $0.45; 12% of $45 = $4.50 + $0.45 + $0.45 = $5.40; $45 + $5.40 = $50.40)
3. 1 (any number to the power of zero is 1)
4. $6x^2 - 10x - 4x - 8 = 6x^2 - 14x - 8$
5. 5^4 ($625 = 25 \times 35 = 5^2 \times 5^2$)
6. 1, 2, 4, 17, 34, 68 (since $68 = 2 \times 2 \times 17$)
7. 8 (let $x = 0$)
8. $-\frac{16}{15}$ $\left(m = \frac{15 - (-1)}{-8 - 7} = \frac{16}{-15} = -\frac{16}{15}\right)$

Part B: Adding and Subtracting Radical Expressions

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Simplify each radical expression as much as possible. Given: $x \geq 0$.

   a) $2\sqrt{3} + 4\sqrt{3} - \sqrt{3}$

   b) $\sqrt{96} - \sqrt{54} - \sqrt{24}$

   c) $3\sqrt{75} - 2\sqrt{48} + \sqrt{12}$

   d) $\sqrt{98} - \sqrt{50} + \sqrt{72}$

   e) $5\sqrt{x} - 3\sqrt{x} + 6\sqrt{x}$

   f) $6\sqrt{5} + 3\sqrt{2} - 4\sqrt{5} + \sqrt{2}$

   g) $\sqrt{x^5} + 3x^2\sqrt{x} - 7x\sqrt{4x^3}$

   h) $2x\sqrt[3]{8x^4} - 4x^2\sqrt[3]{x} + \sqrt[3]{125x^7}$

   Answers:

   a) $5\sqrt{3}$

   b) $\sqrt{16}\sqrt{6} - \sqrt{9}\sqrt{6} - \sqrt{4}\sqrt{6} = 4\sqrt{6} - 3\sqrt{6} - 2\sqrt{6} = -\sqrt{6}$

   c) $3\sqrt{25}\sqrt{3} - 2\sqrt{16}\sqrt{3} + \sqrt{4}\sqrt{3}$
   $= 3(5)\sqrt{3} - 2(4)\sqrt{3} + 2\sqrt{3}$
   $= 15\sqrt{3} - 8\sqrt{3} + 2\sqrt{3}$
   $= 9\sqrt{3}$

   d) $\sqrt{49}\sqrt{2} - \sqrt{25}\sqrt{2} + \sqrt{36}\sqrt{2}$
   $= 7\sqrt{2} - 5\sqrt{2} + 6\sqrt{2}$
   $= 8\sqrt{2}$

   e) $8\sqrt{x}$

   f) $2\sqrt{5} + 4\sqrt{2}$

g) $\sqrt{x^5} + 3x^2\sqrt{x} - 7x\sqrt{4x^3}$

$= \sqrt{x^4 x} + 3x^2\sqrt{x} - 7x\sqrt{4x^2 x}$

$= x^2\sqrt{x} + 3x^2\sqrt{x} - 7x(2)(x)\sqrt{x}$

$= x^2\sqrt{x} + 3x^2\sqrt{x} - 14x^2\sqrt{x}$

$= -10x^2\sqrt{x}$

h) $2x\sqrt[3]{8x^4} - 4x^2\sqrt[3]{x} + \sqrt[3]{125x^7}$

$= 2x\sqrt[3]{8x^3 x} - 4x^2\sqrt[3]{x} + \sqrt[3]{125x^6 x}$

$= 2x(2)(x)\sqrt[3]{x} - 4x^2\sqrt[3]{x} + 5x^2\sqrt[3]{x}$

$= 4x^2\sqrt[3]{x} - 4x^2\sqrt[3]{x} + 5x^2\sqrt[3]{x}$

$= 5x^2\sqrt[3]{x}$

2. Find the values of the missing numerical coefficients A and B.

a) $3\sqrt{2} + 2\sqrt{3} + A\sqrt{12} - B\sqrt{8} = \sqrt{75} - \sqrt{32}$

Answer:

$3\sqrt{2} + 2\sqrt{3} + A\sqrt{4}\sqrt{3} - B\sqrt{4}\sqrt{2} = \sqrt{25}\sqrt{3} - \sqrt{16}\sqrt{2}$

$3\sqrt{2} + 2\sqrt{3} + 2A\sqrt{3} - 2B\sqrt{2} = 5\sqrt{3} - 4\sqrt{2}$

$2A\sqrt{3} - 2B\sqrt{2} = 5\sqrt{3} - 4\sqrt{2} - 3\sqrt{2} - 2\sqrt{3}$

$2A\sqrt{3} - 2B\sqrt{2} = 3\sqrt{3} - 7\sqrt{2}$

$\therefore 2A = 3$ and $-2B = -7$ Equate coefficients of like radicals.

$\therefore A = \frac{3}{2}$ and $B = \frac{7}{2}$

b) $\sqrt{24} + A\sqrt{2} + B\sqrt{150} = 3\sqrt{2}$

Answer:

$\sqrt{4}\sqrt{6} + A\sqrt{2} + B\sqrt{25}\sqrt{6} = 3\sqrt{2}$

$2\sqrt{6} + A\sqrt{2} + 5B\sqrt{6} = 3\sqrt{2}$

$A\sqrt{2} + 5B\sqrt{6} = 3\sqrt{2} - 2\sqrt{6}$

$\therefore A = 3$ and $5B = -2$ Equate coefficients of like radicals.

$B = -\frac{2}{5}$

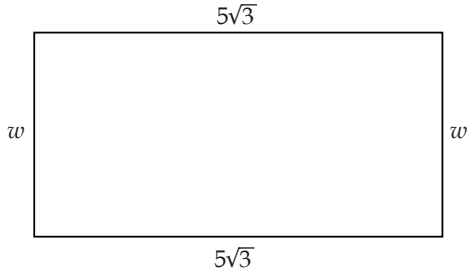
3. Find the width of a rectangle if the length is $5\sqrt{3}$ cm and the perimeter is $\left(16\sqrt{3} - 4\sqrt{2}\right)$ cm.

Answer:

Let w = the width of the rectangle.

Given: length = $5\sqrt{3}$ cm

perimeter = $\left(16\sqrt{3} - 4\sqrt{2}\right)$ cm



Perimeter $= 2l + 2w$

$16\sqrt{3} - 4\sqrt{2} = 2\left(5\sqrt{3}\right) + 2w$

$16\sqrt{3} - 4\sqrt{2} = 10\sqrt{3} + 2w$

$16\sqrt{3} - 10\sqrt{3} - 4\sqrt{2} = 2w$

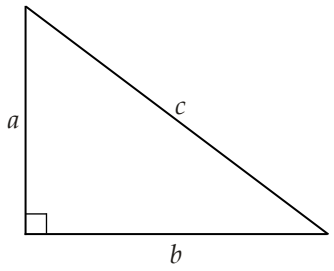
$6\sqrt{3} - 4\sqrt{2} = 2w$

$\frac{6\sqrt{3}}{2} - \frac{4\sqrt{2}}{2} = w$

$3\sqrt{3} - 2\sqrt{2} = w$

The width is $\left(3\sqrt{3} - 2\sqrt{2}\right)$ cm.

4. Given the lengths of two sides of the right triangle, find the length of the third side.



a) $a = 3$ $b = 4$

b) $a = 3$ $b = 5$

c) $a = 15$ $c = 17$

Answers:

a) $a^2 + b^2 = c^2$ (Pythagorean theorem)

$3^2 + 4^2 = c^2$

$25 = c^2$

$5 = c$ (any length is assumed to be *positive*)

b) $a^2 + b^2 = c^2$

$3^2 + 5^2 = c^2$

$9 + 25 = c^2$

$34 = c^2$

$\sqrt{34} = c$

c) $a^2 + b^2 = c^2$

$15^2 + b^2 = 17^2$

$225 + b^2 = 289$

$b^2 = 64$

$b = 8$

5. Why is it important to simplify radical expressions before adding or subtracting? Provide an example to support your explanation.

   Answer:

   Radical expressions may appear to not be like terms until they are simplified. It is therefore necessary to simplify radical expressions before adding or subtracting them to ensure you arrive at the answer in simplest form.

   Consider the following expression.

   $\sqrt{96} - \sqrt{54}$

   These radical expressions are not like terms; therefore, it appears that they cannot be combined, as you cannot subtract radicals that are not like terms. However, if you simplify each radical, the resulting radicals become like terms.

   $\sqrt{96} - \sqrt{54}$

   $= \sqrt{(16)(6)} - \sqrt{(9)(6)}$

   $= 4\sqrt{6} - 3\sqrt{6}$

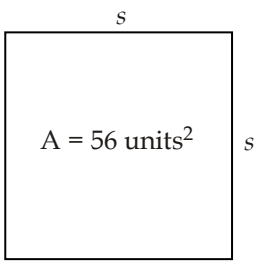
   It is only after simplifying that it is clear that the radical expressions are like terms that can be subtracted.

   $4\sqrt{6} - 3\sqrt{6}$

   $= \sqrt{6}$

6. The perimeter of a square is $8\sqrt{2x}$ units. The area of a square is 56 units^2. Determine the value of x, if $x > 0$.

Answer:



$P = 8\sqrt{2x}$ units

s = length of side

$A = s^2$

$56 = s^2$

$\sqrt{56} = s$

$\sqrt{4}\sqrt{14} = s$

$2\sqrt{14} = s$

If the area of the square is 56 units^2, then the length of one side is $2\sqrt{14}$ units.

Therefore, the perimeter = $(4)\left(2\sqrt{14}\right) = 8\sqrt{14}$.

Thus, $8\sqrt{14}$ has to equal $8\sqrt{2x}$.

$8\sqrt{14} = 8\sqrt{2x}$

$\sqrt{14} = \sqrt{2x}$ Divide by 8.

$\left(\sqrt{14}\right)^2 = \left(\sqrt{2x}\right)^2$

$14 = 2x$

$7 = x$

Learning Activity 5.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. School starts at 8:45 a.m. Hunter spends twenty minutes eating breakfast, fifteen minutes taking a shower, thirty-five minutes watching TV, and ten minutes biking to school. If Hunter arrives just on time for school, what time did he wake up?
2. Simplify: $\left(x + \sqrt{3}\right)^2$
3. Express 0.72 as a fraction in lowest terms.
4. Find the x-intercepts of the following parabola: $y = x^2 + 9x + 14$
5. If three angles in a parallelogram measure 286° in total, what is the measure of the fourth angle in the parallelogram?
6. Determine the distance between the following two points, (9, 24) and (4, 12)
7. Express as a power: $\sqrt[4]{5^8}$
8. Write this percent as a decimal: 3.2%.

Answers:

1. 7:25 a.m. (8:45 − (20 + 15 + 35 + 10) = 8:45 − 80 minutes = 8:45 − 1 hour and 20 minutes = 7:25)
2. $x^2 + 2x\sqrt{3} + 3$ $\left(\left(x + \sqrt{3}\right)\left(x + \sqrt{3}\right) = x^2 + x\sqrt{3} + x\sqrt{3} + \sqrt{9}\right)$
3. $\frac{18}{25}$ $\left(\frac{72}{100} = \frac{36}{50} = \frac{18}{25}\right)$
4. −7 and −2 ($(x + 7)(x + 2) = 0$)
5. 74° (360° − 286°)
6. 13 $\left(d = \sqrt{(-5)^2 + (-12)^2} = \sqrt{25 + 144} = \sqrt{169} = 13\right)$
7. $\sqrt[4]{5^8} = 5^{\frac{8}{4}} = 5^2$
8. 0.032

Part B: Multiplying Radical Expressions

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Simplify each radical expression. Given: $x \geq 0$, $y \geq 0$.

a) $\left(\sqrt{5}\right)\left(\sqrt{3}\right)$

b) $\left(\sqrt{8}\right)\left(\sqrt{2}\right)$

c) $\left(4\sqrt{10}\right)\left(3\sqrt{5}\right)$

d) $\left(2\sqrt{5}\right)\left(-3\sqrt{5}\right)$

e) $\left(-3\sqrt{3}\right)\left(-4\sqrt{6}\right)$

f) $\left(2\sqrt{x}\right)\left(3\sqrt{xy^2}\right)$

g) $\sqrt{3}\left(\sqrt{2}-\sqrt{5}\right)$

h) $-\sqrt{2}\left(3\sqrt{2}+3\sqrt{5}\right)$

i) $\sqrt{6}\left(3\sqrt{2}+5\sqrt{8}\right)$

j) $\left(2\sqrt{5}+3\right)\left(2\sqrt{5}-3\right)$

k) $\left(6\sqrt{5}-3\sqrt{10}\right)\left(4\sqrt{5}+\sqrt{10}\right)$

l) $\left(\sqrt{2}-\sqrt{3}\right)^2$

m) $\left(4\sqrt{7}+3\sqrt{2}\right)^2$

n) $\left(\sqrt{x}+\sqrt{y^3}\right)\left(\sqrt{xy}\right)$

o) $\left(2\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}-3\sqrt{y}\right)$

Answers:

a) $\left(\sqrt{5}\right)\left(\sqrt{3}\right) = \sqrt{15}$

b) $\sqrt{8}\sqrt{2} = \sqrt{16} = 4$

c) $\begin{aligned}\left(4\sqrt{10}\right)\left(3\sqrt{5}\right) &= 12\sqrt{50} \\ &= 12\sqrt{25}\sqrt{2} \\ &= 12(5)\sqrt{2} \\ &= 60\sqrt{2}\end{aligned}$

d) $\begin{aligned}\left(2\sqrt{5}\right)\left(-3\sqrt{5}\right) &= -6\sqrt{25} \\ &= -6(5) \\ &= -30\end{aligned}$

e) $\begin{aligned}\left(-3\sqrt{3}\right)\left(-4\sqrt{6}\right) &= 12\sqrt{18} \\ &= 12\sqrt{9}\sqrt{2} \\ &= 12(3)\sqrt{2} \\ &= 36\sqrt{2}\end{aligned}$

f) $\begin{aligned}\left(2\sqrt{x}\right)\left(3\sqrt{xy^2}\right) &= 6\sqrt{x^2y^2} \\ &= 6xy\end{aligned}$

g) $\sqrt{3}\left(\sqrt{2}-\sqrt{5}\right) = \sqrt{6}-\sqrt{15}$

h) $-\sqrt{2}\left(3\sqrt{2}+3\sqrt{5}\right) = -3\sqrt{4}-3\sqrt{10}$

$= -3(2)-3\sqrt{10}$

$= -6-3\sqrt{10}$

i) $\sqrt{6}\left(3\sqrt{2}+5\sqrt{8}\right) = 3\sqrt{12}+5\sqrt{48}$

$= 3\sqrt{4}\sqrt{3}+5\sqrt{16}\sqrt{3}$

$= 3(2)\sqrt{3}+5(4)\sqrt{3}$

$= 6\sqrt{3}+20\sqrt{3}$

$= 26\sqrt{3}$

j) $\left(2\sqrt{5}+3\right)\left(2\sqrt{5}-3\right) = 2\sqrt{5}\left(2\sqrt{5}-3\right)+3\left(2\sqrt{5}-3\right)$

$= 4\sqrt{25}-6\sqrt{5}+6\sqrt{5}-9$

$= 4(5)-9$

$= 20-9$

$= 11$

k) $\left(6\sqrt{5}-3\sqrt{10}\right)\left(4\sqrt{5}+\sqrt{10}\right) = 6\sqrt{5}\left(4\sqrt{5}+\sqrt{10}\right)-3\sqrt{10}\left(4\sqrt{5}+\sqrt{10}\right)$

$= 24\sqrt{25}+6\sqrt{50}-12\sqrt{50}-3\sqrt{100}$

$= 24(5)-6\sqrt{50}-3(10)$

$= 120-6\sqrt{25}\sqrt{2}-30$

$= 120-6(5)\sqrt{2}-30$

$= 90-30\sqrt{2}$

l) $\left(\sqrt{2}-\sqrt{3}\right)^2 = \left(\sqrt{2}-\sqrt{3}\right)\left(\sqrt{2}-\sqrt{3}\right)$

$= \sqrt{2}\left(\sqrt{2}-\sqrt{3}\right)-\sqrt{3}\left(\sqrt{2}-\sqrt{3}\right)$

$= \sqrt{4}-\sqrt{6}-\sqrt{6}+\sqrt{9}$

$= 2-2\sqrt{6}+3$

$= 5-2\sqrt{6}$

m) $\left(4\sqrt{7}+3\sqrt{2}\right)^2 = \left(4\sqrt{7}+3\sqrt{2}\right)\left(4\sqrt{7}+3\sqrt{2}\right)$

$= 4\sqrt{7}\left(4\sqrt{7}+3\sqrt{2}\right)+3\sqrt{2}\left(4\sqrt{7}+3\sqrt{2}\right)$

$= 16\sqrt{49}+12\sqrt{14}+12\sqrt{14}+9\sqrt{4}$

$= 16(7)+24\sqrt{14}+9(2)$

$= 112+24\sqrt{14}+18$

$= 130+24\sqrt{14}$

n) $\left(\sqrt{x}+\sqrt{y^3}\right)\left(\sqrt{xy}\right) = \sqrt{x^2y}+\sqrt{xy^4}$

$= x\sqrt{y}+y^2\sqrt{x}$

o) $\left(2\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}-3\sqrt{y}\right) = 2\sqrt{x}\left(\sqrt{x}-3\sqrt{y}\right)-\sqrt{y}\left(\sqrt{x}-3\sqrt{y}\right)$

$= 2\sqrt{x^2}-6\sqrt{xy}-\sqrt{xy}+3\sqrt{y^2}$

$= 2x-7\sqrt{xy}+3y$

2. Find the exact value of the area of a square if its perimeter is $\left(4\sqrt{2}+8\right)$ cm.

Answer:

Let s = length of side of square.

$P = 4s$

$P = \left(4\sqrt{2}+8\right)$ cm

$\therefore 4s = \left(4\sqrt{2}+8\right)$

$\frac{4s}{4} = \frac{4\sqrt{2}}{4}+\frac{8}{4}$

$s = \sqrt{2}+2$

Area $= A = s^2$

$A = \left(\sqrt{2}+2\right)^2$

$A = \left(\sqrt{2}+2\right)\left(\sqrt{2}+2\right)$

$= \sqrt{2}\left(\sqrt{2}+2\right)+2\left(\sqrt{2}+2\right)$

$= \sqrt{4}+2\sqrt{2}+2\sqrt{2}+4$

$= 2+4\sqrt{2}+4$

$= 6+4\sqrt{2}$

The exact value of the area of the square is $\left(6+4\sqrt{2}\right)$ cm^2.

Note: The exact value cannot be found using a calculator.

3. Identify the greater number:

a) $2\sqrt{80}$ or $5\sqrt{8}\left(2\sqrt{3}\right)$?

Answer:

$$\left(5\sqrt{8}\right)\left(2\sqrt{3}\right) = 10\sqrt{24}$$
$$= \sqrt{10^2}\sqrt{24}$$
$$= \sqrt{(100)(24)}$$
$$= \sqrt{2400}$$

$$2\sqrt{80} = \sqrt{2^2}\sqrt{80}$$
$$= \sqrt{(4)(80)}$$
$$= \sqrt{320}$$

$\therefore \left(5\sqrt{8}\right)\left(2\sqrt{3}\right)$ is greater.

b) $\left(3\sqrt{2}\right)\left(\sqrt{75}\right)$ or $\left(5\sqrt{2}\right)\left(3\sqrt{2}\right)$?

Answer:

$$\left(3\sqrt{2}\right)\left(\sqrt{75}\right) = 3\sqrt{150}$$
$$= \sqrt{3^2}\sqrt{150}$$
$$= \sqrt{9 \times 150}$$
$$= \sqrt{1350}$$

$$\left(5\sqrt{2}\right)\left(3\sqrt{2}\right) = 15\sqrt{4}$$
$$= \sqrt{15^2}\sqrt{4}$$
$$= \sqrt{225 \times 4}$$
$$= \sqrt{900}$$

$\therefore \left(3\sqrt{2}\right)\left(\sqrt{75}\right)$ is greater.

4. Every hour, a spider climbs up a distance in metres of $\left(2\sqrt{3}+5\sqrt{2}\right)$ and then is pushed down a distance of $\left(\sqrt{3}-\sqrt{2}\right)$. If the spider begins to climb at 12 noon, what is his distance above his starting point at 4:00 p.m. sharp?

Answer:

Time	Distance Above Starting Point
12 noon	0
1 p.m.	$2\sqrt{3}+5\sqrt{2}$
just past 1 p.m.	$\left(2\sqrt{3}+5\sqrt{2}\right)-\left(\sqrt{3}-\sqrt{2}\right)=\sqrt{3}+6\sqrt{2}$
2 p.m.	$\left(\sqrt{3}+6\sqrt{2}\right)+\left(2\sqrt{3}+5\sqrt{2}\right)=3\sqrt{3}+11\sqrt{2}$
just past 2 p.m.	$\left(3\sqrt{3}+11\sqrt{2}\right)-\left(\sqrt{3}-\sqrt{2}\right)=2\sqrt{3}+12\sqrt{2}$
3 p.m.	$\left(2\sqrt{3}+12\sqrt{2}\right)+\left(2\sqrt{3}+5\sqrt{2}\right)=4\sqrt{3}+17\sqrt{2}$
just past p.m.	$\left(4\sqrt{3}+17\sqrt{2}\right)-\left(\sqrt{3}-\sqrt{2}\right)=3\sqrt{3}+18\sqrt{2}$
4 p.m.	$\left(3\sqrt{3}+18\sqrt{2}\right)+\left(2\sqrt{3}+5\sqrt{2}\right)=5\sqrt{3}+23\sqrt{2}$

A more efficient method:

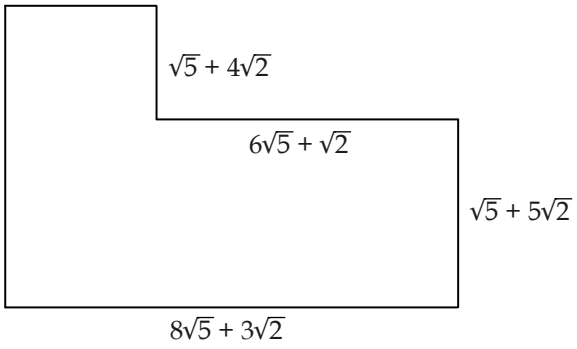
From 12 noon to 4 p.m., the spider climbs up $4\left(2\sqrt{3}+5\sqrt{2}\right)=8\sqrt{3}+20\sqrt{2}$.

The spider is pushed down 3 times = $3\left(\sqrt{3}-\sqrt{2}\right)=3\sqrt{3}-3\sqrt{2}$.

$\therefore$ at 4 p.m., the spider is $\left(8\sqrt{3}+20\sqrt{2}\right)-\left(3\sqrt{3}-3\sqrt{2}\right)=5\sqrt{3}+23\sqrt{2}$.

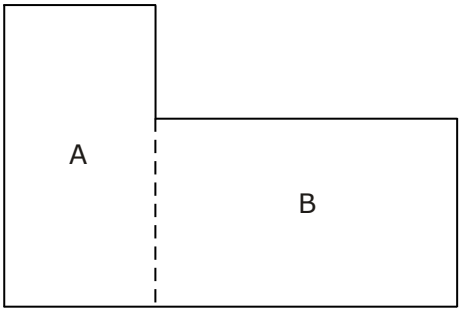
$\therefore$ the spider is $\left(5\sqrt{3}+23\sqrt{2}\right)$ m above his starting point.

5. Find the area of the figure drawn below.



Answer:

Subdivide the figure into two rectangles.



Area of a rectangle = lw

$$\text{Area of A} = \left[\left(\sqrt{5} + 5\sqrt{2}\right) + \left(\sqrt{5} + 4\sqrt{2}\right)\right]\left[\left(8\sqrt{5} + 3\sqrt{2}\right) - \left(6\sqrt{5} + \sqrt{2}\right)\right]$$

$$= \left(2\sqrt{5} + 9\sqrt{2}\right)\left(2\sqrt{5} + 2\sqrt{2}\right)$$

$$= 2\sqrt{5}\left(2\sqrt{5} + 2\sqrt{2}\right) + 9\sqrt{2}\left(2\sqrt{5} + 2\sqrt{2}\right)$$

$$= 4\sqrt{25} + 4\sqrt{10} + 18\sqrt{10} + 18\sqrt{4}$$

$$= 4(5) + 22\sqrt{10} + 18(2)$$

$$= 20 + 22\sqrt{10} + 36$$

$$= 56 + 22\sqrt{10}$$

Area of B $= \left(6\sqrt{5} + \sqrt{2}\right)\left(\sqrt{5} + 5\sqrt{2}\right)$

$= 6\sqrt{5}\left(\sqrt{5} + 5\sqrt{2}\right) + \sqrt{2}\left(\sqrt{5} + 5\sqrt{2}\right)$

$= 6\sqrt{25} + 30\sqrt{10} + \sqrt{10} + 5\sqrt{4}$

$= 6(5) + 31\sqrt{10} + 5(2)$

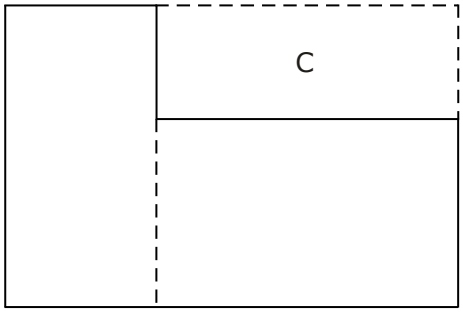
$= 30 + 31\sqrt{10} + 10$

$= 40 + 31\sqrt{10}$

Total area = area of A + Area of B

$= \left(56 + 22\sqrt{10}\right) + \left(40 + 31\sqrt{10}\right) = 96 + 53\sqrt{10}$

Alternatively, you could complete the large rectangle:



Required area = area of large rectangle − area of C.

$= \left(8\sqrt{5} + 3\sqrt{2}\right)\left(2\sqrt{5} + 9\sqrt{2}\right) - \left(6\sqrt{5} + \sqrt{2}\right)\left(\sqrt{5} + 4\sqrt{2}\right) = 96 + 53\sqrt{10}$

6. Determine whether $3 - \sqrt{2}$ is a root of the equation $x^2 - 6x + 7 = 0$.

Answer:

Substitute $3 - \sqrt{2}$ for x in the equation.

$\left(3 - \sqrt{2}\right)^2 - 6\left(3 - \sqrt{2}\right) + 7 = \left(3 - \sqrt{2}\right)\left(3 - \sqrt{2}\right) - 18 + 6\sqrt{2} + 7$

$= 3\left(3 - \sqrt{2}\right) - \sqrt{2}\left(3 - \sqrt{2}\right) - 18 + 6\sqrt{2} + 7$

$= 9 - 3\sqrt{2} - 3\sqrt{2} + \sqrt{4} - 18 + 6\sqrt{2} + 7$

$= 9 - 6\sqrt{2} + 2 - 18 + 6\sqrt{2} + 7$

$= 0$

$\therefore$ $3 - \sqrt{2}$ is a root of the equation $x^2 - 6x + 7 = 0$, since $3 - \sqrt{2}$ satisfies the equation.

7. The operation $\square$ is defined as follows: $a \square b = a^2 + ab$.

Evaluate each of the following:

a) $(2\sqrt{3}) \square (3 + \sqrt{3})$

Answer:

$$\begin{aligned}(2\sqrt{3}) \square (3 + \sqrt{3}) &= (2\sqrt{3})^2 + 2\sqrt{3}(3 + \sqrt{3}) \\ &= (2\sqrt{3})(2\sqrt{3}) + 2\sqrt{3}(3 + \sqrt{3}) \\ &= 4\sqrt{9} + 6\sqrt{3} + 2\sqrt{9} \\ &= 4(3) + 6\sqrt{3} + 2(3) \\ &= 12 + 6\sqrt{3} + 6 \\ &= 18 + 6\sqrt{3}\end{aligned}$$

b) $(2 + \sqrt{3}) \square (2 - \sqrt{3})$

Answer:

$$\begin{aligned}(2 + \sqrt{3}) \square (2 - \sqrt{3}) &= (2 + \sqrt{3})^2 + (2 + \sqrt{3})(2 - \sqrt{3}) \\ &= (2 + \sqrt{3})(2 + \sqrt{3}) + (2 + \sqrt{3})(2 - \sqrt{3}) \\ &= 4 + 2\sqrt{3} + 2\sqrt{3} + \sqrt{9} + 4 - 2\sqrt{3} + 2\sqrt{3} - \sqrt{9} \\ &= 4 + 2\sqrt{3} + 2\sqrt{3} + 3 + 4 - 2\sqrt{3} + 2\sqrt{3} - 3 \\ &= 8 + 4\sqrt{3}\end{aligned}$$

c) $(\sqrt{2} - 5) \square (3\sqrt{5})$

Answer:

$$\begin{aligned}(\sqrt{2} - 5) \square (3\sqrt{5}) &= (\sqrt{2} - 5)^2 + (\sqrt{2} - 5)(3\sqrt{5}) \\ &= (\sqrt{2} - 5)(\sqrt{2} - 5) + (3\sqrt{5})(\sqrt{2} - 5) \\ &= \sqrt{4} - 5\sqrt{2} - 5\sqrt{2} + 25 + 3\sqrt{10} - 15\sqrt{5} \\ &= 2 - 5\sqrt{2} - 5\sqrt{2} + 25 + 3\sqrt{10} - 15\sqrt{5} \\ &= 27 - 10\sqrt{2} + 3\sqrt{10} - 15\sqrt{5}\end{aligned}$$

8. Use playing cards to create four unique radical expression multiplication problems and simplify. Take out all the face cards from your deck of cards before creating your radical expression problems. Deal out at least four cards to create each problem. Black suits (spades and clubs) represent natural numbers, while red suits (diamonds and hearts) represent radicands. If you deal out a 3 of diamonds, a 2 of clubs, a 6 of clubs, and an 8 of hearts, you will have the following expression:

   3 of diamonds = $\sqrt{3}$

   2 of clubs = 2

   6 of clubs = 6

   6 of hearts = $\sqrt{6}$

   Expression: $\sqrt{3} \cdot 2 \cdot 6 \cdot \sqrt{6}$

   Simplifying the expression: $\sqrt{3} \cdot 2 \cdot 6 \cdot \sqrt{6}$

   $= 12\sqrt{18}$

   $= 12\sqrt{(9)(2)}$

   $= 12(3)\sqrt{2}$

   $= 36\sqrt{2}$

   You can try a variety of random cards and then test yourself using the following examples. Check your work against the solutions given in the answer key.

   a) 5C, 10S, 3H, 5D

   b) 4S, 3S, 2H, 5D

   c) 8H, 3C, AceD, 10S

   d) 7D, 2H, 8D, 2C

Answers:

Answers will vary. Four possible radical expression multiplication problems are shown below.

a) 5C, 10S, 3H, 5D

$= 5 \cdot 10 \cdot \sqrt{3} \cdot \sqrt{5}$

$= 50\sqrt{15}$

b) 4S, 3S, 2H, 5D

$= 4 \cdot 3 \cdot \sqrt{2} \cdot \sqrt{5}$

$= 12\sqrt{10}$

c) 8H, 3C, 1D, 10S

$= \sqrt{8} \cdot 3 \cdot \sqrt{1} \cdot 10$

$= 30\sqrt{(4)(2)}$

$= 30(2)\sqrt{2}$

$= 60\sqrt{2}$

d) 7D, 2H, 8D, 2C

$= \sqrt{7} \cdot \sqrt{2} \cdot \sqrt{8} \cdot 2$

$= 2\sqrt{112}$

$= 2\sqrt{16(7)}$

$= 2 \cdot 4\sqrt{7}$

$= 8\sqrt{7}$

Learning Activity 5.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Marissa loves to eat cookie dough so she bought a tub of cookie dough that was supposed to make 270 cookies. If Marissa and her brother each eat a scoop of cookie dough that is equivalent to three cookies every day, how long will it take them to eat the entire tub of cookie dough?
2. Write 32 as a power with a base of 2.
3. Bill went on a biking trip where he biked at 20 km/h for two hours and then 30 km/h for three hours. How far did Bill bike?
4. Factor: $2x(x^2 + 1) - 3(x^2 + 1)$
5. Simplify: $16^{\frac{1}{2}}$
6. Simplify: $-32m^{12} \div 8m^4$
7. Rationalize the denominator: $\dfrac{20}{\sqrt{3} + 1}$
8. Determine the length of the hypotenuse of a right-angled triangle when the other two sides measure 7 cm and 9 cm respectively.

Answers:

1. $45 \left(\dfrac{270}{(3 + 3)} = \dfrac{270}{6} = 45 \right)$
2. 2^5
3. 130 km (20 km/h for 2 hours = 40 km; 30 km/h for 3 hours = 90 km; 90 + 40 = 130 km)
4. $(2x - 3)(x^2 + 1)$
5. $4 \left(16^{\frac{1}{2}} = \sqrt{16} = 4 \right)$
6. $-4m^8 \left(\dfrac{-32m^{12}}{8m^4} = -4m^8 \right)$

7. $10\sqrt{3} - 10 \left(\dfrac{20\left(\sqrt{3}-1\right)}{\left(\sqrt{3}+1\right)\left(\sqrt{3}-1\right)} = \dfrac{20\sqrt{3}-20}{\sqrt{9}-1} = \dfrac{20\sqrt{3}-20}{2} \right)$

8. $\sqrt{130}$ cm $\left(\sqrt{7^2+9^2} = \sqrt{49+81} = \sqrt{130} \text{ cm} \right)$

Part B: Dividing Radical Expressions

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Simplify. Present your answer with no radicals in the denominators. If the expression contains variables, state for which values of the variables the radical expression is defined.

   a) $\dfrac{\sqrt{15}}{\sqrt{3}}$

   b) $\dfrac{\sqrt{2}}{2\sqrt{3}}$

   c) $\dfrac{10\sqrt{2}}{\sqrt{5}}$

   d) $\dfrac{\sqrt{2}+5}{\sqrt{3}}$

   e) $\dfrac{8\sqrt{10}-7\sqrt{5}}{2\sqrt{5}}$

   f) $\dfrac{7}{3-\sqrt{2}}$

   g) $\dfrac{\sqrt{5}}{9\sqrt{3}-4}$

   h) $\dfrac{\sqrt{7}-\sqrt{3}}{\sqrt{7}+\sqrt{3}}$

   i) $\dfrac{4-3\sqrt{3}}{6+\sqrt{2}}$

   j) $\dfrac{2\sqrt{3}-\sqrt{2}}{2\sqrt{3}+\sqrt{2}}$

   k) $\dfrac{\sqrt{x}}{\sqrt{x}-y}$

   l) $\dfrac{\sqrt{x}+1}{\sqrt{x}}$

   m) $\dfrac{\sqrt{x^2y^7}}{\sqrt{8y^6}}$

   n) $\dfrac{3-\sqrt{x}}{\sqrt{2x}-\sqrt{3}}$

   Answers:

   a) $\sqrt{\dfrac{15}{3}} = \sqrt{5}$

   b) $\dfrac{\sqrt{2}}{2\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{6}}{2(3)} = \dfrac{\sqrt{6}}{6}$

c) $\dfrac{10\sqrt{2}}{\sqrt{5}} \cdot \dfrac{\sqrt{5}}{\sqrt{5}} = \dfrac{10\sqrt{10}}{5} = 2\sqrt{10}$

d) $\dfrac{\sqrt{2}+5}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{6}+5\sqrt{3}}{3}$

e) $\dfrac{8\sqrt{10}-7\sqrt{5}}{2\sqrt{5}} \cdot \dfrac{\sqrt{5}}{\sqrt{5}} = \dfrac{8\sqrt{50}-7\sqrt{25}}{2\sqrt{25}}$

$= \dfrac{8\sqrt{25}\sqrt{2}-7(5)}{2(5)}$

$= \dfrac{8(5)\sqrt{2}-35}{10}$

$= \dfrac{40\sqrt{2}-35}{10}$

$= \dfrac{5\left(8\sqrt{2}-7\right)}{5(2)}$

$= \dfrac{8\sqrt{2}-7}{2}$

f) $\dfrac{7}{3-\sqrt{2}} \cdot \dfrac{3+\sqrt{2}}{3+\sqrt{2}} = \dfrac{21+7\sqrt{2}}{9+3\sqrt{2}-3\sqrt{2}-\sqrt{4}}$

$= \dfrac{21+7\sqrt{2}}{9-2}$

$= \dfrac{21+7\sqrt{2}}{7}$

$= \dfrac{7\left(3+\sqrt{2}\right)}{7}$

$= 3+\sqrt{2}$

g) $\dfrac{\sqrt{5}}{9\sqrt{3}-4} \cdot \dfrac{9\sqrt{3}+4}{9\sqrt{3}+4} = \dfrac{9\sqrt{15}+4\sqrt{5}}{81\sqrt{9}+36\sqrt{3}-36\sqrt{3}-16}$

$= \dfrac{9\sqrt{15}+4\sqrt{5}}{81(3)-16}$

$= \dfrac{9\sqrt{15}+4\sqrt{5}}{227}$

h) $\dfrac{\sqrt{7}-\sqrt{3}}{\sqrt{7}+\sqrt{3}} \cdot \dfrac{\sqrt{7}-\sqrt{3}}{\sqrt{7}-\sqrt{3}} = \dfrac{\sqrt{49}-\sqrt{21}-\sqrt{21}+\sqrt{9}}{\sqrt{49}-\sqrt{21}+\sqrt{21}-\sqrt{9}}$

$= \dfrac{7-2\sqrt{21}+3}{7-3}$

$= \dfrac{10-2\sqrt{21}}{4}$

$= \dfrac{2\left(5-\sqrt{21}\right)}{4}$

$= \dfrac{5-\sqrt{21}}{2}$

i) $\dfrac{4-3\sqrt{3}}{6+\sqrt{2}} \cdot \dfrac{6-\sqrt{2}}{6-\sqrt{2}} = \dfrac{24-4\sqrt{2}-18\sqrt{3}+3\sqrt{6}}{36-6\sqrt{2}+6\sqrt{2}-\sqrt{4}}$

$= \dfrac{24-4\sqrt{2}-18\sqrt{3}+3\sqrt{6}}{36-2}$

$= \dfrac{24-4\sqrt{2}-18\sqrt{3}+3\sqrt{6}}{34}$

j) $\dfrac{2\sqrt{3}-\sqrt{2}}{2\sqrt{3}+\sqrt{2}} \cdot \dfrac{2\sqrt{3}-\sqrt{2}}{2\sqrt{3}-\sqrt{2}} = \dfrac{4\sqrt{9}-2\sqrt{6}-2\sqrt{6}+\sqrt{4}}{4\sqrt{9}-2\sqrt{6}+2\sqrt{6}-\sqrt{4}}$

$= \dfrac{4(3)-4\sqrt{6}+2}{4(3)-2}$

$= \dfrac{14-4\sqrt{6}}{10}$

$= \dfrac{2\left(7-2\sqrt{6}\right)}{10}$

$= \dfrac{7-2\sqrt{6}}{5}$

k) $\dfrac{\sqrt{x}}{\sqrt{x}-y} \cdot \dfrac{\sqrt{x}+y}{\sqrt{x}+y} = \dfrac{\sqrt{x^2}+y\sqrt{x}}{\sqrt{x^2}+y\sqrt{x}-y\sqrt{x}-y^2}$

$= \dfrac{x+y\sqrt{x}}{x-y^2}$

This expression is only defined for $x \geq 0$. You cannot take the square root of a negative number. Also, $y \neq \sqrt{x}$ because the denominator cannot equal zero.

l) $\dfrac{\sqrt{x}+1}{\sqrt{x}} \cdot \dfrac{\sqrt{x}}{\sqrt{x}} = \dfrac{x+\sqrt{x}}{x}$

This expression is only defined for $x > 0$. You cannot take the square root of a negative number and the denominator cannot equal zero. Thus, x has to be greater than zero.

m) $$\frac{\sqrt{x^2y^7}}{\sqrt{8y^6}} = \sqrt{\frac{x^2y^7}{8y^6}}$$

$$= \sqrt{\frac{x^2y}{8}}$$

$$= \frac{\sqrt{x^2y}}{\sqrt{(4)(2)}}$$

$$= \frac{x\sqrt{y}}{2\sqrt{2}}$$

$$= \frac{x\sqrt{y}}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{x\sqrt{2y}}{2\left(\sqrt{4}\right)} = \frac{x\sqrt{2y}}{(2)(2)}$$

$$= \frac{x\sqrt{2y}}{4}$$

This expression is defined for $y > 0$ and $x \in \Re$. You cannot take the square root of a negative number nor have a zero in the denominator.

n) $$\frac{3-\sqrt{x}}{\sqrt{2x}-\sqrt{3}} = \frac{3-\sqrt{x}}{\sqrt{2x}-\sqrt{3}} \cdot \frac{\sqrt{2x}+\sqrt{3}}{\sqrt{2x}+\sqrt{3}}$$

$$= \frac{3\sqrt{2x}+3\sqrt{3}-\sqrt{2x^2}-\sqrt{3x}}{\sqrt{4x^2}+\sqrt{6x}-\sqrt{6x}-\sqrt{9}}$$

$$= \frac{3\sqrt{2x}+3\sqrt{3}-x\sqrt{2}-\sqrt{3x}}{2x-3}$$

This expression is only defined for $x \geq 0$ because you cannot take the square root of a negative number. Also, $x \neq \frac{3}{2}$ because the denominator cannot equal zero. Thus, the expression is defined for $x \geq 0$ and $x \neq \frac{3}{2}$.

2. The area of a rectangle is $\left(4+3\sqrt{3}\right)\text{m}^2$. If the length is $\left(\sqrt{3}+2\right)\text{m}$, find the

a) width

b) perimeter

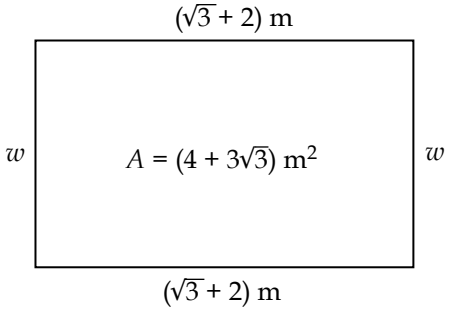
c) length of its diagonal

Answers:

a) Given: $A=\left(4+3\sqrt{3}\right)\text{m}^2$

$$l=\left(\sqrt{3}+2\right)\text{m}$$

Let w = width



$$A=lw$$

$$4+3\sqrt{3}=\left(\sqrt{3}+2\right)w$$

$$w=\frac{4+3\sqrt{3}}{\sqrt{3}+2}$$

$$w=\frac{4+3\sqrt{3}}{\sqrt{3}+2}\cdot\frac{\sqrt{3}-2}{\sqrt{3}-2}$$

$$=\frac{4\sqrt{3}-8+3\sqrt{9}-6\sqrt{3}}{\sqrt{9}-2\sqrt{3}+2\sqrt{3}-4}$$

$$=\frac{-2\sqrt{3}-8+3(3)}{3-4}$$

$$=\frac{-2\sqrt{3}+1}{-1}$$

$$=\frac{-2\sqrt{3}+1}{-1}\cdot\frac{(-1)}{(-1)}$$

$$=2\sqrt{3}-1$$

The width is $\left(2\sqrt{3}-1\right)\text{m}$.

b) Perimeter $= 2l + 2w$

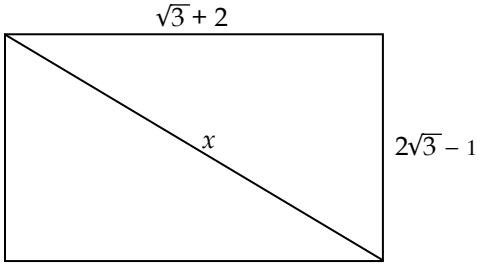
$$=2\left(\sqrt{3}+2\right)+2\left(2\sqrt{3}-1\right)$$

$$=2\sqrt{3}+4+4\sqrt{3}-2$$

$$=6\sqrt{3}+2$$

$\therefore$ perimeter $=\left(6\sqrt{3}+2\right)$ m

c) Let x = length of the diagonal.



By the Pythagorean Theorem:

$$x^2=\left(\sqrt{3}+2\right)^2+\left(2\sqrt{3}-1\right)^2$$

$$x^2=\left(\sqrt{3}+2\right)\left(\sqrt{3}+2\right)+\left(2\sqrt{3}-1\right)\left(2\sqrt{3}-1\right)$$

$$x^2=\sqrt{9}+2\sqrt{3}+2\sqrt{3}+4+4\sqrt{9}-2\sqrt{3}-2\sqrt{3}+1$$

$$x^2=3+2\sqrt{3}+2\sqrt{3}+4+12-2\sqrt{3}-2\sqrt{3}+1$$

$$x^2=20$$

$x=\pm\sqrt{20}$ but $-\sqrt{20}$ cannot be the length of a diagonal

$$x=\sqrt{20}=\sqrt{4}\sqrt{5}=2\sqrt{5}$$

The diagonal is $2\sqrt{5}$ m long.

3. Observing that both these numbers are positive, is it possible that they are equal?

$$\frac{\sqrt{2+\sqrt{3}}}{2};\ \frac{\sqrt{2}+\sqrt{6}}{4}$$

(**Hint:** Square both numbers and compare the results.)

Answer:

Let $x = \dfrac{\sqrt{2+\sqrt{3}}}{2}$ and $y = \dfrac{\sqrt{2}+\sqrt{6}}{4}$

$$x^2 = \frac{\sqrt{2+\sqrt{3}}}{2} \cdot \frac{\sqrt{2+\sqrt{3}}}{2}$$

$$x^2 = \frac{\sqrt{\left(2+\sqrt{3}\right)^2}}{4}$$

$$x^2 = \frac{2+\sqrt{3}}{4}$$

$$y^2 = \frac{\sqrt{2}+\sqrt{6}}{4} \cdot \frac{\sqrt{2}+\sqrt{6}}{4}$$

$$= \frac{\sqrt{4}+\sqrt{12}+\sqrt{12}+\sqrt{36}}{16}$$

$$= \frac{2+2\sqrt{12}+6}{16}$$

$$= \frac{8+2\sqrt{4}\sqrt{3}}{16}$$

$$= \frac{8+4\sqrt{3}}{16}$$

$$= \frac{4\left(2+\sqrt{3}\right)}{16}$$

$$= \frac{2+\sqrt{3}}{4}$$

Hence, $x^2 = y^2$.

Furthermore, since x and y are positive, it follows that $x = y$—that is, $\dfrac{\sqrt{2+\sqrt{3}}}{2} = \dfrac{\sqrt{2}+\sqrt{6}}{4}$.

Although some radicals do not look the same, they may still be equal.

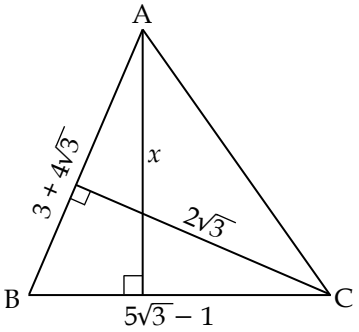
In the above question, it is important that x and y are both positive. Otherwise, it is possible that $x = -y$, since the squares of the positive and the negative numbers are equal (e.g., $(5)^2 = 25$ and $(-5)^2 = 25$. However, $5 \neq -5$.) Hence, x and y must both be positive numbers when you are comparing their squares.

4. Find the length of the altitude from A, if the length of the altitude from C is $2\sqrt{3}$.

Answer:

The altitude of a triangle is the height of the triangle. The formula for the area of a triangle is $A = \frac{1}{2}bh$.

Let x = height from A to BC.



Area = $\frac{1}{2}bh$ can be found using BC as the base or using AB as the base.

$\therefore$ Area $= \frac{1}{2}\left(5\sqrt{3}-1\right)x$ or

Area $= \frac{1}{2}\left(3+4\sqrt{3}\right)\left(2\sqrt{3}\right)$.

The two areas must be equal.

$$\frac{1}{2}\left(5\sqrt{3}-1\right)x = \frac{1}{2}\left(3+4\sqrt{3}\right)\left(2\sqrt{3}\right)$$

$$\left(5\sqrt{3}-1\right)x = \left(3+4\sqrt{3}\right)\left(2\sqrt{3}\right)$$ Divide both sides by $\frac{1}{2}$.

$$\left(5\sqrt{3}-1\right)x = 6\sqrt{3}+8\sqrt{9}$$ Multiply.

$$x = \frac{6\sqrt{3}+8(3)}{5\sqrt{3}-1}$$ Divide both sides by $5\sqrt{3}-1$.

$$x = \frac{6\sqrt{3}+24}{5\sqrt{3}-1}\cdot\frac{5\sqrt{3}+1}{5\sqrt{3}+1}$$ Rationalize the denominator.

$$x = \frac{30\sqrt{9}+6\sqrt{3}+120\sqrt{3}+24}{25\sqrt{9}+5\sqrt{3}-5\sqrt{3}-1}$$

$$x = \frac{30(3)+126\sqrt{3}+24}{25(3)-1}$$

$$x = \frac{114+126\sqrt{3}}{74}$$

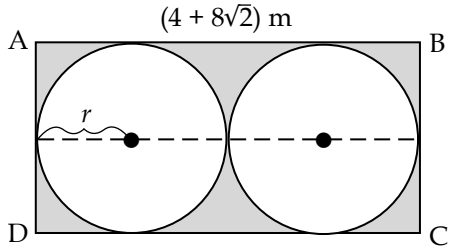
$$x = \frac{2\left(57+63\sqrt{3}\right)}{2(37)}$$

$$x = \frac{57+63\sqrt{3}}{37}$$

The length of the altitude from $A = \dfrac{57 + 63\sqrt{3}}{37}$.

5. Find the area of the shaded region if two identical circles are cut out of rectangle ABCD, as illustrated below. The length of $AB = \left(4 + 8\sqrt{2}\right)$ m.

   Leave your answer as an exact value in terms of π.



Answer:

Since $AB = 4 + 8\sqrt{2}$ and the circles are identical, the diameter of each circle is $\dfrac{4 + 8\sqrt{2}}{2} = 2 + 4\sqrt{2}$ and the radius of each circle is $\dfrac{2 + 4\sqrt{2}}{2} = 1 + 2\sqrt{2}$.

The width of the rectangle is $2 + 4\sqrt{2}$, which is the diameter of the circle.

Shaded area = area of rectangle $-$ areas of the two circles.

Area = $lw - 2(\pi r^2)$

$$A = \left(4 + 8\sqrt{2}\right)\left(2 + 4\sqrt{2}\right) - 2\pi\left(1 + 2\sqrt{2}\right)^2$$

$$A = \left(4 + 8\sqrt{2}\right)\left(2 + 4\sqrt{2}\right) - 2\pi\left(1 + 2\sqrt{2}\right)\left(1 + 2\sqrt{2}\right)$$

$$A = 8 + 16\sqrt{2} + 16\sqrt{2} + 32\sqrt{4} - 2\pi\left[1 + 2\sqrt{2} + 2\sqrt{2} + 4\sqrt{4}\right]$$

$$A = 8 + 32\sqrt{2} + 32(2) - 2\pi\left[1 + 4\sqrt{2} + 4(2)\right]$$

$$A = 72 + 32\sqrt{2} - 2\pi\left(9 + 4\sqrt{2}\right)$$

$A = 72 + 32\sqrt{2} - 18\pi - 8\pi\sqrt{2}$ (The answer is acceptable in this form.)

$$A = 72 - 18\pi + 32\sqrt{2} - 8\pi\sqrt{2}$$

$$A = (72 - 18\pi) + (32 - 8\pi)\sqrt{2}$$

The area of the shaded region is $\left[(72 - 18\pi) + (32 - 8\pi)\sqrt{2}\right]$ m^2.

6. What is the value of A if $\dfrac{2+\sqrt{3}}{\mathrm{A}} = 5-\sqrt{3}$?

Answer:

$$2+\sqrt{3} = \left(5-\sqrt{3}\right)A$$

$$A = \frac{2+\sqrt{3}}{5-\sqrt{3}}$$

$$A = \frac{2+\sqrt{3}}{5-\sqrt{3}} \cdot \frac{5+\sqrt{3}}{5+\sqrt{3}}$$

$$A = \frac{10+2\sqrt{3}+5\sqrt{3}+\sqrt{9}}{25+5\sqrt{3}-5\sqrt{3}-\sqrt{9}}$$

$$A = \frac{10+7\sqrt{3}+3}{25-3}$$

$$A = \frac{13+7\sqrt{3}}{22}$$

7. Simplify: $\dfrac{2}{\sqrt{3}-1} + \left(2+\sqrt{3}\right)\left(5-\sqrt{3}\right) - \left(2+\sqrt{3}\right)^2$

Answer:

Perhaps the easiest way to simplify such a complicated expression is to break it into parts. The first part needs its denominator rationalized; the other two parts require multiplication.

$$\frac{2}{\sqrt{3}-1} = \frac{2}{\sqrt{3}-1} \cdot \frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$= \frac{2\sqrt{3}+2}{\sqrt{9}+\sqrt{3}-\sqrt{3}-1}$$

$$= \frac{2\sqrt{3}+2}{2}$$

$$= \sqrt{3}+1$$

$$\left(2+\sqrt{3}\right)\left(5-\sqrt{3}\right) = 10-2\sqrt{3}+5\sqrt{3}-3$$

$$= 7+3\sqrt{3}$$

$$\left(2+\sqrt{3}\right)^2 = \left(2+\sqrt{3}\right)\left(2+\sqrt{3}\right)$$

$$= 4+2\sqrt{3}+2\sqrt{3}+3$$

$$= 7+4\sqrt{3}$$

Now combine the parts.

$$\frac{2}{\sqrt{3}-1}+\left(2+\sqrt{3}\right)\left(5-\sqrt{3}\right)+\left(2+\sqrt{3}\right)^2$$

$$=\left(\sqrt{3}+1\right)+\left(7+3\sqrt{3}\right)-\left(7+4\sqrt{3}\right)$$

$$=\sqrt{3}+1+7+3\sqrt{3}-7-4\sqrt{3}$$

$$=1$$

8. Write these numbers in descending order.

$$2,\ \frac{3}{\sqrt{5}},\text{ and }\sqrt{3}$$

Answer:

These numbers are all positive. Compare their squares.

$$2^2=4;\left(\frac{3}{\sqrt{5}}\right)^2=\frac{9}{5}=1\frac{4}{5};\left(\sqrt{3}\right)^2=3$$

Since $4>3>1\frac{4}{5}$, it follows that the positive numbers

$\sqrt{4}>\sqrt{3}>\sqrt{1\frac{4}{5}}$ – that is, $2>\sqrt{3}>\frac{3}{\sqrt{5}}$.

Learning Activity 5.5

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Maddox has 130 quarters, 72 dimes, and 26 nickels. How much money does he have in total?
2. Evaluate: $\frac{3}{5} \times \frac{10}{7}$
3. If today is Friday and your math exam is in 17 days, what day of the week is your math exam on?
4. Evaluate: $52 - (-3 - 4(-26 - (-28)))$.
5. Simplify: $\frac{3x^5y^2z^8}{9x^7y^3z^2}$
6. Estimate: $\sqrt{140}$
7. Trey and Tanisha are discussing their solutions to the linear inequality $-3x > 6$. Trey believes the answer is $x > -2$, while Tanisha believes the answer is $x < -2$. Who is right?
8. Simplify: $(x - 2)^3$

Answers:

1. \$41 (\$32.50 + \$7.20 + \$1.30 = \$41; 130 quarters = \$32.50; 72 dimes = \$7.20; 26 nickels = \$1.30)
2. $\frac{6}{7}\left(\frac{3}{\cancel{8}_1}, \frac{\cancel{10}^2}{7}\right)$
3. Monday (in 14 days it will be Friday again; three days past Friday is Monday)
4. $63\ \big(52 - \big(-3 - 4\big(-26 - (-28)\big)\big); 52 - \big(-3 - 4\big(-26 + 28\big)\big);$

   $52 - \big(-3 - 4(2)\big); 52 - (-3 - 8); 52 - (-11); 52 + 11 = 63\big)$
5. $\frac{z^6}{3x^2y}$
6. 11.8 (140 is between perfect squares 121 and 144; therefore, $\sqrt{140}$ is between 11 and 12 and is closer to 12)

7. Tanisha is right.

   $(-3x > 6$

   $x < -2$

   When you divide by a negative in an inequality, you need to switch the sign of the inequality.

   To show this is true, substitute a value of x less than -2 into the original inequality.

   Let $x = -4$
   LS = $-3x$
   $= -3(-4)$
   $= 12$
   RS = 6
   LS > RS
   True.)

8. $x^3 - 6x^2 + 12x - 8$ $[(x - 2)^3 = (x - 2)(x - 2)(x - 2) = (x^2 - 4x + 4)(x - 2)$
   $= x^3 - 2x^2 - 4x^2 + 8x + 4x - 8]$

Part B: Solving Radical Equations

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. State whether each statement is true or false. If the statement is false, rewrite it so that it is true.

   a) For each real number x, $\sqrt{x}$ is a real number only when $x \geq 0$.

   b) For each real number x, $\sqrt{x-3}$ is a real number only when $x \geq 0$.

   c) For each real number x, $\sqrt{x^2} = x$.

   d) The real number solution set of $\sqrt{x^2} = -3$ is $\{\sqrt{3}, -\sqrt{3}\}$.

   e) The equation $(x-1)^2 = \left(\sqrt{2x-1}\right)^2$ is equivalent to the sentence $(x-1) = \sqrt{2x-1}$ or $(x-1) = -\sqrt{2x-1}$.

   f) For each real number x, if $x^2 = 16$, then $x = \sqrt{16}$.

   g) Squaring both sides of $\sqrt{x} + 2 = 3x$ yields $x + 4 = 9x^2$.

Answers:

a) True.

b) False. For each real number x, $\sqrt{x-3}$ is a real number when $x \geq 3$.

c) False. For each real number x, $\sqrt{x^2} = |x|$ because $\sqrt{x^2}$ is the principal square root and is a positive number or zero. It is never negative. For example, if $x = -3$, then $\sqrt{(-3)^2} = |-3| = 3$.

d) False. The real number solution set of $\sqrt{x^2} = -3$ is the empty set. Saying the solution set is the empty set is the same as saying there is no solution to the problem. The principal square root is positive or zero, never negative.

e) True.

f) False. For each real number x, if $x^2 = 16$, then $x = \sqrt{16}$ or $x = -\sqrt{16}$.

g) False. Squaring both sides of $\sqrt{x} + 2 = 3x$ yields $x + 4\sqrt{x} + 4 = 9x^2$. The left side of the equation is a binomial. Its square is the product: $\left(\sqrt{x} + 2\right)\left(\sqrt{x} + 2\right)$.

2. Simplify each of the following expressions.

a) $\left(\sqrt{2x-1}\right)^2$

b) $\left(5 + \sqrt{x}\right)^2$

c) $\left(2 + \sqrt{x-5}\right)^2$

Answers:

a) $\left(\sqrt{2x-1}\right)^2 = 2x - 1$

b) $\left(5 + \sqrt{x}\right)^2 = 25 + 10\sqrt{x} + x$

c) $\left(2 + \sqrt{x-5}\right)^2 = \left(2 + \sqrt{x-5}\right)\left(2 + \sqrt{x-5}\right)$

$= 4 + 4\sqrt{x-5} + x - 5$

$= 4\sqrt{x-5} + x - 1$

3. Find the solutions for each of the following equations. Check your solutions for extraneous roots.

a) $x + 2 = \sqrt{2x + 7}$

b) $x = 2 - \sqrt{2x - 5}$

c) $\sqrt{2x + 3} - \sqrt{x + 1} = 1$

d) $\sqrt{x^2 - 3} + 1 = 0$

e) $x = \sqrt{3x - 2} + 2$

f) $\sqrt{\sqrt{x^2 + 6x}} = 2$

g) $\sqrt{3x + 2} = 3\sqrt{x} - \sqrt{2}$

h) $\sqrt{1 - x} + \sqrt{x} = \sqrt{x + 1}$

Answers:

a) $x + 2 = \sqrt{2x + 7}$

Solution:

$$(x + 2)^2 = \left(\sqrt{2x + 7}\right)^2$$

$$x^2 + 4x + 4 = 2x + 7$$

$$x^2 + 2x - 3 = 0$$

$$(x + 3)(x - 1) = 0$$

$$x = -3, 1$$

Check: $x + 2 = \sqrt{2x + 7}$

$x = -3$	$x = 1$
LHS $= x + 2$	LHS $= x + 2$
$= -3 + 2$	$= 1 + 2$
$= -1$	$= 3$
RHS $= \sqrt{2x + 7}$	RHS $= \sqrt{2x + 7}$
$= \sqrt{2(-3) + 7}$	$= \sqrt{2(1) + 7}$
$= \sqrt{1}$	$= \sqrt{9}$
$= 1$	$= 3$
LHS $\neq$ RHS	LHS $=$ RHS

$x = -3$ is not a solution. The only solution is $x = 1$.

b) $x = 2 - \sqrt{2x - 5}$

Solution:

$$(x - 2)^2 = \left(-\sqrt{2x - 5}\right)^2$$

$$x^2 - 4x + 4 = 2x - 5$$

$$x^2 - 6x + 9 = 0$$

$$(x - 3)^2 = 0$$

$$x = 3$$

Check: $x = 2 - \sqrt{2x - 5}$

$$x = 3$$

$$\text{LHS} = x$$

$$= 3$$

$$\text{RHS} = 2 - \sqrt{2x - 5}$$

$$= 2 - \sqrt{2(3) - 5}$$

$$= 2 - \sqrt{1}$$

$$= 2 - 1$$

$$= 1$$

$$\text{LHS} \neq \text{RHS}$$

There is no solution.

c) $\sqrt{2x+3} - \sqrt{x+1} = 1$

Solution:

$$\left(\sqrt{2x+3}\right)^2 = \left(\sqrt{x+1}+1\right)^2$$

$$2x+3 = x+1+2\sqrt{x+1}+1$$

$$2x+3 = x+2+2\sqrt{x+1}$$

$$x+1 = 2\sqrt{x+1}$$

$$(x+1)^2 = \left(2\sqrt{x+1}\right)^2$$

$$(x+1)^2 = 4(x+1)$$

$$x^2+2x+1 = 4x+4$$

$$x^2-2x-3 = 0$$

$$(x-3)(x+1) = 0$$

$$x = 3 \text{ or } -1$$

Check: $\sqrt{2x+3} - \sqrt{x+1} = 1$

$x = 3$

LHS $= \sqrt{2x+3} - \sqrt{x+1}$

$= \sqrt{2(3)+3} - \sqrt{3+1}$

$= \sqrt{9} - \sqrt{4}$

$= 3-2$

$= 1$

RHS $= 1$

LHS $=$ RHS

$x = -1$

LHS $= \sqrt{2x+3} - \sqrt{x+1}$

$= \sqrt{2(-1)+3} - \sqrt{-1+1}$

$= \sqrt{1} - \sqrt{0}$

$= 1$

RHS $= 1$

LHS $=$ RHS

There are two solutions, $x = 3$ or $x = -1$.

d) $\sqrt{x^2-3}+1=0$

Solution:

$$\left(\sqrt{x^2-3}\right)^2=(-1)^2$$

$$x^2-3=1$$

$$x^2=4$$

$$x=\pm 2$$

Check: $\sqrt{x^2-3}+1=0$

$x=2$

$\text{LHS}=\sqrt{x^2-3}+1$

$=\sqrt{2^2-3}+1$

$=\sqrt{1}+1$

$=2$

$\text{RHS}=0$

$\text{LHS}\neq\text{RHS}$

$x=-2$

$\text{LHS}=\sqrt{(-2)^2-3}+1$

$=\sqrt{4-3}+1$

$=2$

$\text{RHS}=0$

$\text{LHS}\neq\text{RHS}$

There is no solution.

Note: On the LHS of the initial question, the principal root of x^2-3, which is positive, is added to positive one. The result will never equal the RHS, which is zero. So the answer of no solution makes good sense.

e) $x = \sqrt{3x-2} + 2$

Solution:

$$(x-2)^2 = \left(\sqrt{3x-2}\right)^2$$

$$x^2 - 4x + 4 = 3x - 2$$

$$x^2 - 7x + 6 = 0$$

$$(x-6)(x-1) = 0$$

$$x = 6 \text{ or } 1$$

Check: $x = \sqrt{3x-2} + 2$

$x = 6$	$x = 1$
LHS $= x$	LHS $= x$
$= 6$	$= 1$
RHS $= \sqrt{3x-2} + 2$	RHS $= \sqrt{3x-2} + 2$
$= \sqrt{3(6)-2} + 2$	$= \sqrt{3(2)-2} + 2$
$= \sqrt{16} + 2$	$= \sqrt{1} + 2$
$= 4 + 2$	$= 3$
$= 6$	LHS $\neq$ RHS
LHS = RHS	

The only solution is $x = 6$.

f) $\sqrt{\sqrt{x^2+6x}} = 2$

Solution:

$$\left(\sqrt{\sqrt{x^2+6x}}\right)^2 = (2)^2$$

$$\sqrt{x^2+6x} = 4$$

$$\left(\sqrt{x^2+6x}\right)^2 = (4)^2$$

$$x^2+6x = 16$$

$$x^2+6x-16 = 0$$

$$(x+8)(x-2) = 0$$

$$x = -8 \text{ or } x = 2$$

Check: $\sqrt{\sqrt{x^2+6x}} = 2$

$x = -8$	$x = 2$
$\text{LHS} = \sqrt{\sqrt{x^2+6x}}$	$\text{LHS} = \sqrt{\sqrt{x^2+6x}}$
$= \sqrt{\sqrt{(-8)^2+6(-8)}}$	$= \sqrt{\sqrt{2^2+6(2)}}$
$= \sqrt{\sqrt{64-48}}$	$= \sqrt{\sqrt{4+12}}$
$= \sqrt{\sqrt{16}}$	$= \sqrt{\sqrt{16}}$
$= \sqrt{4}$	$= \sqrt{4}$
$= 2$	$= 2$
$\text{RHS} = 2$	$\text{RHS} = 2$
$\text{LHS} = \text{RHS}$	$\text{LHS} = \text{RHS}$

There are two solutions: $x = -8$ or $x = 2$.

g) $\sqrt{3x+2} = 3\sqrt{x} - \sqrt{2}$

Solution:

$$\left(\sqrt{3x+2}\right)^2 = \left(3\sqrt{x} - \sqrt{2}\right)^2$$

$$3x + 2 = 9x - 6\sqrt{2x} + 2$$

$$3x - 9x + 2 - 2 = -6\sqrt{2x}$$

$$\frac{-6x}{-6} = \frac{-6\sqrt{2x}}{-6}$$

$$x = \sqrt{2x}$$

$$(x)^2 = \left(\sqrt{2x}\right)^2$$

$$x^2 = 2x$$

$$x^2 - 2x = 0$$

$$x(x-2) = 0$$

$$x = 0 \text{ or } x - 2 = 0$$

$$x = 2$$

Check: $\sqrt{3x+2} = 3\sqrt{x} - \sqrt{2}$

$x = 0$

$\text{LHS} = \sqrt{3x+2}$

$= \sqrt{3(0)+2}$

$= \sqrt{2}$

$\text{RHS} = 3\sqrt{x} - \sqrt{2}$

$= 3\sqrt{0} - \sqrt{2}$

$= -\sqrt{2}$

$\text{LHS} \neq \text{RHS}$

$x = 2$

$\text{LHS} = \sqrt{3x+2}$

$= \sqrt{3(2)+2}$

$= \sqrt{8}$

$= 2\sqrt{2}$

$\text{RHS} = 3\sqrt{x} - \sqrt{2}$

$= 3\sqrt{2} - \sqrt{2}$

$= 2\sqrt{2}$

$\text{LHS} = \text{RHS}$

The only solution is $x = 2$.

h) $\sqrt{1-x}+\sqrt{x}=\sqrt{x+1}$

Solution:

$$\left(\sqrt{1-x}+\sqrt{x}\right)^2=\left(\sqrt{x+1}\right)^2$$

$$(1-x)+2\sqrt{x}\sqrt{1-x}+x=x+1$$

$$1-x+2\sqrt{x-x^2}+x=x+1$$

$$2\sqrt{x-x^2}=x$$

$$\left(2\sqrt{x-x^2}\right)^2=(x)^2$$

$$4\left(x-x^2\right)=x^2$$

$$4x-4x^2=x^2$$

$$4x-5x^2=0$$

$$x(4-5x)=0$$

$$x=0 \text{ or } 4-5x=0$$

$$x=\frac{4}{5}$$

Check: $\sqrt{1-x}+\sqrt{x}=\sqrt{x+1}$

$x=0$

$\text{LHS}=\sqrt{1-x}+\sqrt{x}$

$=\sqrt{1-0}+\sqrt{0}$

$=\sqrt{1}+0$

$=1$

$\text{RHS}=\sqrt{x+1}$

$=\sqrt{0+1}$

$=1$

LHS = RHS

$x=\frac{4}{5}$

$\text{LHS}=\sqrt{1-\frac{4}{5}}+\sqrt{\frac{4}{5}}$

$=\sqrt{\frac{1}{5}}+\sqrt{\frac{4}{5}}$

$=\frac{\sqrt{1}}{\sqrt{5}}+\frac{\sqrt{4}}{\sqrt{5}}$

$=\frac{1}{\sqrt{5}}+\frac{2}{\sqrt{5}}$

$=\frac{3}{\sqrt{5}}$

$\text{RHS}=\sqrt{x+1}$

$=\sqrt{\frac{4}{5}+1}$

$=\sqrt{\frac{4}{5}+\frac{5}{5}}$

$=\sqrt{\frac{9}{5}}$

$=\frac{3}{\sqrt{5}}$

LHS = RHS

There are two solutions: $x = 0$ or $x=\frac{4}{5}$.

4. Solve each of the following equations.

   a) $\sqrt[5]{x} = 4$

   b) $\sqrt[3]{x} = 0.5$

   c) $x^{\frac{2}{3}} = 81$

   d) $2x^{0.25} = 3$

   Answers:

   a) $\sqrt[5]{x} = 4$

   $\left(\sqrt[5]{x}\right)^5 = (4)^5$

   $x = 1024$

   Check for extraneous roots is not necessary; you are raising both sides to the fifth power, an odd number.

   b) $\sqrt[3]{x} = 0.5$

   $\left(\sqrt[3]{x}\right)^3 = (0.5)^3$

   $x = 0.125$

   Check for extraneous roots is not necessary; you are cubing both sides (an odd power).

   c) **Method 1:** Using Radicals

   $x^{\frac{2}{3}} = 81$

   $\sqrt[3]{x^2} = 81$

   $\left(\sqrt[3]{x^2}\right)^3 = (81)^3$

   $x^2 = 531\ 441$

   $\sqrt{x^2} = \pm\sqrt{531\ 441}$

   $x = \pm 729$

   There are two solutions: $x = 729$ or $x = -729$.

   A check for extraneous roots is not necessary since you are cubing both sides (an odd power).

Method 2: Using Rational Exponents

$$x^{\frac{2}{3}} = 81$$

$$\left(x^{\frac{2}{3}}\right)^{\frac{3}{2}} = (81)^{\frac{3}{2}}$$

$$x^{\left(\frac{2}{3}\right)\left(\frac{3}{2}\right)} = \left(\pm\sqrt{81}\right)^3 \quad \text{Take square root of both sides (don't forget the } \pm\text{).}$$

$$x^1 = (9)^3 \qquad \text{or} \qquad x^1 = (-9)^3$$

$$x = 729 \qquad \text{or} \qquad x = -729$$

Notice that with this method you need to be very careful since you are introducing the square root operation with the power of $\frac{1}{2}$. In line 3, you must consider $\pm\sqrt{81}$ inside the brackets, not just $\sqrt{81}$. Then you will get both answers.

d) $2x^{0.25} = 3$

$$\frac{2x^{\frac{1}{4}}}{2} = \frac{3}{2}$$

$$x^{\frac{1}{4}} = \frac{3}{2}$$

$$\sqrt[4]{x} = \frac{3}{2}$$

$$\left(\sqrt[4]{x}\right)^4 = \left(\frac{3}{2}\right)^4 \quad \text{Raise each side to the fourth power.}$$

$$x = \frac{81}{16}$$

A check for extraneous roots is necessary since part of the solution was to raise each side to an even power.

Check in the original equation $2x^{0.25} = 3$.

$$x = \frac{81}{16}$$

$$\text{LHS} = 2x^{0.25}$$

$$= 2\left(\frac{81}{16}\right)^{\frac{1}{4}}$$

$$= 2\sqrt[4]{\frac{81}{16}}$$

$$= (2)\left(\frac{3}{2}\right)$$

$$= 3$$

$$\text{RHS} = 3$$

$$\text{LHS} = \text{RHS}$$

The only solution is $x = \frac{81}{16}$.

5. Solve each of the following equations. Determine any restrictions on the variables.

   a) $\sqrt{x-3} = \sqrt{x} + 3$

   b) $\sqrt{x+2} = 4 - \sqrt{x+2}$

   Answers:

   a) $\sqrt{x-3} = \sqrt{x} + 3$

   This radical equation is only defined when $x - 3$ and x are greater than or equal to zero.

   $$x - 3 \geq 0$$

   $$x \geq 3 \text{ and } x \geq 0$$

   Thus, x has to be greater than or equal to both 3 and 0 in order for this equation to be defined. For both inequalities to be true, this is the same as saying $x \geq 3$ in order for this equation to be defined.

$$\sqrt{x-3} = \sqrt{x} + 3$$

$$\left(\sqrt{x-3}\right)^2 = \left(\sqrt{x} + 3\right)^2$$

$$x - 3 = x + 6\sqrt{x} + 9$$

$$x - 3 - x - 9 = 6\sqrt{x}$$

$$-12 = 6\sqrt{x}$$

$$-2 = \sqrt{x}$$

$$(-2)^2 = \left(\sqrt{x}\right)^2$$

$$4 = x$$

Since 4 is greater than or equal to 3, then $x = 4$ satisfies the restriction, $x \geq 3$.

Check $x = 4$ in the original equation $\sqrt{x-3} = \sqrt{x} + 3$.

LHS	RHS
$= \sqrt{x-3}$	$= \sqrt{x} + 3$
$= \sqrt{4-3}$	$= \sqrt{4} + 3$
$= 1$	$= 2 + 3$
	$= 5$

LHS $\neq$ RHS

Thus, there are no real solutions for this equation.

b) $\sqrt{x+2} = 4 - \sqrt{x+2}$

This radical equation is only defined when $x + 2$ is greater than or equal to zero.

$$x + 2 \geq 0$$
$$x \geq -2$$

Thus, this radical equation is only defined when $x \geq -2$.

$$\sqrt{x+2} = 4 - \sqrt{x+2}$$
$$2\sqrt{x+2} = 4$$
$$\left(2\sqrt{x+2}\right)^2 = 4^2$$
$$4(x+2) = 16$$
$$4x + 8 = 16$$
$$4x = 8$$
$$x = 2$$

$x = 2$ is greater than or equal to -2; thus, this is a possible solution.

Check $x = 2$ in $\sqrt{x+2} = 4 - \sqrt{x+2}$

$$\text{LHS} = \sqrt{x+2}$$
$$= \sqrt{2+2}$$
$$= \sqrt{4}$$
$$= 2$$
$$\text{RHS} = 4 - \sqrt{x+2}$$
$$= 4 - \sqrt{2+2}$$
$$= 4 - \sqrt{4}$$
$$= 4 - 2$$
$$= 2$$
$$\text{LHS} = \text{RHS}$$

Thus, the solution to this radical equation is $x = 2$.

6. Consider a cylindrical drink can. The radius of the can is 3 cm. The height of the can is 12 cm.

a) Find the volume of the pop can. Given: $r = \sqrt{\frac{V}{\pi h}}$. Round your answer to two decimal places. Check your answer for extraneous roots.

Answer:

You have the following information:

$r = 3$ cm

$h = 12$ cm

$$r = \sqrt{\frac{V}{\pi h}}$$

and the restrictions $V \geq 0$, $h > 0$.

$$r = \sqrt{\frac{V}{\pi h}}$$

$$3 = \sqrt{\frac{V}{12\pi}}$$

$$3^2 = \left(\sqrt{\frac{V}{12\pi}}\right)^2$$

$$9 = \frac{V}{12\pi}$$

$$108\pi = V$$

Check:

$$r = \sqrt{\frac{V}{\pi h}}$$

$$\text{RHS} = \sqrt{\frac{108\pi}{12\pi}}$$

$$= \sqrt{9}$$

$$= 3$$

$$\text{LHS} = 3$$

$$\text{LHS} = \text{RHS}$$

Thus, the volume of this pop can is $108\pi = 339.29$ cm^3.

b) If the volume of a typical pop can is 355 mL, how does your answer compare to that value?

Answer:

The volume of the pop can is 108π cm^3, which is approximately 339.29 cm^3. As one cm^3 is equivalent to one mL, the volume of the pop can is a bit smaller than the typical value of a pop can (355 mL).

7. Kinetic energy is calculated using the formula $E_k = \frac{1}{2}mv^2$, where E_k represents the kinetic energy of an object measured in joules, m represents the mass of an object measured in kilograms, and v represents the positive speed of an object. Speed is measured in metres per second. Using the formula, calculate the speed of a baseball that has a mass of 0.145 kg and 32.1 J (joules) of kinetic energy. Round your answer to two decimal places.

Answer:

You have the following information:

$m = 0.145$ kg

$E_k = 32.1$ J

Substitute those values into the equation.

$$E_k = \frac{1}{2}mv^2$$

$$32.1 = \frac{1}{2}(0.145)v^2$$

$$32.1 = 0.0725v^2$$

$$\frac{32.1}{0.0725} = v^2$$

$$v^2 = 442.7586207$$

$$\sqrt{v^2} = \pm\sqrt{442.7586207}$$

$$v = \pm 21.04$$

Since v is positive, use only the principal square root value. Thus, the speed of the baseball is 21.04 m/s.

Grade 11 Pre-Calculus Mathematics (30S)

Module 5
Radicals

Learning Activity Answer Keys

Module 5:
Radicals

Learning Activity 5.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Jenelle's doctor told her to eat an average of 2000 calories a day. If Jenelle ate 520 calories for breakfast, 630 calories for lunch, and had two snacks with 125 calories each, how many calories does Jenelle have left for supper?
2. Multiply: $(x + 3)(x + 4)$
3. Evaluate: $\sqrt{256}$
4. Solve for x: $(2x + 4)(x - 2) = 0$
5. Patrick sleeps for 8 hours a day, attends school for 6 hours a day, and plays sports for 2 hours a day. What fraction of the day (in lowest terms) does Patrick have for free time?
6. Order these values from lowest to highest: $\sqrt[3]{8}$, $2^{\frac{2}{4}}$, $2^{\frac{9}{3}}$, $2^{-\frac{1}{2}}$
7. Are these fractions equivalent? $\frac{1}{\sqrt{2}}$ and $\frac{\sqrt{2}}{2}$
8. Find the slope of the line perpendicular to the line, $3y - 2x + 7 = 0$.

Answers:

1. 600 $(2000 - (520 + 630 + 125 + 125) = 2000 - 1400 = 600)$
2. $x^2 + 7x + 12$ $(x^2 + 4x + 3x + 12)$
3. 16 $\left(\sqrt{2 \times 128} = \sqrt{2 \times 2 \times 64} = \sqrt{4} \times \sqrt{64} = 2 \times 8\right)$
4. -2 and 2
5. $\frac{1}{3}$ $\left(24 - (8 + 6 + 2) = 24 - 16 = 8; \frac{8}{24} = \frac{1}{3}\right)$

6. $\frac{1}{\sqrt{2}} < \sqrt{2} < 2 < 8$ or $2^{-\frac{1}{2}} < 2^{\frac{2}{4}} < \sqrt[3]{8} < 2^{\frac{9}{3}}$

$\left(\sqrt[3]{8} = 2;\ 2^{\frac{2}{4}} = 2^{\frac{1}{2}} = \sqrt{2};\ 2^{\frac{9}{3}} = 2^3 = 8;\ 2^{-\frac{1}{2}} = \frac{1}{2^{\frac{1}{2}}} = \frac{1}{\sqrt{2}}\right)$

7. Yes $\left(\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{\sqrt{2}}{2}\right)$

8. The slope of the perpendicular line is $-\frac{3}{2}$.

$\left(3y = 2x - 7;\ y = \frac{2}{3}x - \frac{7}{3};\ \text{the slope of this line is } \frac{2}{3}\right)$

Part B: Simplifying Radical Expressions

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Simplify each radical.

a) $\sqrt{80}$

b) $\sqrt{18}$

c) $\sqrt{200}$

d) $\sqrt{250}$

e) $\sqrt{1224}$

f) $\sqrt{288}$

g) $\sqrt{512}$

h) $-\sqrt{8}$

i) $-\sqrt{800}$

Answers:

a) $\sqrt{16}\sqrt{5} = 4\sqrt{5}$

b) $\sqrt{9}\sqrt{2} = 3\sqrt{2}$

c) $\sqrt{100}\sqrt{2} = 10\sqrt{2}$

d) $\sqrt{25}\sqrt{10} = 5\sqrt{10}$

e) $\sqrt{36}\sqrt{34} = 6\sqrt{34}$

f) $\sqrt{144}\sqrt{2} = 12\sqrt{2}$

g) $\sqrt{256}\sqrt{2} = 16\sqrt{2}$

h) $-\sqrt{4}\sqrt{2} = -2\sqrt{2}$

i) $-\sqrt{400}\sqrt{2} = -20\sqrt{2}$

2. Simplify each radical, given $x \geq 0$, $y \geq 0$, and $z \geq 0$.

   a) $\sqrt{4x^4y^6}$

   b) $\sqrt{8x^{10}y^{12}}$

   c) $\sqrt{75x^8y^5}$

   d) $\sqrt{x^4yz^2}$

   e) $\sqrt{5x^3y^7}$

   Answers:

   a) $\sqrt{4}\sqrt{x^4}\sqrt{y^6} = 2x^2y^3$

   b) $\sqrt{4}\sqrt{2}\sqrt{x^{10}}\sqrt{y^{12}} = 2x^5y^6\sqrt{2}$

   c) $\sqrt{25}\sqrt{3}\sqrt{x^8}\sqrt{y^4}\sqrt{y} = 5x^4y^2\sqrt{3y}$

   d) $\sqrt{x^4}\sqrt{y}\sqrt{z^2} = x^2z\sqrt{y}$

   e) $\sqrt{5}\sqrt{x^2}\sqrt{x}\sqrt{y^6}\sqrt{y} = xy^3\sqrt{5xy}$

3. Write each mixed radical as an entire radical.

   a) $3\sqrt{15}$

   b) $3x\sqrt{5x}$

   c) $5x^2y\sqrt{3xy}$

   Answers:

   a) $\sqrt{9}\sqrt{15} = \sqrt{135}$

   b) $\sqrt{9}\sqrt{x^2}\sqrt{5x} = \sqrt{45x^3}$

   c) $\sqrt{25}\sqrt{x^4}\sqrt{y^2}\sqrt{3xy} = \sqrt{75x^5y^3}$

4. Identify the greater number. (Do not use a calculator.)

   a) $\sqrt{125}$ or $3\sqrt{20}$?

   Answer:

   $\sqrt{125} = \sqrt{25}\sqrt{5} = 5\sqrt{5}$

   $3\sqrt{20} = 3\sqrt{4}\sqrt{5} = 3(2)\sqrt{5} = 6\sqrt{5}$

   As $6\sqrt{5} > 5\sqrt{5}$, then $3\sqrt{20} > \sqrt{125}$.

   b) $16\sqrt{2}$ or $\sqrt{768}$?

   Answer:

   $\sqrt{768} = \sqrt{256}\sqrt{3} = 16\sqrt{3}$

   As $\sqrt{3} > \sqrt{2}$, then $16\sqrt{3} > 16\sqrt{2}$ and $\sqrt{768} > 16\sqrt{2}$.

5. Order the following radical expressions from least to greatest. (Do not use a calculator.)

   a) $5\sqrt{3}, \sqrt{49}, 6\sqrt{5}$

   Answer:

   $5\sqrt{3} = \sqrt{25}\sqrt{3} = \sqrt{75}$

   $\sqrt{49} = \sqrt{49}$

   $6\sqrt{5} = \sqrt{36}\sqrt{5} = \sqrt{180}$

   Thus, $\sqrt{49} < \sqrt{75} < \sqrt{180}$ and $\sqrt{49} < 5\sqrt{3} < 6\sqrt{5}$.

   b) $3\sqrt{2}, 6\sqrt{2}, \sqrt{32}$

   Answer:

   $\sqrt{32} = \sqrt{16}\sqrt{2} = 4\sqrt{2}$

   Thus, $3\sqrt{2} < 4\sqrt{2} < 6\sqrt{2}$ and $3\sqrt{2} < \sqrt{32} < 6\sqrt{2}$.

6. Solve for x.

   a) $x^2 = 25$
   b) $x^2 = 16$
   c) $x^2 = 48$
   d) $x^2 = 0$
   e) $x^2 = -9$

   Answers:

   a) $x = \pm 5$
   b) $x = \pm 4$
   c) $x = \pm\sqrt{48} = \pm\sqrt{16}\sqrt{3} = \pm 4\sqrt{3}$
   d) $x = \sqrt{0} = 0$
   e) $x = \pm\sqrt{-9}$ = no real number answer

7. By converting the radical to a rational exponent, prove the First Property of Radicals:

   $\sqrt[n]{a^n} = a$ when n is a positive integer greater than 1.

   Answer:

   $\sqrt[n]{a^n}$ can be written using a rational exponent as $a^{\frac{n}{n}}$.

   Since n is a positive integer greater than one, and $a^{\frac{n}{n}} = a^1 = a$,

   thus, $\sqrt[n]{a^n} = a$ when $n > 1$.

8. Identify the values of the variable(s) for which the radical represents a real number.

a) $\sqrt{8-2x}$

Answer:

A square root is only defined when the radicand is greater than or equal to zero. You will need to solve a simple inequality.

$8 - 2x \geq 0$

$8 \geq 2x$

$4 \geq x$

Thus, this radical expression is defined when $x \leq 4$.

b) $\sqrt{x^3}$

Answer:

$\sqrt{x^3}$ has an index of 2. Therefore, as the index is even, this radical expression is only defined when $x^3 \geq 0$ and, thus, $x \geq 0$.

c) $\sqrt[5]{z}$

Answer:

$\sqrt[5]{z}$ has an index of 5. Therefore, as the index is odd, this radical expression is defined for $z \in \Re$.

d) $\sqrt[4]{x^7}$

Answer:

$\sqrt[4]{x^7}$ has an index of 4. Therefore, as the index is even, this radical expression is only defined when $x^7 \geq 0$ and, thus, $x \geq 0$.

e) $\sqrt{64a^3b^6}$

Answer:

$\sqrt{64a^3b^6}$ has an index of 2. Therefore, as the index is even, this radical expression is only defined when $64a^3b^6 \geq 0$. Since the exponent on a is odd, then a must be 0 or positive. However, b can be any value, $b \in \Re$, as b has an even exponent. Thus, this expression is defined when $a \geq 0$ and $b \in \Re$.

f) $\sqrt{-432xy^2}$

Answer:

To answer this question, you have to consider the radicand to be positive. Since the coefficient, -432, is negative, then one of the variables has to be negative. Since y^2 will never be negative, x has to be negative. Also, x or y could be zero. Thus, $x \leq 0$ and $y \in \Re$.

Learning Activity 5.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Alyssa is twice as old as her brother today. When she is 25 years old, her brother will be 20 years old. How many years old is Alyssa today?
2. How much would a sweater cost if it were 25% off the original price of $60 and you had to pay an additional 12% sales tax on the discounted price?
3. Simplify: $\left(\sqrt[3]{2x + 10}\right)^0$
4. Simplify: $2x(3x - 5) - 4(x + 2)$
5. Write 625 as a power with a base of 5.
6. List the factors of 68.
7. The equation of a parabola is, $y = 3x^2 - 5x + 8$. What is the y-intercept?
8. Find the slope of the line that contains the points $(7, -1)$ and $(-8, 15)$.

Answers:

1. Alyssa is 10. (Since Alyssa is 5 years older, she is twice as old when her brother is 5 and she is 10.)
2. $50.40 in total ($60 × 25% = $15; the sweater costs $60 − $15 = $45 without tax; 10% of $45 = $4.50; 1% of $45 = $0.45; 12% of $45 = $4.50 + $0.45 + $0.45 = $5.40; $45 + $5.40 = $50.40)
3. 1 (any number to the power of zero is 1)
4. $6x^2 - 10x - 4x - 8 = 6x^2 - 14x - 8$
5. 5^4 ($625 = 25 \times 35 = 5^2 \times 5^2$)
6. 1, 2, 4, 17, 34, 68 (since $68 = 2 \times 2 \times 17$)
7. 8 (let $x = 0$)
8. $-\frac{16}{15}$ $\left(m = \frac{15 - (-1)}{-8 - 7} = \frac{16}{-15} = -\frac{16}{15}\right)$

Part B: Adding and Subtracting Radical Expressions

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Simplify each radical expression as much as possible. Given: $x \geq 0$.

   a) $2\sqrt{3} + 4\sqrt{3} - \sqrt{3}$

   b) $\sqrt{96} - \sqrt{54} - \sqrt{24}$

   c) $3\sqrt{75} - 2\sqrt{48} + \sqrt{12}$

   d) $\sqrt{98} - \sqrt{50} + \sqrt{72}$

   e) $5\sqrt{x} - 3\sqrt{x} + 6\sqrt{x}$

   f) $6\sqrt{5} + 3\sqrt{2} - 4\sqrt{5} + \sqrt{2}$

   g) $\sqrt{x^5} + 3x^2\sqrt{x} - 7x\sqrt{4x^3}$

   h) $2x\sqrt[3]{8x^4} - 4x^2\sqrt[3]{x} + \sqrt[3]{125x^7}$

   Answers:

   a) $5\sqrt{3}$

   b) $\sqrt{16}\sqrt{6} - \sqrt{9}\sqrt{6} - \sqrt{4}\sqrt{6} = 4\sqrt{6} - 3\sqrt{6} - 2\sqrt{6} = -\sqrt{6}$

   c) $3\sqrt{25}\sqrt{3} - 2\sqrt{16}\sqrt{3} + \sqrt{4}\sqrt{3}$
   $= 3(5)\sqrt{3} - 2(4)\sqrt{3} + 2\sqrt{3}$
   $= 15\sqrt{3} - 8\sqrt{3} + 2\sqrt{3}$
   $= 9\sqrt{3}$

   d) $\sqrt{49}\sqrt{2} - \sqrt{25}\sqrt{2} + \sqrt{36}\sqrt{2}$
   $= 7\sqrt{2} - 5\sqrt{2} + 6\sqrt{2}$
   $= 8\sqrt{2}$

   e) $8\sqrt{x}$

   f) $2\sqrt{5} + 4\sqrt{2}$

g) $\sqrt{x^5} + 3x^2\sqrt{x} - 7x\sqrt{4x^3}$

$= \sqrt{x^4 x} + 3x^2\sqrt{x} - 7x\sqrt{4x^2 x}$

$= x^2\sqrt{x} + 3x^2\sqrt{x} - 7x(2)(x)\sqrt{x}$

$= x^2\sqrt{x} + 3x^2\sqrt{x} - 14x^2\sqrt{x}$

$= -10x^2\sqrt{x}$

h) $2x\sqrt[3]{8x^4} - 4x^2\sqrt[3]{x} + \sqrt[3]{125x^7}$

$= 2x\sqrt[3]{8x^3 x} - 4x^2\sqrt[3]{x} + \sqrt[3]{125x^6 x}$

$= 2x(2)(x)\sqrt[3]{x} - 4x^2\sqrt[3]{x} + 5x^2\sqrt[3]{x}$

$= 4x^2\sqrt[3]{x} - 4x^2\sqrt[3]{x} + 5x^2\sqrt[3]{x}$

$= 5x^2\sqrt[3]{x}$

2. Find the values of the missing numerical coefficients A and B.

a) $3\sqrt{2} + 2\sqrt{3} + A\sqrt{12} - B\sqrt{8} = \sqrt{75} - \sqrt{32}$

Answer:

$3\sqrt{2} + 2\sqrt{3} + A\sqrt{4}\sqrt{3} - B\sqrt{4}\sqrt{2} = \sqrt{25}\sqrt{3} - \sqrt{16}\sqrt{2}$

$3\sqrt{2} + 2\sqrt{3} + 2A\sqrt{3} - 2B\sqrt{2} = 5\sqrt{3} - 4\sqrt{2}$

$2A\sqrt{3} - 2B\sqrt{2} = 5\sqrt{3} - 4\sqrt{2} - 3\sqrt{2} - 2\sqrt{3}$

$2A\sqrt{3} - 2B\sqrt{2} = 3\sqrt{3} - 7\sqrt{2}$

$\therefore 2A = 3$ and $-2B = -7$ Equate coefficients of like radicals.

$\therefore A = \frac{3}{2}$ and $B = \frac{7}{2}$

b) $\sqrt{24} + A\sqrt{2} + B\sqrt{150} = 3\sqrt{2}$

Answer:

$\sqrt{4}\sqrt{6} + A\sqrt{2} + B\sqrt{25}\sqrt{6} = 3\sqrt{2}$

$2\sqrt{6} + A\sqrt{2} + 5B\sqrt{6} = 3\sqrt{2}$

$A\sqrt{2} + 5B\sqrt{6} = 3\sqrt{2} - 2\sqrt{6}$

$\therefore A = 3$ and $5B = -2$ Equate coefficients of like radicals.

$B = -\frac{2}{5}$

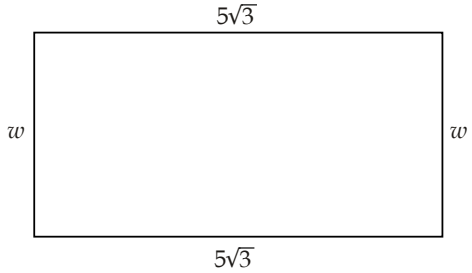
3. Find the width of a rectangle if the length is $5\sqrt{3}$ cm and the perimeter is $\left(16\sqrt{3} - 4\sqrt{2}\right)$ cm.

Answer:

Let w = the width of the rectangle.

Given: length = $5\sqrt{3}$ cm

perimeter = $\left(16\sqrt{3} - 4\sqrt{2}\right)$ cm



Perimeter $= 2l + 2w$

$16\sqrt{3} - 4\sqrt{2} = 2\left(5\sqrt{3}\right) + 2w$

$16\sqrt{3} - 4\sqrt{2} = 10\sqrt{3} + 2w$

$16\sqrt{3} - 10\sqrt{3} - 4\sqrt{2} = 2w$

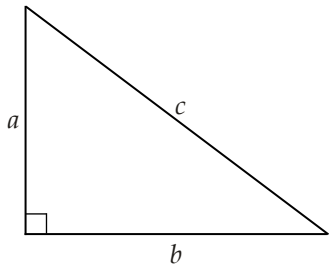
$6\sqrt{3} - 4\sqrt{2} = 2w$

$\frac{6\sqrt{3}}{2} - \frac{4\sqrt{2}}{2} = w$

$3\sqrt{3} - 2\sqrt{2} = w$

The width is $\left(3\sqrt{3} - 2\sqrt{2}\right)$ cm.

4. Given the lengths of two sides of the right triangle, find the length of the third side.



a) $a = 3$ $b = 4$

b) $a = 3$ $b = 5$

c) $a = 15$ $c = 17$

Answers:

a) $a^2 + b^2 = c^2$ (Pythagorean theorem)

$3^2 + 4^2 = c^2$

$25 = c^2$

$5 = c$ (any length is assumed to be *positive*)

b) $a^2 + b^2 = c^2$

$3^2 + 5^2 = c^2$

$9 + 25 = c^2$

$34 = c^2$

$\sqrt{34} = c$

c) $a^2 + b^2 = c^2$

$15^2 + b^2 = 17^2$

$225 + b^2 = 289$

$b^2 = 64$

$b = 8$

5. Why is it important to simplify radical expressions before adding or subtracting? Provide an example to support your explanation.

   Answer:

   Radical expressions may appear to not be like terms until they are simplified. It is therefore necessary to simplify radical expressions before adding or subtracting them to ensure you arrive at the answer in simplest form.

   Consider the following expression.

   $\sqrt{96} - \sqrt{54}$

   These radical expressions are not like terms; therefore, it appears that they cannot be combined, as you cannot subtract radicals that are not like terms. However, if you simplify each radical, the resulting radicals become like terms.

   $\sqrt{96} - \sqrt{54}$

   $= \sqrt{(16)(6)} - \sqrt{(9)(6)}$

   $= 4\sqrt{6} - 3\sqrt{6}$

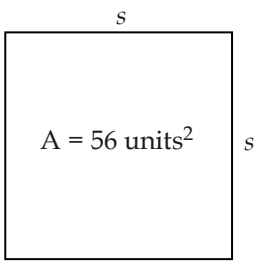
   It is only after simplifying that it is clear that the radical expressions are like terms that can be subtracted.

   $4\sqrt{6} - 3\sqrt{6}$

   $= \sqrt{6}$

6. The perimeter of a square is $8\sqrt{2x}$ units. The area of a square is 56 units^2. Determine the value of x, if $x > 0$.

Answer:



$P = 8\sqrt{2x}$ units

s = length of side

$A = s^2$

$56 = s^2$

$\sqrt{56} = s$

$\sqrt{4}\sqrt{14} = s$

$2\sqrt{14} = s$

If the area of the square is 56 units^2, then the length of one side is $2\sqrt{14}$ units.

Therefore, the perimeter = $(4)\left(2\sqrt{14}\right) = 8\sqrt{14}$.

Thus, $8\sqrt{14}$ has to equal $8\sqrt{2x}$.

$8\sqrt{14} = 8\sqrt{2x}$

$\sqrt{14} = \sqrt{2x}$ Divide by 8.

$\left(\sqrt{14}\right)^2 = \left(\sqrt{2x}\right)^2$

$14 = 2x$

$7 = x$

Learning Activity 5.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. School starts at 8:45 a.m. Hunter spends twenty minutes eating breakfast, fifteen minutes taking a shower, thirty-five minutes watching TV, and ten minutes biking to school. If Hunter arrives just on time for school, what time did he wake up?
2. Simplify: $\left(x + \sqrt{3}\right)^2$
3. Express 0.72 as a fraction in lowest terms.
4. Find the x-intercepts of the following parabola: $y = x^2 + 9x + 14$
5. If three angles in a parallelogram measure 286° in total, what is the measure of the fourth angle in the parallelogram?
6. Determine the distance between the following two points, (9, 24) and (4, 12)
7. Express as a power: $\sqrt[4]{5^8}$
8. Write this percent as a decimal: 3.2%.

Answers:

1. 7:25 a.m. (8:45 − (20 + 15 + 35 + 10) = 8:45 − 80 minutes = 8:45 − 1 hour and 20 minutes = 7:25)
2. $x^2 + 2x\sqrt{3} + 3$ $\left(\left(x + \sqrt{3}\right)\left(x + \sqrt{3}\right) = x^2 + x\sqrt{3} + x\sqrt{3} + \sqrt{9}\right)$
3. $\frac{18}{25}$ $\left(\frac{72}{100} = \frac{36}{50} = \frac{18}{25}\right)$
4. −7 and −2 ($(x + 7)(x + 2) = 0$)
5. 74° (360° − 286°)
6. 13 $\left(d = \sqrt{(-5)^2 + (-12)^2} = \sqrt{25 + 144} = \sqrt{169} = 13\right)$
7. $\sqrt[4]{5^8} = 5^{\frac{8}{4}} = 5^2$
8. 0.032

Part B: Multiplying Radical Expressions

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Simplify each radical expression. Given: $x \geq 0$, $y \geq 0$.

a) $\left(\sqrt{5}\right)\left(\sqrt{3}\right)$

b) $\left(\sqrt{8}\right)\left(\sqrt{2}\right)$

c) $\left(4\sqrt{10}\right)\left(3\sqrt{5}\right)$

d) $\left(2\sqrt{5}\right)\left(-3\sqrt{5}\right)$

e) $\left(-3\sqrt{3}\right)\left(-4\sqrt{6}\right)$

f) $\left(2\sqrt{x}\right)\left(3\sqrt{xy^2}\right)$

g) $\sqrt{3}\left(\sqrt{2}-\sqrt{5}\right)$

h) $-\sqrt{2}\left(3\sqrt{2}+3\sqrt{5}\right)$

i) $\sqrt{6}\left(3\sqrt{2}+5\sqrt{8}\right)$

j) $\left(2\sqrt{5}+3\right)\left(2\sqrt{5}-3\right)$

k) $\left(6\sqrt{5}-3\sqrt{10}\right)\left(4\sqrt{5}+\sqrt{10}\right)$

l) $\left(\sqrt{2}-\sqrt{3}\right)^2$

m) $\left(4\sqrt{7}+3\sqrt{2}\right)^2$

n) $\left(\sqrt{x}+\sqrt{y^3}\right)\left(\sqrt{xy}\right)$

o) $\left(2\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}-3\sqrt{y}\right)$

Answers:

a) $\left(\sqrt{5}\right)\left(\sqrt{3}\right) = \sqrt{15}$

b) $\sqrt{8}\sqrt{2} = \sqrt{16} = 4$

c) $\begin{aligned}\left(4\sqrt{10}\right)\left(3\sqrt{5}\right) &= 12\sqrt{50} \\ &= 12\sqrt{25}\sqrt{2} \\ &= 12(5)\sqrt{2} \\ &= 60\sqrt{2}\end{aligned}$

d) $\begin{aligned}\left(2\sqrt{5}\right)\left(-3\sqrt{5}\right) &= -6\sqrt{25} \\ &= -6(5) \\ &= -30\end{aligned}$

e) $\begin{aligned}\left(-3\sqrt{3}\right)\left(-4\sqrt{6}\right) &= 12\sqrt{18} \\ &= 12\sqrt{9}\sqrt{2} \\ &= 12(3)\sqrt{2} \\ &= 36\sqrt{2}\end{aligned}$

f) $\begin{aligned}\left(2\sqrt{x}\right)\left(3\sqrt{xy^2}\right) &= 6\sqrt{x^2y^2} \\ &= 6xy\end{aligned}$

g) $\sqrt{3}\left(\sqrt{2}-\sqrt{5}\right) = \sqrt{6}-\sqrt{15}$

h) $-\sqrt{2}\left(3\sqrt{2}+3\sqrt{5}\right) = -3\sqrt{4}-3\sqrt{10}$

$= -3(2)-3\sqrt{10}$

$= -6-3\sqrt{10}$

i) $\sqrt{6}\left(3\sqrt{2}+5\sqrt{8}\right) = 3\sqrt{12}+5\sqrt{48}$

$= 3\sqrt{4}\sqrt{3}+5\sqrt{16}\sqrt{3}$

$= 3(2)\sqrt{3}+5(4)\sqrt{3}$

$= 6\sqrt{3}+20\sqrt{3}$

$= 26\sqrt{3}$

j) $\left(2\sqrt{5}+3\right)\left(2\sqrt{5}-3\right) = 2\sqrt{5}\left(2\sqrt{5}-3\right)+3\left(2\sqrt{5}-3\right)$

$= 4\sqrt{25}-6\sqrt{5}+6\sqrt{5}-9$

$= 4(5)-9$

$= 20-9$

$= 11$

k) $\left(6\sqrt{5}-3\sqrt{10}\right)\left(4\sqrt{5}+\sqrt{10}\right) = 6\sqrt{5}\left(4\sqrt{5}+\sqrt{10}\right)-3\sqrt{10}\left(4\sqrt{5}+\sqrt{10}\right)$

$= 24\sqrt{25}+6\sqrt{50}-12\sqrt{50}-3\sqrt{100}$

$= 24(5)-6\sqrt{50}-3(10)$

$= 120-6\sqrt{25}\sqrt{2}-30$

$= 120-6(5)\sqrt{2}-30$

$= 90-30\sqrt{2}$

l) $\left(\sqrt{2}-\sqrt{3}\right)^2 = \left(\sqrt{2}-\sqrt{3}\right)\left(\sqrt{2}-\sqrt{3}\right)$

$= \sqrt{2}\left(\sqrt{2}-\sqrt{3}\right)-\sqrt{3}\left(\sqrt{2}-\sqrt{3}\right)$

$= \sqrt{4}-\sqrt{6}-\sqrt{6}+\sqrt{9}$

$= 2-2\sqrt{6}+3$

$= 5-2\sqrt{6}$

m) $\left(4\sqrt{7}+3\sqrt{2}\right)^2 = \left(4\sqrt{7}+3\sqrt{2}\right)\left(4\sqrt{7}+3\sqrt{2}\right)$

$= 4\sqrt{7}\left(4\sqrt{7}+3\sqrt{2}\right)+3\sqrt{2}\left(4\sqrt{7}+3\sqrt{2}\right)$

$= 16\sqrt{49}+12\sqrt{14}+12\sqrt{14}+9\sqrt{4}$

$= 16(7)+24\sqrt{14}+9(2)$

$= 112+24\sqrt{14}+18$

$= 130+24\sqrt{14}$

n) $\left(\sqrt{x}+\sqrt{y^3}\right)\left(\sqrt{xy}\right) = \sqrt{x^2y}+\sqrt{xy^4}$

$= x\sqrt{y}+y^2\sqrt{x}$

o) $\left(2\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}-3\sqrt{y}\right) = 2\sqrt{x}\left(\sqrt{x}-3\sqrt{y}\right)-\sqrt{y}\left(\sqrt{x}-3\sqrt{y}\right)$

$= 2\sqrt{x^2}-6\sqrt{xy}-\sqrt{xy}+3\sqrt{y^2}$

$= 2x-7\sqrt{xy}+3y$

2. Find the exact value of the area of a square if its perimeter is $\left(4\sqrt{2}+8\right)$ cm.

Answer:

Let s = length of side of square.

$P = 4s$

$P = \left(4\sqrt{2}+8\right)$ cm

$\therefore 4s = \left(4\sqrt{2}+8\right)$

$\frac{4s}{4} = \frac{4\sqrt{2}}{4}+\frac{8}{4}$

$s = \sqrt{2}+2$

Area $= A = s^2$

$A = \left(\sqrt{2}+2\right)^2$

$A = \left(\sqrt{2}+2\right)\left(\sqrt{2}+2\right)$

$= \sqrt{2}\left(\sqrt{2}+2\right)+2\left(\sqrt{2}+2\right)$

$= \sqrt{4}+2\sqrt{2}+2\sqrt{2}+4$

$= 2+4\sqrt{2}+4$

$= 6+4\sqrt{2}$

The exact value of the area of the square is $\left(6+4\sqrt{2}\right)$ cm^2.

Note: The exact value cannot be found using a calculator.

3. Identify the greater number:

a) $2\sqrt{80}$ or $5\sqrt{8}\left(2\sqrt{3}\right)$?

Answer:

$$\left(5\sqrt{8}\right)\left(2\sqrt{3}\right) = 10\sqrt{24}$$
$$= \sqrt{10^2}\sqrt{24}$$
$$= \sqrt{(100)(24)}$$
$$= \sqrt{2400}$$

$$2\sqrt{80} = \sqrt{2^2}\sqrt{80}$$
$$= \sqrt{(4)(80)}$$
$$= \sqrt{320}$$

$\therefore \left(5\sqrt{8}\right)\left(2\sqrt{3}\right)$ is greater.

b) $\left(3\sqrt{2}\right)\left(\sqrt{75}\right)$ or $\left(5\sqrt{2}\right)\left(3\sqrt{2}\right)$?

Answer:

$$\left(3\sqrt{2}\right)\left(\sqrt{75}\right) = 3\sqrt{150}$$
$$= \sqrt{3^2}\sqrt{150}$$
$$= \sqrt{9 \times 150}$$
$$= \sqrt{1350}$$

$$\left(5\sqrt{2}\right)\left(3\sqrt{2}\right) = 15\sqrt{4}$$
$$= \sqrt{15^2}\sqrt{4}$$
$$= \sqrt{225 \times 4}$$
$$= \sqrt{900}$$

$\therefore \left(3\sqrt{2}\right)\left(\sqrt{75}\right)$ is greater.

4. Every hour, a spider climbs up a distance in metres of $\left(2\sqrt{3}+5\sqrt{2}\right)$ and then is pushed down a distance of $\left(\sqrt{3}-\sqrt{2}\right)$. If the spider begins to climb at 12 noon, what is his distance above his starting point at 4:00 p.m. sharp?

Answer:

Time	Distance Above Starting Point
12 noon	0
1 p.m.	$2\sqrt{3}+5\sqrt{2}$
just past 1 p.m.	$\left(2\sqrt{3}+5\sqrt{2}\right)-\left(\sqrt{3}-\sqrt{2}\right)=\sqrt{3}+6\sqrt{2}$
2 p.m.	$\left(\sqrt{3}+6\sqrt{2}\right)+\left(2\sqrt{3}+5\sqrt{2}\right)=3\sqrt{3}+11\sqrt{2}$
just past 2 p.m.	$\left(3\sqrt{3}+11\sqrt{2}\right)-\left(\sqrt{3}-\sqrt{2}\right)=2\sqrt{3}+12\sqrt{2}$
3 p.m.	$\left(2\sqrt{3}+12\sqrt{2}\right)+\left(2\sqrt{3}+5\sqrt{2}\right)=4\sqrt{3}+17\sqrt{2}$
just past p.m.	$\left(4\sqrt{3}+17\sqrt{2}\right)-\left(\sqrt{3}-\sqrt{2}\right)=3\sqrt{3}+18\sqrt{2}$
4 p.m.	$\left(3\sqrt{3}+18\sqrt{2}\right)+\left(2\sqrt{3}+5\sqrt{2}\right)=5\sqrt{3}+23\sqrt{2}$

A more efficient method:

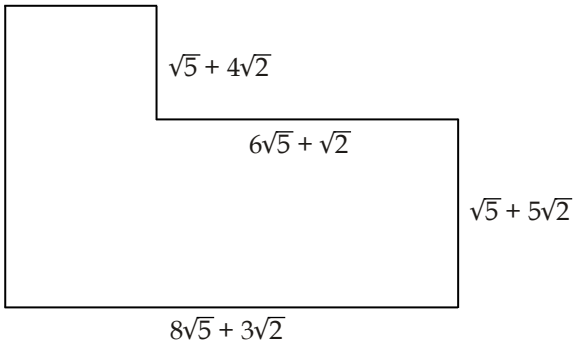
From 12 noon to 4 p.m., the spider climbs up $4\left(2\sqrt{3}+5\sqrt{2}\right)=8\sqrt{3}+20\sqrt{2}$.

The spider is pushed down 3 times = $3\left(\sqrt{3}-\sqrt{2}\right)=3\sqrt{3}-3\sqrt{2}$.

$\therefore$ at 4 p.m., the spider is $\left(8\sqrt{3}+20\sqrt{2}\right)-\left(3\sqrt{3}-3\sqrt{2}\right)=5\sqrt{3}+23\sqrt{2}$.

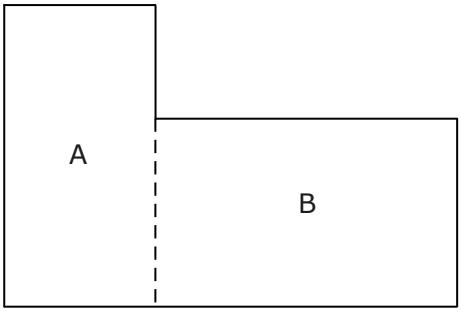
$\therefore$ the spider is $\left(5\sqrt{3}+23\sqrt{2}\right)$ m above his starting point.

5. Find the area of the figure drawn below.



Answer:

Subdivide the figure into two rectangles.



Area of a rectangle = lw

$$\text{Area of A} = \left[\left(\sqrt{5} + 5\sqrt{2}\right) + \left(\sqrt{5} + 4\sqrt{2}\right)\right]\left[\left(8\sqrt{5} + 3\sqrt{2}\right) - \left(6\sqrt{5} + \sqrt{2}\right)\right]$$

$$= \left(2\sqrt{5} + 9\sqrt{2}\right)\left(2\sqrt{5} + 2\sqrt{2}\right)$$

$$= 2\sqrt{5}\left(2\sqrt{5} + 2\sqrt{2}\right) + 9\sqrt{2}\left(2\sqrt{5} + 2\sqrt{2}\right)$$

$$= 4\sqrt{25} + 4\sqrt{10} + 18\sqrt{10} + 18\sqrt{4}$$

$$= 4(5) + 22\sqrt{10} + 18(2)$$

$$= 20 + 22\sqrt{10} + 36$$

$$= 56 + 22\sqrt{10}$$

Area of B $= \left(6\sqrt{5} + \sqrt{2}\right)\left(\sqrt{5} + 5\sqrt{2}\right)$

$= 6\sqrt{5}\left(\sqrt{5} + 5\sqrt{2}\right) + \sqrt{2}\left(\sqrt{5} + 5\sqrt{2}\right)$

$= 6\sqrt{25} + 30\sqrt{10} + \sqrt{10} + 5\sqrt{4}$

$= 6(5) + 31\sqrt{10} + 5(2)$

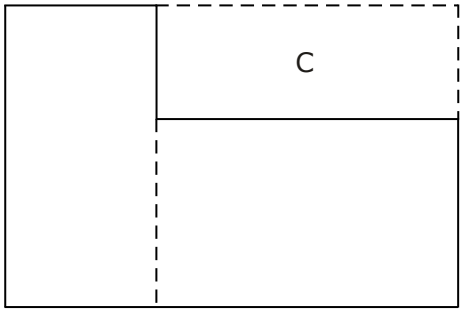
$= 30 + 31\sqrt{10} + 10$

$= 40 + 31\sqrt{10}$

Total area = area of A + Area of B

$= \left(56 + 22\sqrt{10}\right) + \left(40 + 31\sqrt{10}\right) = 96 + 53\sqrt{10}$

Alternatively, you could complete the large rectangle:



Required area = area of large rectangle − area of C.

$= \left(8\sqrt{5} + 3\sqrt{2}\right)\left(2\sqrt{5} + 9\sqrt{2}\right) - \left(6\sqrt{5} + \sqrt{2}\right)\left(\sqrt{5} + 4\sqrt{2}\right) = 96 + 53\sqrt{10}$

6. Determine whether $3 - \sqrt{2}$ is a root of the equation $x^2 - 6x + 7 = 0$.

Answer:

Substitute $3 - \sqrt{2}$ for x in the equation.

$\left(3 - \sqrt{2}\right)^2 - 6\left(3 - \sqrt{2}\right) + 7 = \left(3 - \sqrt{2}\right)\left(3 - \sqrt{2}\right) - 18 + 6\sqrt{2} + 7$

$= 3\left(3 - \sqrt{2}\right) - \sqrt{2}\left(3 - \sqrt{2}\right) - 18 + 6\sqrt{2} + 7$

$= 9 - 3\sqrt{2} - 3\sqrt{2} + \sqrt{4} - 18 + 6\sqrt{2} + 7$

$= 9 - 6\sqrt{2} + 2 - 18 + 6\sqrt{2} + 7$

$= 0$

$\therefore$ $3 - \sqrt{2}$ is a root of the equation $x^2 - 6x + 7 = 0$, since $3 - \sqrt{2}$ satisfies the equation.

7. The operation $\square$ is defined as follows: $a \square b = a^2 + ab$.

Evaluate each of the following:

a) $(2\sqrt{3}) \square (3 + \sqrt{3})$

Answer:

$$\begin{aligned}(2\sqrt{3}) \square (3 + \sqrt{3}) &= (2\sqrt{3})^2 + 2\sqrt{3}(3 + \sqrt{3}) \\ &= (2\sqrt{3})(2\sqrt{3}) + 2\sqrt{3}(3 + \sqrt{3}) \\ &= 4\sqrt{9} + 6\sqrt{3} + 2\sqrt{9} \\ &= 4(3) + 6\sqrt{3} + 2(3) \\ &= 12 + 6\sqrt{3} + 6 \\ &= 18 + 6\sqrt{3}\end{aligned}$$

b) $(2 + \sqrt{3}) \square (2 - \sqrt{3})$

Answer:

$$\begin{aligned}(2 + \sqrt{3}) \square (2 - \sqrt{3}) &= (2 + \sqrt{3})^2 + (2 + \sqrt{3})(2 - \sqrt{3}) \\ &= (2 + \sqrt{3})(2 + \sqrt{3}) + (2 + \sqrt{3})(2 - \sqrt{3}) \\ &= 4 + 2\sqrt{3} + 2\sqrt{3} + \sqrt{9} + 4 - 2\sqrt{3} + 2\sqrt{3} - \sqrt{9} \\ &= 4 + 2\sqrt{3} + 2\sqrt{3} + 3 + 4 - 2\sqrt{3} + 2\sqrt{3} - 3 \\ &= 8 + 4\sqrt{3}\end{aligned}$$

c) $(\sqrt{2} - 5) \square (3\sqrt{5})$

Answer:

$$\begin{aligned}(\sqrt{2} - 5) \square (3\sqrt{5}) &= (\sqrt{2} - 5)^2 + (\sqrt{2} - 5)(3\sqrt{5}) \\ &= (\sqrt{2} - 5)(\sqrt{2} - 5) + (3\sqrt{5})(\sqrt{2} - 5) \\ &= \sqrt{4} - 5\sqrt{2} - 5\sqrt{2} + 25 + 3\sqrt{10} - 15\sqrt{5} \\ &= 2 - 5\sqrt{2} - 5\sqrt{2} + 25 + 3\sqrt{10} - 15\sqrt{5} \\ &= 27 - 10\sqrt{2} + 3\sqrt{10} - 15\sqrt{5}\end{aligned}$$

8. Use playing cards to create four unique radical expression multiplication problems and simplify. Take out all the face cards from your deck of cards before creating your radical expression problems. Deal out at least four cards to create each problem. Black suits (spades and clubs) represent natural numbers, while red suits (diamonds and hearts) represent radicands. If you deal out a 3 of diamonds, a 2 of clubs, a 6 of clubs, and an 8 of hearts, you will have the following expression:

   3 of diamonds = $\sqrt{3}$

   2 of clubs = 2

   6 of clubs = 6

   6 of hearts = $\sqrt{6}$

   Expression: $\sqrt{3} \cdot 2 \cdot 6 \cdot \sqrt{6}$

   Simplifying the expression: $\sqrt{3} \cdot 2 \cdot 6 \cdot \sqrt{6}$

   $= 12\sqrt{18}$

   $= 12\sqrt{(9)(2)}$

   $= 12(3)\sqrt{2}$

   $= 36\sqrt{2}$

   You can try a variety of random cards and then test yourself using the following examples. Check your work against the solutions given in the answer key.

   a) 5C, 10S, 3H, 5D

   b) 4S, 3S, 2H, 5D

   c) 8H, 3C, AceD, 10S

   d) 7D, 2H, 8D, 2C

Answers:

Answers will vary. Four possible radical expression multiplication problems are shown below.

a) 5C, 10S, 3H, 5D

$= 5 \cdot 10 \cdot \sqrt{3} \cdot \sqrt{5}$

$= 50\sqrt{15}$

b) 4S, 3S, 2H, 5D

$= 4 \cdot 3 \cdot \sqrt{2} \cdot \sqrt{5}$

$= 12\sqrt{10}$

c) 8H, 3C, 1D, 10S

$= \sqrt{8} \cdot 3 \cdot \sqrt{1} \cdot 10$

$= 30\sqrt{(4)(2)}$

$= 30(2)\sqrt{2}$

$= 60\sqrt{2}$

d) 7D, 2H, 8D, 2C

$= \sqrt{7} \cdot \sqrt{2} \cdot \sqrt{8} \cdot 2$

$= 2\sqrt{112}$

$= 2\sqrt{16(7)}$

$= 2 \cdot 4\sqrt{7}$

$= 8\sqrt{7}$

Learning Activity 5.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Marissa loves to eat cookie dough so she bought a tub of cookie dough that was supposed to make 270 cookies. If Marissa and her brother each eat a scoop of cookie dough that is equivalent to three cookies every day, how long will it take them to eat the entire tub of cookie dough?
2. Write 32 as a power with a base of 2.
3. Bill went on a biking trip where he biked at 20 km/h for two hours and then 30 km/h for three hours. How far did Bill bike?
4. Factor: $2x(x^2 + 1) - 3(x^2 + 1)$
5. Simplify: $16^{\frac{1}{2}}$
6. Simplify: $-32m^{12} \div 8m^4$
7. Rationalize the denominator: $\dfrac{20}{\sqrt{3} + 1}$
8. Determine the length of the hypotenuse of a right-angled triangle when the other two sides measure 7 cm and 9 cm respectively.

Answers:

1. $45 \left(\dfrac{270}{(3 + 3)} = \dfrac{270}{6} = 45 \right)$
2. 2^5
3. 130 km (20 km/h for 2 hours = 40 km; 30 km/h for 3 hours = 90 km; 90 + 40 = 130 km)
4. $(2x - 3)(x^2 + 1)$
5. $4 \left(16^{\frac{1}{2}} = \sqrt{16} = 4 \right)$
6. $-4m^8 \left(\dfrac{-32m^{12}}{8m^4} = -4m^8 \right)$

7. $10\sqrt{3} - 10 \left(\dfrac{20\left(\sqrt{3}-1\right)}{\left(\sqrt{3}+1\right)\left(\sqrt{3}-1\right)} = \dfrac{20\sqrt{3}-20}{\sqrt{9}-1} = \dfrac{20\sqrt{3}-20}{2} \right)$

8. $\sqrt{130}$ cm $\left(\sqrt{7^2+9^2} = \sqrt{49+81} = \sqrt{130} \text{ cm} \right)$

Part B: Dividing Radical Expressions

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Simplify. Present your answer with no radicals in the denominators. If the expression contains variables, state for which values of the variables the radical expression is defined.

   a) $\dfrac{\sqrt{15}}{\sqrt{3}}$

   b) $\dfrac{\sqrt{2}}{2\sqrt{3}}$

   c) $\dfrac{10\sqrt{2}}{\sqrt{5}}$

   d) $\dfrac{\sqrt{2}+5}{\sqrt{3}}$

   e) $\dfrac{8\sqrt{10}-7\sqrt{5}}{2\sqrt{5}}$

   f) $\dfrac{7}{3-\sqrt{2}}$

   g) $\dfrac{\sqrt{5}}{9\sqrt{3}-4}$

   h) $\dfrac{\sqrt{7}-\sqrt{3}}{\sqrt{7}+\sqrt{3}}$

   i) $\dfrac{4-3\sqrt{3}}{6+\sqrt{2}}$

   j) $\dfrac{2\sqrt{3}-\sqrt{2}}{2\sqrt{3}+\sqrt{2}}$

   k) $\dfrac{\sqrt{x}}{\sqrt{x}-y}$

   l) $\dfrac{\sqrt{x}+1}{\sqrt{x}}$

   m) $\dfrac{\sqrt{x^2y^7}}{\sqrt{8y^6}}$

   n) $\dfrac{3-\sqrt{x}}{\sqrt{2x}-\sqrt{3}}$

   Answers:

   a) $\sqrt{\dfrac{15}{3}} = \sqrt{5}$

   b) $\dfrac{\sqrt{2}}{2\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{6}}{2(3)} = \dfrac{\sqrt{6}}{6}$

c) $\dfrac{10\sqrt{2}}{\sqrt{5}} \cdot \dfrac{\sqrt{5}}{\sqrt{5}} = \dfrac{10\sqrt{10}}{5} = 2\sqrt{10}$

d) $\dfrac{\sqrt{2}+5}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{6}+5\sqrt{3}}{3}$

e) $\dfrac{8\sqrt{10}-7\sqrt{5}}{2\sqrt{5}} \cdot \dfrac{\sqrt{5}}{\sqrt{5}} = \dfrac{8\sqrt{50}-7\sqrt{25}}{2\sqrt{25}}$

$= \dfrac{8\sqrt{25}\sqrt{2}-7(5)}{2(5)}$

$= \dfrac{8(5)\sqrt{2}-35}{10}$

$= \dfrac{40\sqrt{2}-35}{10}$

$= \dfrac{5\left(8\sqrt{2}-7\right)}{5(2)}$

$= \dfrac{8\sqrt{2}-7}{2}$

f) $\dfrac{7}{3-\sqrt{2}} \cdot \dfrac{3+\sqrt{2}}{3+\sqrt{2}} = \dfrac{21+7\sqrt{2}}{9+3\sqrt{2}-3\sqrt{2}-\sqrt{4}}$

$= \dfrac{21+7\sqrt{2}}{9-2}$

$= \dfrac{21+7\sqrt{2}}{7}$

$= \dfrac{7\left(3+\sqrt{2}\right)}{7}$

$= 3+\sqrt{2}$

g) $\dfrac{\sqrt{5}}{9\sqrt{3}-4} \cdot \dfrac{9\sqrt{3}+4}{9\sqrt{3}+4} = \dfrac{9\sqrt{15}+4\sqrt{5}}{81\sqrt{9}+36\sqrt{3}-36\sqrt{3}-16}$

$= \dfrac{9\sqrt{15}+4\sqrt{5}}{81(3)-16}$

$= \dfrac{9\sqrt{15}+4\sqrt{5}}{227}$

h) $\dfrac{\sqrt{7}-\sqrt{3}}{\sqrt{7}+\sqrt{3}} \cdot \dfrac{\sqrt{7}-\sqrt{3}}{\sqrt{7}-\sqrt{3}} = \dfrac{\sqrt{49}-\sqrt{21}-\sqrt{21}+\sqrt{9}}{\sqrt{49}-\sqrt{21}+\sqrt{21}-\sqrt{9}}$

$= \dfrac{7-2\sqrt{21}+3}{7-3}$

$= \dfrac{10-2\sqrt{21}}{4}$

$= \dfrac{2\left(5-\sqrt{21}\right)}{4}$

$= \dfrac{5-\sqrt{21}}{2}$

i) $\dfrac{4-3\sqrt{3}}{6+\sqrt{2}} \cdot \dfrac{6-\sqrt{2}}{6-\sqrt{2}} = \dfrac{24-4\sqrt{2}-18\sqrt{3}+3\sqrt{6}}{36-6\sqrt{2}+6\sqrt{2}-\sqrt{4}}$

$= \dfrac{24-4\sqrt{2}-18\sqrt{3}+3\sqrt{6}}{36-2}$

$= \dfrac{24-4\sqrt{2}-18\sqrt{3}+3\sqrt{6}}{34}$

j) $\dfrac{2\sqrt{3}-\sqrt{2}}{2\sqrt{3}+\sqrt{2}} \cdot \dfrac{2\sqrt{3}-\sqrt{2}}{2\sqrt{3}-\sqrt{2}} = \dfrac{4\sqrt{9}-2\sqrt{6}-2\sqrt{6}+\sqrt{4}}{4\sqrt{9}-2\sqrt{6}+2\sqrt{6}-\sqrt{4}}$

$= \dfrac{4(3)-4\sqrt{6}+2}{4(3)-2}$

$= \dfrac{14-4\sqrt{6}}{10}$

$= \dfrac{2\left(7-2\sqrt{6}\right)}{10}$

$= \dfrac{7-2\sqrt{6}}{5}$

k) $\dfrac{\sqrt{x}}{\sqrt{x}-y} \cdot \dfrac{\sqrt{x}+y}{\sqrt{x}+y} = \dfrac{\sqrt{x^2}+y\sqrt{x}}{\sqrt{x^2}+y\sqrt{x}-y\sqrt{x}-y^2}$

$= \dfrac{x+y\sqrt{x}}{x-y^2}$

This expression is only defined for $x \geq 0$. You cannot take the square root of a negative number. Also, $y \neq \sqrt{x}$ because the denominator cannot equal zero.

l) $\dfrac{\sqrt{x}+1}{\sqrt{x}} \cdot \dfrac{\sqrt{x}}{\sqrt{x}} = \dfrac{x+\sqrt{x}}{x}$

This expression is only defined for $x > 0$. You cannot take the square root of a negative number and the denominator cannot equal zero. Thus, x has to be greater than zero.

m) $$\frac{\sqrt{x^2y^7}}{\sqrt{8y^6}} = \sqrt{\frac{x^2y^7}{8y^6}}$$

$$= \sqrt{\frac{x^2y}{8}}$$

$$= \frac{\sqrt{x^2y}}{\sqrt{(4)(2)}}$$

$$= \frac{x\sqrt{y}}{2\sqrt{2}}$$

$$= \frac{x\sqrt{y}}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{x\sqrt{2y}}{2\left(\sqrt{4}\right)} = \frac{x\sqrt{2y}}{(2)(2)}$$

$$= \frac{x\sqrt{2y}}{4}$$

This expression is defined for $y > 0$ and $x \in \Re$. You cannot take the square root of a negative number nor have a zero in the denominator.

n) $$\frac{3-\sqrt{x}}{\sqrt{2x}-\sqrt{3}} = \frac{3-\sqrt{x}}{\sqrt{2x}-\sqrt{3}} \cdot \frac{\sqrt{2x}+\sqrt{3}}{\sqrt{2x}+\sqrt{3}}$$

$$= \frac{3\sqrt{2x}+3\sqrt{3}-\sqrt{2x^2}-\sqrt{3x}}{\sqrt{4x^2}+\sqrt{6x}-\sqrt{6x}-\sqrt{9}}$$

$$= \frac{3\sqrt{2x}+3\sqrt{3}-x\sqrt{2}-\sqrt{3x}}{2x-3}$$

This expression is only defined for $x \geq 0$ because you cannot take the square root of a negative number. Also, $x \neq \frac{3}{2}$ because the denominator cannot equal zero. Thus, the expression is defined for $x \geq 0$ and $x \neq \frac{3}{2}$.

2. The area of a rectangle is $\left(4+3\sqrt{3}\right)\text{m}^2$. If the length is $\left(\sqrt{3}+2\right)\text{m}$, find the

a) width

b) perimeter

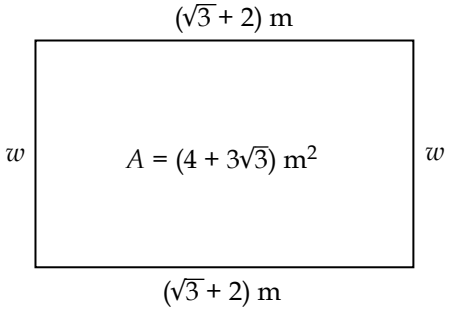
c) length of its diagonal

Answers:

a) Given: $A=\left(4+3\sqrt{3}\right)\text{m}^2$

$$l=\left(\sqrt{3}+2\right)\text{m}$$

Let w = width



$$A=lw$$

$$4+3\sqrt{3}=\left(\sqrt{3}+2\right)w$$

$$w=\frac{4+3\sqrt{3}}{\sqrt{3}+2}$$

$$w=\frac{4+3\sqrt{3}}{\sqrt{3}+2}\cdot\frac{\sqrt{3}-2}{\sqrt{3}-2}$$

$$=\frac{4\sqrt{3}-8+3\sqrt{9}-6\sqrt{3}}{\sqrt{9}-2\sqrt{3}+2\sqrt{3}-4}$$

$$=\frac{-2\sqrt{3}-8+3(3)}{3-4}$$

$$=\frac{-2\sqrt{3}+1}{-1}$$

$$=\frac{-2\sqrt{3}+1}{-1}\cdot\frac{(-1)}{(-1)}$$

$$=2\sqrt{3}-1$$

The width is $\left(2\sqrt{3}-1\right)\text{m}$.

b) Perimeter $= 2l + 2w$

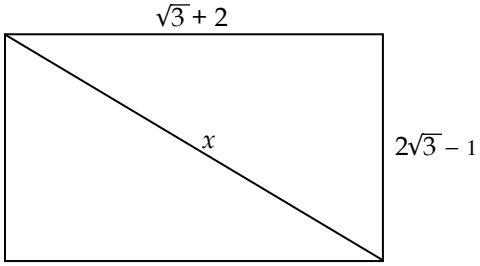
$$=2\left(\sqrt{3}+2\right)+2\left(2\sqrt{3}-1\right)$$

$$=2\sqrt{3}+4+4\sqrt{3}-2$$

$$=6\sqrt{3}+2$$

$\therefore$ perimeter $=\left(6\sqrt{3}+2\right)$ m

c) Let x = length of the diagonal.



By the Pythagorean Theorem:

$$x^2=\left(\sqrt{3}+2\right)^2+\left(2\sqrt{3}-1\right)^2$$

$$x^2=\left(\sqrt{3}+2\right)\left(\sqrt{3}+2\right)+\left(2\sqrt{3}-1\right)\left(2\sqrt{3}-1\right)$$

$$x^2=\sqrt{9}+2\sqrt{3}+2\sqrt{3}+4+4\sqrt{9}-2\sqrt{3}-2\sqrt{3}+1$$

$$x^2=3+2\sqrt{3}+2\sqrt{3}+4+12-2\sqrt{3}-2\sqrt{3}+1$$

$$x^2=20$$

$x=\pm\sqrt{20}$ but $-\sqrt{20}$ cannot be the length of a diagonal

$$x=\sqrt{20}=\sqrt{4}\sqrt{5}=2\sqrt{5}$$

The diagonal is $2\sqrt{5}$ m long.

3. Observing that both these numbers are positive, is it possible that they are equal?

$$\frac{\sqrt{2+\sqrt{3}}}{2};\ \frac{\sqrt{2}+\sqrt{6}}{4}$$

(**Hint:** Square both numbers and compare the results.)

Answer:

Let $x = \dfrac{\sqrt{2+\sqrt{3}}}{2}$ and $y = \dfrac{\sqrt{2}+\sqrt{6}}{4}$

$$x^2 = \frac{\sqrt{2+\sqrt{3}}}{2} \cdot \frac{\sqrt{2+\sqrt{3}}}{2}$$

$$x^2 = \frac{\sqrt{\left(2+\sqrt{3}\right)^2}}{4}$$

$$x^2 = \frac{2+\sqrt{3}}{4}$$

$$y^2 = \frac{\sqrt{2}+\sqrt{6}}{4} \cdot \frac{\sqrt{2}+\sqrt{6}}{4}$$

$$= \frac{\sqrt{4}+\sqrt{12}+\sqrt{12}+\sqrt{36}}{16}$$

$$= \frac{2+2\sqrt{12}+6}{16}$$

$$= \frac{8+2\sqrt{4}\sqrt{3}}{16}$$

$$= \frac{8+4\sqrt{3}}{16}$$

$$= \frac{4\left(2+\sqrt{3}\right)}{16}$$

$$= \frac{2+\sqrt{3}}{4}$$

Hence, $x^2 = y^2$.

Furthermore, since x and y are positive, it follows that $x = y$—that is, $\dfrac{\sqrt{2+\sqrt{3}}}{2} = \dfrac{\sqrt{2}+\sqrt{6}}{4}$.

Although some radicals do not look the same, they may still be equal.

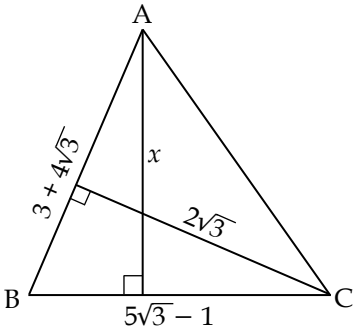
In the above question, it is important that x and y are both positive. Otherwise, it is possible that $x = -y$, since the squares of the positive and the negative numbers are equal (e.g., $(5)^2 = 25$ and $(-5)^2 = 25$. However, $5 \neq -5$.) Hence, x and y must both be positive numbers when you are comparing their squares.

4. Find the length of the altitude from A, if the length of the altitude from C is $2\sqrt{3}$.

Answer:

The altitude of a triangle is the height of the triangle. The formula for the area of a triangle is $A = \frac{1}{2}bh$.

Let x = height from A to BC.



Area = $\frac{1}{2}bh$ can be found using BC as the base or using AB as the base.

$\therefore$ Area $= \frac{1}{2}\left(5\sqrt{3}-1\right)x$ or

Area $= \frac{1}{2}\left(3+4\sqrt{3}\right)\left(2\sqrt{3}\right)$.

The two areas must be equal.

$$\frac{1}{2}\left(5\sqrt{3}-1\right)x = \frac{1}{2}\left(3+4\sqrt{3}\right)\left(2\sqrt{3}\right)$$

$$\left(5\sqrt{3}-1\right)x = \left(3+4\sqrt{3}\right)\left(2\sqrt{3}\right)$$ Divide both sides by $\frac{1}{2}$.

$$\left(5\sqrt{3}-1\right)x = 6\sqrt{3}+8\sqrt{9}$$ Multiply.

$$x = \frac{6\sqrt{3}+8(3)}{5\sqrt{3}-1}$$ Divide both sides by $5\sqrt{3}-1$.

$$x = \frac{6\sqrt{3}+24}{5\sqrt{3}-1}\cdot\frac{5\sqrt{3}+1}{5\sqrt{3}+1}$$ Rationalize the denominator.

$$x = \frac{30\sqrt{9}+6\sqrt{3}+120\sqrt{3}+24}{25\sqrt{9}+5\sqrt{3}-5\sqrt{3}-1}$$

$$x = \frac{30(3)+126\sqrt{3}+24}{25(3)-1}$$

$$x = \frac{114+126\sqrt{3}}{74}$$

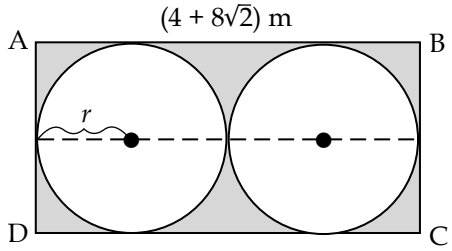
$$x = \frac{2\left(57+63\sqrt{3}\right)}{2(37)}$$

$$x = \frac{57+63\sqrt{3}}{37}$$

The length of the altitude from $A = \dfrac{57 + 63\sqrt{3}}{37}$.

5. Find the area of the shaded region if two identical circles are cut out of rectangle ABCD, as illustrated below. The length of $AB = \left(4 + 8\sqrt{2}\right)$ m.

   Leave your answer as an exact value in terms of π.



Answer:

Since $AB = 4 + 8\sqrt{2}$ and the circles are identical, the diameter of each circle is $\dfrac{4 + 8\sqrt{2}}{2} = 2 + 4\sqrt{2}$ and the radius of each circle is $\dfrac{2 + 4\sqrt{2}}{2} = 1 + 2\sqrt{2}$.

The width of the rectangle is $2 + 4\sqrt{2}$, which is the diameter of the circle.

Shaded area = area of rectangle $-$ areas of the two circles.

Area = $lw - 2(\pi r^2)$

$$A = \left(4 + 8\sqrt{2}\right)\left(2 + 4\sqrt{2}\right) - 2\pi\left(1 + 2\sqrt{2}\right)^2$$

$$A = \left(4 + 8\sqrt{2}\right)\left(2 + 4\sqrt{2}\right) - 2\pi\left(1 + 2\sqrt{2}\right)\left(1 + 2\sqrt{2}\right)$$

$$A = 8 + 16\sqrt{2} + 16\sqrt{2} + 32\sqrt{4} - 2\pi\left[1 + 2\sqrt{2} + 2\sqrt{2} + 4\sqrt{4}\right]$$

$$A = 8 + 32\sqrt{2} + 32(2) - 2\pi\left[1 + 4\sqrt{2} + 4(2)\right]$$

$$A = 72 + 32\sqrt{2} - 2\pi\left(9 + 4\sqrt{2}\right)$$

$A = 72 + 32\sqrt{2} - 18\pi - 8\pi\sqrt{2}$ (The answer is acceptable in this form.)

$$A = 72 - 18\pi + 32\sqrt{2} - 8\pi\sqrt{2}$$

$$A = (72 - 18\pi) + (32 - 8\pi)\sqrt{2}$$

The area of the shaded region is $\left[(72 - 18\pi) + (32 - 8\pi)\sqrt{2}\right]$ m^2.

6. What is the value of A if $\dfrac{2+\sqrt{3}}{\mathrm{A}} = 5-\sqrt{3}$?

Answer:

$$2+\sqrt{3} = \left(5-\sqrt{3}\right)A$$

$$A = \frac{2+\sqrt{3}}{5-\sqrt{3}}$$

$$A = \frac{2+\sqrt{3}}{5-\sqrt{3}} \cdot \frac{5+\sqrt{3}}{5+\sqrt{3}}$$

$$A = \frac{10+2\sqrt{3}+5\sqrt{3}+\sqrt{9}}{25+5\sqrt{3}-5\sqrt{3}-\sqrt{9}}$$

$$A = \frac{10+7\sqrt{3}+3}{25-3}$$

$$A = \frac{13+7\sqrt{3}}{22}$$

7. Simplify: $\dfrac{2}{\sqrt{3}-1} + \left(2+\sqrt{3}\right)\left(5-\sqrt{3}\right) - \left(2+\sqrt{3}\right)^2$

Answer:

Perhaps the easiest way to simplify such a complicated expression is to break it into parts. The first part needs its denominator rationalized; the other two parts require multiplication.

$$\frac{2}{\sqrt{3}-1} = \frac{2}{\sqrt{3}-1} \cdot \frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$= \frac{2\sqrt{3}+2}{\sqrt{9}+\sqrt{3}-\sqrt{3}-1}$$

$$= \frac{2\sqrt{3}+2}{2}$$

$$= \sqrt{3}+1$$

$$\left(2+\sqrt{3}\right)\left(5-\sqrt{3}\right) = 10-2\sqrt{3}+5\sqrt{3}-3$$

$$= 7+3\sqrt{3}$$

$$\left(2+\sqrt{3}\right)^2 = \left(2+\sqrt{3}\right)\left(2+\sqrt{3}\right)$$

$$= 4+2\sqrt{3}+2\sqrt{3}+3$$

$$= 7+4\sqrt{3}$$

Now combine the parts.

$$\frac{2}{\sqrt{3}-1}+\left(2+\sqrt{3}\right)\left(5-\sqrt{3}\right)+\left(2+\sqrt{3}\right)^2$$

$$=\left(\sqrt{3}+1\right)+\left(7+3\sqrt{3}\right)-\left(7+4\sqrt{3}\right)$$

$$=\sqrt{3}+1+7+3\sqrt{3}-7-4\sqrt{3}$$

$$=1$$

8. Write these numbers in descending order.

$$2,\ \frac{3}{\sqrt{5}},\text{ and }\sqrt{3}$$

Answer:

These numbers are all positive. Compare their squares.

$$2^2=4;\left(\frac{3}{\sqrt{5}}\right)^2=\frac{9}{5}=1\frac{4}{5};\left(\sqrt{3}\right)^2=3$$

Since $4>3>1\frac{4}{5}$, it follows that the positive numbers

$\sqrt{4}>\sqrt{3}>\sqrt{1\frac{4}{5}}$ – that is, $2>\sqrt{3}>\frac{3}{\sqrt{5}}$.

Learning Activity 5.5

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Maddox has 130 quarters, 72 dimes, and 26 nickels. How much money does he have in total?
2. Evaluate: $\frac{3}{5} \times \frac{10}{7}$
3. If today is Friday and your math exam is in 17 days, what day of the week is your math exam on?
4. Evaluate: $52 - (-3 - 4(-26 - (-28)))$.
5. Simplify: $\frac{3x^5y^2z^8}{9x^7y^3z^2}$
6. Estimate: $\sqrt{140}$
7. Trey and Tanisha are discussing their solutions to the linear inequality $-3x > 6$. Trey believes the answer is $x > -2$, while Tanisha believes the answer is $x < -2$. Who is right?
8. Simplify: $(x - 2)^3$

Answers:

1. \$41 (\$32.50 + \$7.20 + \$1.30 = \$41; 130 quarters = \$32.50; 72 dimes = \$7.20; 26 nickels = \$1.30)
2. $\frac{6}{7}\left(\frac{3}{\cancel{8}_1}, \frac{\cancel{10}^2}{7}\right)$
3. Monday (in 14 days it will be Friday again; three days past Friday is Monday)
4. $63\ \big(52 - \big(-3 - 4\big(-26 - (-28)\big)\big); 52 - \big(-3 - 4\big(-26 + 28\big)\big);$

   $52 - \big(-3 - 4(2)\big); 52 - (-3 - 8); 52 - (-11); 52 + 11 = 63\big)$
5. $\frac{z^6}{3x^2y}$
6. 11.8 (140 is between perfect squares 121 and 144; therefore, $\sqrt{140}$ is between 11 and 12 and is closer to 12)

7. Tanisha is right.

   $(-3x > 6$

   $x < -2$

   When you divide by a negative in an inequality, you need to switch the sign of the inequality.

   To show this is true, substitute a value of x less than -2 into the original inequality.

   Let $x = -4$
   LS = $-3x$
   $= -3(-4)$
   $= 12$
   RS = 6
   LS > RS
   True.)

8. $x^3 - 6x^2 + 12x - 8$ $[(x - 2)^3 = (x - 2)(x - 2)(x - 2) = (x^2 - 4x + 4)(x - 2)$
   $= x^3 - 2x^2 - 4x^2 + 8x + 4x - 8]$

Part B: Solving Radical Equations

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. State whether each statement is true or false. If the statement is false, rewrite it so that it is true.

   a) For each real number x, $\sqrt{x}$ is a real number only when $x \geq 0$.

   b) For each real number x, $\sqrt{x-3}$ is a real number only when $x \geq 0$.

   c) For each real number x, $\sqrt{x^2} = x$.

   d) The real number solution set of $\sqrt{x^2} = -3$ is $\{\sqrt{3}, -\sqrt{3}\}$.

   e) The equation $(x-1)^2 = \left(\sqrt{2x-1}\right)^2$ is equivalent to the sentence $(x-1) = \sqrt{2x-1}$ or $(x-1) = -\sqrt{2x-1}$.

   f) For each real number x, if $x^2 = 16$, then $x = \sqrt{16}$.

   g) Squaring both sides of $\sqrt{x} + 2 = 3x$ yields $x + 4 = 9x^2$.

Answers:

a) True.

b) False. For each real number x, $\sqrt{x-3}$ is a real number when $x \geq 3$.

c) False. For each real number x, $\sqrt{x^2} = |x|$ because $\sqrt{x^2}$ is the principal square root and is a positive number or zero. It is never negative. For example, if $x = -3$, then $\sqrt{(-3)^2} = |-3| = 3$.

d) False. The real number solution set of $\sqrt{x^2} = -3$ is the empty set. Saying the solution set is the empty set is the same as saying there is no solution to the problem. The principal square root is positive or zero, never negative.

e) True.

f) False. For each real number x, if $x^2 = 16$, then $x = \sqrt{16}$ or $x = -\sqrt{16}$.

g) False. Squaring both sides of $\sqrt{x} + 2 = 3x$ yields $x + 4\sqrt{x} + 4 = 9x^2$. The left side of the equation is a binomial. Its square is the product: $\left(\sqrt{x} + 2\right)\left(\sqrt{x} + 2\right)$.

2. Simplify each of the following expressions.

a) $\left(\sqrt{2x-1}\right)^2$

b) $\left(5 + \sqrt{x}\right)^2$

c) $\left(2 + \sqrt{x-5}\right)^2$

Answers:

a) $\left(\sqrt{2x-1}\right)^2 = 2x - 1$

b) $\left(5 + \sqrt{x}\right)^2 = 25 + 10\sqrt{x} + x$

c) $\left(2 + \sqrt{x-5}\right)^2 = \left(2 + \sqrt{x-5}\right)\left(2 + \sqrt{x-5}\right)$

$= 4 + 4\sqrt{x-5} + x - 5$

$= 4\sqrt{x-5} + x - 1$

3. Find the solutions for each of the following equations. Check your solutions for extraneous roots.

a) $x + 2 = \sqrt{2x + 7}$

b) $x = 2 - \sqrt{2x - 5}$

c) $\sqrt{2x + 3} - \sqrt{x + 1} = 1$

d) $\sqrt{x^2 - 3} + 1 = 0$

e) $x = \sqrt{3x - 2} + 2$

f) $\sqrt{\sqrt{x^2 + 6x}} = 2$

g) $\sqrt{3x + 2} = 3\sqrt{x} - \sqrt{2}$

h) $\sqrt{1 - x} + \sqrt{x} = \sqrt{x + 1}$

Answers:

a) $x + 2 = \sqrt{2x + 7}$

Solution:

$$(x + 2)^2 = \left(\sqrt{2x + 7}\right)^2$$

$$x^2 + 4x + 4 = 2x + 7$$

$$x^2 + 2x - 3 = 0$$

$$(x + 3)(x - 1) = 0$$

$$x = -3, 1$$

Check: $x + 2 = \sqrt{2x + 7}$

$x = -3$	$x = 1$
LHS $= x + 2$	LHS $= x + 2$
$= -3 + 2$	$= 1 + 2$
$= -1$	$= 3$
RHS $= \sqrt{2x + 7}$	RHS $= \sqrt{2x + 7}$
$= \sqrt{2(-3) + 7}$	$= \sqrt{2(1) + 7}$
$= \sqrt{1}$	$= \sqrt{9}$
$= 1$	$= 3$
LHS $\neq$ RHS	LHS $=$ RHS

$x = -3$ is not a solution. The only solution is $x = 1$.

b) $x = 2 - \sqrt{2x - 5}$

Solution:

$$(x - 2)^2 = \left(-\sqrt{2x - 5}\right)^2$$

$$x^2 - 4x + 4 = 2x - 5$$

$$x^2 - 6x + 9 = 0$$

$$(x - 3)^2 = 0$$

$$x = 3$$

Check: $x = 2 - \sqrt{2x - 5}$

$$x = 3$$

$$\text{LHS} = x$$

$$= 3$$

$$\text{RHS} = 2 - \sqrt{2x - 5}$$

$$= 2 - \sqrt{2(3) - 5}$$

$$= 2 - \sqrt{1}$$

$$= 2 - 1$$

$$= 1$$

$$\text{LHS} \neq \text{RHS}$$

There is no solution.

c) $\sqrt{2x+3} - \sqrt{x+1} = 1$

Solution:

$$\left(\sqrt{2x+3}\right)^2 = \left(\sqrt{x+1}+1\right)^2$$

$$2x+3 = x+1+2\sqrt{x+1}+1$$

$$2x+3 = x+2+2\sqrt{x+1}$$

$$x+1 = 2\sqrt{x+1}$$

$$(x+1)^2 = \left(2\sqrt{x+1}\right)^2$$

$$(x+1)^2 = 4(x+1)$$

$$x^2+2x+1 = 4x+4$$

$$x^2-2x-3 = 0$$

$$(x-3)(x+1) = 0$$

$$x = 3 \text{ or } -1$$

Check: $\sqrt{2x+3} - \sqrt{x+1} = 1$

$x = 3$

LHS $= \sqrt{2x+3} - \sqrt{x+1}$

$= \sqrt{2(3)+3} - \sqrt{3+1}$

$= \sqrt{9} - \sqrt{4}$

$= 3-2$

$= 1$

RHS $= 1$

LHS $=$ RHS

$x = -1$

LHS $= \sqrt{2x+3} - \sqrt{x+1}$

$= \sqrt{2(-1)+3} - \sqrt{-1+1}$

$= \sqrt{1} - \sqrt{0}$

$= 1$

RHS $= 1$

LHS $=$ RHS

There are two solutions, $x = 3$ or $x = -1$.

d) $\sqrt{x^2-3}+1=0$

Solution:

$$\left(\sqrt{x^2-3}\right)^2=(-1)^2$$

$$x^2-3=1$$

$$x^2=4$$

$$x=\pm 2$$

Check: $\sqrt{x^2-3}+1=0$

$x=2$

$\text{LHS}=\sqrt{x^2-3}+1$

$=\sqrt{2^2-3}+1$

$=\sqrt{1}+1$

$=2$

$\text{RHS}=0$

$\text{LHS}\neq\text{RHS}$

$x=-2$

$\text{LHS}=\sqrt{(-2)^2-3}+1$

$=\sqrt{4-3}+1$

$=2$

$\text{RHS}=0$

$\text{LHS}\neq\text{RHS}$

There is no solution.

Note: On the LHS of the initial question, the principal root of x^2-3, which is positive, is added to positive one. The result will never equal the RHS, which is zero. So the answer of no solution makes good sense.

e) $x = \sqrt{3x-2} + 2$

Solution:

$$(x-2)^2 = \left(\sqrt{3x-2}\right)^2$$

$$x^2 - 4x + 4 = 3x - 2$$

$$x^2 - 7x + 6 = 0$$

$$(x-6)(x-1) = 0$$

$$x = 6 \text{ or } 1$$

Check: $x = \sqrt{3x-2} + 2$

$x = 6$	$x = 1$
LHS $= x$	LHS $= x$
$= 6$	$= 1$
RHS $= \sqrt{3x-2} + 2$	RHS $= \sqrt{3x-2} + 2$
$= \sqrt{3(6)-2} + 2$	$= \sqrt{3(2)-2} + 2$
$= \sqrt{16} + 2$	$= \sqrt{1} + 2$
$= 4 + 2$	$= 3$
$= 6$	LHS $\neq$ RHS
LHS = RHS	

The only solution is $x = 6$.

f) $\sqrt{\sqrt{x^2+6x}} = 2$

Solution:

$$\left(\sqrt{\sqrt{x^2+6x}}\right)^2 = (2)^2$$

$$\sqrt{x^2+6x} = 4$$

$$\left(\sqrt{x^2+6x}\right)^2 = (4)^2$$

$$x^2+6x = 16$$

$$x^2+6x-16 = 0$$

$$(x+8)(x-2) = 0$$

$$x = -8 \text{ or } x = 2$$

Check: $\sqrt{\sqrt{x^2+6x}} = 2$

$x = -8$	$x = 2$
$\text{LHS} = \sqrt{\sqrt{x^2+6x}}$	$\text{LHS} = \sqrt{\sqrt{x^2+6x}}$
$= \sqrt{\sqrt{(-8)^2+6(-8)}}$	$= \sqrt{\sqrt{2^2+6(2)}}$
$= \sqrt{\sqrt{64-48}}$	$= \sqrt{\sqrt{4+12}}$
$= \sqrt{\sqrt{16}}$	$= \sqrt{\sqrt{16}}$
$= \sqrt{4}$	$= \sqrt{4}$
$= 2$	$= 2$
$\text{RHS} = 2$	$\text{RHS} = 2$
$\text{LHS} = \text{RHS}$	$\text{LHS} = \text{RHS}$

There are two solutions: $x = -8$ or $x = 2$.

g) $\sqrt{3x+2} = 3\sqrt{x} - \sqrt{2}$

Solution:

$$\left(\sqrt{3x+2}\right)^2 = \left(3\sqrt{x} - \sqrt{2}\right)^2$$

$$3x + 2 = 9x - 6\sqrt{2x} + 2$$

$$3x - 9x + 2 - 2 = -6\sqrt{2x}$$

$$\frac{-6x}{-6} = \frac{-6\sqrt{2x}}{-6}$$

$$x = \sqrt{2x}$$

$$(x)^2 = \left(\sqrt{2x}\right)^2$$

$$x^2 = 2x$$

$$x^2 - 2x = 0$$

$$x(x-2) = 0$$

$$x = 0 \text{ or } x - 2 = 0$$

$$x = 2$$

Check: $\sqrt{3x+2} = 3\sqrt{x} - \sqrt{2}$

$x = 0$

$\text{LHS} = \sqrt{3x+2}$

$= \sqrt{3(0)+2}$

$= \sqrt{2}$

$\text{RHS} = 3\sqrt{x} - \sqrt{2}$

$= 3\sqrt{0} - \sqrt{2}$

$= -\sqrt{2}$

$\text{LHS} \neq \text{RHS}$

$x = 2$

$\text{LHS} = \sqrt{3x+2}$

$= \sqrt{3(2)+2}$

$= \sqrt{8}$

$= 2\sqrt{2}$

$\text{RHS} = 3\sqrt{x} - \sqrt{2}$

$= 3\sqrt{2} - \sqrt{2}$

$= 2\sqrt{2}$

$\text{LHS} = \text{RHS}$

The only solution is $x = 2$.

h) $\sqrt{1-x}+\sqrt{x}=\sqrt{x+1}$

Solution:

$$\left(\sqrt{1-x}+\sqrt{x}\right)^2=\left(\sqrt{x+1}\right)^2$$

$$(1-x)+2\sqrt{x}\sqrt{1-x}+x=x+1$$

$$1-x+2\sqrt{x-x^2}+x=x+1$$

$$2\sqrt{x-x^2}=x$$

$$\left(2\sqrt{x-x^2}\right)^2=(x)^2$$

$$4\left(x-x^2\right)=x^2$$

$$4x-4x^2=x^2$$

$$4x-5x^2=0$$

$$x(4-5x)=0$$

$$x=0 \text{ or } 4-5x=0$$

$$x=\frac{4}{5}$$

Check: $\sqrt{1-x}+\sqrt{x}=\sqrt{x+1}$

$x=0$

$\text{LHS}=\sqrt{1-x}+\sqrt{x}$

$=\sqrt{1-0}+\sqrt{0}$

$=\sqrt{1}+0$

$=1$

$\text{RHS}=\sqrt{x+1}$

$=\sqrt{0+1}$

$=1$

LHS = RHS

$x=\frac{4}{5}$

$\text{LHS}=\sqrt{1-\frac{4}{5}}+\sqrt{\frac{4}{5}}$

$=\sqrt{\frac{1}{5}}+\sqrt{\frac{4}{5}}$

$=\frac{\sqrt{1}}{\sqrt{5}}+\frac{\sqrt{4}}{\sqrt{5}}$

$=\frac{1}{\sqrt{5}}+\frac{2}{\sqrt{5}}$

$=\frac{3}{\sqrt{5}}$

$\text{RHS}=\sqrt{x+1}$

$=\sqrt{\frac{4}{5}+1}$

$=\sqrt{\frac{4}{5}+\frac{5}{5}}$

$=\sqrt{\frac{9}{5}}$

$=\frac{3}{\sqrt{5}}$

LHS = RHS

There are two solutions: $x = 0$ or $x=\frac{4}{5}$.

4. Solve each of the following equations.

   a) $\sqrt[5]{x} = 4$

   b) $\sqrt[3]{x} = 0.5$

   c) $x^{\frac{2}{3}} = 81$

   d) $2x^{0.25} = 3$

   Answers:

   a) $\sqrt[5]{x} = 4$

   $\left(\sqrt[5]{x}\right)^5 = (4)^5$

   $x = 1024$

   Check for extraneous roots is not necessary; you are raising both sides to the fifth power, an odd number.

   b) $\sqrt[3]{x} = 0.5$

   $\left(\sqrt[3]{x}\right)^3 = (0.5)^3$

   $x = 0.125$

   Check for extraneous roots is not necessary; you are cubing both sides (an odd power).

   c) **Method 1:** Using Radicals

   $x^{\frac{2}{3}} = 81$

   $\sqrt[3]{x^2} = 81$

   $\left(\sqrt[3]{x^2}\right)^3 = (81)^3$

   $x^2 = 531\ 441$

   $\sqrt{x^2} = \pm\sqrt{531\ 441}$

   $x = \pm 729$

   There are two solutions: $x = 729$ or $x = -729$.

   A check for extraneous roots is not necessary since you are cubing both sides (an odd power).

Method 2: Using Rational Exponents

$$x^{\frac{2}{3}} = 81$$

$$\left(x^{\frac{2}{3}}\right)^{\frac{3}{2}} = (81)^{\frac{3}{2}}$$

$$x^{\left(\frac{2}{3}\right)\left(\frac{3}{2}\right)} = \left(\pm\sqrt{81}\right)^3 \quad \text{Take square root of both sides (don't forget the } \pm\text{).}$$

$$x^1 = (9)^3 \qquad \text{or} \qquad x^1 = (-9)^3$$

$$x = 729 \qquad \text{or} \qquad x = -729$$

Notice that with this method you need to be very careful since you are introducing the square root operation with the power of $\frac{1}{2}$. In line 3, you must consider $\pm\sqrt{81}$ inside the brackets, not just $\sqrt{81}$. Then you will get both answers.

d) $2x^{0.25} = 3$

$$\frac{2x^{\frac{1}{4}}}{2} = \frac{3}{2}$$

$$x^{\frac{1}{4}} = \frac{3}{2}$$

$$\sqrt[4]{x} = \frac{3}{2}$$

$$\left(\sqrt[4]{x}\right)^4 = \left(\frac{3}{2}\right)^4 \quad \text{Raise each side to the fourth power.}$$

$$x = \frac{81}{16}$$

A check for extraneous roots is necessary since part of the solution was to raise each side to an even power.

Check in the original equation $2x^{0.25} = 3$.

$$x = \frac{81}{16}$$

$$\text{LHS} = 2x^{0.25}$$

$$= 2\left(\frac{81}{16}\right)^{\frac{1}{4}}$$

$$= 2\sqrt[4]{\frac{81}{16}}$$

$$= (2)\left(\frac{3}{2}\right)$$

$$= 3$$

$$\text{RHS} = 3$$

$$\text{LHS} = \text{RHS}$$

The only solution is $x = \frac{81}{16}$.

5. Solve each of the following equations. Determine any restrictions on the variables.

   a) $\sqrt{x-3} = \sqrt{x} + 3$

   b) $\sqrt{x+2} = 4 - \sqrt{x+2}$

   Answers:

   a) $\sqrt{x-3} = \sqrt{x} + 3$

   This radical equation is only defined when $x - 3$ and x are greater than or equal to zero.

   $$x - 3 \geq 0$$

   $$x \geq 3 \text{ and } x \geq 0$$

   Thus, x has to be greater than or equal to both 3 and 0 in order for this equation to be defined. For both inequalities to be true, this is the same as saying $x \geq 3$ in order for this equation to be defined.

$$\sqrt{x-3} = \sqrt{x} + 3$$

$$\left(\sqrt{x-3}\right)^2 = \left(\sqrt{x} + 3\right)^2$$

$$x - 3 = x + 6\sqrt{x} + 9$$

$$x - 3 - x - 9 = 6\sqrt{x}$$

$$-12 = 6\sqrt{x}$$

$$-2 = \sqrt{x}$$

$$(-2)^2 = \left(\sqrt{x}\right)^2$$

$$4 = x$$

Since 4 is greater than or equal to 3, then $x = 4$ satisfies the restriction, $x \geq 3$.

Check $x = 4$ in the original equation $\sqrt{x-3} = \sqrt{x} + 3$.

LHS	RHS
$= \sqrt{x-3}$	$= \sqrt{x} + 3$
$= \sqrt{4-3}$	$= \sqrt{4} + 3$
$= 1$	$= 2 + 3$
	$= 5$

LHS $\neq$ RHS

Thus, there are no real solutions for this equation.

b) $\sqrt{x+2} = 4 - \sqrt{x+2}$

This radical equation is only defined when $x + 2$ is greater than or equal to zero.

$$x + 2 \geq 0$$
$$x \geq -2$$

Thus, this radical equation is only defined when $x \geq -2$.

$$\sqrt{x+2} = 4 - \sqrt{x+2}$$
$$2\sqrt{x+2} = 4$$
$$\left(2\sqrt{x+2}\right)^2 = 4^2$$
$$4(x+2) = 16$$
$$4x + 8 = 16$$
$$4x = 8$$
$$x = 2$$

$x = 2$ is greater than or equal to -2; thus, this is a possible solution.

Check $x = 2$ in $\sqrt{x+2} = 4 - \sqrt{x+2}$

$$\text{LHS} = \sqrt{x+2}$$
$$= \sqrt{2+2}$$
$$= \sqrt{4}$$
$$= 2$$
$$\text{RHS} = 4 - \sqrt{x+2}$$
$$= 4 - \sqrt{2+2}$$
$$= 4 - \sqrt{4}$$
$$= 4 - 2$$
$$= 2$$
$$\text{LHS} = \text{RHS}$$

Thus, the solution to this radical equation is $x = 2$.

6. Consider a cylindrical drink can. The radius of the can is 3 cm. The height of the can is 12 cm.

a) Find the volume of the pop can. Given: $r = \sqrt{\frac{V}{\pi h}}$. Round your answer to two decimal places. Check your answer for extraneous roots.

Answer:

You have the following information:

$r = 3$ cm

$h = 12$ cm

$$r = \sqrt{\frac{V}{\pi h}}$$

and the restrictions $V \geq 0$, $h > 0$.

$$r = \sqrt{\frac{V}{\pi h}}$$

$$3 = \sqrt{\frac{V}{12\pi}}$$

$$3^2 = \left(\sqrt{\frac{V}{12\pi}}\right)^2$$

$$9 = \frac{V}{12\pi}$$

$$108\pi = V$$

Check:

$$r = \sqrt{\frac{V}{\pi h}}$$

$$\text{RHS} = \sqrt{\frac{108\pi}{12\pi}}$$

$$= \sqrt{9}$$

$$= 3$$

$$\text{LHS} = 3$$

$$\text{LHS} = \text{RHS}$$

Thus, the volume of this pop can is $108\pi = 339.29$ cm^3.

b) If the volume of a typical pop can is 355 mL, how does your answer compare to that value?

Answer:

The volume of the pop can is 108π cm^3, which is approximately 339.29 cm^3. As one cm^3 is equivalent to one mL, the volume of the pop can is a bit smaller than the typical value of a pop can (355 mL).

7. Kinetic energy is calculated using the formula $E_k = \frac{1}{2}mv^2$, where E_k represents the kinetic energy of an object measured in joules, m represents the mass of an object measured in kilograms, and v represents the positive speed of an object. Speed is measured in metres per second. Using the formula, calculate the speed of a baseball that has a mass of 0.145 kg and 32.1 J (joules) of kinetic energy. Round your answer to two decimal places.

Answer:

You have the following information:

$m = 0.145$ kg

$E_k = 32.1$ J

Substitute those values into the equation.

$$E_k = \frac{1}{2}mv^2$$

$$32.1 = \frac{1}{2}(0.145)v^2$$

$$32.1 = 0.0725v^2$$

$$\frac{32.1}{0.0725} = v^2$$

$$v^2 = 442.7586207$$

$$\sqrt{v^2} = \pm\sqrt{442.7586207}$$

$$v = \pm 21.04$$

Since v is positive, use only the principal square root value. Thus, the speed of the baseball is 21.04 m/s.

Grade 11 Pre-Calculus Mathematics (30S)

Module 6
Systems of Equations and Inequalities

Module 6:
Systems of Equations and Inequalities

Introduction

Welcome to Module 6! In this module, you will learn how to solve systems of equations as well as inequalities. When you solve systems of equations, you need to be familiar with how to solve linear as well as quadratic equations. Therefore, some questions in this module allow you to practice skills you learned in earlier modules, such as solving quadratic equations by factoring and by using the quadratic formula. As you will see, these systems of equations appear in situations involving linear and quadratic motion. Some examples include throwing a baseball or a football, jumping off a diving board, or just travelling in a straight line.

You may not be familiar with inequalities. From previous courses, however, you are familiar with the symbols for *greater than*, *greater than or equal to*, *less than*, and *less than or equal to*. These symbols are very important when you work with inequalities. Inequalities appear when you want to find which values are above or below a certain curve in order to solve a problem. These types of problems often have to do with profit, speed, height, or the weight that can be supported by a particular object.

Assignments in Module 6

When you have completed the assignments for Module 6, submit your completed assignments to the Distance Learning Unit either by mail or electronically through the learning management system (LMS). The staff will forward your work to your tutor/marker.

Lesson	Assignment Number	Assignment Title
2	Assignment 6.1	Solving Systems of Equations
4	Assignment 6.2	Solving Linear Inequalities in Two Variables and Quadratic Inequalities in One Variable
5	Assignment 6.3	Solving Quadratic Inequalities in Two Variables

Resource Sheet

When you write your final examination, you are encouraged to take a Final When you write your final examination, you are encouraged to take a Final Examination Resource Sheet with you into the examination. This sheet will be one letter-sized page, 8½ " by 11", with both sides in your handwriting or typewritten. You will submit it with your examination, but you do not receive any marks for it.

Many students have found that preparing a resource sheet is an excellent way to review. It provides you with a summary of the important facts of each module. You should complete a resource sheet for each module to help with your studying and reviewing. Lesson summaries and module summaries are included for you to use as a guide.

You may use the list of instructions that follows to help you with preparing your resource sheet for the material in Module 6. On this sheet, you should record math terms and definitions, formulas, sample questions, or a list of places where you often make mistakes. You should also identify special areas that require extra attention or review by writing the page numbers.

After you have completed each module's resource sheet, you may summarize the sheets from Modules 1 to 8 to prepare your Final Examination Resource Sheet. The final examination for this course is based on Modules 1 to 8.

Resource Sheet for Module 6

As you go through the lessons of this module, you may want to consider the following suggestions regarding the creation of a resource sheet.

1. List all the important math terms, and define them if necessary.
2. List all the formulas and perhaps a sample problem that shows how the formula is used.
3. If necessary, write the solutions to some problems, showing in detail how you did the calculations.
4. Copy any questions that represent the key points of the lesson, and perhaps include the solutions as well.
5. Identify the problems you found most difficult, and copy the page numbers onto the resource sheet so that you can review them before writing the examination. You may also copy the problems and the solutions onto your resource sheet, and later write them onto your Final Examination Resource Sheet.
6. Write any comments, ideas, shortcuts, or other reminders that may be helpful during an examination.

Lesson 1: Solving Quadratic-Linear Systems of Equations

Lesson Focus

In this lesson, you will

- ❑ learn the meaning of the point of intersection of a system of linear-quadratic equations
- ❑ learn how to solve quadratic-linear systems of equations algebraically, graphically, and by using technology
- ❑ learn how to model a situation using a system of linear-quadratic equations in two variables

Lesson Introduction

In *Grade 10 Introduction to Applied and Pre-Calculus Mathematics*, you learned how to solve systems of linear equations algebraically and graphically. You learned how to solve these systems of equations algebraically using the elimination method and the substitution method. In this lesson, you will build on these skills to learn how to solve systems of equations containing one linear equation and one quadratic equation, using similar methods.

Quadratic–Linear Systems of Equations

Quadratic-linear systems of equations consist of one quadratic equation and one linear equation. As you know, linear equations can have a positive or negative slope.

The linear function can be in the form $y = mx + b$ or $ax + by + c = 0$. In either case, the slope can be positive or negative. A line with equation $y = k$ has 0 slope. A line with equation $x = k$ has undefined slope.

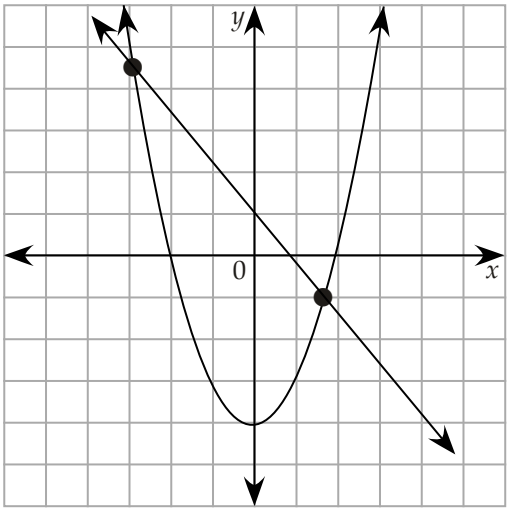
The forms for writing a quadratic function are $y = a(x - p)^2 + q$ or $y = ax^2 + bx + c$.

In regard to quadratic equations, they can have positive or negative a-values. When quadratic equations have positive a-values, they open upwards. When quadratic equations have negative a-values, they open downwards.

When a linear equation intersects a quadratic equation, the **point of intersection** is a point on both functions and is called the solution to the quadratic-linear system of equations. Add this definition to your resource sheet.

Example 1

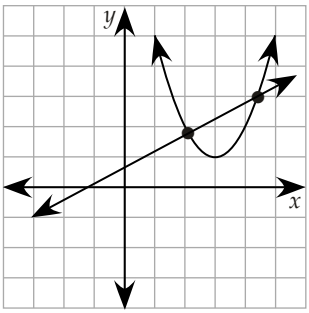
Identify the number of solutions in the following systems of equations.



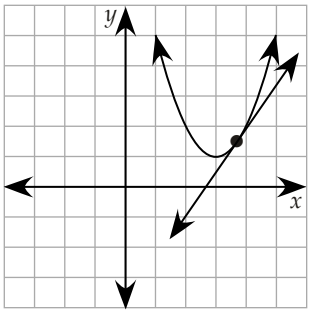
Solution

There are two solutions because there are two points of intersection.

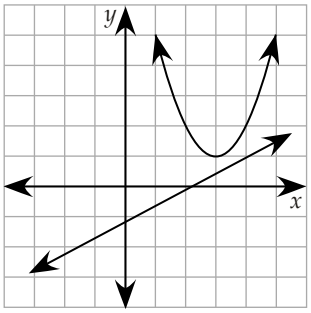
Consider the following possibilities.



two points of intersection
∴ two solutions



one point of intersection
∴ one solution



no point of intersection
∴ no solution

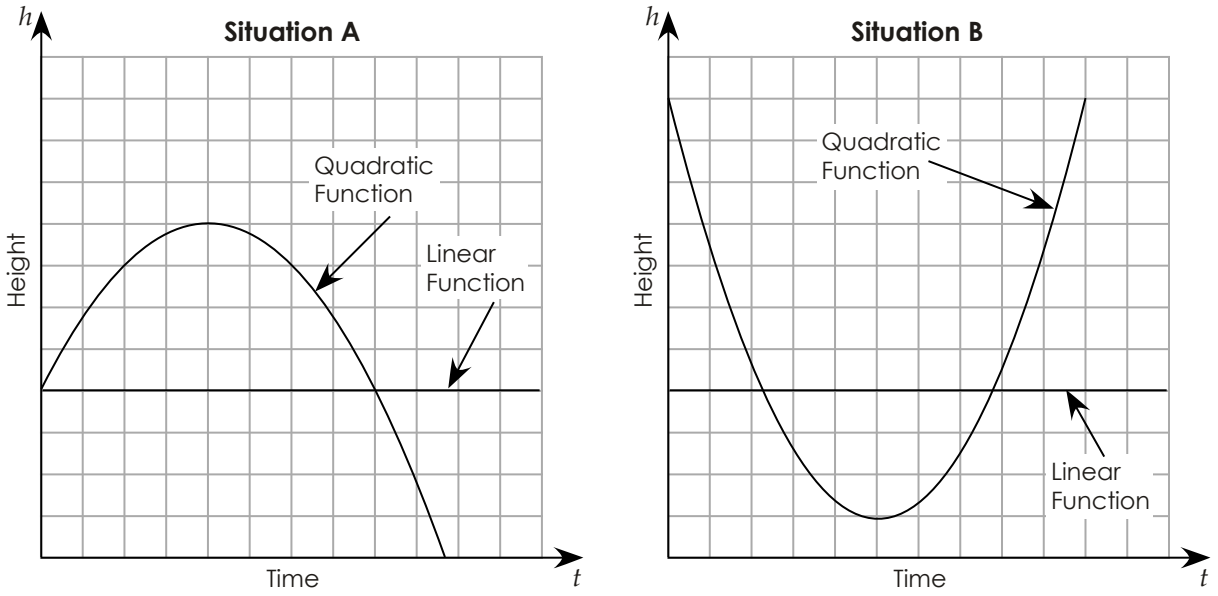
As you can see, there are three different types of solutions with regard to quadratic-linear systems of equations. There can be no solution (no point of intersection), one solution (one point of intersection), or two solutions (two points of intersection). It is possible to find these solutions either graphically or algebraically.

Real-Life Models of Quadratic–Linear Systems of Equations

Many situations can be modelled by a quadratic-linear system of equations, such as jumping off a dock into the water or going down a roller coaster that travels above and below ground level.

Example 2

Consider the two graphs containing systems of equations showing two different situations below.



a) Which graph models a situation involving someone jumping off a dock into the water?

b) Which graph models a situation involving someone riding a roller coaster that travels above and below ground level?

Solution

a) Situation A models the height of someone jumping off a dock, as the height of the dock stays constant while the height of the jumper increases as he jumps up and then decreases as he heads toward the water.

b) Situation B models the height of someone riding a roller coaster as the height of the ground stays constant while the height of the roller coaster decreases until it is below ground and then increases above the ground.

Solving Quadratic–Linear Systems of Equations Graphically

Now that you know two situations where quadratic-linear systems of equations are useful, it is time to learn how to solve these types of equations. To solve quadratic-linear systems of equations by graphing, you need to graph each equation and determine the point of intersection. Consider the following example.

Example 3

Solve the following systems of equations by graphing part (a) on graph paper and part (b) using graphing technology, if you have access to it.

a) $y = x^2 - 4$
$y = -x - 2$

b) $y = x - 2$
$y = x^2 - 4x - 2$

Solution

In order to solve a system of equations graphically, you can either graph the equations by hand or use technology to graph the equations. In (a), you will learn how to solve the system of equations by graphing by hand. In (b), you will learn how to solve the system of equations by graphing using technology.

a) $y = x^2 - 4$
$y = -x - 2$

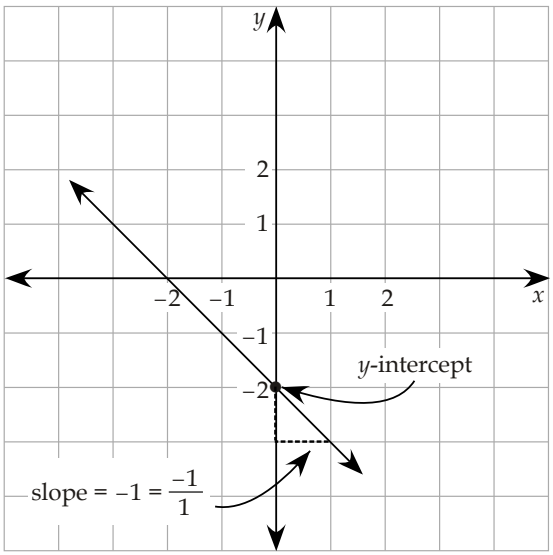
Graph these equations using graph paper.

To graph a linear equation in the form $y = mx + b$, you can find the y-intercept, b. You can then use the slope, m, to determine another point on the graph. Once you have two points on the graph, you can connect them with a straight line to form your line.

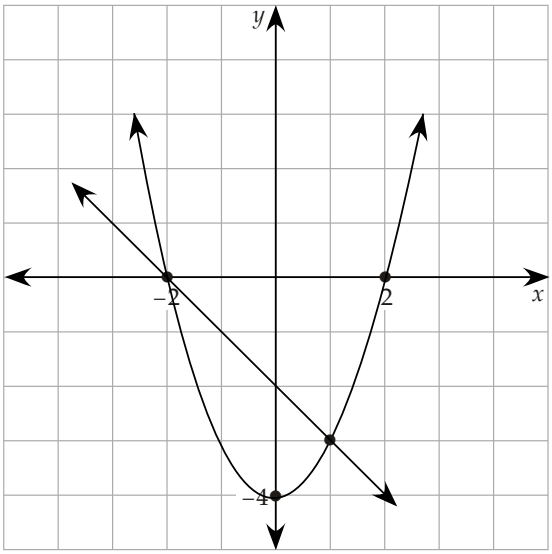
$y = -x - 2$

y-intercept: -2

slope: -1

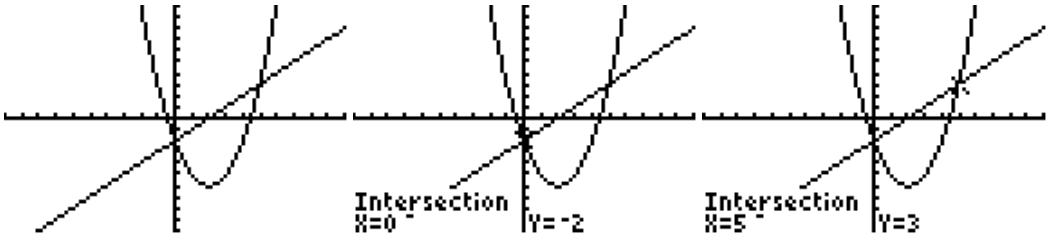


Then you can graph the parabola $y = x^2 - 4$ on the same graph. It has a vertex of $(0, -4)$, opens up and is normal shape.



From the graph, you can see that there are two solutions to this system of equations. The solutions are $(-2, 0)$ and $(1, -3)$.

b) Using technology, such as a graphing calculator (or a computer application such as *Winplot*), graph the system of equations to determine the solution. If you do not have access to technology, omit (b) or graph it by hand.



The solutions to this system of equations are $(0, -2)$ and $(5, 3)$.

Solving Quadratic–Linear Systems of Equations Algebraically

Sometimes, it is not reasonable to solve quadratic-linear systems of equations graphically, as the points of intersection for the system are not easy to determine when the solution contains decimal values. It is, however, possible to determine the solution to a quadratic-linear system of equations algebraically. This can be done either by using the substitution method or the elimination method.

The Substitution Method

To use the substitution method to solve quadratic-linear systems of equations, you substitute the linear equation into the quadratic equation. Consider the following example.

Example 4

Solve the following system of equations algebraically and check your answer algebraically and graphically.

$y = x^2 - 6x + 9$

$5 = x + y$

Solution

Solve the linear equation for y.

$x + y = 5$

$y = 5 - x$

Substitute the expression for y into the first equation.

$y = x^2 - 6x + 9$

$5 - x = x^2 - 6x + 9$ Substitute for y.

$0 = x^2 - 5x + 4$ Solve the resulting quadratic equation.

$0 = (x - 4)(x - 1)$

$x = 4$ or $x = 1$

You are not finished. You **must** find the corresponding y-values.

To find the corresponding y-values, substitute the x-values into the linear equation and solve for y. You can substitute into the quadratic equation too if that is easier.

$x = 4$	$x = 1$
$x + y = 5$	$x + y = 5$
$4 + y = 5$	$1 + y = 5$
$y = 1$	$y = 4$

The solutions are (4, 1) and (1, 4). There are two solutions to this system of equations. Check your answers algebraically and graphically.

Unlike radical equations where you must check for extraneous roots introduced by the method of squaring both sides, when solving a system of linear and quadratic functions, you are only checking to catch mistakes you may have made. So, checking is optional.

Algebraic Check

Equation (1): $y = x^2 - 6x + 9$

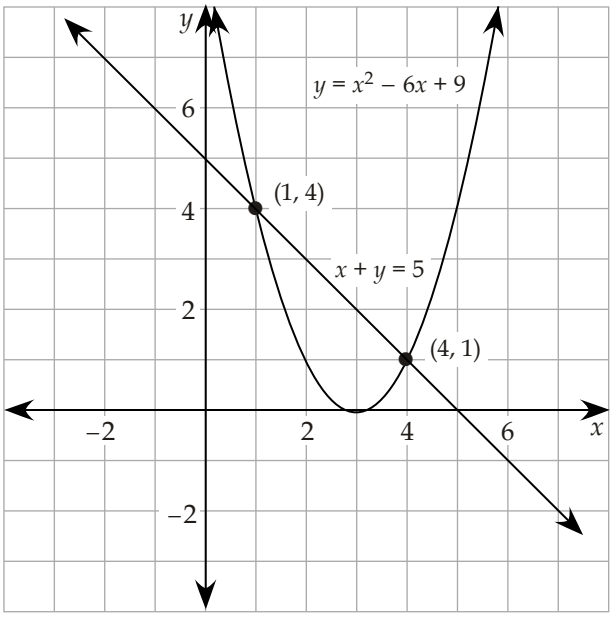
Point (4, 1)	Point (1, 4)
LHS $= y = 1$	LHS $= y = 4$
RHS $= x^2 - 6x + 9$	RHS $= x^2 - 6x + 9$
$= (4)^2 - 6(4) + 9$	$= (1)^2 - 6(1) + 9$
$= 16 - 24 + 9$	$= 1 - 6 + 9$
$= 1$	$= 4$
LHS = RHS	LHS = RHS

Equation (2): $5 = x + y$

Point (4, 1)	Point (1, 4)
LHS = 5	LHS = 5
RHS $= x + y$	RHS $= x + y$
$= 4 + 1$	$= 1 + 4$
$= 5$	$= 5$
LHS = RHS	LHS = RHS

Thus, the points (4, 1) and (1, 4) satisfy both the quadratic equation and the linear equation.

Graphic Check



The points of intersection are (4, 1) and (1, 4).

The Elimination Method

To solve quadratic-linear systems of equations by using the elimination method, first align the terms with the same variable and degree. Then, eliminate the y-term. Consider the following example.

Example 5

Solve the following system of equations by using the elimination method.

$y = 2x + 4$

$y = x^2 + x - 2$

Solution

First, you need to align the terms with the same variable and degree.

$y = \quad 2x + 4$

$y = x^2 + x - 2$

Now, you can multiply the first equation by (-1) and then add the two equations to eliminate the y-term.

$$\begin{array}{rl} -y = & -2x - 4 \\ + \quad y = & x^2 + x - 2 \\ \hline 0 = & x^2 - x - 6 \end{array}$$

Solve the quadratic equation by factoring.

$0 = x^2 - x - 6$

$0 = (x - 3)(x + 2)$

$x = 3$ or $x = -2$

Now you must find the corresponding y-values.

Substitute these values into the linear equation.

$x = 3$	$x = -2$
$y = 2(3) + 4$	$y = 2(-2) + 4$
$y = 10$	$y = 0$

The solutions to this system of equations are $(3, 10)$ and $(-2, 0)$.

To check the solutions, substitute each ordered pair into the original equations.

(3, 10)

$y = 2x + 4$

$\text{LHS} = y = 10$

$\text{RHS} = 2x + 4$

$= 2(3) + 4$

$= 10$

$\text{LHS} = \text{RHS}$

$y = x^2 + x - 2$

$\text{LHS} = y = 10$

$\text{RHS} = 3^3 + 3 - 2$

$= 9 + 3 - 2$

$= 10$

$\text{LHS} = \text{RHS}$

(−2, 0)

$y = 2x + 4$

$\text{LHS} = y = 0$

$\text{RHS} = 2x + 4$

$= 2(-2) + 4$

$= -4 + 4$

$= 0$

$\text{LHS} = \text{RHS}$

$y = x^2 + x - 2$

$\text{LHS} = y = 0$

$\text{RHS} = x^2 + x - 2$

$= (-2)^2 + (-2) - 2$

$= 4 - 2 - 2$

$= 0$

$\text{LHS} = \text{RHS}$

Thus, (3, 10) and (−2, 0) are correct solutions to the system of equations.

Example 6

Solve the following system of equations by using the elimination method.

$y = 5x - 10$

$2y = x^2 + x$

Solution

First, you need to align the terms with the same variable and degree.

$y = \quad 5x - 10$

$2y = x^2 + x$

You want to eliminate the y term. However, the coefficients of each y term are different. In order to make these coefficients the same value with opposite sign, multiply the first equation by (-2).

$$-2y = -10x + 20$$
$$2y = x^2 + x$$

Now, you can add like terms to eliminate the $2y$ term.

$$\begin{array}{rl} & -2y = -10x + 20 \\ + & 2y = x^2 + x \\ \hline & 0 = x^2 - 9x + 20 \end{array}$$

Solve the equation by factoring.

$$0 = x^2 - 9x + 20$$
$$0 = (x - 5)(x - 4)$$
$$x = 5 \text{ or } x = 4$$

Substitute these values into the linear equation to solve for y.

$x = 5$	$x = 4$
$y = 5x - 10$	$y = 5x - 10$
$y = 5(5) - 10$	$y = 5(4) - 10$
$y = 15$	$y = 10$

You can check the solutions if you want to confirm they are correct.

You have now learned how to solve quadratic-linear systems of equations graphically and algebraically. The following learning activity will give you the chance to practice these skills.

Learning Activity 6.1

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. How much tax would you have to pay on an item that costs $49.99 (approximately) if the PST = 8% and the GST = 5%?
2. Simplify: $\sqrt{98}$
3. Find the equation of a line with a slope of $-\frac{1}{2}$ and a y-intercept of 3.
4. Evaluate: $\frac{3}{21} \times \frac{3}{15}$
5. If 10% of 440 is 44, what is 35% of 440?
6. Are these fractions equivalent? $\frac{\sqrt{5}}{\sqrt{7}}$ and $\frac{\sqrt{35}}{7}$
7. Determine the length of the hypotenuse of a right-angled triangle when the two sides measure 10 cm and 13 cm.
8. Find the slope of the line that contains the points (26, −13) and (15, −21).

continued

Learning Activity 6.1 (continued)

Part B: Solving Quadratic-Linear Systems of Equations

Remember, these questions are similar to the ones that will be on your assignments and final exam. So, if you were able to answer them correctly, you are likely to do well on your assignments and final exam. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Verify that (−1, 0) and (4, 15) are solutions to the following system of equations.

   $y = 3x + 3$

   $y = x^2 - 1$

2. Solve each system both graphically and algebraically.

   a) $y = 4x^2$

      $y = 8x$

   b) $x - 2y = 10$

      $y = x^2 + 2x - 15$

3. Solve each system by the substitution method.

   a) $y = 3x + 7$

      $y = x^2 - 11$

   b) $y = 2x + 6$

      $y = x^2 + 4x + 3$

4. Solve each system by the elimination method.

   a) $2y = 2x - 2$

      $y = x^2 - 4x + 3$

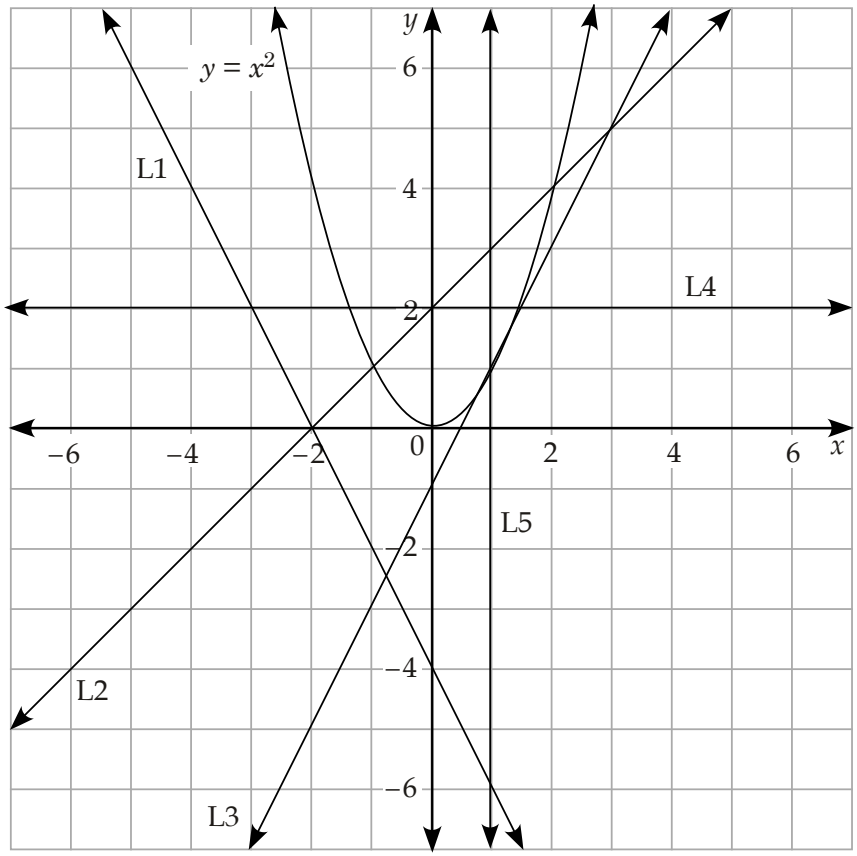
   b) $2x + y = 5$

      $y = x^2 - 6x + 5$

continued

Learning Activity 6.1 (continued)

5. Consider the graph. Use this graph to answer the following questions.



a) State the number of solutions to the system consisting of the parabola and L1.

b) State the number of solutions to the system consisting of the parabola and L2.

c) State the number of solutions to the system consisting of the parabola and L3.

d) State the number of solutions to the system consisting of the parabola and L4.

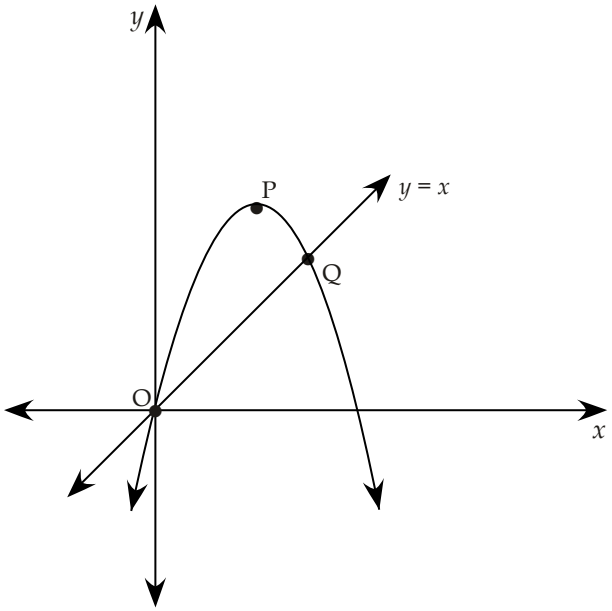
e) State the number of solutions to the system consisting of the parabola and L5.

f) State the solution(s) to the system consisting of the parabola and L2.

g) State the solution(s) to the system consisting of the parabola and L5.

continued

6. The line $y = x$ intersects the parabola $y = 4x - x^2$ at the origin and the point Q. Let the point P be the vertex of the parabola.



   a) Find the coordinates of Q.

   b) Find the coordinates of P.

7. The perimeter of a rectangle is 26 cm and the area is 12 cm^2. Find the dimensions of the rectangle.

continued

Learning Activity 6.1 (continued)

8. Below is Schyler's work for solving a system of equations.

$-2y = -x^2 - x$

$2y = 10x - 20$

Schyler's Solution:

$$\begin{array}{rl} & -2y = -x^2 - x \\ + & \underline{2y = \quad 10x - 20} \\ & 0 = -x^2 + 9x - 20 \end{array}$$

$0 = x^2 - 9x + 20$

$0 = (x - 4)(x - 5)$

$x = 4$ or $x = 5$

When $x = 4$:

$2y = 10x - 20$

$2y = 10(4) - 20$

$2y = 20$

$y = 10$

Thus, one solution is (4, 10).

When $x = 5$:

$2y = 10x - 20$

$2y = 10(5) - 20$

$2y = 30$

$y = 15$

Thus, another solution is (5, 15).

Is his method correct? Explain why or why not.

Lesson Summary

In previous courses, you studied systems of equations resulting from two linear equations. In this lesson, you learned that there are other types of systems of equations. You also learned how to solve one of these types of systems of equations, the quadratic-linear systems of equations. You learned how to solve these systems of equations graphically and algebraically.

In the next lesson, you will learn how to solve an additional type of system of equations called a quadratic-quadratic system of equations.

Lesson 2: Solving Quadratic-Quadratic Systems of Equations

Lesson Focus

In this lesson, you will

- ❑ learn how to solve quadratic-quadratic systems of equations algebraically, graphically, and by using technology
- ❑ learn how to model a situation using a system of quadratic-quadratic equations

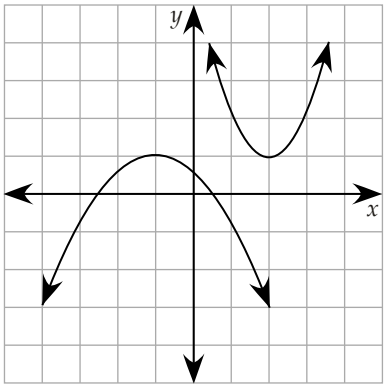
Lesson Introduction

In Lesson 1, you learned how to solve quadratic-linear systems of equations. These systems contain one quadratic equation and one linear equation. It is also possible to solve systems of quadratic-quadratic equations, which contain two quadratic equations in two variables. When you solve these types of equations, you need to remember how to graph quadratic functions in vertex form, and how to use the quadratic formula to solve quadratic equations. For this reason, this lesson serves as an excellent review of how to solve quadratic equations.

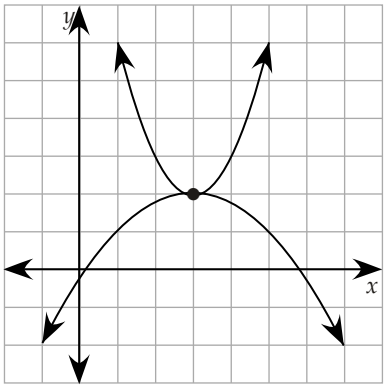
Quadratic–Quadratic Systems of Equations

Quadratic-quadratic systems of equations consist of two quadratic functions. The points of intersection where the two quadratic functions (or two parabolas) have points in common are called the solution to the quadratic-quadratic system of equations. Add this definition to your resource sheet.

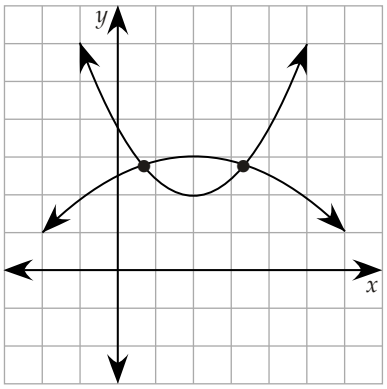
Consider the following diagrams.



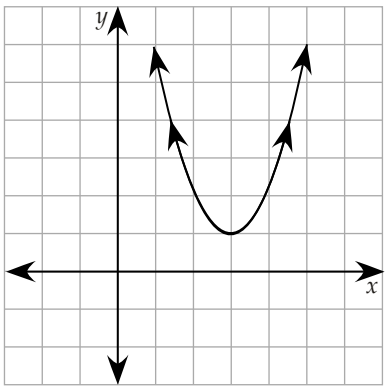
no intersection
∴ no solution



one point of intersection
∴ one solution



two points of intersection
∴ two solutions



The two parabolas are the same and have an infinite number of points of intersection along the curve.
∴ infinite number of solutions

As you can see, there are four different types of solutions with regard to quadratic-quadratic systems of equations. There can be no solution (no point of intersection), one solution (one point of intersection), two solutions (two points of intersection), or an infinite number of solutions. An infinite number of solutions occurs only when the quadratic equations are identical. It is possible to find the solution to a system of quadratic-quadratic systems of equations either graphically or algebraically.

Real-Life Models of Quadratic-Quadratic Systems of Equations

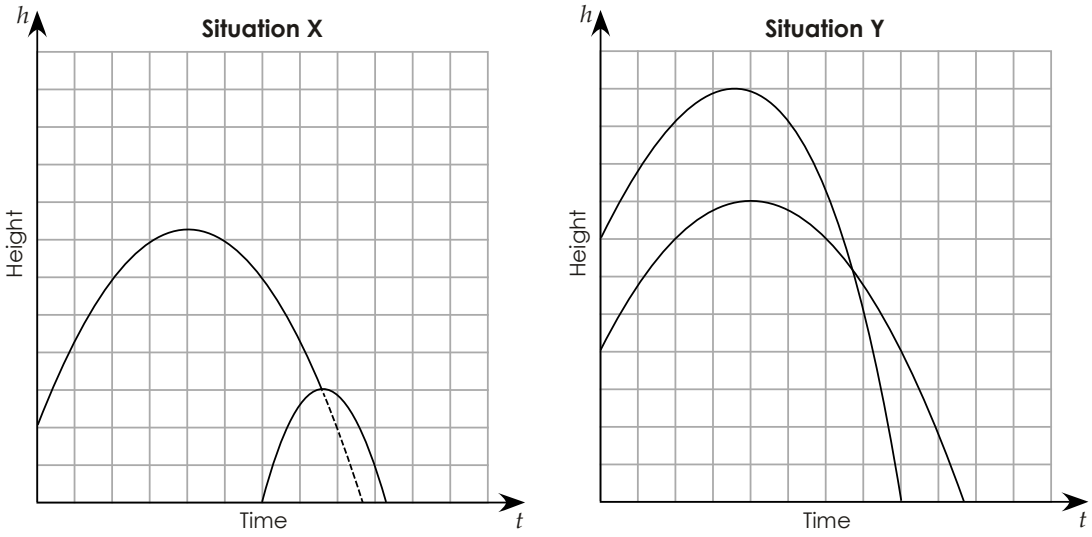
There are a number of situations that can be modelled by a quadratic-quadratic system of equations. You only have to look at sports, such as high jump or diving, that involve jumping or sports, such as football, basketball, and baseball, that involve throwing and jumping to see situations that can be modelled by a quadratic-quadratic system of equations.

Example 1

Consider the two graphs containing systems of equations showing two different situations below.

a) Which graph models a situation involving two people jumping into a lake at the same time from cliffs of different heights?

b) Which graph models a situation involving a football being thrown and a receiver jumping to catch the ball?



Solution

a) Situation Y models a situation where two people jump off cliffs of different heights into the lake. Each of the parabolas starts at a different height, and both end at the x-axis. Therefore, this symbolizes one person starting at a higher height and jumping into the lake while another person starts at a lower height and jumps into the lake. The jumpers will need to stagger their timing so they don't hit each other where the parabolas intersect.

b) Situation X models the situation of a football being thrown and a receiver jumping to catch the ball. The football is thrown not from ground level but a little above, as the quarterback is most likely throwing from chest height. Also, the receiver doesn't start jumping until the ball is near. Therefore, the second parabola doesn't start until the first one is more than halfway finished. They intersect when the player catches the ball.

Solving Quadratic–Quadratic Systems of Equations Graphically

Just as it was possible to solve quadratic-linear systems of equations both graphically and algebraically, it is also possible to solve quadratic-quadratic systems of equations this way. It is possible to solve these types of equations by graphing using either graph paper or technology such as a graphing calculator.

Example 2

Solve each system graphically.

a) Use graph paper:

$y = x^2$

$y = -(x - 2)^2 + 2$

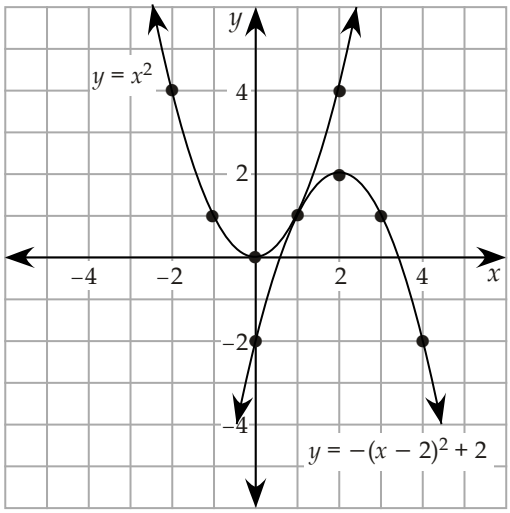
b) Use graphing technology:

$$y = \frac{1}{2}(x + 1)^2 - 4$$

$-6x^2 + x + 3 = y$

Solution

a) You should be able to solve this system of equations by hand using graph paper and the properties of transformations of quadratic functions.

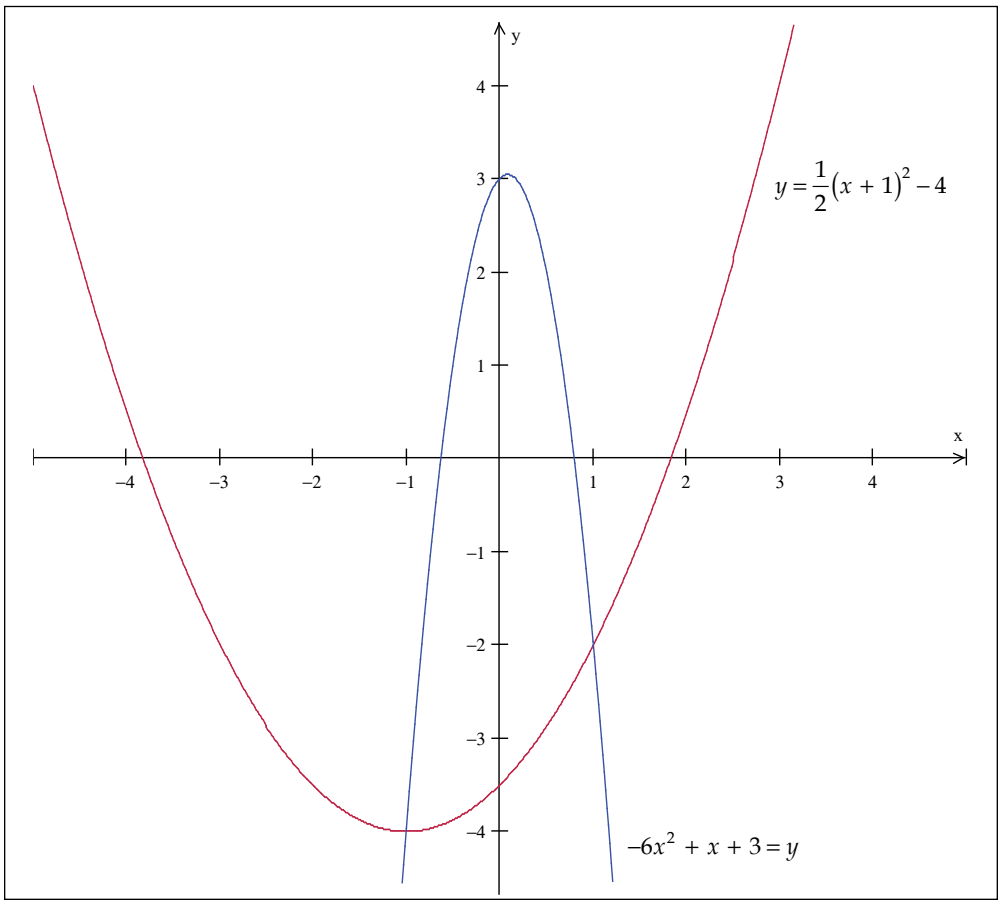


As you can see, $y = x^2$ and $y = -(x - 2)^2 + 2$ have one point of intersection at (1, 1). Therefore, (1, 1) is the solution to this system of equations.

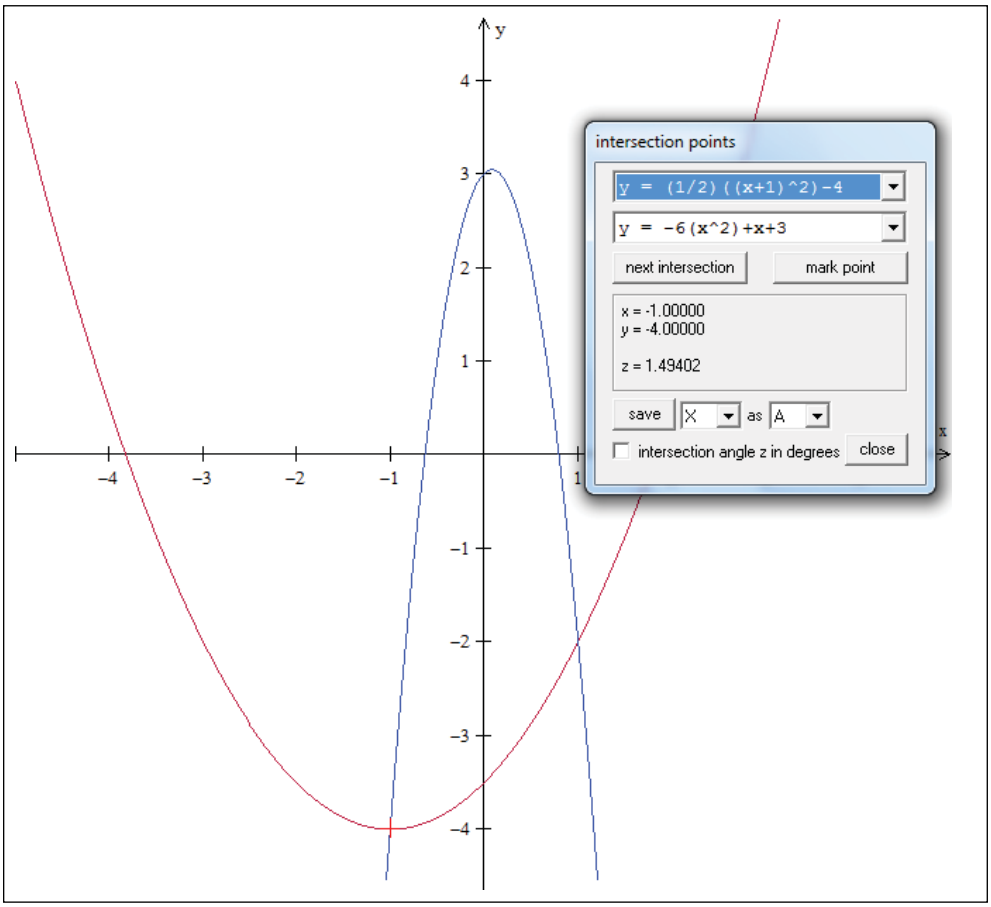
b) $y = \frac{1}{2}(x+1)^2 - 4$

$-6x^2 + x + 3 = y$

Graph both of the above functions using a graphing calculator or a graphing applet such as Winplot. The graph created using Winplot is shown below.



As you can see, there are two points of intersection. To calculate these points, click "Two" in the menu bar and then click "Intersections."



The first intersection occurs at the point (−1, −4).

Click "Next Intersection" to show the next intersection. The next intersection occurs at the point (1, −2).

The two solutions to this system of equations are the points (−1, −4) and (1, −2).

Solving Quadratic–Quadratic Systems of Equations Algebraically

It is possible to solve quadratic-quadratic systems of equations algebraically by using the substitution method and also by using the elimination method. Consider the following examples.

Example 3: The Substitution Method

Solve the following system of equations by using the substitution method. Write your answers rounded to two decimal places.

$y = 2x^2 + 7x - 11$ Equation (1)

$5x^2 - 3x - y = 6$ Equation (2)

Solution

Since Equation (1) is already solved for y, you can substitute the expression from Equation (1) into the y-variable of Equation (2).

$$\begin{aligned} 5x^2 - 3x - y &= 6 \\ 5x^2 - 3x - (2x^2 + 7x - 11) &= 6 \\ 5x^2 - 3x - 2x^2 - 7x + 11 &= 6 \\ 3x^2 - 10x + 5 &= 0 \end{aligned}$$

As this quadratic equation is not factorable, you will need to use the quadratic formula to solve for x, with $a = 3$, $b = -10$, and $c = 5$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{10 \pm \sqrt{100 - 4(3)(5)}}{6}$$

$$x = \frac{10 \pm \sqrt{40}}{6}$$

$$x = \frac{10 \pm 2\sqrt{10}}{6}$$

$$x = \frac{5 \pm \sqrt{10}}{3}$$

$$x = \frac{5 + \sqrt{10}}{3} \text{ or } \frac{5 - \sqrt{10}}{3}$$

Therefore, $x \approx 2.72$ or $x \approx 0.61$.

Now you need to substitute the values for x into one of the original quadratic equations to find the corresponding y-values. To avoid rounding errors, the answers for y are calculated using all the decimal places for x given by the calculator.

When $x = 2.72$.

$y = 2x^2 + 7x - 11$

$y = 2(2.72) + 7(2.72) - 11$

$y = 22.85$

When $x = 0.61$.

$y = 2x^2 + 7x - 11$

$y = 2(0.61)^2 + 7(0.61) - 11$

$y = -5.96$

Thus, the two approximate solutions are (2.72, 22.85) and (0.61, −5.96).

Remember, when calculating with decimals, round off only in the final answer.

Example 4: The Elimination Method

Solve the following system of equations by using the elimination method.

$6x^2 - 3x = 2y - 5$

$2x^2 + x = y - 4$

Solution

As you want to eliminate y from both equations, rearrange to isolate for y, ensure the coefficients of y in each equation are opposites, and then add the like terms of the two equations.

$6x^2 - 3x = 2y - 5$ Equation (1)

$6x^2 - 3x + 5 = 2y$

or

$2y = 6x^2 - 3x + 5$ Equation (1) rearranged

$2x^2 + x = y - 4$ Equation (2)

or

$y = 2x^2 + x + 4$ Equation (2) rearranged

Now multiply Equation (2) by -2 and add the equations.

$$\begin{array}{rll} (1) & 2y = & 6x^2 - 3x + 5 \\ + \ (-2)\times(2) & -2y = & -4x^2 - 2x - 8 \\ \hline & 0 = & 2x^2 - 5x - 3 \\ & 0 = & (2x+1)(x-3) \end{array}$$

$$2x + 1 = 0 \qquad x - 3 = 0$$

$$x = -\frac{1}{2} \qquad x = 3$$

When $x = -\frac{1}{2}$:

$$6x^2 - 3x = 2y - 5$$

$$6\left(-\frac{1}{2}\right)^2 - 3\left(-\frac{1}{2}\right) = 2y - 5$$

$$6\left(\frac{1}{4}\right) + \frac{3}{2} = 2y - 5$$

$$\frac{3}{2} + \frac{3}{2} = 2y - 5$$

$$3 = 2y - 5$$

$$8 = 2y$$

$$4 = y$$

Thus, one solution is $\left(-\frac{1}{2}, 4\right)$.

When $x = 3$:

$$6(3)^2 - 3(3) = 2y - 5$$

$$54 - 9 + 5 = 2y$$

$$50 = 2y$$

$$25 = y$$

Thus, another solution is (3, 25).

The two solutions to this system of equations are $\left(-\frac{1}{2}, 4\right)$ and (3, 25).

Example 5

Two projectiles are shot towards each other. The first projectile follows a path represented by the equation $-x^2 + 6x - y - 1 = 0$, while the second projectile follows a path represented by the equation $y - 11 = -2x^2 - 5x$, where x represents the time the projectiles travel, measured in seconds, and y represents the height the projectiles travel, measured in metres. How much time will elapse before the two projectiles meet and to what height will they travel from the ground before they meet?

Solution

To solve this problem, you need to find the solution(s) to the following system of equations:

$-x^2 + 6x - y - 1 = 0$ (first projectile)

$y - 11 = -2x^2 - 5x$ (second projectile)

$$\begin{aligned} -y &= x^2 - 6x + 1 \\ +\ y &= -2x^2 - 5x + 11 \\ \hline 0 &= -x^2 - 11x + 12 \\ 0 &= x^2 + 11x - 12 \\ 0 &= (x + 12)(x - 1) \end{aligned}$$

$x + 12 = 0$ $\qquad$ $x - 1 = 0$

$x = -12$ $\qquad$ $x = 1$

The projectiles cannot intersect at a negative time, thus, $x = 1$ is the only possible solution for x.

When $x = 1$:

$$\begin{aligned} -x^2 + 6x - y - 1 &= 0 \\ -(1)^2 + 6(1) - y - 1 &= 0 \\ -1 + 6 - y - 1 &= 0 \\ 4 - y &= 0 \\ 4 &= y \end{aligned}$$

The point of intersection of the two equations is (1, 4).

Thus, the two projectiles meet after one second, four metres off the ground.

You should now be prepared to complete the following learning activity. Once you complete Learning Activity 6.2, check your answers in the learning activity answer keys found at the end of this module. If you have difficulty with any of the questions, go back and review those areas in Lesson 2 before you complete the assignment that follows this learning activity.

Learning Activity 6.2

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Between which two whole numbers is $\sqrt{150}$?
2. Simplify: $3x^2 + 4x - 6 - 5x + 4 + 6x^2 + 8 - 9x^2$
3. Pizza Express is offering two medium pizzas for $15.99 while Speedy Pizza is offering three medium pizzas for $22.49. Which is the better buy?
4. Lisa worked 40 hours in one week and made $520. How much does Lisa make an hour?
5. Calculate: $\frac{4}{7} - \frac{2}{3}$
6. Express as an entire radical: $7y^{\frac{8}{3}}$
7. What is the slope of a vertical line?
8. How many centimetres are in 2 kilometres?

Part B: Solving Quadratic-Quadratic Systems of Equations

Remember, these questions are similar to the ones that will be on your assignments and final exam. So, if you were able to answer them correctly, you are likely to do well on your assignments and final exam. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Verify that $\left(\frac{10}{3}, \frac{17}{9}\right)$ and $(1, -2)$ are solutions to the following system of equations.

   $y = 2x^2 - 7x + 3$

   $y = -x^2 + 6x - 7$

continued

Learning Activity 6.2 (continued)

2. Solve the following systems of equations, both graphically and algebraically.

   a) $y = -2x^2$

      $y = 2x^2 - 4$

   b) $x^2 + y - 3 = 0$

      $x^2 - y + 1 = 0$

3. Solve the following system of equations graphically, using technology. Round to three decimal places when necessary. Note: If you do not have access to technology, you can still do this question by hand and estimate the solution. **You will not have questions on your exam requiring the use of technology.**

   $y = 2x^2 - 5x + 1$

   $y = 2x^2 + 6x - 2$

4. Solve the following systems of equations by using the substitution method.

   a) $y = x^2 + 4$

      $y = -x^2 + 12$

   b) $y = 2(x + 4)^2$

      $y = 2x^2 + 16x + 32$

5. Solve the following systems of equations by using the elimination method.

   a) $y = x^2 - 2x + 2$

      $3x^2 - 4y = 7$

   b) $5x^2 + 3y = -3 - x$

      $2x^2 - x = -4 - 2y$

6. Solve the following system of equations using any method you prefer. Explain why you chose that method.

   $8x^2 + 100 = -5y$

   $6x^2 - x - 5 = 3y$

7. Let x represent time, measured in seconds, and let y represent height, measured in feet. A football is thrown and its path is represented by the equation $y = -x^2 + 4x + 2$. A receiver jumps to receive the ball and his path is represented by the equation $y = -(x - 5)^2 + 6$. Does the receiver have any chance of catching the football? Explain.

Lesson Summary

In this lesson, you learned how to solve quadratic-quadratic systems of equations. You learned how these types of systems of equations are useful if you are trying to determine when two objects will meet, such as a football and a receiver, or two projectiles. These types of equations can also be applied to preventing two objects from colliding, such as two airplanes.

In the next lesson, you will learn about linear inequalities. They are helpful in situations where there are restrictions, such as finding the values that are either greater than or less than a certain function.

Notes

Assignment 6.1

Solving Systems of Equations

Total: 40 marks

1. Verify that (−3, 0) and (1, 8) are solutions to the following system of equations. *(3 marks)*

$$y = 2x + 6$$
$$x^2 + 4x = y - 3$$

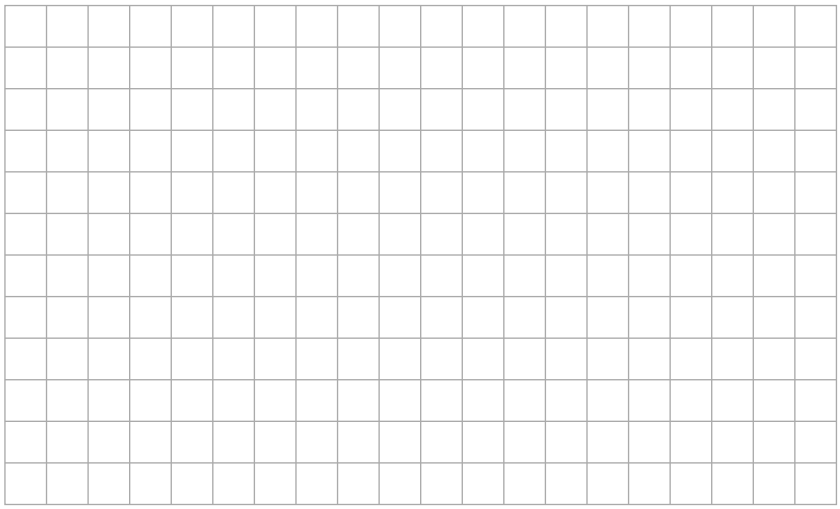
continued

Assignment 6.1: Solving Systems of Equations (continued)

2. Solve the following systems of equations algebraically and graphically.
   (6 marks each, for a total of 12 marks)

   a) $y = x^2 + 4x + 6$

      $y = 2x + 6$

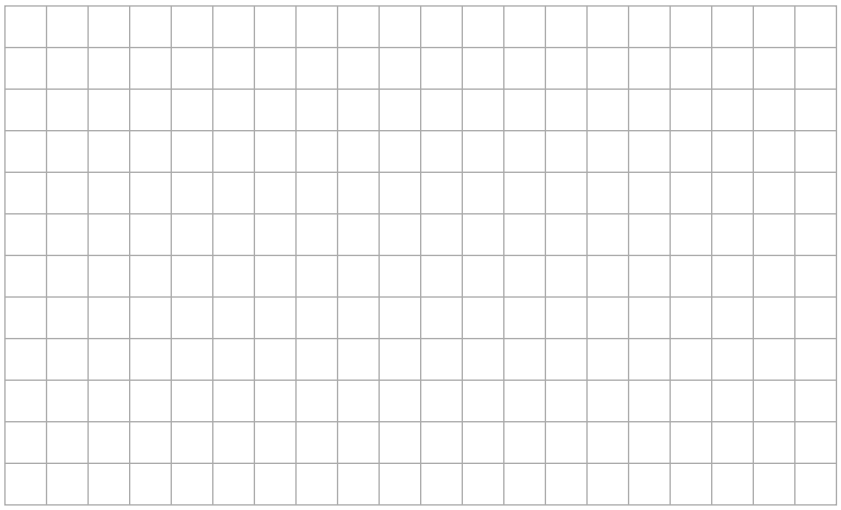


continued

Assignment 6.1: Solving Systems of Equations (continued)

b) $y = \frac{1}{4}(x+5)^2 - 3$

$y = -\frac{1}{3}(x-3)^2 + 4$



continued

Assignment 6.1: Solving Systems of Equations (continued)

3. Solve the following systems of equations using the substitution method.
(4 marks each, for a total of 8 marks)

a) $y = x^2 + 4x + 1$
$y = 5x + 3$

b) $y = x^2 + x - 1$
$y = -2x^2 - 5x + 8$

continued

Assignment 6.1: Solving Systems of Equations (continued)

4. Solve the following systems of equations using the elimination method. *(4 marks each, for a total of 8 marks)*

   a) $y = -x^2 + \frac{8}{3}x - 7$

   $-3y = 2x^2 + 6$

   b) $y = x^2 + 5x - 3$

   $-3y = 2x^2 - 21x + 1$

continued

Assignment 6.1: Solving Systems of Equations (continued)

5. The sum of two numbers is 27. Their product is 126. What are the two numbers? *(4 marks)*

continued

Assignment 6.1: Solving Systems of Equations (continued)

6. Bezait is playing baseball with her friends. When it is her turn at bat, she hits the baseball and its path is modelled by the equation $y = -\frac{1}{4}x^2 + 2x$, where y represents the height of the ball in metres and x represents the horizontal distance from the batter measured in metres. One of her friends is in the outfield and the ball is heading straight for her. Her friend decides to jump and try to catch the ball. Her path is modelled by the equation $y = -\frac{1}{4}(x-6)^2 + 3$.

   a) Solve the following system of equations algebraically. *(4 marks)*

   $y = -\frac{1}{4}x^2 + 2x$

   $y = -\frac{1}{4}(x-6)^2 + 3$

   b) If you were to graph the two equations, what would be the meaning of the point of intersection? What assumption are you making? *(1 mark)*

Notes

Lesson 3: Solving Linear Inequalities in Two Variables

Lesson Focus

In this lesson, you will

- ❑ learn how to graphically find the solution of a linear inequality in two variables
- ❑ learn how to determine a linear inequality when given its graph
- ❑ learn how to apply linear inequalities to solve a word problem

Lesson Introduction

You have now learned several ways of solving systems of equations. You will now investigate the idea of a linear inequality because some of the mathematics that you use in daily life do not come in the form of an equation. Statements such as, "I want to find a job that pays more than the minimum wage" or "I have to spend less than $10 this week" are examples of statements of inequalities.

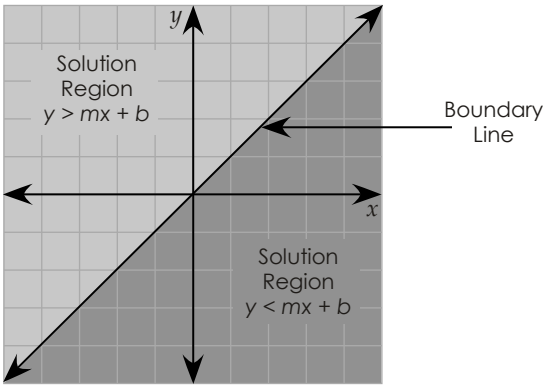
Linear Inequalities in Two Variables

In order to be able to solve linear inequalities, you first need to review what the inequality symbols are.

- ≤ means less than or equal to
- < means less than
- ≥ means greater than or equal to
- > means greater than

Graphing a Linear Inequality

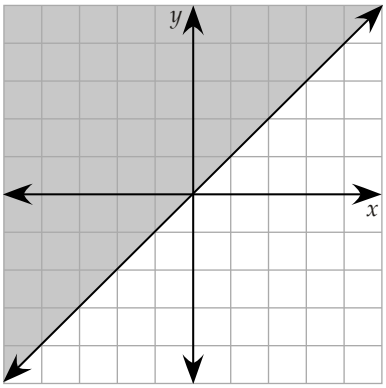
The graph of a line separates the plane into three distinct regions, the line and the set of all points on either side of the line. The line itself is called the **boundary line**.

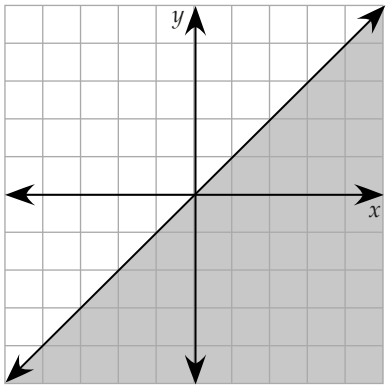


The solution region is above the boundary line when $y > mx + b$ and is below the boundary line when $y < mx + b$.

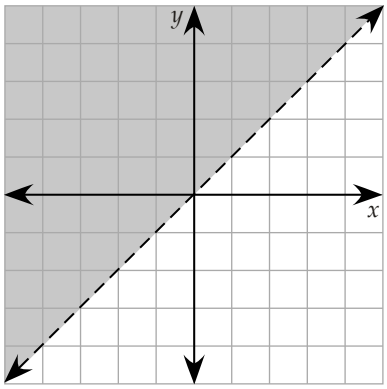
A linear inequality has a **boundary line** that can be expressed in the form $y = mx + b$. The **solution region** to a linear inequality is the set of all ordered pairs that make the inequality true.

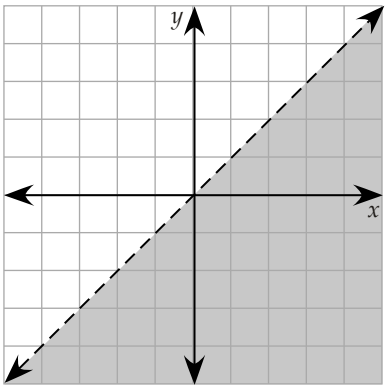
When the inequality is $\leq$ (read *less than or equal to*) or $\geq$ (read *greater than or equal to*), the solution includes the points on the boundary line and the graph has a solid boundary line, as shown below.





When the inequality is < (read *less than*) or > (read *greater than*) the solution does not include the points on the boundary line and the graph will have a dashed boundary line, as shown below.





Example 1

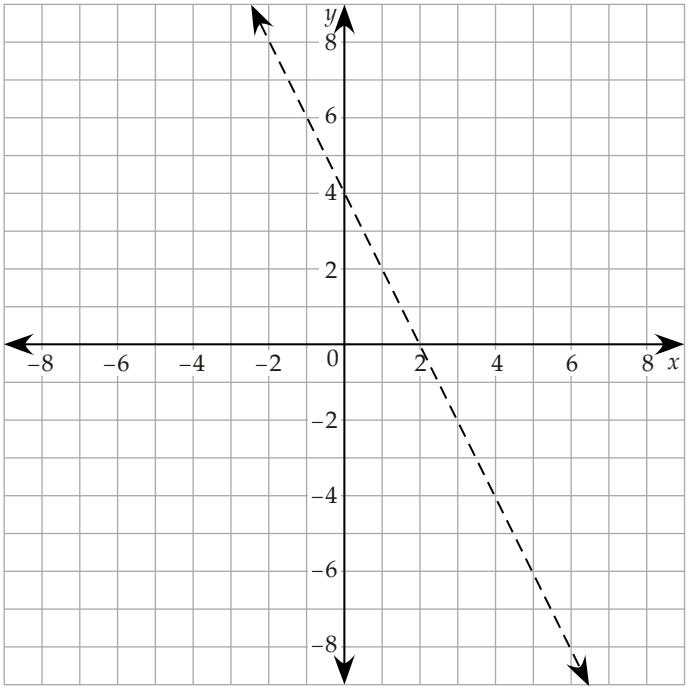
Graph $2x + y < 4$.

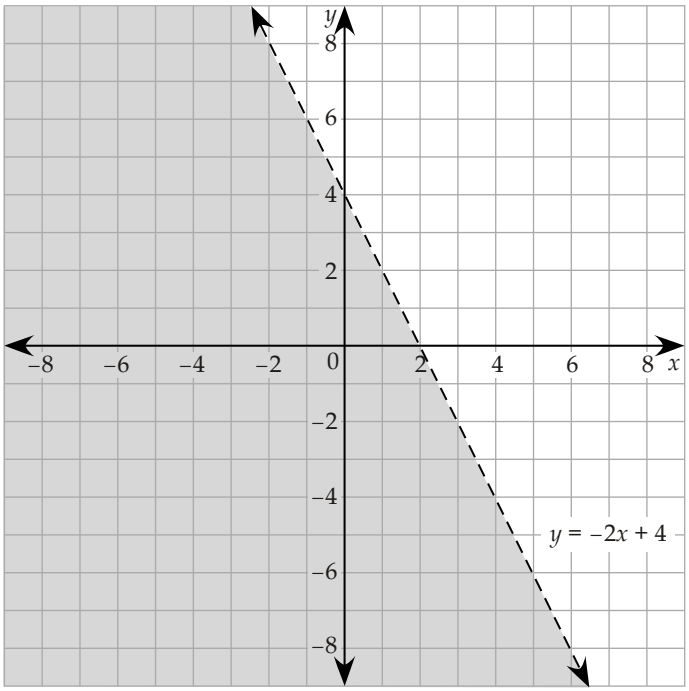
Solution

The boundary line is $2x + y = 4$. You can graph the line by rearranging it into $y = -2x + 4$. You can use the slope-intercept method, with $b = 4$ and the slope -2, to graph the line.

Use a dashed line to show that the points on the line are not solutions to $2x + y < 4$.

To determine which region to shade, you must rearrange the inequality to solve for y. The inequality $y < -2x + 4$ states that you want all y-values less than those on the line $y = -2x + 4$. Thus, the region below the line is shaded.





To verify which region contains the solution, select a point from each region and see which ordered pair makes the inequality true. If the inequality is true for one point in the solution region, it will be true for all points in that region. An easy test point is the origin, (0, 0), as long as the boundary line does not go through it. If it does, then choose a different point, such as (1, 0).

On the bottom side of the line, choose (0, 0) as a test point for $y < -2x + 4$.

LHS	RHS
y	$-2x + 4$
0	$-2(0) + 4$
	4

LHS < RHS

$\therefore$ (0, 0) is in the solution region as expected.

On the top side of the line, choose (4, 0) as a test point for $y < -2x + 4$.

LHS	RHS
y	$-2x + 4$
0	$-2(4) + 4$
	-4

LHS $\not<$ RHS

$\therefore$ (4, 0) is not in the solution region as expected.

Example 2

Graph:

a) $2x + y \leq 4$

b) $2x + y > 4$

c) $2x + y \geq 4$

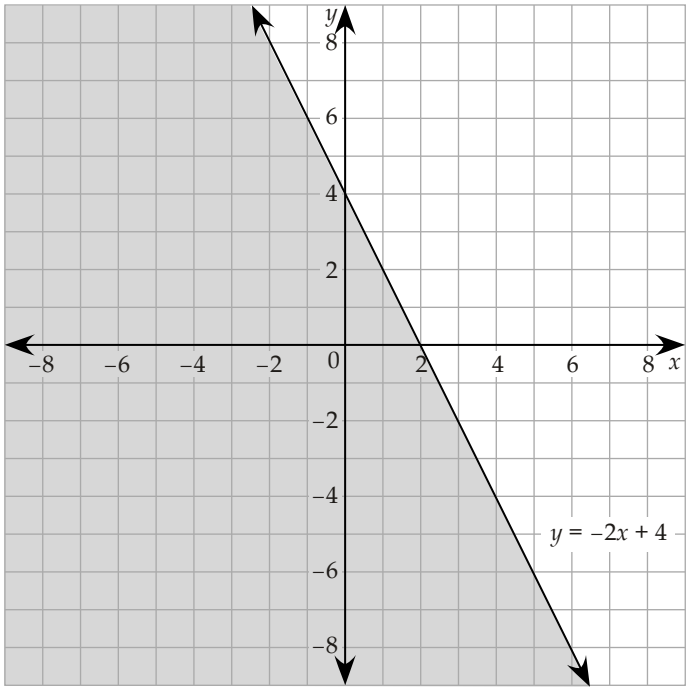
Solutions:

a) $2x + y \leq 4$

becomes $y \leq -2x + 4$

Sketch the boundary line using the related equation, $y = -2x + 4$.

The boundary line would be solid and the shaded region has y-values less than the line on the bottom side.



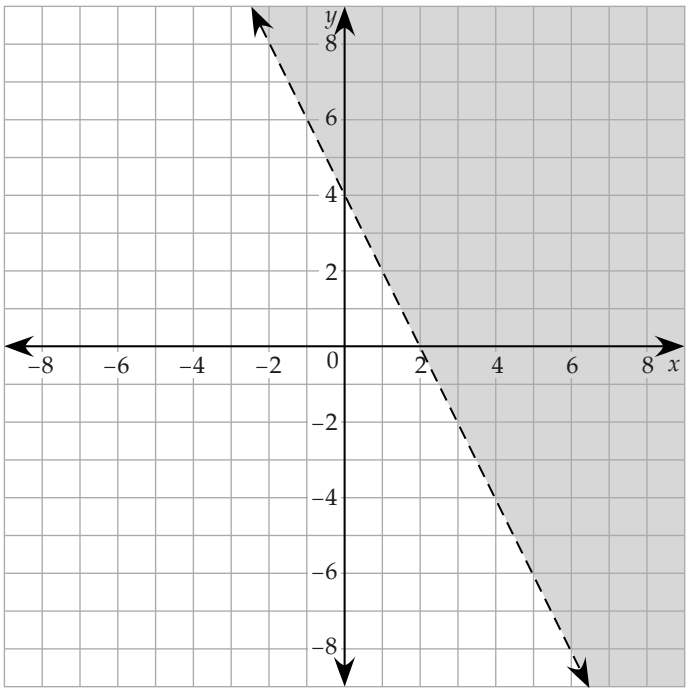
b) $2x + y > 4$

becomes $y > -2x + 4$

Sketch the boundary line using the related equation $y = -2x + 4$.

The boundary line would be dashed to show that the points on the line are not solutions of $y > -2x + 4$.

The shaded region contains points whose y-values are greater than those on the line and are above the line.

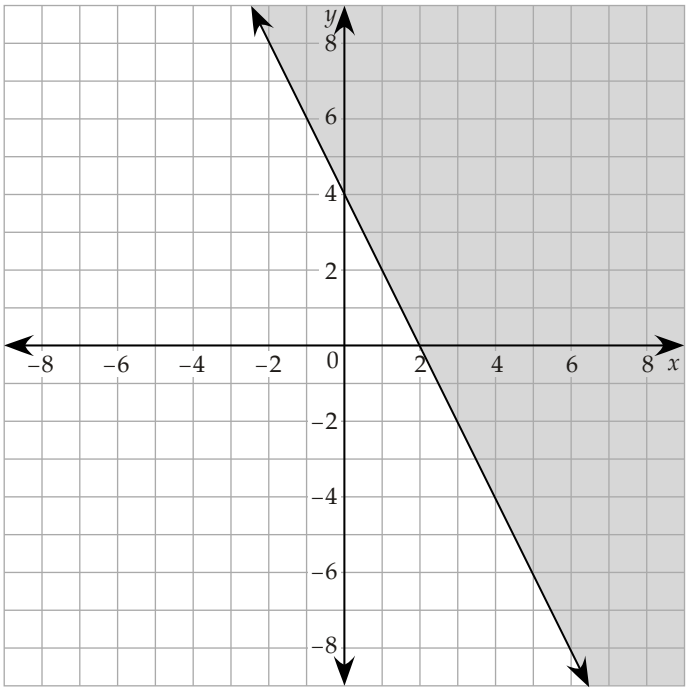


c) $2x + y \geq 4$

becomes $y \geq -2x + 4$

Sketch the boundary line using the related equation $y = -2x + 4$.

The boundary line is solid and the shading is the same as in $y > -2x + 4$.



To summarize:

A linear inequality in x and y is an inequality that can be written in one of the following forms: $y < mx + b$, $y \leq mx + b$, $y > mx + b$, $y \geq mx + b$.

An ordered pair (s, t) is a solution of a linear inequality in x and y if the inequality is true when s and t are substituted for x and y, respectively.

For instance (4, 0) is a solution to $y > -2x + 4$ because upon substituting, 0 is greater than $-2(4) + 4 = -4$, or $0 > -4$ is a true statement.

To sketch the graph of a linear inequality in two variables, first sketch the line given by the corresponding equation. Use a solid line for inequalities with $\geq$ or $\leq$. Use a dashed line for inequalities with $<$ or $>$. This boundary line separates the coordinate plane into two solution regions. In one of the regions are points that are solutions of the inequality. In the other region, no point is a solution.

The solution region is determined by the inequality sign $<$ or $>$. If $y < mx + b$, shade where the y-values are less than the y-values on the line $y = mx + b$. If $y < mx + b$, shade where the y-values are greater than the y-values on the line $y = mx + b$.

You can verify whether the points in an entire solution region satisfy the inequality by testing one point in the solution region.

Example 3

Sketch the graph of the inequality $y \geq 3$.

Solution

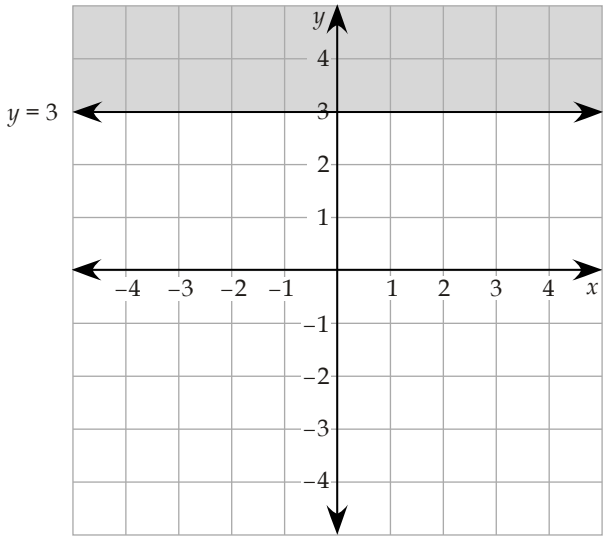
Graph the solid horizontal line $y = 3$ and all points above $y = 3$.

Next, check the point (0, 0) by substituting it into the inequality:

LHS	RHS
y	3
0	3

$$\text{LHS} \not\geq \text{RHS}$$

The origin is not a solution and it is below the line. This confirms the shaded region is all points on or above the line $y = 3$.



Example 4

Sketch the graph of the inequality $x < 2$.

Solution

$x < 2$

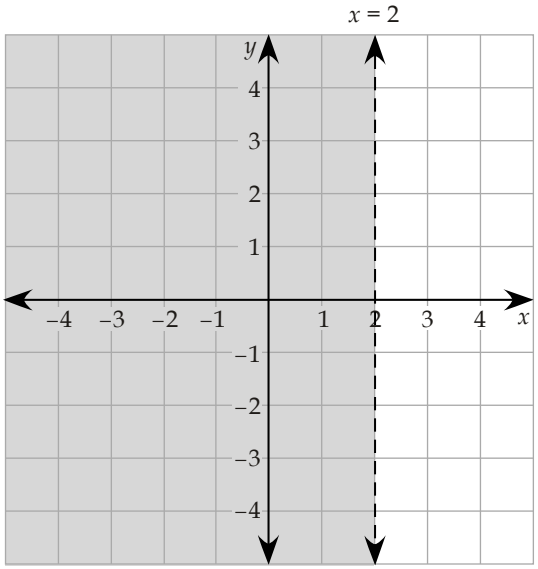
This is a little different since it cannot be written in the form "$y =$." When the inequality is written in terms of x, you will shade left for < and right for >, rather than above or below.

Sketch the dashed vertical line, $x = 2$ and shade the region on its left side. Next, choose a test checkpoint (0, 0). Substitute it into the inequality:

$x < 2$

$0 < 2$ True

This is a solution that represents all points to the left of $x = 2$.



Now that you have some experience with graphing linear inequalities, it is time to add to your resource sheet. Add any definitions or steps that you feel would help you remember how to graph linear inequalities in two variables.

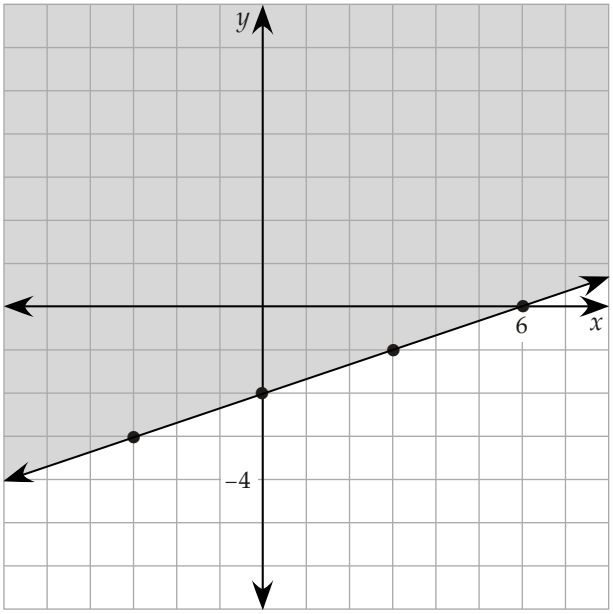
Note that you could also find the x- and y-intercepts to graph each line.

Determining an Inequality When Given Its Graph

Now that you know how to graph an inequality, you can also do the process in reverse. In other words, when given the graph of an inequality, you can determine the corresponding inequality. Consider the following example.

Example 5

Determine the inequality represented by the following graph.



Solution

To find the inequality, you first need to find the equation of the boundary line.

To do this, you can find the slope of the line, and the y-intercept. Then, you can put those values into the slope-intercept form equation of a line.

Slope = $\frac{1}{3}$

y-intercept = -2

slope-intercept equation of a line: $y = mx + b$

Thus, the equation of this line is $y = \frac{1}{3}x - 2$

Now, you need to determine whether you have a greater than or less than symbol.

As you can see, the solution region is above the boundary line. Therefore, the symbol you need is greater than or >.

You now need to determine if the points on the boundary line are included in the solution or not.

As the boundary line is solid, the points on the boundary line are included in the solution to the inequality.

Thus, the inequality becomes $y \geq \frac{1}{3}x - 2$.

To check your inequality, choose a test point and substitute. For example, if you choose (0, 0), it should satisfy the inequality $y \geq \frac{1}{3}x - 2$. Upon substituting, you get $0 \geq \frac{1}{3}(0) - 2$ or $0 \geq -2$, which is true.

If you need to, copy down steps for how to determine the equation of the linear inequality when given its graph on your resource sheet.

Applying and Solving a Linear Inequality In a Word Problem

Sometimes, you will not be given a linear inequality to graph. Instead, you will have to create a linear inequality from the information given in the question. This is similar to creating a linear equation from given information. Consider the following example.

Example 6

A phone company charges 25 cents for daytime minutes and 5 cents for evening/weekend minutes. Find all combinations of daytime and evening/weekend minutes you would have in order to be charged more than $5 in a week.

Solution

Let d = the number of daytime minutes

Let e = the number of evening/weekend minutes

Thus, $25d + 5e$ has to be greater than some number.

You may be tempted to write $25d + 5e > 5$. However, this is not correct, as your units are different. The 25 and 5 are measured in cents; however, the $5 is measured in dollars. Therefore, you need to convert $5 to cents.

$5 = 500 cents

Thus, the linear inequality becomes $25d + 5e > 500$.

To find all the combinations, you first need to solve for a variable and then graph the solution. Let d, the number of daytime minutes, be graphed on the x-axis, and e, the number of evening/weekend minutes, be graphed on the y-axis. You will then need to solve this inequality in terms of e and then graph the corresponding inequality.

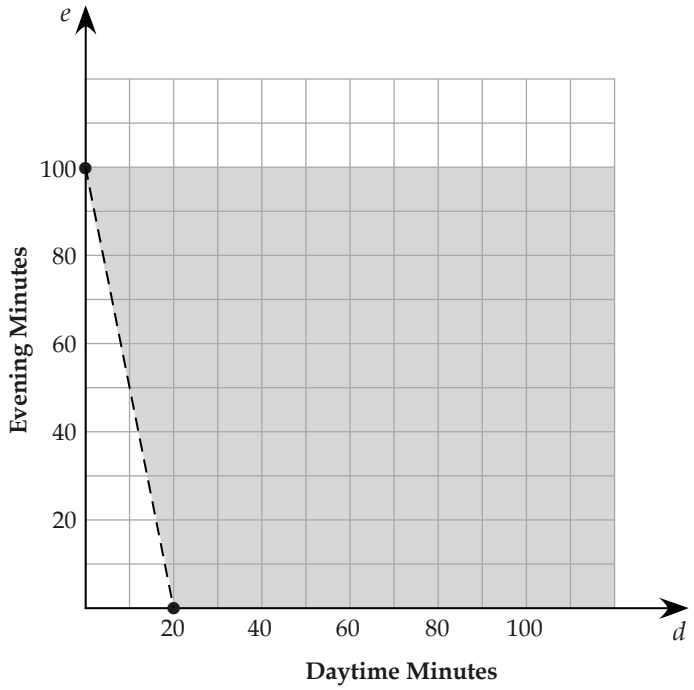
$$5e > 500 - 25d$$
$$e > 100 - 5d$$
$$e > -5d + 100$$

Sketch the dashed line using the related equation $e = -5d + 100$ by finding the x- and y-intercepts and shade the region above the line. The e-intercept is 100 and the d-intercept is 20. Next, choose a checkpoint (0, 0). Substitute it into the inequality.

LHS	RHS
e	$-5d + 100$
0	$-5(0) + 100$
	100

LHS $\not>$ RHS

As expected, the solution region does not contain the origin.



Note: The graph does not contain negative values, as it is not possible to speak for a negative amount of minutes on the phone.

As you can see, there are many combinations of daytime and evening minutes that will satisfy this inequality. Some combinations include:

- 40 daytime minutes and 20 evening/weekend minutes, or the point (40, 20)
- 25 daytime minutes and 10 evening/weekend minutes, or the point (25, 10)

Example 7

Solve for y in the following linear inequality.

$$-12x - 3y > 3$$

Solution

To solve for y, you simply isolate the variable y.

$$-12x - 3y > 3$$

$$-3y > 12x + 3$$

When you get to this step, you may be tempted to just divide by negative three and leave the inequality sign the same. However, when you are multiplying or dividing both sides of the inequality by a negative number, you need to reverse the inequality sign. For example, if the inequality sign was originally >, it would become <. If the inequality sign was originally $\leq$, it would become $\geq$.

Add this information to your resource sheet.

Thus, the inequality becomes:

$$-3y > 12x + 3$$

$$\frac{-3y}{-3} < \frac{12x}{-3} + \frac{3}{-3}$$

$$y < -4x - 1$$

Learning Activity 6.3

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Jessica sells $545 worth of fundraising tickets. If she makes 5% commission on her total sales, how much did Jessica make?
2. Simplify: $(3x + 1)^2$
3. Simplify: $\sqrt{54}$
4. Sam mowed half of the yard on Friday and half of the remaining grass on Saturday. How much grass still needs to be cut?
5. What is the slope of the line defined by the equation $2y = 4x + 6$?
6. Factor completely. $121x^2 - 196y^2$
7. The diameter of a circle is 34 cm. What is the radius of the circle?
8. Is 87 a prime or composite number?

Part B: Solving Linear Inequalities in Two Variables

Remember, these questions are similar to the ones that will be on your assignments and final exam. So, if you were able to answer them correctly, you are likely to do well on your assignments and final exam. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Which of the ordered pairs are solutions to the given inequality?
   a) $4x - 3y > 2$

      $(-4, 8), (1, 2), (3, -6)$

   b) $-8x + 6y \leq 5$

      $(-2, -7), (4, 9), (-1, 8)$

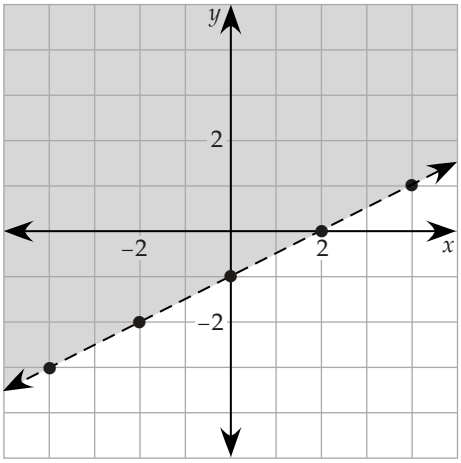
continued

Learning Activity 6.3 (continued)

2. Sketch the graphs of the following linear inequalities.
   a) $y < \frac{1}{2}x + 2$
   b) $y \geq 0$
   c) $3x + 2y \leq 6$
   d) $-2y + 8 > 5x$

3. Match each linear inequality to its graph.
   a) $y \leq 4$
   b) $y \leq -x + 1$
   c) $2y > x - 2$
   d) $y \geq 4$

i)



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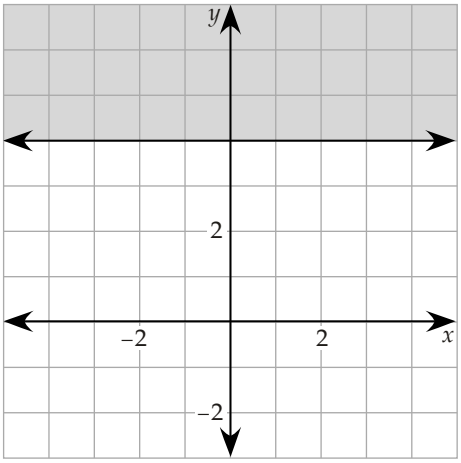

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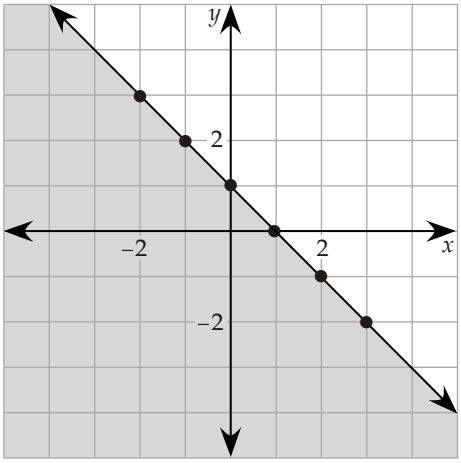

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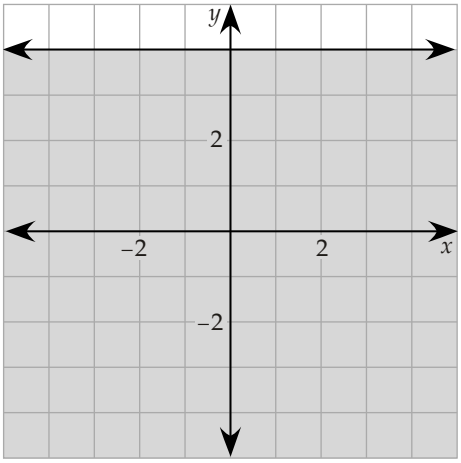

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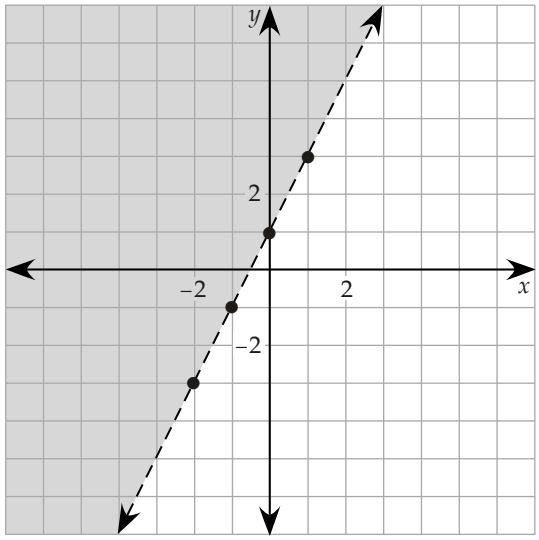
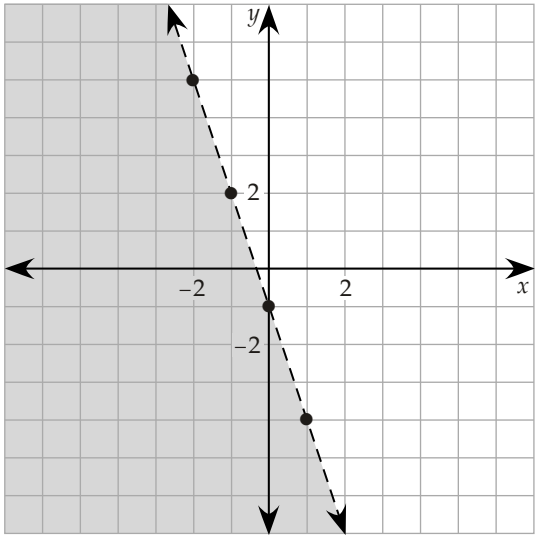
Learning Activity 6.3 (continued)

4. Write a linear inequality to represent each graph.

a)



b)



continued

Learning Activity 6.3 (continued)

5. Hunter is a high school student and he wants to save enough money to pay for his first year at university. He is holding down two part-time jobs. If he wants to save enough money, he needs to make at least $200 a week. One of his part-time jobs is working at a restaurant and he is paid $10 an hour. He makes $12 an hour at his second part-time job as a maintenance worker at his dad's business.
   a) Write an inequality to represent Hunter's situation.
   b) Graph the inequality.
   c) List two different ordered pairs that are solutions to this inequality.
   d) What do these ordered pairs symbolize?

Lesson Summary

In this lesson, you learned how to solve linear inequalities in two variables, x and y, graphically. You learned whether a solid or broken boundary line should be used when graphing an inequality, and how to test points to determine the solution region of a linear inequality. You also discovered how linear inequalities appear in financial situations, such as making a certain amount of money or paying a certain amount for your phone bill.

In the next two lessons, you will be learning about a different type of inequality: quadratic inequalities.

Notes

Lesson 4: Solving Quadratic Inequalities in One Variable

Lesson Focus

In this lesson, you will

- ❑ learn how to solve quadratic inequalities in one variable by graphing the corresponding function
- ❑ learn how to solve quadratic inequalities in one variable by making a sign diagram
- ❑ learn how to solve quadratic inequalities in one variable by case analysis using linear inequality diagrams
- ❑ learn how to solve quadratic inequalities in one variable by sign analysis making a factor diagram

Lesson Introduction

In the last lesson, you learned how to solve linear inequalities graphically. The approach for solving quadratic inequalities in one variable is similar to the strategy you used to solve linear inequalities. However, instead of solution regions, you will be looking for intervals of solutions or solution sets, because there is only one variable, not two.

Quadratic inequalities are useful when you are dealing with situations involving parabolas, such as in sports, water fountains, or fireworks. In these situations, you may want to know when the height of the ball, the water, or the fireworks is above a certain value. If you are able to solve quadratic inequalities, you will be able to find the information you need. You can see from the list of what you are learning in this lesson that diagrams are involved in every method.

Quadratic Inequalities in One Variable

Recall, linear inequalities are inequalities in which the highest exponent of x is one. Quadratic inequalities are inequalities in which the highest exponent of x is two.

For this course, there are four forms of quadratic inequalities in one variable depending upon the inequality sign:

$ax^2 + bx + c < 0$

$ax^2 + bx + c \leq 0$

$ax^2 + bx + c > 0$

$ax^2 + bx + c \geq 0$

where a, b, and c are rational numbers and $a \neq 0$.

When you have an inequality with a greater than or equal to (≥) sign or a greater than (>) sign, you want to find the interval where the expression is positive. When you have an inequality with a less than or equal to (≤) sign or a less than (<) sign, you want to find the interval where the expression is negative. The **solution set** of a quadratic inequality is the interval of points that makes the inequality true. You can express your answer using either set notation or interval notation.

A solution to one of these quadratic inequalities can contain no points, one point, an interval of points, or all real numbers. You will discover when each of these situations occurs in the following examples.

Solving Quadratic Inequalities by Graphing the Corresponding Function

It is possible to solve quadratic inequalities by graphing. However, you may recall from earlier modules that graphing quadratic functions in general form can sometimes require completing the square. Therefore, this method is mainly used when quadratic inequalities are given to you in vertex form. Consider the following example.

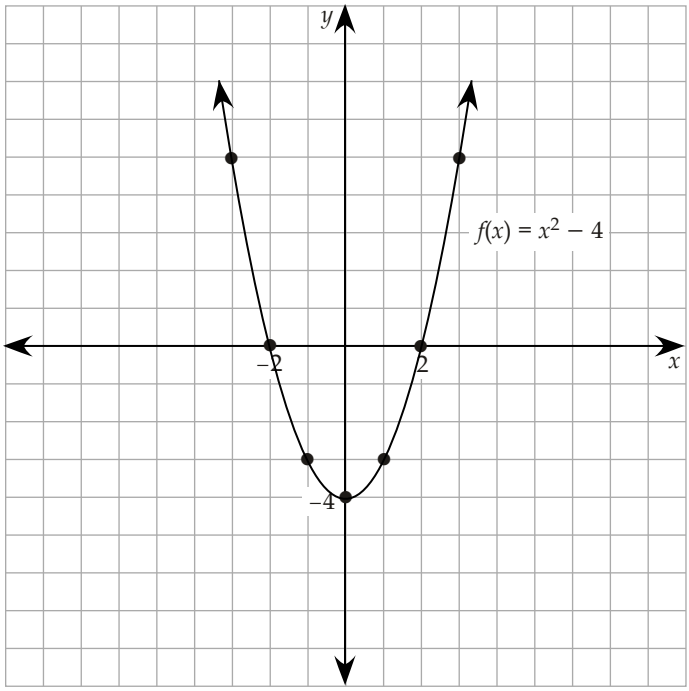
Example 1

Solve the quadratic inequality $x^2 - 4 \leq 0$ by graphing.

Solution

Your first instinct may be to write $x^2 - 4 \leq 0$ as $x^2 \leq 4$ and then take the square root of both sides. However, it is not as meaningful to take square roots in an inequality. It is only a good idea to take the square root of both sides if the equation is an *equality*, not an *inequality*. This is because inequalities are dealing with sets of numbers, not just one or two values.

Therefore, the corresponding function, $f(x) = x^2 - 4$, can be graphed and the solution set read from the graph.



The original inequality asked for values of the expression that are *less than or equal to*. Therefore, you are looking for the x-values at which the graph of $f(x) = x^2 - 4$ lies on or below the x-axis. As you can see, the parabola crosses the x-axis at -2 and 2. Therefore, the parabola lies on the x-axis at these points. Also, the parabola lies below the x-axis between these points. The graph lies on or below the x-axis when $-2 \leq x \leq 2$. Thus, the solution set is $\{x \mid -2 \leq x \leq 2, x \in \Re\}$ or the same thing represented using interval notation is $[-2, 2]$.

Example 2

Solve the quadratic inequality $x^2 - 4 > 0$ by graphing.

Solution

You solved a similar quadratic inequality in Example 1. As both quadratic inequalities have the same corresponding function, $f(x) = x^2 - 4$, you can use the graph from Example 1 to answer this question.

As the inequality asked for values of the expression that are *greater than*, you are looking for the x-values at which the graph lies above the x-axis. This occurs when x is less than -2 and when x is greater than 2. In other words, the solution set is $\{x \mid x < -2 \text{ or } x > 2, x \in \Re\}$. In interval notation, the solution is written as $(-\infty, -2) \cup (2, \infty)$.

As x-values change, a function only changes signs from positive to negative or vice versa, at one of its zeros. The zeros or x-intercepts of a function are the points where the x-value makes the function-value zero. Between two zeros or two x-intercepts, a quadratic function must be entirely positive or entirely negative. Due to the role of the x-intercepts as described here, they are often called **critical points** of the inequality.

Solving Quadratic Inequalities by Making a Sign Diagram

When quadratic inequalities appear in general form, they are more difficult to graph. However, it is possible to solve quadratic inequalities by finding the roots of the corresponding equation and analyzing test points.

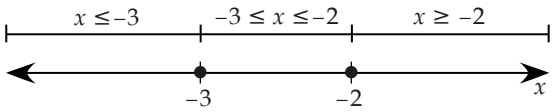
Example 3

Solve the quadratic inequality $x^2 + 5x + 6 \geq 0$ by finding the roots of the corresponding equation and analyzing test points.

Solution

To find the roots of the quadratic inequality, you need to factor the related equation, $x^2 + 5x + 6 = 0$ into $(x + 2)(x + 3) = 0$. Thus, the roots of the equation are -2 and -3.

Next, you need to plot these points on a number line. **Note:** The points are a closed (or solid) dot, because the values -2 and -3 are solutions to the inequality. A solid dot is used for $\leq$ or $\geq$, and an open dot for $>$ and $<$.

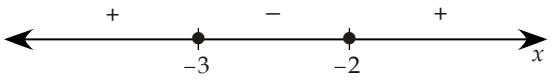


As you can see, the x-axis is divided into three intervals, $x \leq -3$, $-3 \leq x \leq -2$, and $x \geq -2$ by the roots of the equation, -2 and -3.

Choose one point to test from each interval to determine whether the quadratic expression is positive or negative. As the original question asks for values of the expression that are greater than or equal to, you are looking for intervals that are positive. To test these points, you can set up a chart as follows. You can substitute your test points into the general or the factored form of the inequality.

	$x^2 + 5x + 6 \geq 0$		
Interval	$x \leq -3$	$-3 \leq x \leq -2$	$x \geq -2$
Test Point	$x = -4$	$x = -2.5$	$x = 0$
Substituting the Test Point into the Inequality	$(-4)^2 + 5(-4) + 6$ $= 16 - 20 + 6$ $= 2$	$(-2.5)^2 + 5(-2.5) + 6$ $= \frac{25}{4} - \frac{25}{2} + 6$ $= -\frac{1}{4}$	$(0)^2 + 5(0) + 6$ $= 6$
Sign of the Answer	positive	negative	positive

You now can complete the sign diagram using the results from the chart.



Thus, the intervals that make the inequality positive are $x \leq -3$ and $x \geq -2$. As the inequality in the original equation includes "*equal to*," the points -3 and -2 need to be included in the solution set.

Thus, the solution set is $\{x \mid x \geq -2 \text{ or } x \leq -3, x \in \Re\}$.

Solving Quadratic Inequalities by Case Analysis

Solving quadratic inequalities by case analysis is similar to solving quadratic inequalities by finding their roots and testing points. In both instances, you have to factor the corresponding equation first. However, when you are solving quadratic inequalities using the case analysis method, the steps after you factor are different. Consider the following example.

Example 4

Solve the quadratic inequality $x^2 + 2x - 8 \geq 0$ by using the method of case analysis.

Solution

To solve a quadratic inequality using case analysis, first you need to factor the quadratic.

$x^2 + 2x - 8 \geq 0$

$(x + 4)(x - 2) \geq 0$

The quadratic inequality asks you to find where the expression is *greater than or equal to* zero. This will occur when both factors of the quadratic equation have the same sign, either both positive or both negative.

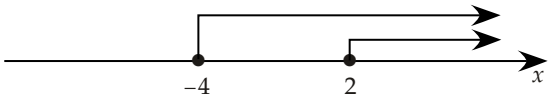
Case 1:

Consider when both factors are positive:

$x + 4 \geq 0$ and $x - 2 \geq 0$

$x \geq -4$ and $x \geq 2$

Now, graph these inequalities on a number line. Any values that satisfy both of these conditions are part of the solution set. Note that "greater than" is to the right on a number line.



As you can see, the values when $x \geq 2$ satisfy both of these conditions because that is where the two sets overlap. Thus, they are part of the solution set, $[2, \infty)$.

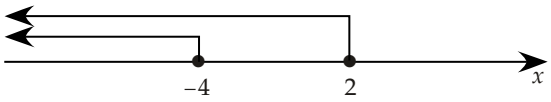
Case 2:

Consider when both factors are negative:

$x + 4 \leq 0$ and $x - 2 \leq 0$

$x \leq -4$ and $x \leq 2$

Any values that satisfy both of these conditions will also be part of the solution set. Note that "less than" is to the left on a number line.



As you can see, the values that satisfy both of the above conditions are when $x \leq -4$ because that is where the two sets overlap. The solution is $\{x \mid x \leq 4, x \in \Re\}$.

Thus, the complete solution set is: $\{x \mid x \leq -4 \text{ or } x \geq 2, x \in \Re\}$. In interval notation, the solution is written $(-\infty, -4] \cup [2, \infty)$.

Example 5

Solve the quadratic inequality $x^2 - 4x + 3 < 0$ using the method of case analysis.

Solution

Factoring the inequality:

$x^2 - 4x + 3 < 0$

$(x - 3)(x - 1) < 0$

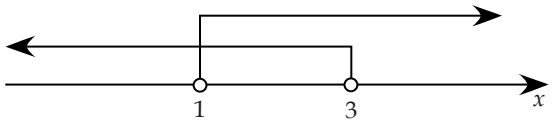
This time, you want to find values such that the quadratic equation is less than zero because the inequality symbol in the original inequality is the *less than* symbol. This will occur when one of the factors is negative and the other is positive.

Case 1:

Consider the case when $x - 3$ is negative and $x - 1$ is positive:

$x - 3 < 0$ and $x - 1 > 0$

$x < 3$ and $x > 1$



Note: The points on the line are represented by open circles, which signifies that those values aren't included in the solution because the inequality symbol is strictly less than or strictly greater than (no "or equal to" is included in either case).

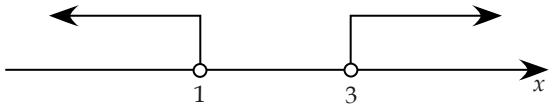
Thus, the solution set contains the points where $1 < x < 3$ because this is the interval where the two lines overlap.

Case 2:

Consider the case when $x - 1$ is negative and $x - 3$ is positive:

$x - 3 > 0$ and $x - 1 < 0$

$x > 3$ and $x < 1$



No points satisfy both conditions indicated by the fact that the lines do not overlap.

Therefore, the solution set for the original inequality from Case 1 is simply $\{x \mid 1 < x < 3, x \in \Re\}$. In interval notation, the solution is written $(1, 3)$.

Solving Quadratic Inequalities by Sign Analysis and Making a Factor Diagram

The fourth way to solve quadratic inequalities is by sign analysis and factor diagrams. When the zeros of a function are put in order and marked on a number line, they divide the number line into intervals. The zeros are referred to as the **critical points** of the inequality. The resulting intervals are the test intervals for the inequality. Using a sign diagram, we can solve the inequality. The following example will show you how this works. This method is very much like the first method, only it has as many sign diagrams as there are factors.

Example 6

Solve the quadratic inequality $x^2 + x - 2 \geq 4$ using the method of sign analysis and factor diagrams.

Solution

$x^2 + x - 2 \geq 4$

$x^2 + x - 6 \geq 0$ Rearrange

$(x + 3)(x - 2) \geq 0$ Factor

Because the inequality sign includes equality, the zeros are part of the solution and are represented by closed dots on the sign diagram.

Now make a sign diagram for each factor and then a summary sign diagram for the product of the factors and their corresponding signs.

Determine where each factor is zero, positive, and negative by substituting test points into each factor.

Consider $x + 3$:

It is zero when $x = -3$.

Test to the left of -3.

Substitute $x = -4$ into $x + 3$

$x + 3$
$= -4 + 3$
$= -1$ negative

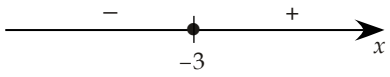
Test to the right of -3:

Substitute $x = -2$ into $x + 3$

$x + 3$
$= -2 + 3$
$= 1$ positive

It is a good idea to do the tests described mentally so you only need to write the "+" and the "−" on the following sign diagram.

The sign diagram for $x + 3$ is:



Consider the factor $x - 2$:

It is zero when $x = 2$.

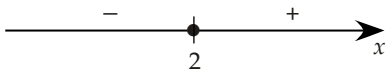
On its left, when $x = 1$:

$x - 2$
$= 1 - 2$
$= -1$ negative

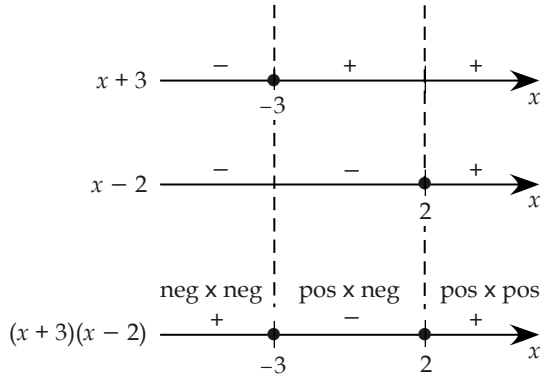
On its right, when $x = 3$

$x - 2$
$= 3 - 2$
$= 1$ positive

The sign diagram for $x - 2$ is:



Now make a summary diagram showing the diagram for each factor and then a third diagram showing the results for the product of the factors. Notice how vertical dotted lines are used to separate the intervals for each factor and are extended to intersect all three diagrams. The third diagram represents the product of the factors.



The solution for the inequality is:

$x^2 + x - 2 \geq 4$

or $(x + 3)(x - 2 \geq 0$

means where its sign diagram includes all x-values that are positive or zero.

$\{x \mid x \leq -3 \text{ or } x \geq 2, x \in \Re\}$

In interval notation, the solution is written $(-\infty, -3] \cup [2, \infty)$.

Before you continue to the following learning activity, it is time to update your resource sheet. Go back through this lesson and consider each of the four methods for solving quadratic inequalities and, in your own words, write down the steps for solving each inequality on your resource sheet. You may also find it helpful to include an example of how to solve a quadratic inequality.

Learning Activity 6.4

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Simplify: $\frac{x^6y^3}{x^8y^2}$
2. Pizza Express delivers pizza with the following charges: a $10.00 delivery and tip charge and $7 per pizza. Your sports team wants to order pizza. Everyone pools their money and they come up with $73.15. What is the maximum number of pizzas they can order?
3. Patricia ate one-third of the red velvet cake her mom bought her for her birthday that day. The next day, she ate one-half of the remaining cake. How much cake is left?
4. Simplify: $2^4 \times 2^2$
5. Reduce the following fraction: $\frac{105}{21}$
6. Simplify: $\left(-\frac{4^7\left(3^2\right)}{16^2}\right)^0$
7. Find the midpoint between the following two points: (26, −13) and (15, −21)
8. Solve the inequality $3 - 7x \geq 0$.

continued

Learning Activity 6.4 (continued)

Part B: Solving Quadratic Inequalities in One Variable

Remember, these questions are similar to the ones that will be on your assignments and final exam. So, if you were able to answer them correctly, you are likely to do well on your assignments and final exam. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Determine whether the given value of x is a solution to the given inequality.
   a) $x = 2,\ x^2 + 3x - 6 \leq 0$
   b) $x = -4,\ 2x^2 + 4x - 9 > 0$
   c) $x = 0,\ 4x^2 - 3x + 4 > 0$
2. Draw the graph of the quadratic function $y = x^2 + 2x + 1$. Then, determine the solution for the following:
   a) $x^2 + 2x + 1 \geq 0$
   b) $x^2 + 2x + 1 > 0$
   c) $x^2 + 2x + 1 \leq 0$
   d) $x^2 + 2x + 1 < 0$
3. Use the graphing method to solve the following quadratic inequalities.
   a) $x^2 + 4x + 4 \geq 0$
   b) $x^2 - 2x > 3$
4. Use the roots and test point method to make a sign diagram and solve the following quadratic inequalities.
   a) $x^2 - 2x + 1 \geq 0$
   b) $-x^2 + x + 1 \geq 0$. Show your answer rounded to three decimal places.
5. Use the case analysis method to solve the following quadratic inequalities.
   a) $3x^2 + 16x \leq -5$
   b) $x^2 + 3x - 180 \geq 0$
6. Use the sign analysis method to make factor diagrams to solve the following quadratic inequalities.
   a) $x^2 + 4x < 0$
   b) $2x^2 - 5x - 7 > 0$

continued

Learning Activity 6.4 (continued)

7. A man is standing on a 50-foot cliff. He can only see things that are higher than or as high as the cliff. Fireworks are being fired from the ground down below, and their height, measured in feet, at time t, measured in seconds, is given by the equation $h = -4t^2 + 28t + 5$.
   a) Solve the inequality $-4t^2 + 28t + 5 \geq 50$.
   b) Interpret the solution to the inequality you solved in (a).
   c) Explain how you chose which strategy to use to solve the quadratic inequality.

Lesson Summary

In this lesson, you learned how to solve quadratic inequalities in one variable. You solved these inequalities using four methods including graphing, roots and test points, case analysis, and sign analysis. In the next lesson, you will learn how to solve quadratic inequalities in two variables using a graphing method.

Assignment 6.2

Solving Linear Inequalities in Two Variables and Quadratic Inequalities in One Variable

Total: 40 marks

1. Which of the ordered pairs are solutions to the given inequality? *(6 marks)*

   $3x - 7 < 2y$

   $(1, 2), (3, -4), (-4, -5)$

continued

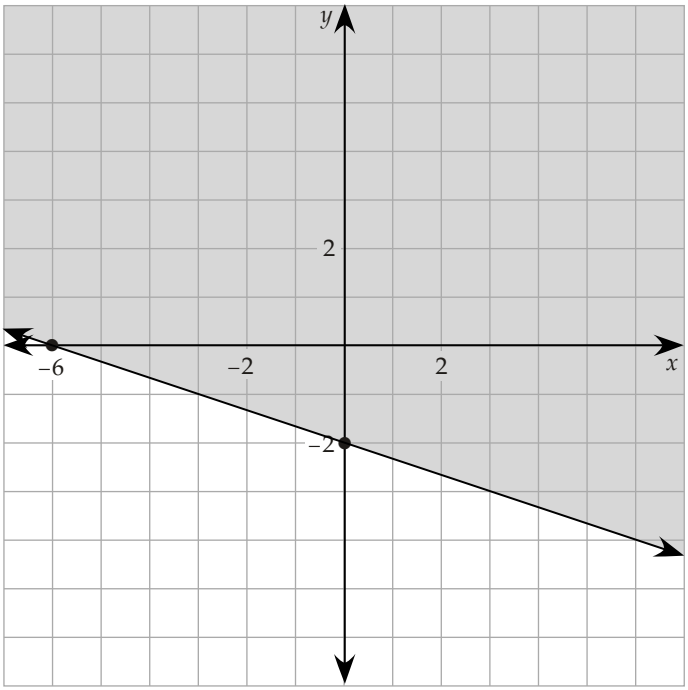
Assignment 6.2: Solving Linear Inequalities in Two Variables and Quadratic Inequalities in One Variable (continued)

2. Determine whether the given value of x is a solution to the given inequality. *(2 marks)*

   $x = 4$; $2x^2 + 4x - 1 \leq 4$

3. Write a linear inequality to represent each graph. *(2 marks each, for a total of 4 marks)*

   a)



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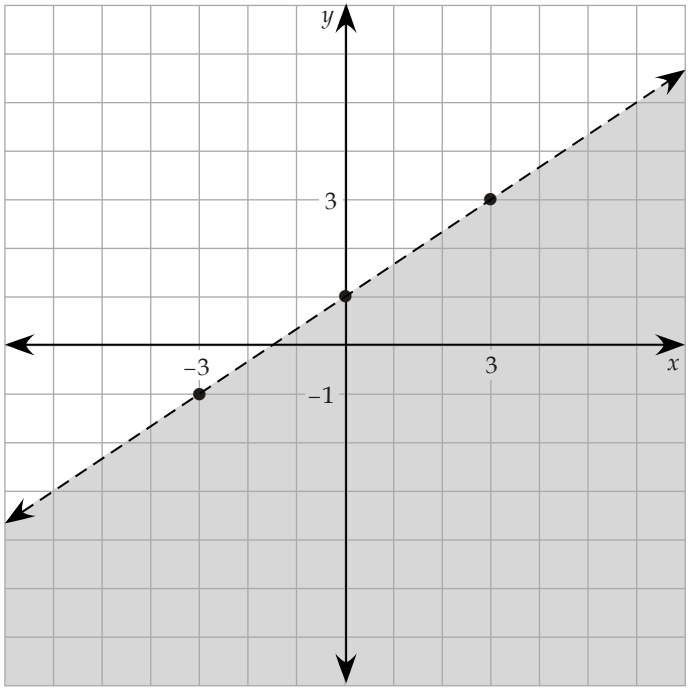
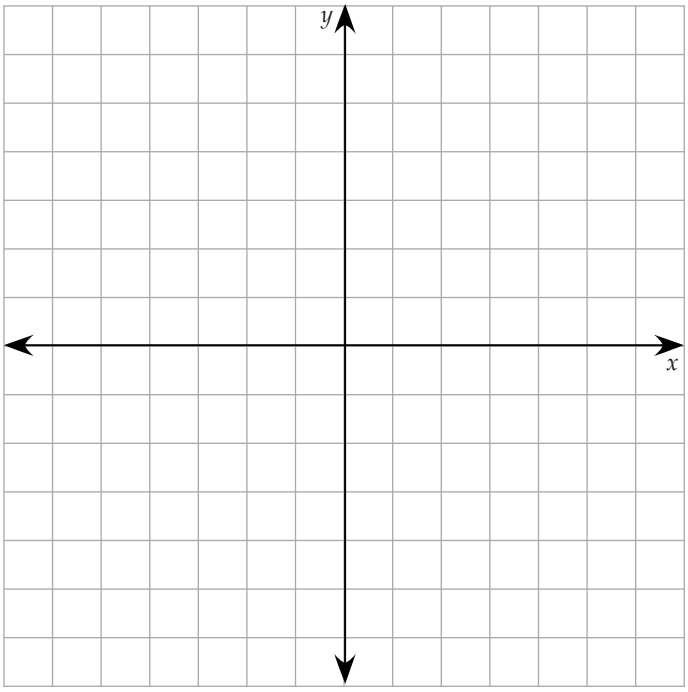
b)



4. Sketch the graph of each linear inequality. *(3 marks each, for a total of 6 marks)*

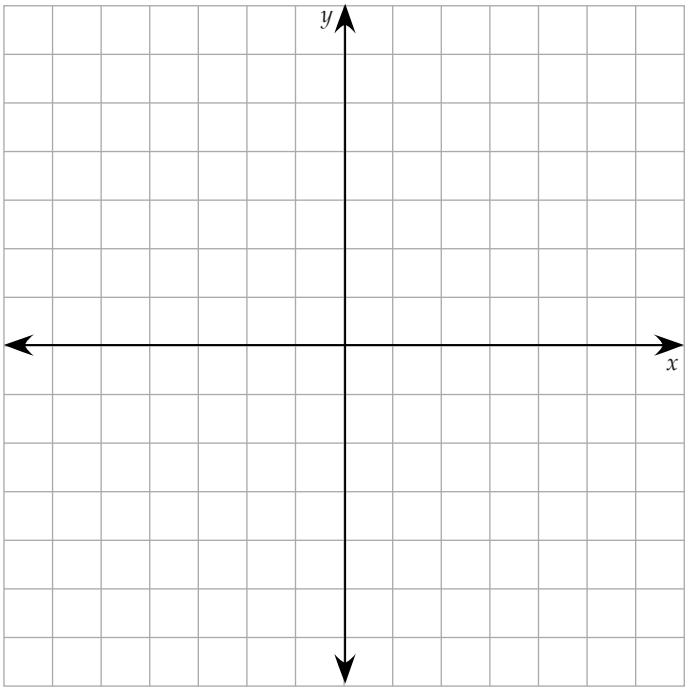
a) $y > \frac{1}{2}x + 3$



continued

Assignment 6.2: Solving Linear Inequalities in Two Variables and Quadratic Inequalities in One Variable (continued)

b) $4x - 6 \geq 2y$

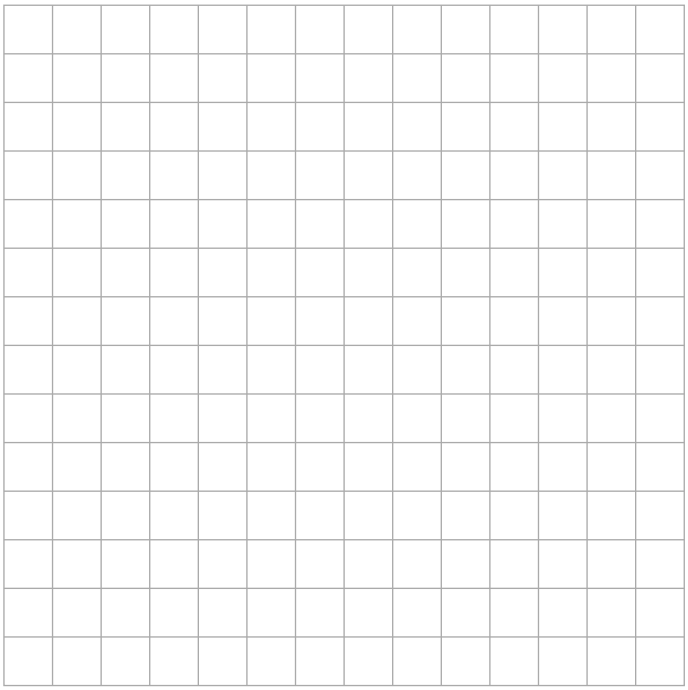


5. Tesa has at most $200 a month to spend on clothes and food. An article of clothing on average costs her $20, and the average cost for a meal is $10.
   a) Write an inequality to represent the number of articles of clothing, c, and the number of meals, m, that Tesa can afford within her budget each month. *(2 marks)*

continued

Assignment 6.2: Solving Linear Inequalities in Two Variables and Quadratic Inequalities in One Variable (continued)

b) Graph the inequality with the number of articles of clothing, c, represented on the x-axis and the number of meals, m, on the y-axis. *(2 marks)*



c) List two different ordered pairs that are solutions to this inequality. *(1 mark)*

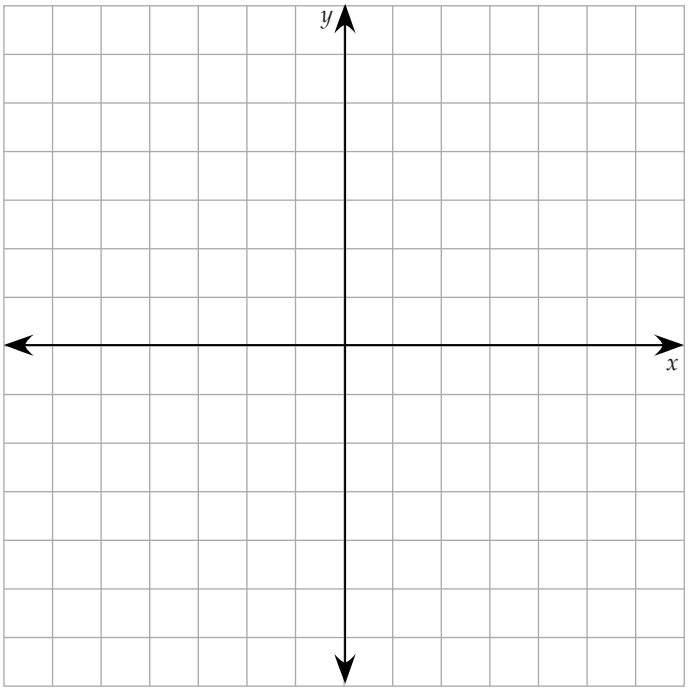
d) What do these ordered pairs symbolize? *(1 mark)*

continued

Assignment 6.2: Solving Linear Inequalities in Two Variables and Quadratic Inequalities in One Variable (continued)

6. Use the graphing method to solve the following quadratic inequality. *(3 marks)*

   $2(x - 2)^2 + 3 < 1$



continued

Assignment 6.2: Solving Linear Inequalities in Two Variables and Quadratic Inequalities in One Variable (continued)

7. Use the roots and test point method to make a sign diagram and solve the following quadratic inequality. *(3 marks)*

   $x^2 + x - 2 \geq 4$

continued

Assignment 6.2: Solving Linear Inequalities in Two Variables and Quadratic Inequalities in One Variable (continued)

8. Use the case analysis method to solve the following quadratic inequality. *(3 marks)*

   $3x^2 + 24x + 43 \geq 7$

continued

Assignment 6.2: Solving Linear Inequalities in Two Variables and Quadratic Inequalities in One Variable (continued)

9. Use the sign analysis method to solve the following quadratic inequality. Include a factor diagram. *(3 marks)*

$0 \geq x^2 - 6x + 5$

continued

Assignment 6.2: Solving Linear Inequalities in Two Variables and Quadratic Inequalities in One Variable (continued)

10. A hairstyling business's profit is represented by the equation $P = -x^2 + 150x - 1500$, where x represents the number of customers and P represents the profit in dollars. How many customers does this hairstyling business need to have a profit greater than $2000? Round to the nearest person. *(4 marks)*

Lesson 5: Solving Quadratic Inequalities in Two Variables

Lesson Focus

In this lesson, you will

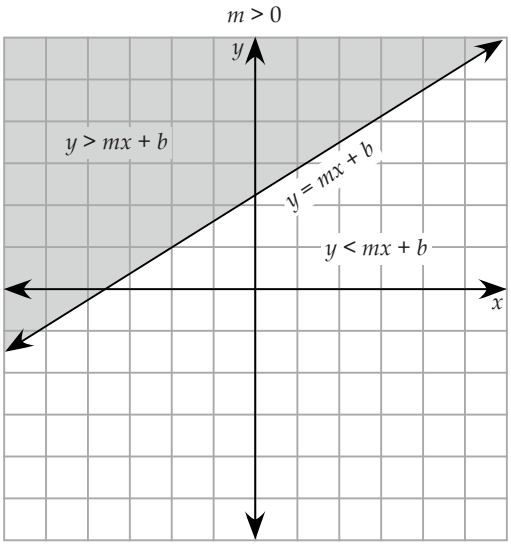
- ❑ learn how to solve quadratic inequalities in two variables by graphing

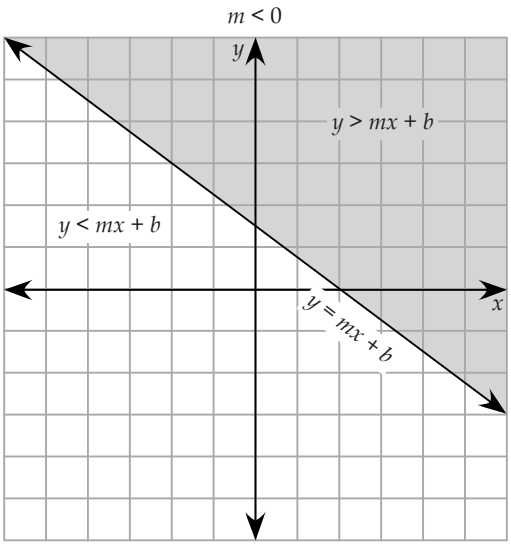
Lesson Introduction

In the last lesson, you learned how to solve quadratic inequalities in one variable. In this lesson, you will learn how to solve quadratic inequalities in two variables. When a quadratic inequality only has one variable, you are looking for the x-interval(s) where the corresponding parabola is either greater than, less than, or equal to zero. When a quadratic inequality has two variables, you are looking for the solution region either *inside* or *outside* the given parabola. You need to draw a graph to show the solution.

Quadratic Inequalities in Two Variables

The graphs of linear and quadratic inequalities in two variables are closely related to the graphs of lines and parabolas. If a line has the equation $y = mx + b$, any point (x, y) above the line satisfies the inequality $y > mx + b$. Any point (x, y) below the line $y = mx + b$ satisfies the inequality $y < mx + b$.

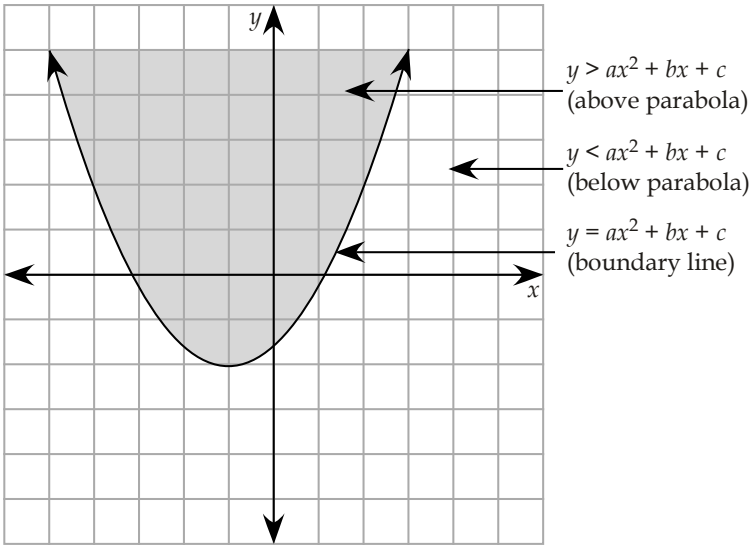




Graphing Quadratic Inequalities in Two Variables

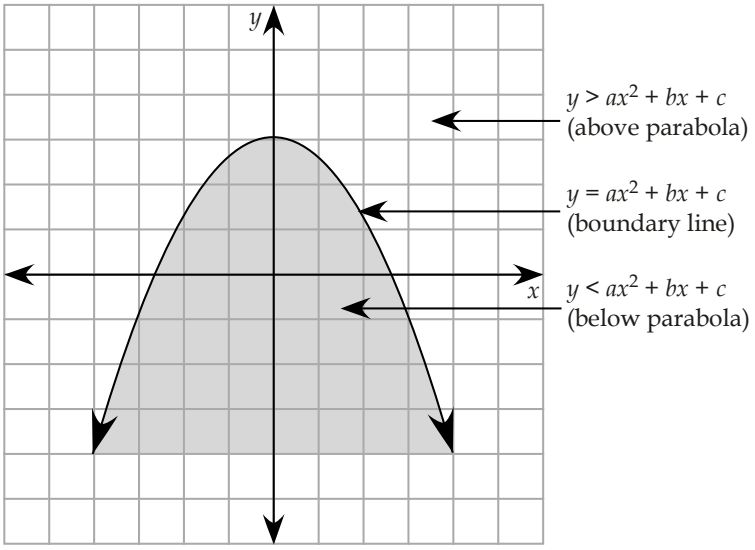
When *a* is positive

If the quadratic has the equation $y = ax^2 + bx + c$, and a is greater than zero, any point, (x, y), above the parabola satisfies the inequality $y > ax^2 + bx + c$. Any point, (x, y), below the parabola $y = ax^2 + bx + c$ satisfies the inequality $y < ax^2 + bx + c$.



When *a* is negative

If the quadratic has the equation $y = ax^2 + bx + c$, and a is less than zero (or negative), any point, (x, y), below the parabola satisfies the inequality $y < ax^2 + bx + c$. Any point, (x, y), above the parabola $y = ax^2 + bx + c$ satisfies the inequality $y > ax^2 + bx + c$.



In this lesson, you will investigate four types of quadratic inequalities:

$y < ax^2 + bx + c$ $\qquad$ $y \leq ax^2 + bx + c$

$y > ax^2 + bx + c$ $\qquad$ $y \geq ax^2 + bx + c$

The steps used to solve a quadratic inequality in two variables are as follows:

1. Sketch the graph of the corresponding parabola $y = ax^2 + bx + c$. Use a dashed boundary line for inequalities with < or > and a solid line for inequalities with ≤ or ≥. A dashed line indicates the points on the curve *are not* included in the solution to the inequality. A solid line indicates the points on the curve *are* included in the solution to the inequality. This is exactly the same notation used for linear inequalities.
2. Shade "above" the parabola for $y > ax^2 + bx + c$ and $y \geq ax^2 + bx + c$. Otherwise shade "below" the parabola.
3. Check your work by testing one point inside and one point outside the parabola. Only one of the test points will be a solution.

The reason for the solutions to be what they are has to do with the y-values in the inequality. If they are greater than the y-values on the equation of the parabola, the solution region must be "above" the parabola.

Example 1

Sketch the graph of $y \geq x^2 - 2x - 8$.

Solution

$y \geq x^2 - 2x - 8$

Graph $y = x^2 - 2x - 8$.

The parabola has its vertex at $x = \dfrac{-b}{2a} = -\dfrac{(-2)}{2(1)} = 1$.

Substitute $x = 1$ for $y = x^2 - 2x - 8$

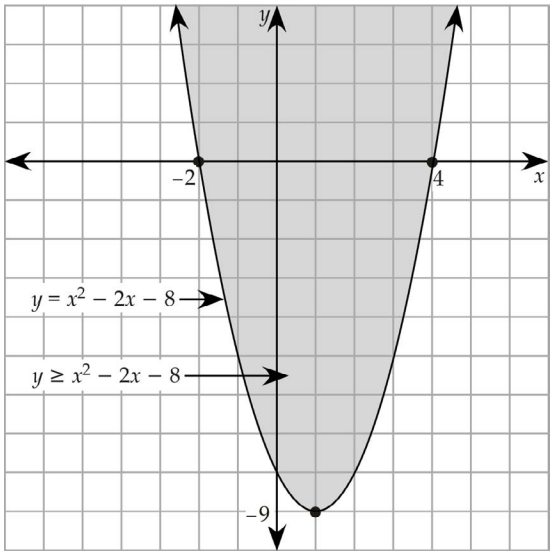
$y = 1^2 - 2(1) - 8$

$y = 1 - 2 - 8$

$y = -9$

Vertex: $(1, -9)$, opens up, normal shape

Draw the sketch using a solid line for the parabola as the points on the parabola are included in the solution to the inequality. Shade above the parabola.



To check your work, choose one point inside the region and one outside the region and substitute into the original inequality.

One point inside the region: (0, 0)

$y \geq x^2 - 2x - 8$

Check (0, 0)

LHS = $y = 0$

RHS = $0^2 - 2(0) - 8$

$= -8$

LHS ≥ RHS

∴ (0, 0) is in the solution region

∴ The shading is correct.

One point outside the region: (5, 0)

Check (5, 0)

LHS = $y = 0$

RHS = $5^2 - 2(5) - 8$

$= 7$

LHS $\not\geq$ RHS

∴ (5, 0) is not in the solution region

Example 2

Sketch the graph of $y < x^2 + 4x - 12$.

Solution

Graph $y = x^2 + 4x - 12$.

The parabola has its vertex at $x = -\frac{b}{2a} = -\frac{4}{2} = -2$.

Substitute $x = -2$ for $y = x^2 + 4x - 12$.

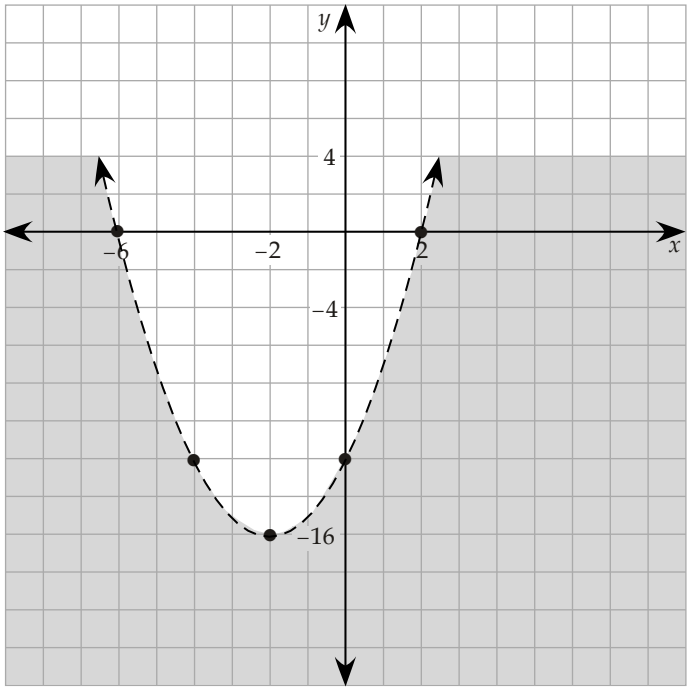
$= (-2)^2 + 4(-2) - 12$

$= 4 - 8 - 12$

$= -16$

Vertex: (−2, −16), opens up, normal shape

Draw the sketch using a broken line for the parabola, as the inequality symbol is *less than* and, thus, the points on the parabola are not included in the solution to the inequality. Shade below the parabola.



Now check your shading. Choose one point in the solution region and one not in the solution region, and substitute into the original inequality.

One point outside the shaded region: (0, 0)

$y < x^2 + 4x - 12$

LHS = $y = 0$

RHS = $x^2 + 4x - 12$

$= 0^2 + 4(0) - 12$

$= -12$

LHS $\not<$ RHS

$\therefore$ (0, 0) does not satisfy the inequality. It is outside the solution region.

$\therefore$ The shading is correct.

One point in the region: (3, 0)

$y < x^2 + 4x - 12$

LHS = $y = 0$

RHS = $x^2 + 4x - 12$

$= 3^2 + 4(3) - 12$

$= 9$

LHS < RHS

$\therefore$ (3, 0) satisfies the inequality and is in the solution region.

Example 3

Sketch the graph of $y \leq -x^2 + 4x + 21$.

Solution

Graph $y = -x^2 + 4x + 21$.

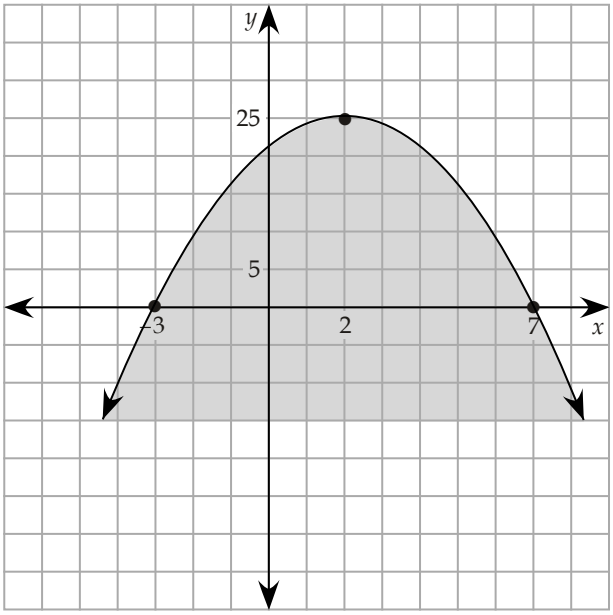
The parabola has its vertex at $x = -\frac{b}{2a} = -\frac{4}{2(-1)} = 2$.

Substitute $x = 2$ for $y = -x^2 + 4x + 21$

$$= -(2)^2 + 4(2) + 21$$
$$= -4 + 8 + 21$$
$$= 25$$

Vertex: (2, 25), opens down, normal shape

Draw the sketch using a solid line for the parabola and shade below the parabola where the y-values are less than those on the parabola itself.



Checkpoint (0, 0)

$y \leq -x^2 + 4x + 21$

LHS = $y = 0$

RHS = $-x^2 + 4x + 21$

$= -0^2 + 4(0) + 21$

$= 21$

LHS ≤ RHS

$\therefore$ (0, 0) is part of the solution region and the shading is correct.

You should now be able to solve quadratic inequalities in two variables. Use the following learning activity to practice. Make sure you check your answers in the Module 6 Learning Activities Answer Key to ensure you are completing the questions correctly before you complete the assignment following the learning activity.

Learning Activity 6.5

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Cassandra pays for $78.83 worth of groceries with a $100 bill. What is her change?
2. What is the vertex of the parabola defined by the equation $y = 2x^2 - 6$?
3. The movie you want to see starts playing at 8:30 p.m. It takes you 45 minutes to drive to the theatre and 30 minutes to get ready. If you want to be at the theatre 30 minutes early, what time should you start getting ready?
4. Find the common difference in the following pattern: 1, 7, 13, 19, . . .
5. Calculate: $\dfrac{\frac{4}{3}}{\frac{5}{8}}$

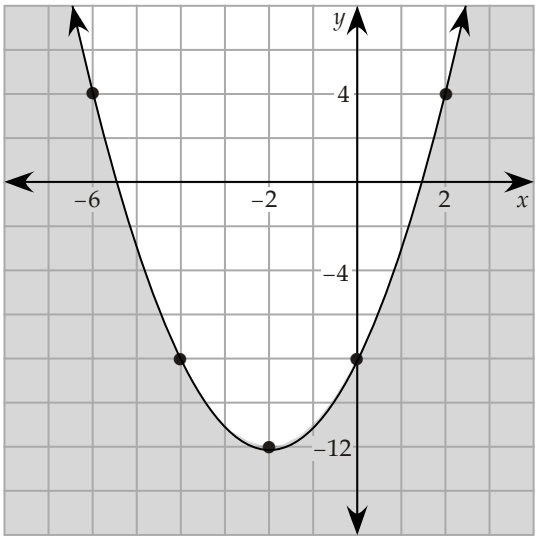
continued

6. Express in exponential form: $\sqrt{y^{\frac{7}{2}}}$
7. Simplify: $(4x^2 + 7x - 3)(-3x + 2)$
8. What does the range of a quadratic function depend on?

Part B: Solving Quadratic Inequalities in Two Variables

Remember, these questions are similar to the ones that will be on your assignments and final exam. So, if you were able to answer them correctly, you are likely to do well on your assignments and final exam. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Determine whether the following statements are true or false. If they are false, explain why they are false and give the correct statement.
   a) To determine if the region outside a parabola is a solution to the inequality, test a point outside the parabola. If that point does not satisfy the inequality, then that region is a solution to the inequality.
   b) If (1, 2) satisfies the inequality $y \geq (x + 2)^2 - 12$, then the following graph represents the inequality.



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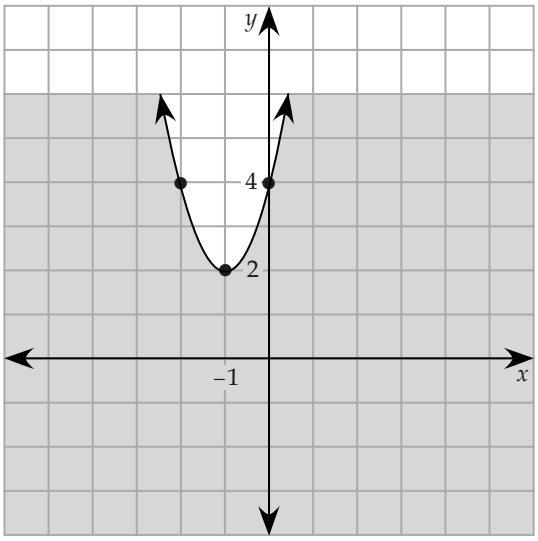
c) If $(-3, -2)$ satisfies the inequality $y \leq 2(x + 1)^2 + 2$, then the following graph represents the inequality.



d) The quadratic inequality $y < 3x^2 - 6x + 7$ includes a solid boundary line.

e) The quadratic inequality $y > -x^2 - 5x + 4$ includes a broken or dashed boundary line.

f) The point (1, 6) is a solution to the inequality $y > -x^2 + x + 4$.

g) The point $(-1, 5)$ is a solution to the inequality $y \leq 2x^2 - x + 1$.

2. Match the inequality with its graph.

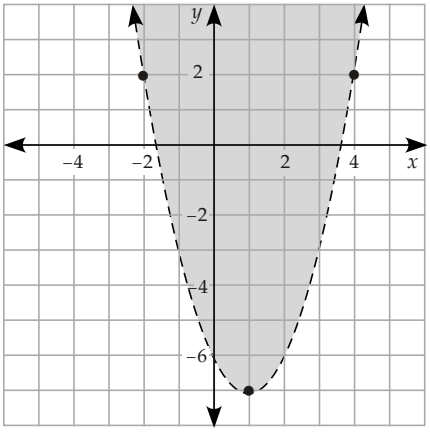
a) $y > -x^2 - 6x - 2$

b) $y \geq 2x^2 + 3x - 5$

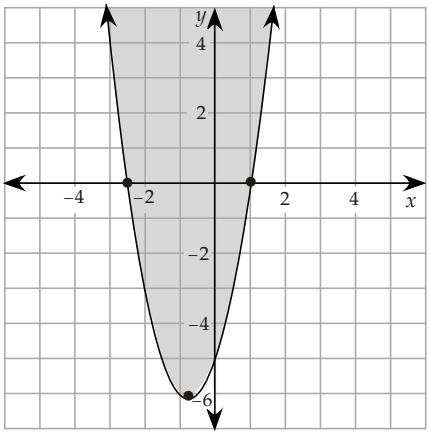
c) $y \leq -x^2 + 3x + 5$

d) $y > x^2 - 2x - 6$

i)



ii)



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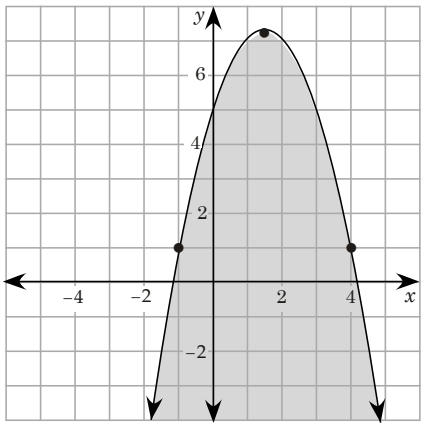
iii)



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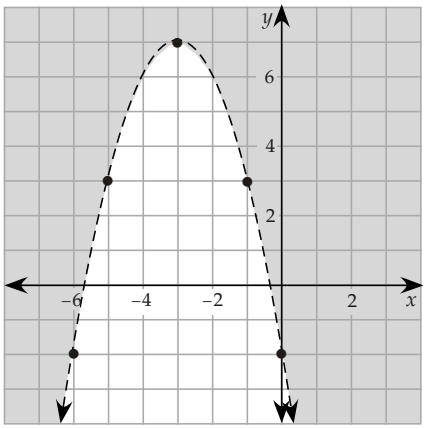
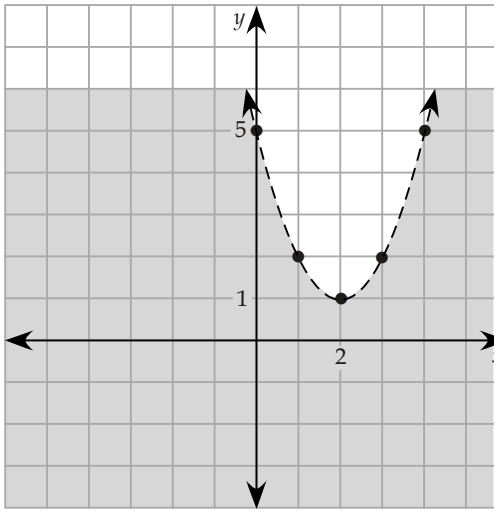
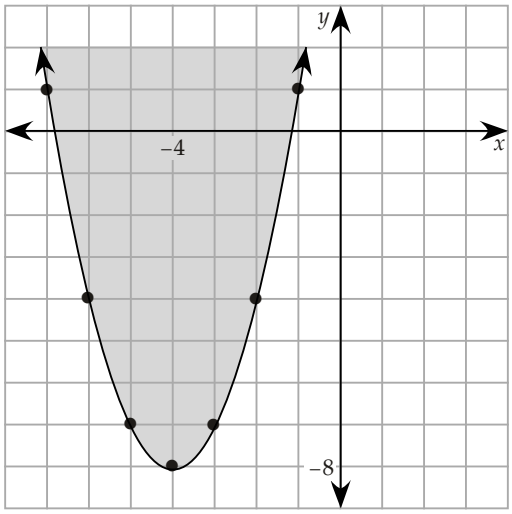


3. Write an inequality to describe each of the following graphs.

a)



b)



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Learning Activity 6.5 (continued)

4. Sketch the graph of the inequalities without using the graphing calculator.

   a) $y \leq x^2 + 4x$

   b) $y > \frac{1}{2}x^2 + 3x + 2$

   c) $y < -3x^2 - 6x$

Lesson Summary

In this lesson, you learned how to solve quadratic inequalities in two variables by graphing. The shading for the solution region to a quadratic inequality in two variables is either inside or outside the graph of the parabola. You learned how to check the solution region by using test points.

Before you complete Assignment 6.3, make sure you have completed Learning Activity 6.5, as the assignment will have similar questions to the learning activity. Also, now would be a good time to update your resource sheet if you haven't already.

Notes

Assignment 6.3

Solving Quadratic Inequalities in Two Variables

Total: 19 marks

1. Which of the ordered pairs are solutions to the following quadratic inequalities? *(6 marks)*

   $y \geq x^2 + 3x - 7$

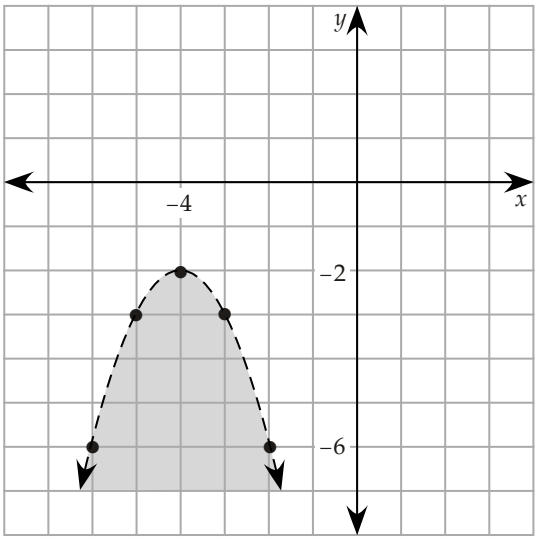
   (1, 2), (−3, 4), (−7, −5)

continued

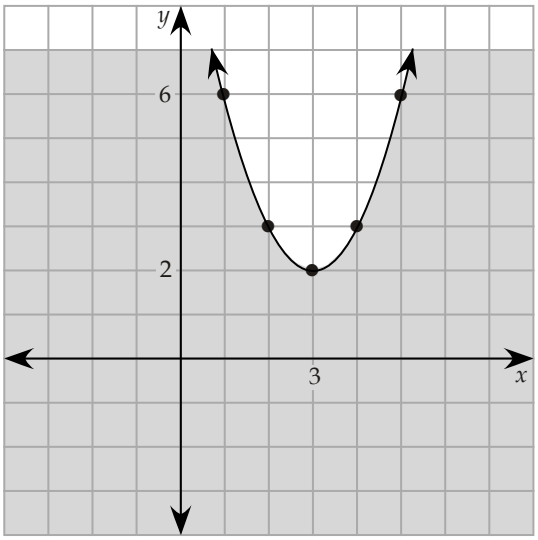
Assignment 6.3: Solving Quadratic Inequalities in Two Variables (continued)

2. Write an inequality to describe each of the following graphs. *(2 marks each, for a total of 4 marks)*

a)



b)



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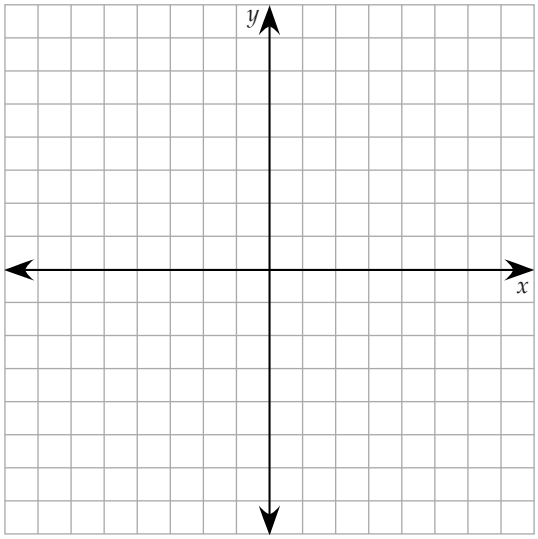
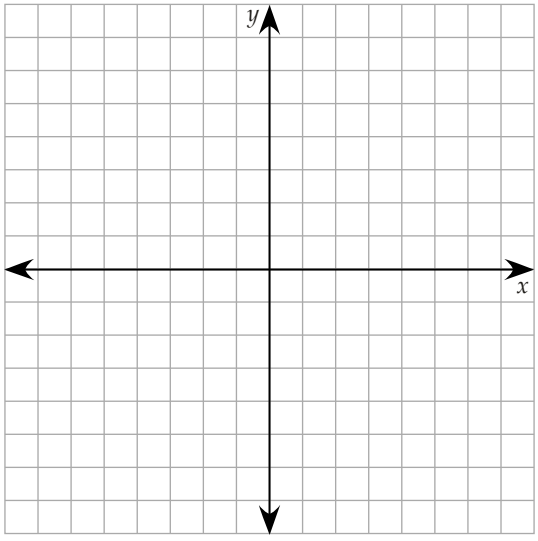
Assignment 6.3: Solving Quadratic Inequalities in Two Variables (continued)

3. Sketch the graph of the following inequalities. *(3 marks each, for a total of 9 marks)*

   a) $y < x^2 - 2x - 24$



   b) $y \geq x^2 + 4x - 5$



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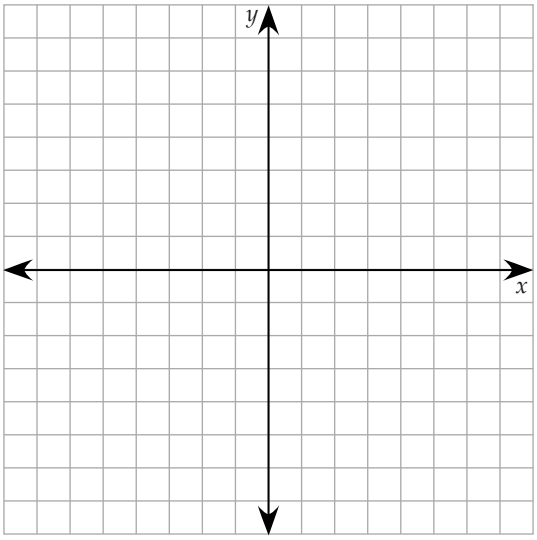
Assignment 6.3: Solving Quadratic Inequalities in Two Variables (continued)

c) $y > -2x^2 + 4x + 6$



Module 6 Summary

In this module, you learned about equations and inequalities. You learned how to solve systems of equations involving one linear equation and one quadratic equation graphically and algebraically. You also learned how to solve systems of equations involving two quadratic equations. You then focused on linear and quadratic inequalities in one or two variables, which can be solved in many ways, including graphically and algebraically. These inequalities can be useful in situations involving budgets, finances, and speed.

In the next module, you will revisit your study of trigonometry. You will learn about angles in standard position, how to solve trigonometric equations, and how to solve non-right triangles when you are given various pieces of information.

Submitting Your Assignments

It is now time for you to submit Assignments 6.1 to 6.3 to the Distance Learning Unit so that you can receive some feedback on how you are doing in this course. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 6 assignments and organize your material in the following order:

- ❑ Module 6 Cover Sheet (found at the end of the course Introduction)
- ❑ Assignment 6.1: Solving Systems of Equations
- ❑ Assignment 6.2: Solving Linear Inequalities in Two Variables and Quadratic Inequalities in One Variable
- ❑ Assignment 6.3: Solving Quadratic Inequalities in Two Variables

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Notes

Grade 11 Pre-Calculus Mathematics (30S)

Module 6
Systems of Equations and Inequalities

Learning Activity Answer Keys

Module 6:
Systems of Equations and Inequalities

Learning Activity 6.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. How much tax would you have to pay on an item that costs $49.99 (approximately) if the PST = 8% and the GST = 5%?
2. Simplify: $\sqrt{98}$
3. Find the equation of a line with a slope of $-\frac{1}{2}$ and a y-intercept of 3.
4. Evaluate: $\frac{3}{21} \times \frac{3}{15}$
5. If 10% of 440 is 44, what is 35% of 440?
6. Are these fractions equivalent? $\frac{\sqrt{5}}{\sqrt{7}}$ and $\frac{\sqrt{35}}{7}$
7. Determine the length of the hypotenuse of a right-angled triangle when the two sides measure 10 cm and 13 cm.
8. Find the slope of the line that contains the points (26, −13) and (15, −21).

Answers:

1. Approximately $6.50 (13% of $100 = $13; 13% of $50 = $6.50)
2. $7\sqrt{2}$ $\left(\sqrt{98} = \sqrt{49 \times 2}\right)$
3. $y = -\frac{1}{2}x + 3$
4. $\frac{1}{35}$ $\left(\frac{1}{7} \times \frac{1}{5}\right)$
5. 154 (10% of 440 = 44; 5% of 440 = 22; 35% of 440 = 44 + 44 + 44 + 22 = 154)

6. Yes $\left(\frac{\sqrt{5}}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{\sqrt{35}}{7}\right)$

7. $\sqrt{269}$ $\left(\sqrt{10^2 + 13^2} = \sqrt{100 + 169}\right)$

8. $\frac{8}{11}$ $\left(m = \frac{-21 - (-13)}{15 - 26} = \frac{-8}{-11}\right)$

Part B: Solving Quadratic-Linear Systems of Equations

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Verify that (−1, 0) and (4, 15) are solutions to the following system of equations.

   $y = 3x + 3$

   $y = x^2 - 1$

   Answer:

   (−1, 0)

   $y = 3x + 3$

   LHS $= y = 0$

   RHS $= 3x + 3$

   $= 3(-1) + 3$

   $= -3 + 3$

   $= 0$

   LHS = RHS

   $y = x^2 - 1$

   LHS $= y = 0$

   RHS $= x^2 - 1$

   $= (-1)^2 - 1$

   $= 1 - 1$

   $= 0$

   LHS = RHS

(4, 15)

$y = 3x + 3$

LHS $= y = 15$

RHS $= 3x + 3$

$= 3(4) + 3$

$= 12 + 3$

$= 15$

LHS = RHS

$y = x^2 - 1$

LHS $= y = 15$

RHS $= x^2 - 1$

$= (4)^2 - 1$

$= 16 - 1$

$= 15$

LHS = RHS

Thus, (−1, 0) and (4, 15) are solutions to the above system of equations.

2. Solve each system both graphically and algebraically.

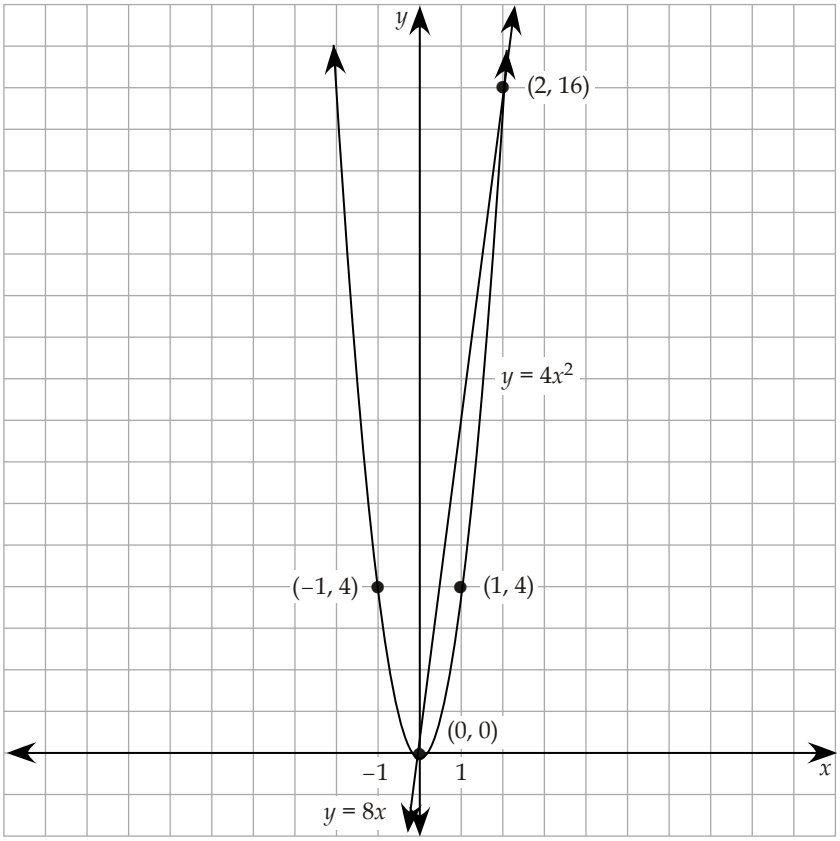
a) $y = 4x^2$

$y = 8x$

Answer:

Graph solution:

The points of intersection are (0, 0) and (2, 16).



Algebraic solution:

$y = 4x^2$

$y = 8x$

Because both equations are equal to y, then:

$4x^2 = 8x$

$4x^2 - 8x = 0$

$4x(x - 2) = 0$

$4x = 0$ or $x - 2 = 0$

$x = 0$ $x = 2$

Substitute the x-values into the linear equation to solve for y.

$y = 8x$	$y = 8x$
$y = 8 \cdot 0$	$y = 8 \cdot 2$
$y = 0$	$y = 16$

The two solutions are (0, 0) and (2, 16), and they are the two points of intersection in the graph solution.

b) $x - 2y = 10$

$y = x^2 + 2x - 15$

Answer:

Graph solution:

To graph $x - 2y = 10$, find the intercepts first.

x-intercept	y-intercept
let $y = 0$	let $x = 0$
$x - 2(0) = 10$	$0 - 2y = 10$
$x = 10$	$y = -5$
Point (10, 0)	Point (0, −5)

To graph $y = x^2 + 2x - 15$, first complete the square to find the vertex coordinates:

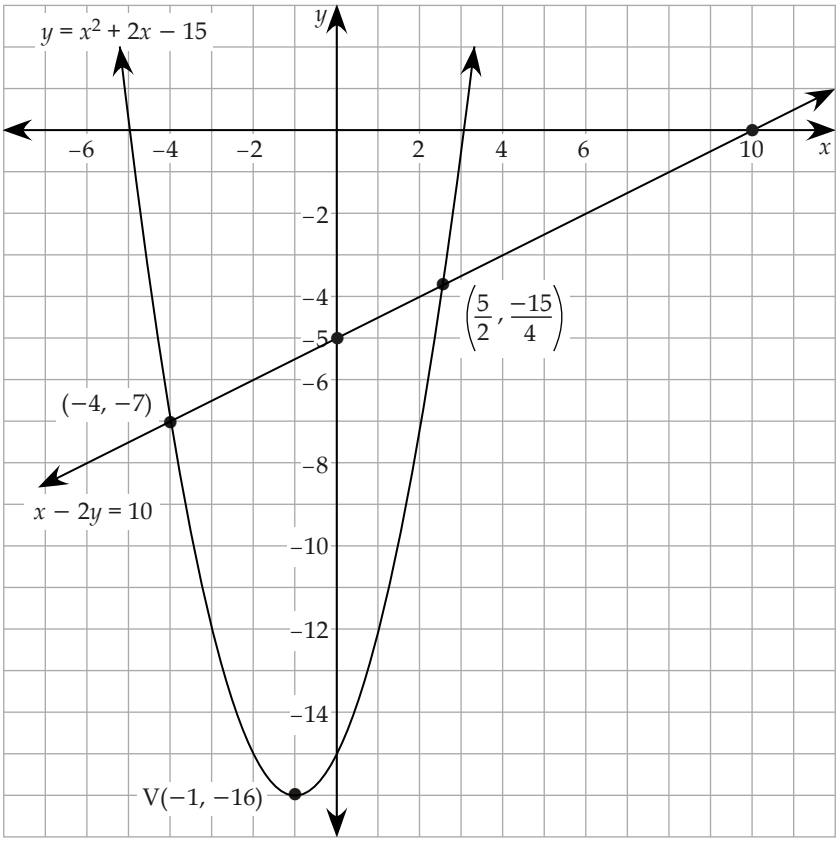
$y = (x^2 + 2x \quad) - 15$

$y = (x^2 + 2x + 1) - 15 - 1$

$y = (x + 1)^2 - 16$

Vertex (−1, −16), opens up, normal shape

One point of intersection is (−4, −7). The other point would need to be estimated near (3, −4).



Algebraic solution:

$x - 2y = 10$ (1)

$y = x^2 + 2x - 15$ (2)

Since Equation (2) is already solved for y, substitute $x^2 + 2x - 15$ for y in Equation (1).

$$x - 2y = 10$$
$$x - 2\left(x^2 + 2x - 15\right) = 10$$
$$x - 2x^2 - 4x + 30 = 10$$
$$0 = 2x^2 + 3x - 20$$
$$0 = (2x - 5)(x + 4)$$

$2x - 5 = 0$ $\qquad$ $x + 4 = 0$

$x = \frac{5}{2}$ $\qquad$ $x = -4$

Find the corresponding y-values using the linear equation.

$x = \frac{5}{2}$ $\qquad$ $x = -4$

$x - 2y = 10$ $\qquad$ $x - 2y = 10$

$\frac{5}{2} - 2y = 10$ $\qquad$ $-4 - 2y = 10$

$5 - 4y = 20$ $\qquad$ $-2y = 14$

$-4y = 15$ $\qquad$ $y = -7$

$y = -\frac{15}{4}$

The solutions are $\left(\frac{5}{2}, -\frac{15}{4}\right)$ and $(-4, -7)$.

The algebraic method is preferred for this example because it is difficult to be precise using the graph.

3. Solve each system by the substitution method.

a) $y = 3x + 7$

$y = x^2 - 11$

Answer:

Using the substitution method:

$3x + 7 = x^2 - 11$

$0 = x^2 - 3x - 18$

$0 = (x - 6)(x + 3)$

$x = 6$ or $x = -3$

When $x = 6$:

$y = 3x + 7$

$y = 3(6) + 7$

$y = 25$

Thus, one solution is (6, 25).

When $x = -3$:

$y = 3x + 7$

$y = 3(-3) + 7$

$y = -2$

Thus, another solution is $(-3, -2)$.

The solutions to this system are (6, 25) and $(-3, -2)$.

b) $y = 2x + 6$

$y = x^2 + 4x + 3$

Answer:

Using the substitution method

$2x + 6 = x^2 + 4x + 3$

$0 = x^2 + 2x - 3$

$0 = (x + 3)(x - 1)$

$x = -3$ or $x = 1$

When $x = -3$:

$y = 2x + 6$

$y = 2(-3) + 6$

$y = 0$

Thus, one solution is $(-3, 0)$.

When $x = 1$:

$y = 2x + 6$

$y = 2(1) + 6$

$y = 8$

Thus, another solution is (1, 8).

The solutions to this system are $(-3, 0)$ and (1, 8).

4. Solve each system by the elimination method.

a) $2y = 2x - 2$

$y = x^2 - 4x + 3$

Answer:

Using the elimination method:

Multiply the first equation by $-\frac{1}{2}$ and then add the like terms.

$$\begin{aligned} -y &= -x + 1 \\ +\ y &= x^2 - 4x + 3 \\ \hline 0 &= x^2 - 5x + 4 \\ 0 &= (x - 4)(x - 1) \end{aligned}$$

$x = 4$ or $x = 1$

When $x = 4$:

$2y = 2x - 2$

$y = x - 1$

$y = 4 - 1$

$y = 3$

Thus, (4, 3) is one solution to this system.

When $x = 1$:

$2y = 2x - 2$

$2y = 2(1) - 2$

$2y = 0$

$y = 0$

Thus, (1, 0) is another solution to this system.

The solutions to this system are (4, 3) and (1, 0).

b) $2x + y = 5$

$y = x^2 - 6x + 5$

Answer:

Using the elimination method, solve the linear equation for y.

$$y = -2x + 5$$

Multiply both sides of the linear equation by -1 and then add like terms.

$$\begin{aligned} -y &= 2x - 5 \\ +\ y &= x^2 - 6x + 5 \\ \hline 0 &= x^2 - 4x \\ 0 &= x(x - 4) \end{aligned}$$

$x = 0$ or $x = 4$

When $x = 0$:

$$\begin{aligned} 2x + y &= 5 \\ 2(0) + y &= 5 \\ y &= 5 \end{aligned}$$

Thus, one solution to this system is (0, 5).

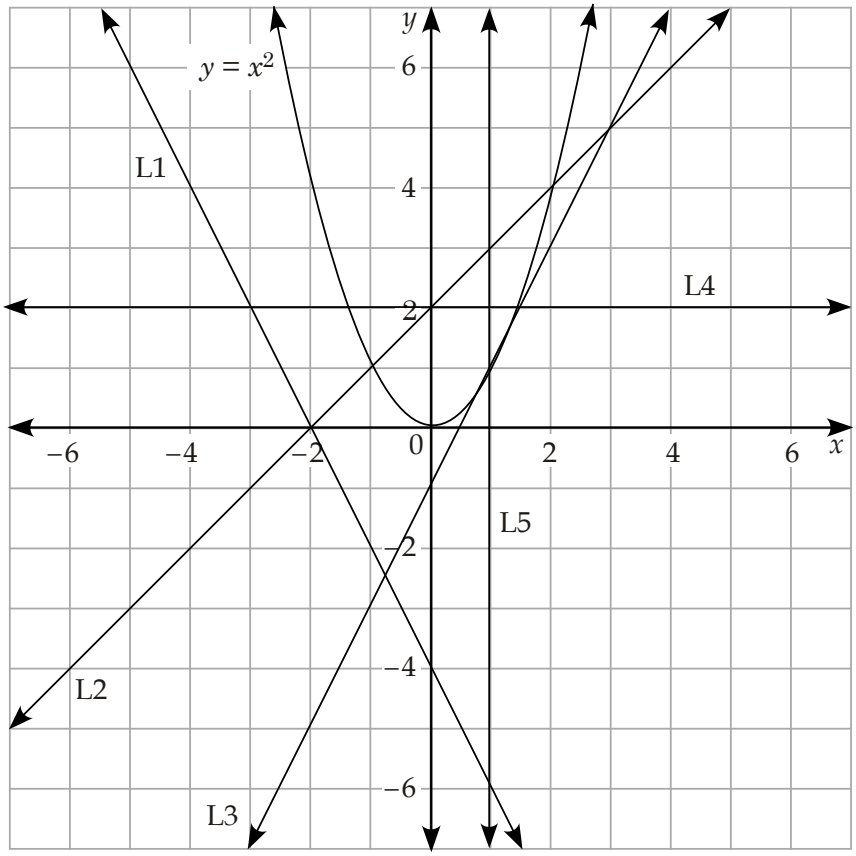
When $x = 4$:

$$\begin{aligned} 2x + y &= 5 \\ 2(4) + y &= 5 \\ 8 + y &= 5 \\ y &= -3 \end{aligned}$$

Thus, another solution to this system is (4, −3).

The solutions to this system are (0, 5) and (4, −3).

5. Consider the graph. Use this graph to answer the following questions.



a) State the number of solutions to the system consisting of the parabola and L1.

b) State the number of solutions to the system consisting of the parabola and L2.

c) State the number of solutions to the system consisting of the parabola and L3.

d) State the number of solutions to the system consisting of the parabola and L4.

e) State the number of solutions to the system consisting of the parabola and L5.

f) State the solution(s) to the system consisting of the parabola and L2.

g) State the solution(s) to the system consisting of the parabola and L5.

Answers:

a) There are no solutions to this system.

b) There are two solutions to this system.

c) There is one solution to this system.

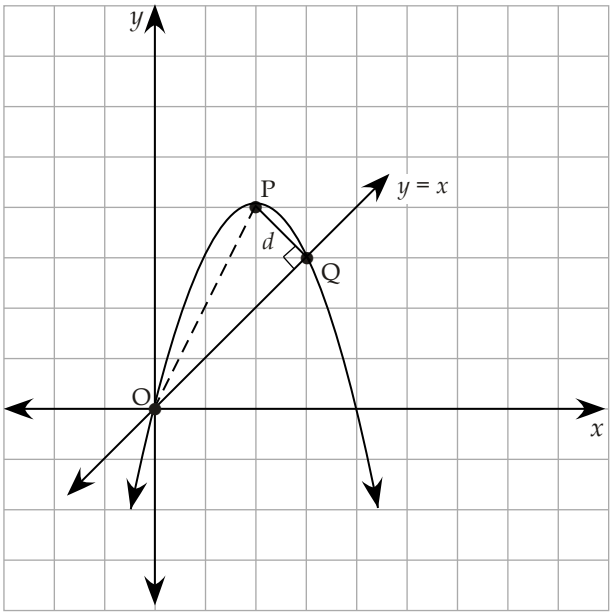
d) There are two solutions to this system.

e) There is one solution to this system.

f) The solutions are (−1, 1) and (2, 4).

g) The solution is (1, 1).

6. The line $y = x$ intersects the parabola $y = 4x - x^2$ at the origin and the point Q. Let the point, P, be the vertex of the parabola.



a) Find the coordinates of Q.

Answer:

$y = x$ (1)

$y = 4x - x^2$ (2)

$x = 4x - x^2$ Substitute y with x from Equation (1)

$x^2 - 3x = 0$

$x(x - 3) = 0$

$x = 0$ or $x = 3$

Find the corresponding values of y.

If $x = 0$, $y = 0$.

If $x = 3$, $y = 3$.

The coordinates of Q are (3, 3).

b) Find the coordinates of P.

Answer:

You could use $x = \frac{-b}{2a}$ to find the x-coordinate of P, the vertex of the parabola.

$$y = -x^2 + 4x$$

$$a = -1,\ b = 4$$

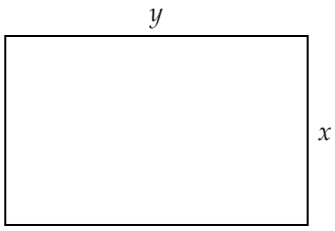
$$x = -\frac{4}{2(-1)} = \frac{4}{2} = 2$$

Substitute $x = 2$ into $y = -x^2 + 4x$ to find the y-coordinate of the vertex.

$$\begin{aligned} y &= -x^2 + 4x \\ &= -(2)^2 + 4(2) \\ &= -4 + 8 \\ &= 4 \end{aligned}$$

The coordinates of vertex P are (2, 4).

7. The perimeter of a rectangle is 26 cm and the area is 12 cm^2. Find the dimensions of the rectangle.



Answer:

Let x = width of the rectangle.

y = length of the rectangle.

Perimeter: $2x + 2y = 26$ (1)

Area: $xy = 12$ (2)

Solve equation (1) for y:

$$\begin{aligned} 2x + 2y &= 26 \\ 2y &= 26 - 2x \\ y &= 13 - x \end{aligned}$$

Substitute $y = 13 - x$ into equation (2):

$$xy = 12$$
$$x(13 - x) = 12$$
$$13x - x^2 = 12$$
$$0 = x^2 - 13x + 12$$
$$0 = (x - 12)(x - 1)$$

If $x = 12$, then $y = 13 - x = 12 - 12 = 1$.

If $x = 1$, then $y = 13 - x = 13 - 1 = 12$.

In both cases, the dimensions of the rectangle are 12 cm × 1 cm.

8. Below is Schyler's work for solving a system of equations.

$$-2y = -x^2 - x$$
$$2y = 10x - 20$$

Schyler's Solution:

$$\begin{array}{rl} & -2y = -x^2 - x \\ + & \underline{2y = \quad 10x - 20} \\ & 0 = -x^2 + 9x - 20 \end{array}$$
$$0 = x^2 - 9x + 20$$
$$0 = (x - 4)(x - 5)$$

$x = 4$ or $x = 5$

When $x = 4$:

$$2y = 10x - 20$$
$$2y = 10(4) - 20$$
$$2y = 20$$
$$y = 10$$

Thus, one solution is (4, 10).

When $x = 5$:

$$2y = 10x - 20$$
$$2y = 10(5) - 20$$
$$2y = 30$$
$$y = 15$$

Thus, another solution is (5, 15).

Is his method correct? Explain why or why not.

Answer:

Schyler used the elimination method to solve this system of equations. This is a correct way to solve a system of equations. Therefore, his work should be correct. To ensure his answers are correct, verify the solutions.

(4, 10)

$-2y = -x^2 - x$

$\begin{aligned} \text{LHS} &= -2y \\ &= -2(10) \\ &= -20 \end{aligned}$

$\begin{aligned} \text{RHS} &= -x^2 - x \\ &= -(4)^2 - 4 \\ &= -16 - 4 \\ &= -20 \end{aligned}$

LHS = RHS

$2y = 10x - 20$

$\begin{aligned} \text{LHS} &= 2y \\ &= 2(10) \\ &= 20 \end{aligned}$

$\begin{aligned} \text{RHS} &= 10x - 20 \\ &= 10(4) - 20 \\ &= 20 \end{aligned}$

LHS = RHS

(5, 15)

$-2y = -x^2 - x$

$\begin{aligned} \text{LHS} &= -2y \\ &= -2(15) \\ &= -30 \end{aligned}$

$\begin{aligned} \text{RHS} &= -x^2 - x \\ &= -(5)^2 - 5 \\ &= -25 - 5 \\ &= -30 \end{aligned}$

LHS = RHS

$2y = 10x - 20$

$\begin{aligned} \text{LHS} &= 2y \\ &= 2(15) \\ &= 30 \end{aligned}$

$\begin{aligned} \text{RHS} &= 10x - 20 \\ &= 10(5) - 20 \\ &= 30 \end{aligned}$

LHS = RHS

Thus, (4, 10) and (5, 15) are both solutions to this system of equations, and Schyler is using a correct method of elimination to solve this system.

Learning Activity 6.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Between which two whole numbers is $\sqrt{150}$?
2. Simplify: $3x^2 + 4x - 6 - 5x + 4 + 6x^2 + 8 - 9x^2$
3. Pizza Express is offering two medium pizzas for $15.99 while Speedy Pizza is offering three medium pizzas for $22.49. Which is the better buy?
4. Lisa worked 40 hours in one week and made $520. How much does Lisa make an hour?
5. Calculate: $\frac{4}{7} - \frac{2}{3}$
6. Express as an entire radical: $7y^{\frac{8}{3}}$
7. What is the slope of a vertical line?
8. How many centimetres are in 2 kilometres?

Answers:

1. 12 and 13 ($12^2 = 144$ and $13^2 = 169$)
2. $-x + 6$
3. Speedy Pizza has the better deal. $\left(\frac{\$15.99}{2} \cong \$8 \text{ and } \frac{\$22.49}{3} = \$7.50\right)$
4. Lisa makes $13 an hour. $\left(\frac{520}{40} = \frac{52}{4} = 13\right)$
5. $-\frac{2}{21}$ $\left(\frac{12}{21} - \frac{14}{21}\right)$
6. $\sqrt[3]{343y^8}$ $\left(7y^{\frac{8}{3}} = 343^{\frac{1}{3}}y^{\frac{8}{3}} = \left(343y^8\right)^{\frac{1}{3}}\right)$
7. The slope of a vertical line is undefined.
8. 200 000 centimetres (There are 1000 metres in 1 kilometre and 100 centimetres in 1 metre. Therefore, there are 100 × 1000 = 100 000 centimetres in 1 kilometre and 200 000 centimetres in 2 kilometres.

Part B: Solving Quadratic-Quadratic Systems of Equations

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Verify that $\left(\frac{10}{3}, \frac{17}{9}\right)$,and (1, −2) are solutions to the following system of equations.

$y = 2x^2 - 7x + 3$

$y = -x^2 + 6x - 7$

Answer:

Verifying: $\left(\frac{10}{3}, \frac{17}{9}\right)$:

$\text{LHS} = y = \frac{17}{9}$

$\text{RHS} = 2x^2 - 7x + 3$

$= 2\left(\frac{10}{3}\right)^2 - 7\left(\frac{10}{3}\right) + 3$

$= 2\left(\frac{100}{9}\right) - \frac{70}{3} + 3$

$= \frac{200}{9} - \frac{210}{9} + \frac{27}{9}$

$= \frac{17}{9}$

$\text{LHS} = \text{RHS}$

$\text{LHS} = y = \frac{17}{9}$

$\text{RHS} = -x^2 + 6x - 7$

$= -\left(\frac{10}{3}\right)^2 + 6\left(\frac{10}{3}\right) - 7$

$= -\frac{100}{9} + \frac{180}{9} - \frac{63}{9}$

$= \frac{17}{9}$

$\text{LHS} = \text{RHS}$

Verifying (1, −2):

$\text{LHS} = y = -2$

$\text{RHS} = 2x^2 - 7x + 3$

$= 2(1)^2 - 7(1) + 3$

$= 2 - 7 + 3$

$= -2$

$\text{LHS} = \text{RHS}$

$\text{LHS} = y = -2$

$\text{RHS} = -x^2 + 6x - 7$

$= -(1)^2 + 6(1) - 7$

$= -1 + 6 - 7$

$= -2$

$\text{LHS} = \text{RHS}$

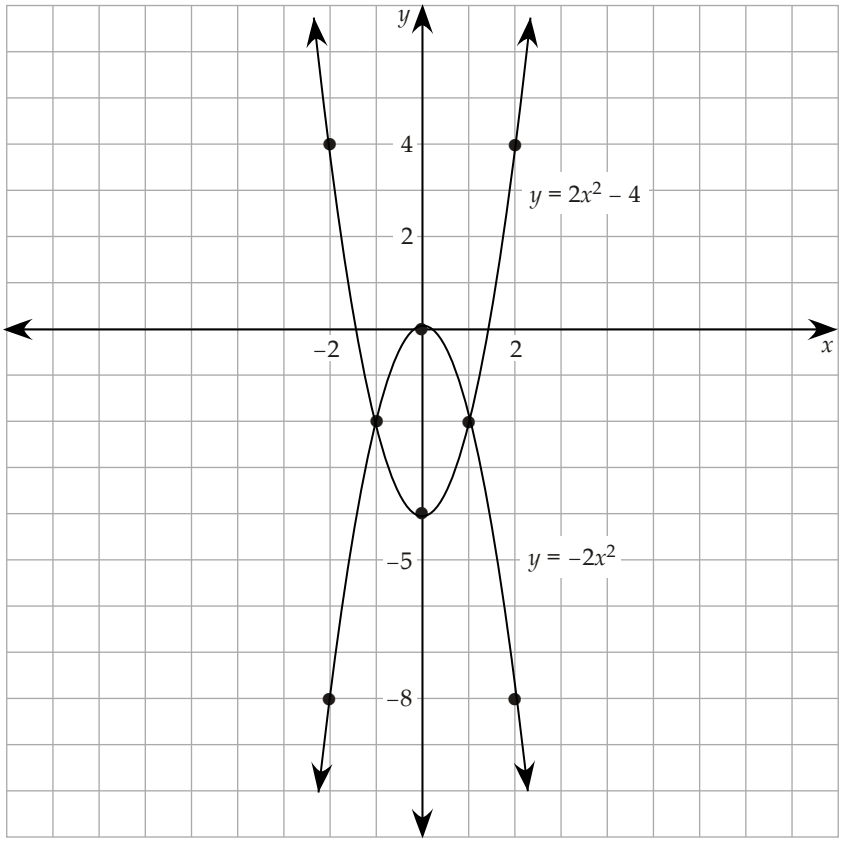
2. Solve the following systems of equations, both graphically and algebraically.

   a) $y = -2x^2$

   $y = 2x^2 - 4$

   Answer:

   Graphically:



   As you can see from the graph, the solutions to the systems of equations are $(-1, -2)$ and $(1, -2)$.

   Algebraically:

   Substitute for y:

   $-2x^2 = 2x^2 - 4$

   $0 = 4x^2 - 4$

   $0 = 4(x^2 - 1)$

   $0 = (x - 1)(x + 1)$

   Thus, $x = 1$ or -1.

When $x = 1$

$y = -2x^2$

$y = -2(1)^2$

$y = -2$

Thus, one solution is $(1, -2)$

When $x = -1$

$y = -2x^2$

$y = -2(-1)^2$

$y = -2(1)$

$y = -2$

Thus, one solution is $(-1, -2)$.

The two solutions are $(-1, -2)$ and $(1, -2)$.

As expected, these are the same solutions as determined graphically.

b) $x^2 + y - 3 = 0$

$x^2 - y + 1 = 0$

Answer:

Graphically:

First rearrange each equation to general form to find the vertex.

$x^2 + y - 3 = 0$ Equation (1)

$y = -x^2 + 3$

Vertex (0, 3), opens down, and normal shape.

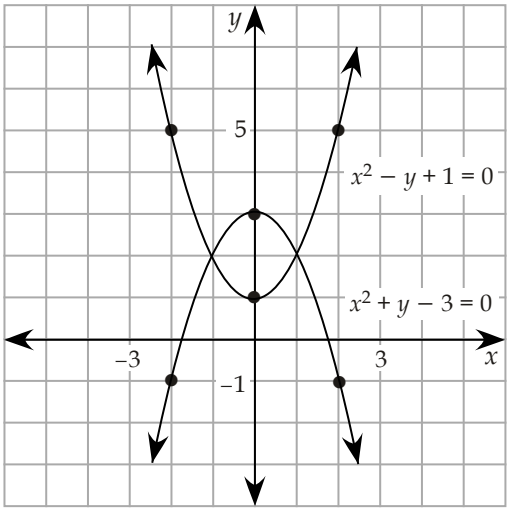
$x^2 - y + 1 = 0$ Equation (2)

$-y = -x^2 - 1$

$y = x^2 + 1$

Vertex (0, 1), opens up, and normal shape.

Now graph and solve.



The solutions to this system of equations are (−1, 2) and (1, 2).

Algebraically:

Use the rearranged equations:

$y = -x^2 + 3$ (1)

$y = x^2 + 1$ (2)

Now substitute for y.

$-x^2 + 3 = x^2 + 1$

$0 = 2x^2 - 2$

$0 = 2(x^2 - 1)$

$0 = (x - 1)(x + 1)$

Thus, $x = 1$ and -1.

When $x = 1$:

$y = x^2 + 1$

$y = (1)^2 + 1$

$y = 2$

Thus, one solution is (1, 2).

When $x = -1$:

$y = x^2 + 1$

$y = (-1)^2 + 1$

$y = 2$

Thus, another solution is (−1, 2).

The two solutions are (−1, 2) and (1, 2) and are the same as those found in the graphing solution.

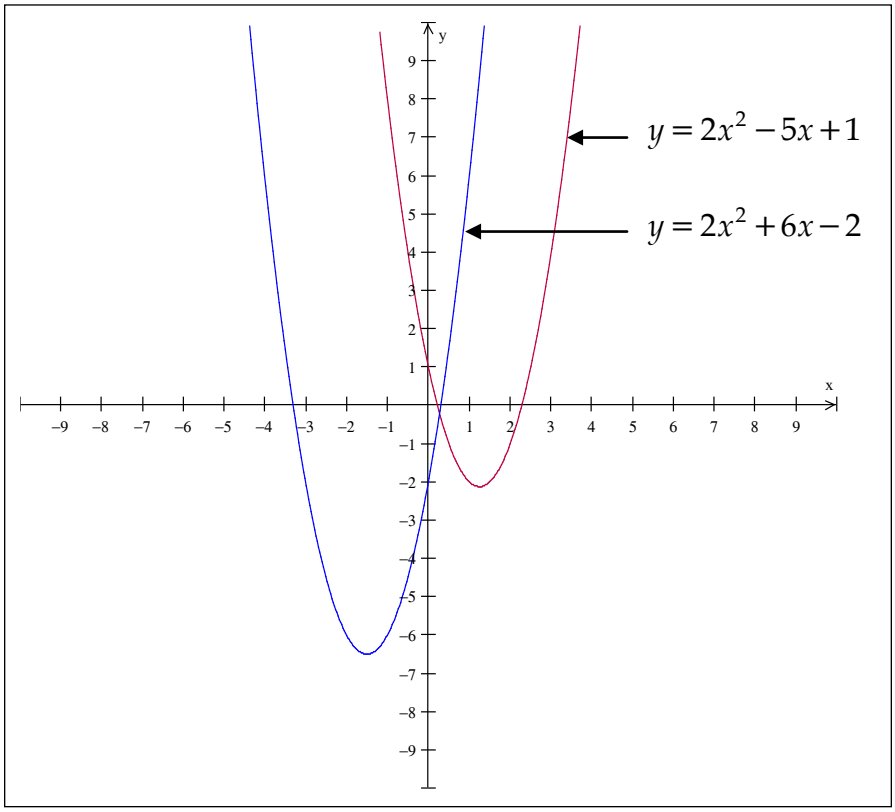
3. Solve the following system of equations graphically, using technology. Round to three decimal places when necessary. **Note:** If you do not have access to technology, you can still do this question by hand and estimate the solution. **You will not have questions on your exam requiring the use of technology.**

$y = 2x^2 - 5x + 1$

$y = 2x^2 + 6x - 2$

Answer:

Graph both of the above functions using a graphing calculator or a graphing applet such as Winplot. The graph created using Winplot is shown below.



Calculate the intersection by clicking "Two" in the menu bar and then "Intersections."

These functions intersect at the point (0.273, −0.215). Since they are the same shape, they will never cross at a second point.

4. Solve the following systems of equations by using the substitution method.

a) $y = x^2 + 4$

$y = -x^2 + 12$

Answer:

$$x^2 + 4 = -x^2 + 12$$
$$2x^2 - 8 = 0$$
$$2(x^2 - 4) = 0$$
$$2(x - 2)(x + 2) = 0$$

Thus, $x = 2$ or -2.

When $x = 2$:

$y = x^2 + 4$

$y = (2)^2 + 4$

$y = 8$

Thus, one solution is (2, 8).

When $x = -2$:

$y = x^2 + 4$

$y = (-2)^2 + 4$

$y = 8$

Thus, one solution is (−2, 8).

The two solutions are (2, 8) and (−2, 8).

b) $y = 2(x + 4)^2$

$y = 2x^2 + 16x + 32$

Answer:

Substitute for y.

$$2(x + 4)^2 = 2x^2 + 16x + 32$$
$$2(x^2 + 8x + 16) = 2x^2 + 16x + 32$$
$$2x^2 + 16x + 32 = 2x^2 + 16x + 32$$

These functions are identical. Therefore, there are an infinite number of solutions to this system including all of the points represented by the function $y = 2(x + 4)^2$. An example is (0, 32).

Thus, the solution is $y = 2(x + 4)^2$ or $\{(x, y) \mid y = 2(x + 4)^2\}$.

5. Solve the following systems of equations by using the elimination method.

a) $y = x^2 - 2x + 2$

$3x^2 - 4y = 7$

Answer:

Rearrange and fix the coefficients of y so that they are the same value with opposite signs. Then add the like terms of the two equations.

(1) $y = x^2 - 2x + 2$

(2) $-4y = -3x^2 + 7$

$4 \times$ (1) $4y = 4x^2 - 8x + 8$

$+$ (2) $-4y = -3x^2 + 7$

$0 = x^2 - 8x + 15$

$0 = (x - 5)(x - 3)$

Thus, $x = 5$ or 3.

When $x = 5$:

$y = x^2 - 2x + 2$

$y = 5^2 - 2(5) + 2$

$y = 17$

When $x = 3$:

$y = x^2 - 2x + 2$

$y = 3^2 - 2(3) + 2$

$y = 5$

Thus, the two solutions to this system of equations are (5, 17) and (3, 5).

b) $5x^2 + 3y = -3 - x$

$2x^2 - x = -4 - 2y$

Answer:

Rearrange and then fix the coefficients of y so that they are the same value with opposite signs. Then add the like terms of the two equations and solve.

(1) $5x^2 + 3y = -3 - x$

$3y = -5x^2 - x - 3$

(2) $2x^2 - x = -4 - 2y$

$2y = -2x^2 + x - 4$

$2 \times (1)$ $\quad 6y = -10x^2 - 2x - 6$

$+ \ -3 \times (2)$ $\quad -6y = 6x^2 - 3x + 12$

$0 = -4x^2 - 5x + 6$

$0 = 4x^2 + 5x - 6$

$0 = (4x - 3)(x + 2)$

$4x - 3 = 0 \qquad x + 2 = 0$

$x = \frac{3}{4} \qquad x = -2$

Thus, $x = \frac{3}{4}$ or -2.

When $x = \frac{3}{4}$:

$2x^2 - x = -4 - 2y$

$2\left(\frac{3}{4}\right)^2 - \frac{3}{4} + 4 = -2y$

$\frac{18}{16} - \frac{12}{16} + \frac{64}{16} = -2y$

$\frac{70}{16} = -2y$

$-\frac{35}{16} = y$

When $x = -2$:

$$\begin{aligned} 2x^2 - x &= -4 - 2y \\ 2(-2)^2 - (-2) + 4 &= -2y \\ 8 + 2 + 4 &= -2y \\ 14 &= -2y \\ -7 &= y \end{aligned}$$

Thus, the two solutions to this system of equations are, $\left(\frac{3}{4}, -\frac{35}{16}\right)$ and $(-2, -7)$.

6. Solve the following system of equations using any method you prefer. Explain why you chose that method.

$8x^2 + 100 = -5y$

$6x^2 - x - 5 = 3y$

Answer:

Fix coefficients of y so that they are opposites.

$$\begin{array}{rrrrrr} & 3 \times (1) & 24x^2 & + 300 & = & -15y \\ + & 5 \times (2) & 30x^2 - 5x - 25 & = & & 15y \\ \hline & & 54x^2 - 5x + 275 & = & & 0 \end{array}$$

Since the coefficients are so large, use the discriminant ($b^2 - 4ac$) to determine the number and types of roots.

$a = 54$, $b = -5$, $c = 275$

$$\begin{aligned} b^2 - 4ac &= (-5)^2 - 4(54)(275) \\ &= 25 - 59400 \\ &= -59375 \end{aligned}$$

As the discriminant is negative, there is no solution to this system of equations. Therefore, the parabolas never cross each other.

The elimination method was chosen because it is sometimes difficult to work with fractions, as would be the case if the substitution method were chosen. The graphing method would not be easy with these equations unless you have graphing technology.

7. Let x represent time, measured in seconds, and let y represent height, measured in feet. A football is thrown and its path is represented by the equation $y = -x^2 + 4x + 2$. A receiver jumps to receive the ball and his path is represented by the equation $y = -(x - 5)^2 + 6$. Does the receiver have any chance of catching the football? Explain.

Answer:

Solving the system of equations by substitution:

$y = -x^2 + 4x + 2$ (football)

$y = -(x - 5)^2 + 6$ (receiver)

$$-x^2 + 4x + 2 = -(x - 5)^2 + 6$$

$$-x^2 + 4x + 2 = -\left(x^2 - 10x + 25\right) + 6$$

$$-x^2 + 4x + 2 = -x^2 + 10x - 25 + 6$$

$$-x^2 + 4x + 2 = -x^2 + 10x - 19$$

$$4x + 2 = 10x - 19$$

$$-6x = -21$$

$$x = \frac{-21}{-6} = \frac{7}{2}$$

Thus, this receiver had one chance to catch the ball, after $\frac{7}{2}$ seconds.

When $x = \frac{7}{2}$:

$$y = -x^2 + 4x + 2$$

$$y = -\frac{49}{4} + \frac{28}{2} + 2$$

$$y = -\frac{49}{4} + \frac{56}{4} + \frac{8}{4}$$

$$y = \frac{15}{4}$$

The two equations intersect at the point $\left(\frac{7}{2}, \frac{15}{4}\right)$.

After $\frac{7}{2}$ seconds, the receiver and the ball were both $\frac{15}{4}$ feet, or 3.75 feet, off the ground and in the same spot. Therefore, this receiver had a chance to catch the ball.

Learning Activity 6.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Jessica sells $545 worth of fundraising tickets. If she makes 5% commission on her total sales, how much did Jessica make?
2. Simplify: $(3x + 1)^2$
3. Simplify: $\sqrt{54}$
4. Sam mowed half of the yard on Friday and half of the remaining grass on Saturday. How much grass still needs to be cut?
5. What is the slope of the line defined by the equation $2y = 4x + 6$?
6. Factor completely. $121x^2 - 196y^2$
7. The diameter of a circle is 34 cm. What is the radius of the circle?
8. Is 87 a prime or composite number?

Answers:

1. $27.25 (10% of $545 = $54.50; 5% of $545 = $27.25)
2. $9x^2 + 6x + 1$
3. $3\sqrt{6}$ $\left(\sqrt{54} = \sqrt{9 \times 6}\right)$
4. $\frac{1}{4}$ $\left(\frac{1}{2} \text{ of } \frac{1}{2}\right)$ (Sam has mowed $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ and she still needs to mow $\frac{1}{4}$.)
5. 2 $(y = 2x + 3)$
6. $(11x - 14y)(11x + 14y)$ [it is a difference of squares]
7. $r = 17$ cm (The radius is half the diameter.)
8. 87 is a composite number; it can be divided by 3 and 29.

Part B: Solving Linear Inequalities in Two Variables

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Which of the ordered pairs are solutions to the given inequality?

   a) $4x - 3y > 2$

   $(-4, 8), (1, 2), (3, -6)$

   Answer:

   $4x - 3y > 2$

   Check: $(-4, 8)$

   $\text{LHS} = 4x - 3y$

   $= 4(-4) - 3(8)$

   $= -16 - 24$

   $= -40$

   $\text{RHS} = 2$

   $\text{LHS} \not> \text{RHS}$

   Check: $(1, 2)$

   $\text{LHS} = 4x - 3y$

   $= 4(1) - 3(2)$

   $= 4 - 6$

   $= -2$

   $\text{RHS} = 2$

   $\text{LHS} \not> \text{RHS}$

   Check: $(3, -6)$

   $\text{LHS} = 4x - 3y$

   $= 4(3) - 3(-6)$

   $= 12 + 18$

   $= 30$

   $\text{RHS} = 2$

   $\text{LHS} > \text{RHS}$

   Thus, $(-4, 8)$ and $(1, 2)$ are not solutions to the inequality, but $(3, -6)$ is a solution.

b) $-8x + 6y \leq 5$

$(-2, -7), (4, 9), (-1, 8)$

Answer:

$-8x + 6y \leq 5$

Check: $(-2, -7)$

LHS $= -8x + 6y$

$= -8(-2) + 6(-7)$

$= 16 - 42$

$= -26$

RHS $= 5$

LHS $\leq$ RHS

Check: $(4, 9)$

LHS $= -8x + 6y$

$= -8(4) + 6(9)$

$= -32 + 54$

$= 22$

RHS $= 5$

LHS $\not\leq$ RHS

Check: $(-1, 8)$

LHS $= -8x + 6y$

$= -8(-1) + 6(8)$

$= 8 + 48$

$= 56$

RHS $= 5$

LHS $\not\leq$ RHS

Thus, $(4, 9)$ and $(-1, 8)$ are not solutions to the inequality, but $(-2, -7)$ is a solution.

2. Sketch the graphs of the following linear inequalities.

   a) $y < \frac{1}{2}x + 2$

   b) $y \geq 0$

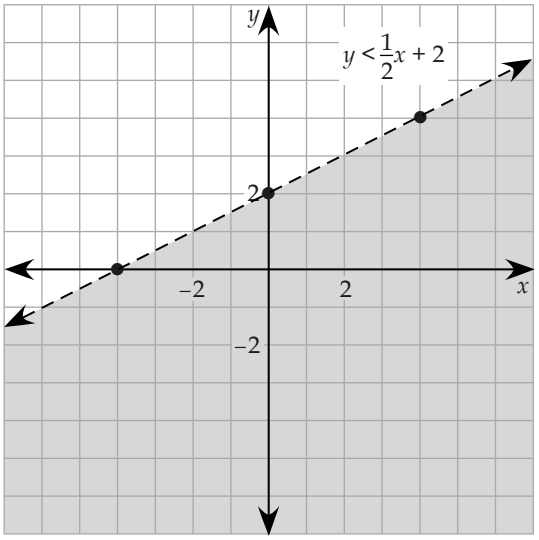
   c) $3x + 2y \leq 6$

   d) $-2y + 8 > 5x$

   Answers:

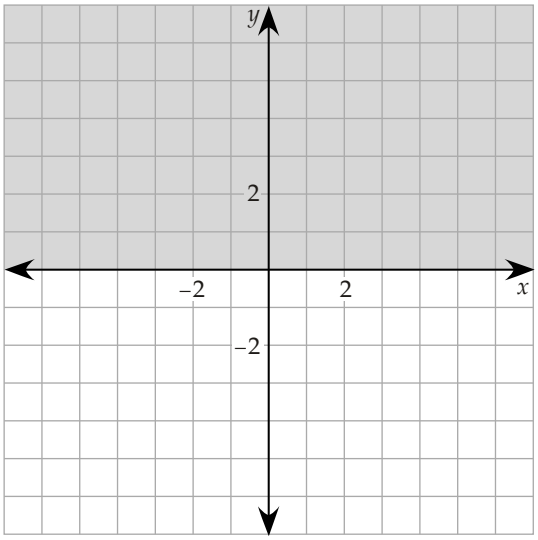
   a) $y < \frac{1}{2}x + 2$

   Graphing the dotted line with slope $\frac{1}{2}$ and y-intercept 2, and shading "underneath" the line where the y-values are smaller than those on the line.



b) $y \geq 0$

This is simply the horizontal line $y = 0$ and all y-values greater than 0.



c) $3x + 2y \leq 6$

Rearrange the inequality into $y = mx + b$ form first, and then graph a solid line and the shading.

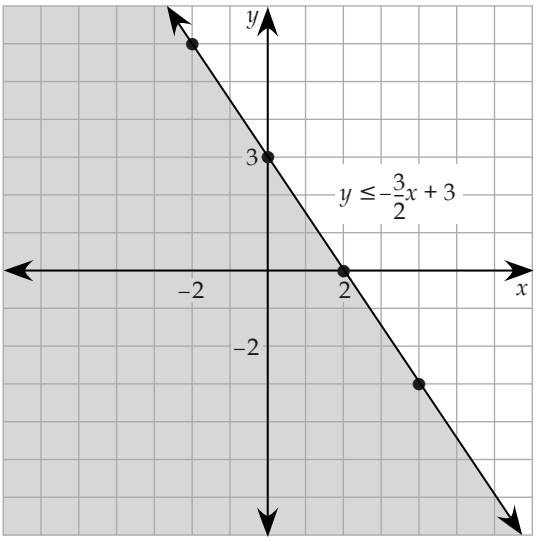
$$3x + 2y \leq 6$$

$$2y \leq -3x + 6$$

$$y \leq -\frac{3}{2}x + 3$$

Graph the related equation $y = -\frac{3}{2}x + 3$.

The line passes through a y-intercept of 3, having a slope of $-\frac{3}{2}$. Draw a solid line since the $\leq$ includes "equal to." The shading is "below" the line, including all y-values that are less than the ones on the line.



d) $-2y + 8 > 5x$

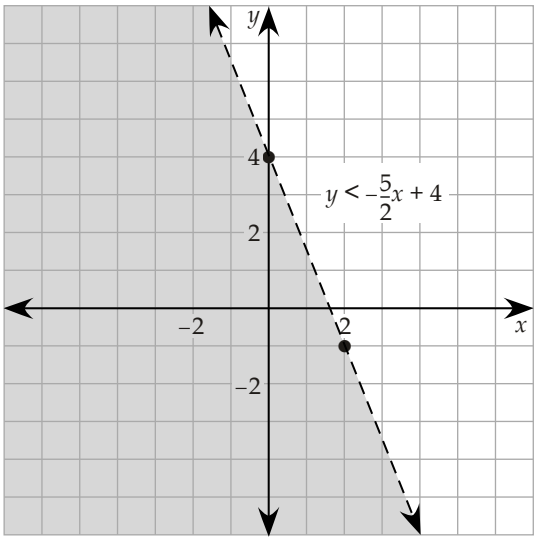
Rearrange to slope-intercept form, graph the related equation as a dotted line.

$$-2y + 8 > 5x$$

$$-2y > 5x - 8$$

$$y < -\frac{5}{2}x + 4 \qquad \text{Dividing by a negative number.}$$

Shade where y-values are less than those on the line.



3. Match each linear inequality to its graph.

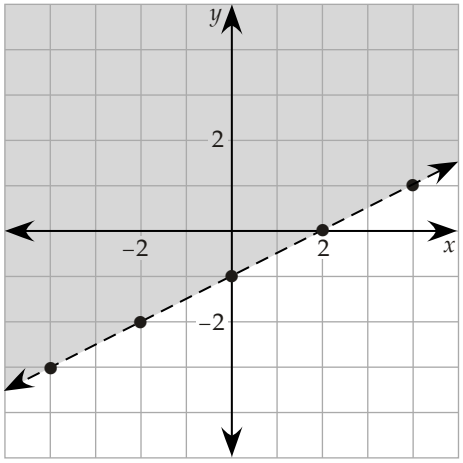
a) $y \leq 4$

b) $y \leq -x + 1$

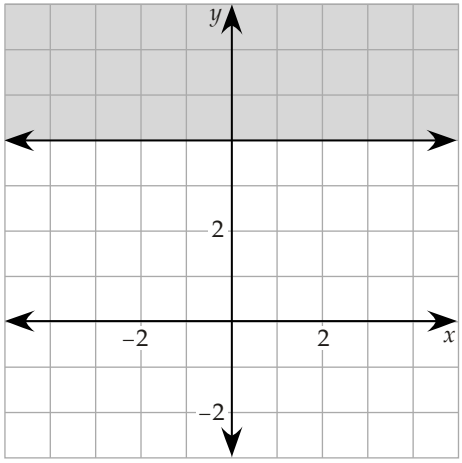
c) $2y > x - 2$

d) $y \geq 4$

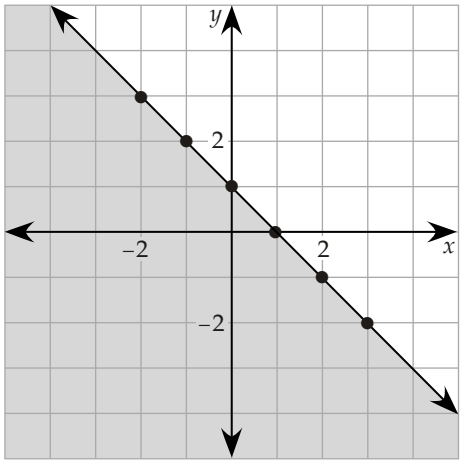
i)



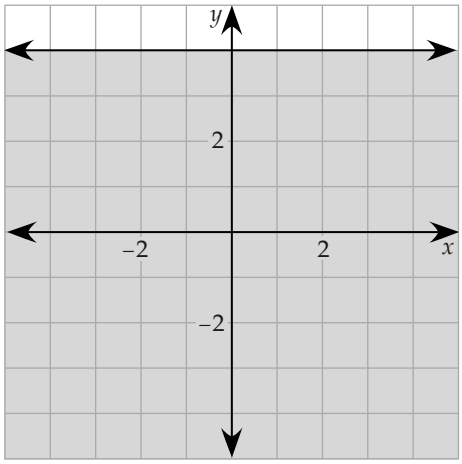
ii)



iii)



iv)



Answer:

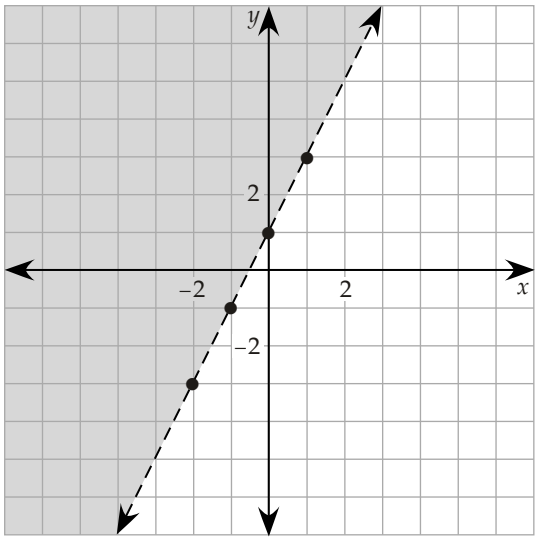
(a) and (iv)

(b) and (iii)

(c) and (i)

(d) and (ii)

4. Write a linear inequality to represent each graph.

a) 

Answer:

To find the equation of the inequality, you first need to find the equation of the boundary line.

To do this, you can find the slope of the line and the y-intercept. Then, you can put those values into the slope-intercept form equation of a line.

slope = 2

y-intercept = 1

slope-intercept equation of a line: $y = mx + b$

Thus, the equation of this line is $y = 2x + 1$.

Now, you need to determine whether you have a *greater than* or *less than* symbol.

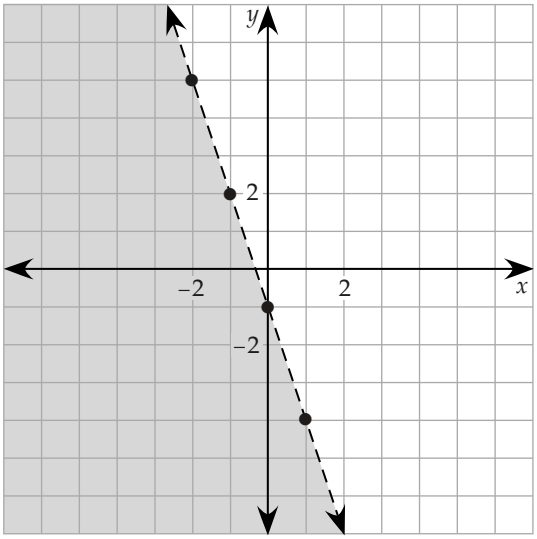
As you can see, the solution region is above the boundary line. Therefore, the symbol you need is *greater than*, or >.

You now need to determine if the points on the boundary line are included in the solution or not.

As the boundary line is broken or dashed, the points on the boundary line are not included in the solution to the inequality.

Thus, the inequality becomes $y > 2x + 1$.

b)



Answer:

To find the equation of the inequality, you first need to find the equation of the boundary line.

To do this, you can find the slope of the line, and the y-intercept. Then, you can put those values into the slope-intercept form equation of a line.

slope = -3

y-intercept = -1

slope-intercept equation of a line: $y = mx + b$

Thus, the equation of this line is $y = -3x - 1$.

Now, you need to determine whether you have a *greater than* or *less than* symbol.

As you can see, the solution region is below the boundary line. Therefore, the symbol you need is *less than*, or <.

You now need to determine if the points on the boundary line are included in the solution or not.

As the boundary line is broken or dashed, the points on the boundary line are not included in the solution to the inequality.

Thus, the inequality becomes $y < -3x - 1$.

5. Hunter is a high school student and he wants to save enough money to pay for his first year at university. He is holding down two part-time jobs. If he wants to save enough money, he needs to make at least $200 a week. One of his part-time jobs is working at a restaurant and he is paid $10 an hour. He makes $12 an hour at his second part-time job as a maintenance worker at his dad's business.

   a) Write an inequality to represent the hours Hunter needs to work.

      Answer:

      Let R = the hours Hunter works at the restaurant

      Let M = the hours Hunter works as a maintenance person

      $10R + 12M \geq 200$

   b) Graph the inequality.

      Answer:

      Rearrange the inequality. Either variable could be placed on the x-axis. This solution shows M on the x-axis. Thus, rearrange to solve for R.

      $10R + 12M \geq 200$

      $10R \geq 200 - 12M$

      $R \geq 20 - \frac{12}{10}M$

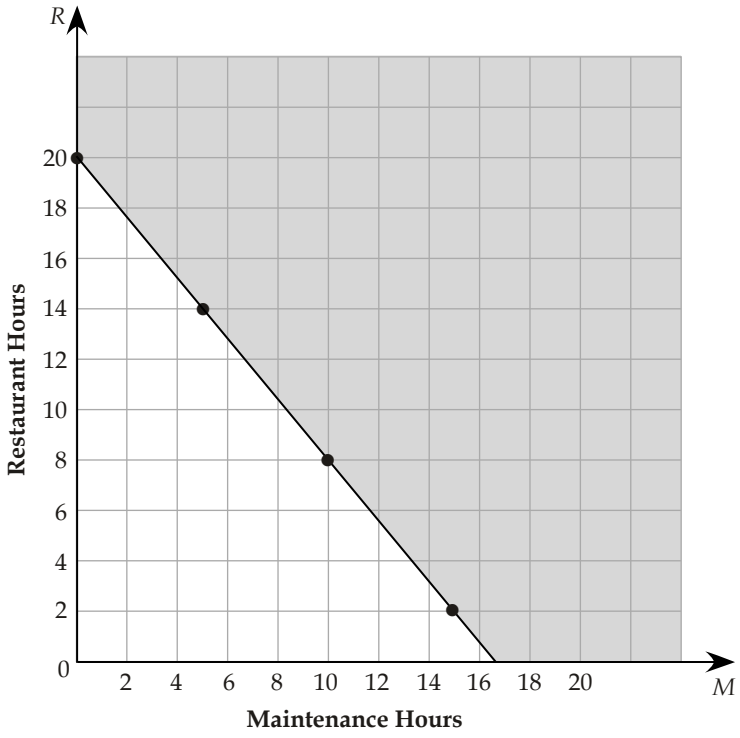
      $R \geq -\frac{6}{5}M + 20$

      slope = $-\frac{6}{5}$

      y-intercept = 20

      solid line

      Shade above the line.



c) List two different ordered pairs that are solutions to this inequality.

Answer:

The ordered pairs for this graph are (M, R). Thus, two solutions could be (10, 12) and (0, 20). Any ordered pairs on the boundary or in the shaded region will work.

d) What do these ordered pairs symbolize?

Answer:

These ordered pairs symbolize how many hours Hunter needs to work at each job in order to make at least $200 a week. Therefore, Hunter can work 10 hours doing maintenance and 12 hours at the restaurant and make at least $200 a week. Also, Hunter can work 20 hours at the restaurant and 0 hours in maintenance in order to make at least $200 a week.

Learning Activity 6.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Simplify: $\frac{x^6y^3}{x^8y^2}$
2. Pizza Express delivers pizza with the following charges: a $10.00 delivery and tip charge and $7 per pizza. Your sports team wants to order pizza. Everyone pools their money and they come up with $73.15. What is the maximum number of pizzas they can order?
3. Patricia ate one-third of the red velvet cake her mom bought her for her birthday that day. The next day, she ate one-half of the remaining cake. How much cake is left?
4. Simplify: $2^4 \times 2^2$
5. Reduce the following fraction: $\frac{105}{21}$
6. Simplify: $\left(-\frac{4^7\left(3^2\right)}{16^2}\right)^0$
7. Find the midpoint between the following two points: (26, −13) and (15, −21)
8. Solve the inequality $3 - 7x \geq 0$.

Answers:

1. $\frac{y}{x^2}$
2. They can order a maximum of 9 pizzas. ($73.15 − $10.00 = $63.15; $\frac{63.15}{7} = 9$)
3. $\frac{1}{3}$ $\left(1 - \frac{1}{3} = \frac{2}{3} \text{ and } \frac{1}{2} \text{ of } \frac{2}{3} = \frac{1}{3}\right)$ $\left(\frac{2}{3} - \frac{1}{3} = \frac{1}{3}\right)$
4. $2^6 = 64$

5. $5\left(\frac{35}{7}=\frac{5}{1}\right)$

6. 1 (any number to the power of zero is 1)

7. $\mathrm{M}=\left(\frac{41}{2},-\frac{34}{2}\right)=\left(\frac{41}{2},-17\right)$

8. $x \leq \frac{3}{7}$

(Method 1:

$3 \geq 7x$

$\frac{3}{7} \geq x$ or $x \leq \frac{3}{7}$

Method 2:

$-7x \geq -3$

$x \leq \frac{-3}{-7}$

$x \leq \frac{3}{7}\Big)$

Part B: Solving Quadratic Inequalities in One Variable

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Determine whether the given value of x is a solution to the given inequality.

   a) $x = 2, x^2 + 3x - 6 \leq 0$

   Answer:

   $x^2 + 3x - 6 \leq 0$

   $x = 2$

   $\mathrm{LHS} = x^2 + 3x - 6$

   $= 2^2 + 3(2) - 6$

   $= 4$

   RHS = 0

   LHS $\not\leq$ RHS

   Thus, 2 is not a solution to the given inequality.

b) $x = -4$, $2x^2 + 4x - 9 > 0$

Answer:

$2x^2 + 4x - 9 > 0$

$x = -4$

LHS $= 2x^2 + 4x - 9$

$= 2(-4)^2 + 4(-4) - 9$

$= 32 - 16 - 9$

$= 7$

RHS $= 0$

LHS > RHS

Thus, $x = -4$ is a solution.

c) $x = 0$, $4x^2 - 3x + 4 > 0$

Answer:

$4x^2 - 3x + 4 > 0$

$x = 0$

LHS $= 4x^2 - 3x + 4$

$= 4(0)^2 - 3(0) + 4$

$= 4$

RHS $= 0$

LHS > RHS

Thus, $x = 0$ is a solution.

2. Draw the graph of the quadratic function $y = x^2 + 2x + 1$. Then, determine the solution for the following:

   a) $x^2 + 2x + 1 \geq 0$

   b) $x^2 + 2x + 1 > 0$

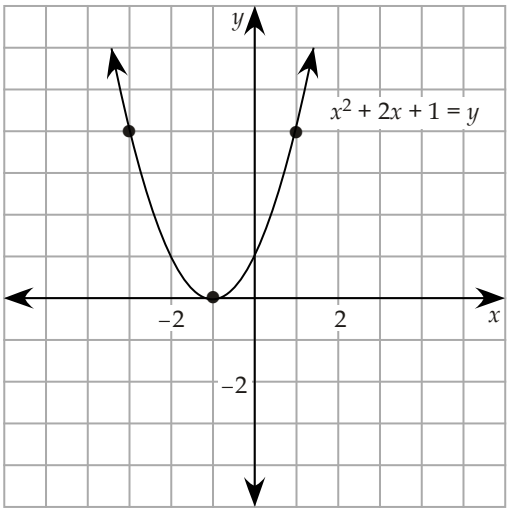
   c) $x^2 + 2x + 1 \leq 0$

   d) $x^2 + 2x + 1 < 0$

   Answer:

   $y = x^2 + 2x + 1$

   $y = (x + 1)^2$



   a) $x^2 + 2x + 1 \geq 0$

      Answer:

      The graph lies on or above the x-axis at every single point. Therefore, the solution set is $\{x \mid x \in \Re\}$.

   b) $x^2 + 2x + 1 > 0$

      Answer:

      The graph lies above the x-axis at every point except when $x = -1$. Therefore, the solution set is $\{x \mid x \in \Re, x \neq -1\}$.

   c) $x^2 + 2x + 1 \leq 0$

      Answer:

      The graph lies on or below the x-axis only at the point $x = -1$ where it touches the x-axis. Therefore, the solution set is $\{x \mid x = -1\}$.

d) $x^2 + 2x + 1 < 0$

Answer:

The graph never lies below the x-axis. Thus, there is no solution to this inequality.

3. Use the graphing method to solve the following quadratic inequalities.

a) $x^2 + 4x + 4 \geq 0$

Answer:

First write the corresponding quadratic function in vertex form.

$x^2 + 4x + 4 \geq 0$

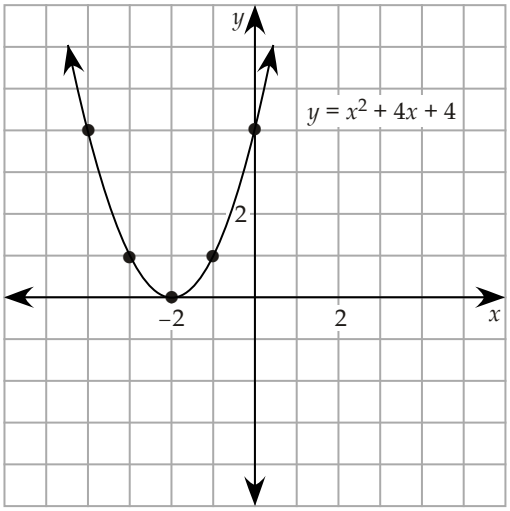
$y = x^2 + 4x + 4$

$y = (x^2 + 4x + 4) + 4 - 4$

$y = (x + 2)^2$

Vertex $(-2, 0)$, opens up, and normal size.

The graph of the function is shown below.



As you can see, the graph lies on or above the x-axis for all values of x. Thus, the solution set is $\{x \mid x \in \Re\}$.

b) $x^2 - 2x > 3$

Answer:

First, find the corresponding function in vertex form.

$x^2 - 2x > 3$

$x^2 - 2x - 3 > 0$

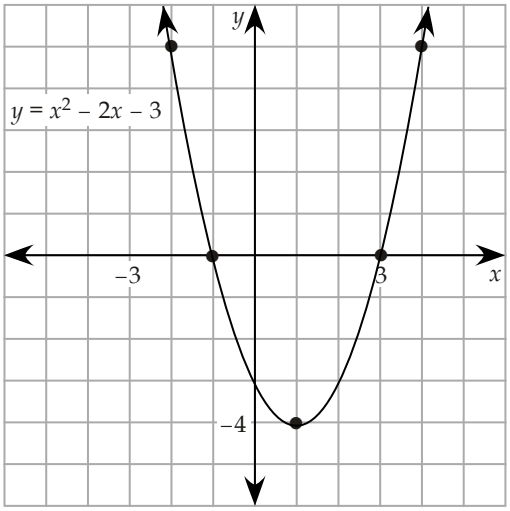
$y = x^2 - 2x - 3$

$y = (x^2 - 2x + 1) - 3 - 1$

$y = (x - 1)^2 - 4$

Vertex $(1, -4)$, opens up, normal shape.

The graph of the function is shown below.



As you can see, the graph lies above the x-axis when x is greater than 3 or when x is less than -1.

Thus, the solution set is $\{x \mid x > 3 \text{ or } x < -1, x \in \Re\}$.

4. Use the roots and test point method to make a sign diagram and solve the following quadratic inequalities.

   a) $x^2 - 2x + 1 \geq 0$

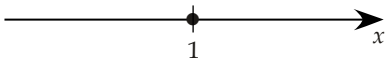
   Answer:

   Factoring the corresponding quadratic equation:

   $x^2 - 2x + 1 = 0$

   $(x - 1)(x - 1) = 0$

   The root of this quadratic equation is $x = 1$. Therefore, the two intervals are $x > 1$ and $x < 1$.



	$x^2 - 2x + 1 \geq 0$	
Interval	$x < 1$	$x > 1$
Test Value	$x = 0$	$x = 2$
Substituting the Test Point into the Inequality	$(0)^2 - 2(0) + 1$ $= 1$	$(2)^2 - 2(2) + 1$ $= 4 - 4 + 1$ $= 1$
Sign	positive	positive

Thus, as the inequality is asking for all points that are greater than or equal to zero, the solution set is all real numbers, or $\{x \mid x \in \Re\}$.

b) $-x^2 + x + 1 \geq 0$. Show your answer rounded to three decimal places.

Answer:

$-x^2 + x + 1 \geq 0$

$x^2 - x - 1 \leq 0$ Multiply by (-1)

This quadratic does not factor easily; thus, you need to use the quadratic formula to find the roots.

$a = 1, b = -1, c = -1$

First, find the discriminant.

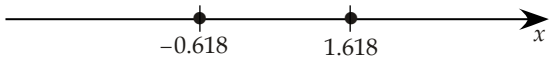
$$b^2 - 4ac = (-1)^2 - 4(1)(-1)$$
$$= 1 + 4$$
$$= 5$$

$\therefore b^2 - 4ac > 0$ and two unequal irrational roots.

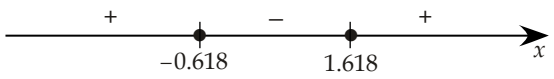
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{1 + \sqrt{5}}{2} = 1.618$$

$$x = \frac{1 - \sqrt{5}}{2} = -0.618$$



	$x^2 - x - 1 \leq 0$		
Interval	$x \leq -0.616$	$-0.616 \leq x \leq 1.618$	$x \geq 1.618$
Test Value	$x = -1$	$x = 0$	$x = 2$
Substituting the Test Point into the Inequality	$(-1)^2 - (-1) - 1$ $= 1 + 1 - 1$ $= 1$	$(0)^2 - 0 - 1$ $= -1$	$(2)^2 - 2 - 1$ $= 4 - 2 - 1$ $= 1$
Sign	positive	negative	positive



The solution consists of all points where the inequality is negative or zero:

$\{x \mid -0.618 \leq x \leq 1.618, x \in \Re\}$

5. Use the case analysis method to solve the following quadratic inequalities.

   a) $3x^2 + 16x \leq -5$

   Answer:

   Factor the quadratic inequality:

   $3x^2 + 16x \leq -5$

   $3x^2 + 16x + 5 \leq 0$

   $(3x + 1)(x + 5) \leq 0$

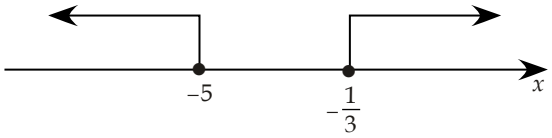
   In order for the quadratic inequality to be less than or equal to zero, one of these factors has to be negative or less than or equal to zero, and the other has to be positive or zero.

   Case 1:

   Consider when $(x + 5)$ is less than or equal to zero and $(3x + 1)$ is greater than or equal to zero:

   $x + 5 \leq 0$ and $3x + 1 \geq 0$

   $x \leq -5$ and $x \geq -\frac{1}{3}$



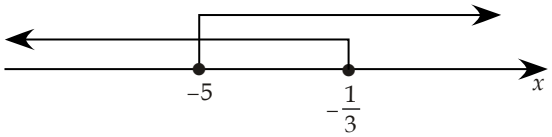
   As you can see from the diagram, there are no points that satisfy both inequalities. There is no solution for this case.

   Case 2:

   Consider when $(3x + 1)$ is less than or equal to zero and $x + 5 \geq 0$:

   $x + 5 \geq 0$ and $3x + 1 \leq 0$

   $x \geq -5$ and $x \leq -\frac{1}{3}$



   As you can see, the interval that satisfies both inequalities is $-5 \leq x \leq -\frac{1}{3}$.

   Thus, the solution set is $\left\{x \mid -5 \leq x \leq -\frac{1}{3}, x \in \Re\right\}$.

b) $x^2 + 3x - 180 \geq 0$

Answer:

Factor the inequality:

$x^2 + 3x - 180 \geq 0$

$(x - 12)(x + 15) \geq 0$

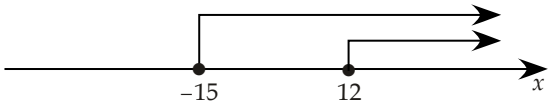
In order for the quadratic inequality to be greater than or equal to zero, both of the factors have to be positive or both factors have to be negative. There are two cases.

Case 1:

Consider both factors are positive:

$(x - 12) \geq 0$ and $(x + 15) \geq 0$

$x \geq 12$ and $x \geq -15$



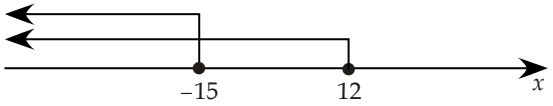
As you can see, both conditions are satisfied when $x \geq 12$.

Case 2:

Consider both factors are negative:

$(x - 12) \leq 0$ and $(x + 15) \leq 0$

$x \leq 12$ and $x \leq -15$



As you can see, both conditions are satisfied when $x \leq -15$.

Considering both cases, the solution set is $\{x \mid x \leq -15 \text{ or } x \geq 12, x \in \Re\}$.

6. Use the sign analysis method to make factor diagrams to solve the following quadratic inequalities.

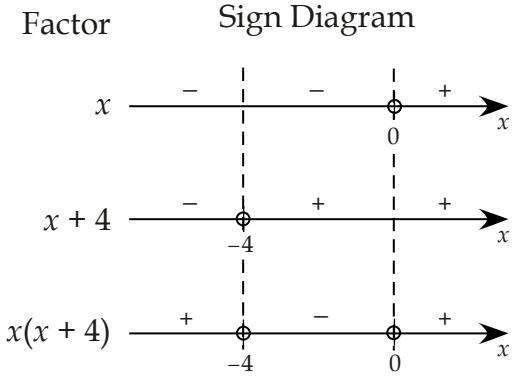
   a) $x^2 + 4x < 0$

   Answer:

   $x^2 + 4x < 0$

   $x(x + 4) < 0$

   The critical points are the roots $x = 0, -4$. By testing x-values on either side of $x = 0$ and $x = -4$, the sign diagram can be made.



   The solution occurs when the product of the factors is negative. The inequality, $x(x + 4) < 0$, is negative in the interval $(-4, 0)$ or can be written using set notation as $\{x \mid -4 < x < 0\}$.

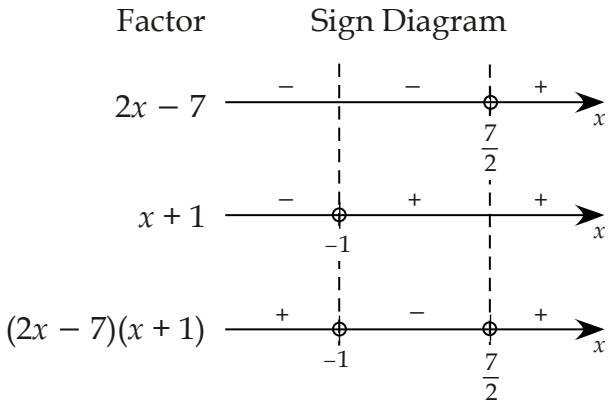
   b) $2x^2 - 5x - 7 > 0$

   Answer:

   $2x^2 - 5x - 7 > 0$

   $(2x - 7)(x + 1) > 0$

   The critical points are the roots $x = \frac{7}{2}$ and $x = -1$. By testing values of x on either side of the critical points, the sign diagram can be made.



The solution occurs where the product of the factors is positive.

Solution set: $\left\{x \mid x < -1 \text{ or } x > \frac{7}{2}, x \in \Re\right\}$

7. A man is standing on a 50-foot cliff. He can only see things that are higher than or as high as the cliff. Fireworks are being fired from the ground down below, and their height, measured in feet, at time t, measured in seconds, is given by the equation $h = -4t^2 + 28t + 5$.

   a) Solve the inequality $-4t^2 + 28t + 5 \geq 50$.

   Answer:

   The quadratic is $-4t^2 + 28t + 5 \geq 50$ or $-4t^2 + 28t - 45 \geq 0$ or $4t^2 - 28t + 45 \leq 0$.

   First, find the discriminant.

   $a = 4, b = -28, c = 45$

   $b^2 = 4ac = (-28)^2 - 4(4)(45)$

   $= 64$

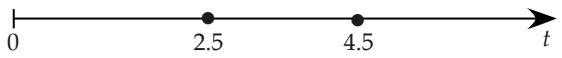
   $\therefore$ two rational roots since the discriminant is a perfect square.

   $\therefore$ factor into (or use the quadratic formula):

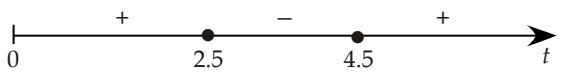
   $(2t - 9)(2t - 5) \leq 0$

   $t = \frac{9}{2}, \frac{5}{2}$

   $t = 4.5, 2.5$



	$4t^2 - 28t + 45 \leq 0$		
Interval	$0 \leq t \leq 2.5$	$2.5 \leq t \leq 4.5$	$t \geq 4.5$
Test Value	$t = 0$	$t = 3$	$t = 5$
Substituting the Test Point into the Inequality	$4(0)^2 - 28(0) + 45 = 45$	$4(3)^2 - 28(3) + 45 = -3$	$4(5)^2 - 28(5) + 45 = 5$
Sign	positive	negative	positive



The solution is where the result is negative or zero:

$\{t \mid 2.5 \leq t \leq 4.5, t \in \Re\}$

b) Interpret the solution to the inequality you solved in (a).

Answer:

This signifies the time at which the fireworks are visible to the man on the cliff. Therefore, the fireworks are visible to the man at 2.5 seconds and until 4.5 seconds after they are fired.

c) Explain how you chose which strategy to use to solve the quadratic inequality.

Answer:

Answers may vary.

The roots and test points method was used because it is efficient. The graphing method would be difficult, as the quadratic equation isn't in vertex form and the solutions may not be integer values. However, case analysis and sign analysis could also be used to solve this inequality.

Learning Activity 6.5

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Cassandra pays for $78.83 worth of groceries with a $100 bill. What is her change?
2. What is the vertex of the parabola defined by the equation $y = 2x^2 - 6$?
3. The movie you want to see starts playing at 8:30 p.m. It takes you 45 minutes to drive to the theatre and 30 minutes to get ready. If you want to be at the theatre 30 minutes early, what time should you start getting ready?
4. Find the common difference in the following pattern: 1, 7, 13, 19, . . .
5. Calculate: $\dfrac{\frac{4}{3}}{\frac{5}{8}}$
6. Express in exponential form: $\sqrt{y^{\frac{7}{2}}}$
7. Simplify: $(4x^2 + 7x - 3)(-3x + 2)$
8. What does the range of a quadratic function depend on?

Answers:

1. $21.17 ($100 − $78.83)
2. Vertex = $(0, -6)$ $\left(x = -\dfrac{b}{2a} = -\dfrac{0}{4} = 0;\ y = 2(0)^2 - 6 = 0 - 6 = -6\right)$
3. 6:45 pm (You need to leave 45 + 30 + 30 = 1 hour and 45 minutes before the movie starts. One hour before 8:30 is 7:30 and 45 minutes before 7:30 is 6:45. Therefore, you need to start getting ready at 6:45 pm.)
4. 6
5. $\dfrac{32}{15}$ $\left(\dfrac{4}{3} \times \dfrac{8}{5}\right)$

6. $y^{\frac{7}{4}}\left(\sqrt{y^{\frac{7}{2}}} = \left(y^{\frac{7}{2}}\right)^{\frac{1}{2}}\right)$

7. $-12x^3 + 8x^2 - 21x^2 + 14x + 9x - 6 = -12x^3 - 13x^2 + 23x - 6$

8. The range of a quadratic function depends on the direction of opening of the parabola and the location of the vertex.

Part B: Solving Quadratic Inequalities in Two Variables

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

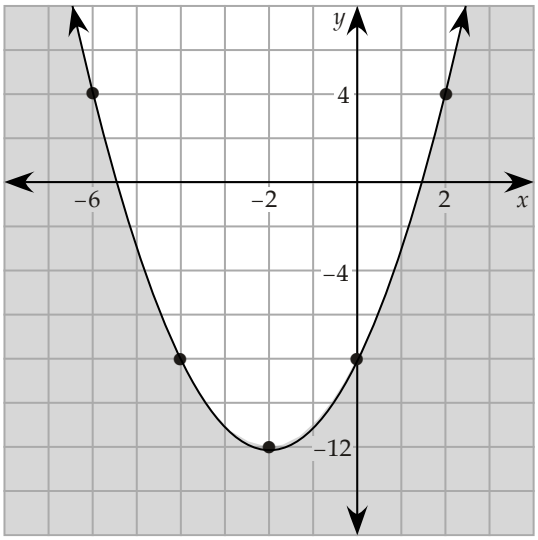
1. Determine whether the following statements are true or false. If they are false, explain why they are false and give the correct statement.

   a) To determine if the region outside a parabola is a solution to an inequality, test a point outside the parabola. If that point does not satisfy the inequality, then that region is a solution to the inequality.

   Answer:

   False. The correct statement should be "To determine if the region outside a parabola is a solution to an inequality, test a point outside the parabola. If that point **satisfies** the inequality, then that region is a solution to the inequality."

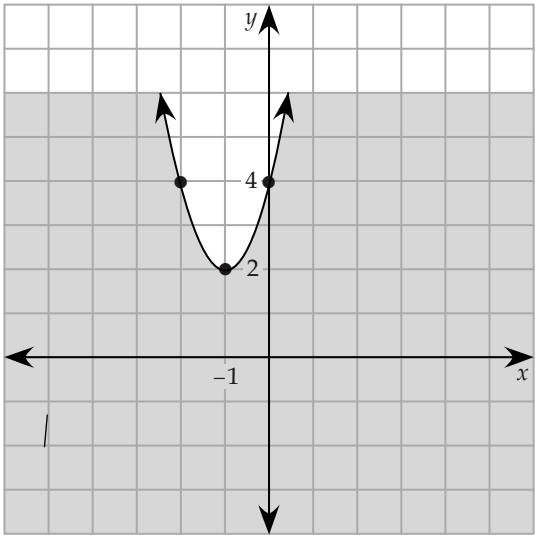
b) If (1, 2) satisfies the inequality $y \geq (x + 2)^2 - 12$, then the following graph represents the inequality.



Answer:

False. The shaded region should be below the parabola. The graph shows the inequality $y \leq (x + 2)^2 - 12$.

c) If $(-3, -2)$ satisfies the inequality $y \leq 2(x + 1)^2 + 2$, then the following graph represents the inequality.



Answer:

True. The point $(-3, -2)$ is in the shaded region below the function curve.

d) The quadratic inequality $y < 3x^2 - 6x + 7$ includes a solid boundary line.

Answer:

False. The boundary line should be dashed or broken because the inequality sign is strictly "less than."

e) The quadratic inequality $y > -x^2 - 5x + 4$ includes a broken or dashed boundary line.

Answer:

True. The inequality does not include "equal to."

f) The point (1, 6) is a solution to the inequality $y > -x^2 + x + 4$.

Answer:

$y > -x^2 + x + 4$

Point (1, 6)

LHS = $y = 6$

RHS = $-x^2 + x + 4$

$= -(1)^2 + (1) + 4$

$= 4$

LHS > RHS

$\therefore$ The statement is true.

g) The point (–1, 5) is a solution to the inequality $y \leq 2x^2 - x + 1$.

Answer:

$y \leq 2x^2 - x + 1$

Point (−1, 5)

LHS = $y = 5$

RHS = $2x^2 - x + 1$

$= 2(-1)^2 - (-1) + 1$

$= 4$

LHS ≤ RHS

$\therefore$ The statement is false.

A true statement would be the point (−1, 5) is a solution to the inequality $y \geq 2x^2 - x + 1$.

2. Match the inequality with its graph.

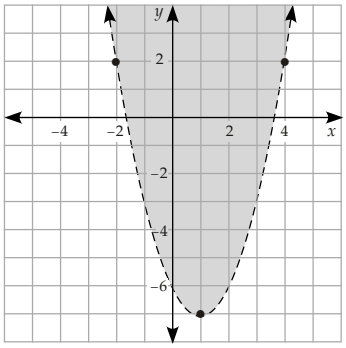
a) $y > -x^2 - 6x - 2$

b) $y \geq 2x^2 + 3x - 5$

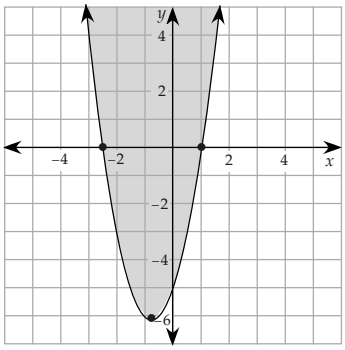
c) $y \leq -x^2 + 3x + 5$

d) $y > x^2 - 2x - 6$

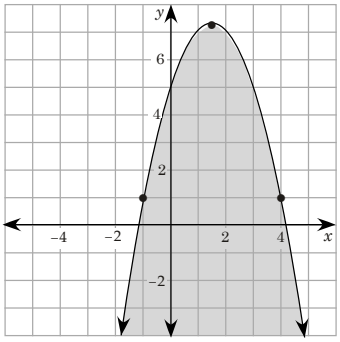
i)



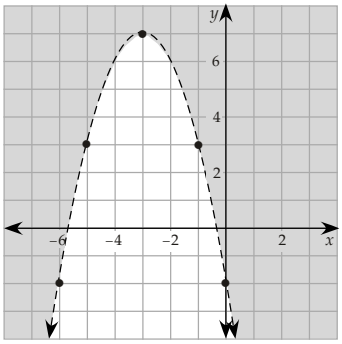
ii)



iii)



iv)



Answers:

a) $y > -x^2 - 6x - 2$

opens down; shading above parabola, dotted boundary

$\therefore$ Answer is (iv).

b) $y \geq 2x^2 + 3x - 5$

opens up, shading is above parabola, boundary is solid

$\therefore$ Answer is (ii).

c) $y \leq -x^2 + 3x + 5$

opens down, shading is below the parabola, boundary is solid

$\therefore$ Answer is (iii)

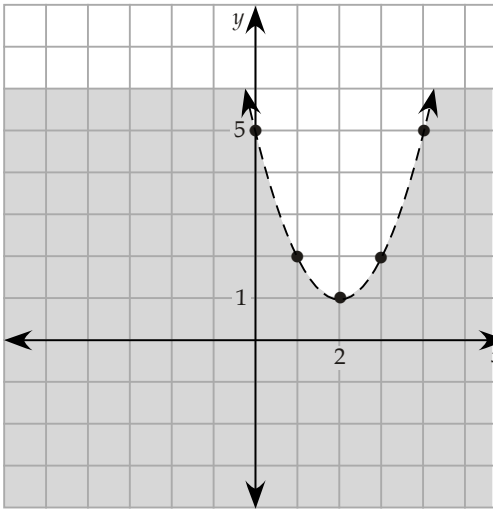
d) $y > x^2 - 2x - 6$

opens up, shading is above parabola, boundary is dotted.

$\therefore$ Answer is (i).

3. Write an inequality to describe each of the following graphs.

a)



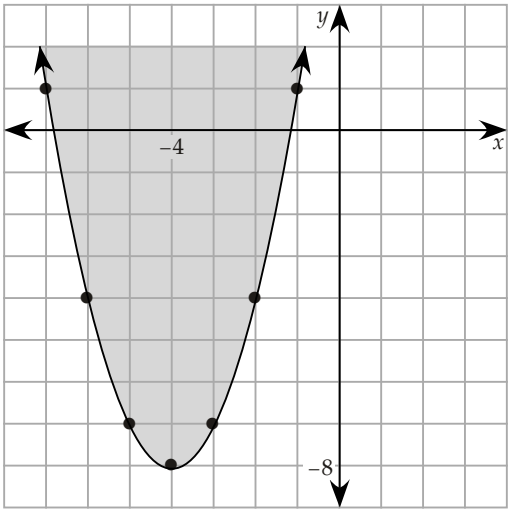
Answer:

The vertex is at (2, 1) and it is the normal shape. So, the equation of the parabola is $y = (x - 2)^2 + 1$.

The symbol is "less than" as the region shaded is below the parabola and the boundary line is broken.

Thus, the inequality is $y < (x - 2)^2 + 1$.

b)



Answer:

The vertex is at $(-4, -8)$ and it is the normal shape. So, the equation of the parabola is $y = (x + 4)^2 - 8$.

The symbol is "greater than or equal to" as the region shaded is inside the parabola and the boundary line is solid.

Thus, the inequality is $y \geq (x + 4)^2 - 8$.

4. Sketch the graph of the inequalities without using the graphing calculator.

   a) $y \leq x^2 + 4x$

   b) $y > \frac{1}{2}x^2 + 3x + 2$

   c) $y < -3x^2 - 6x$

   Answers:

   a) For $y \leq x^2 + 4x$

   Sketch the related equation $y = x^2 + 4x$.

   x-coordinate of vertex

   $$x = \frac{-b}{2a}$$

   $$x = \frac{-4}{2(1)}$$

   $$x = -2$$

   y-coordinate of vertex

   $y = x^2 + 4x$

   $y = (-2)^2 + 4(-2)$

   $y = 4 - 8$

   $y = -4$

   Zeros: Let $y = 0$

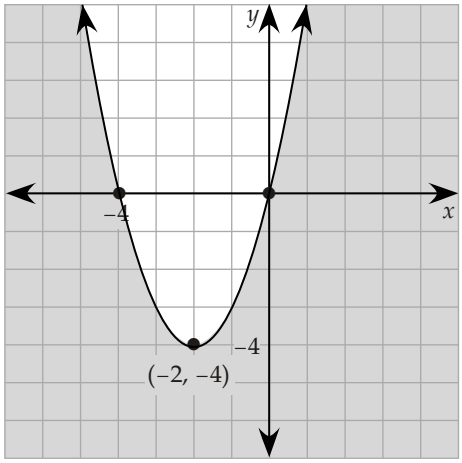
   $x^2 + 4x = 0$

   $x(x + 4) = 0$

   $x = 0$ and $x = -4$

   $\therefore$ Vertex $(-2, -4)$. The curve opens up and is normal shape.

   The parabola is represented by a solid line and the region below the parabola is shaded.



   Checkpoint (1, 0)

   LHS = $y = 0$

   RHS = $x^2 + 4x$

   $= 1^2 + 4(1)$

   $= 5$

   LHS $\leq$ RHS

   $\therefore$ (1, 0) is in the solution region.

b) For $y > \frac{1}{2}x^2 + 3x + 2$

Sketch the related equation $y = \frac{1}{2}x^2 + 3x + 2$.

x-coordinate of vertex

$$x = -\frac{b}{2a}$$

$$x = \frac{-3}{2 \cdot \frac{1}{2}}$$

$$x = -3$$

y-coordinate of vertex

$$y = \frac{1}{2}x^2 + 3x + 2$$

$$y = \frac{1}{2}(-3)^2 + 3(-3) + 2$$

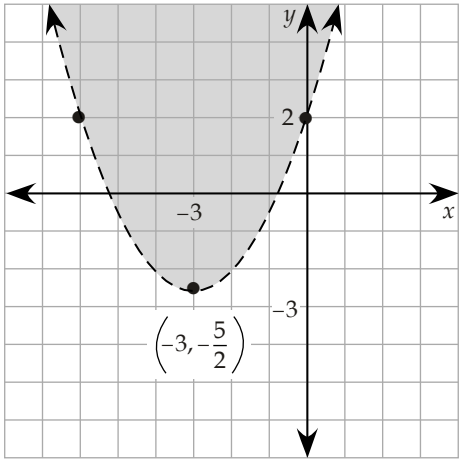
$$y = \frac{9}{2} + (-9) + 2$$

$$y = \frac{-5}{2}$$

$\therefore$ Vertex $\left(-3, -\frac{5}{2}\right)$

y-intercept: 2

The region above $y = \frac{1}{2}x^2 + 3x + 2$ is shaded and the outline of the border curve is dashed.



Checkpoint $(-4, 0)$

LHS = $y = 0$

$$\text{RHS} = \frac{1}{2}(-4)^2 + 3(-4) + 2$$

$$= 8 - 12 + 2$$

$$= -2$$

LHS > RHS

$\therefore$ $(-4, 0)$ is in the solution region.

c) For $y < -3x^2 - 6x$

Graph the related equation $y = -3x^2 - 6x$ and use a broken line curve.

x-coordinate of vertex

$$x = -\frac{b}{2a}$$

$$x = -\frac{-6}{2(-3)}$$

$$x = -1$$

y-coordinate of vertex

$y = -3x^2 - 6x$

$y = -3(-1)^2 - 6(-1)$

$y = -3 + 6$

$y = 3$

Zeros:

$-3x^2 - 6x = 0$

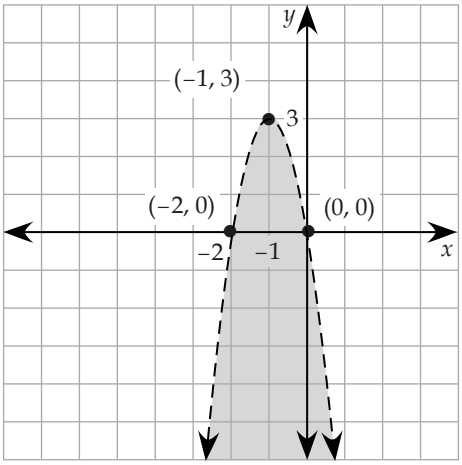
$-3x(x + 2) = 0$

$x = 0$ and $x + 2 = 0$

$x = 0$ and $x = -2$

Vertex $(-1, 3)$

$\therefore$ The shading is below the curve.



Checkpoint $(-1, 0)$

LHS = $y = 0$

RHS = $-3x^2 - 6x$

$= -3(-1)^2 - 6(-1)$

$= -3 + 6$

$= 3$

LHS < RHS

$\therefore$ $(-1, 0)$ is in the solution region.

Grade 11 Pre-Calculus Mathematics (30S)

Module 6
Systems of Equations and Inequalities

Learning Activity Answer Keys

Module 6:
Systems of Equations and Inequalities

Learning Activity 6.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. How much tax would you have to pay on an item that costs $49.99 (approximately) if the PST = 8% and the GST = 5%?
2. Simplify: $\sqrt{98}$
3. Find the equation of a line with a slope of $-\frac{1}{2}$ and a y-intercept of 3.
4. Evaluate: $\frac{3}{21} \times \frac{3}{15}$
5. If 10% of 440 is 44, what is 35% of 440?
6. Are these fractions equivalent? $\frac{\sqrt{5}}{\sqrt{7}}$ and $\frac{\sqrt{35}}{7}$
7. Determine the length of the hypotenuse of a right-angled triangle when the two sides measure 10 cm and 13 cm.
8. Find the slope of the line that contains the points (26, −13) and (15, −21).

Answers:

1. Approximately $6.50 (13% of $100 = $13; 13% of $50 = $6.50)
2. $7\sqrt{2}$ $\left(\sqrt{98} = \sqrt{49 \times 2}\right)$
3. $y = -\frac{1}{2}x + 3$
4. $\frac{1}{35}$ $\left(\frac{1}{7} \times \frac{1}{5}\right)$
5. 154 (10% of 440 = 44; 5% of 440 = 22; 35% of 440 = 44 + 44 + 44 + 22 = 154)

6. Yes $\left(\frac{\sqrt{5}}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{\sqrt{35}}{7}\right)$

7. $\sqrt{269}$ $\left(\sqrt{10^2 + 13^2} = \sqrt{100 + 169}\right)$

8. $\frac{8}{11}$ $\left(m = \frac{-21 - (-13)}{15 - 26} = \frac{-8}{-11}\right)$

Part B: Solving Quadratic-Linear Systems of Equations

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Verify that (−1, 0) and (4, 15) are solutions to the following system of equations.

   $y = 3x + 3$

   $y = x^2 - 1$

   Answer:

   (−1, 0)

   $y = 3x + 3$

   LHS $= y = 0$

   RHS $= 3x + 3$

   $= 3(-1) + 3$

   $= -3 + 3$

   $= 0$

   LHS = RHS

   $y = x^2 - 1$

   LHS $= y = 0$

   RHS $= x^2 - 1$

   $= (-1)^2 - 1$

   $= 1 - 1$

   $= 0$

   LHS = RHS

(4, 15)

$y = 3x + 3$

LHS $= y = 15$

RHS $= 3x + 3$

$= 3(4) + 3$

$= 12 + 3$

$= 15$

LHS = RHS

$y = x^2 - 1$

LHS $= y = 15$

RHS $= x^2 - 1$

$= (4)^2 - 1$

$= 16 - 1$

$= 15$

LHS = RHS

Thus, (−1, 0) and (4, 15) are solutions to the above system of equations.

2. Solve each system both graphically and algebraically.

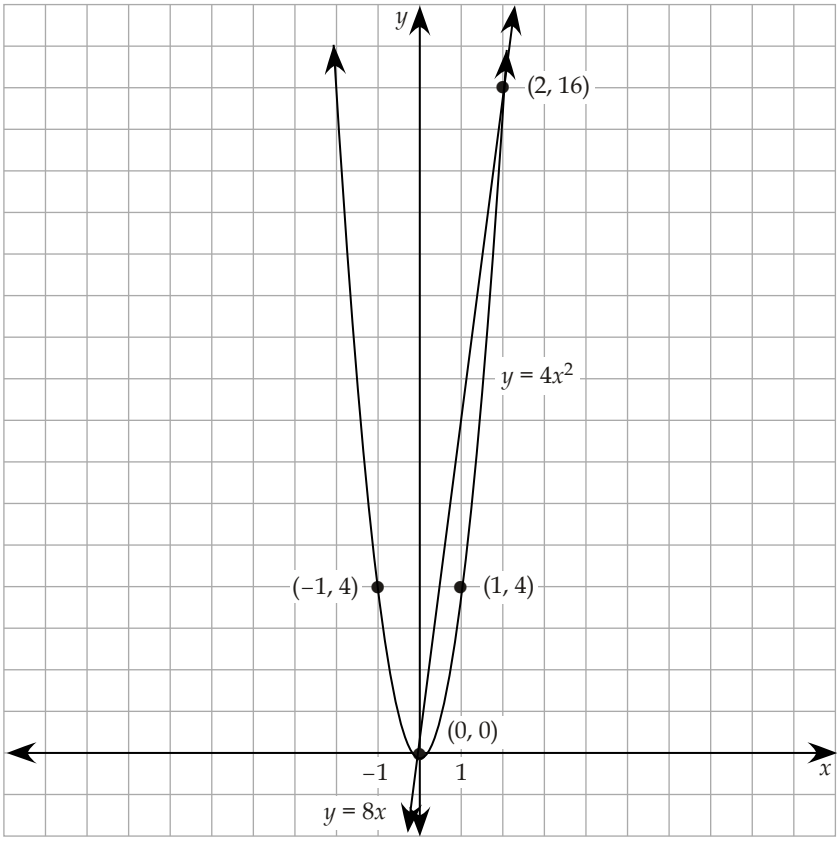
a) $y = 4x^2$

$y = 8x$

Answer:

Graph solution:

The points of intersection are (0, 0) and (2, 16).



Algebraic solution:

$y = 4x^2$

$y = 8x$

Because both equations are equal to y, then:

$4x^2 = 8x$

$4x^2 - 8x = 0$

$4x(x - 2) = 0$

$4x = 0$ or $x - 2 = 0$

$x = 0$ $x = 2$

Substitute the x-values into the linear equation to solve for y.

$y = 8x$	$y = 8x$
$y = 8 \cdot 0$	$y = 8 \cdot 2$
$y = 0$	$y = 16$

The two solutions are (0, 0) and (2, 16), and they are the two points of intersection in the graph solution.

b) $x - 2y = 10$

$y = x^2 + 2x - 15$

Answer:

Graph solution:

To graph $x - 2y = 10$, find the intercepts first.

x-intercept	y-intercept
let $y = 0$	let $x = 0$
$x - 2(0) = 10$	$0 - 2y = 10$
$x = 10$	$y = -5$
Point (10, 0)	Point (0, −5)

To graph $y = x^2 + 2x - 15$, first complete the square to find the vertex coordinates:

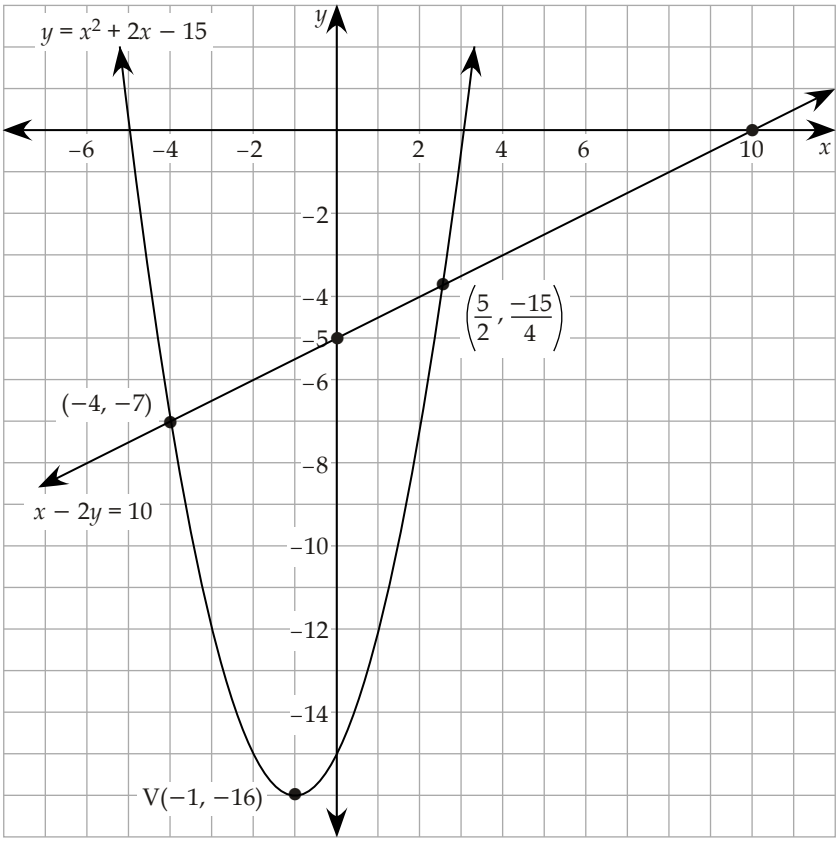
$y = (x^2 + 2x \quad) - 15$

$y = (x^2 + 2x + 1) - 15 - 1$

$y = (x + 1)^2 - 16$

Vertex (−1, −16), opens up, normal shape

One point of intersection is (−4, −7). The other point would need to be estimated near (3, −4).



Algebraic solution:

$x - 2y = 10$ (1)

$y = x^2 + 2x - 15$ (2)

Since Equation (2) is already solved for y, substitute $x^2 + 2x - 15$ for y in Equation (1).

$$x - 2y = 10$$
$$x - 2\left(x^2 + 2x - 15\right) = 10$$
$$x - 2x^2 - 4x + 30 = 10$$
$$0 = 2x^2 + 3x - 20$$
$$0 = (2x - 5)(x + 4)$$

$2x - 5 = 0$ $\qquad$ $x + 4 = 0$

$x = \frac{5}{2}$ $\qquad$ $x = -4$

Find the corresponding y-values using the linear equation.

$x = \frac{5}{2}$ $\qquad$ $x = -4$

$x - 2y = 10$ $\qquad$ $x - 2y = 10$

$\frac{5}{2} - 2y = 10$ $\qquad$ $-4 - 2y = 10$

$5 - 4y = 20$ $\qquad$ $-2y = 14$

$-4y = 15$ $\qquad$ $y = -7$

$y = -\frac{15}{4}$

The solutions are $\left(\frac{5}{2}, -\frac{15}{4}\right)$ and $(-4, -7)$.

The algebraic method is preferred for this example because it is difficult to be precise using the graph.

3. Solve each system by the substitution method.

a) $y = 3x + 7$

$y = x^2 - 11$

Answer:

Using the substitution method:

$3x + 7 = x^2 - 11$

$0 = x^2 - 3x - 18$

$0 = (x - 6)(x + 3)$

$x = 6$ or $x = -3$

When $x = 6$:

$y = 3x + 7$

$y = 3(6) + 7$

$y = 25$

Thus, one solution is (6, 25).

When $x = -3$:

$y = 3x + 7$

$y = 3(-3) + 7$

$y = -2$

Thus, another solution is $(-3, -2)$.

The solutions to this system are (6, 25) and $(-3, -2)$.

b) $y = 2x + 6$

$y = x^2 + 4x + 3$

Answer:

Using the substitution method

$2x + 6 = x^2 + 4x + 3$

$0 = x^2 + 2x - 3$

$0 = (x + 3)(x - 1)$

$x = -3$ or $x = 1$

When $x = -3$:

$y = 2x + 6$

$y = 2(-3) + 6$

$y = 0$

Thus, one solution is $(-3, 0)$.

When $x = 1$:

$y = 2x + 6$

$y = 2(1) + 6$

$y = 8$

Thus, another solution is (1, 8).

The solutions to this system are $(-3, 0)$ and (1, 8).

4. Solve each system by the elimination method.

a) $2y = 2x - 2$

$y = x^2 - 4x + 3$

Answer:

Using the elimination method:

Multiply the first equation by $-\frac{1}{2}$ and then add the like terms.

$$\begin{aligned} -y &= -x + 1 \\ +\ y &= x^2 - 4x + 3 \\ \hline 0 &= x^2 - 5x + 4 \\ 0 &= (x - 4)(x - 1) \end{aligned}$$

$x = 4$ or $x = 1$

When $x = 4$:

$2y = 2x - 2$

$y = x - 1$

$y = 4 - 1$

$y = 3$

Thus, (4, 3) is one solution to this system.

When $x = 1$:

$2y = 2x - 2$

$2y = 2(1) - 2$

$2y = 0$

$y = 0$

Thus, (1, 0) is another solution to this system.

The solutions to this system are (4, 3) and (1, 0).

b) $2x + y = 5$

$y = x^2 - 6x + 5$

Answer:

Using the elimination method, solve the linear equation for y.

$$y = -2x + 5$$

Multiply both sides of the linear equation by -1 and then add like terms.

$$\begin{aligned} -y &= 2x - 5 \\ +\ y &= x^2 - 6x + 5 \\ \hline 0 &= x^2 - 4x \\ 0 &= x(x - 4) \end{aligned}$$

$x = 0$ or $x = 4$

When $x = 0$:

$$\begin{aligned} 2x + y &= 5 \\ 2(0) + y &= 5 \\ y &= 5 \end{aligned}$$

Thus, one solution to this system is (0, 5).

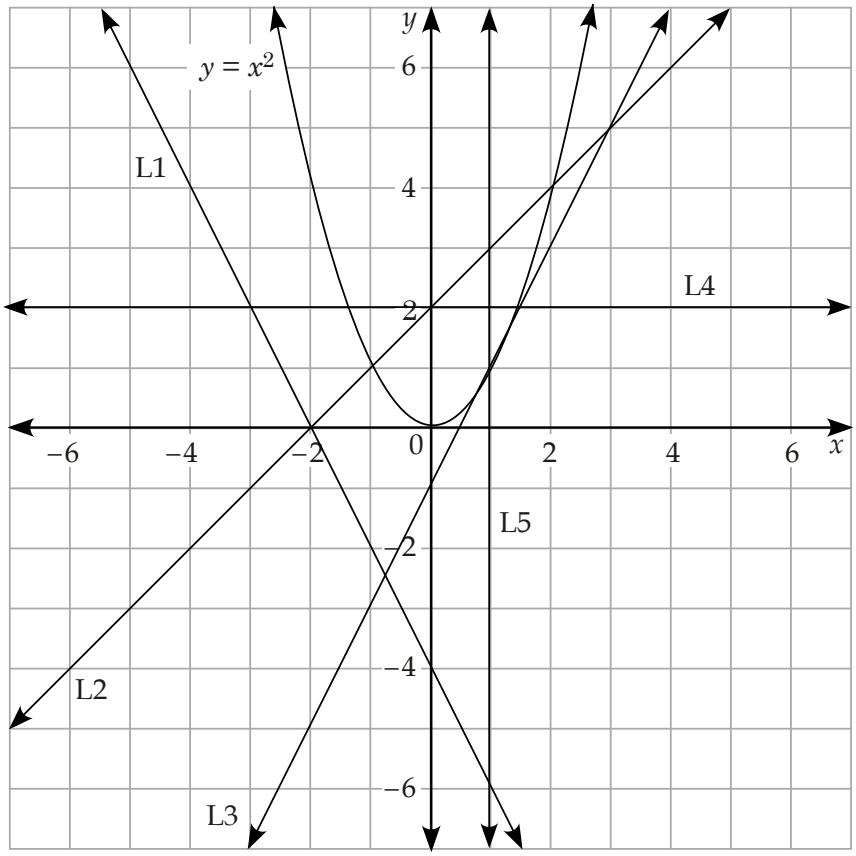
When $x = 4$:

$$\begin{aligned} 2x + y &= 5 \\ 2(4) + y &= 5 \\ 8 + y &= 5 \\ y &= -3 \end{aligned}$$

Thus, another solution to this system is (4, −3).

The solutions to this system are (0, 5) and (4, −3).

5. Consider the graph. Use this graph to answer the following questions.



a) State the number of solutions to the system consisting of the parabola and L1.

b) State the number of solutions to the system consisting of the parabola and L2.

c) State the number of solutions to the system consisting of the parabola and L3.

d) State the number of solutions to the system consisting of the parabola and L4.

e) State the number of solutions to the system consisting of the parabola and L5.

f) State the solution(s) to the system consisting of the parabola and L2.

g) State the solution(s) to the system consisting of the parabola and L5.

Answers:

a) There are no solutions to this system.

b) There are two solutions to this system.

c) There is one solution to this system.

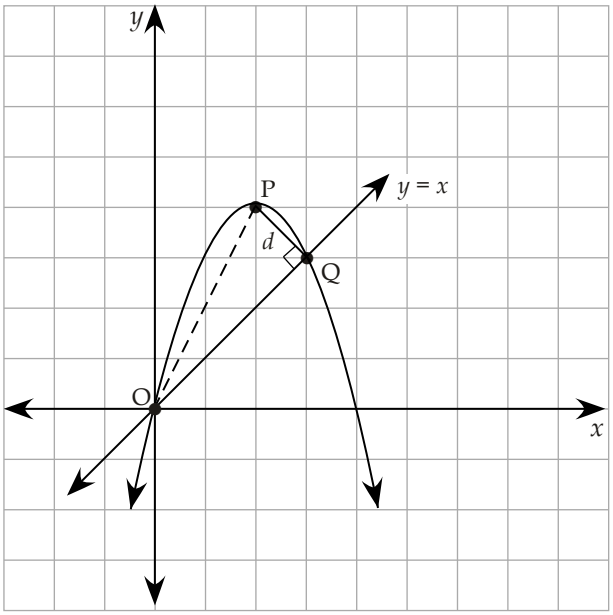
d) There are two solutions to this system.

e) There is one solution to this system.

f) The solutions are (−1, 1) and (2, 4).

g) The solution is (1, 1).

6. The line $y = x$ intersects the parabola $y = 4x - x^2$ at the origin and the point Q. Let the point, P, be the vertex of the parabola.



a) Find the coordinates of Q.

Answer:

$y = x$ (1)

$y = 4x - x^2$ (2)

$x = 4x - x^2$ Substitute y with x from Equation (1)

$x^2 - 3x = 0$

$x(x - 3) = 0$

$x = 0$ or $x = 3$

Find the corresponding values of y.

If $x = 0$, $y = 0$.

If $x = 3$, $y = 3$.

The coordinates of Q are (3, 3).

b) Find the coordinates of P.

Answer:

You could use $x = \frac{-b}{2a}$ to find the x-coordinate of P, the vertex of the parabola.

$$y = -x^2 + 4x$$

$$a = -1,\ b = 4$$

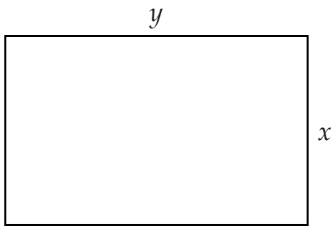
$$x = -\frac{4}{2(-1)} = \frac{4}{2} = 2$$

Substitute $x = 2$ into $y = -x^2 + 4x$ to find the y-coordinate of the vertex.

$$\begin{aligned} y &= -x^2 + 4x \\ &= -(2)^2 + 4(2) \\ &= -4 + 8 \\ &= 4 \end{aligned}$$

The coordinates of vertex P are (2, 4).

7. The perimeter of a rectangle is 26 cm and the area is 12 cm^2. Find the dimensions of the rectangle.



Answer:

Let x = width of the rectangle.

y = length of the rectangle.

Perimeter: $2x + 2y = 26$ (1)

Area: $xy = 12$ (2)

Solve equation (1) for y:

$$\begin{aligned} 2x + 2y &= 26 \\ 2y &= 26 - 2x \\ y &= 13 - x \end{aligned}$$

Substitute $y = 13 - x$ into equation (2):

$$xy = 12$$
$$x(13 - x) = 12$$
$$13x - x^2 = 12$$
$$0 = x^2 - 13x + 12$$
$$0 = (x - 12)(x - 1)$$

If $x = 12$, then $y = 13 - x = 12 - 12 = 1$.

If $x = 1$, then $y = 13 - x = 13 - 1 = 12$.

In both cases, the dimensions of the rectangle are 12 cm × 1 cm.

8. Below is Schyler's work for solving a system of equations.

$$-2y = -x^2 - x$$
$$2y = 10x - 20$$

Schyler's Solution:

$$\begin{array}{rl} & -2y = -x^2 - x \\ + & \underline{2y = \quad 10x - 20} \\ & 0 = -x^2 + 9x - 20 \end{array}$$
$$0 = x^2 - 9x + 20$$
$$0 = (x - 4)(x - 5)$$

$x = 4$ or $x = 5$

When $x = 4$:

$$2y = 10x - 20$$
$$2y = 10(4) - 20$$
$$2y = 20$$
$$y = 10$$

Thus, one solution is (4, 10).

When $x = 5$:

$$2y = 10x - 20$$
$$2y = 10(5) - 20$$
$$2y = 30$$
$$y = 15$$

Thus, another solution is (5, 15).

Is his method correct? Explain why or why not.

Answer:

Schyler used the elimination method to solve this system of equations. This is a correct way to solve a system of equations. Therefore, his work should be correct. To ensure his answers are correct, verify the solutions.

(4, 10)

$-2y = -x^2 - x$

$\begin{aligned} \text{LHS} &= -2y \\ &= -2(10) \\ &= -20 \end{aligned}$

$\begin{aligned} \text{RHS} &= -x^2 - x \\ &= -(4)^2 - 4 \\ &= -16 - 4 \\ &= -20 \end{aligned}$

LHS = RHS

$2y = 10x - 20$

$\begin{aligned} \text{LHS} &= 2y \\ &= 2(10) \\ &= 20 \end{aligned}$

$\begin{aligned} \text{RHS} &= 10x - 20 \\ &= 10(4) - 20 \\ &= 20 \end{aligned}$

LHS = RHS

(5, 15)

$-2y = -x^2 - x$

$\begin{aligned} \text{LHS} &= -2y \\ &= -2(15) \\ &= -30 \end{aligned}$

$\begin{aligned} \text{RHS} &= -x^2 - x \\ &= -(5)^2 - 5 \\ &= -25 - 5 \\ &= -30 \end{aligned}$

LHS = RHS

$2y = 10x - 20$

$\begin{aligned} \text{LHS} &= 2y \\ &= 2(15) \\ &= 30 \end{aligned}$

$\begin{aligned} \text{RHS} &= 10x - 20 \\ &= 10(5) - 20 \\ &= 30 \end{aligned}$

LHS = RHS

Thus, (4, 10) and (5, 15) are both solutions to this system of equations, and Schyler is using a correct method of elimination to solve this system.

Learning Activity 6.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Between which two whole numbers is $\sqrt{150}$?
2. Simplify: $3x^2 + 4x - 6 - 5x + 4 + 6x^2 + 8 - 9x^2$
3. Pizza Express is offering two medium pizzas for $15.99 while Speedy Pizza is offering three medium pizzas for $22.49. Which is the better buy?
4. Lisa worked 40 hours in one week and made $520. How much does Lisa make an hour?
5. Calculate: $\frac{4}{7} - \frac{2}{3}$
6. Express as an entire radical: $7y^{\frac{8}{3}}$
7. What is the slope of a vertical line?
8. How many centimetres are in 2 kilometres?

Answers:

1. 12 and 13 ($12^2 = 144$ and $13^2 = 169$)
2. $-x + 6$
3. Speedy Pizza has the better deal. $\left(\frac{\$15.99}{2} \cong \$8 \text{ and } \frac{\$22.49}{3} = \$7.50\right)$
4. Lisa makes $13 an hour. $\left(\frac{520}{40} = \frac{52}{4} = 13\right)$
5. $-\frac{2}{21}$ $\left(\frac{12}{21} - \frac{14}{21}\right)$
6. $\sqrt[3]{343y^8}$ $\left(7y^{\frac{8}{3}} = 343^{\frac{1}{3}}y^{\frac{8}{3}} = \left(343y^8\right)^{\frac{1}{3}}\right)$
7. The slope of a vertical line is undefined.
8. 200 000 centimetres (There are 1000 metres in 1 kilometre and 100 centimetres in 1 metre. Therefore, there are 100 × 1000 = 100 000 centimetres in 1 kilometre and 200 000 centimetres in 2 kilometres.

Part B: Solving Quadratic-Quadratic Systems of Equations

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Verify that $\left(\frac{10}{3}, \frac{17}{9}\right)$,and (1, −2) are solutions to the following system of equations.

$y = 2x^2 - 7x + 3$

$y = -x^2 + 6x - 7$

Answer:

Verifying: $\left(\frac{10}{3}, \frac{17}{9}\right)$:

$\text{LHS} = y = \frac{17}{9}$

$\text{RHS} = 2x^2 - 7x + 3$

$= 2\left(\frac{10}{3}\right)^2 - 7\left(\frac{10}{3}\right) + 3$

$= 2\left(\frac{100}{9}\right) - \frac{70}{3} + 3$

$= \frac{200}{9} - \frac{210}{9} + \frac{27}{9}$

$= \frac{17}{9}$

$\text{LHS} = \text{RHS}$

$\text{LHS} = y = \frac{17}{9}$

$\text{RHS} = -x^2 + 6x - 7$

$= -\left(\frac{10}{3}\right)^2 + 6\left(\frac{10}{3}\right) - 7$

$= -\frac{100}{9} + \frac{180}{9} - \frac{63}{9}$

$= \frac{17}{9}$

$\text{LHS} = \text{RHS}$

Verifying (1, −2):

$\text{LHS} = y = -2$

$\text{RHS} = 2x^2 - 7x + 3$

$= 2(1)^2 - 7(1) + 3$

$= 2 - 7 + 3$

$= -2$

$\text{LHS} = \text{RHS}$

$\text{LHS} = y = -2$

$\text{RHS} = -x^2 + 6x - 7$

$= -(1)^2 + 6(1) - 7$

$= -1 + 6 - 7$

$= -2$

$\text{LHS} = \text{RHS}$

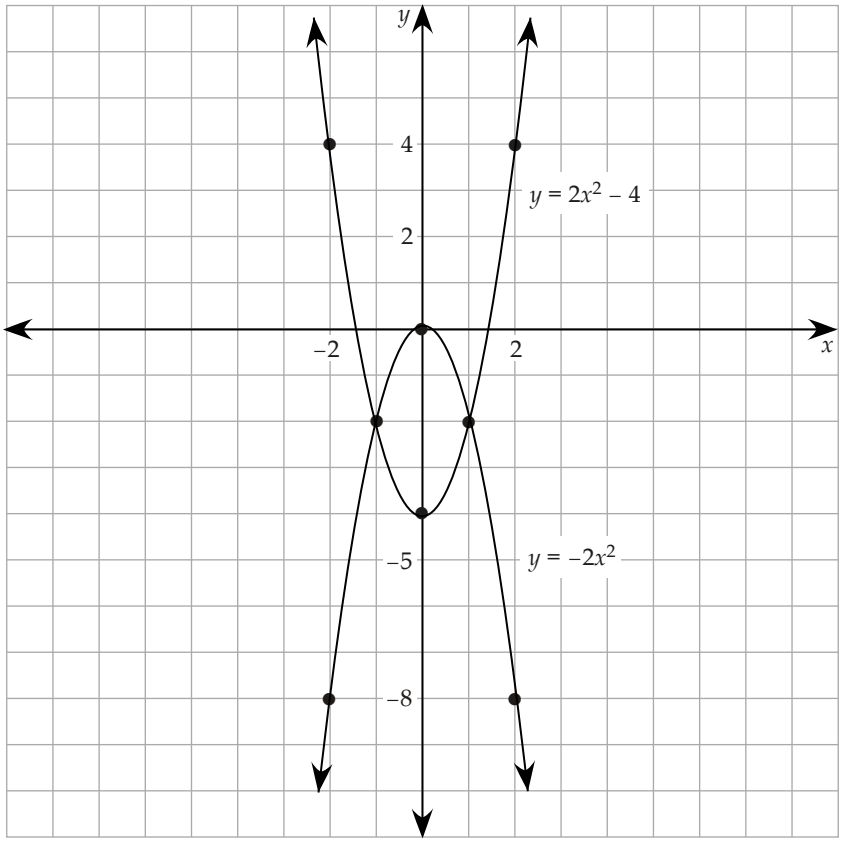
2. Solve the following systems of equations, both graphically and algebraically.

   a) $y = -2x^2$

   $y = 2x^2 - 4$

   Answer:

   Graphically:



   As you can see from the graph, the solutions to the systems of equations are $(-1, -2)$ and $(1, -2)$.

   Algebraically:

   Substitute for y:

   $-2x^2 = 2x^2 - 4$

   $0 = 4x^2 - 4$

   $0 = 4(x^2 - 1)$

   $0 = (x - 1)(x + 1)$

   Thus, $x = 1$ or -1.

When $x = 1$

$y = -2x^2$

$y = -2(1)^2$

$y = -2$

Thus, one solution is $(1, -2)$

When $x = -1$

$y = -2x^2$

$y = -2(-1)^2$

$y = -2(1)$

$y = -2$

Thus, one solution is $(-1, -2)$.

The two solutions are $(-1, -2)$ and $(1, -2)$.

As expected, these are the same solutions as determined graphically.

b) $x^2 + y - 3 = 0$

$x^2 - y + 1 = 0$

Answer:

Graphically:

First rearrange each equation to general form to find the vertex.

$x^2 + y - 3 = 0$ Equation (1)

$y = -x^2 + 3$

Vertex (0, 3), opens down, and normal shape.

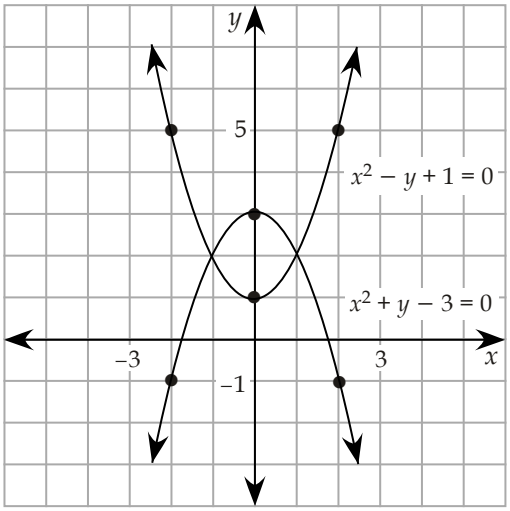
$x^2 - y + 1 = 0$ Equation (2)

$-y = -x^2 - 1$

$y = x^2 + 1$

Vertex (0, 1), opens up, and normal shape.

Now graph and solve.



The solutions to this system of equations are (−1, 2) and (1, 2).

Algebraically:

Use the rearranged equations:

$y = -x^2 + 3$ (1)

$y = x^2 + 1$ (2)

Now substitute for y.

$-x^2 + 3 = x^2 + 1$

$0 = 2x^2 - 2$

$0 = 2(x^2 - 1)$

$0 = (x - 1)(x + 1)$

Thus, $x = 1$ and -1.

When $x = 1$:

$y = x^2 + 1$

$y = (1)^2 + 1$

$y = 2$

Thus, one solution is (1, 2).

When $x = -1$:

$y = x^2 + 1$

$y = (-1)^2 + 1$

$y = 2$

Thus, another solution is (−1, 2).

The two solutions are (−1, 2) and (1, 2) and are the same as those found in the graphing solution.

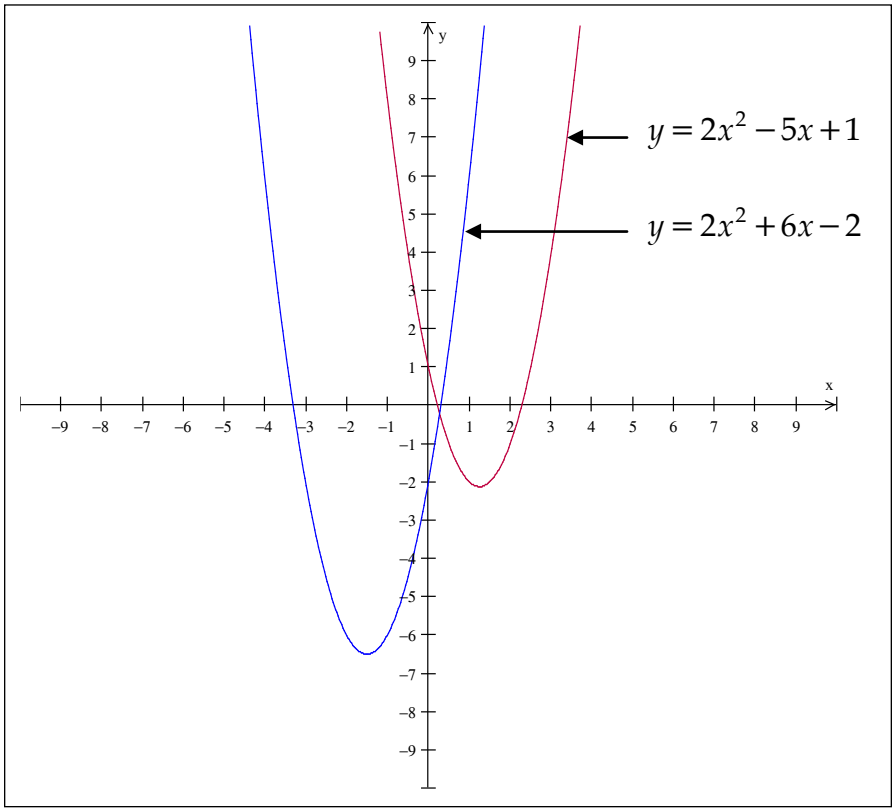
3. Solve the following system of equations graphically, using technology. Round to three decimal places when necessary. **Note:** If you do not have access to technology, you can still do this question by hand and estimate the solution. **You will not have questions on your exam requiring the use of technology.**

$y = 2x^2 - 5x + 1$

$y = 2x^2 + 6x - 2$

Answer:

Graph both of the above functions using a graphing calculator or a graphing applet such as Winplot. The graph created using Winplot is shown below.



Calculate the intersection by clicking "Two" in the menu bar and then "Intersections."

These functions intersect at the point (0.273, −0.215). Since they are the same shape, they will never cross at a second point.

4. Solve the following systems of equations by using the substitution method.

a) $y = x^2 + 4$

$y = -x^2 + 12$

Answer:

$$x^2 + 4 = -x^2 + 12$$
$$2x^2 - 8 = 0$$
$$2(x^2 - 4) = 0$$
$$2(x - 2)(x + 2) = 0$$

Thus, $x = 2$ or -2.

When $x = 2$:

$y = x^2 + 4$

$y = (2)^2 + 4$

$y = 8$

Thus, one solution is (2, 8).

When $x = -2$:

$y = x^2 + 4$

$y = (-2)^2 + 4$

$y = 8$

Thus, one solution is (−2, 8).

The two solutions are (2, 8) and (−2, 8).

b) $y = 2(x + 4)^2$

$y = 2x^2 + 16x + 32$

Answer:

Substitute for y.

$$2(x + 4)^2 = 2x^2 + 16x + 32$$
$$2(x^2 + 8x + 16) = 2x^2 + 16x + 32$$
$$2x^2 + 16x + 32 = 2x^2 + 16x + 32$$

These functions are identical. Therefore, there are an infinite number of solutions to this system including all of the points represented by the function $y = 2(x + 4)^2$. An example is (0, 32).

Thus, the solution is $y = 2(x + 4)^2$ or $\{(x, y) \mid y = 2(x + 4)^2\}$.

5. Solve the following systems of equations by using the elimination method.

a) $y = x^2 - 2x + 2$

$3x^2 - 4y = 7$

Answer:

Rearrange and fix the coefficients of y so that they are the same value with opposite signs. Then add the like terms of the two equations.

(1) $y = x^2 - 2x + 2$

(2) $-4y = -3x^2 + 7$

$4 \times$ (1) $4y = 4x^2 - 8x + 8$

$+$ (2) $-4y = -3x^2 + 7$

$0 = x^2 - 8x + 15$

$0 = (x - 5)(x - 3)$

Thus, $x = 5$ or 3.

When $x = 5$:

$y = x^2 - 2x + 2$

$y = 5^2 - 2(5) + 2$

$y = 17$

When $x = 3$:

$y = x^2 - 2x + 2$

$y = 3^2 - 2(3) + 2$

$y = 5$

Thus, the two solutions to this system of equations are (5, 17) and (3, 5).

b) $5x^2 + 3y = -3 - x$

$2x^2 - x = -4 - 2y$

Answer:

Rearrange and then fix the coefficients of y so that they are the same value with opposite signs. Then add the like terms of the two equations and solve.

(1) $5x^2 + 3y = -3 - x$

$3y = -5x^2 - x - 3$

(2) $2x^2 - x = -4 - 2y$

$2y = -2x^2 + x - 4$

$2 \times (1)$ $\quad 6y = -10x^2 - 2x - 6$

$+ \ -3 \times (2)$ $\quad -6y = 6x^2 - 3x + 12$

$0 = -4x^2 - 5x + 6$

$0 = 4x^2 + 5x - 6$

$0 = (4x - 3)(x + 2)$

$4x - 3 = 0 \qquad x + 2 = 0$

$x = \frac{3}{4} \qquad x = -2$

Thus, $x = \frac{3}{4}$ or -2.

When $x = \frac{3}{4}$:

$2x^2 - x = -4 - 2y$

$2\left(\frac{3}{4}\right)^2 - \frac{3}{4} + 4 = -2y$

$\frac{18}{16} - \frac{12}{16} + \frac{64}{16} = -2y$

$\frac{70}{16} = -2y$

$-\frac{35}{16} = y$

When $x = -2$:

$$\begin{aligned} 2x^2 - x &= -4 - 2y \\ 2(-2)^2 - (-2) + 4 &= -2y \\ 8 + 2 + 4 &= -2y \\ 14 &= -2y \\ -7 &= y \end{aligned}$$

Thus, the two solutions to this system of equations are, $\left(\frac{3}{4}, -\frac{35}{16}\right)$ and $(-2, -7)$.

6. Solve the following system of equations using any method you prefer. Explain why you chose that method.

$8x^2 + 100 = -5y$

$6x^2 - x - 5 = 3y$

Answer:

Fix coefficients of y so that they are opposites.

$$\begin{array}{rrrrrr} & 3 \times (1) & 24x^2 & + 300 & = & -15y \\ + & 5 \times (2) & 30x^2 - 5x - 25 & = & & 15y \\ \hline & & 54x^2 - 5x + 275 & = & & 0 \end{array}$$

Since the coefficients are so large, use the discriminant ($b^2 - 4ac$) to determine the number and types of roots.

$a = 54$, $b = -5$, $c = 275$

$$\begin{aligned} b^2 - 4ac &= (-5)^2 - 4(54)(275) \\ &= 25 - 59400 \\ &= -59375 \end{aligned}$$

As the discriminant is negative, there is no solution to this system of equations. Therefore, the parabolas never cross each other.

The elimination method was chosen because it is sometimes difficult to work with fractions, as would be the case if the substitution method were chosen. The graphing method would not be easy with these equations unless you have graphing technology.

7. Let x represent time, measured in seconds, and let y represent height, measured in feet. A football is thrown and its path is represented by the equation $y = -x^2 + 4x + 2$. A receiver jumps to receive the ball and his path is represented by the equation $y = -(x - 5)^2 + 6$. Does the receiver have any chance of catching the football? Explain.

Answer:

Solving the system of equations by substitution:

$y = -x^2 + 4x + 2$ (football)

$y = -(x - 5)^2 + 6$ (receiver)

$$-x^2 + 4x + 2 = -(x - 5)^2 + 6$$

$$-x^2 + 4x + 2 = -\left(x^2 - 10x + 25\right) + 6$$

$$-x^2 + 4x + 2 = -x^2 + 10x - 25 + 6$$

$$-x^2 + 4x + 2 = -x^2 + 10x - 19$$

$$4x + 2 = 10x - 19$$

$$-6x = -21$$

$$x = \frac{-21}{-6} = \frac{7}{2}$$

Thus, this receiver had one chance to catch the ball, after $\frac{7}{2}$ seconds.

When $x = \frac{7}{2}$:

$$y = -x^2 + 4x + 2$$

$$y = -\frac{49}{4} + \frac{28}{2} + 2$$

$$y = -\frac{49}{4} + \frac{56}{4} + \frac{8}{4}$$

$$y = \frac{15}{4}$$

The two equations intersect at the point $\left(\frac{7}{2}, \frac{15}{4}\right)$.

After $\frac{7}{2}$ seconds, the receiver and the ball were both $\frac{15}{4}$ feet, or 3.75 feet, off the ground and in the same spot. Therefore, this receiver had a chance to catch the ball.

Learning Activity 6.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Jessica sells $545 worth of fundraising tickets. If she makes 5% commission on her total sales, how much did Jessica make?
2. Simplify: $(3x + 1)^2$
3. Simplify: $\sqrt{54}$
4. Sam mowed half of the yard on Friday and half of the remaining grass on Saturday. How much grass still needs to be cut?
5. What is the slope of the line defined by the equation $2y = 4x + 6$?
6. Factor completely. $121x^2 - 196y^2$
7. The diameter of a circle is 34 cm. What is the radius of the circle?
8. Is 87 a prime or composite number?

Answers:

1. $27.25 (10% of $545 = $54.50; 5% of $545 = $27.25)
2. $9x^2 + 6x + 1$
3. $3\sqrt{6}$ $\left(\sqrt{54} = \sqrt{9 \times 6}\right)$
4. $\frac{1}{4}$ $\left(\frac{1}{2} \text{ of } \frac{1}{2}\right)$ (Sam has mowed $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ and she still needs to mow $\frac{1}{4}$.)
5. 2 $(y = 2x + 3)$
6. $(11x - 14y)(11x + 14y)$ [it is a difference of squares]
7. $r = 17$ cm (The radius is half the diameter.)
8. 87 is a composite number; it can be divided by 3 and 29.

Part B: Solving Linear Inequalities in Two Variables

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Which of the ordered pairs are solutions to the given inequality?

   a) $4x - 3y > 2$

   $(-4, 8), (1, 2), (3, -6)$

   Answer:

   $4x - 3y > 2$

   Check: $(-4, 8)$

   $\text{LHS} = 4x - 3y$

   $= 4(-4) - 3(8)$

   $= -16 - 24$

   $= -40$

   $\text{RHS} = 2$

   $\text{LHS} \not> \text{RHS}$

   Check: $(1, 2)$

   $\text{LHS} = 4x - 3y$

   $= 4(1) - 3(2)$

   $= 4 - 6$

   $= -2$

   $\text{RHS} = 2$

   $\text{LHS} \not> \text{RHS}$

   Check: $(3, -6)$

   $\text{LHS} = 4x - 3y$

   $= 4(3) - 3(-6)$

   $= 12 + 18$

   $= 30$

   $\text{RHS} = 2$

   $\text{LHS} > \text{RHS}$

   Thus, $(-4, 8)$ and $(1, 2)$ are not solutions to the inequality, but $(3, -6)$ is a solution.

b) $-8x + 6y \leq 5$

$(-2, -7), (4, 9), (-1, 8)$

Answer:

$-8x + 6y \leq 5$

Check: $(-2, -7)$

LHS $= -8x + 6y$

$= -8(-2) + 6(-7)$

$= 16 - 42$

$= -26$

RHS $= 5$

LHS $\leq$ RHS

Check: $(4, 9)$

LHS $= -8x + 6y$

$= -8(4) + 6(9)$

$= -32 + 54$

$= 22$

RHS $= 5$

LHS $\not\leq$ RHS

Check: $(-1, 8)$

LHS $= -8x + 6y$

$= -8(-1) + 6(8)$

$= 8 + 48$

$= 56$

RHS $= 5$

LHS $\not\leq$ RHS

Thus, $(4, 9)$ and $(-1, 8)$ are not solutions to the inequality, but $(-2, -7)$ is a solution.

2. Sketch the graphs of the following linear inequalities.

   a) $y < \frac{1}{2}x + 2$

   b) $y \geq 0$

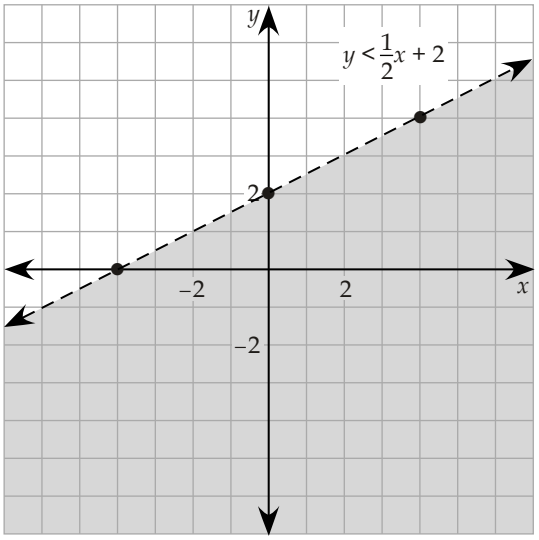
   c) $3x + 2y \leq 6$

   d) $-2y + 8 > 5x$

   Answers:

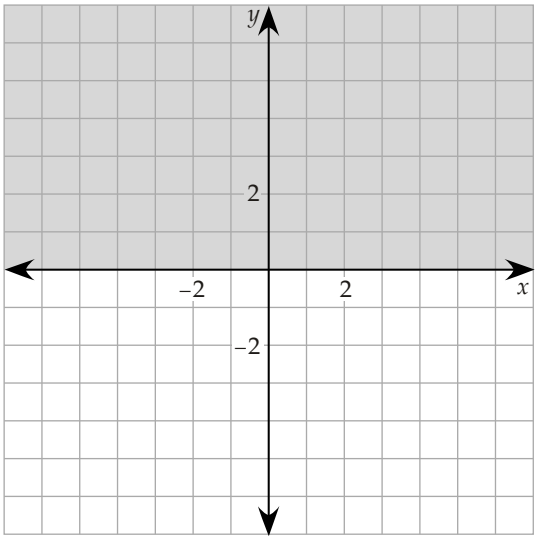
   a) $y < \frac{1}{2}x + 2$

   Graphing the dotted line with slope $\frac{1}{2}$ and y-intercept 2, and shading "underneath" the line where the y-values are smaller than those on the line.



b) $y \geq 0$

This is simply the horizontal line $y = 0$ and all y-values greater than 0.



c) $3x + 2y \leq 6$

Rearrange the inequality into $y = mx + b$ form first, and then graph a solid line and the shading.

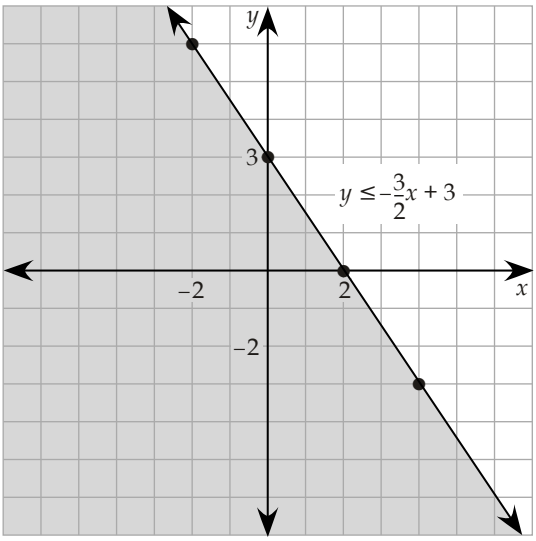
$$3x + 2y \leq 6$$

$$2y \leq -3x + 6$$

$$y \leq -\frac{3}{2}x + 3$$

Graph the related equation $y = -\frac{3}{2}x + 3$.

The line passes through a y-intercept of 3, having a slope of $-\frac{3}{2}$. Draw a solid line since the $\leq$ includes "equal to." The shading is "below" the line, including all y-values that are less than the ones on the line.



d) $-2y + 8 > 5x$

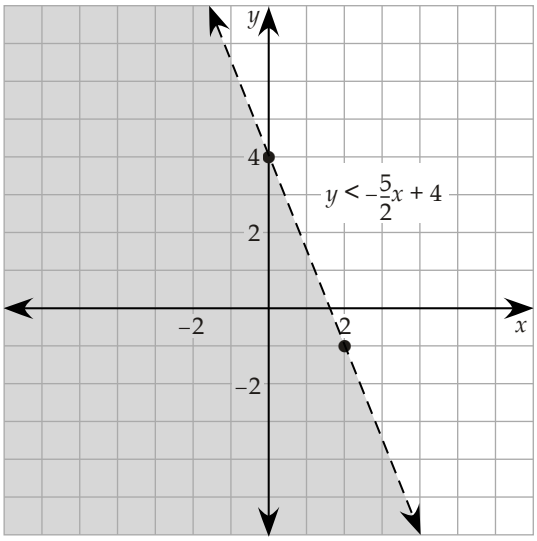
Rearrange to slope-intercept form, graph the related equation as a dotted line.

$$-2y + 8 > 5x$$

$$-2y > 5x - 8$$

$$y < -\frac{5}{2}x + 4 \qquad \text{Dividing by a negative number.}$$

Shade where y-values are less than those on the line.



3. Match each linear inequality to its graph.

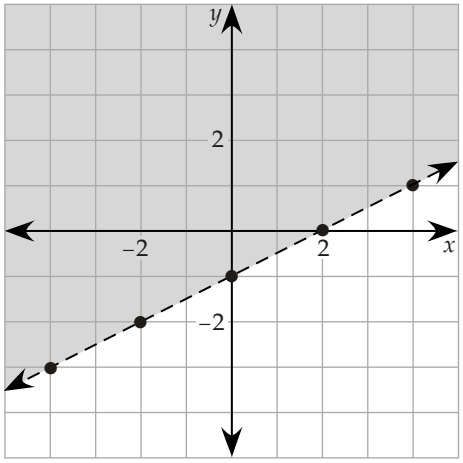
a) $y \leq 4$

b) $y \leq -x + 1$

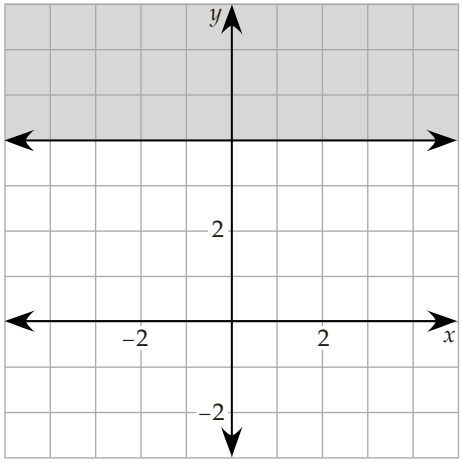
c) $2y > x - 2$

d) $y \geq 4$

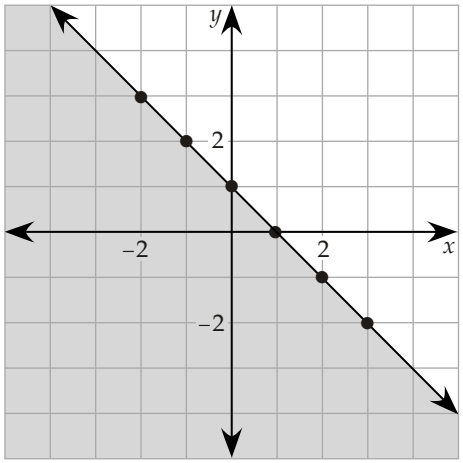
i)



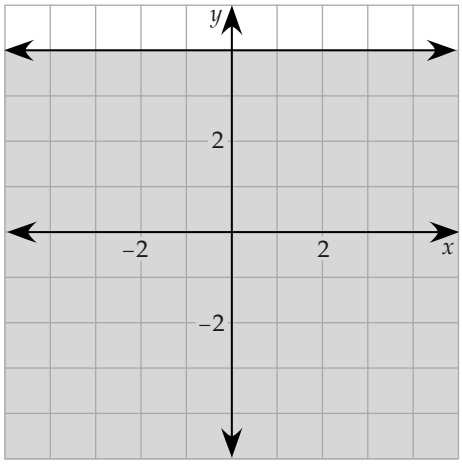
ii)



iii)



iv)



Answer:

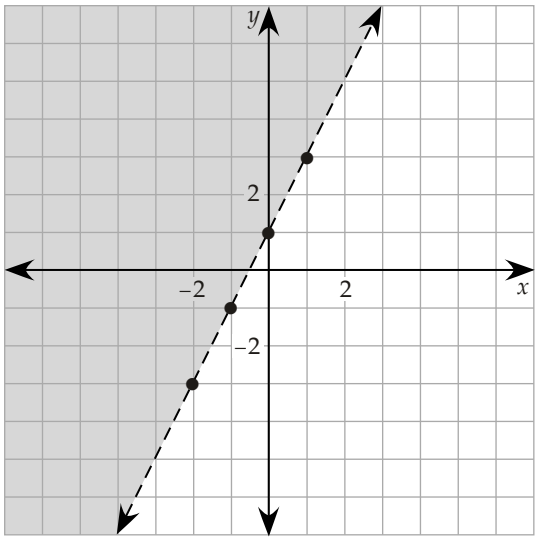
(a) and (iv)

(b) and (iii)

(c) and (i)

(d) and (ii)

4. Write a linear inequality to represent each graph.

a) 

Answer:

To find the equation of the inequality, you first need to find the equation of the boundary line.

To do this, you can find the slope of the line and the y-intercept. Then, you can put those values into the slope-intercept form equation of a line.

slope = 2

y-intercept = 1

slope-intercept equation of a line: $y = mx + b$

Thus, the equation of this line is $y = 2x + 1$.

Now, you need to determine whether you have a *greater than* or *less than* symbol.

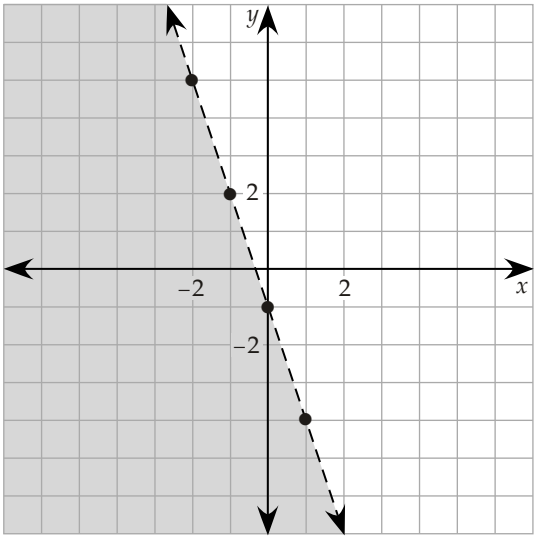
As you can see, the solution region is above the boundary line. Therefore, the symbol you need is *greater than*, or >.

You now need to determine if the points on the boundary line are included in the solution or not.

As the boundary line is broken or dashed, the points on the boundary line are not included in the solution to the inequality.

Thus, the inequality becomes $y > 2x + 1$.

b)



Answer:

To find the equation of the inequality, you first need to find the equation of the boundary line.

To do this, you can find the slope of the line, and the y-intercept. Then, you can put those values into the slope-intercept form equation of a line.

slope = -3

y-intercept = -1

slope-intercept equation of a line: $y = mx + b$

Thus, the equation of this line is $y = -3x - 1$.

Now, you need to determine whether you have a *greater than* or *less than* symbol.

As you can see, the solution region is below the boundary line. Therefore, the symbol you need is *less than*, or <.

You now need to determine if the points on the boundary line are included in the solution or not.

As the boundary line is broken or dashed, the points on the boundary line are not included in the solution to the inequality.

Thus, the inequality becomes $y < -3x - 1$.

5. Hunter is a high school student and he wants to save enough money to pay for his first year at university. He is holding down two part-time jobs. If he wants to save enough money, he needs to make at least $200 a week. One of his part-time jobs is working at a restaurant and he is paid $10 an hour. He makes $12 an hour at his second part-time job as a maintenance worker at his dad's business.

   a) Write an inequality to represent the hours Hunter needs to work.

      Answer:

      Let R = the hours Hunter works at the restaurant

      Let M = the hours Hunter works as a maintenance person

      $10R + 12M \geq 200$

   b) Graph the inequality.

      Answer:

      Rearrange the inequality. Either variable could be placed on the x-axis. This solution shows M on the x-axis. Thus, rearrange to solve for R.

      $10R + 12M \geq 200$

      $10R \geq 200 - 12M$

      $R \geq 20 - \frac{12}{10}M$

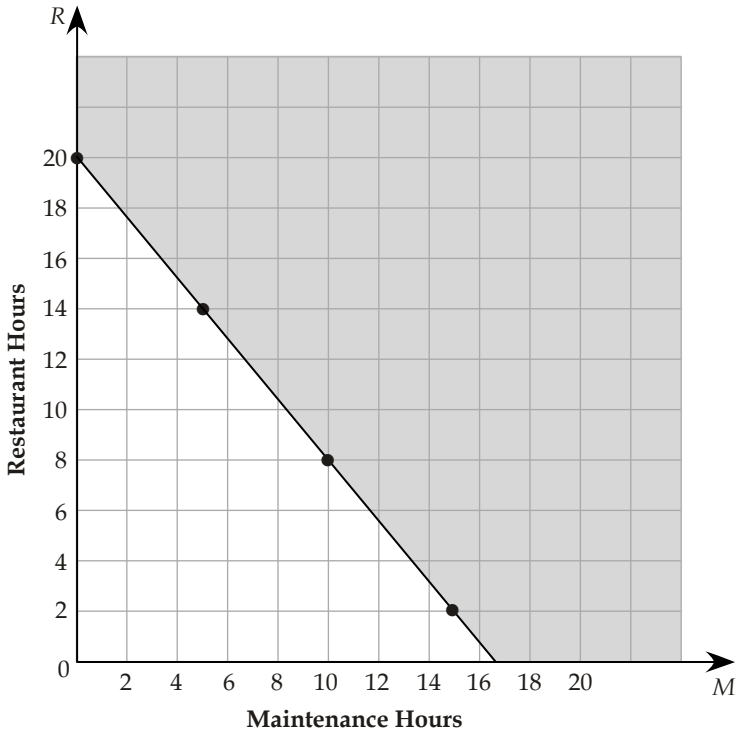
      $R \geq -\frac{6}{5}M + 20$

      slope = $-\frac{6}{5}$

      y-intercept = 20

      solid line

      Shade above the line.



c) List two different ordered pairs that are solutions to this inequality.

Answer:

The ordered pairs for this graph are (M, R). Thus, two solutions could be (10, 12) and (0, 20). Any ordered pairs on the boundary or in the shaded region will work.

d) What do these ordered pairs symbolize?

Answer:

These ordered pairs symbolize how many hours Hunter needs to work at each job in order to make at least $200 a week. Therefore, Hunter can work 10 hours doing maintenance and 12 hours at the restaurant and make at least $200 a week. Also, Hunter can work 20 hours at the restaurant and 0 hours in maintenance in order to make at least $200 a week.

Learning Activity 6.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Simplify: $\frac{x^6y^3}{x^8y^2}$
2. Pizza Express delivers pizza with the following charges: a $10.00 delivery and tip charge and $7 per pizza. Your sports team wants to order pizza. Everyone pools their money and they come up with $73.15. What is the maximum number of pizzas they can order?
3. Patricia ate one-third of the red velvet cake her mom bought her for her birthday that day. The next day, she ate one-half of the remaining cake. How much cake is left?
4. Simplify: $2^4 \times 2^2$
5. Reduce the following fraction: $\frac{105}{21}$
6. Simplify: $\left(-\frac{4^7\left(3^2\right)}{16^2}\right)^0$
7. Find the midpoint between the following two points: (26, −13) and (15, −21)
8. Solve the inequality $3 - 7x \geq 0$.

Answers:

1. $\frac{y}{x^2}$
2. They can order a maximum of 9 pizzas. ($73.15 − $10.00 = $63.15; $\frac{63.15}{7} = 9$)
3. $\frac{1}{3}$ $\left(1 - \frac{1}{3} = \frac{2}{3} \text{ and } \frac{1}{2} \text{ of } \frac{2}{3} = \frac{1}{3}\right)$ $\left(\frac{2}{3} - \frac{1}{3} = \frac{1}{3}\right)$
4. $2^6 = 64$

5. $5\left(\frac{35}{7}=\frac{5}{1}\right)$

6. 1 (any number to the power of zero is 1)

7. $\mathrm{M}=\left(\frac{41}{2},-\frac{34}{2}\right)=\left(\frac{41}{2},-17\right)$

8. $x \leq \frac{3}{7}$

(Method 1:

$3 \geq 7x$

$\frac{3}{7} \geq x$ or $x \leq \frac{3}{7}$

Method 2:

$-7x \geq -3$

$x \leq \frac{-3}{-7}$

$x \leq \frac{3}{7}\Big)$

Part B: Solving Quadratic Inequalities in One Variable

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Determine whether the given value of x is a solution to the given inequality.

   a) $x = 2, x^2 + 3x - 6 \leq 0$

   Answer:

   $x^2 + 3x - 6 \leq 0$

   $x = 2$

   $\mathrm{LHS} = x^2 + 3x - 6$

   $= 2^2 + 3(2) - 6$

   $= 4$

   RHS = 0

   LHS $\not\leq$ RHS

   Thus, 2 is not a solution to the given inequality.

b) $x = -4$, $2x^2 + 4x - 9 > 0$

Answer:

$2x^2 + 4x - 9 > 0$

$x = -4$

LHS $= 2x^2 + 4x - 9$

$= 2(-4)^2 + 4(-4) - 9$

$= 32 - 16 - 9$

$= 7$

RHS $= 0$

LHS > RHS

Thus, $x = -4$ is a solution.

c) $x = 0$, $4x^2 - 3x + 4 > 0$

Answer:

$4x^2 - 3x + 4 > 0$

$x = 0$

LHS $= 4x^2 - 3x + 4$

$= 4(0)^2 - 3(0) + 4$

$= 4$

RHS $= 0$

LHS > RHS

Thus, $x = 0$ is a solution.

2. Draw the graph of the quadratic function $y = x^2 + 2x + 1$. Then, determine the solution for the following:

   a) $x^2 + 2x + 1 \geq 0$

   b) $x^2 + 2x + 1 > 0$

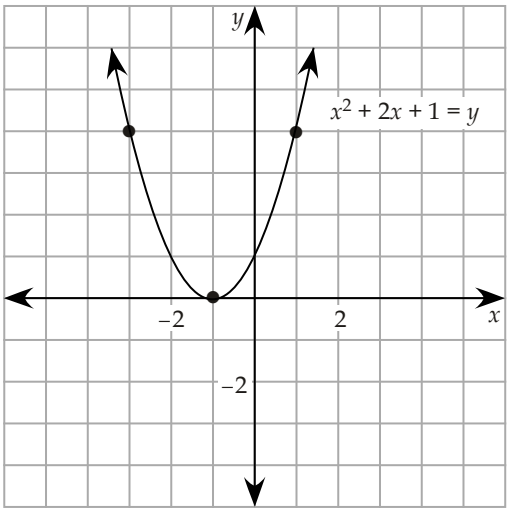
   c) $x^2 + 2x + 1 \leq 0$

   d) $x^2 + 2x + 1 < 0$

   Answer:

   $y = x^2 + 2x + 1$

   $y = (x + 1)^2$



   a) $x^2 + 2x + 1 \geq 0$

      Answer:

      The graph lies on or above the x-axis at every single point. Therefore, the solution set is $\{x \mid x \in \Re\}$.

   b) $x^2 + 2x + 1 > 0$

      Answer:

      The graph lies above the x-axis at every point except when $x = -1$. Therefore, the solution set is $\{x \mid x \in \Re, x \neq -1\}$.

   c) $x^2 + 2x + 1 \leq 0$

      Answer:

      The graph lies on or below the x-axis only at the point $x = -1$ where it touches the x-axis. Therefore, the solution set is $\{x \mid x = -1\}$.

d) $x^2 + 2x + 1 < 0$

Answer:

The graph never lies below the x-axis. Thus, there is no solution to this inequality.

3. Use the graphing method to solve the following quadratic inequalities.

a) $x^2 + 4x + 4 \geq 0$

Answer:

First write the corresponding quadratic function in vertex form.

$x^2 + 4x + 4 \geq 0$

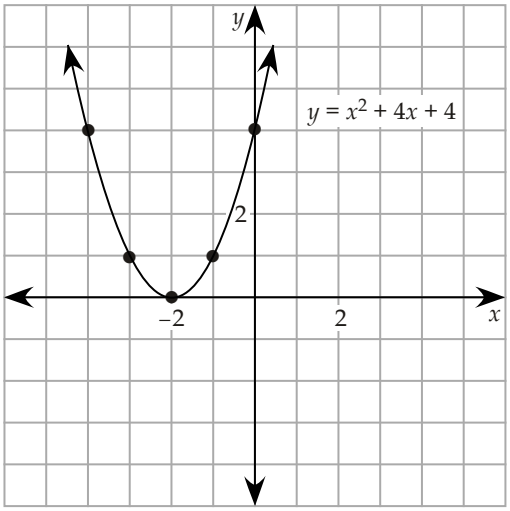
$y = x^2 + 4x + 4$

$y = (x^2 + 4x + 4) + 4 - 4$

$y = (x + 2)^2$

Vertex $(-2, 0)$, opens up, and normal size.

The graph of the function is shown below.



As you can see, the graph lies on or above the x-axis for all values of x. Thus, the solution set is $\{x \mid x \in \Re\}$.

b) $x^2 - 2x > 3$

Answer:

First, find the corresponding function in vertex form.

$x^2 - 2x > 3$

$x^2 - 2x - 3 > 0$

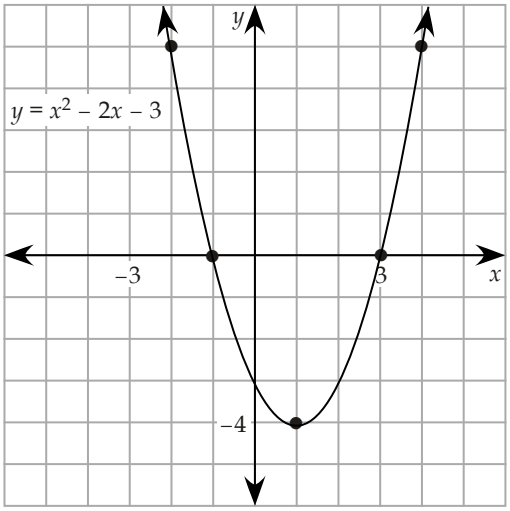
$y = x^2 - 2x - 3$

$y = (x^2 - 2x + 1) - 3 - 1$

$y = (x - 1)^2 - 4$

Vertex $(1, -4)$, opens up, normal shape.

The graph of the function is shown below.



As you can see, the graph lies above the x-axis when x is greater than 3 or when x is less than -1.

Thus, the solution set is $\{x \mid x > 3 \text{ or } x < -1, x \in \Re\}$.

4. Use the roots and test point method to make a sign diagram and solve the following quadratic inequalities.

   a) $x^2 - 2x + 1 \geq 0$

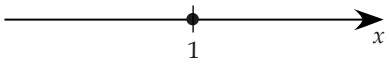
   Answer:

   Factoring the corresponding quadratic equation:

   $x^2 - 2x + 1 = 0$

   $(x - 1)(x - 1) = 0$

   The root of this quadratic equation is $x = 1$. Therefore, the two intervals are $x > 1$ and $x < 1$.



	$x^2 - 2x + 1 \geq 0$	
Interval	$x < 1$	$x > 1$
Test Value	$x = 0$	$x = 2$
Substituting the Test Point into the Inequality	$(0)^2 - 2(0) + 1$ $= 1$	$(2)^2 - 2(2) + 1$ $= 4 - 4 + 1$ $= 1$
Sign	positive	positive

Thus, as the inequality is asking for all points that are greater than or equal to zero, the solution set is all real numbers, or $\{x \mid x \in \Re\}$.

b) $-x^2 + x + 1 \geq 0$. Show your answer rounded to three decimal places.

Answer:

$-x^2 + x + 1 \geq 0$

$x^2 - x - 1 \leq 0$ Multiply by (-1)

This quadratic does not factor easily; thus, you need to use the quadratic formula to find the roots.

$a = 1, b = -1, c = -1$

First, find the discriminant.

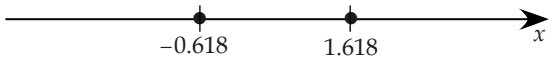
$$b^2 - 4ac = (-1)^2 - 4(1)(-1)$$
$$= 1 + 4$$
$$= 5$$

$\therefore b^2 - 4ac > 0$ and two unequal irrational roots.

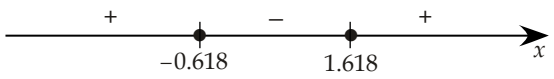
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{1 + \sqrt{5}}{2} = 1.618$$

$$x = \frac{1 - \sqrt{5}}{2} = -0.618$$



	$x^2 - x - 1 \leq 0$		
Interval	$x \leq -0.616$	$-0.616 \leq x \leq 1.618$	$x \geq 1.618$
Test Value	$x = -1$	$x = 0$	$x = 2$
Substituting the Test Point into the Inequality	$(-1)^2 - (-1) - 1$ $= 1 + 1 - 1$ $= 1$	$(0)^2 - 0 - 1$ $= -1$	$(2)^2 - 2 - 1$ $= 4 - 2 - 1$ $= 1$
Sign	positive	negative	positive



The solution consists of all points where the inequality is negative or zero:

$\{x \mid -0.618 \leq x \leq 1.618, x \in \Re\}$

5. Use the case analysis method to solve the following quadratic inequalities.

   a) $3x^2 + 16x \leq -5$

   Answer:

   Factor the quadratic inequality:

   $3x^2 + 16x \leq -5$

   $3x^2 + 16x + 5 \leq 0$

   $(3x + 1)(x + 5) \leq 0$

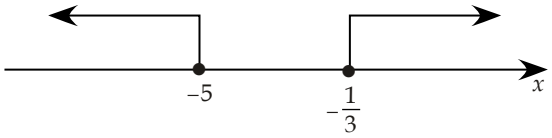
   In order for the quadratic inequality to be less than or equal to zero, one of these factors has to be negative or less than or equal to zero, and the other has to be positive or zero.

   Case 1:

   Consider when $(x + 5)$ is less than or equal to zero and $(3x + 1)$ is greater than or equal to zero:

   $x + 5 \leq 0$ and $3x + 1 \geq 0$

   $x \leq -5$ and $x \geq -\frac{1}{3}$



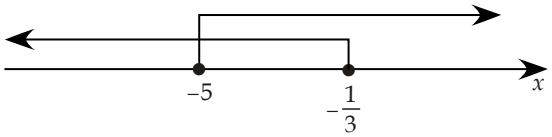
   As you can see from the diagram, there are no points that satisfy both inequalities. There is no solution for this case.

   Case 2:

   Consider when $(3x + 1)$ is less than or equal to zero and $x + 5 \geq 0$:

   $x + 5 \geq 0$ and $3x + 1 \leq 0$

   $x \geq -5$ and $x \leq -\frac{1}{3}$



   As you can see, the interval that satisfies both inequalities is $-5 \leq x \leq -\frac{1}{3}$.

   Thus, the solution set is $\left\{x \mid -5 \leq x \leq -\frac{1}{3}, x \in \Re\right\}$.

b) $x^2 + 3x - 180 \geq 0$

Answer:

Factor the inequality:

$x^2 + 3x - 180 \geq 0$

$(x - 12)(x + 15) \geq 0$

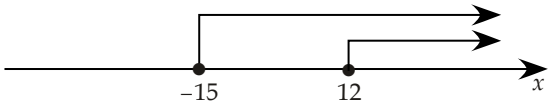
In order for the quadratic inequality to be greater than or equal to zero, both of the factors have to be positive or both factors have to be negative. There are two cases.

Case 1:

Consider both factors are positive:

$(x - 12) \geq 0$ and $(x + 15) \geq 0$

$x \geq 12$ and $x \geq -15$



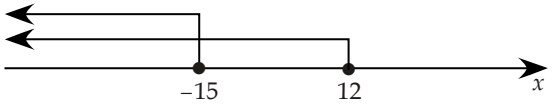
As you can see, both conditions are satisfied when $x \geq 12$.

Case 2:

Consider both factors are negative:

$(x - 12) \leq 0$ and $(x + 15) \leq 0$

$x \leq 12$ and $x \leq -15$



As you can see, both conditions are satisfied when $x \leq -15$.

Considering both cases, the solution set is $\{x \mid x \leq -15 \text{ or } x \geq 12, x \in \Re\}$.

6. Use the sign analysis method to make factor diagrams to solve the following quadratic inequalities.

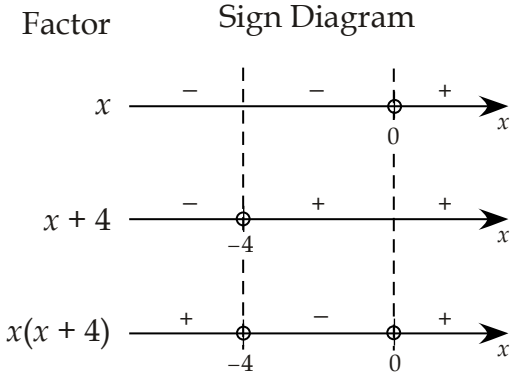
   a) $x^2 + 4x < 0$

   Answer:

   $x^2 + 4x < 0$

   $x(x + 4) < 0$

   The critical points are the roots $x = 0, -4$. By testing x-values on either side of $x = 0$ and $x = -4$, the sign diagram can be made.



   The solution occurs when the product of the factors is negative. The inequality, $x(x + 4) < 0$, is negative in the interval $(-4, 0)$ or can be written using set notation as $\{x \mid -4 < x < 0\}$.

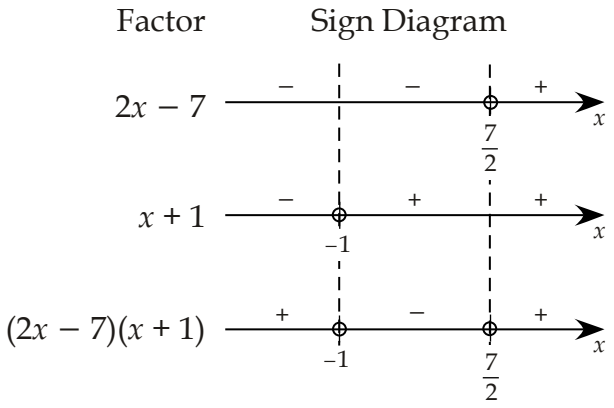
   b) $2x^2 - 5x - 7 > 0$

   Answer:

   $2x^2 - 5x - 7 > 0$

   $(2x - 7)(x + 1) > 0$

   The critical points are the roots $x = \frac{7}{2}$ and $x = -1$. By testing values of x on either side of the critical points, the sign diagram can be made.



The solution occurs where the product of the factors is positive.

Solution set: $\left\{x \mid x < -1 \text{ or } x > \frac{7}{2}, x \in \Re\right\}$

7. A man is standing on a 50-foot cliff. He can only see things that are higher than or as high as the cliff. Fireworks are being fired from the ground down below, and their height, measured in feet, at time t, measured in seconds, is given by the equation $h = -4t^2 + 28t + 5$.

   a) Solve the inequality $-4t^2 + 28t + 5 \geq 50$.

   Answer:

   The quadratic is $-4t^2 + 28t + 5 \geq 50$ or $-4t^2 + 28t - 45 \geq 0$ or $4t^2 - 28t + 45 \leq 0$.

   First, find the discriminant.

   $a = 4, b = -28, c = 45$

   $b^2 = 4ac = (-28)^2 - 4(4)(45)$

   $= 64$

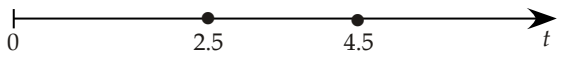
   $\therefore$ two rational roots since the discriminant is a perfect square.

   $\therefore$ factor into (or use the quadratic formula):

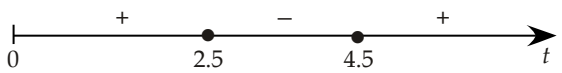
   $(2t - 9)(2t - 5) \leq 0$

   $t = \frac{9}{2}, \frac{5}{2}$

   $t = 4.5, 2.5$



	$4t^2 - 28t + 45 \leq 0$		
Interval	$0 \leq t \leq 2.5$	$2.5 \leq t \leq 4.5$	$t \geq 4.5$
Test Value	$t = 0$	$t = 3$	$t = 5$
Substituting the Test Point into the Inequality	$4(0)^2 - 28(0) + 45 = 45$	$4(3)^2 - 28(3) + 45 = -3$	$4(5)^2 - 28(5) + 45 = 5$
Sign	positive	negative	positive



The solution is where the result is negative or zero:

$\{t \mid 2.5 \leq t \leq 4.5, t \in \Re\}$

b) Interpret the solution to the inequality you solved in (a).

Answer:

This signifies the time at which the fireworks are visible to the man on the cliff. Therefore, the fireworks are visible to the man at 2.5 seconds and until 4.5 seconds after they are fired.

c) Explain how you chose which strategy to use to solve the quadratic inequality.

Answer:

Answers may vary.

The roots and test points method was used because it is efficient. The graphing method would be difficult, as the quadratic equation isn't in vertex form and the solutions may not be integer values. However, case analysis and sign analysis could also be used to solve this inequality.

Learning Activity 6.5

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Cassandra pays for $78.83 worth of groceries with a $100 bill. What is her change?
2. What is the vertex of the parabola defined by the equation $y = 2x^2 - 6$?
3. The movie you want to see starts playing at 8:30 p.m. It takes you 45 minutes to drive to the theatre and 30 minutes to get ready. If you want to be at the theatre 30 minutes early, what time should you start getting ready?
4. Find the common difference in the following pattern: 1, 7, 13, 19, . . .
5. Calculate: $\dfrac{\frac{4}{3}}{\frac{5}{8}}$
6. Express in exponential form: $\sqrt{y^{\frac{7}{2}}}$
7. Simplify: $(4x^2 + 7x - 3)(-3x + 2)$
8. What does the range of a quadratic function depend on?

Answers:

1. $21.17 ($100 − $78.83)
2. Vertex = $(0, -6)$ $\left(x = -\dfrac{b}{2a} = -\dfrac{0}{4} = 0;\ y = 2(0)^2 - 6 = 0 - 6 = -6\right)$
3. 6:45 pm (You need to leave 45 + 30 + 30 = 1 hour and 45 minutes before the movie starts. One hour before 8:30 is 7:30 and 45 minutes before 7:30 is 6:45. Therefore, you need to start getting ready at 6:45 pm.)
4. 6
5. $\dfrac{32}{15}$ $\left(\dfrac{4}{3} \times \dfrac{8}{5}\right)$

6. $y^{\frac{7}{4}}\left(\sqrt{y^{\frac{7}{2}}} = \left(y^{\frac{7}{2}}\right)^{\frac{1}{2}}\right)$

7. $-12x^3 + 8x^2 - 21x^2 + 14x + 9x - 6 = -12x^3 - 13x^2 + 23x - 6$

8. The range of a quadratic function depends on the direction of opening of the parabola and the location of the vertex.

Part B: Solving Quadratic Inequalities in Two Variables

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

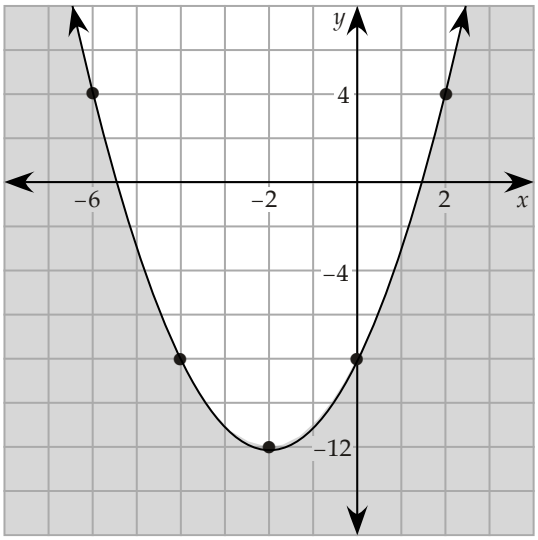
1. Determine whether the following statements are true or false. If they are false, explain why they are false and give the correct statement.

   a) To determine if the region outside a parabola is a solution to an inequality, test a point outside the parabola. If that point does not satisfy the inequality, then that region is a solution to the inequality.

   Answer:

   False. The correct statement should be "To determine if the region outside a parabola is a solution to an inequality, test a point outside the parabola. If that point **satisfies** the inequality, then that region is a solution to the inequality."

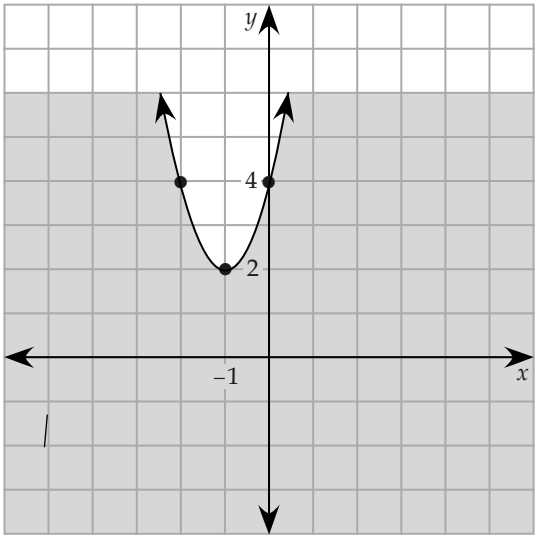
b) If (1, 2) satisfies the inequality $y \geq (x + 2)^2 - 12$, then the following graph represents the inequality.



Answer:

False. The shaded region should be below the parabola. The graph shows the inequality $y \leq (x + 2)^2 - 12$.

c) If $(-3, -2)$ satisfies the inequality $y \leq 2(x + 1)^2 + 2$, then the following graph represents the inequality.



Answer:

True. The point $(-3, -2)$ is in the shaded region below the function curve.

d) The quadratic inequality $y < 3x^2 - 6x + 7$ includes a solid boundary line.

Answer:

False. The boundary line should be dashed or broken because the inequality sign is strictly "less than."

e) The quadratic inequality $y > -x^2 - 5x + 4$ includes a broken or dashed boundary line.

Answer:

True. The inequality does not include "equal to."

f) The point (1, 6) is a solution to the inequality $y > -x^2 + x + 4$.

Answer:

$y > -x^2 + x + 4$

Point (1, 6)

LHS = $y = 6$

RHS = $-x^2 + x + 4$

$= -(1)^2 + (1) + 4$

$= 4$

LHS > RHS

$\therefore$ The statement is true.

g) The point (–1, 5) is a solution to the inequality $y \leq 2x^2 - x + 1$.

Answer:

$y \leq 2x^2 - x + 1$

Point (−1, 5)

LHS = $y = 5$

RHS = $2x^2 - x + 1$

$= 2(-1)^2 - (-1) + 1$

$= 4$

LHS ≤ RHS

$\therefore$ The statement is false.

A true statement would be the point (−1, 5) is a solution to the inequality $y \geq 2x^2 - x + 1$.

2. Match the inequality with its graph.

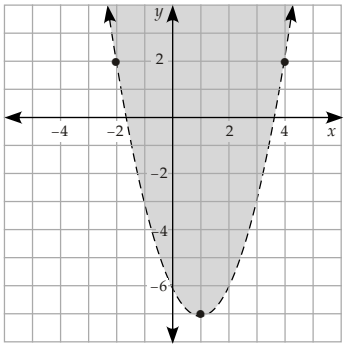
a) $y > -x^2 - 6x - 2$

b) $y \geq 2x^2 + 3x - 5$

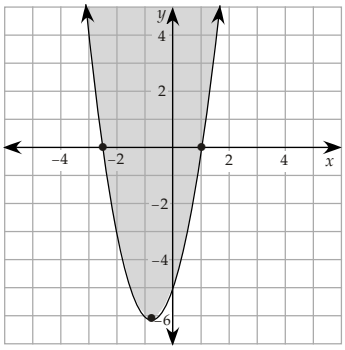
c) $y \leq -x^2 + 3x + 5$

d) $y > x^2 - 2x - 6$

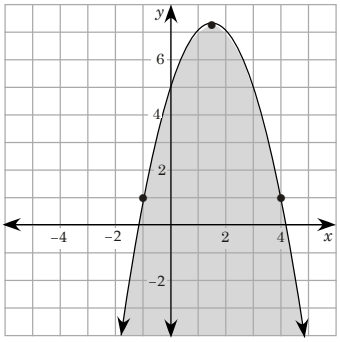
i)



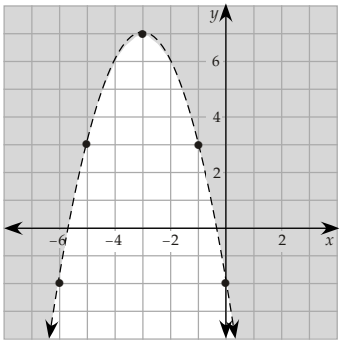
ii)



iii)



iv)



Answers:

a) $y > -x^2 - 6x - 2$

opens down; shading above parabola, dotted boundary

$\therefore$ Answer is (iv).

b) $y \geq 2x^2 + 3x - 5$

opens up, shading is above parabola, boundary is solid

$\therefore$ Answer is (ii).

c) $y \leq -x^2 + 3x + 5$

opens down, shading is below the parabola, boundary is solid

$\therefore$ Answer is (iii)

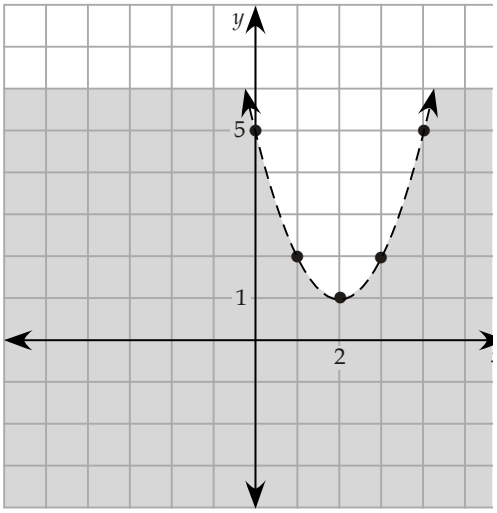
d) $y > x^2 - 2x - 6$

opens up, shading is above parabola, boundary is dotted.

$\therefore$ Answer is (i).

3. Write an inequality to describe each of the following graphs.

a)



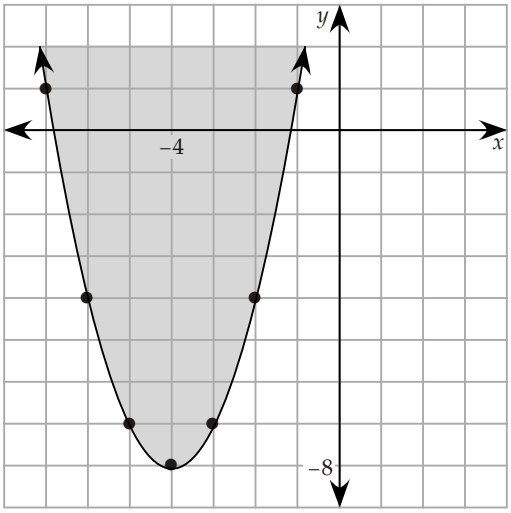
Answer:

The vertex is at (2, 1) and it is the normal shape. So, the equation of the parabola is $y = (x - 2)^2 + 1$.

The symbol is "less than" as the region shaded is below the parabola and the boundary line is broken.

Thus, the inequality is $y < (x - 2)^2 + 1$.

b)



Answer:

The vertex is at $(-4, -8)$ and it is the normal shape. So, the equation of the parabola is $y = (x + 4)^2 - 8$.

The symbol is "greater than or equal to" as the region shaded is inside the parabola and the boundary line is solid.

Thus, the inequality is $y \geq (x + 4)^2 - 8$.

4. Sketch the graph of the inequalities without using the graphing calculator.

   a) $y \leq x^2 + 4x$

   b) $y > \frac{1}{2}x^2 + 3x + 2$

   c) $y < -3x^2 - 6x$

   Answers:

   a) For $y \leq x^2 + 4x$

   Sketch the related equation $y = x^2 + 4x$.

   x-coordinate of vertex

   $$x = \frac{-b}{2a}$$

   $$x = \frac{-4}{2(1)}$$

   $$x = -2$$

   y-coordinate of vertex

   $y = x^2 + 4x$

   $y = (-2)^2 + 4(-2)$

   $y = 4 - 8$

   $y = -4$

   Zeros: Let $y = 0$

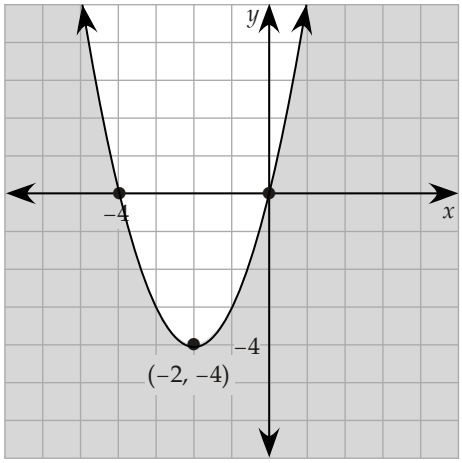
   $x^2 + 4x = 0$

   $x(x + 4) = 0$

   $x = 0$ and $x = -4$

   $\therefore$ Vertex $(-2, -4)$. The curve opens up and is normal shape.

   The parabola is represented by a solid line and the region below the parabola is shaded.



   Checkpoint (1, 0)

   LHS = $y = 0$

   RHS = $x^2 + 4x$

   $= 1^2 + 4(1)$

   $= 5$

   LHS $\leq$ RHS

   $\therefore$ (1, 0) is in the solution region.

b) For $y > \frac{1}{2}x^2 + 3x + 2$

Sketch the related equation $y = \frac{1}{2}x^2 + 3x + 2$.

x-coordinate of vertex

$$x = -\frac{b}{2a}$$

$$x = \frac{-3}{2 \cdot \frac{1}{2}}$$

$$x = -3$$

y-coordinate of vertex

$$y = \frac{1}{2}x^2 + 3x + 2$$

$$y = \frac{1}{2}(-3)^2 + 3(-3) + 2$$

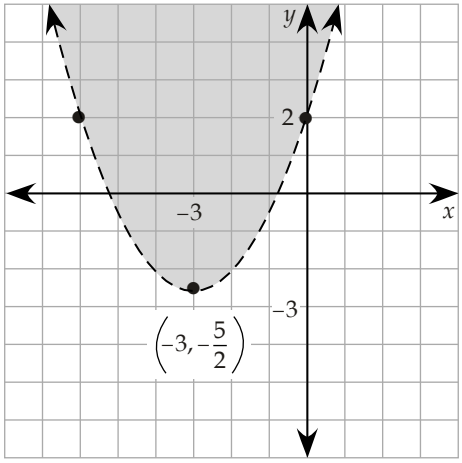
$$y = \frac{9}{2} + (-9) + 2$$

$$y = \frac{-5}{2}$$

$\therefore$ Vertex $\left(-3, -\frac{5}{2}\right)$

y-intercept: 2

The region above $y = \frac{1}{2}x^2 + 3x + 2$ is shaded and the outline of the border curve is dashed.



Checkpoint $(-4, 0)$

LHS = $y = 0$

$$\text{RHS} = \frac{1}{2}(-4)^2 + 3(-4) + 2$$

$$= 8 - 12 + 2$$

$$= -2$$

LHS > RHS

$\therefore$ $(-4, 0)$ is in the solution region.

c) For $y < -3x^2 - 6x$

Graph the related equation $y = -3x^2 - 6x$ and use a broken line curve.

x-coordinate of vertex

$$x = -\frac{b}{2a}$$

$$x = -\frac{-6}{2(-3)}$$

$$x = -1$$

y-coordinate of vertex

$y = -3x^2 - 6x$

$y = -3(-1)^2 - 6(-1)$

$y = -3 + 6$

$y = 3$

Zeros:

$-3x^2 - 6x = 0$

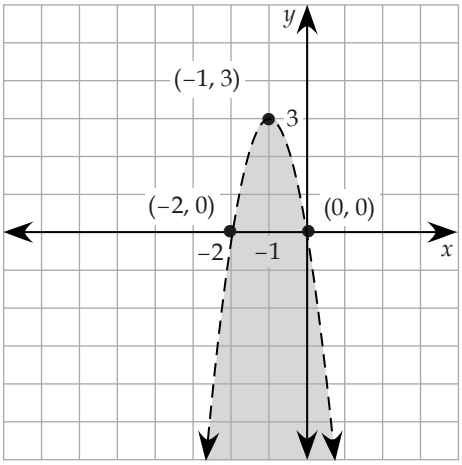
$-3x(x + 2) = 0$

$x = 0$ and $x + 2 = 0$

$x = 0$ and $x = -2$

Vertex $(-1, 3)$

$\therefore$ The shading is below the curve.



Checkpoint $(-1, 0)$

LHS = $y = 0$

RHS = $-3x^2 - 6x$

$= -3(-1)^2 - 6(-1)$

$= -3 + 6$

$= 3$

LHS < RHS

$\therefore$ $(-1, 0)$ is in the solution region.

Grade 11 Pre-Calculus Mathematics (30S)

Module 7
Trigonometry

Module 7: Trigonometry

Introduction

This module incorporates some of your previous knowledge about trigonometry, including the three primary trigonometric ratios: sine, cosine, and tangent. Trigonometry, including the Sine Law and the Cosine Law, allows you to calculate the length of the sides and the measure of the angles of triangles. Trigonometry has many applications in the real world. Have you ever noticed how bridges are made up of triangular shapes? Triangles are used in construction because of their strong shape. The more triangles included in the construction of a bridge, the stronger the bridge will be. For this reason, trigonometry is helpful in the design of bridges and other buildings.

In previous courses, you studied triangles and angles with measures of less than 180°. In this module, you will extend that knowledge to learn how trigonometry applies to angles greater than 180°. You will learn where these angles are located when graphed in standard position on a coordinate plane. You will also learn that there are families of angles with the same reference angle that have related trigonometric ratio values.

Assignments in Module 7

When you have completed the assignments for Module 7, submit your completed assignments to the Distance Learning Unit either by mail or electronically through the learning management system (LMS). The staff will forward your work to your tutor/marker.

Lesson	Assignment Number	Assignment Title
2	Assignment 7.1	Trigonometric Ratios, Reference Angles, and Trigonometric Equations
4	Assignment 7.2	Patterns in Trigonometric Ratios and the Sine Law
6	Assignment 7.3	The Cosine Law and the Ambiguous Case

Resource Sheet

When you write your final examination, you are encouraged to take a Final When you write your final examination, you are encouraged to take a Final Examination Resource Sheet with you into the examination. This sheet will be one letter-sized page, 8½ " by 11", with both sides in your handwriting or typewritten. You will submit it with your examination, but you do not receive any marks for it.

Many students have found that preparing a resource sheet is an excellent way to review. It provides you with a summary of the important facts of each module. You should complete a resource sheet for each module to help with your studying and reviewing. Lesson summaries and module summaries are included for you to use as a guide.

You may use the list of instructions that follows to help you with preparing your resource sheet for the material in Module 7. On this sheet, you should record math terms and definitions, formulas, sample questions, or a list of places where you often make mistakes. You should also identify special areas that require extra attention or review by writing the page numbers.

After you have completed each module's resource sheet, you may summarize the sheets from Modules 1 to 8 to prepare your Final Examination Resource Sheet. The final examination for this course is based on Modules 1 to 8.

Resource Sheet for Module 7

As you go through the lessons of this module, you may want to consider the following suggestions regarding the creation of a resource sheet.

1. List all the important math terms, and define them if necessary.
2. List all the formulas and perhaps a sample problem that shows how the formula is used.
3. If necessary, write the solutions to some problems, showing in detail how you did the calculations.
4. Copy any questions that represent the key points of the lesson, and perhaps include the solutions as well.
5. Identify the problems you found most difficult, and copy the page numbers onto the resource sheet so that you can review them before writing the examination. You may also copy the problems and the solutions onto your resource sheet, and later write them onto your Final Examination Resource Sheet.
6. Write any comments, ideas, shortcuts, or other reminders that may be helpful during an examination.

Lesson 1: The Trigonometric Ratios of Angles $0° \leq \theta \leq 360°$

Lesson Focus

In this lesson, you will

- ❑ learn how to draw a given angle in standard position
- ❑ learn how to recognize angles in standard position
- ❑ learn how to state the sine, cosine, and tangent of the angle in standard position in terms of the coordinates of a given point on its terminal side
- ❑ learn how to determine the distance from the origin to a point $P(x, y)$ on the terminal arm of an angle, the radius vector

Lesson Introduction

In your study of trigonometry, you studied angles between 0° and 180°. In this lesson, you will review this and extend your knowledge of angles up to 360°.

In *Grade 10 Introduction to Applied and Pre-Calculus Mathematics*, you learned about the three primary trigonometric ratios: sine, cosine, and tangent. You used these ratios to help solve right-angled triangles for missing angles or missing side lengths. Right-angled triangles can be drawn in the coordinate plane. Thus, you can use some of your previous knowledge to find the missing coordinates or distance to the origin when angles are located in standard position in the coordinate plane.

Trigonometric Ratios

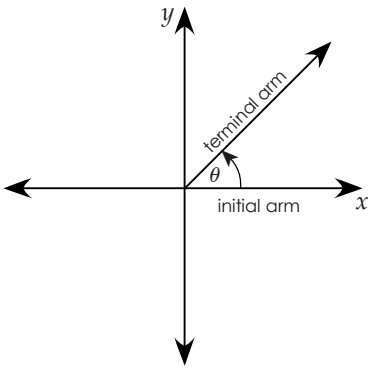
The first step towards using the trigonometric ratios to find the coordinates of P(x, y), or the distance from the origin to P(x, y), is to understand what is meant by the term *an angle in standard position*.

Angles in Standard Position

An angle is in **standard position** if:

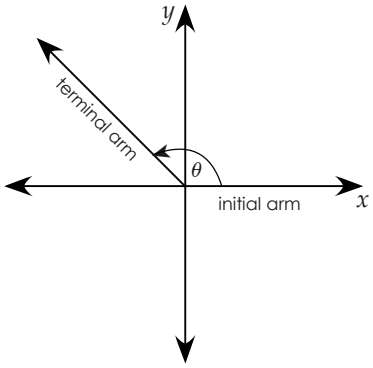
- its first arm, called the initial arm, is a ray from the origin along the positive x-axis
- its second arm, called the terminal arm, is any ray from the origin
- the angle is measured from the initial to the terminal arm and has a positive value if it is rotated in a counter-clockwise direction
- the angle is measured from the initial to the terminal arm and has a negative value if it is rotated in a clockwise direction

Quadrant I



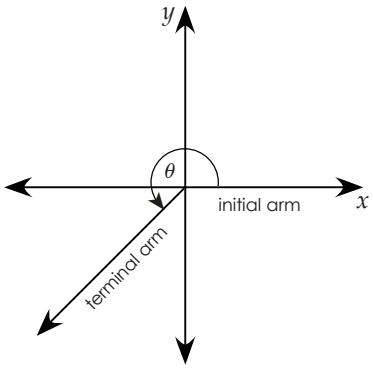
Angle θ is in standard position.
The measure of θ is between 0° and 90°.

Quadrant II



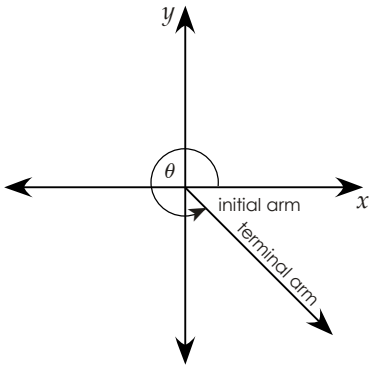
Angle θ is in standard position.
The measure of θ is between 90° and 180°.

Quadrant III



Angle θ is in standard position.
The measure of θ is between 180° and 270°.

Quadrant IV



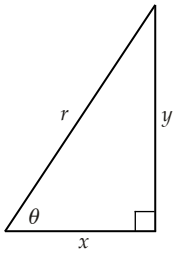
Angle θ is in standard position.
The measure of θ is between 270° and 360°.

Add the definition for an angle in standard position to your resource sheet. You may also find it helpful to include a diagram of an angle in standard position in each quadrant.

The Trigonometric Ratios

Now that you know what an angle in standard position looks like, you can define trigonometric ratios in terms of the coordinates of any point P(x, y) on the terminal arm of an angle in standard position. Compare your knowledge of trigonometric ratios in a right triangle to those defined by a point P(x, y) in a coordinate plane. The application of trigonometric ratios in these two diagrams is identical. The difference is that the (x, y) values in a coordinate system may sometimes be positive and sometimes negative.

Known Definitions

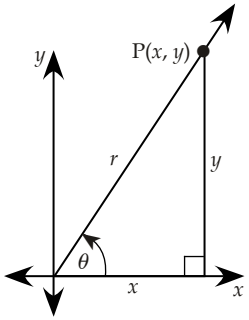


$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{y}{r}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{x}{r}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{y}{x}$$

Coordinates Definition



The line segment, with angle, θ, in standard position, from the origin to P(x, y) is called the radius vector, r. The length of r is always positive and can be found using the Pythagorean theorem, $r = \sqrt{x^2 + y^2}$.

You may remember these formulas using the acronym SOH CAH TOA.

Using these known definitions, if P(x, y) is any point on the terminal arm of an angle in standard position, then:

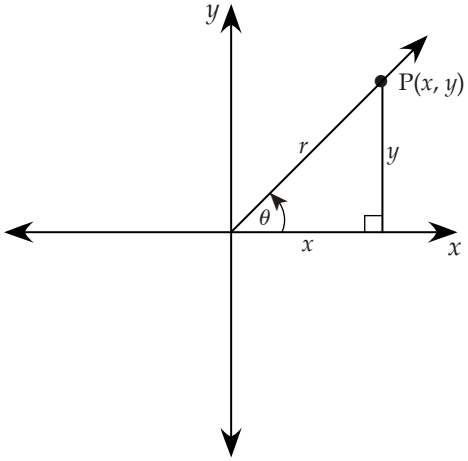
$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{y}{r} = \frac{y\text{-coordinate}}{\text{radius vector}}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{x}{r} = \frac{x\text{-coordinate}}{\text{radius vector}}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{y}{x} = \frac{y\text{-coordinate}}{x\text{-coordinate}}$$

Since the value of r is always positive, the sign of the trigonometric ratio depends upon the sign of the x- and y-coordinates of P(x, y).

If P(x, y) is in Quadrant I, then the measure of angle $\theta < 90°$, x and y are positive, and all trigonometric ratios of angles are positive.



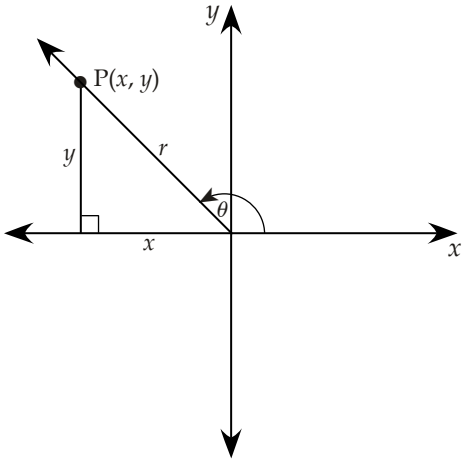
For any angle θ where $\theta < 90°$ and $x > 0$ and $y > 0$,

$\sin\theta = \frac{y}{r}$ will be positive

$\cos\theta = \frac{x}{r}$ will be positive

$\tan\theta = \frac{y}{x}$ will be positive

If P(x, y) is in Quadrant II, the measure of angle θ is between 90° and 180°, and the x-value of P(x, y) is negative. Any trigonometric ratio in Quadrant II whose ratio is defined using x will be negative.



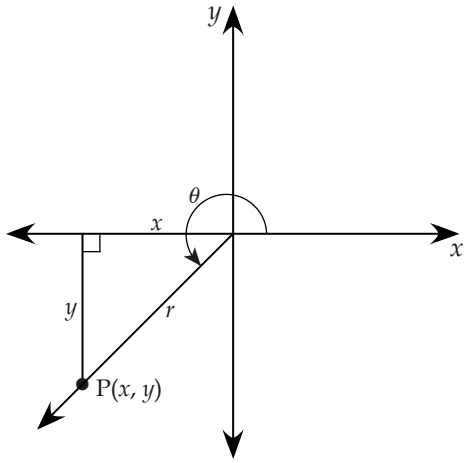
For any angle where $90° < \theta < 180°$ and $x < 0$ and $y > 0$,

$\sin \theta = \frac{y}{r}$ will be positive

$\cos \theta = \frac{x}{r}$ will be negative

$\tan \theta = \frac{y}{x}$ will be negative

If P(x, y) is in Quadrant III, then the measure of angle θ is between 180° and 270°, and the x- and y-values of P(x, y) are negative.



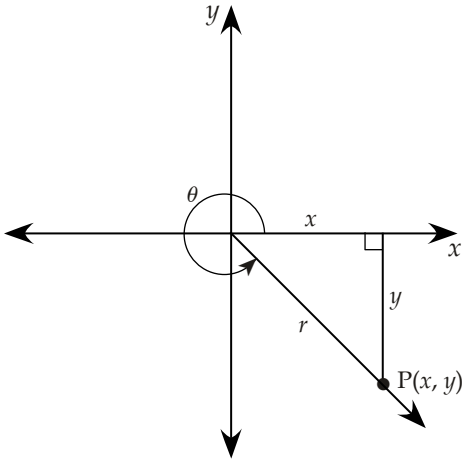
For any angle where $180° < \theta < 270°$ and $x < 0$ and $y < 0$,

$\sin \theta = \frac{y}{r}$ will be negative

$\cos \theta = \frac{x}{r}$ will be negative

$\tan \theta = \frac{y}{x}$ will be positive

If P(x, y) is in Quadrant IV, then the measure of angle θ is between 270° and 360°, and the x-value is positive and the y-value is negative.



For any angle where $270° < \theta < 360°$ and $x > 0$ and $y < 0$,

$\sin\theta = \frac{y}{r}$ will be negative

$\cos\theta = \frac{x}{r}$ will be positive

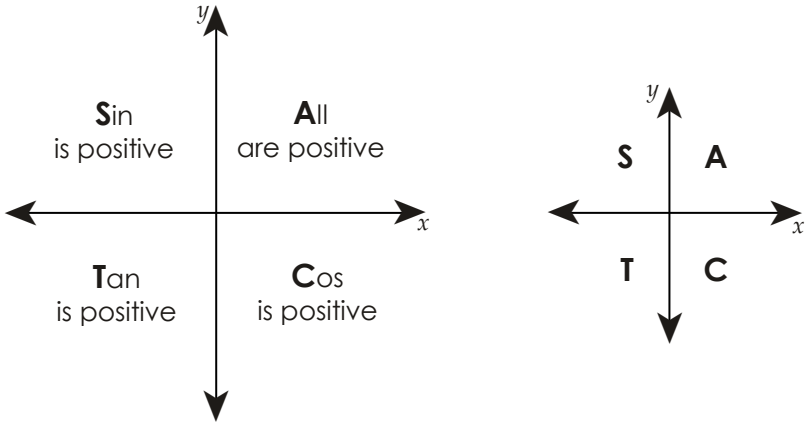
$\tan\theta = \frac{y}{x}$ will be negative

You can generalize this information about the signs of the values of the trigonometric functions using the trigonometric ratio definitions as you were shown. Since r is always positive, the sign of the value of $\sin\theta$ in each quadrant matches the sign of y for the quadrant of θ. Similarly, the sign of the value of $\cos\theta$ in each quadrant matches the sign of x for the quadrant of θ. Finally, the sign of the value of $\tan\theta$ in each quadrant matches the sign of the quotient of $\frac{y}{x}$ for the quadrant of θ.

To summarize:

- sign of $\sin\theta$ matches sign of y for the quadrant of θ
- sign of $\cos\theta$ matches sign of x for the quadrant of θ

Some students prefer to memorize the signs of the trigonometric ratios in each quadrant with the aid of the acronym CAST. The following diagram below show how the CAST acronym can be used to summarize where the trigonometric ratios are positive.



Notice that cos θ is positive in the quadrants where x is positive, sin θ is positive wherever y is positive, and tan θ is positive wherever x and y have the same sign.

Before you continue with the rest of your lesson, go back and reread the material you have just read. Then, add to your resource sheet any definitions, descriptions, or images that you feel will help you remember the material. This material may appear difficult at first, so it is a good idea to read it again to make sure you understand.

Example 1

P(x, y) is a point on the terminal arm of angle θ in standard position. Determine sin θ, cos θ, and tan θ of the following points and draw a sketch. Show your work to determine the length of the radius vector from the origin to P(x, y).

a) P(3, 4)

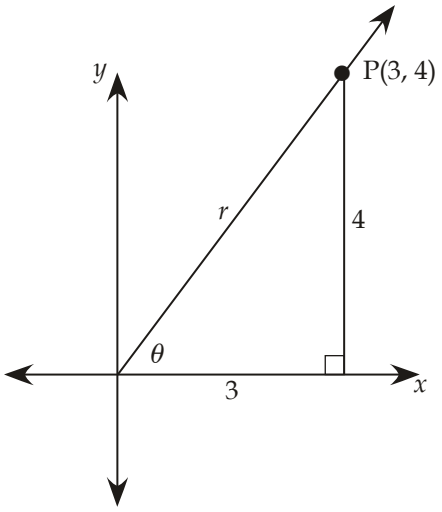
b) P(−3, 4)

c) P(−3, −4)

d) P(3, −4)

Solutions

a) P(3, 4)



$$r = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$\sin\theta = \frac{y}{r} = \frac{4}{5}$$

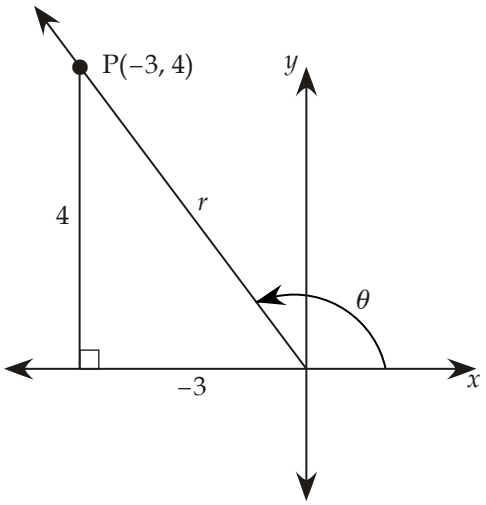
$$\cos\theta = \frac{x}{r} = \frac{3}{5}$$

$$\tan\theta = \frac{y}{x} = \frac{4}{3}$$

Notice that even after you have drawn the diagram, you can still use the trigonometric ratio definition as:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}};\ \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}};\ \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

b) P(−3, 4)



$$r = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5$$

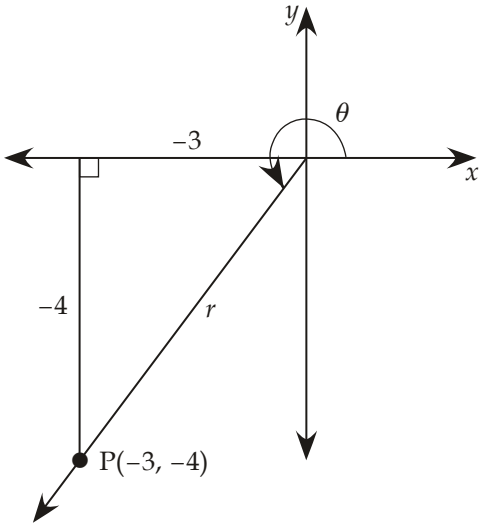
$$\sin\theta = \frac{y}{r} = \frac{4}{5}$$

$$\cos\theta = \frac{x}{r} = \frac{-3}{5}$$

$$\tan\theta = \frac{y}{x} = \frac{4}{-3} = -\frac{4}{3}$$

Notice the signs of the trigonometric values in Quadant II. The sign of sin θ matches the sign of y and is positive. The sign of cos θ matches the sign of x and is negative. The sign of tan θ matches the sign of the quotient, $\frac{y}{x}$, and is negative.

c) P(−3, −4)



$$r = \sqrt{(-3)^2 + (-4)^2} = \sqrt{25} = 5$$

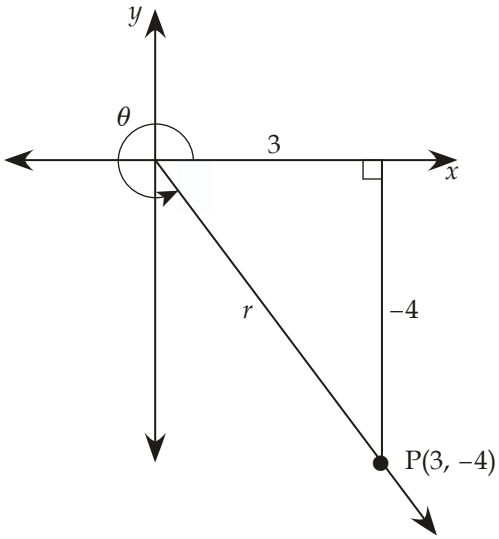
$$\sin\theta = \frac{y}{r} = \frac{-4}{5}$$

$$\cos\theta = \frac{x}{r} = \frac{-3}{5}$$

$$\tan\theta = \frac{y}{x} = \frac{-4}{-3} = \frac{4}{3}$$

Notice the signs of sin θ, cos θ, and tan θ in Quadrant III. Both sin θ and cos θ are negative because y and x are negative in Quadrant III. Tan θ is positive because it matches the sign of the quotient $\frac{y}{x}$.

d) P(3, −4)



$$r = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$$

$$\sin\theta = \frac{y}{r} = \frac{-4}{5}$$

$$\cos\theta = \frac{x}{r} = \frac{3}{5}$$

$$\tan\theta = \frac{y}{x} = \frac{-4}{3}$$

Notice the signs again. Sin θ is negative in Quadrant IV because it matches the sign of y. Cos θ is positive in Quadrant IV because it matches the sign of x.

Example 2

Use your calculator to find each of the following ratios and comment on the sign of the ratio. Round to five decimal places, if necessary.

a) cos 200°
b) tan 315°
c) sin 150°
d) tan 240°
e) tan 275°
f) sin 250°
g) cos 150°
h) sin 315°

Solutions

a) cos 200° = −0.939 69

The terminal arm of 200° is in Quadrant III and the x-coordinate is negative in Quadrant III.

b) tan 315° = −1

The terminal arm of 315° is in Quadrant IV and the y-coordinate is negative in Quadrant IV.

c) sin 150° = 0.5

The terminal arm of 150° is in Quadrant II and the y-coordinate is positive in Quadrant II.

d) tan 240° = 1.732 05

The terminal arm of 240° is in Quadrant III and the x- and y-coordinates are both negative in Quadrant III, so the ratio will be positive.

e) tan 275° = −11.430 05

The terminal arm of 275° is in Quadrant IV and the y-coordinate is negative in Quadrant IV.

f) sin 250° = −0.939 69

The terminal arm of 250° is in Quadrant III and the y-coordinate is negative in Quadrant III.

g) cos 150° = −0.866 03

The terminal arm of 150° is in Quadrant II and the x-coordinate is negative in Quadrant II.

h) sin 315° = −0.707 11

The terminal arm of 315° is in Quadrant IV and the y-coordinate is negative in Quadrant IV.

Example 3

Angle θ is either an acute or an obtuse angle. From the sign of the trigonometric ratio, determine whether θ is acute or obtuse, if:

a) $\cos \theta > 0$

b) $\tan \theta < 0$

c) $\cos \theta < 0$

Solutions

An acute angle has a measure between 0° and 90°. An obtuse angle has a measure between 90° and 180°. Therefore, angle θ will be either in Quadrant I or Quadrant II. Assume that P(x, y) is a point on the terminal arm of θ.

a) The sign of $\cos \theta$ depends upon the sign of the x-coordinate of P(x, y). Since $\cos \theta > 0$, then the x-coordinate must be positive. Thus, the terminal arm of θ must be in Quadrant I, and θ must be an acute angle.

b) The sign of $\tan \theta$ depends upon the signs of both the x- and y-coordinates of P(x, y). Since $\tan \theta < 0$, either $x < 0$ or $y < 0$ but not both. This occurs in Quadrant II where $x < 0$ and $y > 0$ so the angle is obtuse.

c) The sign of $\cos \theta$ depends upon the sign of the x-coordinate of P(x, y). Since $\cos \theta < 0$, then the x-coordinate must be negative. Thus, the terminal arm of θ must be in Quadrant II and θ must be an obtuse angle.

Learning Activity 7.1

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Jacob went to three stores. He spent $76 at the first store, $47 at the second store, and $152 at the third store. If Jacob had $300, how much money does he have left?
2. If two angles in a triangle measure 137° in total, what is the measure of the third angle in the triangle?
3. Solve for g: $(2g + 1)(g - 5) = 0$

continued

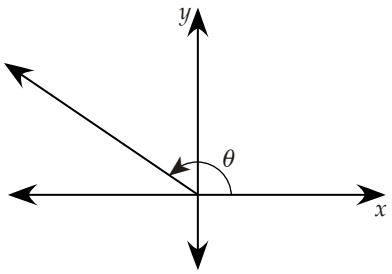
4. You and your brother are partial owners of the family vehicle. If you own half as much as your brother, how much of the car do you own?
5. Simplify: $\sqrt{108}$
6. Order these values from lowest to highest: $5^{\frac{1}{5}}$, $\sqrt{5}$, $5^{\frac{1}{4}}$, and $\sqrt[3]{5}$
7. Rationalize the denominator: $\dfrac{7\sqrt{3}}{3\sqrt{2}}$
8. Find the slope of the line parallel to the line $2x + y = 1$.

Part B: Trigonometric Ratios

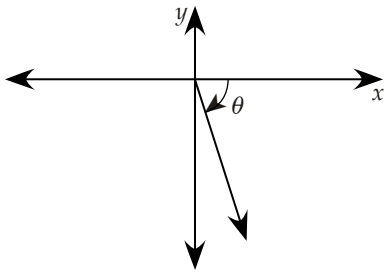
Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Determine which of the following angles are in standard position. Explain.

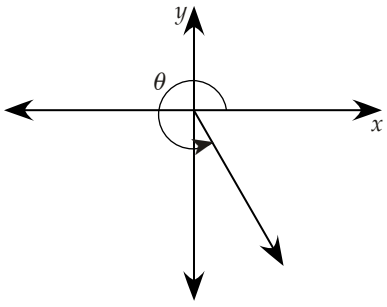
a)



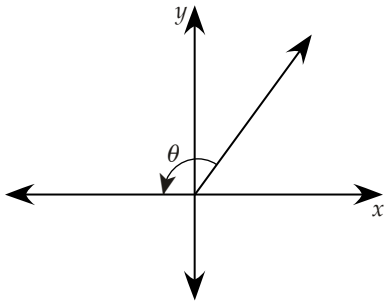
b)



c)



d)



continued

Learning Activity 7.1 (continued)

2. P(x, y) is a point on the terminal side of angle θ in standard position. Determine sin θ, cos θ, and tan θ for the following points. Draw a sketch. Show your work for determining the distance from the origin to the point P(x, y). Rationalize any denominators.
   a) $(-5, 12)$
   b) $(24, -7)$
   c) $(-4, -3)$
   d) $(5, 3)$
3. Without using a calculator, determine if each value should be positive or negative by drawing the angle in standard position.
   a) cos 191°
   b) sin 255°
   c) tan 340.4°
   d) sin 161°
4. Determine the quadrants in which P(x, y) may lie under the following conditions.
   a) $\sin \theta > 0$
   b) $\cos \theta > 0$
   c) $\tan \theta > 0$
   d) $\cos \theta > 0$ and $\sin \theta < 0$

continued

5. Fill in the following chart by indicating which quadrant(s) each angle is in by considering the sign of each ratio or by using the CAST rule.

Ratio Value	Quadrant
$\cos\theta = \frac{2}{5}$	
$\sin\theta = \frac{4}{5}$	
$\tan\theta = -\frac{2}{3}$	
$\sin\theta = \frac{-1}{5}$	
$\tan\theta = \frac{3}{5}$	

6. Given the following angles in standard position, determine if the sine, cosine, and tangent ratios are positive or negative. Explain your answers.
   a) $\theta = 26°$
   b) $\theta = 305°$
   c) $\theta = 152°$
   d) $\theta = 206°$

Lesson Summary

In this lesson, you learned how the three primary trigonometric ratios (sine, cosine, and tangent) could be used on the coordinate plane. As well, you learned how to recognize angles in standard position. When angles are in standard position and you are given a point on the terminal arm, you learned how to draw the corresponding right triangle and calculate the three trigonometric ratios related to that particular triangle. In the next lesson, you will learn about reference angles and how to solve linear trigonometric equations.

Lesson 2: Reference Angles and Solving Trigonometric Equations in Linear Form

Lesson Focus

In this lesson, you will

- ❑ learn how to determine the reference angle of an angle in standard position
- ❑ learn how to determine which angles have the same reference angle
- ❑ learn what quadrantal angles are and how to determine their trig ratio values
- ❑ learn how to solve trigonometric equations in linear form

Lesson Introduction

In the last lesson, you learned how to recognize angles that were in standard position. All angles in standard position have a reference angle. In fact, there is a relationship among all angles having the same reference angle. In this lesson, you will learn about the value of reference angles and how to find them. You will also learn how to solve trigonometric equations in linear form, which often appear in applications such as robotic arms, amusement park rides, and global positioning systems (GPSs).

Reference Angles

Before you can determine the reference angle for a given angle in standard position, you first have to draw the angle in standard position.

Sketch an Angle in Standard Position

Example 1

Sketch each angle in standard position. State the quadrant in which the terminal arm of the angle lies.

a) 82°

b) 328°

c) 164°

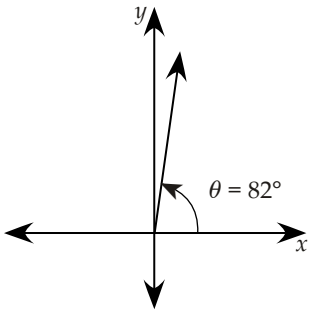
d) 221°

Solution

In order to draw an angle in standard position, first determine in which quadrant the angle lies. You can draw the angle by using a protractor or you can sketch a reasonable approximation for the angle.

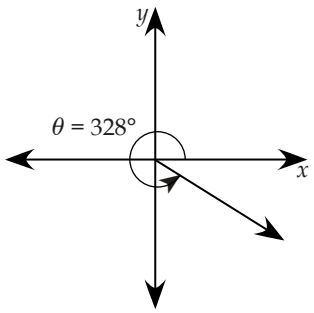
a) Since $0° < 82° < 90°$, $\theta = 82°$ lies in Quadrant I.

As 82° is very close to 90°, this angle will be very close to the positive y-axis.



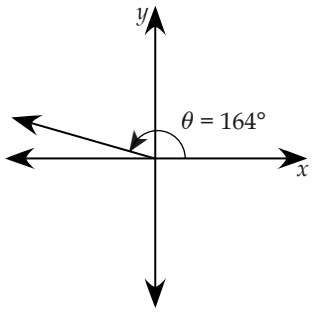
b) Since 270° < 328° < 360°, θ lies in Quadrant IV.

As 328° is a little more than halfway from 270° to 360°, this angle will lie a little bit more than halfway between the negative y-axis and the positive x-axis.



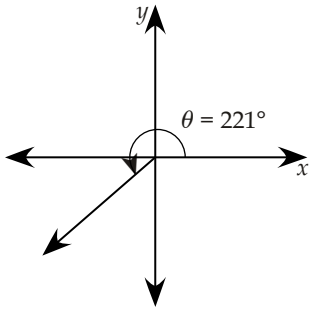
c) Since 90° < 164° < 180°, θ lies in Quadrant II.

164° is very close to 180°, and thus the terminal arm should lie close to the negative x-axis.



d) Since 180° < 221° < 270°, θ lies in Quadrant III.

221° is about halfway between 180° and 270°, and thus this angle should lie halfway between the negative x-axis and the negative y-axis.



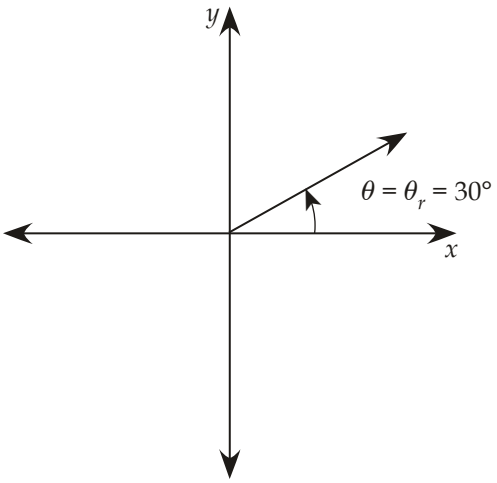
Determine the Reference Angle of an Angle in Standard Position

Let θ be an angle in standard position. Its **reference angle** is the acute angle formed between the terminal arm of θ and the nearer portion of the x-axis. The reference angle is denoted by θ_r and is always a positive acute angle. It is important to learn how to find reference angles because the trigonometric ratio values of all angles with the same reference angle are related.

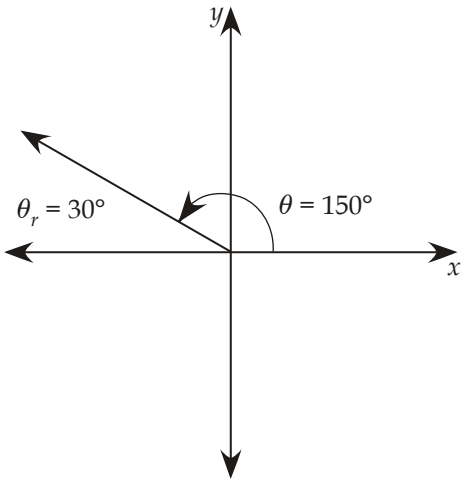
Include this definition on your resource sheet.

In each of the following diagrams, the reference angle is 30°.

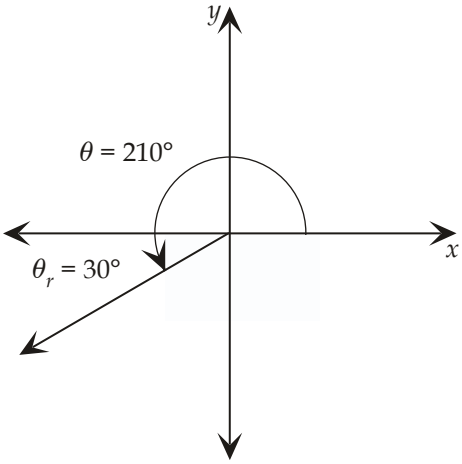
Quadrant I



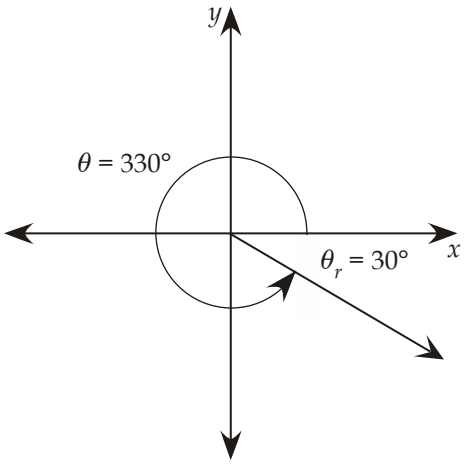
Quadrant II



Quadrant III



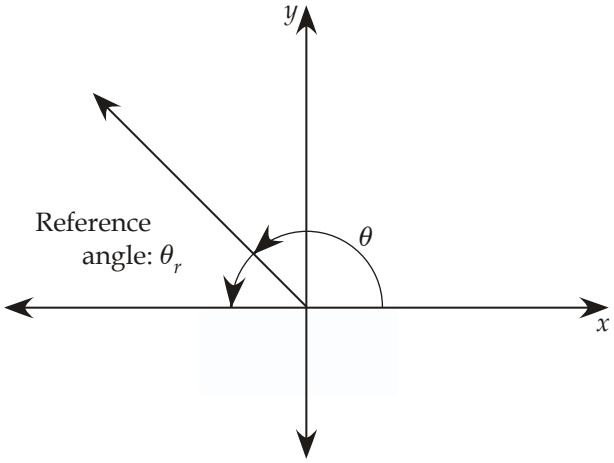
Quadrant IV



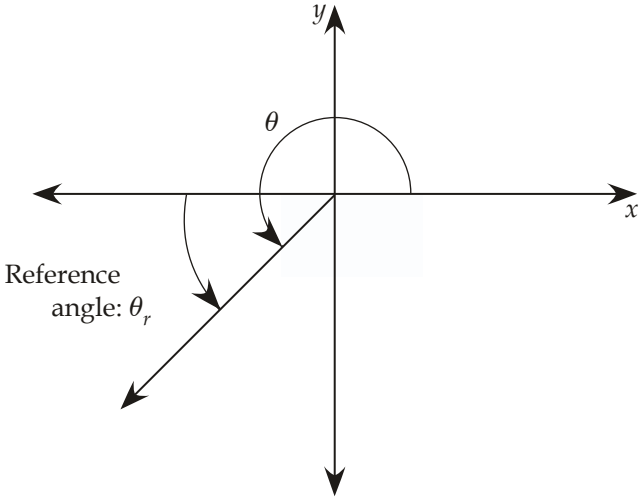
Can you determine how the reference angle (θ_r) is found in each quadrant?

In Quadrant I, any angle is acute so the angle of rotation, θ, and the reference angle, θ_r, are the same angle.

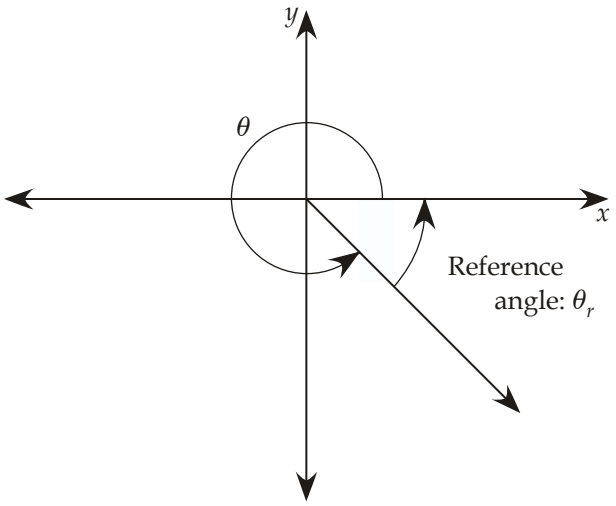
In Quadrant II, $\theta_r = 180° - \theta$.



In Quadrant III, $\theta_r = \theta - 180°$.



In Quadrant IV, $\theta_r = 360° - \theta$.



Example 2

Find the reference angle for the following.

a) 226°

b) 147°

c) 25°

d) 340°

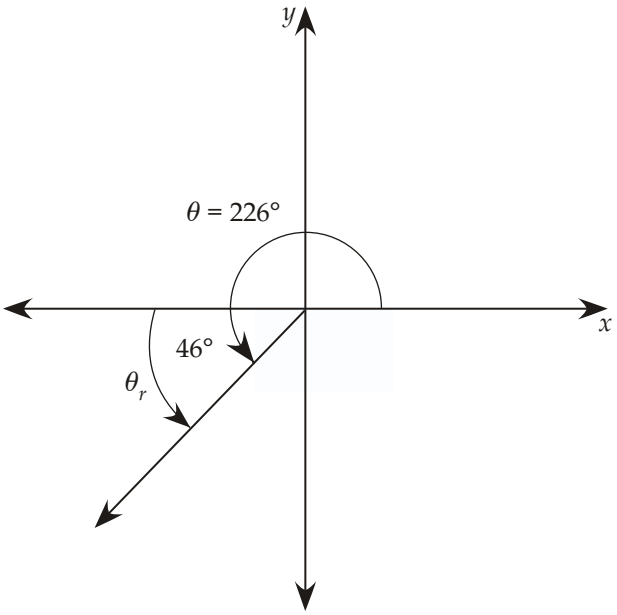
Solution

a) The terminal arm of 226° lies in Quadrant III.

$\therefore \theta_r = \theta - 180°$

$= 226° - 180°$

$= 46°$

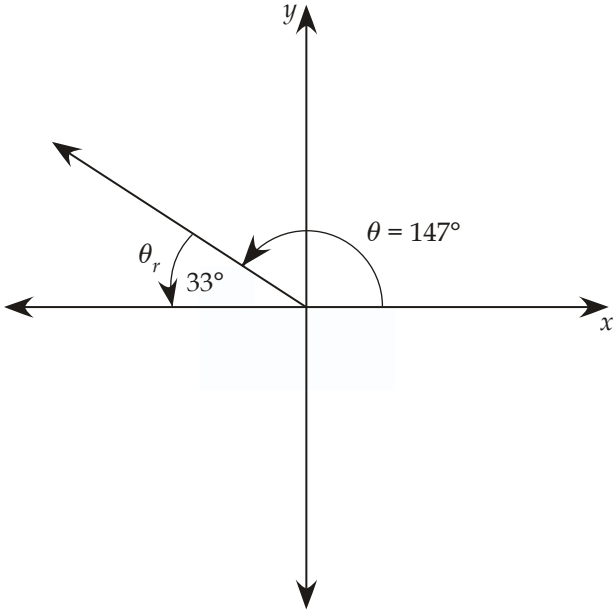


b) The terminal arm of 147° lies in Quadrant II.

$\therefore \theta_r = 180° - \theta$

$= 180° - 147°$

$= 33°$



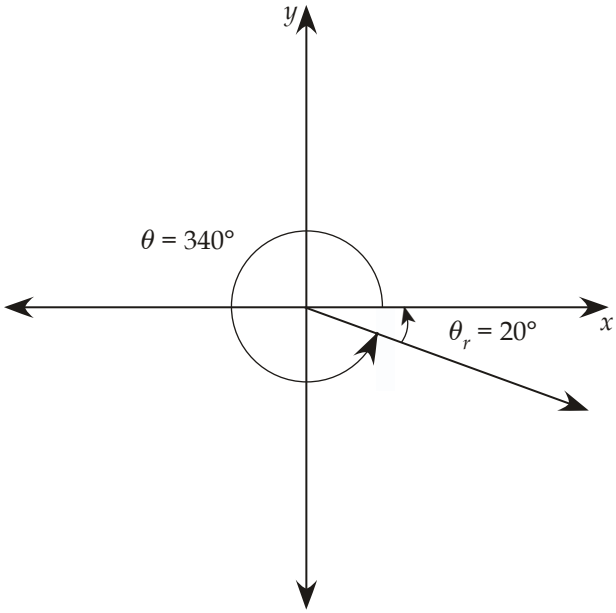
c) The terminal arm of 25° is in Quadrant I, so it is already an acute angle. Thus, $\theta_r = 25°$.

d) The terminal arm of 340° is in Quadrant IV.

$\therefore \theta_r = 180° - \theta$

$= 360° - 340°$

$= 20°$



Determining Which Angles Have the Same Reference Angle

As you noticed, each reference angle has four corresponding angles of rotation: one between 0° and 90°, one between 90° and 180°, one between 180° and 270°, and one between 270° and 360°. Consider the following example.

Example 3

Determine the angles from 0° to 360° that have 20° as a reference angle.

Solution

In Quadrant I, the equation is: $\theta = \theta_r$.

Thus, one angle of rotation that has 20° as a reference angle is 20°.

In Quadrant II, the equation is: $\theta_r = 180° - \theta$.

If you substitute 20° in for θ_r, $20° = 180° - \theta$

$$\theta = 180° - 20°$$

$$\theta = 160°$$

Thus, another angle that has 20° as a reference angle is 160°.

In Quadrant III, the equation is: $\theta_r = \theta - 180°$.

If you substitute 20° in for θ_r, $20° = \theta - 180°$

$$\theta = 20° + 180°$$

$$\theta = 200°$$

Thus, another angle that has 20° as a reference angle is 200°.

In Quadrant IV, the equation is: $\theta_r = 360° - \theta$.

If you substitute 20° in for θ_r, $20° = 360° - \theta$

$$\theta = 360° - 20°$$

$$\theta = 340°$$

Thus, the fourth angle between 0° and 360° that has 20° as a reference angle is 340°.

Rather than memorizing the equations given for each quadrant, it is a good idea to draw a diagram of the angle of rotation or the reference angle and determine how to find the desired angle using reasoning skills. Eventually, you will be able to visualize the diagram in your mind and you won't need to draw it.

All of the angles you have just found are reflections in the x- and/or the y-axis of the reference angle. Consider the following example.

Example 4

Determine the corresponding angle of rotation, θ, in standard position, when the reference angle of 30° is reflected

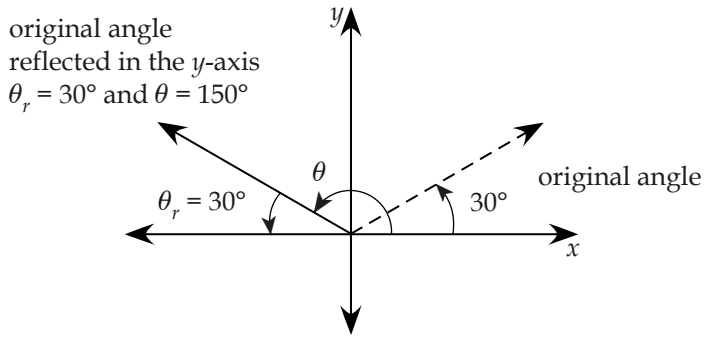
a) in the y-axis

b) in the x-axis

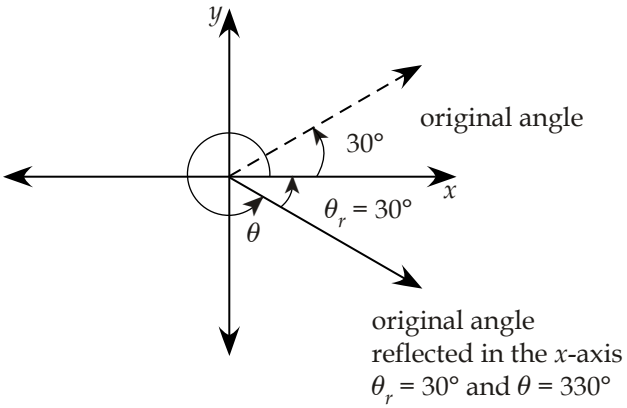
c) in the y-axis and then the x-axis

Solution

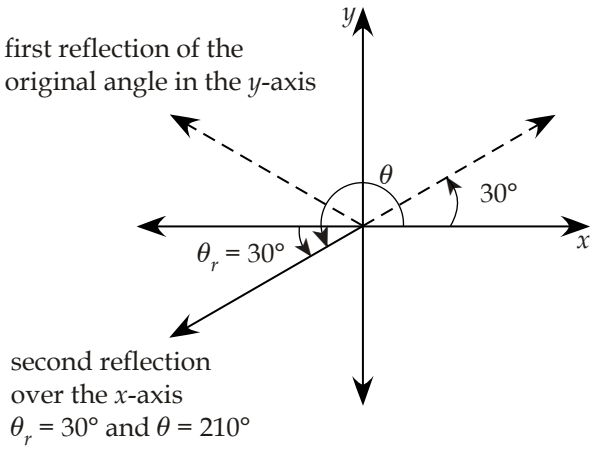
a) When you reflect an angle of 30° in the y-axis, the result is a reference angle of 30° in Quadrant II. The measure of the angle in standard position for Quadrant II is $180° - 30° = 150°$.



b) When you reflect an angle of 30° in the x-axis, this will result in a reference angle of 30° in Quadrant IV. The measure of the angle in standard position for Quadrant IV is $360° - 30° = 330°$.



c) When you reflect an angle of 30° in the y-axis and then the x-axis, this will result in a reference angle of 30° in Quadrant III. The measure of the angle in standard position for Quadrant III is 180° + 30° = 210°.



For every angle between 90° and 360°, its reference angle is a reflection in the x- and/or y- axis of its reference angle from the first quadrant.

Quadrantal Angles

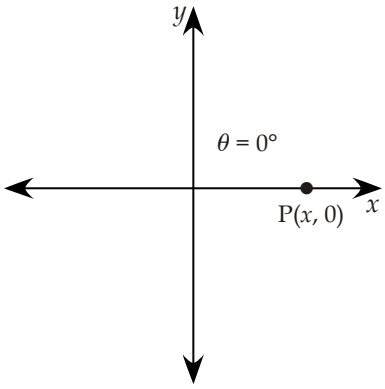
Until now, we have defined angles to be in the four quadrants. Their measures have been in the following intervals: (0°, 90°), (90°, 180°), (180°, 270°), or (270°, 360°).

Sometimes the terminal arm of an angle coincides with the x- or y-axis. In other words, the angle in standard position is exactly 0°, 90°, 180°, 270°, or 360°. These angles are called **quadrantal angles**. A **quadrantal angle** has as its initial arm as a ray from the origin along the positive x-axis and its terminal arm along one of the axes. If measured in a counter-clockwise direction, it has a positive value.

Include this definition on your resource sheet.

These angles are not in any quadrant. However, you can easily choose a point on the axis and then determine the trigonometric ratios for each of these quadrantal angles. In each of the following, assume x is positive and y is positive.

An angle of 0°



The initial side and the terminal side are the same. The measure of the angle is 0°. Since a point P(x, y) is on the positive x-axis, then P has coordinates (x, 0) and $r = \sqrt{x^2 + 0^2} = x$.

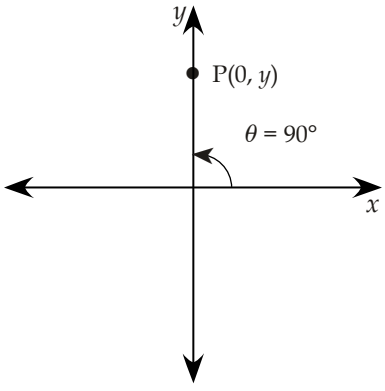
Now you can write the trigonometric ratios:

$$\cos 0° = \frac{x\text{-coordinate}}{\text{radius vector}} = \frac{x}{x} = 1$$

$$\sin 0° = \frac{y\text{-coordinate}}{\text{radius vector}} = \frac{0}{x} = 0$$

$$\tan 0° = \frac{y\text{-coordinate}}{x\text{-coordinate}} = \frac{0}{x} = 0$$

An angle of 90°



The terminal side lies along the positive y-axis. The measure of the angle is 90°. Since the point P(x, y) is on the positive y-axis, then P has coordinates (0, y) and $r = \sqrt{0^2 + y^2} = y$.

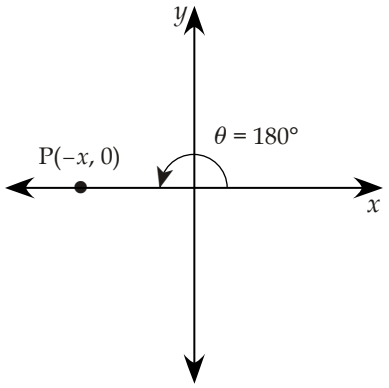
Now you can write the trigonometric ratios:

$$\cos 90° = \frac{x\text{-coordinate}}{\text{radius vector}} = \frac{0}{y} = 0$$

$$\sin 90° = \frac{y\text{-coordinate}}{\text{radius vector}} = \frac{y}{y} = 1$$

$$\tan 90° = \frac{y\text{-coordinate}}{x\text{-coordinate}} = \frac{y}{0} = \text{undefined since division by 0 is impossible}$$

An angle of 180°



The terminal side lies along the negative x-axis. The measure of the angle is 180°. Since the point P(x, y) is on the negative x-axis, then P has coordinates $(-x, 0)$ and $r = \sqrt{(-x)^2 + 0^2} = |-x| = x$.

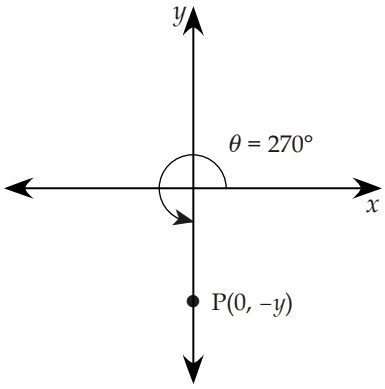
Now you can write the trigonometric ratios:

$$\cos 180° = \frac{x\text{-coordinate}}{\text{radius vector}} = \frac{-x}{x} = -1$$

$$\sin 180° = \frac{y\text{-coordinate}}{\text{radius vector}} = \frac{0}{x} = 0$$

$$\tan 180° = \frac{y\text{-coordinate}}{x\text{-coordinate}} = \frac{0}{-x} = 0$$

An angle of 270°



The terminal side lies along the negative y-axis. The measure of the angle is 270°. Since the point P(x, y) is on the negative y-axis, then P has coordinates $(0, -y)$ and $r = \sqrt{(0)^2 + (-y)^2} = |-y| = y$.

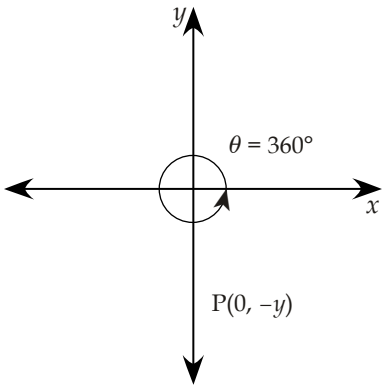
Now you can write the trigonometric ratios:

$$\cos 270° = \frac{x\text{-coordinate}}{\text{radius vector}} = \frac{0}{y} = 0$$

$$\sin 270° = \frac{y\text{-coordinate}}{\text{radius vector}} = \frac{-y}{y} = -1$$

$$\tan 270° = \frac{y\text{-coordinate}}{x\text{-coordinate}} = \frac{y}{0} = \text{undefined since division by 0 is impossible}$$

An angle of 360°

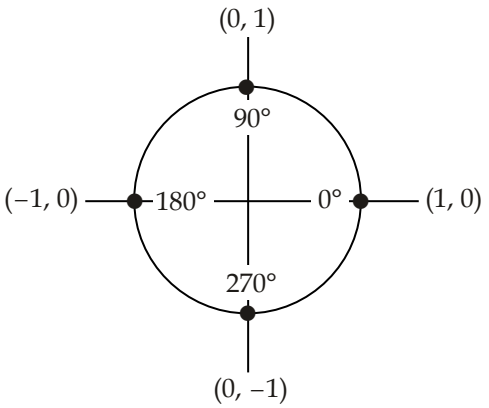


You have already considered this case because the terminal side of an angle of 360° coincides with the positive x-axis. The trigonometric ratios for 360° will be the same as the trigonometric ratios for an angle of 0°.

Summary

θ	$\cos\theta$	$\sin\theta$	$\tan\theta$
0°	1	0	0
90°	0	1	undefined
180°	−1	0	0
270°	0	−1	undefined
360°	1	0	0

An easy way to remember all these values of the trigonometric ratios for the quadrantal angles of θ uses the (x, y) coordinates of points on a circle with radius 1, called the unit circle. This concept will be developed further in Grade 12 pre-calculus.



Since $r = 1$, the ratios become

$$\cos\theta = \frac{x}{r} = \frac{x}{1} = x\text{-coordinate}$$

$$\sin\theta = \frac{y}{r} = \frac{y}{1} = y\text{-coordinate}$$

$$\tan\theta = \frac{y}{x} = \frac{y\text{-coordinate}}{x\text{-coordinate}}$$

For example:

$\cos 90° = 0$, since that is the x-coordinate of $(0, 1)$

$\sin 270° = -1$, since that is the y-coordinate of $(0, -1)$

$\tan 180° = 0$, since that is the $\dfrac{y\text{-coordinate}}{x\text{-coordinate}}$ of $(-1, 0)$

Challenge yourself and see if you can write down, from memory, all of the trigonometric ratio values in the table using the coordinates on the x- and y-axes of the unit circle.

Solving Linear Trigonometric Equations

To solve a trigonometric equation means to determine the values of the unknown angle that satisfy the equation. In other words, θ is the variable you need to isolate.

To solve trigonometric equations where the angles are between 0° and 360°, reference angles are used.

You have developed the following formulas for finding the reference angle for any angle, θ.

In Quadrant I $\theta_r = \theta$

In Quadrant II $\theta_r = 180° - \theta$

In Quadrant III $\theta_r = \theta - 180°$

In Quadrant IV $\theta_r = 360° - \theta$

Now, you can rearrange these formulas and solve for θ. This is similar to what you did in Example 3.

In Quadrant I $\theta = \theta_r$

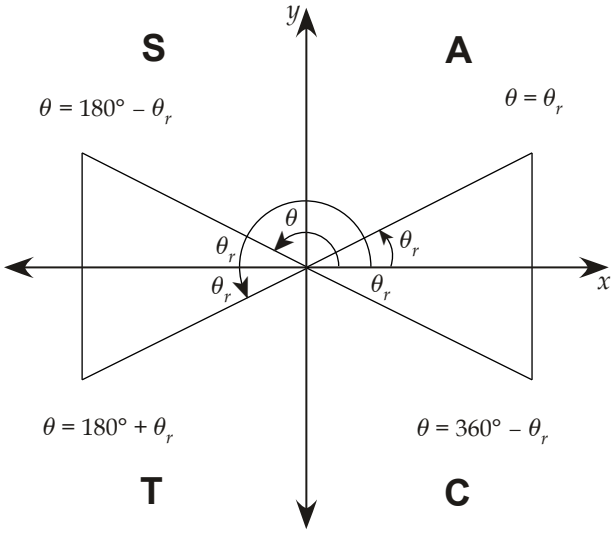
In Quadrant II $\theta = 180° - \theta_r$

In Quadrant III $\theta = 180° + \theta_r$

In Quadrant IV $\theta = 360° - \theta_r$

It is best if you do not try to memorize these formulas. Instead, draw a triangle diagram in a quadrant and use reasoning to determine how θ and θ_r are related in each quadrant.

The following diagram summarizes the CAST rule and the relationship between θ and θ_r in the four quadrants.



Your calculator has limitations when it comes to solving trigonometric equations. Use your calculator to find the following values that all have a reference angle of 30°:

$\cos 30° = \quad 0.8660$

$\cos 150° = -0.8660$

$\cos 210° = -0.8660$

$\cos 330° = -0.8660$

You probably noticed that the number is the same for each $\cos \theta$ value when $\theta_r = 30°$, but the sign depends on the quadrant for θ.

Use your calculator to find all these trigonometric ratios for angles of rotation having a reference angle of 40°:

$\sin 40° = \quad 0.6427$

$\sin 140° = \quad 0.6427$

$\sin 220° = -0.6427$

$\sin 320° = -0.6427$

Again, the $\sin \theta$ values are all the same but signs depend on the quadrant for θ.

The limitations of your calculator show up when you try to find θ when given the value of a trigonometric ratio. This is what is required when you solve a trigonometric equation.

For example, use your calculator to find θ given $\sin \theta = 0.6427$.

$\theta = \sin^{-1}(0.6427)$

$\theta = 39.99°$

You know from the work above that there should be two answers for θ between 0° and 360° but your calculator only gave you one. The other θ value should be very near 140°.

Now try solving $\cos \theta = -0.8660$ to find all values of θ between 0° and 360°.

$\theta = \cos^{-1}(-0.8660)$

$\theta = 150.00°$

Again, the calculator only found one of the two possible answers. The other answer should be very close to 210°.

You will use your knowledge of reference angles to find all possible values of θ between 0° and 360°.

Your calculator can find the reference angle, θ_r, when given a trigonometric ratio. You will use the inverse trigonometric function to find θ and always use the corresponding positive trigonometric ratio value, if it is given as negative, since the reference angle is an acute angle equivalent to θ in Quadrant I.

That is, given $\sin\theta = 0.6427$

$$\theta_r = \sin^{-1}(0.6427)$$
$$\theta_r = 39.99°$$

Now find the angles of rotation with that reference angle in Quadrants I and II where $\sin\theta > 0$.

Also, given $\cos\theta = -0.8660$

$\theta_r = \cos^{-1}(0.8660)$ ⟵ (notice the positive ratio is used)

$\theta_r = 30.00°$

Now find the angles of rotation with that reference angle in Quadrants II and III where $\cos\theta < 0$.

Example 5

Use your calculator to solve for θ if $2\sin\theta - 1 = 0$ and θ lies in the interval $[0°, 360°]$.

Solution

$$2\sin\theta - 1 = 0$$
$$2\sin\theta = 1$$
$$\sin\theta = \frac{1}{2}$$

Now use your calculator to find the reference angle, θ_r.

$$\sin\theta_r = \frac{1}{2}$$

To isolate θ, find inverse sin of both sides of the equation.

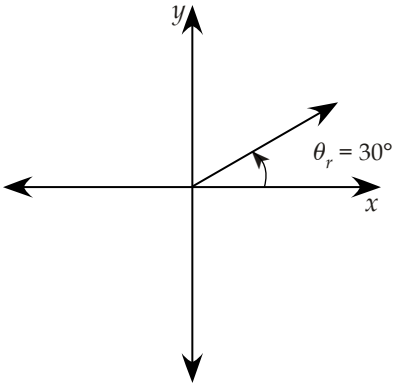
$$\theta_r = \sin^{-1}\left(\frac{1}{2}\right)$$
$$\theta_r = 30°$$

Now find θ.

Since $\sin\theta = \frac{1}{2}$, then $\sin\theta > 0$ and θ must be in Quadrants I or II.

S✓	A✓
T	C

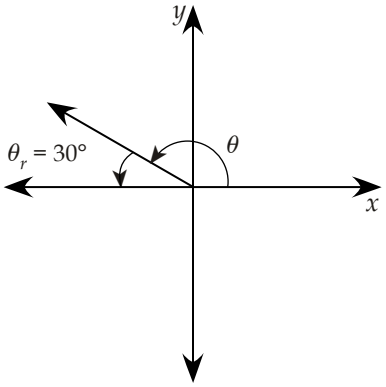
Quadrant I



$\theta = \theta_r$

$\therefore \theta = 30°$

Quadrant II



$\theta = 180° - \theta_r$

$\theta = 180° - 30°$

$\theta = 150°$

The solutions are $\theta = 30°, 150°$.

Example 6

Solve for θ if $2\tan\theta + 5 = 0$. Round to the nearest degree.

Solution

$$2\tan\theta + 5 = 0$$

$$2\tan\theta = -5$$

$$\tan\theta = \frac{-5}{2}$$

$$\theta_r = \tan^{-1}\left(\frac{5}{2}\right)$$

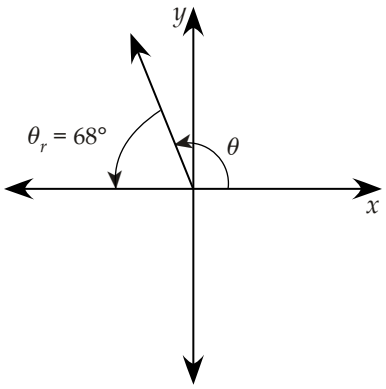
Ignore the negative sign since the reference angle is the same as θ in Quadrant I.

$$\theta_r = 68°$$

Since $\tan\theta = \frac{-5}{2}$, then $\tan\theta < 0$. This occurs in Quadrants II and IV.

S✓	A
T	C✓

Quadrant II

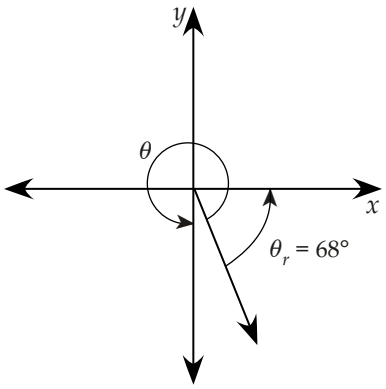


$\theta = 180 - \theta_r$

$\theta = 180° - 68°$

$\theta = 112°$

Quadrant IV



$\theta = 360° - \theta_r$

$\theta = 360° - 68°$

$\theta = 292°$

The solutions are $\theta = 112°, 292°$.

Steps for solving linear trigonometric equations:

1. Rearrange the equation to isolate the trigonometric function.
2. Solve for the reference angle using the appropriate inverse trigonometric function (and a positive trigonometric ratio).
3. Determine in which quadrants the solutions for θ will lie, depending upon the sign of the trigonometric function (or use the CAST rule).
4. Solve for θ using the reference angle relationships in the quadrants.

Learning Activity 7.2

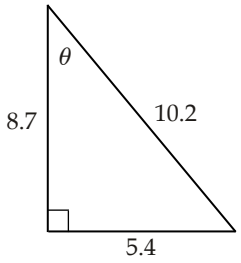
Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Gwen planted 120 seeds of green beans. If one-quarter of the beans did not grow, how many beans did grow?
2. The minute hand of a clock starts at 12 and travels 60% of the way around the clock. How many degrees from the 12 has the minute hand travelled?
3. Calculate: $\frac{4}{3} + \frac{5}{7}$
4. The two legs of a right-angled triangle measure 3 cm and 4 cm. What is the measure of the hypotenuse?
5. If you do homework for half an hour every night from Monday to Friday, as well as 45 minutes on both Saturday and Sunday, how many hours of homework do you do in a week?

Use the following right-angled triangle to answer the next three questions. Do not simplify.



6. Identify the sine ratio for θ.
7. Identify the cosine ratio for θ.
8. Identify the tangent ratio for θ.

continued

Learning Activity 7.2 (continued)

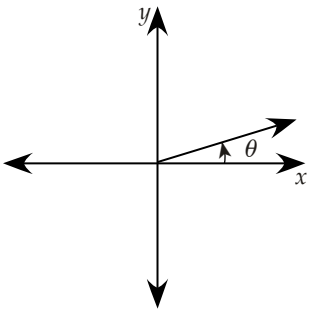
Part B: Reference Angles and Solving Trigonometric Equations in Linear Form

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

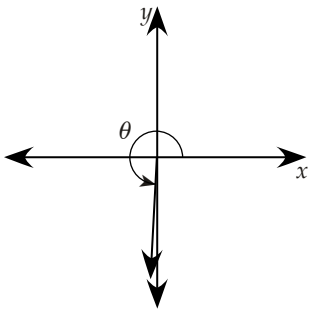
1. In which quadrant does the terminal arm of each angle lie?
   a) 67°
   b) 340°
   c) 241°
   d) 123°

2. Without measuring, match each angle to its corresponding graph.
   a) 140°
   b) 267°
   c) 316°
   d) 17°

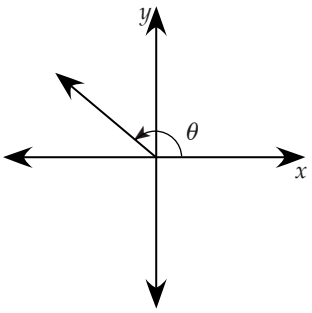
i)



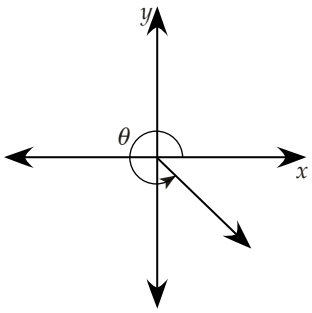
ii)



iii)



iv)



continued

Learning Activity 7.2 (continued)

3. Sketch each angle in standard position and find its reference angle. Write the measure of the reference angle.
   a) 98°
   b) 120°
   c) 352°
   d) 263°
4. Determine the measure of the 3 other angles in standard position that have the same reference angle as 58°.
5. Determine the angle in standard position when an angle of 65° is reflected
   a) in the y-axis. Draw the diagram.
   b) in the x-axis. Draw the diagram.
   c) in the y-axis and then the x-axis. Draw the diagram.
6. Determine the solution set for each of the following trigonometric equations over the interval [0°, 360°]. Round to one decimal place if necessary.
   a) $\cos\theta = -\frac{2}{3}$
   b) $\sin\theta + 1 = 0$
   c) $\tan\theta - 2 = 5$
   d) $2\cos\theta = 2$
   e) $-3\sin\theta = 2$
   f) $\frac{\tan\theta}{2} = 5$
   g) $5\cos\theta - 2 = 0$
   h) $5\tan\theta + 4 = 0$
   i) $\frac{\tan\theta}{6} - 1 = 0$
   j) $2\cos\theta + 1 = \frac{1}{2}$
   k) $4\tan\theta - 7 = 5\tan\theta - 6$

Lesson Summary

In this lesson, you continued to learn about the three primary trigonometric ratios. You learned how to find the reference angle for a given angle, as well as how to determine which angles have the same reference angle. Using this knowledge, you were able to solve trigonometric equations. In the next lesson, you will look at trigonometric ratios again and try to find patterns among them. Before moving onto the next lesson, complete the assignment that follows.

Assignment 7.1

Trigonometric Ratios, Reference Angles, and Trigonometric Equations

Total: 58 marks

1. P(x, y) is a point on the terminal side of angle θ in standard position. Determine $\sin\theta$, $\cos\theta$, and $\tan\theta$ for the following points. Draw a sketch. Show your work. Rationalize any denominators. *(5 marks each, for a total of 20 marks)*

   a) (1, 3)

   b) (5, −7)

continued

Assignment 7.1: Trigonometric Ratios, Reference Angles, and Trigonometric Equations (continued)

c) $(-3, 14)$

d) $(-7, -4)$

continued

Assignment 7.1: Trigonometric Ratios, Reference Angles, and Trigonometric Equations (continued)

2. Without using a calculator, determine whether each value should be positive or negative. Explain. *(1 mark each, for a total of 4 marks)*

   a) cos 113°

   b) sin 265°

   c) tan 179°

   d) sin 81°

3. Given the following angles in standard position, determine in which quadrant the terminal arm of each angle lies. Determine if the sine, cosine, and tangent ratios are positive or negative. Explain your answers. *(2 marks each, for a total of 8 marks)*

   a) $\theta = 96°$

   b) $\theta = 73°$

   c) $\theta = 194°$

   d) $\theta = 303°$

continued

Assignment 7.1: Trigonometric Ratios, Reference Angles, and Trigonometric Equations (continued)

4. Sketch each angle in standard position and find its reference angle. Write the measure of the reference angle. *(2 marks each, for a total of 8 marks)*

   a) 108°

   b) 76°

continued

Assignment 7.1: Trigonometric Ratios, Reference Angles, and Trigonometric Equations (continued)

c) 346°

d) 191°

continued

Assignment 7.1: Trigonometric Ratios, Reference Angles, and Trigonometric Equations (continued)

5. Determine the measure of the 3 other angles in standard position that have the same reference angle as 75°. *(3 marks)*

6. Determine the corresponding angle, θ, in standard position when the reference angle of 15° is reflected in each of the following. Include a diagram in your solution. *(1 mark each, for a total of 3 marks)*

   a) In the y-axis

continued

b) In the x-axis

c) In the y-axis and then the x-axis.

continued

Assignment 7.1: Trigonometric Ratios, Reference Angles, and Trigonometric Equations (continued)

7. Determine the solution for each of the following trigonometric equations over the interval [0°, 360°]. Round to one decimal place. *(2 marks each, for a total of 12 marks)*

   a) $\cos\theta = \frac{-4}{7}$

   b) $\sin\theta - 1 = 0$

   c) $\tan\theta + 2 = 3$

continued

Assignment 7.1: Trigonometric Ratios, Reference Angles, and Trigonometric Equations (continued)

d) $5 \cos \theta = 2$

e) $7 \sin \theta = -1$

f) $\dfrac{\tan \theta}{3} = -2$

Notes

Lesson 3: Patterns in the Sine, Cosine, and Tangent Ratios

Lesson Focus

In this lesson, you will

- ❑ learn how to determine the exact values of any angle with reference angles of 30°, 45°, and 60°
- ❑ learn about the relationship among the coordinates of the points P(x, y), P($-x$, y), P($-x$, $-y$), and P(x, $-y$)
- ❑ learn how to solve application problems involving trigonometric ratios in right triangles

Lesson Introduction

In Lessons 1 and 2, you learned how to determine the sine, cosine, and tangent ratios at a given point in any of the four quadrants. You noticed some patterns between the ratios and their signs. In this lesson, you will begin to discover more patterns among the sine, cosine, and tangent ratios.

Special Right Triangles

For some angles between 0° and 360°, it is possible to find the exact values of the trigonometric ratios. An **exact value** is an answer that does not contain rounded off decimal values. Instead, the answer may contain *radicals*, which are exact value representations of infinite non-repeating decimals (irrational numbers). Consider the following example.

Example 1

Sketch a triangle situated in Quadrant I for the following lengths of x, y, and r. Recall that the value of r is the measure of the hypotenuse or the distance from the origin to the point P(x, y), and $x^2 + y^2 = r^2$. Find the measure of the missing side length of the triangle.

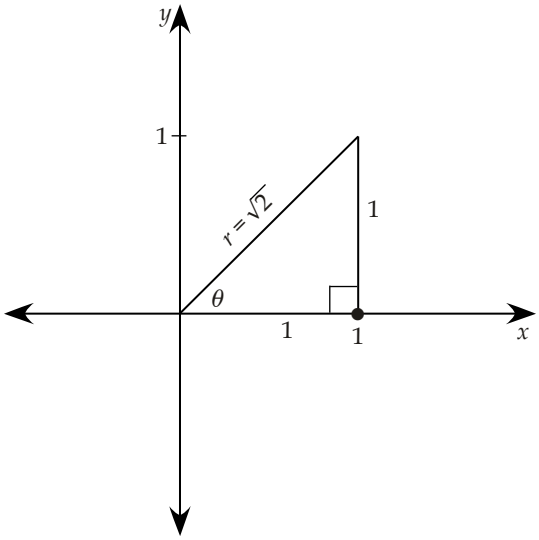
a) $x = 1$, $y = 1$

b) $x = 1$, $r = 2$

c) $y = 1$, $r = 2$

Solution

a) $x = 1, y = 1$

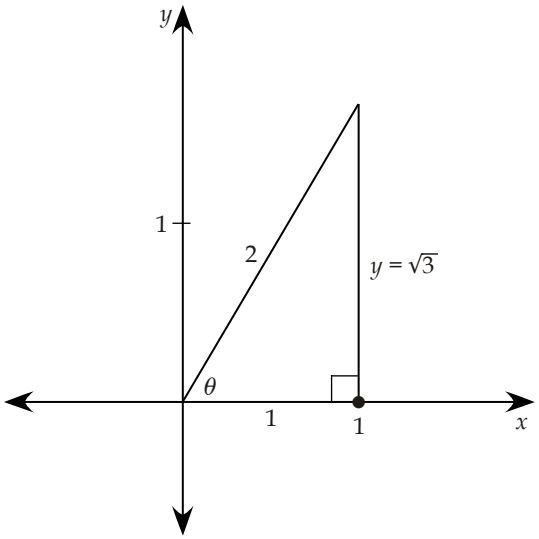


$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{1 + 1}$$
$$r = \sqrt{2}$$

What is the value of θ for this special right triangle? You may have recognized that the triangle is isosceles so the two angles must be equal. The angles of the triangles are 45°, 45°, and 90°. Therefore, $\theta = 45°$.

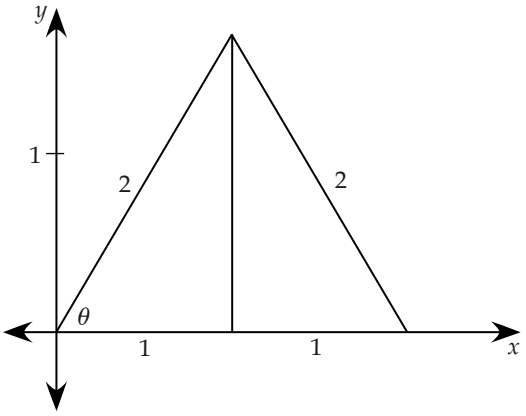
b) $x = 1, r = 2$

One way to sketch the triangle is to find the approximate value of $\sqrt{3}$. Note the difference between "sketching" the length of $\sqrt{3}$ and writing an "exact" value for $\sqrt{3}$.



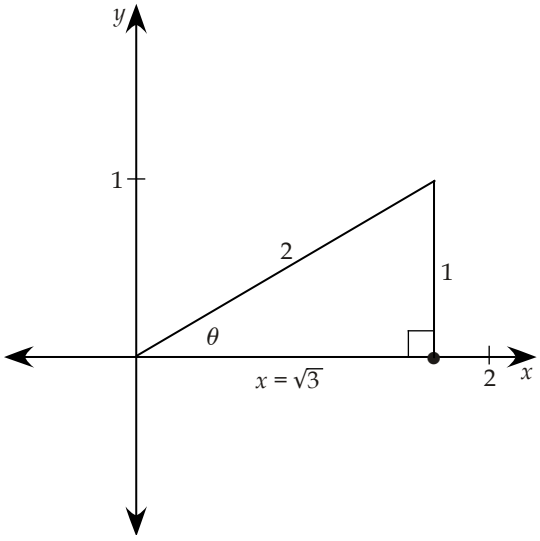
$$y^2 = r^2 - x^2$$
$$y = \sqrt{r^2 - x^2}$$
$$y = \sqrt{4 - 1}$$
$$y = \sqrt{3}$$

What is the value of θ for this special right triangle? This is easier to determine if you think of reflecting the triangle over its vertical side with length of $\sqrt{3}$.



Now you may recognize that the new large triangle is equilateral with all sides that have a length of 2. Therefore, $\theta = 60°$.

c) $y = 1$, $r = 2$



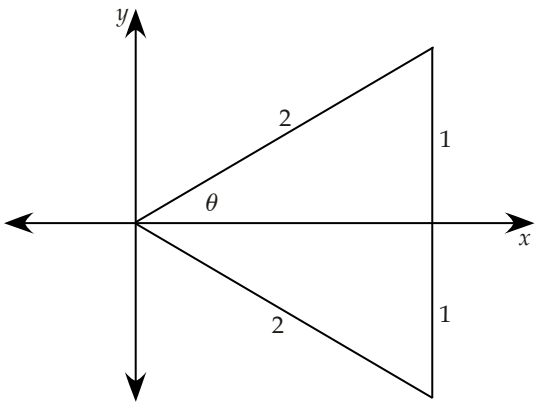
$$x^2 = r^2 - y^2$$

$$x = \sqrt{r^2 - y^2}$$

$$x = \sqrt{4 - 1}$$

$$x = \sqrt{3}$$

What is the value of θ for this special right triangle? This is easier to determine if you think of creating an equilateral triangle by reflecting the triangle over the x-axis.



Now you may recognize that the value of θ is half of one angle of the equilateral triangle. Therefore, $\theta = 30°$.

Now that you have drawn the triangles and determined the missing side lengths, it is possible to find the correct values of the trigonometric ratios for each angle without using your calculator! Consider the following example.

Example 2

Using the diagrams from Example 1, and θ in standard position, find the exact values of trigonometric ratios for $\theta = 45°$, 60°, and 30°.

a) $x = 1, y = 1, r = \sqrt{2}$

b) $x = 1, y = \sqrt{3}, r = 2$

c) $x = \sqrt{3}, y = 1, r = 2$

Solution

a) From Example 1 (a) above,

$$\cos 45° = \frac{x}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\sin 45° = \frac{y}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45° = \frac{y}{x} = \frac{1}{1} = 1$$

Use your calculator to compare the values of cos 45°, sin 45°, and tan 45° to the exact values you found using the triangles.

b) From the Example 1 (b) above,

$$\cos 60° = \frac{x}{r} = \frac{1}{2}$$

$$\sin 60° = \frac{y}{r} = \frac{\sqrt{3}}{2}$$

$$\tan 60° = \frac{y}{x} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

c) From the Example 1 (c) above,

$$\cos 30° = \frac{x}{r} = \frac{\sqrt{3}}{2}$$

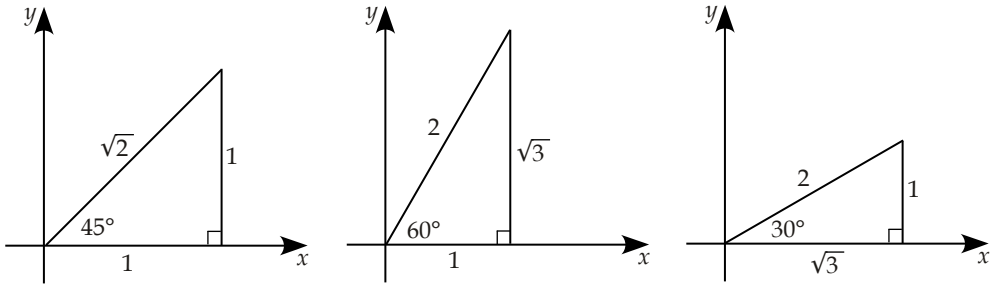
$$\sin 30° = \frac{y}{r} = \frac{1}{2}$$

$$\tan 30° = \frac{y}{x} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

As you can see, it is possible to connect the exact values for the trigonometric ratios to angles 30°, 45°, and 60°. This information can be summarized using the following chart.

θ	$\cos\theta$	$\sin\theta$	$\tan\theta$
30°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$	1
60°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$

To help you find the exact values for any angles in standard position with a reference angle of 30°, 45°, and 60°, remember the side lengths of the three special right triangles.



Can you write all the values in the table above using only these three triangles?

You may want to add information about the three special right triangles to your resource sheet. It is expected that you will remember the triangle side lengths so you can determine exact trigonometric values for $\theta = 30°, 45°$, and 60° and their related angles without your calculator.

Example 3

a) Which angles in the interval [0°, 360°] have a reference angle of:

i) 30°?

ii) 45°?

iii) 60°?

b) Do you think it is possible to find exact trigonometric ratios for each of these special angles?

Solution

a) i) $\theta_r = 30°$

In Quadrant II: $\theta = 180° - \theta_r$

$\theta = 180° - 30°$

$\theta = 150°$

In Quadrant III: $\theta = 180° + \theta_r$

$\theta = 180° + 30°$

$\theta = 210°$

In Quadrant IV: $\theta = 360° - \theta_r$

$\theta = 360° - 30°$

$\theta = 330°$

ii) $\theta_r = 45°$

In Quadrant II: $\theta = 180° - 45° = 135°$

In Quadrant III: $\theta = 180° + 45° = 225°$

In Quadrant IV: $\theta = 360° - 45° = 315°$

iii) $\theta_r = 60°$

In Quadrant II: $\theta = 180° - 60° = 120°$

In Quadrant III: $\theta = 180° + 60° = 240°$

In Quadrant IV: $\theta = 360° - 60° = 300°$

Summary

The "families" of special angles that are related by their reference angles are:

i) 30°, 150°, 210°, 330°

ii) 45°, 135°, 225°, 315°

iii) 60°, 120°, 240°, 300°

b) Yes, it is possible to find exact trigonometric ratios for these special angles as well. This is because the value of the ratio stays the same in each quadrant. The only aspect that changes is the sign of the ratio. This is the same concept that you used in the last lesson.

Consider the following example as it relates to the previous three examples.

Example 4

Draw the following angles in standard position. Label x, y, and r on the corresponding triangle with exact values. Then, complete the following chart for each trigonometric ratio.

a) 120°

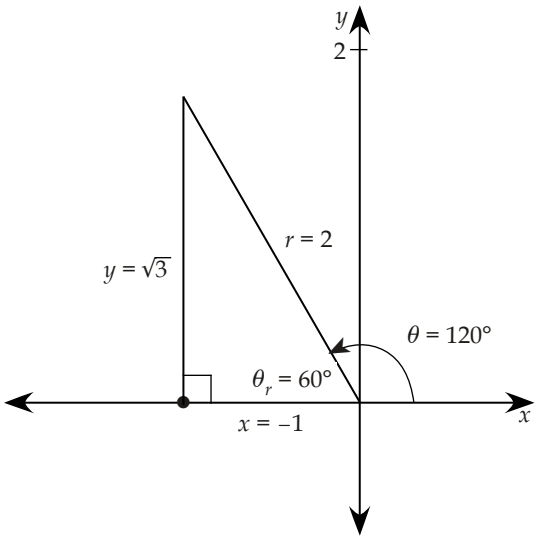
b) 135°

c) 150°

	θ	θ_r	$\cos\theta$	$\sin\theta$	$\tan\theta$
(a)	120°				
(b)	135°				
(c)	150°				

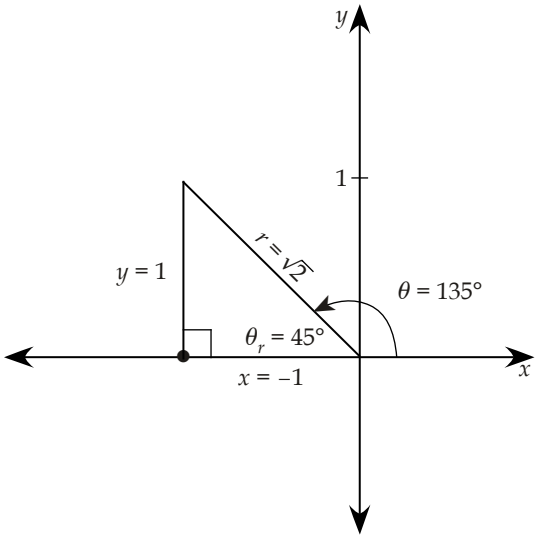
Solution

a)



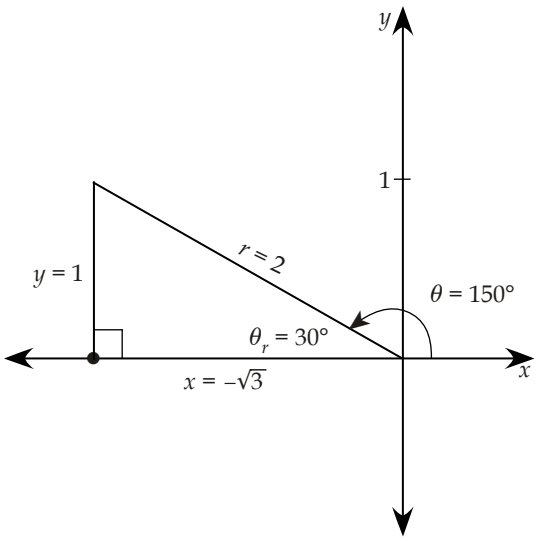
As 120° has 60° as its reference angle, this triangle will have the same side lengths as the triangle drawn in Quadrant I that had an angle of 60° except for the sign of x.

b)



As 135° has 45° as its reference angle, this triangle will have the same side lengths as the triangle drawn in Quadrant I that had an angle of 45° except for the sign of x.

c)



As 150° has 30° as its reference angle, this triangle will have the same side lengths to the triangle drawn in Quadrant I that had an angle of 30° except for the sign of x.

Recall: In Quadrant II, sine is positive while cosine and tangent are negative. As you can see on the diagrams you have drawn, this is because the cosine and tangent ratios contain the variable x.

	θ	θ_r	$\cos\theta$	$\sin\theta$	$\tan\theta$
(a)	120°	60°	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\sqrt{3}$
(b)	135°	45°	$-\frac{1}{\sqrt{2}}$ or $-\frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$	-1
(c)	150°	30°	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\frac{1}{\sqrt{3}}$

If you compare this chart to the chart you completed in Example 2, you will notice that the ratios stayed the same as those of the reference angles, except for their signs. This is one of the many patterns that can be found in the trigonometric ratios.

P(x, y), P($-x$, y), P($-x$, $-y$), and P(x, $-y$)

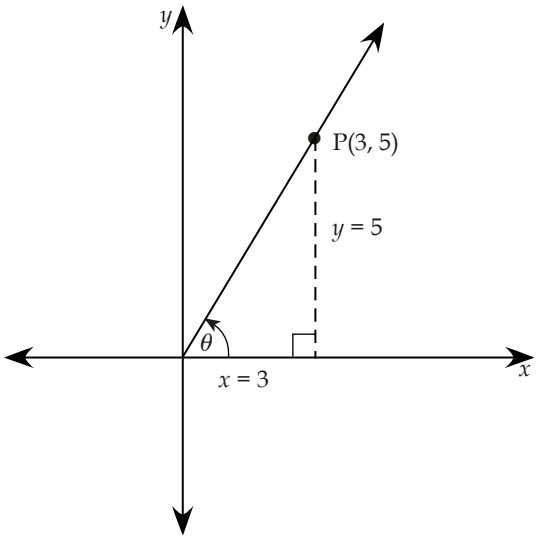
If you are given a point P(x, y), you can determine the three other points that have the same reference angle as the point P(x, y).

Example 5

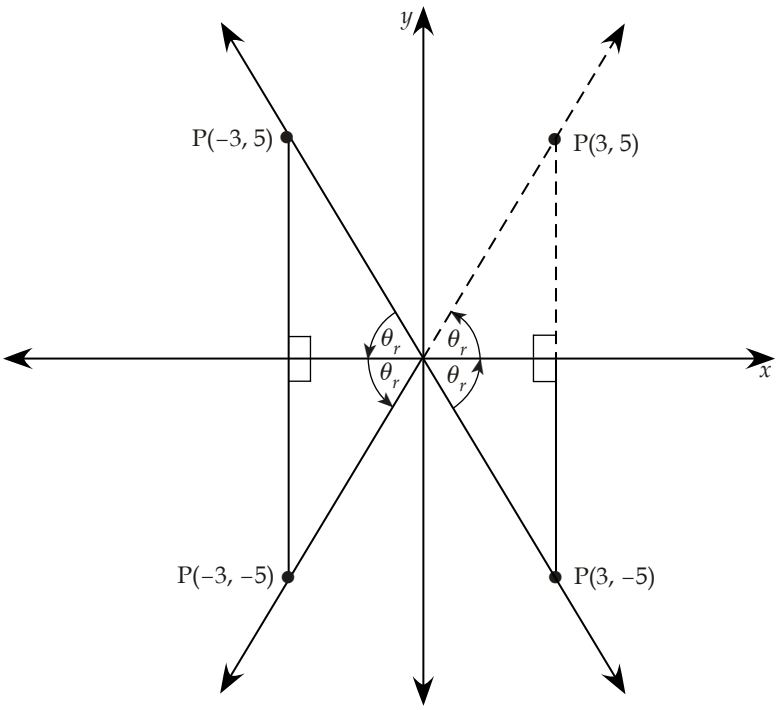
Determine the points on the terminal sides of the angles in standard position that have the same reference angle as the point P(3, 5).

Solution

The point P(3, 5) is in Quadrant I. The angle, θ, and its reference angle, θ_r, are equal. The corresponding right triangle is shown.



In the next diagram, an equivalent triangle is drawn in each of the other three quadrants, showing the corresponding x- and y-coordinates.



Application Problems of Trigonometric Ratios

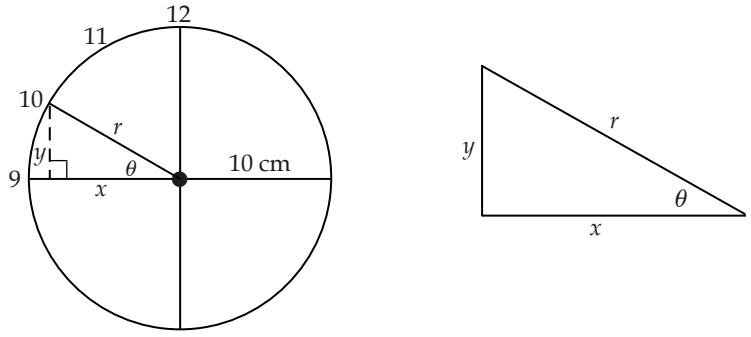
In *Grade 10 Introduction to Applied and Pre-Calculus Mathematics*, you studied right triangle trigonometry and some of its uses. Trigonometric ratios in the coordinate plane have many similar applications. Consider the following example.

Example 6

Determine the exact vertical distance the tip of the hour hand on a clock must travel between 9:00 p.m. and 10:00 p.m. if its length is 10 cm.

Solution

First, you need to sketch a diagram to figure out what values you have and what values you need to find.



From the diagram, you can see that you are looking for the y-value. To find this distance, you first have to realize the hour hand travels clockwise and sweeps out at an angle of 30° every hour. This is because one complete revolution is 360° and there are 12 hours on the clock; 360° ÷ 12° = 30°.

In order to find the vertical distance the hour hand travels, you need to find the y-value. You know the angle and you know the r-value. The y- and r-values both appear in the sine ratio. Thus, you can use the sine ratio to determine the y-value.

$$\sin\theta = \frac{y}{r}$$

$$\sin 30° = \frac{y}{10} \qquad \theta \text{ is } 30°$$

$$y = 10 \sin 30°$$

$$y = 10\left(\frac{1}{2}\right) \qquad 30° \text{ is one of the angles in the special triangles for which you can find an exact sine value without a calculator.}$$

$$y = 5$$

The total vertical distance the hour hand travels is 5 cm.

Learning Activity 7.3

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Jason had 37 hits in 45 times at bat during a baseball season. Sherri had 12 hits in 15 times at bat. Who is the better hitter?
2. Factor: $x^2 + 5x - 6$
3. Evaluate: $\left(6^2\right)^{\frac{1}{2}}$
4. In which quadrants is $\sin\theta > 0$?
5. Find the sum of 12 382 and 14 593.
6. Simplify: $(-3x^2 - 6x + 7) + (2x^2 - 5x)$
7. Two angles in an isosceles triangle measure 40° and 100°. What is the measure of the third angle?
8. Write this decimal as a percent: 0.103

Part B: Patterns in the Sine, Cosine, and Tangent Ratios

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Determine the angles in Quadrants III and IV that have 30°, 45°, and 60° as reference angles.

continued

Learning Activity 7.3 (continued)

2. Fill in the following chart with the exact values for sine, cosine, and tangent. Use the trigonometric ratio for the corresponding reference angle and attach a negative sign appropriately. Try to draw the special triangles from memory and determine the exact values for the trigonometric ratios using the appropriate special triangle.

θ	θ_r	$\cos\theta$	$\sin\theta$	$\tan\theta$
180°				
210°				
225°				
240°				
270°				
300°				
315°				
330°				
360°				

continued

Learning Activity 7.3 (continued)

3. Complete the table below. List the sine, cosine, and tangent ratios for θ. Use your calculator and round to four decimal places.

Angle	cos θ	sin θ	tan θ
0°			
15°			
30°			
45°			
60°			
75°			
90°			

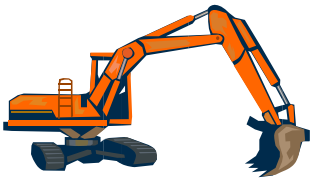
a) What do you notice about the three ratios as the angle increases?

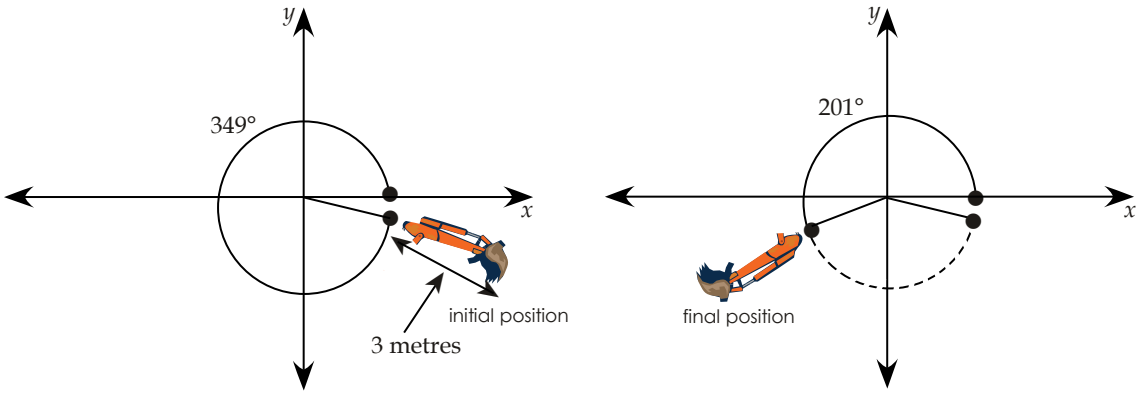
b) What do you notice about the sine and cosine ratios for each angle?

c) Determine the signs of the ratios in Quadrant II.

4. Determine the coordinates of the points on the terminal arm of the angles having 30° as a reference angle if $r = 4$. Write exact values. $0° \leq \theta \leq 360°$

5. A backhoe operator is digging a basement for a new house. To do this, on the first day he starts at the far end of the hole at angle of 349° and digs dirt backwards to end at an angle of 201°. If the length of the arm of the backhoe is 3 metres, what is the horizontal length of the hole that the backhoe operator is digging? Round your answer to 3 decimal places.





Lesson Summary

In this lesson, you discovered patterns among the sine, cosine, and tangent ratios. You discovered that the sign of the tangent ratio depended on the sign of the sine and cosine ratios. You also discovered trigonometric ratios of angles having the same related angle. Using your knowledge of the trigonometric ratios, you also solved application problems involving these ratios. In the next lesson, you will learn how to solve non-right triangles by using the Sine Law.

Notes

Lesson 4: The Sine Law

Lesson Focus

In this lesson, you will

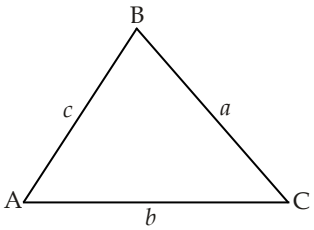
- ❑ learn how to use the Sine Law to solve oblique triangles when you are given the measure of two angles and one side of the triangle
- ❑ learn how to use the Sine Law to solve oblique triangles when you are given the measure of two sides, and one angle opposite one of the sides
- ❑ learn how to prove the Sine Law

Lesson Introduction

In Lessons 1 to 3, you used the definitions of the trigonometric ratios to find formulas for the sine, cosine, and tangent ratios of angles in standard position. All of the triangles included right angles. Triangles that have no right angles are called **oblique triangles**. In this lesson, you will learn how to solve oblique triangles when you are given two angles and one side of a triangle using the Sine Law. As well, you will also learn how to solve oblique triangles when you are given two sides and an angle opposite one of the sides.

Labelling Triangles

As standard notation, we label the angles of a triangle with capital letters, A, B, and C, and their opposite sides with lower-case letters, *a*, *b*, and *c*, as shown in the diagram.



A triangle has six parts: three sides and three angles. To solve an oblique triangle, you need to know the measure of at least one side and any two other parts of the triangle. To solve a triangle means to find the measures of all three of the other missing parts.

The Sine Law can be used to solve the triangles if you are given a side and the angle opposite that side.

The Sine Law

The Sine Law is stated in the form:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Include this formula on your resource sheet.

As you can see from the formula, each ratio has a matching pair, a side and its opposite angle. When you are using this formula, you will only use two ratios at a time. You will never use all three ratios at once. The formula is simply showing that the ratio between any side and the sine of its opposite angle is the same regardless of which side and opposite angle are used.

When solving triangles, a careful sketch is useful as a quick test for the reliability of an answer. **When using a calculator, do not round off until you have reached the end of all calculations.** Try to show as many steps as you can before having to use your calculator.

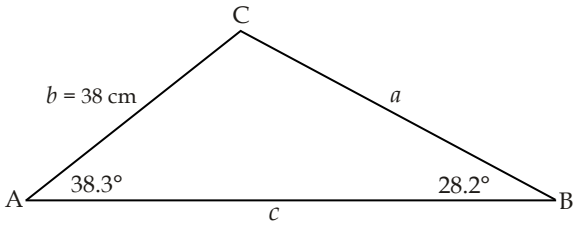
Remember that the longest side is opposite the largest angle and the shortest side is opposite the smallest angle of the triangle.

Example 1

Given ΔABC with $\angle A = 38.3°$, $\angle B = 28.2°$, and $b = 38$ cm, find the remaining angle and sides. Round to 2 decimal places.

Solution

The given information for this oblique triangle includes an angle and its opposite side (that is, angle B and side b). You can use the Sine Law.



$\angle C = 180° - (\angle A + \angle B)$

$\angle C = 180° - (38.3° + 28.2°)$

$\angle C = 113.5°$

Solving for a:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin 38.3°} = \frac{38}{\sin 28.2°}$$

$$a = \frac{38}{\sin 28.2°} \times \sin 38.3° \quad \text{Multiply both sides by } \sin 38.3°.$$

$$a = 49.84 \text{ cm}$$

Solving for c:

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

$$\frac{c}{\sin 113.5°} = \frac{38}{\sin 28.2°}$$

$$c = \frac{38}{\sin 28.2°} \times \sin 113.5°$$

$$c = 73.75 \text{ cm}$$

The measure of the remaining angle is 113.5° and sides a and c are 49.84 cm and 73.75 cm, respectively.

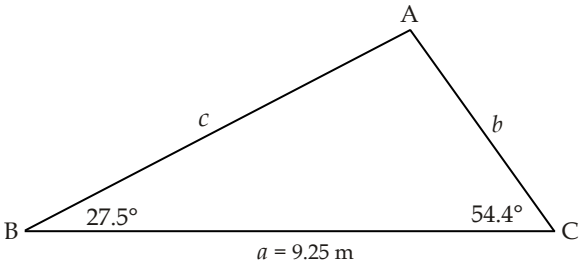
Example 2

Given ΔABC with ∠B = 27.5°, ∠C = 54.4°, and a = 9.25 m, find the remaining angle and sides. Round the measures of the sides to two decimal places and the angle to one decimal place.

Solution

You do not know both a side and its opposite angle so it may appear that you cannot use the Sine Law. However, it is possible to find angle A and you are given the value of its opposite side, a.

$$\angle A = 180° - (27.5° + 54.4°)$$
$$= 98.1°$$



Using the Sine Law and solving for b:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{9.25}{\sin 98.1°} = \frac{b}{\sin 27.5°}$$

$$b = \frac{9.25 \sin 27.5°}{\sin 98.1°}$$

$$b = 4.31 \text{ m}$$

Solving for c:

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

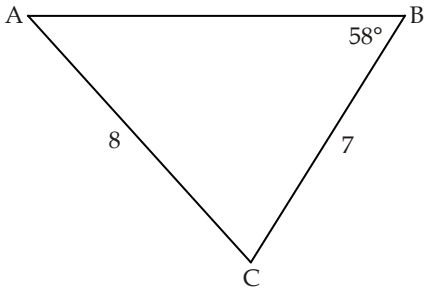
$$\frac{9.25}{\sin 98.1°} = \frac{c}{\sin 54.4°}$$

$$c = \frac{9.25 \sin 54.4°}{\sin 98.1°}$$

$$c = 7.60 \text{ m}$$

Example 3

Solve acute-angled ΔABC by finding the measure of the other two angles of ΔABC and the missing side length, given the diagram below. Round to one decimal place.



Solution

As you are given the measure of an angle that is opposite one of the given side lengths, you can use the Sine Law to solve this triangle.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{7}{\sin A} = \frac{8}{\sin 58°}$$

$$7 \sin 58° = 8 \sin A$$

$$\sin A = \frac{(7)(\sin 58°)}{8}$$

$$\angle A_r = \sin^{-1}\left(\frac{7 \sin 58°}{8}\right)$$

$$\angle A_r = 47.9°$$

Since you are given that ΔABC is acute-angled, then you don't need to find the measure of the obtuse angle for ∠A.

$\therefore \angle A = 47.9°$

$\angle C = 180° - (\angle A + \angle B)$

$\angle C = 180° - (47.9° + 58°)$

$\angle C = 74.1°$

To find the missing side *c*, use Sine Law again, using ∠C.

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{8}{\sin 58°} = \frac{c}{\sin 74.1°}$$

$$c = \frac{8 \sin 74.1°}{\sin 58°}$$

$$c = 9.1$$

The missing values are:

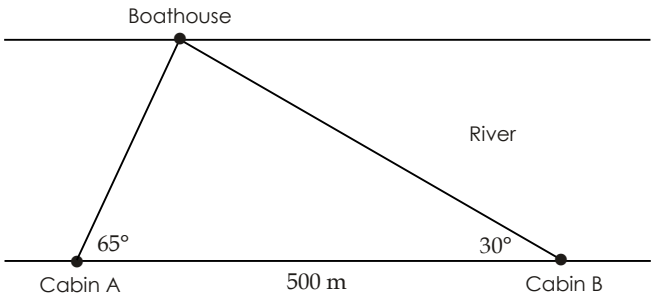
$\angle A = 47.9°$

$\angle C = 74.1°$

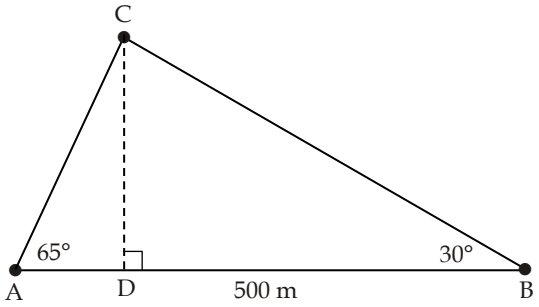
$c = 9.1$

Example 4

Two cabins are located on the same side of a river at its edge. The cabins are 500 m apart. A boathouse, also at the edge of this river, is located on the opposite side of the river. Use the illustration below to find the width of the river to the nearest metre.



Solution



You need to find the distance CD. You can find ∠ACB and then use the Sine Law as part of your solution.

$$\angle ACB = 180° - (65° + 30°) = 85°$$

In ΔACB: $\dfrac{AC}{\sin 30°} = \dfrac{500}{\sin 85°}$

$$AC = \frac{500 \sin 30°}{\sin 85°}$$

$$AC = 250.954\ 959\ 4$$

Now use right triangle ACD to find CD.

$$\sin 65° = \frac{CD}{AC}$$

$$CD = AC \sin 65°$$

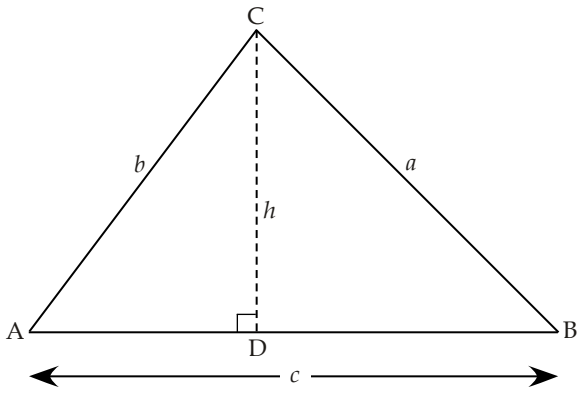
$$CD = 250.954\ 959\ 4 \sin 65°$$

$$CD = 227.442\ 433\ 9$$

The river is 227 m wide.

Proof of the Sine Law

You can prove the Sine Law by using the area of ΔABC with height, h.



In right ΔACD,

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{h}{b}$$

Thus, $h = b \sin A$.

In right ΔBCD,

$$\sin B = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{h}{a}$$

Thus, $h = a \sin B$.

Since the same variable, h, is defined in two different ways,

$h = b \sin A$ and $h = a \sin B$

then we can write:

$b \sin A = a \sin B$.

Now make an equivalent equation by dividing both sides by (sin A)(sin B).

$$\frac{b \sin A}{(\sin A)(\sin B)} = \frac{a \sin B}{(\sin A)(\sin B)}$$

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

If you use either AC or BC to be the base instead, then you will get the third ratio, $\frac{c}{\sin C}$, as shown below.

Using AC as the base, you can drawn an altitude perpendicular to AC through B. You will find there are two equivalent expressions for this altitude:

$h_2 = a \sin C$

$h_2 = c \sin A$

Therefore, a sin C = c sin A.

$$\frac{a \sin C}{\sin A \sin C} = \frac{c \sin A}{\sin A \sin C}$$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

Thus, all three rations are equivalent:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

and the Sine Law has been shown to be true.

To complete the proof of the Sine Law in the following learning activity that follows (question 5), you will be asked to explain the steps in the proof.

Remember: To use the Sine Law, you must know at least one side and the angle opposite that side. That is, in the formula, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$, one of the ratios must be complete.

Learning Activity 7.4

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. You received $\frac{22}{25}$ on a math test. What was your percentage?
2. Find the common difference in the following pattern: 8, 2, −4, −10, . . .
3. In which quadrants is tan > 0?
4. What is the price of one box of Kleenex if 8 boxes of Kleenex cost $16.64?
5. The side lengths of a right-angled triangle are 15, 17, and 8. Which side length is the hypotenuse?
6. Simplify: $\frac{3^3 + 3}{3^2}$
7. Express as a radical: $9^{\frac{2}{3}}$. Do not simplify.
8. Find the slope of the line that contains the points (8, −3) and (−6, 12).

continued

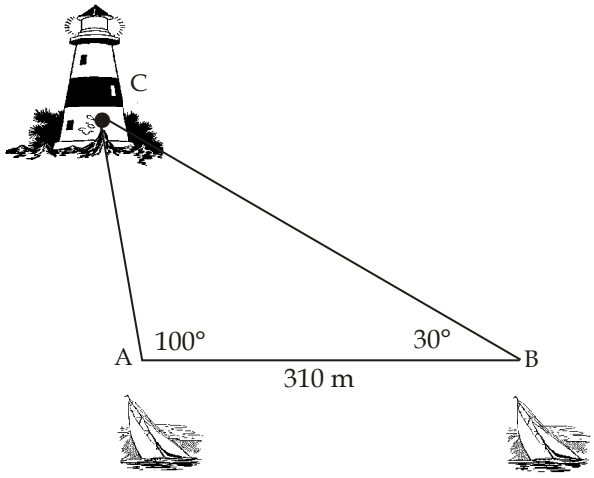
Learning Activity 7.4 (continued)

Part B: The Sine Law

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Solve the following triangles for all missing sides and angles. Round to 2 decimal places.
   a) In ΔABC, $\angle A = 67°$, $\angle B = 73°$, and $b = 42$.
   b) In ΔABC, $\angle A = 150°$, $a = 200$, and $\angle C = 20°$.

2. A lighthouse is visible to two sailboats 310 m apart. Use the angle measures shown to find the distance of each sailboat from the lighthouse. Round to two decimal places.



3. A kite is spotted simultaneously by two observers at points A and B, 3 km apart. The angles of elevation of the kite from A and B are 25° and 75°, respectively. How high above the ground to the nearest tenth is the kite if the kite is in the same vertical plane as A and B and if the point on the ground directly below the kite is:
   a) between A and B? (Include a diagram.)
   b) not between A and B? (Include a diagram.)

4. To calculate the width of the Assiniboine River south of the Legislative Building, a surveyor marks a baseline, $\overline{AB}$, 200 m in length along the north riverbank. From each end of the baseline, an apartment, C, is sighted on the south bank of the river, making angles of 40° and 20° with the baseline. How wide is the river? Round to the nearest hundredth.

continued

Learning Activity 7.4 (continued)

5. Explain the steps in the given proof of the Sine Law.
   a) $h_1 = b \sin A$ and $h_1 = a \sin B$
   b) $b \sin A = a \sin B$
   c) $\dfrac{b}{\sin B} = \dfrac{a}{\sin A}$
   d) $h_2 = a \sin C$ and $h_2 = c \sin A$
   e) $a \sin C = c \sin A$
   f) $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$
   g) $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Lesson Summary

In this lesson, you learned how to find the missing sides and angles of a triangle using the Sine Law when you were given at least one side and the angle opposite that side. You also saw how the Sine Law could be used in application problems involving triangles. In the next lesson, you will be learning about the Cosine Law. The Cosine Law will help you solve triangles for their missing angles and sides where the Sine Law does not apply.

Notes

Assignment 7.2

Patterns in Trigonometric Ratios and the Sine Law

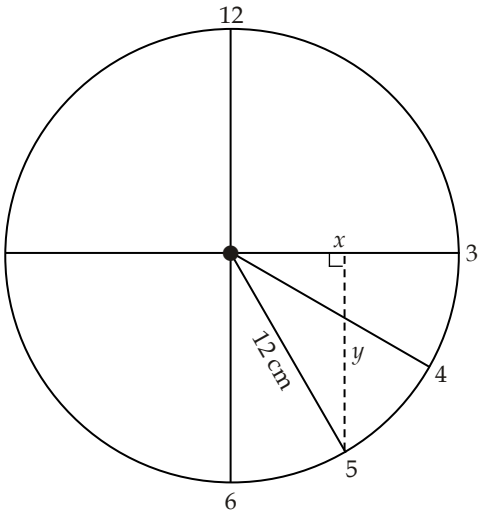
Total: 25 marks

1. Determine the coordinates of the points on the terminal arm of all angles that have $\theta_r = 25°$ as a reference angle if the distance from the point to the origin is 16. Round to two decimal places. Sketch a diagram. *(5 marks)*

continued

Assignment 7.2: Patterns in Trigonometric Ratios and the Sine Law (continued)

2. Determine the ***exact*** vertical distance the hour hand on a clock must travel between 3 p.m. and 5 p.m. if the radius of the clock is 12 cm. *(5 marks)*



continued

Assignment 7.2: Patterns in Trigonometric Ratios and the Sine Law (continued)

3. Solve the following triangles for all missing sides and angles. Round to 2 decimal places. *(5 marks each, for a total of 10 marks)*

   a) In ΔABC, $\angle A = 54°$, $\angle B = 38°$, and $a = 21$.

   b) In ΔABC, $\angle B = 78°$, $b = 37$, and $\angle C = 54°$.

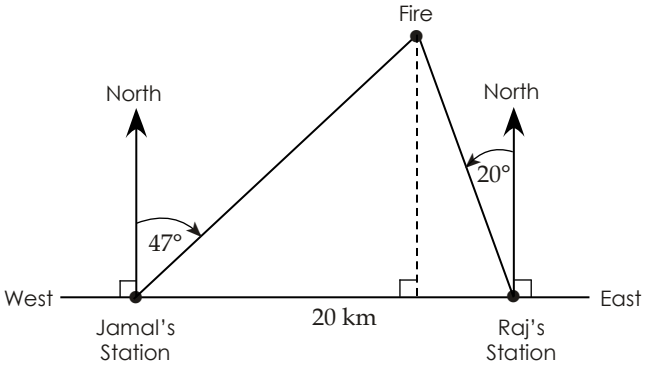
continued

Assignment 7.2: Patterns in Trigonometric Ratios and the Sine Law (continued)

4. Fire ranger Jamal is 20 km due west of fire ranger Raj. They spot a fire and report the sightings as follows:

   Jamal reports that the fire is N 47° E of his station.

   Raj reports that the fire is N 20° W of his station.



   a) How far is the fire from Jamal's station? Round to the nearest tenth. *(2.5 marks)*

continued

Assignment 7.2: Patterns in Trigonometric Ratios and the Sine Law (continued)

b) What is the shortest distance from the fire to the east-west highway joining Jamal's and Raj's stations? Round to the nearest tenth. *(2.5 marks)*

Notes

Lesson 5: The Cosine Law

Lesson Focus

In this lesson, you will

- ❑ learn how to use the Cosine Law to find the length of a side of a triangle when you are given the measure of two sides and the angle between these sides
- ❑ learn how to use the Cosine Law to find the measure of any angle of a triangle when you are given the measure of each of the three sides of the triangle

Lesson Introduction

In the last lesson, you learned how to solve triangles using the Sine Law. You learned that you could use the Sine Law when you were given at least one side and the angle opposite that side. The Cosine Law is helpful in solving triangles when you are not given that information about the triangle. Thus, the Cosine Law can help you solve triangles where the Sine Law cannot.

The Cosine Law

If you are given three sides or two sides and the included angle between the two sides, then only one part of each ratio included in the Sine Law would be known. You would not know both a side and its opposite angle. Therefore, you cannot rely on the Sine Law to solve these two cases. Instead, you need the Cosine Law. The proof of the Cosine Law is shown in Learning Activity 7.5, Question 7.

The Cosine Law can be stated in the following way:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

If $\angle A = 90°$, then $\cos 90° = 0$ and the expression becomes $2bc \cos 90°$ is zero.

Notice that the first part of the formula resembles the Pythagorean theorem. The last part, $2bc \cos A$, calculates an adjustment when $\angle A \neq 90°$.

The placing of the letters for triangle ABC are somewhat arbitrary. The formula could be written as:

$c^2 = a^2 + b^2 - 2ab \cos C$

or

$b^2 = a^2 + c^2 - 2ac \cos B$

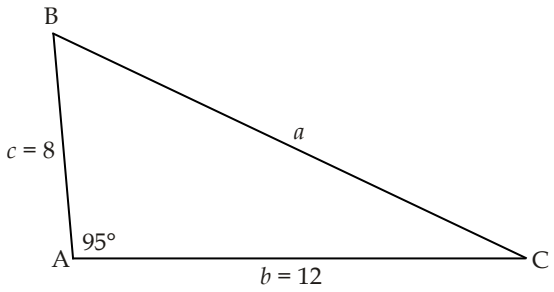
It is important that the side opposite to the angle in the Cosine Law is the side on the LHS of the equal sign.

Include this formula on your resource sheet. You will practice using this formula in Example 1.

Example 1

In ΔABC, AC = 12, AB = 8, $\angle A = 95°$. Find the length of $\overline{BC}$ rounded to two decimal places.

Solution



From the diagram, you can see there is no matching pair of a side and its opposite angle. Thus, cannot use the Sine Law but you can use the Cosine Law to find the side.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$a^2 = 12^2 + 8^2 - 2(12)(8) \cos 95°$$

$$a^2 = 224.733\ 902\ 6$$

$$\sqrt{a^2} = \sqrt{224.733\ 902\ 6}$$

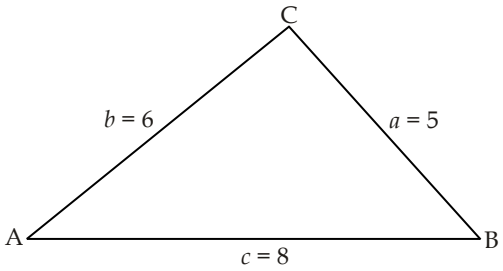
$$a = 14.99$$

Notice that the Cosine Law was used to find the side opposite the given angle of 95°.

Example 2

Given three sides of ΔABC to be $a = 5$, $b = 6$, and $c = 8$, find the three angles of the triangle. Round to the nearest degree.

Solution:



Although it doesn't matter which angle you find first, solving for the largest may give you more information. Thus, solve for ∠C first. You will need to use the Cosine Law for angles.

To find angle C opposite the side of length 8, write:

$8^2 = 6^2 + 5^2 = 2(6)(5) \cos C$

Now carefully use algebra skills to solve for cos C:

$$8^2 - 6^2 - 5^2 - 2(6)(5) \cos C$$

$$3 = -60 \cos C$$

$$\frac{3}{-60} = \cos C$$

$$-0.05 = \cos C$$

$$C = \cos^{-1}(-0.05)$$

$$C = 92.9°$$

Before continuing, notice cos C is negative, which means it is a Quadrant II angle so must be obtuse. Do not make the mistake of changing the cos inverse argument to a positive value as you did when finding a reference angle for an angle of rotation.

The side on the LHS of the Cosine Law must be the side opposite the angle used in the Cosine Law.

To find angle A opposite the smallest side of length 5, write:

$5^2 = 8^2 + 6^2 - 2(8)(6) \cos A$

Use algebra to solve for cos A:

$$5^2 - 8^2 - 6^2 = -2(8)(6)\cos A$$
$$-75 = -96\cos A$$
$$+0.78125 = \cos A$$
$$A = \cos^{-1}(0.78125)$$
$$A = 38.6°$$

Note that cos A > 0 so ∠A is acute.

Solving for ∠B:

$$\angle B = 180° - \angle A - \angle C$$
$$\angle B = 180° - 38.6° - 92.9°$$
$$\angle B = 48.5°$$

Thus, the three angles of ΔABC are ∠A = 38.6°, ∠B = 48.5°, and ∠C = 92.9°.

The formula for the Cosine Law can be rearranged using your algebra skills when you are solving for an angle.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$a^2 + 2bc\cos A = b^2 + c^2$ Add $2bc$ cos A to both sides.

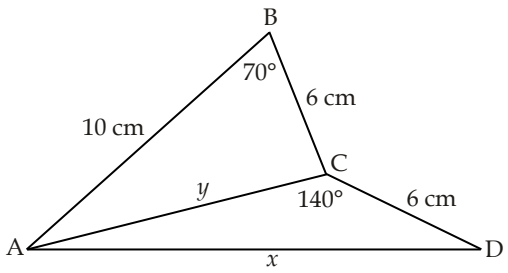
$2bc\cos A = b^2 + c^2 - a^2$ Subtract a^2 from both sides.

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ Divide both sides by $2bc$.

Now let's try more complicated examples.

Example 3

Find the length of x in the diagram. Round to two decimal places.



Solution

In ΔABC, you do not have a matching pair of a side and its opposite angle. Thus, use the Cosine Law for sides to find y.

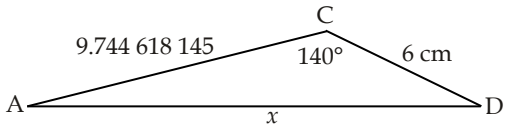
$$y^2 = 6^2 + 10^2 - 2(6)(10)\left(\cos 70°\right)$$

$$y^2 = 94.957\ 582\ 8$$

$$\sqrt{y^2} = \sqrt{94.957\ 582\ 8}$$

$$y = 9.744\ 618\ 145$$

Now use the value of y to solve for x in ΔACD. Since there is no matching pair of a side and its opposite angle, use the Cosine Law for sides.



$$x^2 = 9.744\ 618\ 145^2 + 6^2 - 2\left(9.744\ 618\ 145\right)(6)\left(\cos 140°\right)$$

$$x^2 = 220.535\ 309\ 8$$

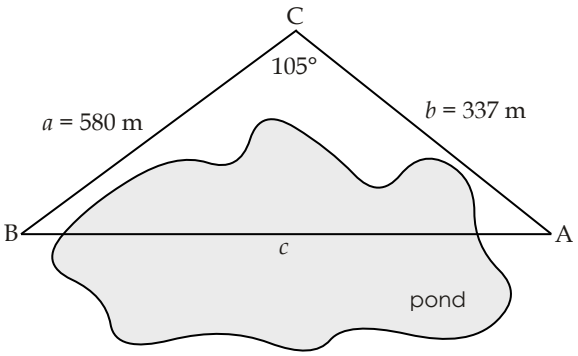
$$x = 14.850\ 431\ 3$$

The length of x is 14.85 cm.

Example 4

An electric transmission line is planned to go directly over a pond. The power line will be supported by posts at points A and B. A surveyor measures the distance from B to point C as 580 m, the distance from A to point C as 337 m, and ∠BCA as 105°. What is the distance in metres from post A to post B rounded to the nearest hundredth?

Solution



You are not given a matching pair of a side and its opposite angle. Therefore, you can use the Cosine Law for sides to solve for side c.

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$c^2 = 580^2 + 337^2 - 2(580)(337) \cos 105°$$

$$c^2 = 551\,146.541\,1$$

$$c = \sqrt{551\,146.541\,1}$$

$$c = 742.392\,444\,1$$

Thus, the distance from post A to post B is 742.39 m.

Learning Activity 7.5

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. In which quadrant are $\sin\theta$, $\cos\theta$, and $\tan\theta$ all greater than zero?
2. What is the probability of rolling an even number on a number cube?
3. Calculate the tax (13%) on an item that costs $43.99.
4. The two legs of a right-angled triangle measure 6 cm and 8 cm. What is the measure of the hypotenuse?
5. What is the value of the discriminant of the quadratic equation $2x^2 - 3x + 6 = 0$?
6. Express in exponential form: $\sqrt{12x^3y^7}$
7. Find the midpoint between the two points $(7, -1)$ and $(-8, 15)$.
8. State the independent and dependent variables for the following situation: "Jace discovers that the more he studies for a test, the higher his mark is."

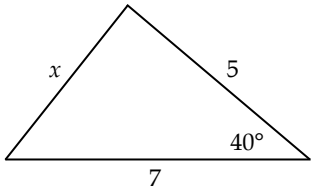
continued

Part B: The Cosine Law

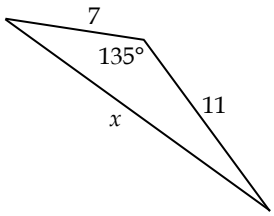
Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Find the length of x in each diagram. Round to the nearest tenth.

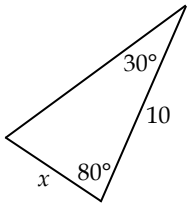
a)



b)



c)



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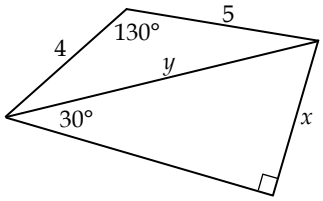
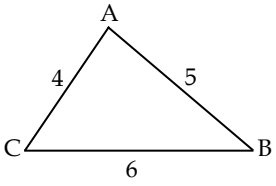
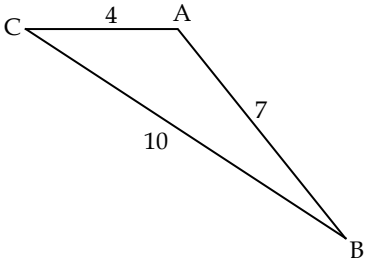


2. Find the measure of ∠A. Round to the nearest tenth.

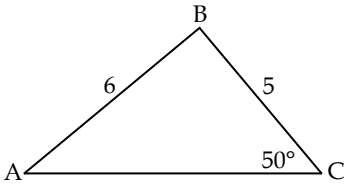
a)



b)



c)

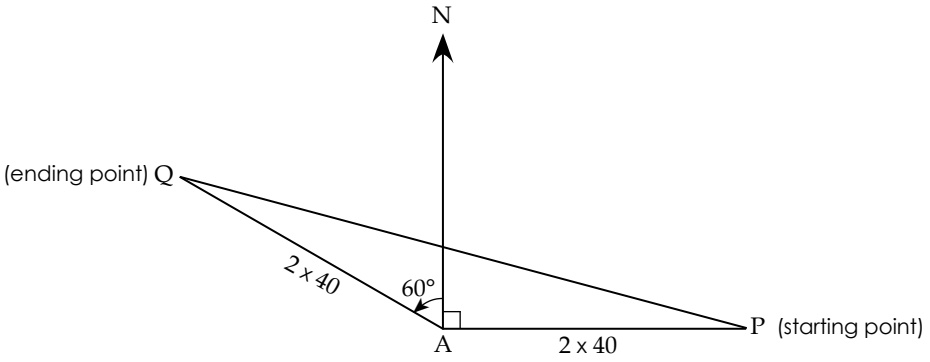


3. An isosceles triangle has sides of length 10 cm, 10 cm, and 6 cm. What is the measure of each angle of this triangle? Round to the nearest tenth.
4. Janice spots a bird on the roof of her apartment block with an angle of elevation of 35°. She moves 15 m closer to the apartment block and now the angle of elevation is 40°. How far above the ground is the bird? (Assume the ground to be level and Janice walked directly toward the bird on the roof.) Round to the nearest tenth.

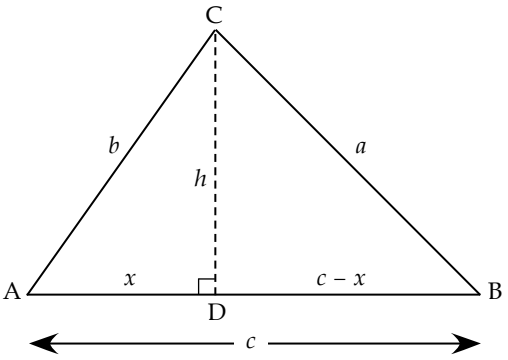
continued

Learning Activity 7.5 (continued)

5. A ship leaves port at 12 noon and sails west at 40 km/h. At 2:00 p.m., it changes direction and sails N 60° W for another two hours. How far is the ship from the port? Round to the nearest tenth.



6. The pitcher's mound on a slo-pitch softball diamond is 15.2 m from home plate and the distance between the bases is 19.8 m. How far is the pitcher's mound from first base if the angle at home plate between the pitcher's mound and first base is 45°? Round to the nearest hundredth.
7. Prove the Law of Cosines for $a^2 = b^2 + c^2 - 2bc \cos A$ using the diagram below.



Proof:

The proof begins by using the Pythagorean theorem for each right triangle:

in ΔBCD, $a^2 = h^2 + (c - x)^2$

in ΔACD, $b^2 = h^2 + x^2$

The next step is to subtract the two equations to eliminate h^2.

Now, finish the proof.

Lesson Summary

In this lesson, you learned how to use the Cosine Law. The Cosine Law is useful in solving triangles where the Sine Law cannot be used. In the next lesson, you will be learning about the ambiguous case. Sometimes when you are solving oblique triangles, there may be no solution, one solution, or two solutions. You will learn about this in more detail in the next lesson.

Lesson 6: The Ambiguous Case

Lesson Focus

In this lesson, you will

- learn how to solve any triangle when two sides of the triangle and an angle opposite one of the sides are given

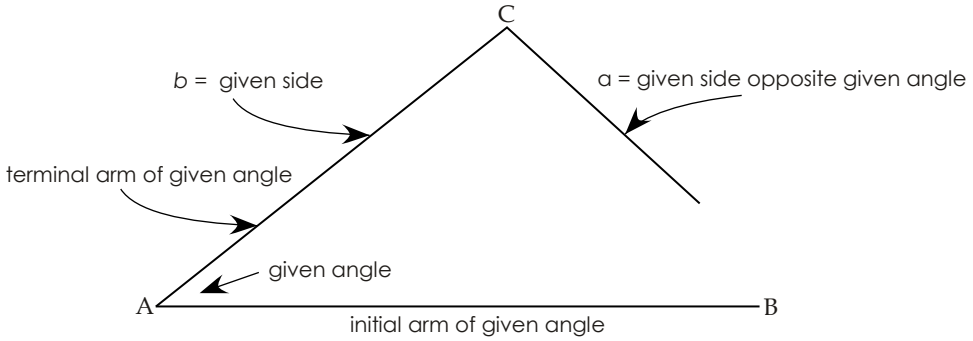
Lesson Introduction

In Lesson 4, you used the Sine Law to solve triangles when you are given two sides of the triangle and an angle opposite one of those sides. You were also told that the unknown angle(s) was acute. However, when you are given this type of information without knowing if the unknown angle is acute or obtuse, the ambiguous case may occur. In the ambiguous case, there are three possible outcomes: no solution, one solution, or two solutions. This means that there could be no possible triangles with the given measurements, only one possible triangle with the given measurements, or two possible triangles with the given measurements. Since the number of triangles possible is not clear, the case involving side-side-angle is called the ambiguous case.

The Ambiguous Case

To understand why the ambiguous case occurs, it is easiest to look at different cases.

In each example, you are given two sides and an angle opposite one of those sides. The information will be placed on the triangle in the same way so that you can make comparisons more easily.



- The given angle, ∠A, will be placed in the lower left corner.
- The side opposite this angle will be partially drawn and its measure indicated, side *a*.
- The base of the triangle will be considered the initial arm of the given angle. You don't know how long this side is or the measure of its opposite angle, ∠C.
- The terminal arm of the given angle will be the second given side of the triangle, side *b*.

When the Given Angle Is Acute

As you can see by the previous diagram, if the side opposite the given angle is too short, then no triangle is possible. You may also be able to visualize that one or two triangles may be possible depending on the length of the side opposite the given angle.

Each of the first four examples illustrates the possible solution(s) for such a triangle when the given angle is **acute**.

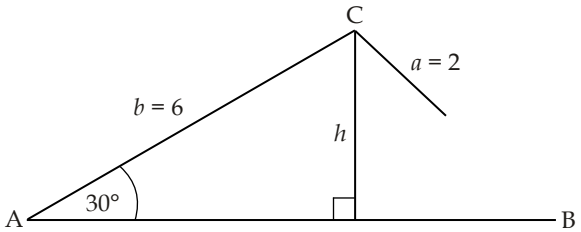
As you will see from the examples, a problem will occur when you try to determine whether the angle at B is acute, right, or obtuse.

Example 1: No Solution

In ΔABC, $a = 2$, $b = 6$, ∠A = 30°. Solve the triangle.

Solution

You are not given the length of $\overline{AB}$, so you do not know where to place B. Also, you do not know where to join C and B. Thus, you need to make a sketch with the side opposite the given angle being incomplete.



Since $a = 2$ and $b = 6$, we have the case where $a < b$.

From this figure, it appears that side *a* is too short to form any triangle. To verify this, we solve for *h* (the height of the triangle, ΔABC).

$$\sin 30° = \frac{h}{6}$$

$$h = 6 \sin 30°$$

$$h = 3$$

Since side a, opposite the given angle A, is shorter than the height of the triangle, it is not possible to draw a triangle. Thus, there is no solution.

Conclusion: If the side opposite the given angle is smaller than the calculated height of the triangle, then no such triangle exists.

Note: In general, the height of the triangle can be found using the formula:

$$\sin A = \frac{h}{b} \text{ or } h = b \sin A$$

Add the above conclusion as well as the formula for finding the height of a triangle to your resource sheet.

An alternate method to reach this conclusion involves the use of the Sine Law.

For the parameters given, check:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{2}{\sin 30°} = \frac{6}{\sin B}$$

$$\sin B = \frac{6 \sin 30°}{2}$$

$$\sin B = 1.5$$

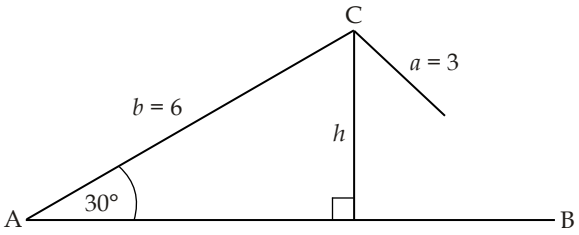
$$B = \sin^{-1}(1.5)$$

You get an error on your calculator since the maximum value for $\sin \theta$ is 1. Thus, no triangle is possible.

Example 2: Right Triangle; One Solution; ∠B = 90°

In ΔABC, a = 3, b = 6, ∠A = 30°. Solve the triangle.

Solution

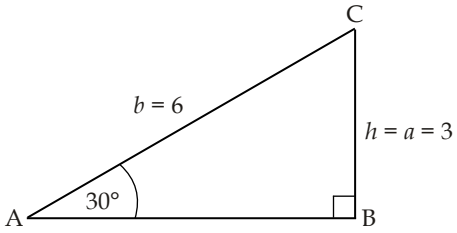


Since a = 3 and b = 6, we have $a < b$.

To determine if a solution is possible, find the height of the triangle.

$h = b \sin A = 6 \sin 30° = 3$

Then, the height is the same length as the side opposite the given angle. Thus, the triangle must be a right triangle and ∠B must be 90°. You can now draw the triangle with the sides completed.



You can now solve for the remaining parts of the triangle.

$\angle C = 180° - 90° - 30° = 60°$

$\cos 30° = \frac{c}{6}$

$c = 6 \cos 30°$

$c = 5.2$

Since it is a right triangle, use $\cos A = \frac{\text{adjacent}}{\text{hypotenuse}}$.

Conclusion: The solution will be one right triangle if the side opposite the given angle is equal to the height of the triangle.

Include the above conclusion on your resource sheet.

An alternate method to reach this conclusion involves the use of the Sine Law.

For the parameters given, check:

$$\frac{3}{\sin 30°} = \frac{6}{\sin B}$$

$$\sin B = \frac{6 \sin 30°}{3}$$

$$\sin B = 1$$

$$B = \sin^{-1}(1)$$

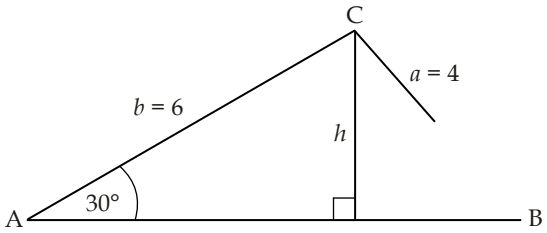
$$B = 90°$$

You get a right angle, thus one right triangle is possible.

Example 3: Oblique Triangle; Two Solutions; ∠B Is Acute or Obtuse

In ΔABC, $a = 4$, $b = 6$, ∠A = 30°. Solve the triangle.

Solution



Since $a = 4$ and $b = 6$, we have $a < b$.

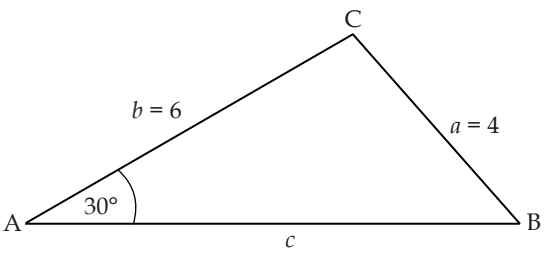
To determine if we have a solution, we will find the height.

$h = b \sin A = 6 \sin 30° = 3$

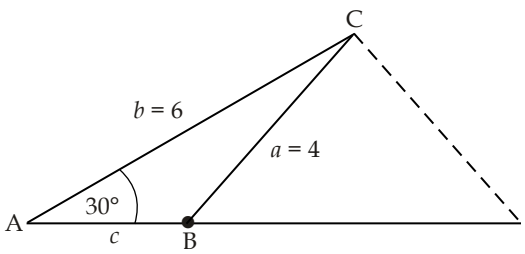
Thus, we have the relationship:

$a < b$ and $a > h$

Since the side opposite the given angle is now bigger than the height but still smaller than the second given side, we will have two possible triangles.



or



To solve the triangle, we use the Sine Law:

$$\frac{4}{\sin 30°} = \frac{6}{\sin B}$$

$$4 \sin B = 6 \sin 30°$$

$$\sin B = \frac{6 \sin 30°}{4} = \frac{3}{4}$$

$$\angle B_r = \sin^{-1}\left(\frac{3}{4}\right)$$

$$\angle B_r = 48.6°$$

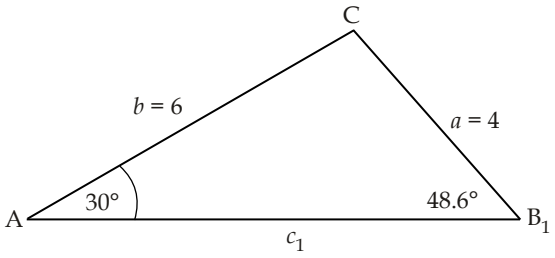
When sin B > 0, you know that ∠B can be acute (Quadrant I), or it can be obtuse (Quadrant II). Thus, there are two possible solutions for ∠B and two possible triangles.

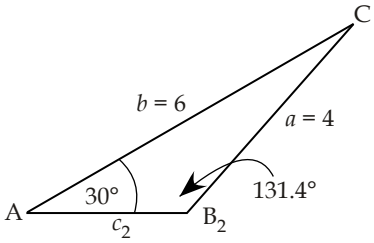
$\angle B_1 = \angle B_r$ or $\angle B_2 = 180° - \angle B_r$

$\angle B_1 = 48.6°$ or $\angle B_2 = 180° - 48.6°$

$\angle B_2 = 131.4°$

Sketch the two possible triangles. One of them will be the triangle drawn with $\angle B_1$ acute, and the other will have $\angle B_2$ as an obtuse angle. Notice how short c_2 must be in comparison with c_1.





Now, finish solving each triangle separately.

$\angle C_1 = 180° - 30° - 48.6° = 101.4°$ $\angle C_2 = 180° - 30° - 131.4° = 18.6°$

$$\frac{4}{\sin 30°} = \frac{c_1}{\sin 101.4°} \qquad \frac{4}{\sin 30°} = \frac{c_2}{\sin 18.6°}$$

$$c_1 = \frac{4 \sin 101.4°}{\sin 30°} \qquad c_1 = \frac{4 \sin 18.6°}{\sin 30°}$$

$c_1 = 7.8$ $c_1 = 2.6$

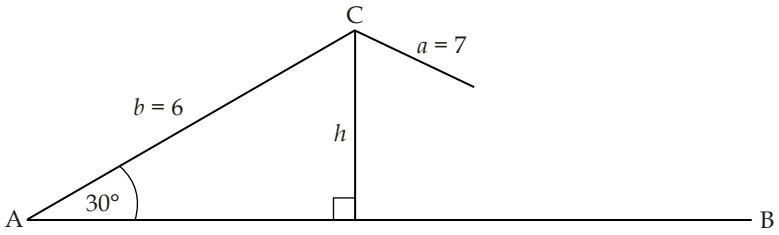
Conclusion: There will be two solutions if the side opposite the given angle is smaller than the second given side and bigger than the height of the triangle.

Include the above conclusion on your resource sheet.

Example 4: Oblique Triangle; One Solution; ∠B Is Acute

In ΔABC, $a = 7$, $b = 6$, ∠A = 30°. Solve the triangle.

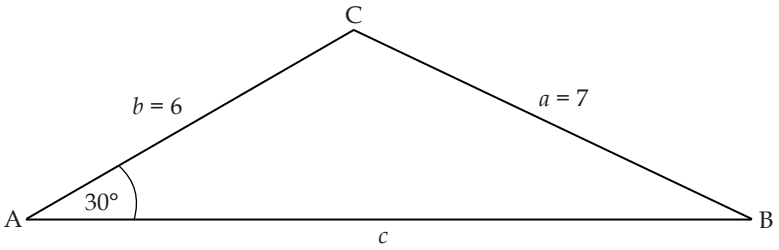
Solution



Since $a = 7$ and $b = 6$, we have $a > b$.

In this case, there will be only one solution and ∠B will be acute. If you tried drawing the triangle with an obtuse angle at B, then side b would need to be the biggest side, opposite the biggest angle, which is not the case.

Thus, you can immediately draw the completed triangle and solve it.



To solve for ∠B:

$$\frac{7}{\sin 30°} = \frac{6}{\sin B}$$

$$\sin B = \frac{6 \sin 30°}{7}$$

$$\angle B = \sin^{-1}\left(\frac{6 \sin 30°}{7}\right)$$

$$\angle B = 25.4°$$

To solve for *c:*

$$\frac{7}{\sin 30°} = \frac{c}{\sin 124.6°}$$

$$c = \frac{7 \sin 124.6°}{\sin 30°}$$

$$c = 11.5$$

To solve for ∠C

$$\angle C = 180° - (30° + 25.4°)$$

$$\angle C = 124.6°$$

Conclusion: There will be only one solution if the side opposite the given angle is larger than the other given side.

Include the above conclusion on your resource sheet.

Note that in Example 4 above, if you were to find both the acute and obtuse angle values for ∠B, you could quickly determine why there is only one solution.

If $\angle B_r = 25.4°$, then $\angle B_1 = 25.4°$ as the acute angle and $\angle B_2 = 180° - 25.4° = 154.6°$ as the obtuse angle. Now solve for $\angle C_2$ using obtuse angle $\angle B_2$. $\angle C_2 = 180° - 154.6° - 30° = -4.6°$.

Since $\angle C_2$ cannot be negative, using obtuse $\angle B_2$ is not possible.

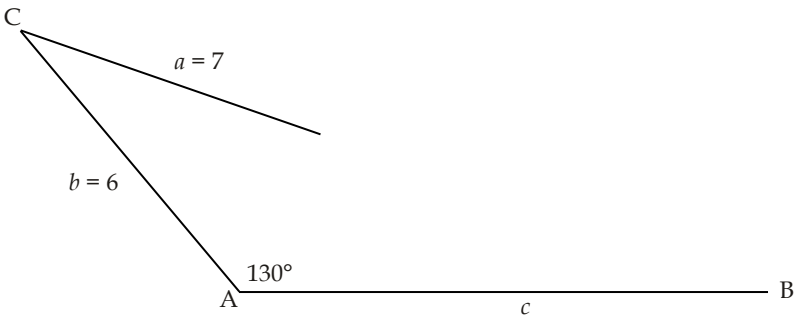
When the Given Angle is Obtuse

If the given angle is **obtuse**, there are two possible cases, as shown in the examples below.

Example 5: Oblique Triangle; One Solution; ∠B Is Acute

In ΔABC, $a = 7$, $b = 6$, $\angle A = 130°$. Solve the triangle.

Solution



Since $a = 7$ and $b = 6$, then $a > b$.

There can only be one obtuse angle in a triangle. Thus, the angle at B must be acute and there is one solution.

To solve for ∠B:

$$\frac{7}{\sin 130°} = \frac{6}{\sin B}$$

$$\sin B = \frac{6 \sin 130°}{7}$$

$$\angle B = 41.0°$$

To solve for ∠C:

$$\angle C = 180° - (41.0° + 130°)$$

$$\angle C = 9.0°$$

To solve for c

$$\frac{7}{\sin 130°} = \frac{c}{\sin 9°}$$

$$c = \frac{7 \sin 9°}{\sin 130°}$$

$$c = 1.4$$

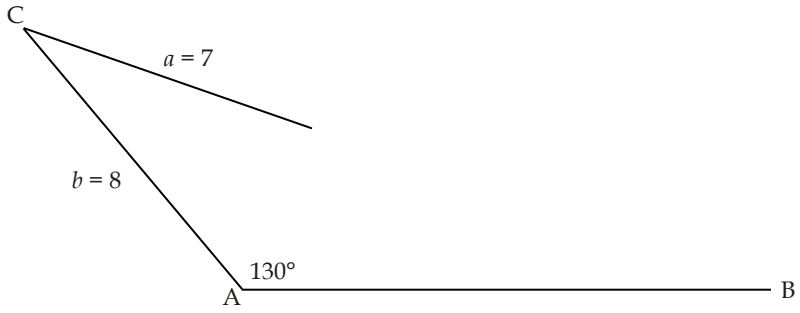
Conclusion: If the given angle is obtuse and the side opposite the given side is bigger than the second given side, there is only one possible triangle.

Include this conclusion on your resource sheet.

Example 6: Oblique Triangle; No Solution

In ΔABC, $a = 7$, $b = 8$, $\angle A = 130°$. Solve the triangle.

Solution



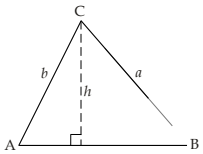
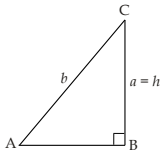
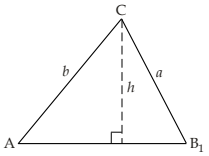
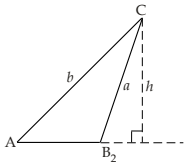
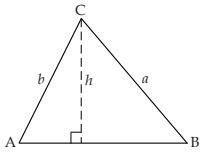
As you can see from the diagram, the longer side is not opposite the obtuse angle. There cannot be a solution.

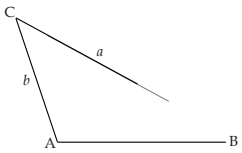
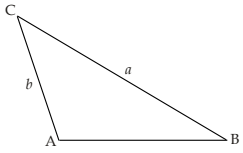
Conclusion: If the given angle is obtuse and the side opposite the given angle is shorter than the second given side, there is no possible triangle.

Include this conclusion on your resource sheet.

In each of the above examples, you were given triangles with two sides and the angle opposite one of those sides. Two solutions may occur when solving a triangle if the given angle is opposite the shorter of the two sides. When the given angle is opposite the longer of the two given sides, there is only one solution.

The following chart summarizes the various possibilities involved in the ambiguous case. Note that a represents the side opposite the given angle, b represents the other given side, not the base, and h represents the height of the triangle (i.e., $h = b \sin A$).

∠A is Acute					
Necessary Conditions	$a < b$ AND $a < h$	$a < b$ AND $a = h$	$a < b$ AND $a > h$	$a < b$ AND $a > h$	$a > b$
Sketch					
Number of Solutions	none	one	two		one
Size of ∠B	none	∠B = 90°	$\angle B_1$ is acute	$\angle B_2$ is obtuse	∠B is acute

∠A is Obtuse		
Necessary Conditions	$a \leq b$	$a > b$
Sketch		
Number of Solutions	none	one
Size of ∠B	none	∠B is acute

You may find it helpful to include the above two charts on your resource sheet.

The Sine Law can help you determine the number of solutions and it is used to solve the question, as you saw in the examples.

Learning Activity 7.6

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Calculate the tax (13%) on an item that costs $72.
2. How many 4-piece bags of candy can you make to give out on Halloween if you have 288 pieces of candy?
3. In which quadrants is $\cos\theta > 0$?
4. The length of the hypotenuse of a right triangle is 13 and one of its legs is length 12. What is the length of the other leg?
5. Does the parabola $y + 5x^2 = -2x + 4$ open up or down?
6. Express as a radical: $x^{\frac{13}{2}} y^{\frac{5}{2}}$
7. $(-56, 27)$ is on a line with a slope of $\frac{6}{8}$. Find another point on this line.
8. Simplify: $(x^2 - 1)(2x^2 + 5x - 6)$

continued

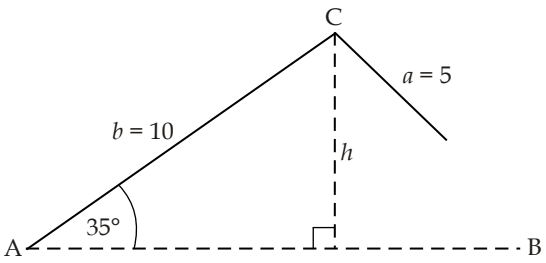
Part B: The Ambiguous Case

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

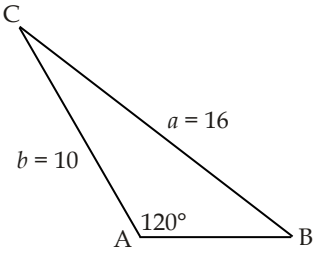
1. Decide whether there are no solutions, exactly one solution, or two solutions for each ΔABC. You do not need to solve the triangle, but you must justify your choice and include a diagram.

   Note: Always draw the given angle in the lower left corner of the triangle. Mark the given side opposite it with a dotted line. Do not mark the second given side as the base. (Reason: It makes solving the ambiguous case easier if you always use the same pattern.)

   a)



   b)



   c) ∠C = 60°, c = 43, b = 32

   d) ∠A = 18°, a = 38, c = 50

   e) ∠B = 60°, b = 14, c = 15

   f) ∠C = 170°, c = 10, b = 15

continued

Learning Activity 7.6 (continued)

2. Find the possible values of the indicated side. Round answers to the nearest tenth.
   a) In ΔABC, $\angle B = 34°$, $a = 4$, and $b = 3$. Find c.
   b) In ΔXYZ, $\angle X = 13°$, $x = 12$, and $y = 15$. Find z.
   c) In ΔABC, $\angle B = 34°$, $a = 4$, and $b = 5$. Find c.
   d) In ΔRST, $\angle R = 130°$, $r = 20$, and $t = 16$. Find s.
   e) In ΔMBT, $\angle M = 170°$, $m = 19$, and $t = 11$. Find b.
   f) In ΔABC, $\angle B = 34°$, $a = 4$, and $b = 2$. Find c.
3. Find all possible values of the indicated angle measure. Round to one decimal place.
   a) In ΔABC, $\angle A = 19°$, $a = 25$, and $c = 30$. Find $\angle C$.
   b) In ΔBDJ, $\angle B = 28°$, $b = 50$, and $d = 20$. Find $\angle D$.
   c) In ΔXYZ, $\angle X = 58°$, $x = 9.3$, and $z = 7.5$. Find $\angle Z$.
   d) In ΔBIG, $\angle B = 39°$, $b = 900$, and $g = 1000$. Find $\angle I$.

Lesson Summary

In this lesson, you learned that when you are given two sides and one angle opposite one of those sides in a triangle, this is called the ambiguous case because three situations may occur. There may be no solution, one solution, or two solutions. You learned to determine when each of these types of solutions occurred by comparing the lengths of the given sides, the height of the triangle, as well as whether the given angle was obtuse or acute. Make sure you remember that when you are given two sides and one angle opposite one of the given sides of a triangle, you always need to check to see how many solutions there are.

Notes

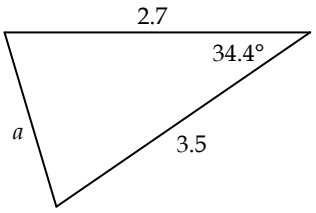
Assignment 7.3

The Cosine Law and the Ambiguous Case

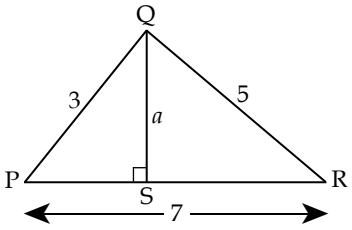
Total: 41 marks

1. Find the length of side *a* in each diagram. Round to three decimal places.

a) *(3 marks)*



b) *(4 marks)*



continued

Assignment 7.3: The Cosine Law and the Ambiguous Case (continued)

2. Find all of the missing angles and the length of all missing sides for the following triangles. Round to the nearest tenth. *(6 marks each, for a total of 12 marks)*

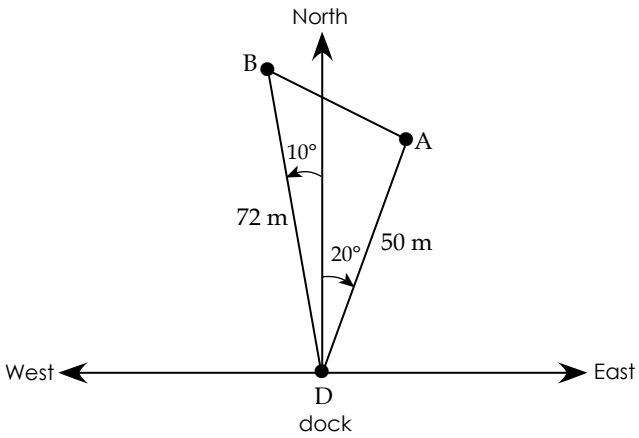
   a) In ΔABC, $a = 7$, $b = 5$ and $c = 11$.

   b) In ΔABC, $a = 15$, $b = 5$ and $\angle C = 62°$.

continued

Assignment 7.3: The Cosine Law and the Ambiguous Case (continued)

3. From a dock, the bearings of two boats A and B are N 20° E and N 10° W, respectively. If boat A is 50 m from the dock and boat B is 72 m from the dock, how far is boat A from boat B? Round to the nearest metre. *(4 marks)*



continued

Assignment 7.3: The Cosine Law and the Ambiguous Case (continued)

4. Find all the possible missing values in the following triangles. Round your answers to one decimal place.

   a) Find all the possible measures of $\angle B$, $\angle C$, and c in ΔABC if $a = 23$, $b = 47$, and $\angle A = 52°$. *(2 marks)*

   b) Find all the possible measures of $\angle A$, $\angle C$, and a in ΔABC if $b = 16$, $c = 20$, and $\angle B = 42°$. Round to the nearest tenth. *(8 marks)*

continued

c) Find all the possible measures of ∠A, ∠C, and c in ΔABC if $a = 5$, $b = 6$, and ∠B = 34°. Round to the nearest tenth. *(4 marks)*

d) Find all the possible measures of ∠B, ∠C, and c in ΔABC if $a = 20$, $b = 15$, and ∠A = 152°. Round to the nearest tenth. *(4 marks)*

Notes

Module 7 Summary

Congratulations, you are now well on your way to finishing this course. You only have to complete one more module.

In this module, you learned more about trigonometry. You extended your knowledge of right-angled trigonometry to learn about angles from 0° to 360° in Quadrants I, II, III, and IV. You learned how each angle in standard position has a reference angle and you also learned how to find that reference angle. Once you learned about angles, you studied triangles that do not have a 90° angle. These oblique triangles can be solved by using either the Sine Law or the Cosine Law, depending upon what information is provided. However, when you are given two sides and an angle opposite one of those sides, the ambiguous case occurs. In the ambiguous case, there can be either no solution, one solution, or two solutions. Make sure you always consider the number of possible solutions when you are solving such a triangle.

In the next module, you will be learning about two new types of functions: absolute value functions and reciprocal functions. Absolute value functions take each y-value, whether positive or negative, and make it positive. Reciprocal functions take the reciprocal of a given function.

Submitting Your Assignments

It is now time for you to submit Assignments 7.1 to 7.3 to the Distance Learning Unit so that you can receive some feedback on how you are doing in this course. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 7 assignments and organize your material in the following order:

- ❑ Module 7 Cover Sheet (found at the end of the course Introduction)
- ❑ Assignment 7.1: Trigonometric Ratios, Reference Angles, and Trigonometric Equations
- ❑ Assignment 7.2: Patterns in Trigonometric Ratios and the Sine Law
- ❑ Assignment 7.3: The Cosine Law and the Ambiguous Case

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Notes

Grade 11 Pre-Calculus Mathematics (30S)

Module 7
Trigonometry

Learning Activity Answer Keys

MODULE 7: TRIGONOMETRY

Learning Activity 7.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Jacob went to three stores. He spent \$76 at the first store, \$47 at the second store, and \$152 at the third store. If Jacob had \$300, how much money does he have left?
2. If two angles in a triangle measure 137° in total, what is the measure of the third angle in the triangle?
3. Solve for g: $(2g + 1)(g - 5) = 0$
4. You and your brother are partial owners of the family vehicle. If you own half as much as your brother, how much of the car do you own?
5. Simplify: $\sqrt{108}$
6. Order these values from lowest to highest: $5^{\frac{1}{5}}$, $\sqrt{5}$, $5^{\frac{1}{4}}$, and $\sqrt[3]{5}$
7. Rationalize the denominator: $\dfrac{7\sqrt{3}}{3\sqrt{2}}$
8. Find the slope of the line parallel to the line $2x + y = 1$.

Answers:

1. \$25 (\$76 + \$47 + \$152 = \$275; \$300 − \$275 = \$25)
2. 43° (180° − 137°)
3. $g = -\dfrac{1}{2}$ or $g = 5$
4. $\dfrac{1}{3}$ (Your brother owns two parts and you own one part. So you can think of there being a total of 3 parts.)
5. $6\sqrt{3}$ $\left(\sqrt{4}\sqrt{9}\sqrt{3}\right)$

6. $5^{\frac{1}{5}} < 5^{\frac{1}{4}} < \sqrt[3]{5} < \sqrt{5}$ $\left(\sqrt{5} = 5^{\frac{1}{2}};\ \sqrt[3]{5} = 5^{\frac{1}{3}}\right)$

7. $\frac{7\sqrt{6}}{6}$ $\left(\frac{7\sqrt{3}}{3\sqrt{2}} = \frac{7\sqrt{3}\sqrt{2}}{3(2)}\right)$

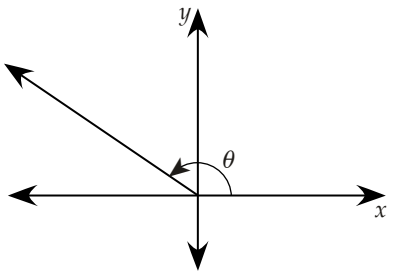
8. -2 $(y = -2x + 1$

Part B: Trigonometric Ratios

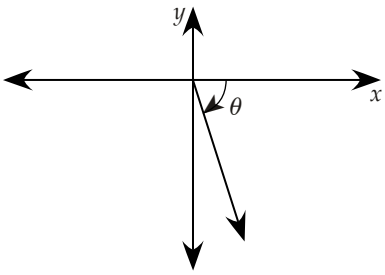
Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Determine which of the following angles are in standard position. Explain.

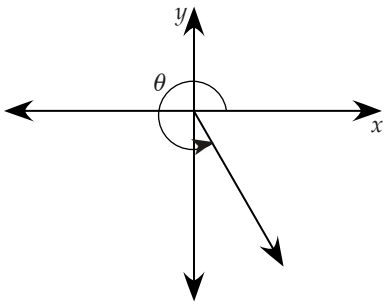
a)



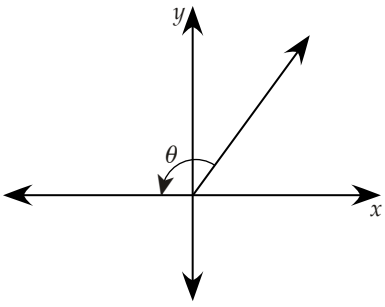
b)



c)



d)



Answers:

a) This angle is in standard position as the initial side is along the positive x-axis and the angle is measured from the initial to the terminal arm, and the vertex is at the origin. The angle is positive because its rotation is in a counter-clockwise direction.

b) This angle is in standard position and the vertex is at the origin. The initial side is along the positive x-axis. It is rotated to the terminal arm in a clockwise direction and its measure is considered a negative value.

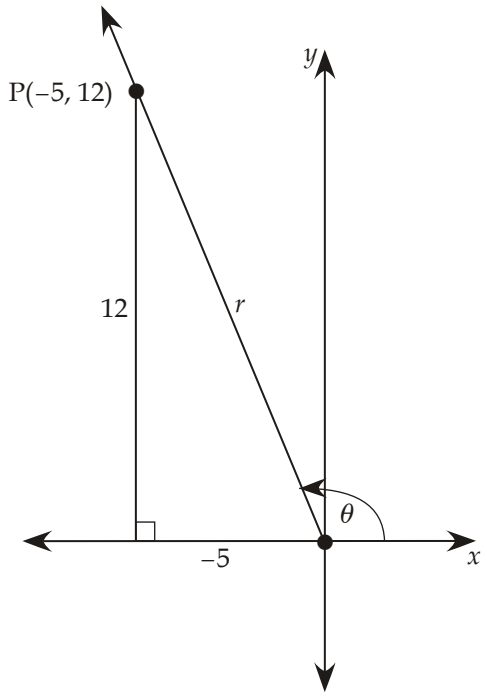
c) This angle is in standard position. This angle is the reflex angle to the angle in (b). The angle is measured counter-clockwise from the initial side to the terminal arm so it is a positive angle.

d) This angle is not in standard position, as the initial side is not along the positive x-axis.

2. P(x, y) is a point on the terminal side of angle θ in standard position. Determine sin θ, cos θ, and tan θ for the following points. Draw a sketch. Show your work for determining the distance from the origin to the point P(x, y). Rationalize any denominators.

a) (−5, 12)

Answer:



$$r = \sqrt{x^2 + y^2}$$
$$= \sqrt{(-5)^2 + 12^2}$$
$$= \sqrt{169}$$
$$= 13$$

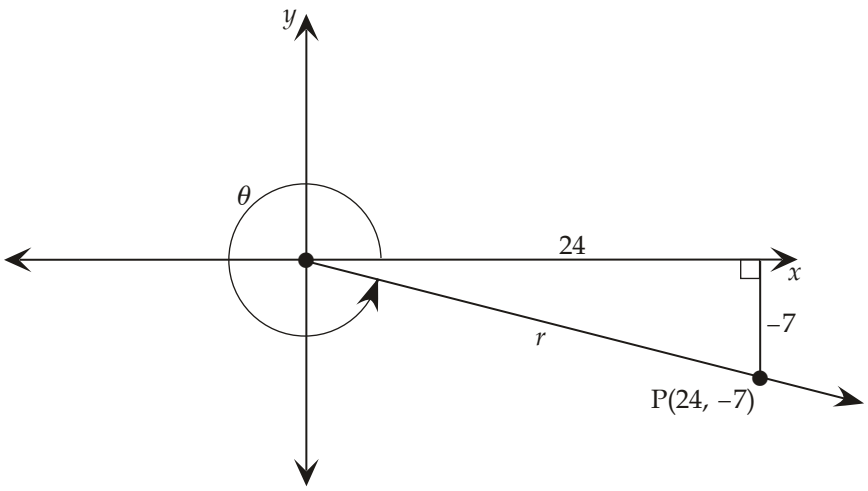
$$\sin\theta = \frac{y}{r} = \frac{12}{13}$$

$$\cos\theta = \frac{x}{r} = \frac{-5}{13} = -\frac{5}{13}$$

$$\tan\theta = \frac{y}{x} = \frac{12}{-5} = -\frac{12}{5}$$

b) (24, −7)

Answer:



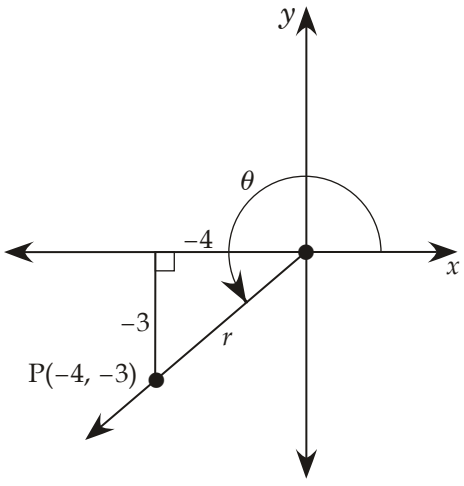
Notice that these triangles in quadrant diagrams do not need to be drawn to scale.

$$r = \sqrt{x^2 + y^2}$$
$$= \sqrt{(24)^2 + (-7)^2}$$
$$= \sqrt{625}$$
$$= 25$$

$$\sin\theta = \frac{y}{r} = \frac{-7}{25} = -\frac{7}{25}$$
$$\cos\theta = \frac{x}{r} = \frac{24}{25}$$
$$\tan\theta = \frac{y}{x} = \frac{-7}{24} = -\frac{7}{24}$$

c) (−4, −3)

Answer:

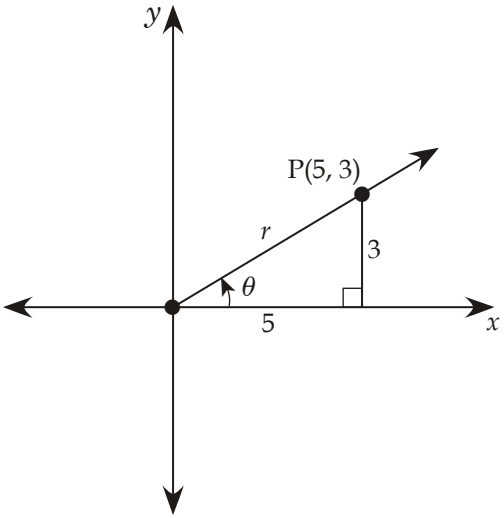


$$r = \sqrt{x^2 + y^2}$$
$$= \sqrt{(-4)^2 + (-3)^2}$$
$$= \sqrt{25}$$
$$= 5$$

$$\sin\theta = \frac{y}{r} = \frac{-3}{5} = -\frac{3}{5}$$
$$\cos\theta = \frac{x}{r} = \frac{-4}{5} = -\frac{4}{5}$$
$$\tan\theta = \frac{y}{x} = \frac{-3}{-4} = \frac{3}{4}$$

d) (5, 3)

Answer:



$$r = \sqrt{x^2 + y^2}$$
$$= \sqrt{5^2 + 3^2}$$
$$= \sqrt{34}$$

$$\sin\theta = \frac{y}{r} = \frac{3}{\sqrt{34}} = \frac{3\sqrt{34}}{34}$$

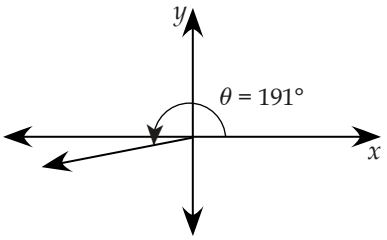
$$\cos\theta = \frac{x}{r} = \frac{5}{\sqrt{34}} = \frac{5\sqrt{34}}{34}$$

$$\tan\theta = \frac{y}{x} = \frac{3}{5}$$

3. Without using a calculator, determine if each value should be positive or negative by drawing the angle in standard position.

a) cos 191°

Answer:



θ is in Quadrant III where x-coordinates are negative; or, you can use the CAST rule:

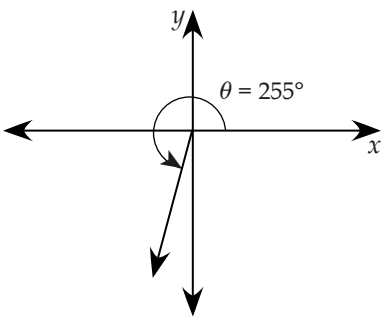
S	A
T	C

cos θ is negative in Quadrant III.

$\therefore$ cos 191° is negative.

b) sin 255°

Answer:



θ = 255° is located in Quadrant III where y-coordinates are negative; or you can use the CAST rule:

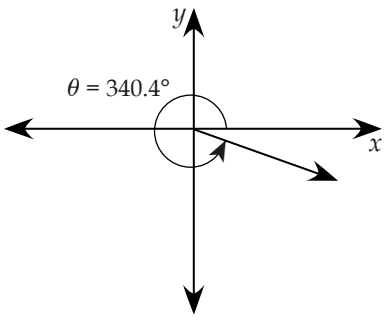
S	A
T	C

sin $\theta < 0$ in this quadrant.

$\therefore$ sin 255° is negative.

c) tan 340.4°

Answer:



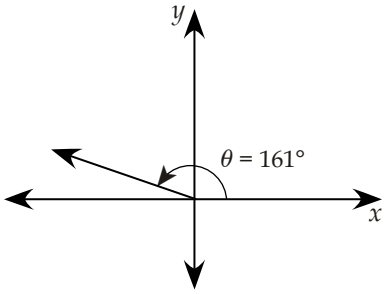
$\theta = 340.4°$ is located in Quadrant IV where x-values are positive and y-values are negative.

$$\therefore \tan 340.4° = \frac{y\text{-value}}{x\text{-value}}$$
$$= \frac{\text{negative}}{\text{positive}}$$
$$= \text{negative}$$

Or, use the CAST rule and only cosine is positive in Quadrant IV. So tan 340.4° is negative.

d) sin 161°

Answer:



$\theta = 161°$ is located in Quadrant II where the y-values are positive.

$$\therefore \sin\theta = \frac{y\text{-value}}{r}$$
$$= \frac{\text{positive}}{\text{positive}}$$
$$= \text{positive}$$

Or, use the CAST rule and $\sin\theta$ is positive in Quadrant II. So sin 161° is positive.

4. Determine the quadrants in which P(x, y) may lie under the following conditions.

a) $\sin\theta > 0$

Answer:

The ratio for $\sin\theta$ is $\frac{y\text{-value}}{r\text{-value}}$. Thus, $\sin\theta$ is positive when the y-values are positive because the r-value is always positive. This occurs in Quadrant I or II.

b) $\cos\theta > 0$

Answer:

The ratio for $\cos\theta$ is $\frac{x\text{-value}}{r\text{-value}}$. Thus, $\cos\theta$ is positive when the x-values are positive. This occurs in Quadrant I or IV.

c) $\tan\theta > 0$

Answer:

The ratio for $\tan\theta$ is $\frac{y\text{-value}}{x\text{-value}}$. Thus, $\tan\theta$ is positive when the x- and y-values are both positive or both negative. This occurs in Quadrant I or III.

d) $\cos\theta > 0$ and $\sin\theta < 0$

Answer:

Since the x-value must be positive and the y-value negative, the only quadrant where this occurs is Quadrant IV.

5. Fill in the following chart by indicating which quadrant(s) each angle is in by considering the sign of each ratio or by using the CAST rule.

Answers:

Ratio Value	Quadrant
$\cos\theta = \frac{2}{5}$	I or IV
$\sin\theta = \frac{4}{5}$	I or II
$\tan\theta = -\frac{2}{3}$	II or IV
$\sin\theta = \frac{-1}{5}$	III or IV
$\tan\theta = \frac{3}{5}$	I or III

6. Given the following angles in standard position, determine if the sine, cosine, and tangent ratios are positive or negative. Explain your answers.

   a) $\theta = 26°$

   Answer:

   This angle is located in Quadrant 1. All of the ratios, including the sine, cosine, and tangent ratios, are positive because all ratios in Quadrant I are positive.

   b) $\theta = 305°$

   Answer:

   This angle is located in Quadrant IV where cosine is positive and both sine and tangent are negative. In Quadrant IV, $x > 0$, $y < 0$; thus, $\frac{x}{r}$ is positive, $\frac{y}{r}$ is negative, and $\frac{y}{x}$ is negative.

   c) $\theta = 152°$

   Answer:

   This angle is located in Quadrant II where sine is positive and both cosine and tangent are negative. In Quadrant II, $x < 0$ and $y > 0$.

   d) $\theta = 206°$

   Answer:

   This angle is located in Quadrant III. Sine and cosine are both negative in Quadrant III and tangent is positive.

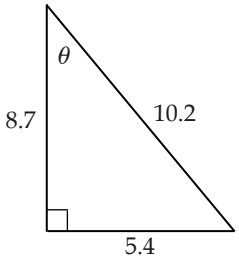
Learning Activity 7.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Gwen planted 120 seeds of green beans. If one-quarter of the beans did not grow, how many beans did grow?
2. The minute hand of a clock starts at 12 and travels 60% of the way around the clock. How many degrees from the 12 has the minute hand travelled?
3. Calculate: $\frac{4}{3} + \frac{5}{7}$
4. The two legs of a right-angled triangle measure 3 cm and 4 cm. What is the measure of the hypotenuse?
5. If you do homework for half an hour every night from Monday to Friday, as well as 45 minutes on both Saturday and Sunday, how many hours of homework do you do in a week?

Use the following right-angled triangle to answer the next three questions. Do not simplify.



6. Identify the sine ratio for θ.
7. Identify the cosine ratio for θ.
8. Identify the tangent ratio for θ.

Answers:

1. 90 (one-quarter of 120 is 30, and 120 − 30 = 90)
2. 216° (60% of 360 degrees = 50% of 360 + 10% of 360 = 180° + 36°)
3. $\frac{43}{21}\left(\frac{28}{21} + \frac{15}{21}\right)$
4. 5 ($3^2 + 4^2 = x^2$; $9 + 16 = x^2$; $25 = x^2$; $5 = x$)

5. 4 hours (5 × half an hour = 2.5 hours; 45 minutes × 2 = 1.5 hours)
6. $\frac{5.4}{10.2}$
7. $\frac{8.7}{10.2}$
8. $\frac{5.4}{8.7}$

Part B: Reference Angles and Solving Trigonometric Equations in Linear Form

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. In which quadrant does the terminal arm of each angle lie?
   a) 65°

      Answer:

      As this angle lies between 0° and 90°, the terminal arm of this angle lies in Quadrant I.

   b) 340°

      Answer:

      As this angle lies between 270° and 360°, the terminal arm of this angle lies in Quadrant IV.

   c) 241°

      Answer:

      As this angle lies between 180° and 270°, the terminal arm of this angle lies in Quadrant III.

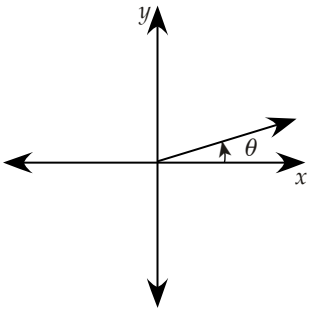
   d) 123°

      Answer:

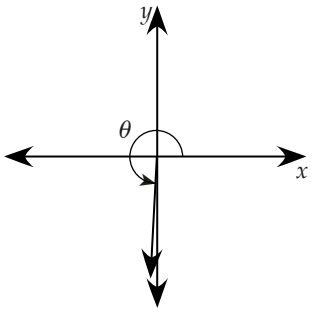
      As this angle lies between 90° and 180°, the terminal arm of this angle lies in Quadrant II.

2. Without measuring, match each angle to its corresponding graph.
   a) 140°
   b) 267°
   c) 316°
   d) 17°

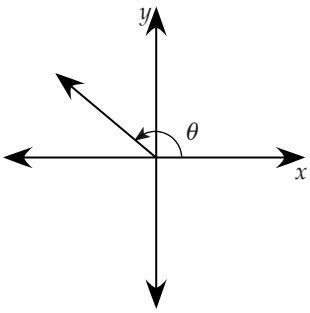
i)



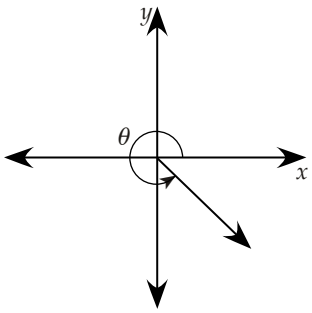
ii)



iii)



iv)



Answers:

(a) and (iii)

(b) and (ii)

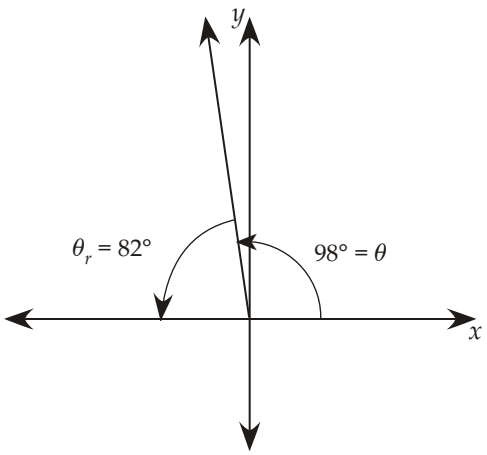
(c) and (iv)

(d) and (i)

3. Sketch each angle in standard position and find its reference angle. Write the measure of the reference angle.

a) 98°

Answer:

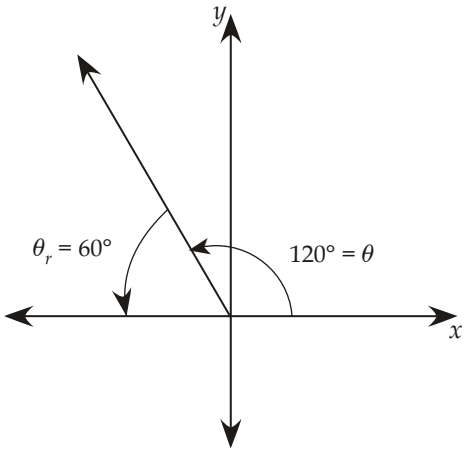


This work is optional.

$\theta_r = 180° - \theta$
$\theta_r = 180° - 98°$
$\theta_r = 82°$

b) 120°

Answer:

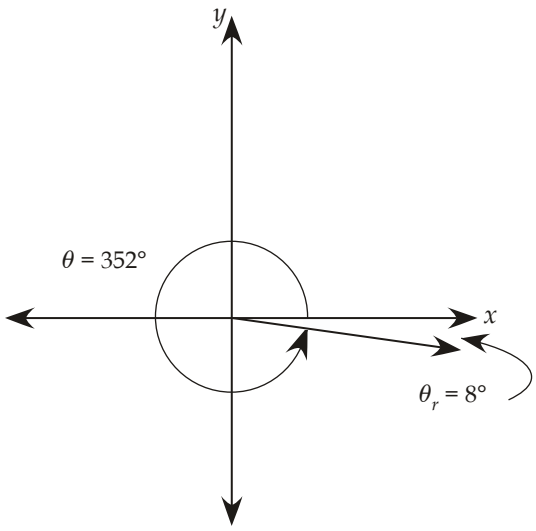


This work is optional.

$\theta_r = 180° - \theta$
$\theta_r = 180° - 120°$
$\theta_r = 60°$

c) 352°

Answer:

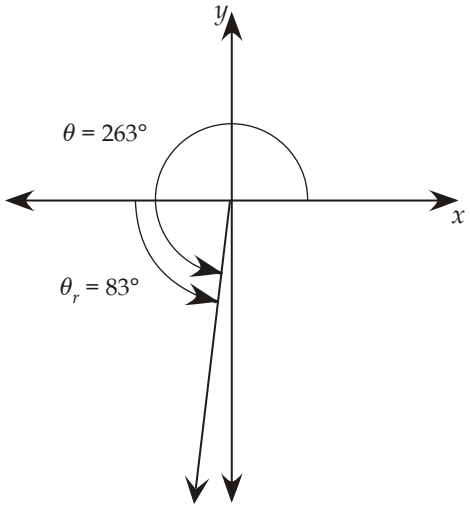


This work is optional.

$\theta_r = 360° - \theta$
$\theta_r = 360° - 352°$
$\theta_r = 8°$

d) 263°

Answer:



This work is optional.

$\theta_r = \theta - 180°$
$\theta_r = 263° - 180°$
$\theta_r = 83°$

4. Determine the measure of the 3 other angles in standard position that have the same reference angle as an angle in standard position of 58°.

   Answer:

   In Quadrant II, the angle is: $\theta = 180° - \theta_r$

   $\theta = 180° - 58°$

   $\theta = 122°$

   In Quadrant III, the angle is: $\theta = 180° + \theta_r$

   $\theta = 180° + 58°$

   $\theta = 238°$

   In Quadrant IV, the angle is: $\theta = 360° - \theta_r$

   $\theta = 360° - 58°$

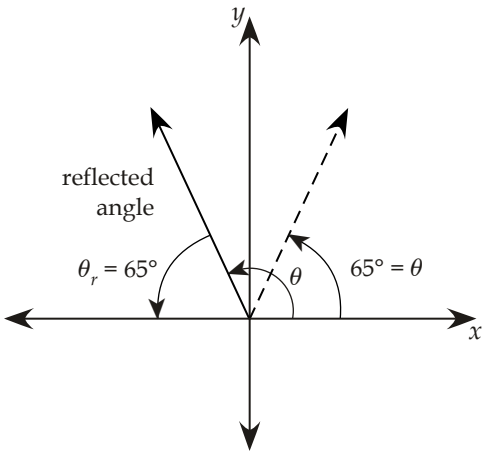
   $\theta = 302°$

   The other three angles are 122°, 238°, and 302°.

5. Determine the angle in standard position when an angle of 65° is reflected

   a) in the *y*-axis. Draw the diagram.

   Answer:

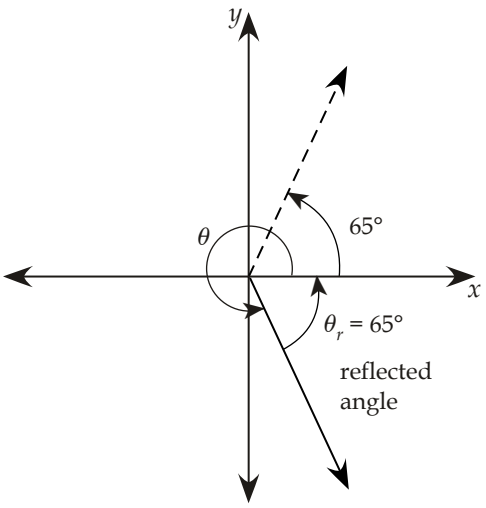


   Thus, the angle in standard position is: $\theta = 180° - 65°$

   $\theta = 115°$

b) in the x-axis. Draw the diagram.

Answer:

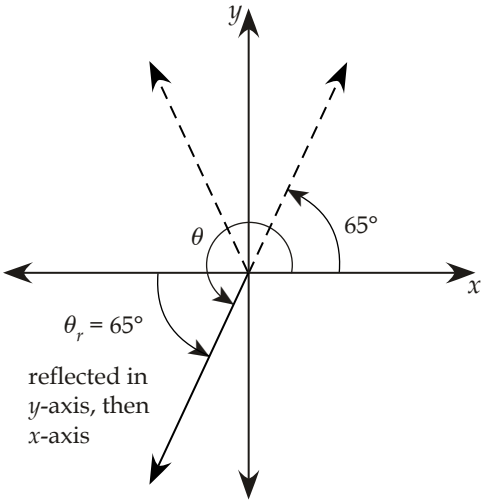


Thus, the angle in standard position is: $\theta = 360° - 65°$

$\theta = 295°$

c) in the y-axis and then the x-axis. Draw the diagram.

Answer:



Thus, the angle in standard position is: $\theta = 180° + 65°$

$\theta = 245°$

6. Determine the solution set for each of the following trigonometric equations over the interval [0°, 360°]. Round to one decimal place if necessary.

a) $\cos\theta = -\frac{2}{3}$

b) $\sin\theta + 1 = 0$

c) $\tan\theta - 2 = 5$

d) $2\cos\theta = 2$

e) $-3\sin\theta = 2$

f) $\frac{\tan\theta}{2} = 5$

g) $5\cos\theta - 2 = 0$

h) $5\tan\theta + 4 = 0$

i) $\frac{\tan\theta}{6} - 1 = 0$

j) $2\cos\theta + 1 = \frac{1}{2}$

k) $4\tan\theta - 7 = 5\tan\theta - 6$

Answers:

a) $\cos\theta = -\frac{2}{3}$

$\theta_r = \cos^{-1}\left(\frac{2}{3}\right)$ Use the positive ratio to find θ_r.

$= 48.2°$

$\cos\theta < 0$ in Quadrants II and III

In Quadrant II, $\theta = 180° - 48.2° = 131.8°$.

In Quadrant III, $\theta = 180° + 48.2° = 228.2°$.

b) $\sin\theta = -1$

This is a quadrantal angle through $(0, -1)$ that lies on the negative y-axis.

$\therefore \theta = 270°$

c) $\tan\theta - 2 = 5$

$\tan\theta = 7$

$\theta_r = \tan^{-1}(7)$

$= 81.9°$

$\tan\theta > 0$ in Quadrants 1 and III

In Quadrant I, $\theta = 81.9°$.

In Quadrant III, $\theta = 180° + 81.9° = 261.9°$.

d) $2 \cos \theta = 2$

$\cos \theta = 1$

This is a quadrantal angle through (1, 0) that lies on the positive x-axis.

$\therefore \theta = 0°$ or $360°$.

e) $-3 \sin \theta = 2$

$$\sin \theta = -\frac{2}{3}$$

$$\theta_r = \sin^{-1}\left(\frac{2}{3}\right) \text{ Use the positive ratio to find } \theta_r.$$

$$= 41.8°$$

$\sin \theta < 0$ in Quadrants III and IV

In Quadrant III, $\theta = 180° + 41.8° = 221.8°$.

In Quadrant IV, $\theta = 360° - 41.8° = 318.2°$.

f) $\frac{\tan \theta}{2} = 5$

$\tan \theta = 10$

$$\theta_r = \tan^{-1}(10)$$

$$= 84.3°$$

$\tan \theta > 0$ in Quadrants I and III

In Quadrant I, $\theta = 84.3°$.

In Quadrant III, $\theta = 180° + 84.3° = 264.3°$.

g) $5 \cos \theta - 2 = 0$

$$\cos \theta = \frac{2}{5}$$

$$\theta_r = \cos^{-1}\left(\frac{2}{5}\right)$$

$$= 66.4°$$

$\cos \theta > 0$ in Quadrants I and IV

In Quadrant I, $\theta = 66.4°$.

In Quadrant IV, $\theta = 360° - 66.4° = 293.6°$.

h) $5 \tan \theta + 4 = 0$

$$\tan \theta = -\frac{4}{5}$$

$$\theta_r = \tan^{-1}\left(\frac{4}{5}\right)$$

$$= 38.7°$$

$\tan \theta < 0$ in Quadrants II and IV

In Quadrant II, $\theta = 180° - 38.7° = 141.3°$.

In Quadrant IV, $\theta = 360° - 38.7° = 321.3°$.

i) $\frac{\tan \theta}{6} - 1 = 0$

$$\frac{\tan \theta}{6} = 1$$

$$\tan \theta = 6$$

$$\theta_r = \tan^{-1} 6$$

$$= 80.5°$$

$\tan \theta > 0$ in Quadrants I and III

In Quadrant I, $\theta = 80.5°$.

In Quadrant III, $\theta = 180° + 80.5° = 260.5°$.

j) $2 \cos \theta + 1 = \frac{1}{2}$

$$2 \cos \theta = -\frac{1}{2}$$

$$\cos \theta = -\frac{1}{4}$$

$$\theta_r = \cos^{-1}\left(\frac{1}{4}\right)$$

$$= 75.5°$$

$\cos \theta < 0$ in Quadrants II and III

In Quadrant II, $\theta = 180° - 75.5° = 104.5°$.

In Quadrant III, $\theta = 180° + 75.5° = 255.5°$.

k) $4 \tan \theta - 7 = 5 \tan \theta - 6$

$-\tan \theta = 1$

$\tan \theta = -1$

$\theta_r = \tan^{-1}(1)$

$\theta_r = 45°$

$\tan \theta < 0$ in Quadrants II and IV

In Quadrant II, $\theta = 180° - 45° = 135°$.

In Quadrant IV, $\theta = 360° - 45° = 315°$.

Learning Activity 7.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Jason had 37 hits in 45 times at bat during a baseball season. Sherri had 12 hits in 15 times at bat. Who is the better hitter?
2. Factor: $x^2 + 5x - 6$
3. Evaluate: $\left(6^2\right)^{\frac{1}{2}}$
4. In which quadrants is $\sin\theta > 0$?
5. Find the sum of 12 382 and 14 593.
6. Simplify: $(-3x^2 - 6x + 7) + (2x^2 - 5x)$
7. Two angles in an isosceles triangle measure 40° and 100°. What is the measure of the third angle?
8. Write this decimal as a percent: 0.103

Answers:

1. Jason (Sherri's ratio is $\frac{12}{15} = \frac{36}{45}$; Jason's ratio is $\frac{37}{45}$)
2. $(x + 6)(x - 1)$
3. 6
4. Quadrants I and II.
5. 26 975 (12 382 + 14 593)
6. $-3x^2 - 6x + 7 + 2x^2 - 5x = -x^2 - 11x + 7$
7. 40°
   (Method 1: The sum of the angles in a triangle must be 180°. Therefore, the third angle must measure 40°.

   Method 2: An isosceles triangle will have two angles that are the same. Therefore, the third angle must measure 40°.)
8. 10.3% (multiply by 100)

Part B: Patterns in the Sine, Cosine, and Tangent Ratios

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Determine the angles in Quadrants III and IV that have 30°, 45°, and 60° as reference angles.

   Answer:

   The formula for finding angles in standard position that have θ_r as a reference angle in Quadrant III is $\theta = 180° + \theta_r$.

   Thus, the three angles in Quadrant III are:

   $\theta = 180° + 30° = 210°$

   $\theta = 180° + 45° = 225°$

   $\theta = 180° + 60° = 240°$

   The formula for finding angles in standard position that have θ_r as a reference angle in Quadrant IV is $\theta = 360° - \theta_r$.

   Thus, the three angles in Quadrant IV are:

   $\theta = 360° - 30° = 330°$

   $\theta = 360° - 45° = 315°$

   $\theta = 360° - 60° = 300°$

2. Fill in the following chart with the exact values for sine, cosine, and tangent. Use the trigonometric ratio for the corresponding reference angle and attach a negative sign appropriately. Try to draw the special triangles from memory and determine the exact values for the trigonometric ratios using the appropriate special triangle.

θ	θ_r	$\cos\theta$	$\sin\theta$	$\tan\theta$
180°	(quadrantal angle)	-1	0	0
210°	30°	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{3}}$
225°	45°	$\frac{-1}{\sqrt{2}}$ or $\frac{-\sqrt{2}}{2}$	$\frac{-1}{\sqrt{2}}$ or $\frac{-\sqrt{2}}{2}$	1
240°	60°	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\sqrt{3}$
270°	(quadrantal angle)	0	-1	undefined
300°	60°	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$
315°	45°	$\frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$	$\frac{-1}{\sqrt{2}}$ or $\frac{-\sqrt{2}}{2}$	-1
330°	30°	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{1}{\sqrt{3}}$
360°	(quadrantal angle)	1	0	0

3. Complete the table below. List the sine, cosine, and tangent ratios for θ. Use your calculator and round to four decimal places.

Angle	cos θ	sin θ	tan θ
0°	1	0	0
15°	0.9659	0.2588	0.2679
30°	0.8660	0.5	0.5774
45°	0.7071	0.7071	1
60°	0.5	0.8660	1.7321
75°	0.2588	0.9659	3.7321
90°	0	1	undefined

a) What do you notice about the three ratios as the angle increases?

Answer:

As the angle increases, sine increases from 0 to 1, cosine decreases from 1 to 0, and tangent increases from 0 to a large number.

b) What do you notice about the sine and cosine ratios for each angle?

Answer:

The cosine ratio has the same value as the sine ratio of its complement. (The complement of θ is $90° - \theta$.) For example, sin 15° = cos 75°.

c) Determine the signs of the ratios in Quadrant II.

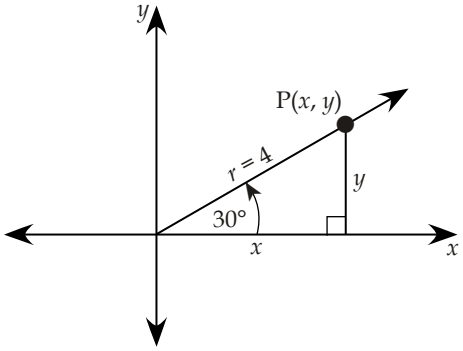
Answer:

In Quadrant II, sine will be positive while cosine and tangent are negative.

4. Determine the coordinates of the points on the terminal arm of the angles having 30° as a reference angle if $r = 4$. Write exact values. $0° \leq \theta \leq 360°$

Answer:

To solve for the coordinates of the point P(x, y), you can use the exact values of the trigonometric ratios for 30°.



$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r}$$

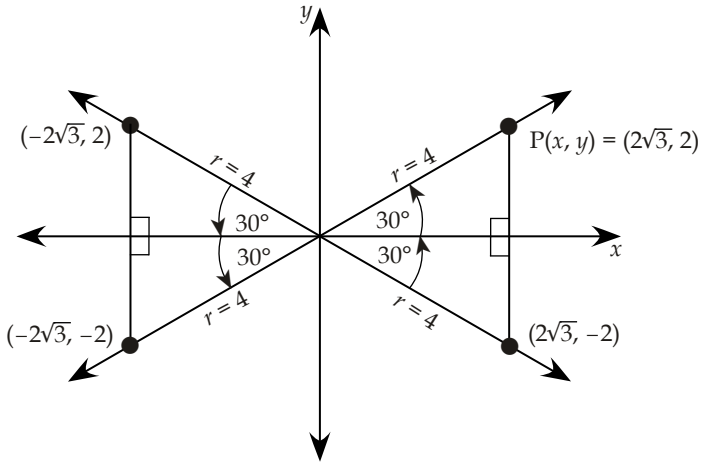
$$\cos 30° = \frac{x}{4} \qquad \sin 30° = \frac{y}{4}$$

$$x = 4\cos 30° \qquad y = 4\sin 30°$$

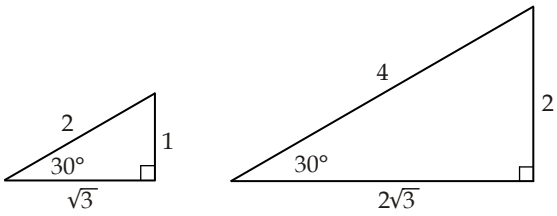
$$x = 4\left(\frac{\sqrt{3}}{2}\right) \qquad y = 4\left(\frac{1}{2}\right)$$

$$x = 2\sqrt{3} \qquad y = 2$$

The coordinates of P(x, y) are $\left(2\sqrt{3}, 2\right)$. The coordinates of the points in the other three quadrants are $\left(-2\sqrt{3}, 2\right)$, $\left(-2\sqrt{3}, -2\right)$, and $\left(2\sqrt{3}, -2\right)$.



Notice that since the r-value in this question is twice the r-value in the basic triangle for 30° that you have memorized, the x- and y-values are also doubled. They are similar triangles.



$r = 2$

coordinates

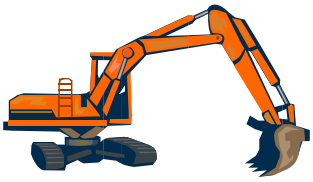
$P(x, y) = \left(\sqrt{3}, 1\right)$

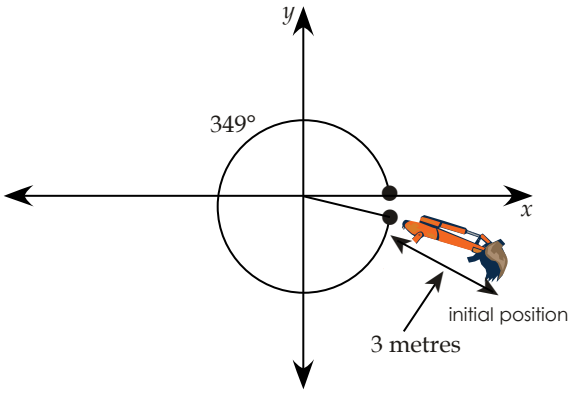
$r = 4$

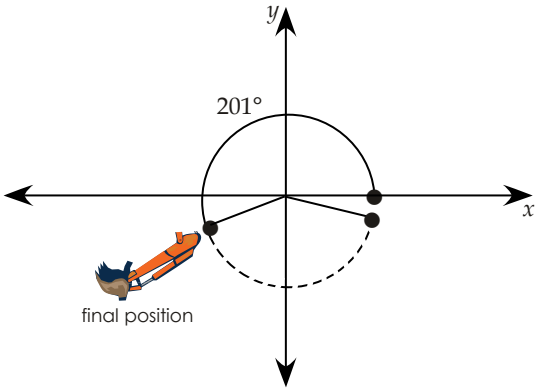
coordinates

$P(x, y) = \left(2\sqrt{3}, 2\right)$

5. A backhoe operator is digging a basement for a new house. To do this, on the first day he starts at the far end of the hole at angle of 349° and digs dirt backwards to end at an angle of 201°. If the length of the arm of the backhoe is 3 metres, what is the horizontal length of the hole that the backhoe operator is digging? Round your answer to 3 decimal places.



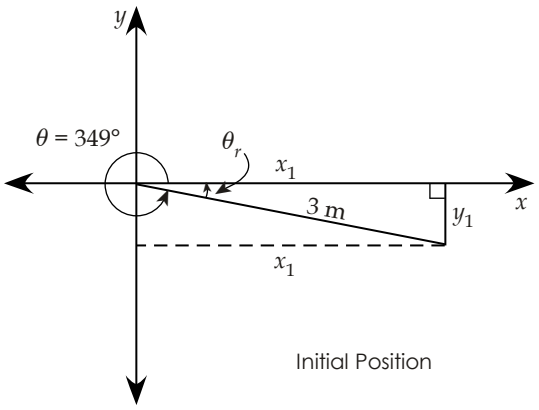




Answer:

You will use your calculator to find approximate answers since the angles in question do not have reference angles of 30°, 45°, or 60°.

To find the horizontal distance, you need to create two different triangles: one in Quadrant IV and one in Quadrant III. These triangles will represent the initial and final positions of the backhoe. Once these triangles are drawn, if you find the x-component of each triangle and add these two values together, you will find the length of the hole.



Initial Position

To find x_1, you need to find the horizontal distance, or the x-component, when the angle is 349° and the arm length is 3 m. To find the value of θ_r, you need to find the reference angle for 349°.

$$\theta_r = 360° - 349° = 11°$$

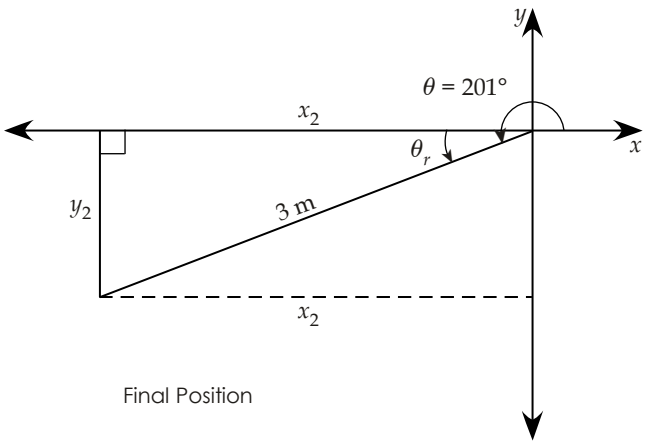
You now have the values of θ_r and r; thus, you need to use to cosine ratio in order to determine the x-value.

$$\cos\theta = \frac{x}{r}$$

$$\cos 11° = \frac{x}{3}$$

$$x = 3\cos 11°$$

$$x = 2.945 \text{ metres}$$



Final Position

To find x_2, you need to find the horizontal distance, or the x-component, when the angle is 201° and the boom length is 3 m.

The reference angle of 201° is:

$$201° - 180° = 21°$$

$$\cos\theta = \frac{x}{r}$$

$$\cos 21° = \frac{x}{3}$$

$$x = 3 \cos 21°$$

$$x = 2.801 \text{ metres}$$

Now, to find the length of the hole, you can add these values together. Thus, the length of the hole is 2.945 + 2.801 = 5.746 metres.

Learning Activity 7.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. You received $\frac{22}{25}$ on a math test. What was your percentage?
2. Find the common difference in the following pattern: 8, 2, −4, −10, . . .
3. In which quadrants is tan > 0?
4. What is the price of one box of Kleenex if 8 boxes of Kleenex cost $16.64?
5. The side lengths of a right-angled triangle are 15, 17, and 8. Which side length is the hypotenuse?
6. Simplify: $\frac{3^3 + 3}{3^2}$
7. Express as a radical: $9^{\frac{2}{3}}$. Do not simplify.
8. Find the slope of the line that contains the points (8, −3) and (−6, 12).

Answers:

1. 88% $\left(\frac{22}{25} = \text{multiply top and bottom by } 4 \text{ to get a denominator of } 100\right)$
2. −6
3. Quadrants I and III
4. \$2.08 $\left(\frac{\$16}{8} = \$2;\ \frac{\$0.64}{8} = \$0.08;\ \$2 + \$0.08 = \$2.08\right)$
5. 17 (The hypotenuse is always the longest side.)
6. $\frac{10}{3}$ $\left(\frac{3^3 + 3}{3^2} = \frac{27 + 3}{9} = \frac{30}{9}\right)$
7. $\sqrt[3]{9^2}$
8. $\frac{15}{-14}$ $\left(m = \frac{12 - (-3)}{-6 - 8} = \frac{15}{-14}\right)$

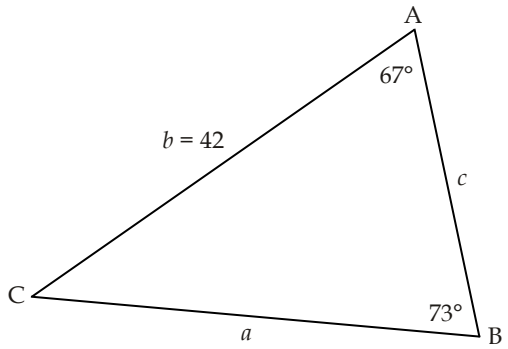
Part B: The Sine Law

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Solve the following triangles for all missing sides and angles. Round to 2 decimal places.

   a) In ΔABC, $\angle A = 67°$, $\angle B = 73°$, and $b = 42$.

   Answer:



$\angle C = 180° - 67° - 73° = 40°$

To find a:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin 67°} = \frac{42}{\sin 73°}$$

$$a = \frac{42 \sin 67°}{\sin 73°}$$

$$a = 40.43$$

To find c:

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

$$\frac{c}{\sin 40°} = \frac{42}{\sin 73°}$$

$$c = \frac{42 \sin 40°}{\sin 73°}$$

$$c = 28.23$$

Summary:

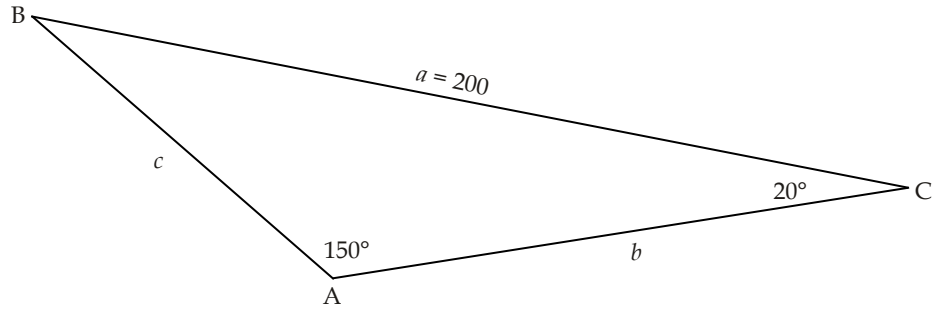
$\angle C = 40°$

$a = 40.43$

$c = 28.23$

b) In ΔABC, $\angle A = 150°$, $a = 200$, and $\angle C = 20°$.

Answer:



$\angle B = 180° - 150° - 20° = 10°$

To find b:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{200}{\sin 150°} = \frac{b}{\sin 10°}$$

$$b = \frac{200 \sin 10°}{\sin 150°}$$

$$b = 69.46$$

To find c:

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{200}{\sin 150°} = \frac{c}{\sin 20°}$$

$$c = \frac{200 \sin 20°}{\sin 150°}$$

$$c = 136.81$$

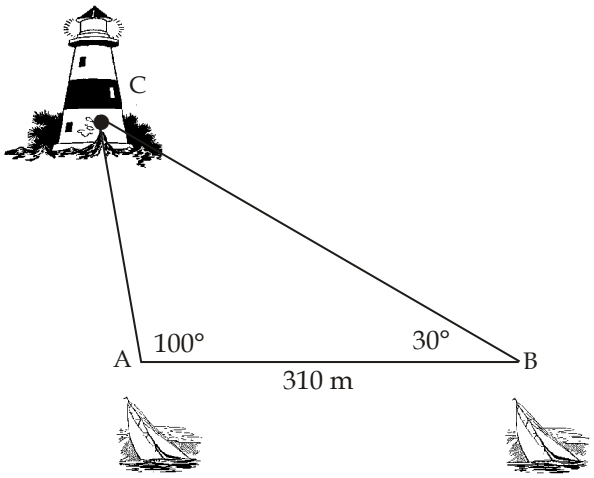
Summary:

$\angle B = 10°$

$b = 69.46$

$c = 136.81$

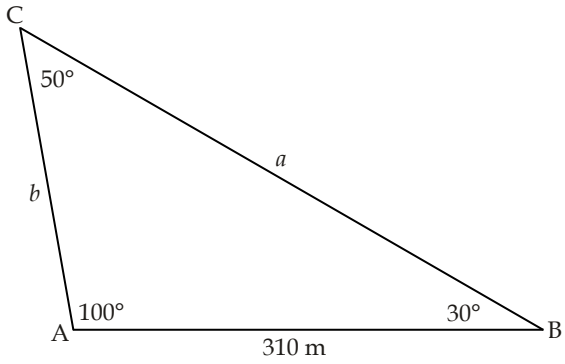
2. A lighthouse is visible to two sailboats 310 m apart. Use the angle measures shown to find the distance of each sailboat from the lighthouse. Round to two decimal places.



Answer:

$\angle C = 180° - (100° + 30°)$

$= 50°$



$$\frac{b}{\sin B} = \frac{c}{\sin C} \qquad \frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{b}{\sin 30°} = \frac{310}{\sin 50°} \qquad \frac{a}{\sin 100°} = \frac{310}{\sin 50°}$$

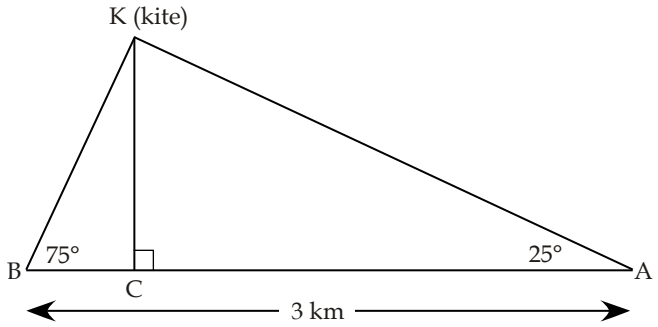
$$b = \frac{310 \sin 30°}{\sin 50°} \qquad a = \frac{310 \sin 100°}{\sin 50°}$$

$b = 202.34$ m $\qquad a = 398.53$ m

3. A kite is spotted simultaneously by two observers at points A and B, 3 km apart. The angles of elevation of the kite from A and B are 25° and 75°, respectively. How high above the ground to the nearest tenth is the kite if the kite is in the same vertical plane as A and B and if the point on the ground directly below the kite is:

   a) between A and B? (Include a diagram.)

   Answer:



$\angle BKA = 180° - (75° + 25°) = 80°$

In ΔBKA, by the Sine Law

$$\frac{BK}{\sin 25°} = \frac{3}{\sin 80°}$$

$$BK = \frac{3 \sin 25°}{\sin 80°}$$

$$BK = 1.287\ 413\ 489$$

In right ΔBCK, $\angle C = 90°$

$$\sin 75° = \frac{KC}{BK}$$

$$KC = BK \sin 75°$$

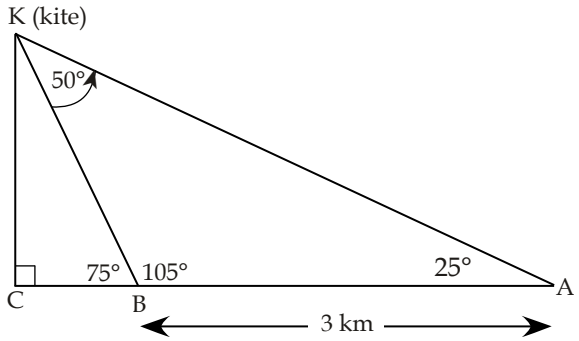
$$KC = 1.287\ 413\ 489 \sin 75°$$

$$= 1.243\ 545\ 938$$

$\therefore$ The kite is approximately 1.2 km above the ground.

b) not between A and B? (Include a diagram.)

Answer:



$\angle KBA = 180° - 75° = 105°$

$\angle BKA = 180° - (105° + 25°) = 50°$

In ΔABK, using the Sine Law:

$$\frac{BK}{\sin 25°} = \frac{3}{\sin 50°}$$

$$BK = \frac{3 \sin 25°}{\sin 50°}$$

$$BK = 1.655\ 066\ 878$$

In right ΔCKB, $\angle C = 90°$

$$\sin 75° = \frac{KC}{BK}$$

$$KC = BK \sin 75°$$

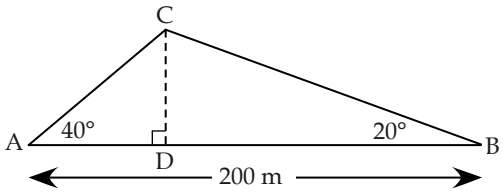
$$KC = 1.655\ 066\ 878 \sin 75°$$

$$= 1.598\ 671\ 842$$

In this case, the kite is 1.6 m above the ground.

4. To calculate the width of the Assiniboine River south of the Legislative Building, a surveyor marks a baseline, $\overline{AB}$, 200 m in length along the north riverbank. From each end of the baseline, an apartment, C, is sighted on the south bank of the river, making angles of 40° and 20° with the baseline. How wide is the river? Round to the nearest hundredth.

   Answer:



In ΔACB,

$$\angle C = 180° - (40° + 20°) = 120°$$

Use ΔABC and the Sine Law to find BC.

$$k\frac{200}{\sin 120°} = \frac{BC}{\sin 40°}$$

$$BC = \frac{200 \sin 40°}{\sin 120°}$$

$$BC = 148.445\ 439\ 8$$

In right ΔCDB, $\angle D = 90°$

$$\sin 20° = \frac{CD}{BC}$$

$$CD = BC \sin 20°$$

$$CD = 148.445\ 439\ 8 \sin 20°$$

$$CD = 50.77$$

The Assiniboine River is about 50.77 m wide at the Legislative Building.

5. Explain the steps in the given proof of the Sine Law.

a) $h_1 = b \sin A$ and $h_1 = a \sin B$

b) $b \sin A = a \sin B$

c) $\dfrac{b}{\sin B} = \dfrac{a}{\sin A}$

d) $h_2 = a \sin C$ and $h_2 = c \sin A$

e) $a \sin C = c \sin A$

f) $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$

g) $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Answer:

a) Determine two expressions for the height through C.

b) Equate two equal expressions for the same height.

c) Divide both sides by sin A sin B.

d) Determine two expressions for the height through B.

e) Equate two equal expressions for the same height.

f) Divide both sides by sin A sin C.

g) Equate the formulas found in (c) and (f) since they both equal $\dfrac{a}{\sin A}$.

Learning Activity 7.5

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. In which quadrant are $\sin\theta$, $\cos\theta$, and $\tan\theta$ all greater than zero?
2. What is the probability of rolling an even number on a number cube?
3. Calculate the tax (13%) on an item that costs $43.99.
4. The two legs of a right-angled triangle measure 6 cm and 8 cm. What is the measure of the hypotenuse?
5. What is the value of the discriminant of the quadratic equation $2x^2 - 3x + 6 = 0$?
6. Express in exponential form: $\sqrt{12x^3y^7}$
7. Find the midpoint between the two points (7, −1) and (−8, 15).
8. State the independent and dependent variables for the following situation: "Jace discovers that the more he studies for a test, the higher his mark is."

Answers:

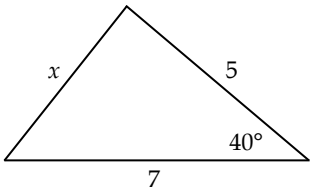
1. Quadrant I
2. $\frac{3}{6} = \frac{1}{2}$
3. $5.28 (10% of 43.99 = $4.40; 1% of 43.99 = $0.44; 13% of $43.99 = $4.40 + $0.44 + $0.44 + $0.44 = $5.72)
4. 10 ($6^2 + 8^2 = x^2$; $36 + 64 = x^2$; $100 = x^2$; $10 = x$)
5. −39 ($b^2 - 4ac = 9 - 4(2)(6) = 9 - 48 = -39$)
6. $12^{\frac{1}{2}}x^{\frac{3}{2}}y^{\frac{7}{2}}$ $\left(\sqrt{12x^3y^7} = \left(12x^3y^7\right)^{\frac{1}{2}}\right)$
7. $\left(-\frac{1}{2}, 7\right)$ $\left(\text{M} = \left(\frac{7+(-8)}{2}, \frac{-1+15}{2}\right) = \left(-\frac{1}{2}, \frac{14}{2}\right) = \left(-\frac{1}{2}, 7\right)\right)$
8. Independent Variable: The amount of time spent studying
   Dependent Variable: The mark on the test

Part B: The Cosine Law

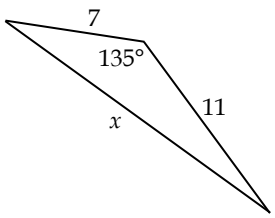
Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Find the length of x in each diagram. Round to the nearest tenth.

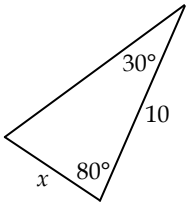
a)



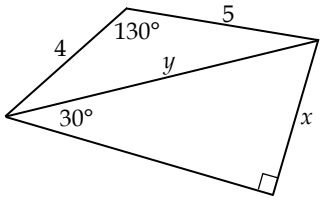
b)



c)



d)



Answers:

a) The triangle has no matching pair of a side and its opposing angle. Use the Cosine Law for sides.

$$x^2 = 7^2 + 5^2 - 2(7)(5)\left(\cos 40^\circ\right)$$

$$x^2 = 20.376\ 888\ 98$$

$$\sqrt{x^2} = \sqrt{20.376\ 888\ 98}$$

$$x = 4.5$$

b) The triangle has no matching pair of a side and its opposing angle. Use the Cosine Law for sides.

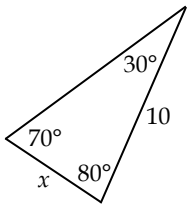
$$x^2 = 7^2 + 11^2 - 2(7)(11)\left(\cos 135^\circ\right)$$

$$x^2 = 278.894\ 444\ 3$$

$$\sqrt{x^2} = \sqrt{278.894\ 444\ 3}$$

$$x = 16.7$$

c) The triangle has a matching pair of side and its opposing angle once ∠A is found. Use the Sine Law.



$$\angle A = 180° - 80° - 30°$$

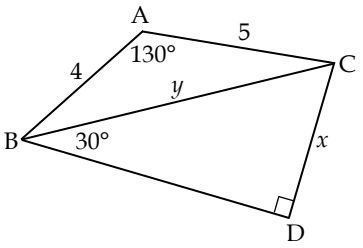
$$\angle A = 70°$$

$$\frac{x}{\sin 30°} = \frac{10}{\sin 70°}$$

$$x = \frac{10 \sin 30°}{\sin 70°}$$

$$x = 5.3$$

d) First solve ΔABC for y using the Cosine Law for sides. There is no matching pair of a side and its opposite angle.



$$y^2 = 4^2 + 5^2 - 2(4)(5)\left(\cos 130°\right)$$

$$y^2 = 66.711\ 504\ 39$$

$$\sqrt{y^2} = \sqrt{66.711\ 504\ 39}$$

$$y = 8.167\ 711\ 086$$

In ΔBCD, there is a right angle. Use the sine ratio.

$$\sin 30° = \frac{x}{y}$$

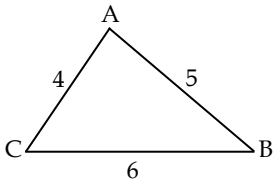
$$x = y \sin 30°$$

$$x = 8.167\ 711\ 086 \sin 30°$$

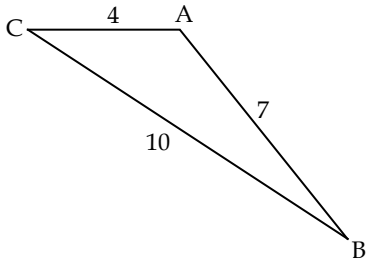
$$x = 4.1$$

2. Find the measure of ∠A. Round to the nearest tenth.

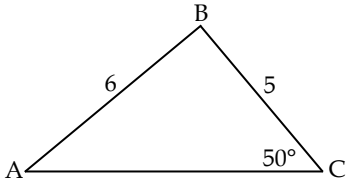
a)



b)



c)



Answers:

a) There is no matching pair for the Sine Law. Use the Cosine Law for angles.

Method 1: Write the Cosine Law as:

$$6^2 = 4^2 + 5^2 - 2(4)(5)\cos A$$

$$6^2 - 4^2 - 5^2 = -2(4)(5)\cos A$$

$$-5 = -40\cos A$$

$$0.125 = \cos A$$

$$A = \cos^{-1}(0.125)$$

$$A = 82.8°$$

Method 2: Use the rearranged formula:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{4^2 + 5^2 - 6^2}{2(4)(5)}$$

$$\cos A = 0.125$$

$$\angle A = \cos^{-1}(0.125)$$

$$\angle A = 82.8°$$

b) There is no matching pair to use the Sine Law. Use the Cosine Law for angles.

Using the rearranged formula:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{4^2 + 7^2 - 10^2}{2(4)(7)}$$

$$\cos A = -0.625$$

(Notice the negative value for cos A; so, ∠A will be obtuse.)

$$\angle A = 128.7°$$

c) There is a matching pair of an angle and its corresponding opposite side. Use the Sine Law.

$$\frac{5}{\sin A} = \frac{6}{\sin 50°}$$

$$\sin A = \frac{5 \sin 50°}{6}$$

$$\sin A = 0.638\ 370\ 369$$

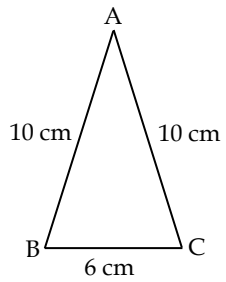
$$\therefore \angle A_r = \sin^{-1}(0.638\ 370\ 369)$$

$$\angle A_r = 39.7°$$

Since your calculator only gives one of two possible angles when using sine inverse, the solution for ∠A could be an acute or an obtuse angle. However, you are given the diagram and it shows ∠A to be acute. Thus, the only solution is ∠A = 39.7°.

3. An isosceles triangle has sides of length 10 cm, 10 cm, and 6 cm. What is the measure of each angle of this triangle? Round to the nearest tenth.

Answer:



Since there is no matching angle and its opposite side, solve for an angle using the Cosine Law for angles.

Using the rearranged formula:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{10^2 + 10^2 - 6^2}{2(10)(10)}$$

$$\cos A = 0.82$$

$$\angle A = \cos^{-1}(0.82)$$

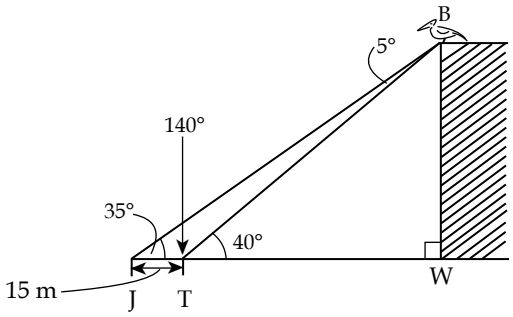
$$\angle A = 34.9°$$

Since ΔABC is isosceles, $\angle B = \angle C$. Thus, $\angle B = \frac{180° - 34.9°}{2} = 72.5°$.

The measure of the angles of the triangle are 34.9°, 72.5°, and 72.5°.

4. Janice spots a bird on the roof of her apartment block with an angle of elevation of 35°. She moves 15 m closer to the apartment block and now the angle of elevation is 40°. How far above the ground is the bird? (Assume the ground to be level and Janice walked directly toward the bird on the roof.) Round to the nearest tenth.

   Answer:



One method to solve for BW is to solve first for BT in ΔBJT and then use the sine ratio in the right triangle, ΔBTW.

For ΔBJT, $\angle JTB = 180° - 40° = 140°$ and $\angle JBT = 180° - 35° - 140° = 5°$.

$$\frac{BT}{\sin 35°} = \frac{15}{\sin 5°}$$

$$BT = \frac{15 \sin 35°}{\sin 5°}$$

$$BT = 98.715\,773\,33$$

In ΔBJW,

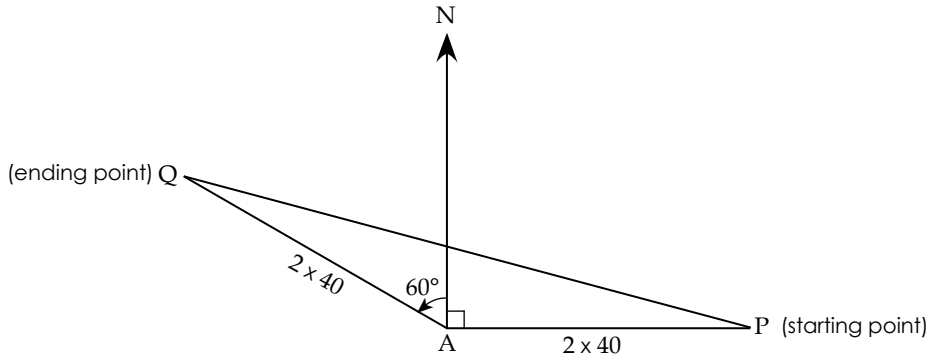
$$\sin 40° = \frac{BW}{BT}$$

$$BW = BT \sin 40°$$

$$BW = (98.715\ 773\ 33)(\sin 40°)$$

$$BW = 63.5 \text{ m}$$

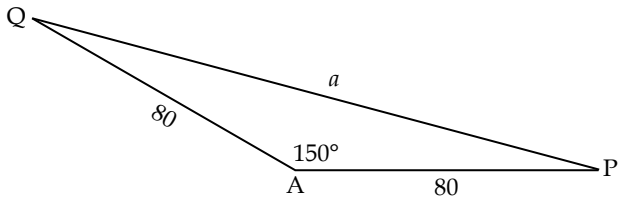
5. A ship leaves port at 12 noon and sails west at 40 km/h. At 2:00 p.m., it changes direction and sails N 60° W for another two hours. How far is the ship from the port? Round to the nearest tenth.



Answer:

The ship is travelling from noon until 2:00 p.m. (2 hours) for 40 km in each hour. It travels 80 km due west. When it changes direction, it travels for the same time at the same speed, and thus travels another 80 km.

The measure of ∠QAP = 60° + 90° = 150°.



There is no matching pair of side and opposite angles. Use the Law of Cosines for sides.

$$a^2 = 80^2 + 80^2 - 2(80)(80)(\cos 150°)$$

$$a^2 = 23\ 885.125\ 17$$

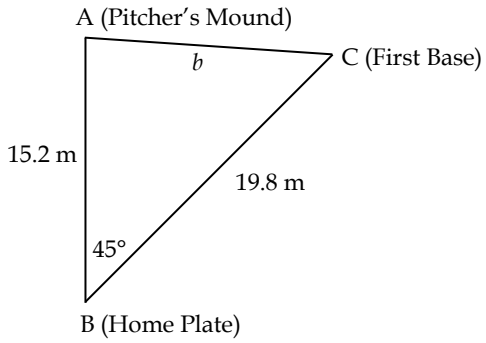
$$\sqrt{a^2} = \sqrt{23\ 885.125\ 17}$$

$$a = 154.5$$

The ship is 154.5 km from port at 4:00 p.m.

6. The pitcher's mound on a slo-pitch softball diamond is 15.2 m from home plate and the distance between the bases is 19.8 m. How far is the pitcher's mound from first base if the angle at home plate between the pitcher's mound and first base is 45°? Round to the nearest hundredth.

Answer:



There is no matching pair of a side and its opposite angle. Thus, use the Cosine Law.

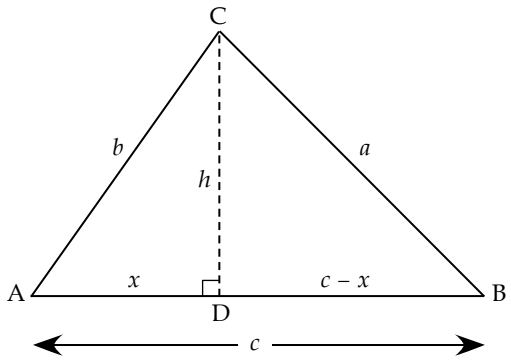
$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$= (19.8)^2 + (15.2)^2 - 2(19.8)(15.2) \cos 45°$$

$$b = 14.05$$

Thus, the distance between the pitcher's mound and first base is 14.05 m.

7. Prove the Law of Cosines for $a^2 = b^2 + c^2 - 2bc \cos A$ using the diagram below.



Proof:

The proof begins by using the Pythagorean theorem for each right triangle:

in ΔBCD, $a^2 = h^2 + (c - x)^2$

in ΔACD, $b^2 = h^2 + x^2$

The next step is to subtract the two equations to eliminate h^2.

Now finish the proof.

Answer:

Subtract the two equations to eliminate h^2:

$$a^2 - b^2 = (c - x)^2 - x^2$$

$$a^2 - b^2 = c^2 - 2cx + \cancel{x^2} - \cancel{x^2} \qquad \text{Simplify}$$

$$a^2 - b^2 = c^2 - 2cx \qquad \text{Simplify}$$

$$a^2 = b^2 + c^2 - 2cx \qquad \text{Rearrange}$$

Now, write x in terms of the sides and angles of triangle ABC, and use ΔACD and find cos A.

$$\cos A = \frac{x}{b}$$

$$x = b \cos A$$

Substitute $x = b \cos A$ into $a^2 = b^2 + c^2 - 2cx$ to get:

$a^2 = b^2 + c^2 - 2c(b \cos A)$

$a^2 = b^2 + c^2 - 2bc \cos A$

Note to Student: You are not expected to know the proof of either the Sine Law or the Cosine Law for the exam.

Learning Activity 7.6

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Calculate the tax (13%) on an item that costs $72.
2. How many 4-piece bags of candy can you make to give out on Halloween if you have 288 pieces of candy?
3. In which quadrants is $\cos\theta > 0$?
4. The length of the hypotenuse of a right triangle is 13 and one of its legs is length 12. What is the length of the other leg?
5. Does the parabola $y + 5x^2 = -2x + 4$ open up or down?
6. Express as a radical: $x^{\frac{13}{2}}y^{\frac{5}{2}}$
7. $(-56, 27)$ is on a line with a slope of $\frac{6}{8}$. Find another point on this line.
8. Simplify: $(x^2 - 1)(2x^2 + 5x - 6)$

Answers:

1. $9.36 (10% of 72 = $7.20; 1% of 72 = $0.72; so 13% of $72 = $7.20 + $0.72 + $0.72 + $0.72 = $9.36)
2. $72\ \left(\frac{288}{4} = \frac{200}{4} + \frac{88}{4} = 50 + 22 = 72\right)$
3. Quadrants I and IV
4. 5 ($x^2 + 12^2 = 13^2$; $x^2 = 169 - 144$; $x^2 = 25$; $x = 5$)
5. Down ($y = -5x^2 - 2x + 4$)
6. $\sqrt{x^{13}y^5}\ \left(x^{\frac{13}{2}}y^{\frac{5}{2}} = \left(x^{13}y^5\right)^{\frac{1}{2}} = \sqrt{x^{13}y^5}\right)$
7. $(-48, 33)$ (A slope of $\frac{6}{8}$ refers to a rise of 6, which affects the y-values, and a run of 8, which affects the x-values, so $(-56 + 8, 27 + 6)$)
8. $2x^4 + 5x^3 - 6x^2 - 2x^2 - 5x + 6 = 2x^4 + 5x^3 - 8x^2 - 5x + 6$

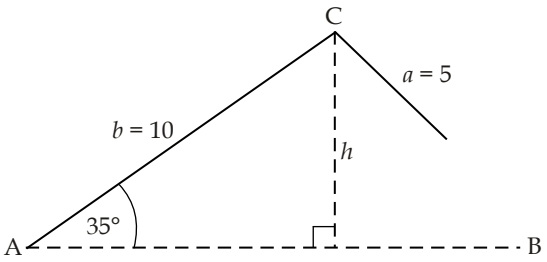
Part B: The Ambiguous Case

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Decide whether there are no solutions, exactly one solution, or two solutions for each ΔABC. You do not need to solve the triangle, but you must justify your choice and include a diagram.

   Note: Always draw the given angle in the lower left corner of the triangle. Mark the given side opposite it with a dotted line. Do not mark the second given side as the base. (Reason: It makes solving the ambiguous case easier if you always use the same pattern.)

   a)



   Answer:

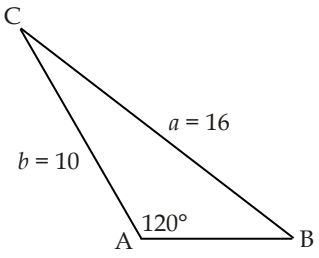
   The side opposite the given angle is smaller than the other given side. We need to find the height.

   $h = 10 \sin 35°$

   $h = 5.7$

   Since $5 < h$, there is no triangle.

   b)



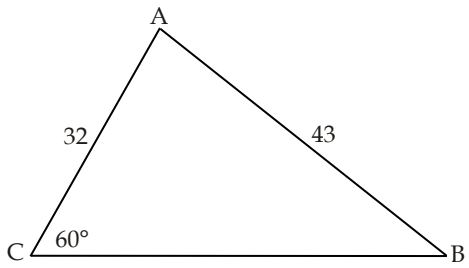
   Answer:

   Because the side opposite the given angle is greater than the other given side, and ∠A is obtuse, there is only one triangle.

c) $\angle C = 60°$, $c = 43$, $b = 32$

Answer:

The longer side is opposite the given angle. There is only one triangle.



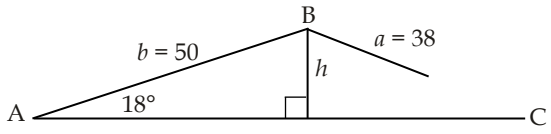
d) $\angle A = 18°$, $a = 38$, $c = 50$

Answer:

The side opposite the given angle is smaller than the other given side. We need to find the height.

$h = 50 \sin 18° = 15.45$

Since $38 > h$, there are two solutions.



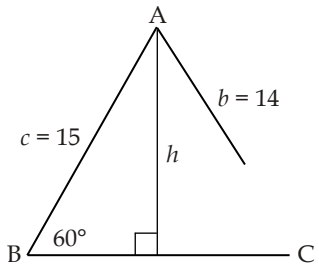
e) $\angle B = 60°$, $b = 14$, $c = 15$

Answer:

The side opposite the given angle is smaller than the other given side. We need to find the height.

$h = 15 \sin 60° = 12.99$

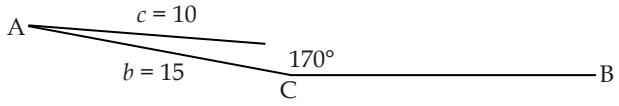
Since $14 > h$, we have two solutions.



f) $\angle C = 170°$, $c = 10$, $b = 15$

Answer:

The side opposite the given angle is shorter than the other given side and the given angle is obtuse. There is no solution.



2. Find the possible values of the indicated side. Round answers to the nearest tenth.

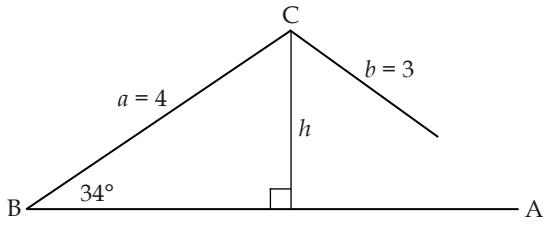
a) In ΔABC, $\angle B = 34°$, $a = 4$, and $b = 3$. Find c.

Answer:

Since the side opposite the given angle is smaller than the other given side, we need to find the height.

$h = 4 \sin 34° = 2.2$

$3 > h$, so there are two triangles.



We will now solve for the two measures of $\angle A$, using the Sine Law.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{4}{\sin A} = \frac{3}{\sin 34°}$$

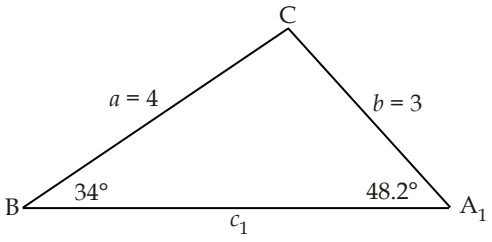
$$\sin A = \frac{4 \sin 34°}{3}$$

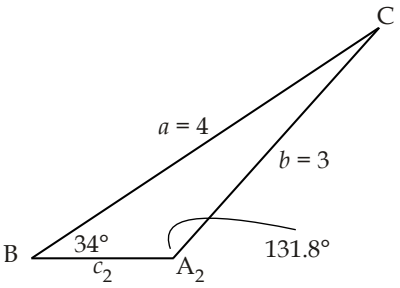
$$\angle A_r = \sin^{-1}\left(\frac{4 \sin 34°}{3}\right)$$

$$\angle A_1 = 48.2° \text{ (acute angle)}$$

$$\angle A_2 = 180° - 48.2° = 131.8° \text{ (obtuse angle)}$$

Now we can draw the two triangles and solve each of them.





$\angle C = 180° - (34° + 48.2°)$

$\angle C = 97.8°$

To solve for c_1:

$$\frac{c_1}{\sin 97.8°} = \frac{3}{\sin 34°}$$

$$c_1 = 5.3$$

$\angle C = 180° - (34° + 131.8°)$

$\angle C = 14.2°$

To solve for c_2:

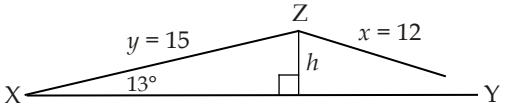
$$\frac{3}{\sin 34°} = \frac{c_2}{\sin 14.2°}$$

$$c_2 = 1.3$$

b) In ΔXYZ, $\angle X = 13°$, $x = 12$, and $y = 15$. Find z.

Answer:

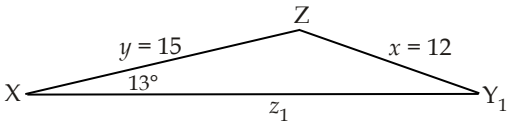
Since the side opposite the given angle is smaller than the other given side, we need to find the height.

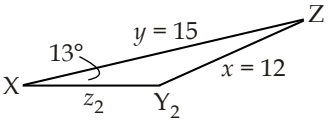


$h = 15 \sin 13° = 3.37$

$12 > h$, so there are two triangles

We will now solve for the two measures for $\angle Y$ using the Sine Law.





$$\frac{x}{\sin X} = \frac{y}{\sin Y}$$

$$\frac{12}{\sin 13°} = \frac{15}{\sin Y}$$

$$\sin Y = \frac{15 \sin 13°}{12}$$

$$\angle Y_r = 16.3°$$

Thus, $\angle Y_1 = 16.3°$ and $\angle Y_2 = 180° - 16.3° = 163.7°$.

In ΔXY_1Z:

If $\angle Y_1 = 16.3°$

then $\angle Z = 180° - (16.3° + 13°)$

$= 150.7°$

In ΔXY_2Z:

If $\angle Y_2 = 163.7°$

then $\angle Z = 180° - (13° + 163.7°)$

$= 3.3°$

In ΔXY_1Z:

For z_1:

$$\frac{x}{\sin X} = \frac{z_1}{\sin Z}$$

$$\frac{12}{\sin 13°} = \frac{z_1}{\sin 150.7°}$$

$$z_1 = \frac{12 \sin 150.7°}{\sin 13°}$$

$$z_1 = 26.1$$

In ΔXY_2Z:

For z_2:

$$\frac{x}{\sin X} = \frac{z_2}{\sin Z}$$

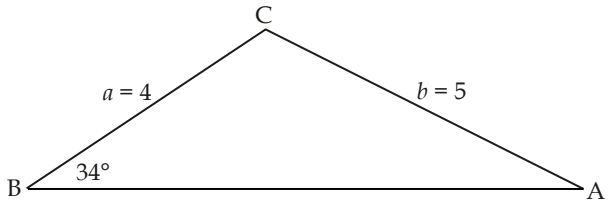
$$\frac{12}{\sin 13°} = \frac{z_2}{\sin 3.3°}$$

$$z_2 = \frac{12 \sin 3.3°}{\sin 13°}$$

$$z_2 = 3.1$$

c) In ΔABC, $\angle B = 34°$, $a = 4$, and $b = 5$. Find c.

Answer:



Since $b > a$, only one triangle exists.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{4}{\sin A} = \frac{5}{\sin 34°}$$

$$\sin A = \frac{4 \sin 34°}{5}$$

$$\angle A = 26.6°$$

$$\angle C = 180° - (34° + 26.6°)$$

$$= 119.4°$$

side c

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{5}{\sin 34°} = \frac{c}{\sin 119.4°}$$

$$c = \frac{5 \sin 119.4°}{\sin 34°}$$

$$c = 7.8$$

Note: You can verify that there is only one value for $\angle A$.

If there were two triangles, then the second $\angle A$ is:

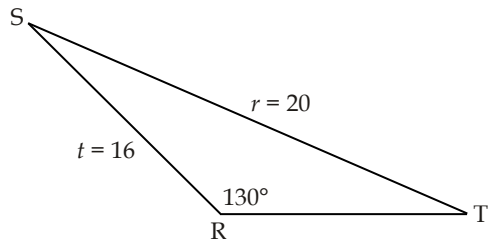
$\angle A_2 = 180° - 26.6° = 153.4°$.

$\therefore \angle C_2 = 180° - (34° + 153.4°) = -7.4°$

This is impossible. An angle in a triangle cannot have a negative measure.

d) In ΔRST, $\angle R = 130°$, $r = 20$, and $t = 16$. Find s.

Answer:



Because $r > t$, there is only one solution.

To find the measure of $\angle T$:

$$\frac{r}{\sin R} = \frac{t}{\sin T}$$

$$\frac{20}{\sin 130°} = \frac{16}{\sin T}$$

$$\sin T = \frac{16 \sin 130°}{20}$$

$$\angle T = 37.8°$$

$$\angle S = 180° - (37.8° + 130°)$$

$$= 12.2°$$

To find the measure of s:

$$\frac{s}{\sin S} = \frac{r}{\sin R}$$

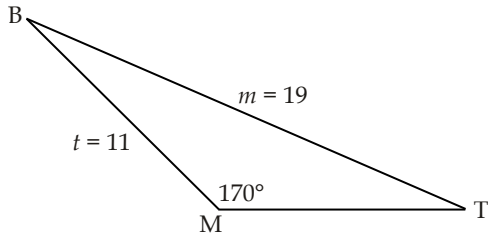
$$\frac{s}{\sin 12.2°} = \frac{20}{\sin 130°}$$

$$s = \frac{20 \sin 12.2°}{\sin 130°}$$

$$s = 5.5$$

e) In ΔMBT, ∠M = 170°, m = 19, and t = 11. Find b.

Answer:



Because $m > t$, there is only one solution.

To find the measure of ∠T:

$$\frac{t}{\sin T} = \frac{m}{\sin M}$$

$$\frac{11}{\sin T} = \frac{19}{\sin 170°}$$

$$\sin T = \frac{11 \sin 170°}{19}$$

$$\angle T = 5.8°$$

$$\angle B = 180° - (170° + 5.8°)$$

$$= 4.2°$$

To find the measure of b:

$$\frac{b}{\sin B} = \frac{m}{\sin M}$$

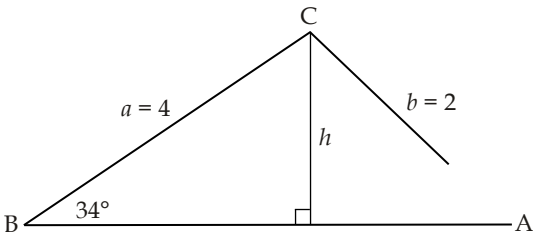
$$\frac{b}{\sin 4.2°} = \frac{19}{\sin 170°}$$

$$b = \frac{19 \sin 4.2°}{\sin 170°}$$

$$b = 8.0$$

f) In ΔABC, ∠B = 34°, a = 4, and b = 2. Find c.

Answer:



Since the side opposite the given angle is smaller than the other given side, we need to find the height.

$h = 4 \sin 34°$

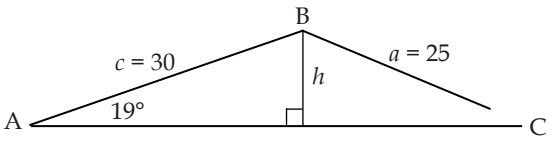
$h = 2.2$

Since the side opposite the given angle is smaller than the height of the triangle, there is no solution.

3. Find all possible values of the indicated angle measure. Round to one decimal place.

a) In ΔABC, ∠A = 19°, a = 25, and c = 30. Find ∠C.

Answer:



Since the side opposite the given angle is smaller than the other given side, we need to find the height.

$h = 30 \sin 19°$

$h = 9.8$

Since the side opposite the given angle is larger than the height of the triangle, there are two solutions of ∠C. We will find the two measures for ∠C using the Sine Law.

$$\frac{30}{\sin C} = \frac{25}{\sin 19°}$$

$$\sin C = \frac{30 \sin 19°}{25}$$

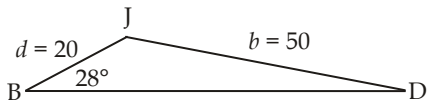
$$\angle C_r = \sin^{-1}\left(\frac{30 \sin 19°}{25}\right)$$

$$\angle C_1 = 23.0° \text{ (acute angle)}$$

$$\angle C_2 = 180° - 23.0° = 157.0° \text{ (obtuse angle)}$$

b) In ΔBDJ, ∠B = 28°, b = 50, and d = 20. Find ∠D.

Answer:



Because $b > d$, there is only one value for ∠D.

$$\frac{d}{\sin D} = \frac{b}{\sin B}$$

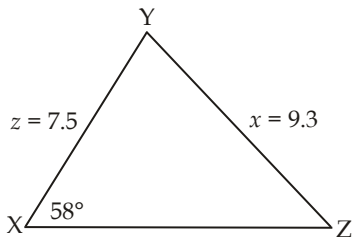
$$\frac{20}{\sin D} = \frac{50}{\sin 28°}$$

$$\sin D = \frac{20 \sin 28°}{50}$$

$$\angle D = 10.8°$$

c) In ΔXYZ, ∠X = 58°, x = 9.3, and z = 7.5. Find ∠Z.

Answer:



Because $x > z$, there is only one solution.

$$\frac{x}{\sin X} = \frac{z}{\sin Z}$$

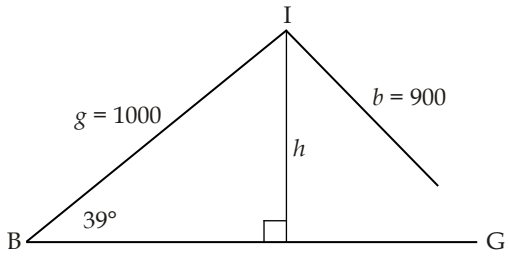
$$\frac{9.3}{\sin 58°} = \frac{7.5}{\sin Z}$$

$$\sin Z = \frac{7.5 \sin 58°}{9.3}$$

$$\angle Z = 43.1°$$

d) In ΔBIG, $\angle B = 39°$, $b = 900$, and $g = 1000$. Find $\angle I$.

Answer:



Since the side opposite the given angle is smaller than the other given side, we need to find the height.

$h = 1000 \sin 39°$

$h = 629.3$

Since the side opposite the given angle is larger than the height of the triangle, there are two solutions. We will find the two measures of $\angle G$ first.

$$\frac{900}{\sin 39°} = \frac{1000}{\sin G}$$

$$\angle G_r = \sin^{-1}\left(\frac{1000 \sin 39°}{900}\right)$$

$\angle G_1 = 44.4°$ (acute)

$\angle G_2 = 180° - 44.4° = 135.6°$ (obtuse)

Now we can find the corresponding measures of the third angle in the triangle, $\angle I$.

$\angle I_1 = 180° - (39° + 44.4°) = 96.6°$

$\angle I_2 = 180° - (39° + 135.6°) = 5.4°$

Grade 11 Pre-Calculus Mathematics (30S)

Module 7
Trigonometry

Learning Activity Answer Keys

MODULE 7: TRIGONOMETRY

Learning Activity 7.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Jacob went to three stores. He spent \$76 at the first store, \$47 at the second store, and \$152 at the third store. If Jacob had \$300, how much money does he have left?
2. If two angles in a triangle measure 137° in total, what is the measure of the third angle in the triangle?
3. Solve for g: $(2g + 1)(g - 5) = 0$
4. You and your brother are partial owners of the family vehicle. If you own half as much as your brother, how much of the car do you own?
5. Simplify: $\sqrt{108}$
6. Order these values from lowest to highest: $5^{\frac{1}{5}}$, $\sqrt{5}$, $5^{\frac{1}{4}}$, and $\sqrt[3]{5}$
7. Rationalize the denominator: $\dfrac{7\sqrt{3}}{3\sqrt{2}}$
8. Find the slope of the line parallel to the line $2x + y = 1$.

Answers:

1. \$25 (\$76 + \$47 + \$152 = \$275; \$300 − \$275 = \$25)
2. 43° (180° − 137°)
3. $g = -\dfrac{1}{2}$ or $g = 5$
4. $\dfrac{1}{3}$ (Your brother owns two parts and you own one part. So you can think of there being a total of 3 parts.)
5. $6\sqrt{3}$ $\left(\sqrt{4}\sqrt{9}\sqrt{3}\right)$

6. $5^{\frac{1}{5}} < 5^{\frac{1}{4}} < \sqrt[3]{5} < \sqrt{5}$ $\left(\sqrt{5} = 5^{\frac{1}{2}};\ \sqrt[3]{5} = 5^{\frac{1}{3}}\right)$

7. $\frac{7\sqrt{6}}{6}$ $\left(\frac{7\sqrt{3}}{3\sqrt{2}} = \frac{7\sqrt{3}\sqrt{2}}{3(2)}\right)$

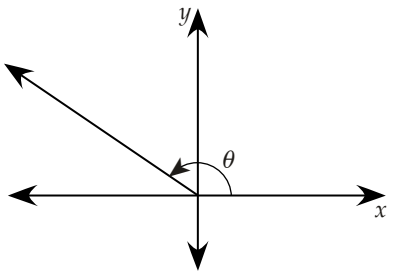
8. -2 $(y = -2x + 1$

Part B: Trigonometric Ratios

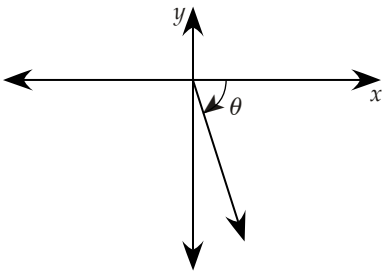
Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Determine which of the following angles are in standard position. Explain.

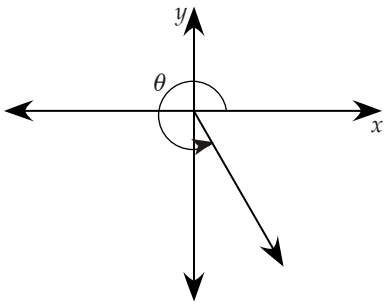
a)



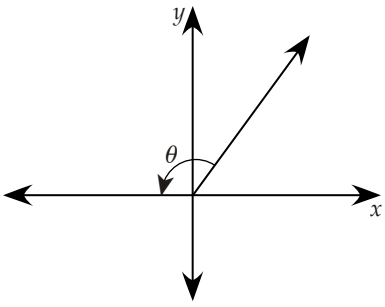
b)



c)



d)



Answers:

a) This angle is in standard position as the initial side is along the positive x-axis and the angle is measured from the initial to the terminal arm, and the vertex is at the origin. The angle is positive because its rotation is in a counter-clockwise direction.

b) This angle is in standard position and the vertex is at the origin. The initial side is along the positive x-axis. It is rotated to the terminal arm in a clockwise direction and its measure is considered a negative value.

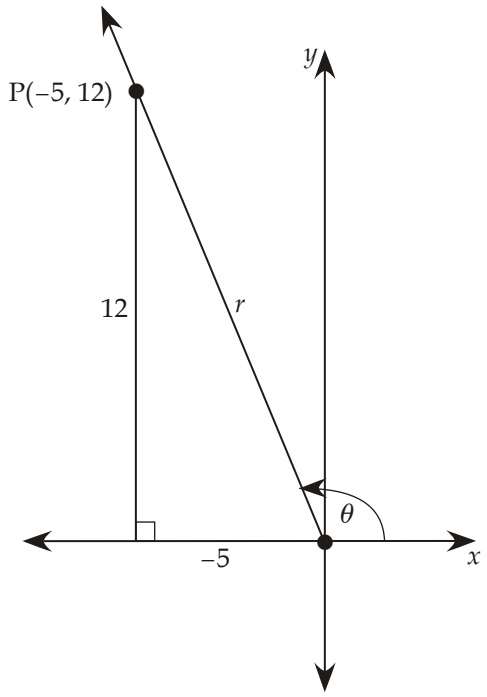
c) This angle is in standard position. This angle is the reflex angle to the angle in (b). The angle is measured counter-clockwise from the initial side to the terminal arm so it is a positive angle.

d) This angle is not in standard position, as the initial side is not along the positive x-axis.

2. P(x, y) is a point on the terminal side of angle θ in standard position. Determine sin θ, cos θ, and tan θ for the following points. Draw a sketch. Show your work for determining the distance from the origin to the point P(x, y). Rationalize any denominators.

a) (−5, 12)

Answer:



$$r = \sqrt{x^2 + y^2}$$
$$= \sqrt{(-5)^2 + 12^2}$$
$$= \sqrt{169}$$
$$= 13$$

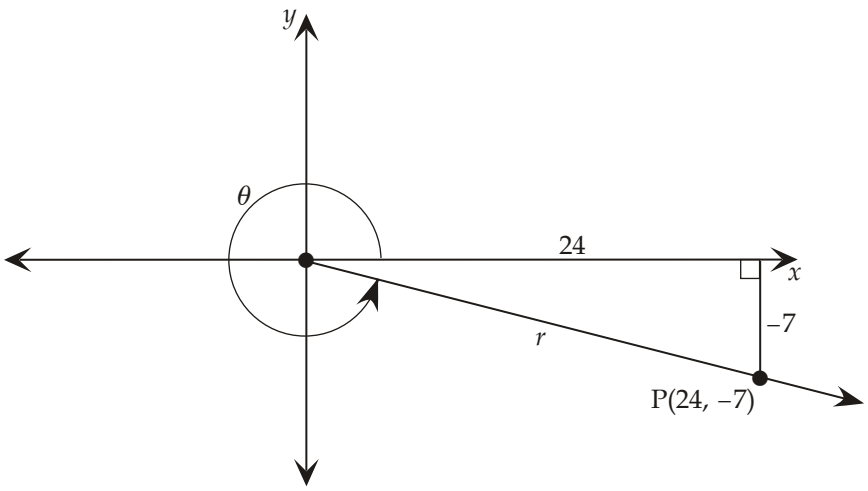
$$\sin\theta = \frac{y}{r} = \frac{12}{13}$$

$$\cos\theta = \frac{x}{r} = \frac{-5}{13} = -\frac{5}{13}$$

$$\tan\theta = \frac{y}{x} = \frac{12}{-5} = -\frac{12}{5}$$

b) (24, −7)

Answer:



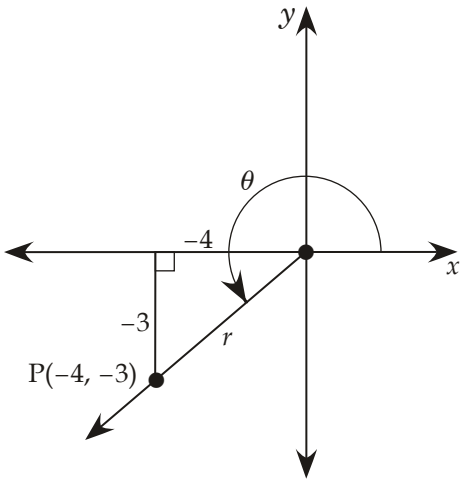
Notice that these triangles in quadrant diagrams do not need to be drawn to scale.

$$r = \sqrt{x^2 + y^2}$$
$$= \sqrt{(24)^2 + (-7)^2}$$
$$= \sqrt{625}$$
$$= 25$$

$$\sin\theta = \frac{y}{r} = \frac{-7}{25} = -\frac{7}{25}$$
$$\cos\theta = \frac{x}{r} = \frac{24}{25}$$
$$\tan\theta = \frac{y}{x} = \frac{-7}{24} = -\frac{7}{24}$$

c) (−4, −3)

Answer:

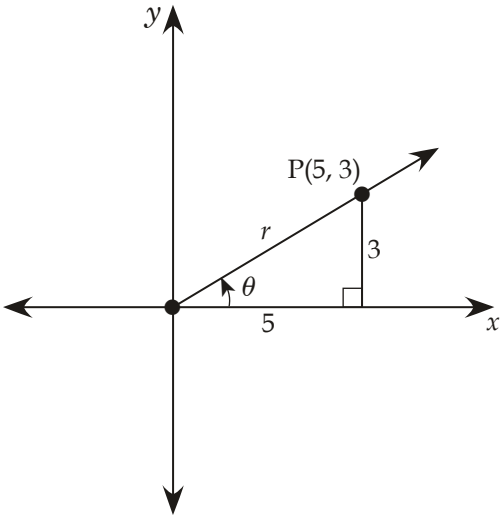


$$r = \sqrt{x^2 + y^2}$$
$$= \sqrt{(-4)^2 + (-3)^2}$$
$$= \sqrt{25}$$
$$= 5$$

$$\sin\theta = \frac{y}{r} = \frac{-3}{5} = -\frac{3}{5}$$
$$\cos\theta = \frac{x}{r} = \frac{-4}{5} = -\frac{4}{5}$$
$$\tan\theta = \frac{y}{x} = \frac{-3}{-4} = \frac{3}{4}$$

d) (5, 3)

Answer:



$$r = \sqrt{x^2 + y^2}$$
$$= \sqrt{5^2 + 3^2}$$
$$= \sqrt{34}$$

$$\sin\theta = \frac{y}{r} = \frac{3}{\sqrt{34}} = \frac{3\sqrt{34}}{34}$$

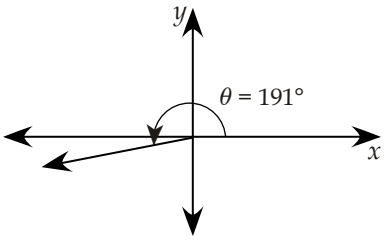
$$\cos\theta = \frac{x}{r} = \frac{5}{\sqrt{34}} = \frac{5\sqrt{34}}{34}$$

$$\tan\theta = \frac{y}{x} = \frac{3}{5}$$

3. Without using a calculator, determine if each value should be positive or negative by drawing the angle in standard position.

a) cos 191°

Answer:



θ is in Quadrant III where x-coordinates are negative; or, you can use the CAST rule:

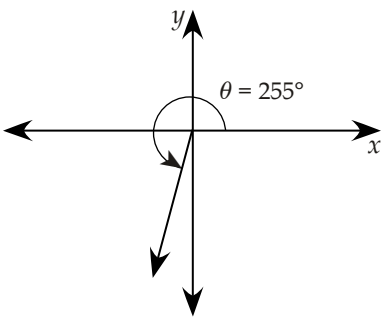
S	A
T	C

cos θ is negative in Quadrant III.

$\therefore$ cos 191° is negative.

b) sin 255°

Answer:



θ = 255° is located in Quadrant III where y-coordinates are negative; or you can use the CAST rule:

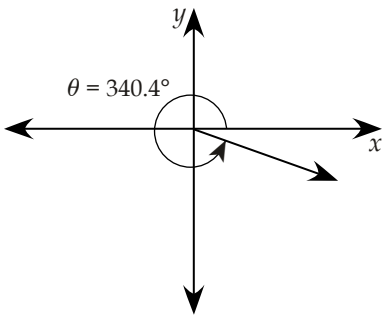
S	A
T	C

sin $\theta < 0$ in this quadrant.

$\therefore$ sin 255° is negative.

c) tan 340.4°

Answer:



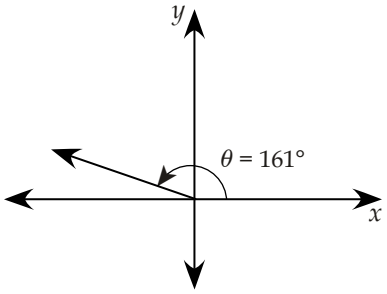
$\theta = 340.4°$ is located in Quadrant IV where x-values are positive and y-values are negative.

$$\therefore \tan 340.4° = \frac{y\text{-value}}{x\text{-value}}$$
$$= \frac{\text{negative}}{\text{positive}}$$
$$= \text{negative}$$

Or, use the CAST rule and only cosine is positive in Quadrant IV. So tan 340.4° is negative.

d) sin 161°

Answer:



$\theta = 161°$ is located in Quadrant II where the y-values are positive.

$$\therefore \sin\theta = \frac{y\text{-value}}{r}$$
$$= \frac{\text{positive}}{\text{positive}}$$
$$= \text{positive}$$

Or, use the CAST rule and $\sin\theta$ is positive in Quadrant II. So sin 161° is positive.

4. Determine the quadrants in which P(x, y) may lie under the following conditions.

a) $\sin\theta > 0$

Answer:

The ratio for $\sin\theta$ is $\frac{y\text{-value}}{r\text{-value}}$. Thus, $\sin\theta$ is positive when the y-values are positive because the r-value is always positive. This occurs in Quadrant I or II.

b) $\cos\theta > 0$

Answer:

The ratio for $\cos\theta$ is $\frac{x\text{-value}}{r\text{-value}}$. Thus, $\cos\theta$ is positive when the x-values are positive. This occurs in Quadrant I or IV.

c) $\tan\theta > 0$

Answer:

The ratio for $\tan\theta$ is $\frac{y\text{-value}}{x\text{-value}}$. Thus, $\tan\theta$ is positive when the x- and y-values are both positive or both negative. This occurs in Quadrant I or III.

d) $\cos\theta > 0$ and $\sin\theta < 0$

Answer:

Since the x-value must be positive and the y-value negative, the only quadrant where this occurs is Quadrant IV.

5. Fill in the following chart by indicating which quadrant(s) each angle is in by considering the sign of each ratio or by using the CAST rule.

Answers:

Ratio Value	Quadrant
$\cos\theta = \frac{2}{5}$	I or IV
$\sin\theta = \frac{4}{5}$	I or II
$\tan\theta = -\frac{2}{3}$	II or IV
$\sin\theta = \frac{-1}{5}$	III or IV
$\tan\theta = \frac{3}{5}$	I or III

6. Given the following angles in standard position, determine if the sine, cosine, and tangent ratios are positive or negative. Explain your answers.

   a) $\theta = 26°$

   Answer:

   This angle is located in Quadrant 1. All of the ratios, including the sine, cosine, and tangent ratios, are positive because all ratios in Quadrant I are positive.

   b) $\theta = 305°$

   Answer:

   This angle is located in Quadrant IV where cosine is positive and both sine and tangent are negative. In Quadrant IV, $x > 0$, $y < 0$; thus, $\frac{x}{r}$ is positive, $\frac{y}{r}$ is negative, and $\frac{y}{x}$ is negative.

   c) $\theta = 152°$

   Answer:

   This angle is located in Quadrant II where sine is positive and both cosine and tangent are negative. In Quadrant II, $x < 0$ and $y > 0$.

   d) $\theta = 206°$

   Answer:

   This angle is located in Quadrant III. Sine and cosine are both negative in Quadrant III and tangent is positive.

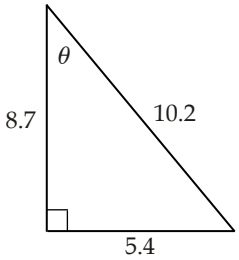
Learning Activity 7.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Gwen planted 120 seeds of green beans. If one-quarter of the beans did not grow, how many beans did grow?
2. The minute hand of a clock starts at 12 and travels 60% of the way around the clock. How many degrees from the 12 has the minute hand travelled?
3. Calculate: $\frac{4}{3} + \frac{5}{7}$
4. The two legs of a right-angled triangle measure 3 cm and 4 cm. What is the measure of the hypotenuse?
5. If you do homework for half an hour every night from Monday to Friday, as well as 45 minutes on both Saturday and Sunday, how many hours of homework do you do in a week?

Use the following right-angled triangle to answer the next three questions. Do not simplify.



6. Identify the sine ratio for θ.
7. Identify the cosine ratio for θ.
8. Identify the tangent ratio for θ.

Answers:

1. 90 (one-quarter of 120 is 30, and 120 − 30 = 90)
2. 216° (60% of 360 degrees = 50% of 360 + 10% of 360 = 180° + 36°)
3. $\frac{43}{21}\left(\frac{28}{21} + \frac{15}{21}\right)$
4. 5 ($3^2 + 4^2 = x^2$; $9 + 16 = x^2$; $25 = x^2$; $5 = x$)

5. 4 hours (5 × half an hour = 2.5 hours; 45 minutes × 2 = 1.5 hours)
6. $\frac{5.4}{10.2}$
7. $\frac{8.7}{10.2}$
8. $\frac{5.4}{8.7}$

Part B: Reference Angles and Solving Trigonometric Equations in Linear Form

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. In which quadrant does the terminal arm of each angle lie?
   a) 65°

      Answer:

      As this angle lies between 0° and 90°, the terminal arm of this angle lies in Quadrant I.

   b) 340°

      Answer:

      As this angle lies between 270° and 360°, the terminal arm of this angle lies in Quadrant IV.

   c) 241°

      Answer:

      As this angle lies between 180° and 270°, the terminal arm of this angle lies in Quadrant III.

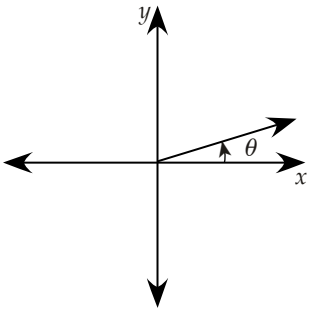
   d) 123°

      Answer:

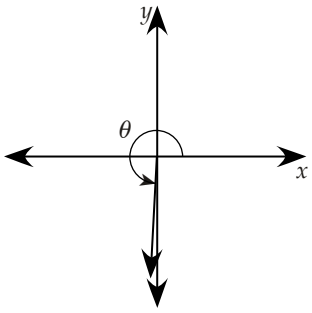
      As this angle lies between 90° and 180°, the terminal arm of this angle lies in Quadrant II.

2. Without measuring, match each angle to its corresponding graph.
   a) 140°
   b) 267°
   c) 316°
   d) 17°

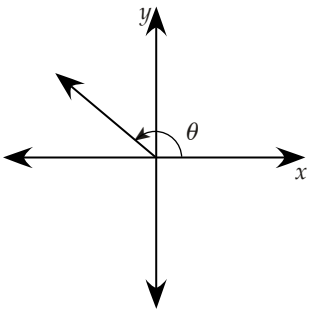
i)



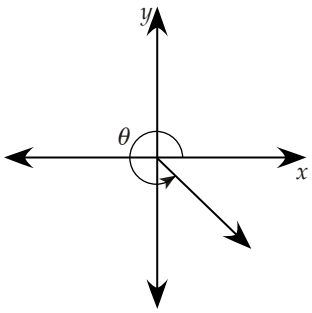
ii)



iii)



iv)



Answers:

(a) and (iii)

(b) and (ii)

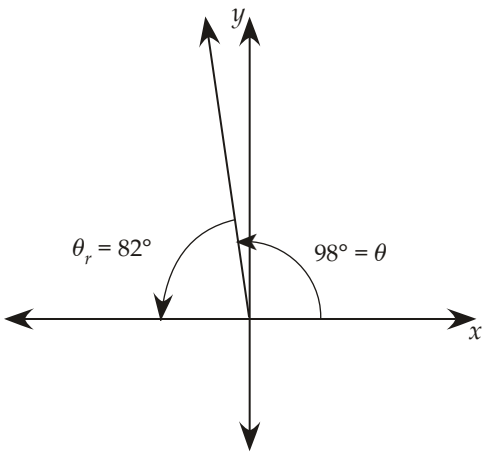
(c) and (iv)

(d) and (i)

3. Sketch each angle in standard position and find its reference angle. Write the measure of the reference angle.

a) 98°

Answer:

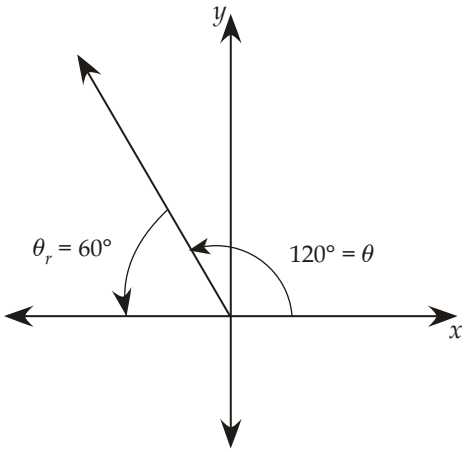


This work is optional.

$\theta_r = 180° - \theta$
$\theta_r = 180° - 98°$
$\theta_r = 82°$

b) 120°

Answer:

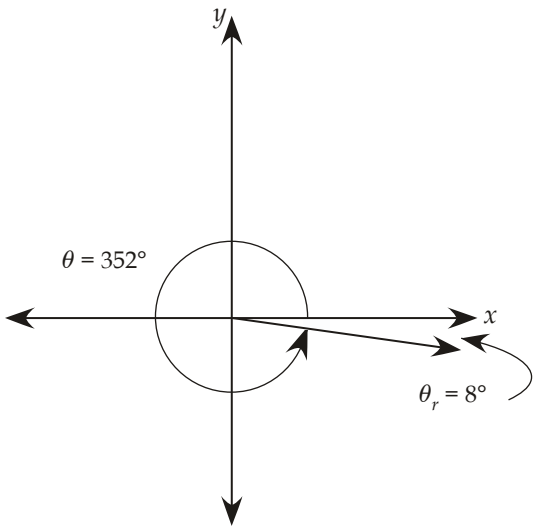


This work is optional.

$\theta_r = 180° - \theta$
$\theta_r = 180° - 120°$
$\theta_r = 60°$

c) 352°

Answer:

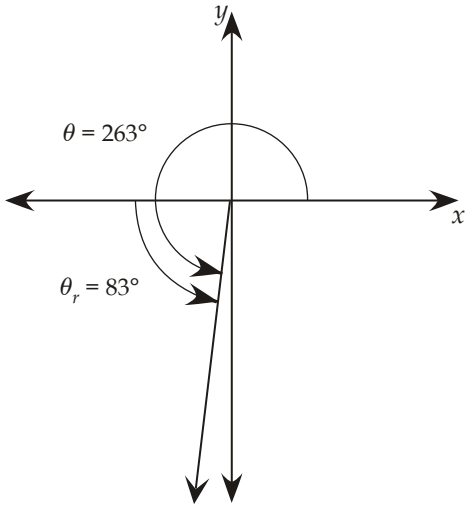


This work is optional.

$\theta_r = 360° - \theta$
$\theta_r = 360° - 352°$
$\theta_r = 8°$

d) 263°

Answer:



This work is optional.

$\theta_r = \theta - 180°$
$\theta_r = 263° - 180°$
$\theta_r = 83°$

4. Determine the measure of the 3 other angles in standard position that have the same reference angle as an angle in standard position of 58°.

   Answer:

   In Quadrant II, the angle is: $\theta = 180° - \theta_r$

   $\theta = 180° - 58°$

   $\theta = 122°$

   In Quadrant III, the angle is: $\theta = 180° + \theta_r$

   $\theta = 180° + 58°$

   $\theta = 238°$

   In Quadrant IV, the angle is: $\theta = 360° - \theta_r$

   $\theta = 360° - 58°$

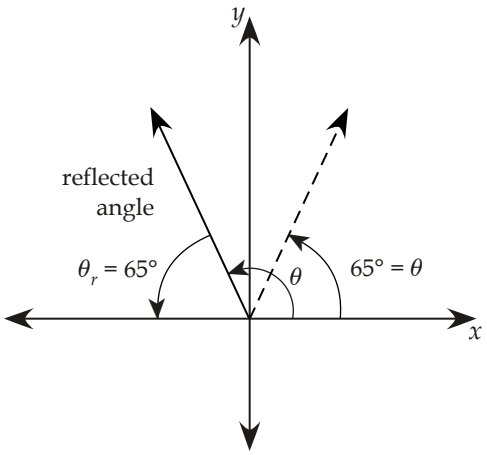
   $\theta = 302°$

   The other three angles are 122°, 238°, and 302°.

5. Determine the angle in standard position when an angle of 65° is reflected

   a) in the *y*-axis. Draw the diagram.

   Answer:

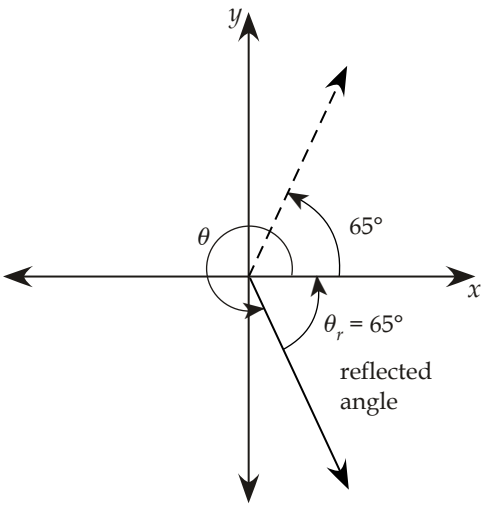


   Thus, the angle in standard position is: $\theta = 180° - 65°$

   $\theta = 115°$

b) in the x-axis. Draw the diagram.

Answer:

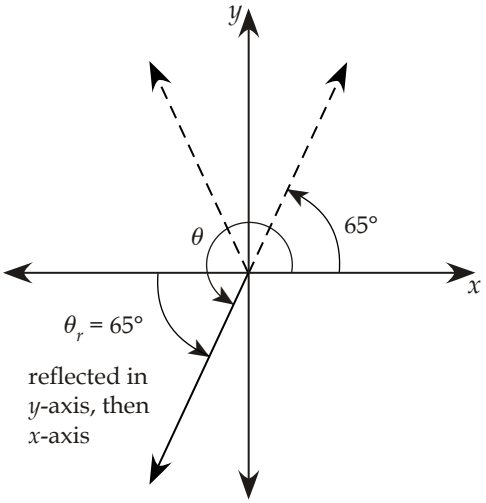


Thus, the angle in standard position is: $\theta = 360° - 65°$

$\theta = 295°$

c) in the y-axis and then the x-axis. Draw the diagram.

Answer:



Thus, the angle in standard position is: $\theta = 180° + 65°$

$\theta = 245°$

6. Determine the solution set for each of the following trigonometric equations over the interval [0°, 360°]. Round to one decimal place if necessary.

a) $\cos\theta = -\frac{2}{3}$

b) $\sin\theta + 1 = 0$

c) $\tan\theta - 2 = 5$

d) $2\cos\theta = 2$

e) $-3\sin\theta = 2$

f) $\frac{\tan\theta}{2} = 5$

g) $5\cos\theta - 2 = 0$

h) $5\tan\theta + 4 = 0$

i) $\frac{\tan\theta}{6} - 1 = 0$

j) $2\cos\theta + 1 = \frac{1}{2}$

k) $4\tan\theta - 7 = 5\tan\theta - 6$

Answers:

a) $\cos\theta = -\frac{2}{3}$

$\theta_r = \cos^{-1}\left(\frac{2}{3}\right)$ Use the positive ratio to find θ_r.

$= 48.2°$

$\cos\theta < 0$ in Quadrants II and III

In Quadrant II, $\theta = 180° - 48.2° = 131.8°$.

In Quadrant III, $\theta = 180° + 48.2° = 228.2°$.

b) $\sin\theta = -1$

This is a quadrantal angle through $(0, -1)$ that lies on the negative y-axis.

$\therefore \theta = 270°$

c) $\tan\theta - 2 = 5$

$\tan\theta = 7$

$\theta_r = \tan^{-1}(7)$

$= 81.9°$

$\tan\theta > 0$ in Quadrants 1 and III

In Quadrant I, $\theta = 81.9°$.

In Quadrant III, $\theta = 180° + 81.9° = 261.9°$.

d) $2 \cos \theta = 2$

$\cos \theta = 1$

This is a quadrantal angle through (1, 0) that lies on the positive x-axis.

$\therefore \theta = 0°$ or $360°$.

e) $-3 \sin \theta = 2$

$$\sin \theta = -\frac{2}{3}$$

$$\theta_r = \sin^{-1}\left(\frac{2}{3}\right) \text{ Use the positive ratio to find } \theta_r.$$

$$= 41.8°$$

$\sin \theta < 0$ in Quadrants III and IV

In Quadrant III, $\theta = 180° + 41.8° = 221.8°$.

In Quadrant IV, $\theta = 360° - 41.8° = 318.2°$.

f) $\frac{\tan \theta}{2} = 5$

$\tan \theta = 10$

$$\theta_r = \tan^{-1}(10)$$

$$= 84.3°$$

$\tan \theta > 0$ in Quadrants I and III

In Quadrant I, $\theta = 84.3°$.

In Quadrant III, $\theta = 180° + 84.3° = 264.3°$.

g) $5 \cos \theta - 2 = 0$

$$\cos \theta = \frac{2}{5}$$

$$\theta_r = \cos^{-1}\left(\frac{2}{5}\right)$$

$$= 66.4°$$

$\cos \theta > 0$ in Quadrants I and IV

In Quadrant I, $\theta = 66.4°$.

In Quadrant IV, $\theta = 360° - 66.4° = 293.6°$.

h) $5 \tan \theta + 4 = 0$

$$\tan \theta = -\frac{4}{5}$$

$$\theta_r = \tan^{-1}\left(\frac{4}{5}\right)$$

$$= 38.7°$$

$\tan \theta < 0$ in Quadrants II and IV

In Quadrant II, $\theta = 180° - 38.7° = 141.3°$.

In Quadrant IV, $\theta = 360° - 38.7° = 321.3°$.

i) $\frac{\tan \theta}{6} - 1 = 0$

$$\frac{\tan \theta}{6} = 1$$

$$\tan \theta = 6$$

$$\theta_r = \tan^{-1} 6$$

$$= 80.5°$$

$\tan \theta > 0$ in Quadrants I and III

In Quadrant I, $\theta = 80.5°$.

In Quadrant III, $\theta = 180° + 80.5° = 260.5°$.

j) $2 \cos \theta + 1 = \frac{1}{2}$

$$2 \cos \theta = -\frac{1}{2}$$

$$\cos \theta = -\frac{1}{4}$$

$$\theta_r = \cos^{-1}\left(\frac{1}{4}\right)$$

$$= 75.5°$$

$\cos \theta < 0$ in Quadrants II and III

In Quadrant II, $\theta = 180° - 75.5° = 104.5°$.

In Quadrant III, $\theta = 180° + 75.5° = 255.5°$.

k) $4 \tan \theta - 7 = 5 \tan \theta - 6$

$-\tan \theta = 1$

$\tan \theta = -1$

$\theta_r = \tan^{-1}(1)$

$\theta_r = 45°$

$\tan \theta < 0$ in Quadrants II and IV

In Quadrant II, $\theta = 180° - 45° = 135°$.

In Quadrant IV, $\theta = 360° - 45° = 315°$.

Learning Activity 7.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Jason had 37 hits in 45 times at bat during a baseball season. Sherri had 12 hits in 15 times at bat. Who is the better hitter?
2. Factor: $x^2 + 5x - 6$
3. Evaluate: $\left(6^2\right)^{\frac{1}{2}}$
4. In which quadrants is $\sin\theta > 0$?
5. Find the sum of 12 382 and 14 593.
6. Simplify: $(-3x^2 - 6x + 7) + (2x^2 - 5x)$
7. Two angles in an isosceles triangle measure 40° and 100°. What is the measure of the third angle?
8. Write this decimal as a percent: 0.103

Answers:

1. Jason (Sherri's ratio is $\frac{12}{15} = \frac{36}{45}$; Jason's ratio is $\frac{37}{45}$)
2. $(x + 6)(x - 1)$
3. 6
4. Quadrants I and II.
5. 26 975 (12 382 + 14 593)
6. $-3x^2 - 6x + 7 + 2x^2 - 5x = -x^2 - 11x + 7$
7. 40°
   (Method 1: The sum of the angles in a triangle must be 180°. Therefore, the third angle must measure 40°.

   Method 2: An isosceles triangle will have two angles that are the same. Therefore, the third angle must measure 40°.)
8. 10.3% (multiply by 100)

Part B: Patterns in the Sine, Cosine, and Tangent Ratios

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Determine the angles in Quadrants III and IV that have 30°, 45°, and 60° as reference angles.

   Answer:

   The formula for finding angles in standard position that have θ_r as a reference angle in Quadrant III is $\theta = 180° + \theta_r$.

   Thus, the three angles in Quadrant III are:

   $\theta = 180° + 30° = 210°$

   $\theta = 180° + 45° = 225°$

   $\theta = 180° + 60° = 240°$

   The formula for finding angles in standard position that have θ_r as a reference angle in Quadrant IV is $\theta = 360° - \theta_r$.

   Thus, the three angles in Quadrant IV are:

   $\theta = 360° - 30° = 330°$

   $\theta = 360° - 45° = 315°$

   $\theta = 360° - 60° = 300°$

2. Fill in the following chart with the exact values for sine, cosine, and tangent. Use the trigonometric ratio for the corresponding reference angle and attach a negative sign appropriately. Try to draw the special triangles from memory and determine the exact values for the trigonometric ratios using the appropriate special triangle.

θ	θ_r	$\cos\theta$	$\sin\theta$	$\tan\theta$
180°	(quadrantal angle)	-1	0	0
210°	30°	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{3}}$
225°	45°	$\frac{-1}{\sqrt{2}}$ or $\frac{-\sqrt{2}}{2}$	$\frac{-1}{\sqrt{2}}$ or $\frac{-\sqrt{2}}{2}$	1
240°	60°	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\sqrt{3}$
270°	(quadrantal angle)	0	-1	undefined
300°	60°	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$
315°	45°	$\frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$	$\frac{-1}{\sqrt{2}}$ or $\frac{-\sqrt{2}}{2}$	-1
330°	30°	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{1}{\sqrt{3}}$
360°	(quadrantal angle)	1	0	0

3. Complete the table below. List the sine, cosine, and tangent ratios for θ. Use your calculator and round to four decimal places.

Angle	cos θ	sin θ	tan θ
0°	1	0	0
15°	0.9659	0.2588	0.2679
30°	0.8660	0.5	0.5774
45°	0.7071	0.7071	1
60°	0.5	0.8660	1.7321
75°	0.2588	0.9659	3.7321
90°	0	1	undefined

a) What do you notice about the three ratios as the angle increases?

Answer:

As the angle increases, sine increases from 0 to 1, cosine decreases from 1 to 0, and tangent increases from 0 to a large number.

b) What do you notice about the sine and cosine ratios for each angle?

Answer:

The cosine ratio has the same value as the sine ratio of its complement. (The complement of θ is $90° - \theta$.) For example, sin 15° = cos 75°.

c) Determine the signs of the ratios in Quadrant II.

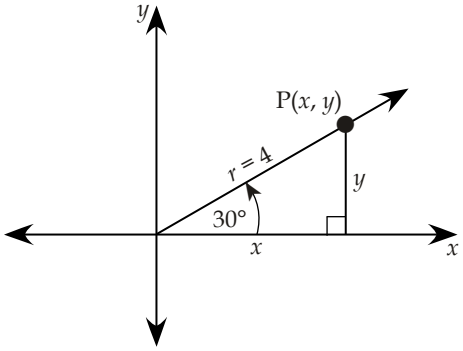
Answer:

In Quadrant II, sine will be positive while cosine and tangent are negative.

4. Determine the coordinates of the points on the terminal arm of the angles having 30° as a reference angle if $r = 4$. Write exact values. $0° \leq \theta \leq 360°$

Answer:

To solve for the coordinates of the point P(x, y), you can use the exact values of the trigonometric ratios for 30°.



$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r}$$

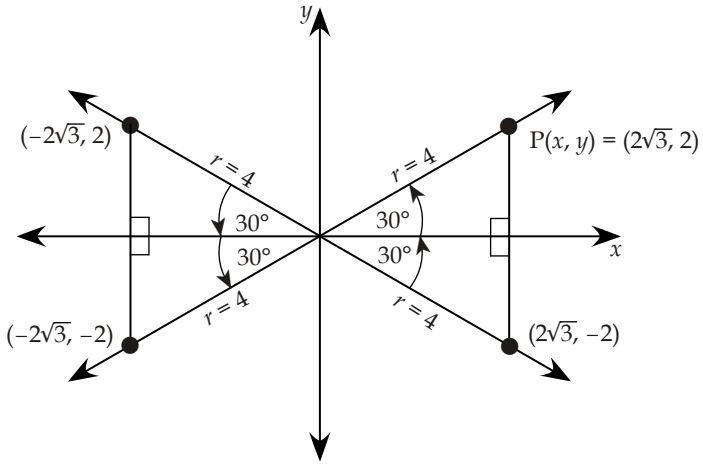
$$\cos 30° = \frac{x}{4} \qquad \sin 30° = \frac{y}{4}$$

$$x = 4\cos 30° \qquad y = 4\sin 30°$$

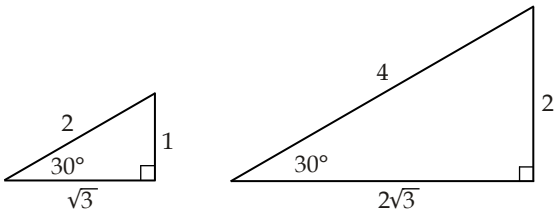
$$x = 4\left(\frac{\sqrt{3}}{2}\right) \qquad y = 4\left(\frac{1}{2}\right)$$

$$x = 2\sqrt{3} \qquad y = 2$$

The coordinates of P(x, y) are $\left(2\sqrt{3}, 2\right)$. The coordinates of the points in the other three quadrants are $\left(-2\sqrt{3}, 2\right)$, $\left(-2\sqrt{3}, -2\right)$, and $\left(2\sqrt{3}, -2\right)$.



Notice that since the r-value in this question is twice the r-value in the basic triangle for 30° that you have memorized, the x- and y-values are also doubled. They are similar triangles.



$r = 2$

coordinates

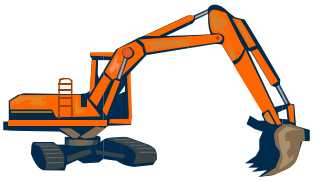
$P(x, y) = \left(\sqrt{3}, 1\right)$

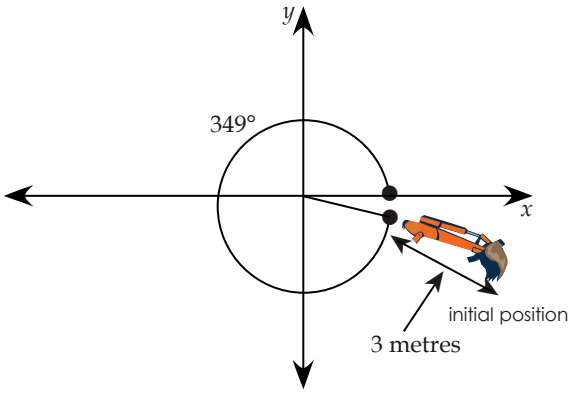
$r = 4$

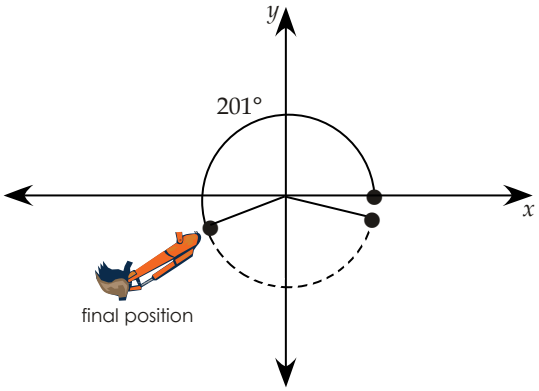
coordinates

$P(x, y) = \left(2\sqrt{3}, 2\right)$

5. A backhoe operator is digging a basement for a new house. To do this, on the first day he starts at the far end of the hole at angle of 349° and digs dirt backwards to end at an angle of 201°. If the length of the arm of the backhoe is 3 metres, what is the horizontal length of the hole that the backhoe operator is digging? Round your answer to 3 decimal places.



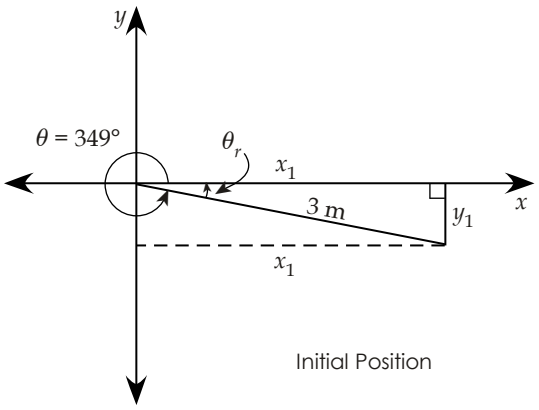




Answer:

You will use your calculator to find approximate answers since the angles in question do not have reference angles of 30°, 45°, or 60°.

To find the horizontal distance, you need to create two different triangles: one in Quadrant IV and one in Quadrant III. These triangles will represent the initial and final positions of the backhoe. Once these triangles are drawn, if you find the x-component of each triangle and add these two values together, you will find the length of the hole.



Initial Position

To find x_1, you need to find the horizontal distance, or the x-component, when the angle is 349° and the arm length is 3 m. To find the value of θ_r, you need to find the reference angle for 349°.

$$\theta_r = 360° - 349° = 11°$$

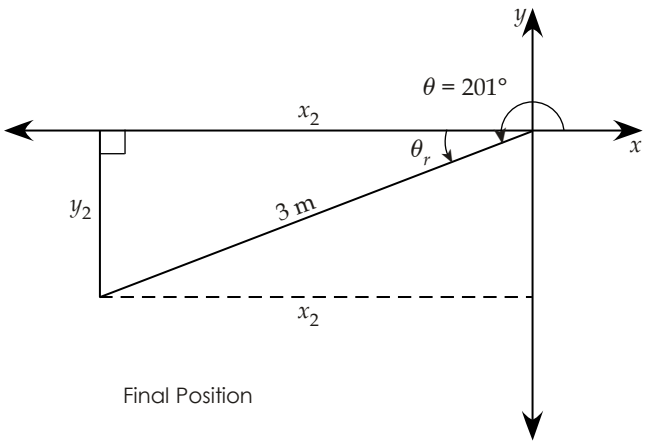
You now have the values of θ_r and r; thus, you need to use to cosine ratio in order to determine the x-value.

$$\cos\theta = \frac{x}{r}$$

$$\cos 11° = \frac{x}{3}$$

$$x = 3\cos 11°$$

$$x = 2.945 \text{ metres}$$



Final Position

To find x_2, you need to find the horizontal distance, or the x-component, when the angle is 201° and the boom length is 3 m.

The reference angle of 201° is:

$$201° - 180° = 21°$$

$$\cos\theta = \frac{x}{r}$$

$$\cos 21° = \frac{x}{3}$$

$$x = 3 \cos 21°$$

$$x = 2.801 \text{ metres}$$

Now, to find the length of the hole, you can add these values together. Thus, the length of the hole is 2.945 + 2.801 = 5.746 metres.

Learning Activity 7.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. You received $\frac{22}{25}$ on a math test. What was your percentage?
2. Find the common difference in the following pattern: 8, 2, −4, −10, . . .
3. In which quadrants is tan > 0?
4. What is the price of one box of Kleenex if 8 boxes of Kleenex cost $16.64?
5. The side lengths of a right-angled triangle are 15, 17, and 8. Which side length is the hypotenuse?
6. Simplify: $\frac{3^3 + 3}{3^2}$
7. Express as a radical: $9^{\frac{2}{3}}$. Do not simplify.
8. Find the slope of the line that contains the points (8, −3) and (−6, 12).

Answers:

1. 88% $\left(\frac{22}{25} = \text{multiply top and bottom by } 4 \text{ to get a denominator of } 100\right)$
2. −6
3. Quadrants I and III
4. \$2.08 $\left(\frac{\$16}{8} = \$2;\ \frac{\$0.64}{8} = \$0.08;\ \$2 + \$0.08 = \$2.08\right)$
5. 17 (The hypotenuse is always the longest side.)
6. $\frac{10}{3}$ $\left(\frac{3^3 + 3}{3^2} = \frac{27 + 3}{9} = \frac{30}{9}\right)$
7. $\sqrt[3]{9^2}$
8. $\frac{15}{-14}$ $\left(m = \frac{12 - (-3)}{-6 - 8} = \frac{15}{-14}\right)$

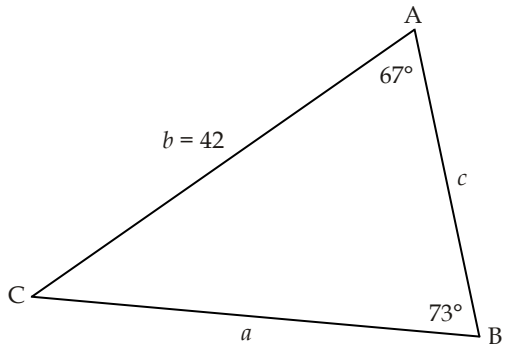
Part B: The Sine Law

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Solve the following triangles for all missing sides and angles. Round to 2 decimal places.

   a) In ΔABC, $\angle A = 67°$, $\angle B = 73°$, and $b = 42$.

   Answer:



$\angle C = 180° - 67° - 73° = 40°$

To find a:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin 67°} = \frac{42}{\sin 73°}$$

$$a = \frac{42 \sin 67°}{\sin 73°}$$

$$a = 40.43$$

To find c:

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

$$\frac{c}{\sin 40°} = \frac{42}{\sin 73°}$$

$$c = \frac{42 \sin 40°}{\sin 73°}$$

$$c = 28.23$$

Summary:

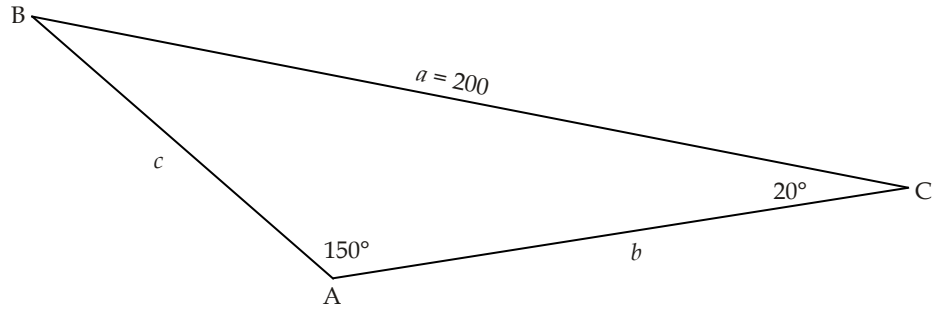
$\angle C = 40°$

$a = 40.43$

$c = 28.23$

b) In ΔABC, $\angle A = 150°$, $a = 200$, and $\angle C = 20°$.

Answer:



$\angle B = 180° - 150° - 20° = 10°$

To find b:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{200}{\sin 150°} = \frac{b}{\sin 10°}$$

$$b = \frac{200 \sin 10°}{\sin 150°}$$

$$b = 69.46$$

To find c:

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{200}{\sin 150°} = \frac{c}{\sin 20°}$$

$$c = \frac{200 \sin 20°}{\sin 150°}$$

$$c = 136.81$$

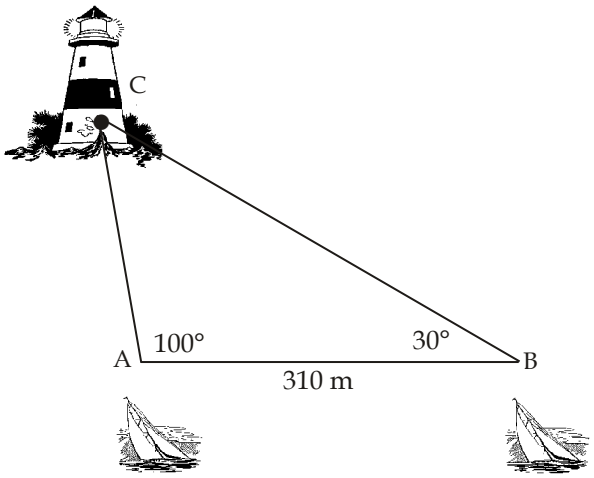
Summary:

$\angle B = 10°$

$b = 69.46$

$c = 136.81$

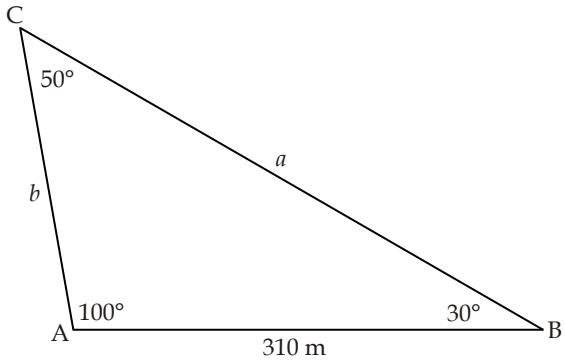
2. A lighthouse is visible to two sailboats 310 m apart. Use the angle measures shown to find the distance of each sailboat from the lighthouse. Round to two decimal places.



Answer:

$\angle C = 180° - (100° + 30°)$

$= 50°$



$$\frac{b}{\sin B} = \frac{c}{\sin C} \qquad \frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{b}{\sin 30°} = \frac{310}{\sin 50°} \qquad \frac{a}{\sin 100°} = \frac{310}{\sin 50°}$$

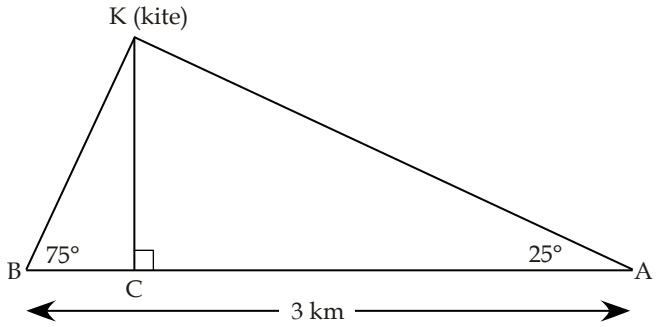
$$b = \frac{310 \sin 30°}{\sin 50°} \qquad a = \frac{310 \sin 100°}{\sin 50°}$$

$b = 202.34$ m $\qquad a = 398.53$ m

3. A kite is spotted simultaneously by two observers at points A and B, 3 km apart. The angles of elevation of the kite from A and B are 25° and 75°, respectively. How high above the ground to the nearest tenth is the kite if the kite is in the same vertical plane as A and B and if the point on the ground directly below the kite is:

   a) between A and B? (Include a diagram.)

   Answer:



$\angle BKA = 180° - (75° + 25°) = 80°$

In ΔBKA, by the Sine Law

$$\frac{BK}{\sin 25°} = \frac{3}{\sin 80°}$$

$$BK = \frac{3 \sin 25°}{\sin 80°}$$

$$BK = 1.287\ 413\ 489$$

In right ΔBCK, $\angle C = 90°$

$$\sin 75° = \frac{KC}{BK}$$

$$KC = BK \sin 75°$$

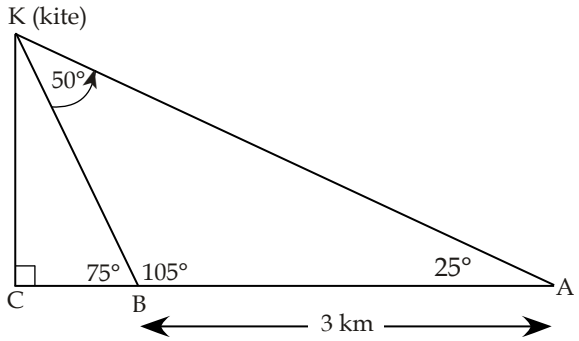
$$KC = 1.287\ 413\ 489 \sin 75°$$

$$= 1.243\ 545\ 938$$

$\therefore$ The kite is approximately 1.2 km above the ground.

b) not between A and B? (Include a diagram.)

Answer:



$\angle KBA = 180° - 75° = 105°$

$\angle BKA = 180° - (105° + 25°) = 50°$

In ΔABK, using the Sine Law:

$$\frac{BK}{\sin 25°} = \frac{3}{\sin 50°}$$

$$BK = \frac{3 \sin 25°}{\sin 50°}$$

$$BK = 1.655\ 066\ 878$$

In right ΔCKB, $\angle C = 90°$

$$\sin 75° = \frac{KC}{BK}$$

$$KC = BK \sin 75°$$

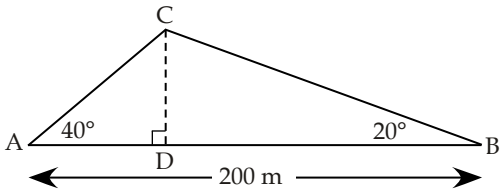
$$KC = 1.655\ 066\ 878 \sin 75°$$

$$= 1.598\ 671\ 842$$

In this case, the kite is 1.6 m above the ground.

4. To calculate the width of the Assiniboine River south of the Legislative Building, a surveyor marks a baseline, $\overline{AB}$, 200 m in length along the north riverbank. From each end of the baseline, an apartment, C, is sighted on the south bank of the river, making angles of 40° and 20° with the baseline. How wide is the river? Round to the nearest hundredth.

   Answer:



In ΔACB,

$$\angle C = 180° - (40° + 20°) = 120°$$

Use ΔABC and the Sine Law to find BC.

$$k\frac{200}{\sin 120°} = \frac{BC}{\sin 40°}$$

$$BC = \frac{200 \sin 40°}{\sin 120°}$$

$$BC = 148.445\ 439\ 8$$

In right ΔCDB, $\angle D = 90°$

$$\sin 20° = \frac{CD}{BC}$$

$$CD = BC \sin 20°$$

$$CD = 148.445\ 439\ 8 \sin 20°$$

$$CD = 50.77$$

The Assiniboine River is about 50.77 m wide at the Legislative Building.

5. Explain the steps in the given proof of the Sine Law.

a) $h_1 = b \sin A$ and $h_1 = a \sin B$

b) $b \sin A = a \sin B$

c) $\dfrac{b}{\sin B} = \dfrac{a}{\sin A}$

d) $h_2 = a \sin C$ and $h_2 = c \sin A$

e) $a \sin C = c \sin A$

f) $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$

g) $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Answer:

a) Determine two expressions for the height through C.

b) Equate two equal expressions for the same height.

c) Divide both sides by sin A sin B.

d) Determine two expressions for the height through B.

e) Equate two equal expressions for the same height.

f) Divide both sides by sin A sin C.

g) Equate the formulas found in (c) and (f) since they both equal $\dfrac{a}{\sin A}$.

Learning Activity 7.5

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. In which quadrant are $\sin\theta$, $\cos\theta$, and $\tan\theta$ all greater than zero?
2. What is the probability of rolling an even number on a number cube?
3. Calculate the tax (13%) on an item that costs $43.99.
4. The two legs of a right-angled triangle measure 6 cm and 8 cm. What is the measure of the hypotenuse?
5. What is the value of the discriminant of the quadratic equation $2x^2 - 3x + 6 = 0$?
6. Express in exponential form: $\sqrt{12x^3y^7}$
7. Find the midpoint between the two points (7, −1) and (−8, 15).
8. State the independent and dependent variables for the following situation: "Jace discovers that the more he studies for a test, the higher his mark is."

Answers:

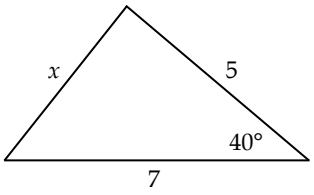
1. Quadrant I
2. $\frac{3}{6} = \frac{1}{2}$
3. $5.28 (10% of 43.99 = $4.40; 1% of 43.99 = $0.44; 13% of $43.99 = $4.40 + $0.44 + $0.44 + $0.44 = $5.72)
4. 10 ($6^2 + 8^2 = x^2$; $36 + 64 = x^2$; $100 = x^2$; $10 = x$)
5. −39 ($b^2 - 4ac = 9 - 4(2)(6) = 9 - 48 = -39$)
6. $12^{\frac{1}{2}}x^{\frac{3}{2}}y^{\frac{7}{2}}$ $\left(\sqrt{12x^3y^7} = \left(12x^3y^7\right)^{\frac{1}{2}}\right)$
7. $\left(-\frac{1}{2}, 7\right)$ $\left(\text{M} = \left(\frac{7+(-8)}{2}, \frac{-1+15}{2}\right) = \left(-\frac{1}{2}, \frac{14}{2}\right) = \left(-\frac{1}{2}, 7\right)\right)$
8. Independent Variable: The amount of time spent studying
   Dependent Variable: The mark on the test

Part B: The Cosine Law

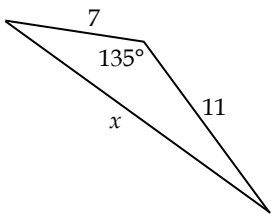
Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Find the length of x in each diagram. Round to the nearest tenth.

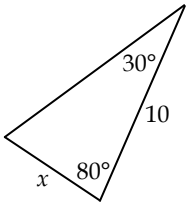
a)



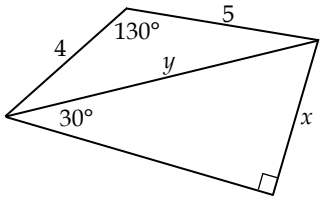
b)



c)



d)



Answers:

a) The triangle has no matching pair of a side and its opposing angle. Use the Cosine Law for sides.

$$x^2 = 7^2 + 5^2 - 2(7)(5)\left(\cos 40^\circ\right)$$

$$x^2 = 20.376\ 888\ 98$$

$$\sqrt{x^2} = \sqrt{20.376\ 888\ 98}$$

$$x = 4.5$$

b) The triangle has no matching pair of a side and its opposing angle. Use the Cosine Law for sides.

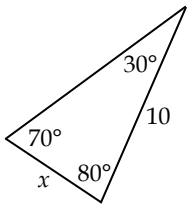
$$x^2 = 7^2 + 11^2 - 2(7)(11)\left(\cos 135^\circ\right)$$

$$x^2 = 278.894\ 444\ 3$$

$$\sqrt{x^2} = \sqrt{278.894\ 444\ 3}$$

$$x = 16.7$$

c) The triangle has a matching pair of side and its opposing angle once ∠A is found. Use the Sine Law.



$$\angle A = 180° - 80° - 30°$$

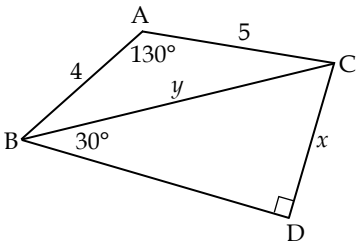
$$\angle A = 70°$$

$$\frac{x}{\sin 30°} = \frac{10}{\sin 70°}$$

$$x = \frac{10 \sin 30°}{\sin 70°}$$

$$x = 5.3$$

d) First solve ΔABC for y using the Cosine Law for sides. There is no matching pair of a side and its opposite angle.



$$y^2 = 4^2 + 5^2 - 2(4)(5)\left(\cos 130°\right)$$

$$y^2 = 66.711\ 504\ 39$$

$$\sqrt{y^2} = \sqrt{66.711\ 504\ 39}$$

$$y = 8.167\ 711\ 086$$

In ΔBCD, there is a right angle. Use the sine ratio.

$$\sin 30° = \frac{x}{y}$$

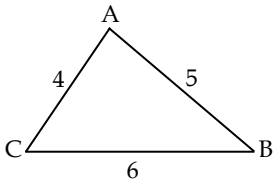
$$x = y \sin 30°$$

$$x = 8.167\ 711\ 086 \sin 30°$$

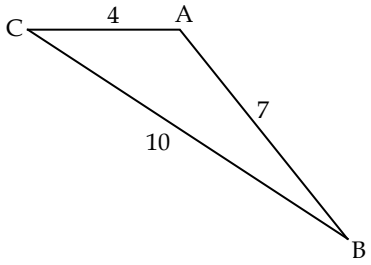
$$x = 4.1$$

2. Find the measure of ∠A. Round to the nearest tenth.

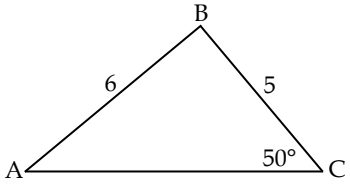
a)



b)



c)



Answers:

a) There is no matching pair for the Sine Law. Use the Cosine Law for angles.

Method 1: Write the Cosine Law as:

$$6^2 = 4^2 + 5^2 - 2(4)(5)\cos A$$

$$6^2 - 4^2 - 5^2 = -2(4)(5)\cos A$$

$$-5 = -40\cos A$$

$$0.125 = \cos A$$

$$A = \cos^{-1}(0.125)$$

$$A = 82.8°$$

Method 2: Use the rearranged formula:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{4^2 + 5^2 - 6^2}{2(4)(5)}$$

$$\cos A = 0.125$$

$$\angle A = \cos^{-1}(0.125)$$

$$\angle A = 82.8°$$

b) There is no matching pair to use the Sine Law. Use the Cosine Law for angles.

Using the rearranged formula:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{4^2 + 7^2 - 10^2}{2(4)(7)}$$

$$\cos A = -0.625$$

(Notice the negative value for cos A; so, ∠A will be obtuse.)

$$\angle A = 128.7°$$

c) There is a matching pair of an angle and its corresponding opposite side. Use the Sine Law.

$$\frac{5}{\sin A} = \frac{6}{\sin 50°}$$

$$\sin A = \frac{5 \sin 50°}{6}$$

$$\sin A = 0.638\ 370\ 369$$

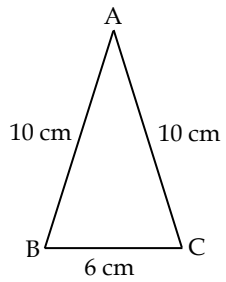
$$\therefore \angle A_r = \sin^{-1}(0.638\ 370\ 369)$$

$$\angle A_r = 39.7°$$

Since your calculator only gives one of two possible angles when using sine inverse, the solution for ∠A could be an acute or an obtuse angle. However, you are given the diagram and it shows ∠A to be acute. Thus, the only solution is ∠A = 39.7°.

3. An isosceles triangle has sides of length 10 cm, 10 cm, and 6 cm. What is the measure of each angle of this triangle? Round to the nearest tenth.

Answer:



Since there is no matching angle and its opposite side, solve for an angle using the Cosine Law for angles.

Using the rearranged formula:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{10^2 + 10^2 - 6^2}{2(10)(10)}$$

$$\cos A = 0.82$$

$$\angle A = \cos^{-1}(0.82)$$

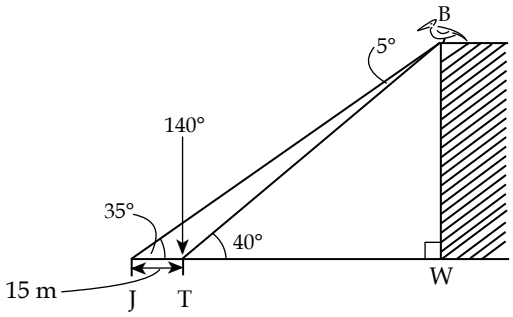
$$\angle A = 34.9°$$

Since ΔABC is isosceles, $\angle B = \angle C$. Thus, $\angle B = \frac{180° - 34.9°}{2} = 72.5°$.

The measure of the angles of the triangle are 34.9°, 72.5°, and 72.5°.

4. Janice spots a bird on the roof of her apartment block with an angle of elevation of 35°. She moves 15 m closer to the apartment block and now the angle of elevation is 40°. How far above the ground is the bird? (Assume the ground to be level and Janice walked directly toward the bird on the roof.) Round to the nearest tenth.

   Answer:



One method to solve for BW is to solve first for BT in ΔBJT and then use the sine ratio in the right triangle, ΔBTW.

For ΔBJT, $\angle JTB = 180° - 40° = 140°$ and $\angle JBT = 180° - 35° - 140° = 5°$.

$$\frac{BT}{\sin 35°} = \frac{15}{\sin 5°}$$

$$BT = \frac{15 \sin 35°}{\sin 5°}$$

$$BT = 98.715\,773\,33$$

In ΔBJW,

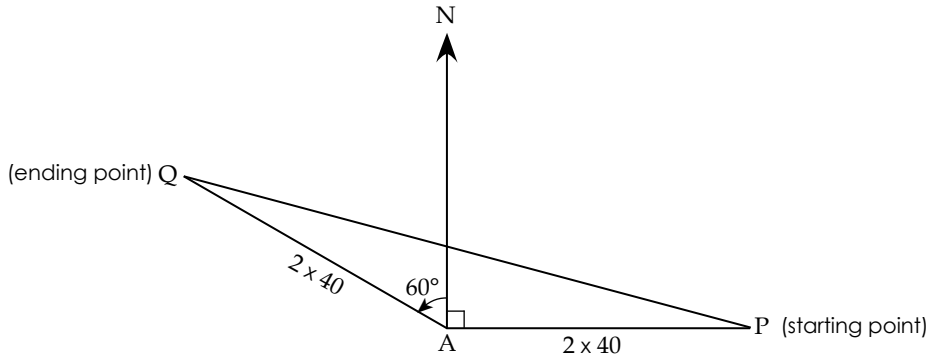
$$\sin 40° = \frac{BW}{BT}$$

$$BW = BT \sin 40°$$

$$BW = (98.715\ 773\ 33)(\sin 40°)$$

$$BW = 63.5 \text{ m}$$

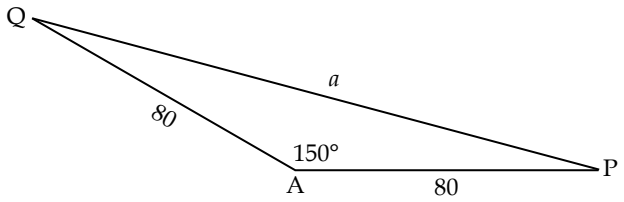
5. A ship leaves port at 12 noon and sails west at 40 km/h. At 2:00 p.m., it changes direction and sails N 60° W for another two hours. How far is the ship from the port? Round to the nearest tenth.



Answer:

The ship is travelling from noon until 2:00 p.m. (2 hours) for 40 km in each hour. It travels 80 km due west. When it changes direction, it travels for the same time at the same speed, and thus travels another 80 km.

The measure of ∠QAP = 60° + 90° = 150°.



There is no matching pair of side and opposite angles. Use the Law of Cosines for sides.

$$a^2 = 80^2 + 80^2 - 2(80)(80)(\cos 150°)$$

$$a^2 = 23\ 885.125\ 17$$

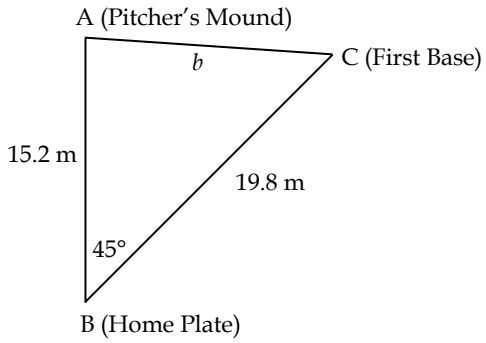
$$\sqrt{a^2} = \sqrt{23\ 885.125\ 17}$$

$$a = 154.5$$

The ship is 154.5 km from port at 4:00 p.m.

6. The pitcher's mound on a slo-pitch softball diamond is 15.2 m from home plate and the distance between the bases is 19.8 m. How far is the pitcher's mound from first base if the angle at home plate between the pitcher's mound and first base is 45°? Round to the nearest hundredth.

Answer:



There is no matching pair of a side and its opposite angle. Thus, use the Cosine Law.

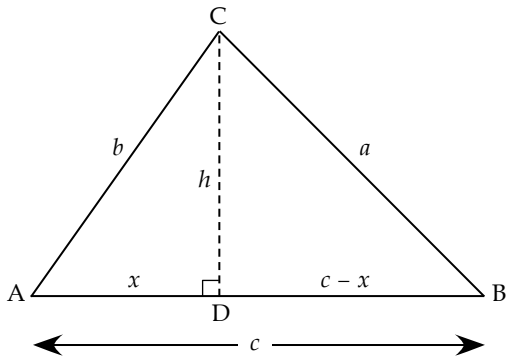
$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$= (19.8)^2 + (15.2)^2 - 2(19.8)(15.2) \cos 45°$$

$$b = 14.05$$

Thus, the distance between the pitcher's mound and first base is 14.05 m.

7. Prove the Law of Cosines for $a^2 = b^2 + c^2 - 2bc \cos A$ using the diagram below.



Proof:

The proof begins by using the Pythagorean theorem for each right triangle:

in ΔBCD, $a^2 = h^2 + (c - x)^2$

in ΔACD, $b^2 = h^2 + x^2$

The next step is to subtract the two equations to eliminate h^2.

Now finish the proof.

Answer:

Subtract the two equations to eliminate h^2:

$$a^2 - b^2 = (c - x)^2 - x^2$$

$$a^2 - b^2 = c^2 - 2cx + \cancel{x^2} - \cancel{x^2} \qquad \text{Simplify}$$

$$a^2 - b^2 = c^2 - 2cx \qquad \text{Simplify}$$

$$a^2 = b^2 + c^2 - 2cx \qquad \text{Rearrange}$$

Now, write x in terms of the sides and angles of triangle ABC, and use ΔACD and find cos A.

$$\cos A = \frac{x}{b}$$

$$x = b \cos A$$

Substitute $x = b \cos A$ into $a^2 = b^2 + c^2 - 2cx$ to get:

$a^2 = b^2 + c^2 - 2c(b \cos A)$

$a^2 = b^2 + c^2 - 2bc \cos A$

Note to Student: You are not expected to know the proof of either the Sine Law or the Cosine Law for the exam.

Learning Activity 7.6

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Calculate the tax (13%) on an item that costs $72.
2. How many 4-piece bags of candy can you make to give out on Halloween if you have 288 pieces of candy?
3. In which quadrants is $\cos\theta > 0$?
4. The length of the hypotenuse of a right triangle is 13 and one of its legs is length 12. What is the length of the other leg?
5. Does the parabola $y + 5x^2 = -2x + 4$ open up or down?
6. Express as a radical: $x^{\frac{13}{2}}y^{\frac{5}{2}}$
7. $(-56, 27)$ is on a line with a slope of $\frac{6}{8}$. Find another point on this line.
8. Simplify: $(x^2 - 1)(2x^2 + 5x - 6)$

Answers:

1. $9.36 (10% of 72 = $7.20; 1% of 72 = $0.72; so 13% of $72 = $7.20 + $0.72 + $0.72 + $0.72 = $9.36)
2. $72\ \left(\frac{288}{4} = \frac{200}{4} + \frac{88}{4} = 50 + 22 = 72\right)$
3. Quadrants I and IV
4. 5 ($x^2 + 12^2 = 13^2$; $x^2 = 169 - 144$; $x^2 = 25$; $x = 5$)
5. Down ($y = -5x^2 - 2x + 4$)
6. $\sqrt{x^{13}y^5}\ \left(x^{\frac{13}{2}}y^{\frac{5}{2}} = \left(x^{13}y^5\right)^{\frac{1}{2}} = \sqrt{x^{13}y^5}\right)$
7. $(-48, 33)$ (A slope of $\frac{6}{8}$ refers to a rise of 6, which affects the y-values, and a run of 8, which affects the x-values, so $(-56 + 8, 27 + 6)$)
8. $2x^4 + 5x^3 - 6x^2 - 2x^2 - 5x + 6 = 2x^4 + 5x^3 - 8x^2 - 5x + 6$

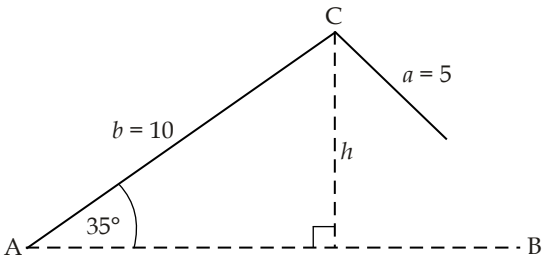
Part B: The Ambiguous Case

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Decide whether there are no solutions, exactly one solution, or two solutions for each ΔABC. You do not need to solve the triangle, but you must justify your choice and include a diagram.

   Note: Always draw the given angle in the lower left corner of the triangle. Mark the given side opposite it with a dotted line. Do not mark the second given side as the base. (Reason: It makes solving the ambiguous case easier if you always use the same pattern.)

   a)



   Answer:

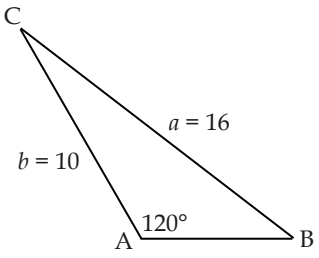
   The side opposite the given angle is smaller than the other given side. We need to find the height.

   $h = 10 \sin 35°$

   $h = 5.7$

   Since $5 < h$, there is no triangle.

   b)



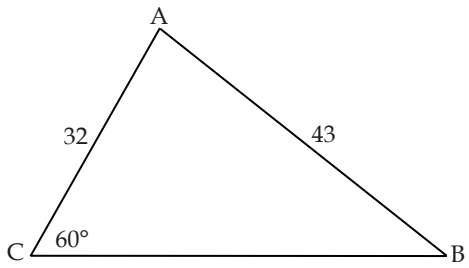
   Answer:

   Because the side opposite the given angle is greater than the other given side, and ∠A is obtuse, there is only one triangle.

c) $\angle C = 60°$, $c = 43$, $b = 32$

Answer:

The longer side is opposite the given angle. There is only one triangle.



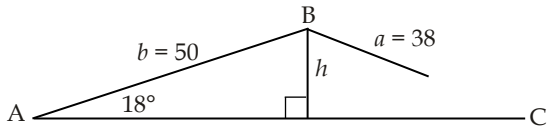
d) $\angle A = 18°$, $a = 38$, $c = 50$

Answer:

The side opposite the given angle is smaller than the other given side. We need to find the height.

$h = 50 \sin 18° = 15.45$

Since $38 > h$, there are two solutions.



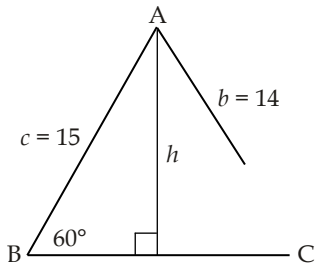
e) $\angle B = 60°$, $b = 14$, $c = 15$

Answer:

The side opposite the given angle is smaller than the other given side. We need to find the height.

$h = 15 \sin 60° = 12.99$

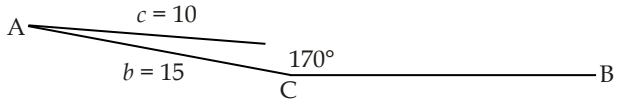
Since $14 > h$, we have two solutions.



f) $\angle C = 170°$, $c = 10$, $b = 15$

Answer:

The side opposite the given angle is shorter than the other given side and the given angle is obtuse. There is no solution.



2. Find the possible values of the indicated side. Round answers to the nearest tenth.

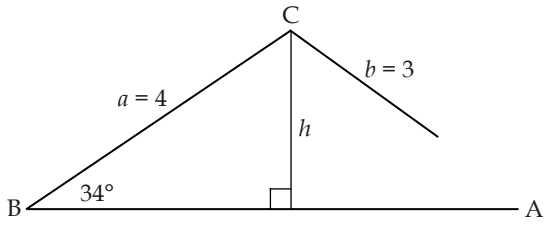
a) In ΔABC, $\angle B = 34°$, $a = 4$, and $b = 3$. Find c.

Answer:

Since the side opposite the given angle is smaller than the other given side, we need to find the height.

$h = 4 \sin 34° = 2.2$

$3 > h$, so there are two triangles.



We will now solve for the two measures of $\angle A$, using the Sine Law.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{4}{\sin A} = \frac{3}{\sin 34°}$$

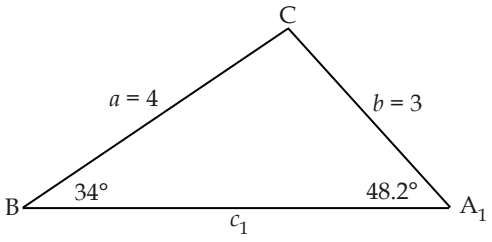
$$\sin A = \frac{4 \sin 34°}{3}$$

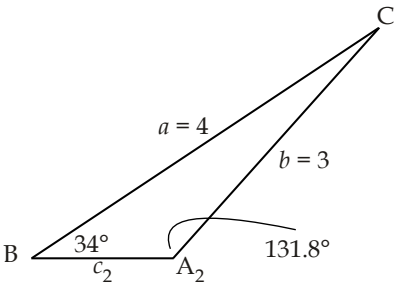
$$\angle A_r = \sin^{-1}\left(\frac{4 \sin 34°}{3}\right)$$

$$\angle A_1 = 48.2° \text{ (acute angle)}$$

$$\angle A_2 = 180° - 48.2° = 131.8° \text{ (obtuse angle)}$$

Now we can draw the two triangles and solve each of them.





$\angle C = 180° - (34° + 48.2°)$

$\angle C = 97.8°$

To solve for c_1:

$$\frac{c_1}{\sin 97.8°} = \frac{3}{\sin 34°}$$

$$c_1 = 5.3$$

$\angle C = 180° - (34° + 131.8°)$

$\angle C = 14.2°$

To solve for c_2:

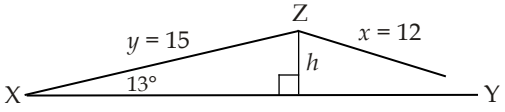
$$\frac{3}{\sin 34°} = \frac{c_2}{\sin 14.2°}$$

$$c_2 = 1.3$$

b) In ΔXYZ, $\angle X = 13°$, $x = 12$, and $y = 15$. Find z.

Answer:

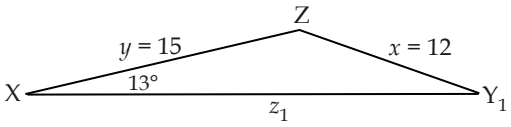
Since the side opposite the given angle is smaller than the other given side, we need to find the height.

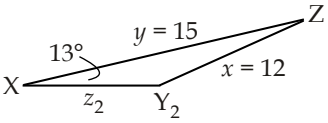


$h = 15 \sin 13° = 3.37$

$12 > h$, so there are two triangles

We will now solve for the two measures for $\angle Y$ using the Sine Law.





$$\frac{x}{\sin X} = \frac{y}{\sin Y}$$

$$\frac{12}{\sin 13°} = \frac{15}{\sin Y}$$

$$\sin Y = \frac{15 \sin 13°}{12}$$

$$\angle Y_r = 16.3°$$

Thus, $\angle Y_1 = 16.3°$ and $\angle Y_2 = 180° - 16.3° = 163.7°$.

In ΔXY_1Z:

If $\angle Y_1 = 16.3°$

then $\angle Z = 180° - (16.3° + 13°)$

$= 150.7°$

In ΔXY_2Z:

If $\angle Y_2 = 163.7°$

then $\angle Z = 180° - (13° + 163.7°)$

$= 3.3°$

In ΔXY_1Z:

For z_1:

$$\frac{x}{\sin X} = \frac{z_1}{\sin Z}$$

$$\frac{12}{\sin 13°} = \frac{z_1}{\sin 150.7°}$$

$$z_1 = \frac{12 \sin 150.7°}{\sin 13°}$$

$$z_1 = 26.1$$

In ΔXY_2Z:

For z_2:

$$\frac{x}{\sin X} = \frac{z_2}{\sin Z}$$

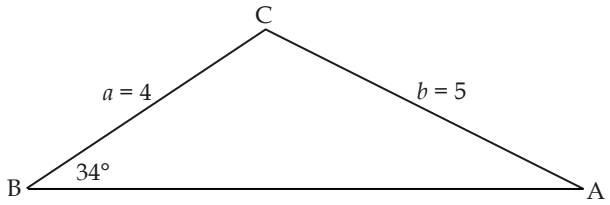
$$\frac{12}{\sin 13°} = \frac{z_2}{\sin 3.3°}$$

$$z_2 = \frac{12 \sin 3.3°}{\sin 13°}$$

$$z_2 = 3.1$$

c) In ΔABC, $\angle B = 34°$, $a = 4$, and $b = 5$. Find c.

Answer:



Since $b > a$, only one triangle exists.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{4}{\sin A} = \frac{5}{\sin 34°}$$

$$\sin A = \frac{4 \sin 34°}{5}$$

$$\angle A = 26.6°$$

$$\angle C = 180° - (34° + 26.6°)$$

$$= 119.4°$$

side c

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{5}{\sin 34°} = \frac{c}{\sin 119.4°}$$

$$c = \frac{5 \sin 119.4°}{\sin 34°}$$

$$c = 7.8$$

Note: You can verify that there is only one value for $\angle A$.

If there were two triangles, then the second $\angle A$ is:

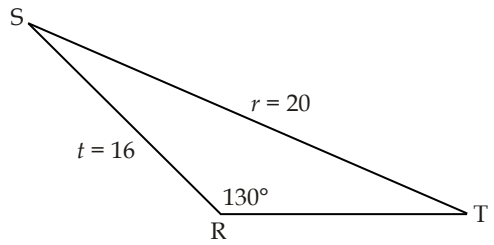
$\angle A_2 = 180° - 26.6° = 153.4°$.

$\therefore \angle C_2 = 180° - (34° + 153.4°) = -7.4°$

This is impossible. An angle in a triangle cannot have a negative measure.

d) In ΔRST, $\angle R = 130°$, $r = 20$, and $t = 16$. Find s.

Answer:



Because $r > t$, there is only one solution.

To find the measure of $\angle T$:

$$\frac{r}{\sin R} = \frac{t}{\sin T}$$

$$\frac{20}{\sin 130°} = \frac{16}{\sin T}$$

$$\sin T = \frac{16 \sin 130°}{20}$$

$$\angle T = 37.8°$$

$$\angle S = 180° - (37.8° + 130°)$$

$$= 12.2°$$

To find the measure of s:

$$\frac{s}{\sin S} = \frac{r}{\sin R}$$

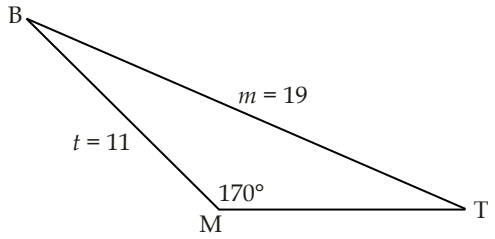
$$\frac{s}{\sin 12.2°} = \frac{20}{\sin 130°}$$

$$s = \frac{20 \sin 12.2°}{\sin 130°}$$

$$s = 5.5$$

e) In ΔMBT, ∠M = 170°, m = 19, and t = 11. Find b.

Answer:



Because $m > t$, there is only one solution.

To find the measure of ∠T:

$$\frac{t}{\sin T} = \frac{m}{\sin M}$$

$$\frac{11}{\sin T} = \frac{19}{\sin 170°}$$

$$\sin T = \frac{11 \sin 170°}{19}$$

$$\angle T = 5.8°$$

$$\angle B = 180° - (170° + 5.8°)$$

$$= 4.2°$$

To find the measure of b:

$$\frac{b}{\sin B} = \frac{m}{\sin M}$$

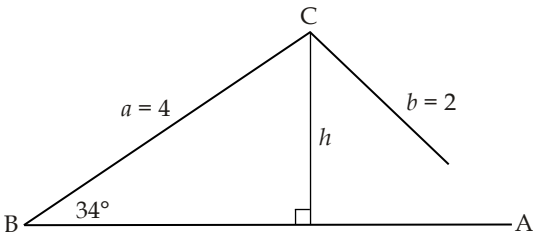
$$\frac{b}{\sin 4.2°} = \frac{19}{\sin 170°}$$

$$b = \frac{19 \sin 4.2°}{\sin 170°}$$

$$b = 8.0$$

f) In ΔABC, ∠B = 34°, a = 4, and b = 2. Find c.

Answer:



Since the side opposite the given angle is smaller than the other given side, we need to find the height.

$h = 4 \sin 34°$

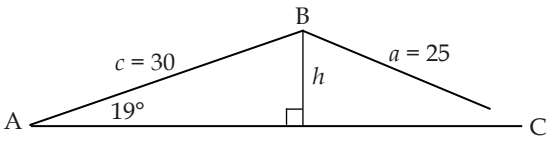
$h = 2.2$

Since the side opposite the given angle is smaller than the height of the triangle, there is no solution.

3. Find all possible values of the indicated angle measure. Round to one decimal place.

a) In ΔABC, ∠A = 19°, a = 25, and c = 30. Find ∠C.

Answer:



Since the side opposite the given angle is smaller than the other given side, we need to find the height.

$h = 30 \sin 19°$

$h = 9.8$

Since the side opposite the given angle is larger than the height of the triangle, there are two solutions of ∠C. We will find the two measures for ∠C using the Sine Law.

$$\frac{30}{\sin C} = \frac{25}{\sin 19°}$$

$$\sin C = \frac{30 \sin 19°}{25}$$

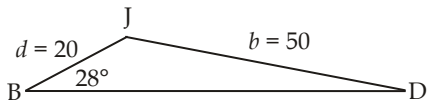
$$\angle C_r = \sin^{-1}\left(\frac{30 \sin 19°}{25}\right)$$

$$\angle C_1 = 23.0° \text{ (acute angle)}$$

$$\angle C_2 = 180° - 23.0° = 157.0° \text{ (obtuse angle)}$$

b) In ΔBDJ, ∠B = 28°, b = 50, and d = 20. Find ∠D.

Answer:



Because $b > d$, there is only one value for ∠D.

$$\frac{d}{\sin D} = \frac{b}{\sin B}$$

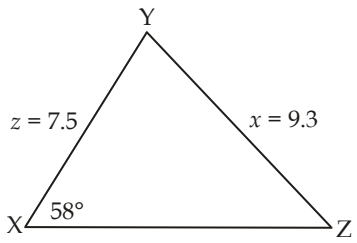
$$\frac{20}{\sin D} = \frac{50}{\sin 28°}$$

$$\sin D = \frac{20 \sin 28°}{50}$$

$$\angle D = 10.8°$$

c) In ΔXYZ, ∠X = 58°, x = 9.3, and z = 7.5. Find ∠Z.

Answer:



Because $x > z$, there is only one solution.

$$\frac{x}{\sin X} = \frac{z}{\sin Z}$$

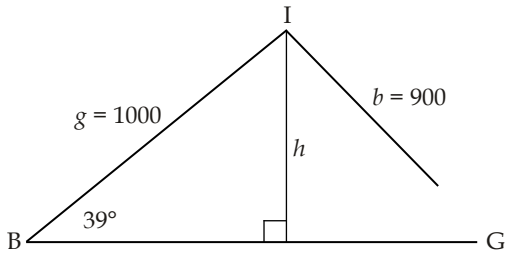
$$\frac{9.3}{\sin 58°} = \frac{7.5}{\sin Z}$$

$$\sin Z = \frac{7.5 \sin 58°}{9.3}$$

$$\angle Z = 43.1°$$

d) In ΔBIG, $\angle B = 39°$, $b = 900$, and $g = 1000$. Find $\angle I$.

Answer:



Since the side opposite the given angle is smaller than the other given side, we need to find the height.

$h = 1000 \sin 39°$

$h = 629.3$

Since the side opposite the given angle is larger than the height of the triangle, there are two solutions. We will find the two measures of $\angle G$ first.

$$\frac{900}{\sin 39°} = \frac{1000}{\sin G}$$

$$\angle G_r = \sin^{-1}\left(\frac{1000 \sin 39°}{900}\right)$$

$\angle G_1 = 44.4°$ (acute)

$\angle G_2 = 180° - 44.4° = 135.6°$ (obtuse)

Now we can find the corresponding measures of the third angle in the triangle, $\angle I$.

$\angle I_1 = 180° - (39° + 44.4°) = 96.6°$

$\angle I_2 = 180° - (39° + 135.6°) = 5.4°$

Grade 11 Pre-Calculus Mathematics (30S)

Module 8
Absolute Value and Reciprocal Functions

Module 8:
Absolute Value and Reciprocal Functions

Introduction

Welcome to the last module of Grade 11 Pre-Calculus Mathematics! In this module, you will be learning about two different types of functions and their graphs.

You will first learn about absolute value functions, which occur when only positive y-values are desired. Absolute value is especially important in distance, as you would never say you are a negative distance away from your destination. You could be 10 kilometres east or 10 kilometres west, but either way you are still 10 kilometres away.

Once you learn about absolute value functions, you will learn about reciprocal functions. Reciprocal functions are used to define many relationships, including gravitational forces between planets, currency exchange, and the amount of time a diver can take to rise to the surface of the water depending on the depth. As you have already studied linear and quadratic functions, you will now learn how to graph the reciprocals of these types of functions in this module.

Assignments in Module 8

When you have completed the assignments for Module 8, submit your completed assignments to the Distance Learning Unit either by mail or electronically through the learning management system (LMS). The staff will forward your work to your tutor/marker.

Lesson	Assignment Number	Assignment Title
2	Assignment 8.1	Absolute Value
4	Assignment 8.2	Absolute Value Equations and Graphing Reciprocal Functions: Part 1
5	Assignment 8.3	Reciprocal Functions: Part 2

Resource Sheet

When you write your final examination, you are encouraged to take a Final When you write your final examination, you are encouraged to take a Final Examination Resource Sheet with you into the examination. This sheet will be one letter-sized page, 8½ " by 11", with both sides in your handwriting or typewritten. You will submit it with your examination, but you do not receive any marks for it.

Many students have found that preparing a resource sheet is an excellent way to review. It provides you with a summary of the important facts of each module. You should complete a resource sheet for each module to help with your studying and reviewing. Lesson summaries and module summaries are included for you to use as a guide.

You may use the list of instructions that follows to help you with preparing your resource sheet for the material in Module 8. On this sheet, you should record math terms and definitions, formulas, sample questions, or a list of places where you often make mistakes. You should also identify special areas that require extra attention or review by writing the page numbers.

After you have completed each module's resource sheet, you may summarize the sheets from Modules 1 to 8 to prepare your Final Examination Resource Sheet. The final examination for this course is based on Modules 1 to 8.

Resource Sheet for Module 8

As you go through the lessons of this module, you may want to consider the following suggestions regarding the creation of a resource sheet.

1. List all the important math terms, and define them if necessary.
2. List all the formulas and perhaps a sample problem that shows how the formula is used.
3. If necessary, write the solutions to some problems, showing in detail how you did the calculations.
4. Copy any questions that represent the key points of the lesson, and perhaps include the solutions as well.
5. Identify the problems you found most difficult, and copy the page numbers onto the resource sheet so that you can review them before writing the examination. You may also copy the problems and the solutions onto your resource sheet, and later write them onto your Final Examination Resource Sheet.
6. Write any comments, ideas, shortcuts, or other reminders that may be helpful during an examination.

Writing Your Final Examination

You will write the final examination when you have completed Module 8 of this course. The final examination is based on Modules 1 to 8, and is worth 25 percent of your final mark in the course. The examination content will consist of 20% from Modules 1 to 4 and 80% from Modules 5 to 8. You will write the final examination when you have completed Module 8. To do well on the final examination, you should review all the work you complete in Modules 1 to 8, including all the learning activities and assignments. You will write the final examination under supervision.

Notes

Lesson 1: Absolute Value of a Number

Lesson Focus

In this lesson, you will

- ❑ learn how to determine the absolute value of a positive or negative real number
- ❑ learn how the distance between two points on a number line is related to absolute value
- ❑ learn how to determine the absolute value of a numerical expression

Lesson Introduction

Consider the following situation. You and your friend are meeting at a movie theatre. You are both coming from opposite directions; however, you are both 3 km away from the theatre. This situation could be represented as 3 km in one direction and negative 3 km (−3 km) in the other direction. However, since distance is usually represented using a positive value, you would, therefore, normally say that you are both 3 km away from the movie theatre and 6 km away from each other. This idea is related to the concept of absolute value.

Absolute Value of a Number

The absolute value of a real number is always positive or zero. To denote the absolute value of a number, a vertical bar is placed on each side of the number. For example, $|5|$ denotes the absolute value of 5.

Determining the Absolute Value of a Number

The absolute value of a positive number is always the positive number. For instance, $|6| = 6$.

The absolute value of zero is always zero, $|0| = 0$.

The absolute value of a negative number is the negative of that number. For example, $|-6| = -(-6) = 6$.

Definition: The absolute value of a real number x, $|x|$, is defined by

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

This method of defining the absolute value of a number is very important, very useful, and is used in solving absolute value equations. This kind of notation, piece-wise, is studied more thoroughly in Lesson 2. In this lesson, some experience using the notation is provided.

Example 1

Evaluate:

a) $|8|$

b) $|-2|$

c) $|1|$

d) $|-(-9)|$

Solution

a) $|8| = 8$ as $|x| = x$ for all $x \geq 0$.

b) $|-2| = -(-2) = 2$ as $|x| = -x$ for all $x < 0$.

c) $|1| = 1$ as $|x| = x$ for all $x \geq 0$.

d) $|-(-9)| = |9| = 9$. Notice that you can simplify the expression inside the absolute value symbols first, before finding the answer.

Relating Absolute Value to Distance

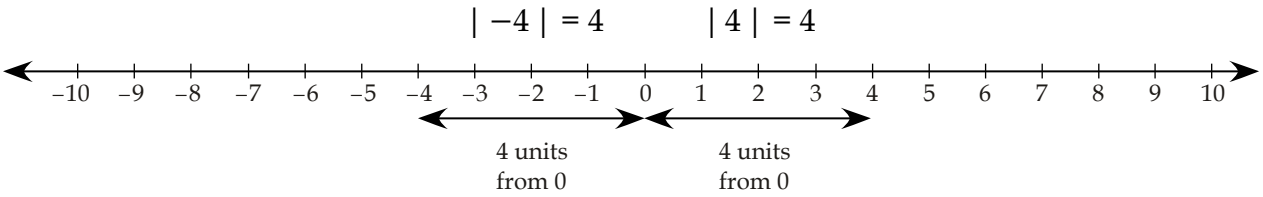
Consider the absolute value of negative 4 and positive 4.

$|-4| = 4$ $\qquad\qquad$ $|4| = 4$

These answers are equivalent. Where else do you encounter a negative value and a positive value being equal?

Distance! But their values aren't equal; rather, their absolute values are equal.

As mentioned in the introduction to this lesson, distance is usually measured in positive values. For example, you would never say you were a negative distance away from somewhere. This same concept can be related to distance from zero on a number line.



From the number line, you can see how $|-4| = |4| = 4$. As negative 4 and positive 4 are both four units away from zero, the absolute values of 4 and –4 are equivalent.

Example 2

a) Determine the distance of -8 and 8 from 0 on the number line.

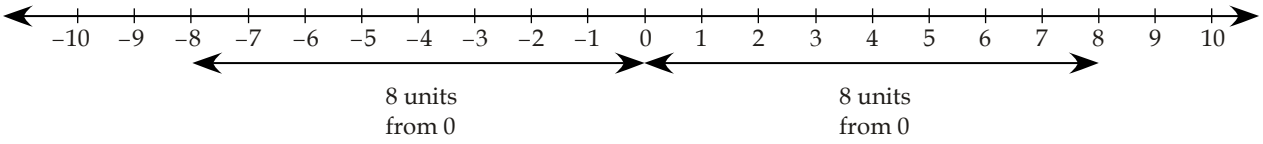
b) Evaluate the following:

i) $|-8|$

ii) $|8|$

Solution

a)



As you can see, both negative 8 and positive 8 are 8 units from 0. Any positive and negative number in the form $\pm a$ will be the same distance away from zero. In other words, any positive number and its negative will be the same distance away from zero.

b) i) $|-8| = -(-8) = 8$

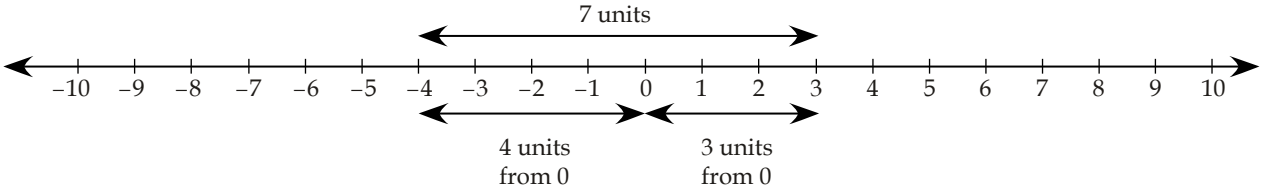
ii) $|8| = 8$

Both negative 8 and positive 8 have an absolute value of 8. From part (a) above, you can see how this relates to the distance from 0 of each value.

Example 3

Determine the distance from −4 to 3 using a number line and an equation.

Solution



Using the number line, you can see that the distance from −4 to 3 is 7 units.

To find a positive value for distance, you can subtract the two values from each other (in any order) and then find the absolute value of the answer.

$| -4 - 3 | = | -7 | = -(-7) = 7$

$| 3 - (-4) | = | 3 + 4 | = | 7 | = 7$

You will get the same answer of 7 units in either case and a positive answer for distance.

Example 4

Determine the distance between the following points using subtraction and absolute values.

a) 5 and −12

b) −8 and −5

Solution

a) $| 5 - (-12) | = | 5 + 12 | = 17$

or

$| -12 - 5 | = | -17 | = -(-17) = 17$

b) $| -8 - (-5) | = | -8 + 5 | = | -3 | = -(-3) = 3$

or

$| -5 - (-8) | = | -5 + 8 | = | 3 | = 3$

Example 5

Order the following values from least to greatest using a number line.

$$\left|-\frac{1}{2}\right|,\ \frac{2}{5},\ -4,\ |-4|,\ \left|\frac{3}{5}\right|,\ \left|\frac{1}{2}\right|,\ \frac{1}{8},\ -\frac{7}{2}$$

Solution

First, you need to evaluate each expression and express the final answer in decimal form. This will allow you to compare the values easily.

$$\left|-\frac{1}{2}\right| = \frac{1}{2} = 0.5$$

$$\frac{2}{5} = 0.4$$

$$-4 = -4$$

$$|-4| = -(-4) = 4$$

$$\left|\frac{3}{5}\right| = \frac{3}{5} = 0.6$$

$$\left|\frac{1}{2}\right| = \frac{1}{2} = 0.5$$

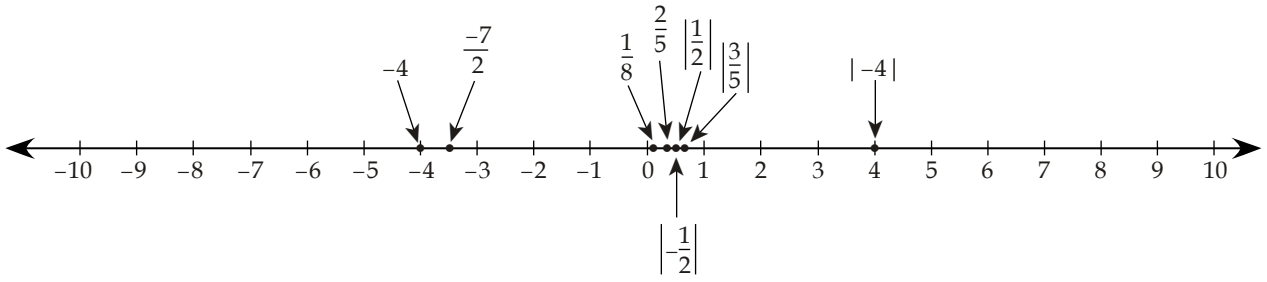
$$\frac{1}{8} = 0.125$$

$$-\frac{7}{2} = -3.5$$

Now, you need to order the values from least to greatest.

$-4,\ -3.5,\ 0.125,\ 0.4,\ 0.5,\ 0.5,\ 0.6,\ 4$

Place the corresponding original expression of each value on the number line in its correct position.



Evaluating Absolute Value Expressions

Just as you can evaluate linear or quadratic expressions, it is also possible to evaluate absolute value expressions. Consider the following example.

Example 6

Evaluate the following expressions.

a) $|4| - |-9|$

b) $5 - 3|-2 - 7|$

c) $|-3 - 5(3 - 6)^2|$

Solution

The first step for evaluating absolute value expressions is to evaluate everything inside the absolute value symbols. Once the absolute value symbols are only surrounding one term, you can then take the absolute value of that term. Remember the order of operations: inside the **B**rackets, **E**xponents, **D**ivision, **M**ultiplication, **A**ddition, **S**ubtraction, or the mnemonic BEDMAS.

a) $|4| - |-9|$

As each set of absolute value symbols only contain one term, you can take the absolute value of each term right away and then simplify.

$|4| - |-9| = 4 - [-(-9)] = 4 - 9 = -5$

b) $5 - 3|-2 - 7|$

As the set of absolute value symbols does not contain only one term, you need to simplify the expression inside the absolute value symbols first.

$5 - 3|-2 - 7| = 5 - 3|-9|$

You can now take the absolute value and then simplify.

$5 - 3|-9| = 5 - 3(-(-9)) = 5 - 3(9) = 5 - 27 = -22$

c) $|-3 - 5(3 - 6)^2|$

Following the order of operations, you need to simplify the expression inside the brackets first, and then deal with the exponent.

$|-3 - 5(3 - 6)^2| = |-3 - 5(-3)^2| = |-3 - 5(9)|$

Now, you can compute the multiplication and then the subtraction following the BEDMAS rule.

$|-3 - 5(9)| = |-3 - 45| = |-48|$

Finally, you can take the absolute value.

$|-48| = -(-48) = 48$

It is also possible to evaluate an absolute value expression for a variable. Consider the following examples.

Example 7

Evaluate the following expressions for $x = -3$.

a) $2|-2x + 5| - 1$

b) $3|-x^2 - 6x + 3| + 4$

Solution

To evaluate each of these expressions, simply substitute $x = -3$ in for the variable x and evaluate as you did in Example 6 using BEDMAS.

a) $2|-2x + 5| - 1$

$= 2|-2(-3) + 5| - 1$

$= 2|6 + 5| - 1$

$= 2|11| - 1$

$= 2(11) - 1$

$= 22 - 1$

$= 21$

b) $3|-x^2 - 6x + 3| + 4$

$= 3|-(-3)^2 - 6(-3) + 3| + 4$

$= 3|-9 + 18 + 3| + 4$

$= 3|12| + 4$

$= 3(12) + 4$

$= 36 + 4$

$= 40$

Before you complete the following learning activity to practice the skills you learned in this lesson, update your resource sheet. Include any examples that you found difficult or helped you understand the concept of absolute value better, as well as its definition.

Learning Activity 8.1

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. In the three courses you are taking this semester, you receive final marks of 72%, 83%, and 91%. What is your average?
2. Find the equation of a line with a slope of 2 and a y-intercept of -6.
3. Evaluate and express as a mixed fraction: $\frac{5}{6} + \frac{7}{4}$
4. You have a job at an electronics store. As a bonus for excellent sales this month, your employer offers you either $\frac{1}{20}$ or 7% of your annual salary. Which one is the better choice?
5. Evaluate: 2^6
6. Estimate: $\sqrt{220}$
7. Are these fractions equivalent? $\frac{3}{\sqrt{2}-4}$ and $\frac{3\sqrt{2}+12}{18}$
8. In which quadrant is an angle of 294° located?

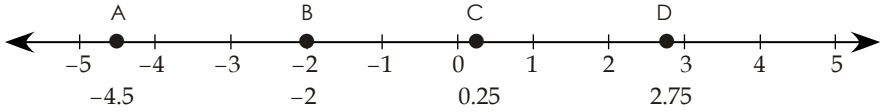
continued

Part B: Absolute Value

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Evaluate.
   a) $|-13|$
   b) $|4|$
   c) $|0|$
   d) $|-(52)|$
   e) $2|5-3(2)|+1$
   f) $3|4(3+2)^2-42|-2$

2. Use absolute value symbols to write an expression for the distance between each pair of points on the number line. Using this expression, determine the distance between the two points.



   a) A and D
   b) B and D
   c) C and B

3. Describe how distance can be related to absolute value.

4. Determine the distance between each of the following sets of points. Include a graph for (b), (c), and (d).
   a) -45 and -23
   b) $(5, 3)$ and $(-2, 3)$
   c) $(-1, 6)$ and $(-1, -5)$
   d) An extension of finding absolute value on the coordinate graph is to find the distance of a point from the origin—in other words, its absolute value. Find the absolute value for the point $(2, 3)$.

5. Order the following set of absolute values from least to greatest.

$$\left|-\frac{1}{4}\right|,\ \frac{1}{3},\ -3,\ |-3|,\ \left|\frac{2}{9}\right|,\ \left|\frac{3}{2}\right|,\ \frac{1}{2},\ -\frac{7}{3}$$

continued

Learning Activity 8.1 (continued)

6. Evaluate the following expressions for $x = -2$.
   a) $3|-2x-5|+4$
   b) $3|x^2-7x+2|-5$

7. Consider the expression $\frac{x}{|x|}$.
   a) What is the value of $\frac{x}{|x|}$ when x is positive?
   b) What is the value of $\frac{x}{|x|}$ when x is negative?

8. In Manitoba, the temperature could go as low as $-40°$ C in the winter. In the summer, the temperature may reach a high of 30° C. Determine the difference in the range of these two temperatures.

9. Charlize works in an office building and uses an electronic key card in order to travel to different floors of the building by elevator. This electronic key card tracks her movements throughout the day. The floors Charlize travelled to throughout one day are displayed in the chart below.

Time	Floor
8 a.m.	1
9:13 a.m.	4
11:41 a.m.	2
12:51 p.m.	7
2:00 p.m.	4
2:10 p.m.	6
2:30 p.m.	4
4 p.m.	1

   a) Write and evaluate an absolute value expression to determine the total amount of floor changes throughout Charlize's day.
   b) Write and evaluate an absolute value expression to determine the net change in Charlize's location from the beginning of the day to the end of the day.
   c) Compare the value you calculated in (a) to the value you found in (b). How does the total change compare to the net change of Charlize's location?

Lesson Summary

In this lesson, you were introduced to the concept of absolute value. You learned how the absolute value of a number is always positive or zero. The concept of absolute value can also be related to distance. For example, the distance between any two points on a number line or any two places on a map is always positive. In the next lesson, you will be learning about absolute value graphs.

Notes

Lesson 2: Graphing Absolute Value Functions

Lesson Focus

In this lesson, you will

- ❑ learn how to express an absolute value function using piece-wise notation
- ❑ learn how to graph the absolute value of a given function
- ❑ learn how the absolute value operation affects the graph of a function as well as its intercepts, domain, and range

Lesson Introduction

In Lesson 1, you learned that the absolute value of a number is always positive or zero. This idea leads to using a new notation for defining functions called **piece-wise functions**. In this lesson, you will learn how to graph absolute value functions and how to express the graph of an absolute value function as a piece-wise function.

Absolute Value Functions

All absolute value functions can be expressed as piece-wise functions.

Piece-wise Functions

Piece-wise functions are defined by a formula for each different part of their domains. The absolute value is just one example of a function that can be defined using piece-wise notation.

Include this definition on your resource sheet.

Piece-wise functions break up the domain of the graph. In different parts of the domain, different functions are graphed. In Lesson 1, you saw that the definition of absolute value consists of two pieces, when x is greater than or equal to zero and when x is less than zero. The definition for an absolute value function is similar.

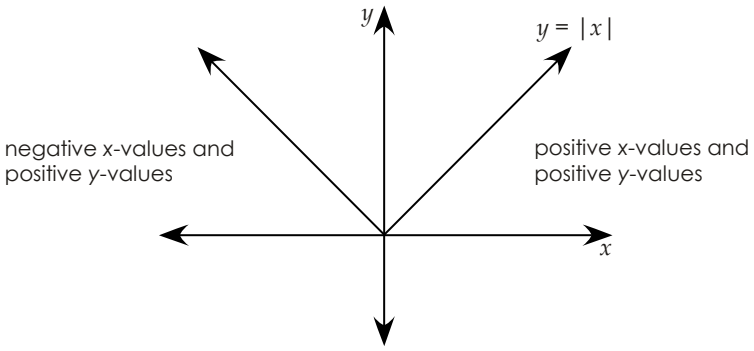
Definition: The absolute value function $y = |x|$ is defined by

$$y = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

This definition states that the y-value remains the same as the x-value as long as the x-value is non-negative and the y-value is the opposite of the x-value if the x-value is negative.

Graphing Absolute Value Functions

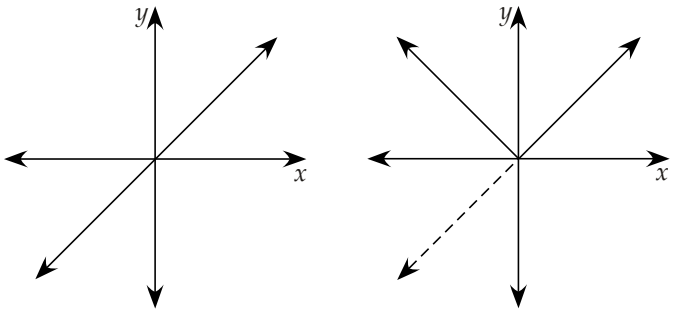
The sketch of $y = |x|$ is shown below.



The graph shows a piece of the line, $y = x$, when $x \geq 0$, and a piece of the line $y = -x$, when $x < 0$. Which piece of each line is graphed depends on how the domain of the function is given.

Another interpretation of piece-wise functions involves transformations. The absolute value can be thought of as an operation on a function. The absolute value operation results in a partial reflection on the graph of a function by changing the negative y-values to positive y-values and, thus, reflecting the points below the x-axis to points above the x-axis. You could think of the graph of $y = |x|$ as the graph of $y = x$ with the portion below the x-axis reflected above the x-axis. The x-axis becomes the line of reflection.

The absolute value transformation can be defined as $(x, y) \rightarrow (x, |y|)$. A graph of this transformation is shown below.



Example 1

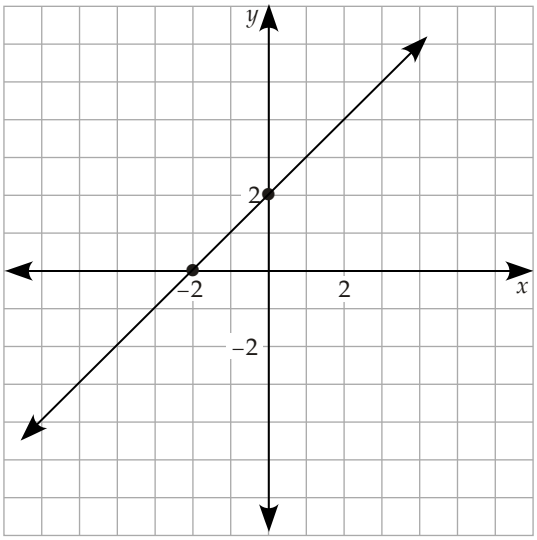
Sketch the graph of $y = x + 2$ and then the graph of $y = |x + 2|$.

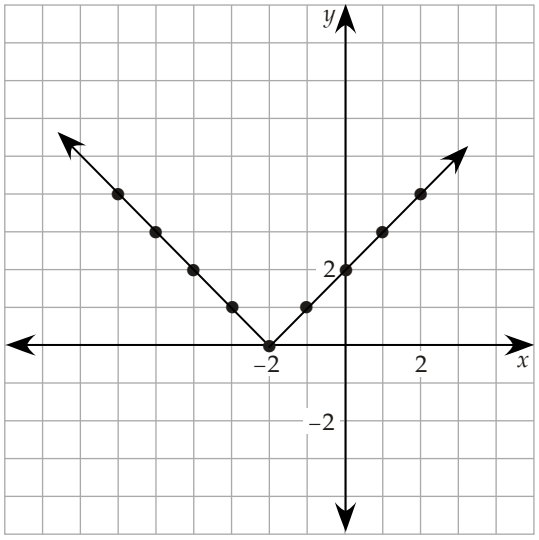
Solution

Method 1

Use a table of values:

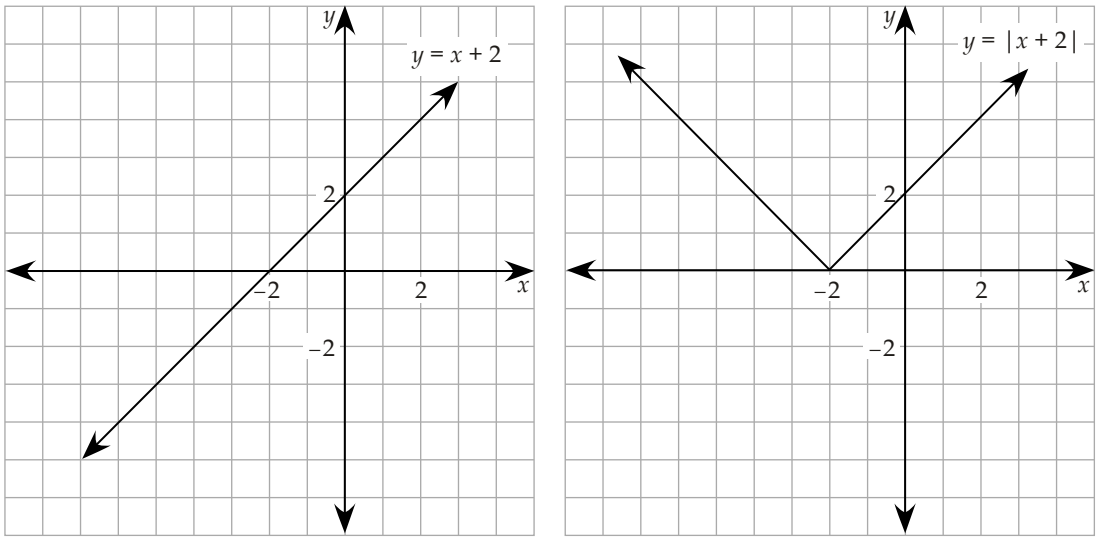
x	$x + 2$	$\lvert x + 2 \rvert$
−5	−3	3
−4	−2	2
−3	−1	1
−2	0	0
−1	1	1
0	2	2
1	3	3
2	4	4
3	5	5





Method 2

Use your knowledge of the absolute value transformation.



Note that the x-intercept, $x = -2$, in the original function is the x-intercept of the absolute value function as well. In other words, the point $(-2, 0)$ is not changed in the absolute value transformation. Such a point is called an **invariant point.**

In absolute value transformations, the x-intercept of the original function also becomes a **pivot point** or a turning point, where the graph changes direction.

All the segments of the graph that are below the x-axis get reflected over the x-axis at its pivot points. The negative y-values are changed to positive y-values.

Add the definition of *pivot point* and *invariant point* to your resource sheet.

Example 2

Write $y = |x - 2|$ as a piece-wise function.

Solution

By definition of an absolute value function, you have

$$y = \begin{cases} x - 2, & x - 2 \geq 0 \\ -(x - 2), & x - 2 < 0 \end{cases}$$

There are two cases to consider, because there are two pieces of the domain.

Case 1: $y = x - 2$ and $x - 2 \geq 0$

This case can be simplified to $y = x - 2$ and $x \geq 2$.

This means the expression inside the absolute value symbols is positive or zero when x is greater than or equal to two. Thus, the function, $y = |x - 2|$, can be written as $y = x - 2$. The absolute value signs can be removed. You have a piece of the line $y = x - 2$ as long as $x \geq 2$.

Case 2: $y = -(x - 2)$ and $x - 2 < 0$

This case can be simplified to $y = -x + 2$ and $x < 2$.

Since the expression inside the absolute value brackets is now negative, the function values need changing to make them positive. Thus, when the absolute value symbols are removed, $y = |x - 2|$ must be written as $y = -(x - 2) = -x + 2$. Now you have a piece of the line, $y = -x + 2$, as long as $x < 2$.

In summary, you can write the absolute value function $y = |x - 2|$ in piece-wise notation as

$$y = \begin{cases} x - 2, & x \geq 2 \\ -x + 2, & x < 2 \end{cases}$$

Example 3

Sketch each pair of functions using the absolute value transformation, and find the x- and y-intercepts.

a) $f(x) = -2x + 1$ and $F(x) = |-2x + 1|$

b) $g(x) = x^2 - 4$ and $G(x) = |x^2 - 4|$

c) Provide the following information for the functions in (a) and (b), summarized in a chart: domain, range, intercepts.

Solution

a) $f(x) = -2x + 1$

To graph $F(x)$, it is easiest to find the pivot point of $f(x)$ by solving for the x-intercept of $f(x)$.

$$\begin{aligned} f(x) &= -2x + 1 \\ 0 &= -2x + 1 \\ -1 &= -2x \\ \frac{1}{2} &= x \end{aligned}$$

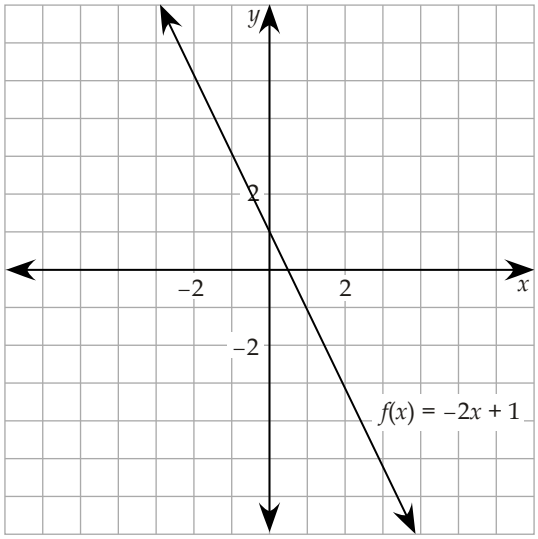
Now, find the y-intercept of $F(x)$.

$$F(x) = |-2x + 1|$$

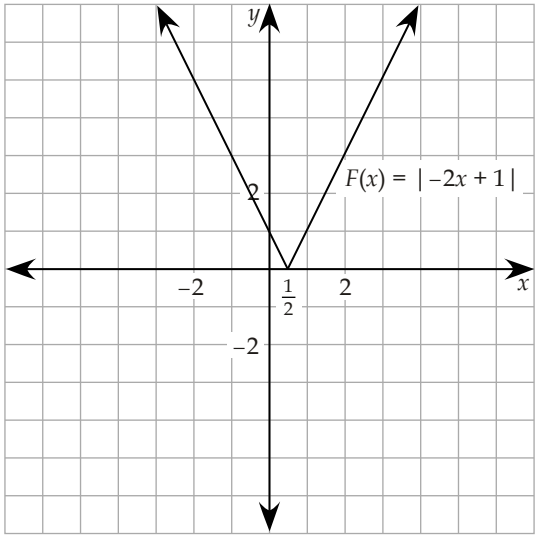
$$F(0) = |-2(0) + 1|$$

$$F(0) = 1$$

Graph these two points on the coordinate plane. Then, using the slope of $f(x)$ as negative 2, you can find additional points on the line.



Then, reflect the part of $f(x)$ that is below the x-axis to above the x-axis, starting at the x-intercept or pivot point $\left(\frac{1}{2}, 0\right)$ to get the function $F(x) = |-2x + 1|$.



b) $g(x) = x^2 - 4$

The pivot points are the zeros of the function. Therefore, you need to solve for x when the function value is zero in order to find the pivot points.

$$g(x) = x^2 - 4$$
$$x^2 - 4 = 0$$
$$(x - 2)(x + 2) = 0$$
$$x = \pm 2$$

Now, find the y-intercept of $G(x)$.

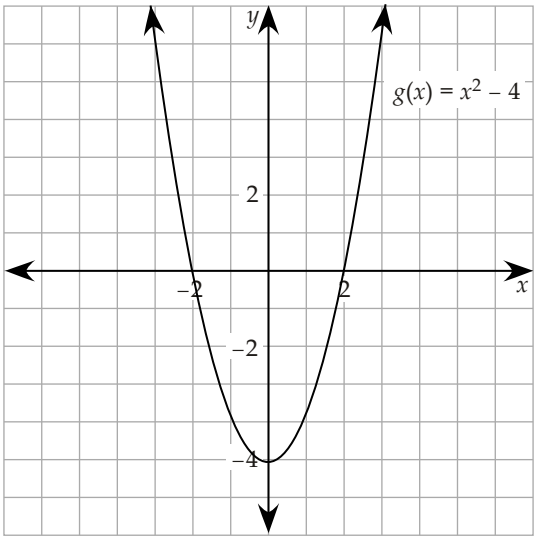
$G(x) = | x^2 - 4 |$

$G(0) = | 0^2 - 4 |$

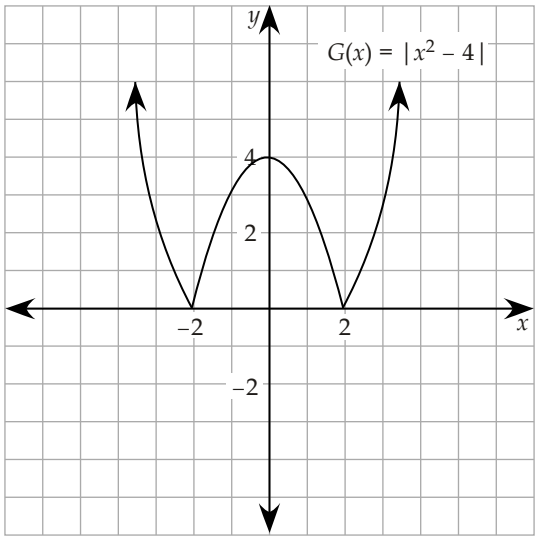
$G(0) = 4$

Also, the vertex of $g(x)$ is (0, 4); the curve opens up and is normal size.

Using this information, you can draw the graph of $g(x) = x^2 - 4$.



Using this graph, you can reflect the portion below the x-axis to above the x-axis to get $y = |x^2 - 4|$.



Notice how the curve now has sharp corners or "cusps" at the pivot points.

c)

Function	Domain	Range	x-intercept(s)	y-intercept
$f(x) = -2x + 1$	$\Re$	$\Re$	$\frac{1}{2}$	1
$F(x) = \lvert -2x + 1 \rvert$	$\Re$	$[0, \infty)$	$\frac{1}{2}$	1
$g(x) = x^2 - 4$	$\Re$	$[-4, \infty)$	± 2	-4
$G(x) = \lvert x^2 - 4 \rvert$	$\Re$	$[0, \infty)$	± 2	4

Example 4

Write a simplified piece–wise function for the two absolute value functions in Example 3.

Solution

There are two methods to answer this question. An algebraic method is shown for $F(x)$. A graphing method is shown for $G(x)$.

a) $F(x) = |-2x + 1|$

By definition of an absolute value function,

$$F(x) = \begin{cases} -2x + 1, -2x + 1 \geq 0 \\ -(-2x + 1), -2x + 1 < 0 \end{cases}$$

There are two cases to consider.

Case 1: $F(x) = -2x + 1$ and $-2x + 1 \geq 0$

First, simplify the domain:

$$-2x + 1 \geq 0$$

$$-2x \geq -1$$

$$x \leq \frac{1}{2}$$ Reverse the direction of the inequality when dividing by a negative.

This means the expression inside the absolute value signs is positive or zero when x is less than or equal to $\frac{1}{2}$. Thus, the function, $F(x) = |-2x + 1|$ can be written as $F(x) = -2x + 1$. The absolute value signs can be removed. You have a piece of the line $y = -2x + 1$ as long as $x \leq \frac{1}{2}$.

Case 2: $F(x) = -(-2x + 1)$ and $-2x + 1 < 0$

Simplify the domain:

$$-2x + 1 < 0$$

$$-2x < -1$$

$$x > \frac{1}{2}$$

Since the expression inside the absolute value brackets is negative, the function values need changing to make them positive when x is greater than $\frac{1}{2}$. Thus, when the absolute value signs are removed, $F(x) = |-2x + 1|$ must be written as $F(x) = -(-2x + 1) = 2x - 1$. Now you have a piece of the line, $y = 2x - 1$, as long as $x > \frac{1}{2}$.

In summary, you can write the absolute value function $F(x) = |-2x + 1|$ in piece-wise notation as

$$F(x) = \begin{cases} -2x + 1, & x \leq \frac{1}{2} \\ 2x - 1, & x > \frac{1}{2} \end{cases}$$

b) $G(x) = |x^2 - 4|$

By looking at the graph of $g(x) = x^2 - 4$, you can see that $g(x) \geq 0$ when $x \leq -2$ and when $x \geq 2$. Therefore, these segments of the graph are always positive and the function can be written as $G(x) = x^2 - 4$ when $x \leq -2$ and when $x \geq 2$.

Between -2 and 2, the graph of $g(x) = x^2 - 4$ is below the x-axis. When the absolute value of this function is taken, this part of the graph is reflected over the x-axis. Therefore, $G(x) = |x^2 - 4|$ can be written as $G(x) = -(x^2 - 4)$ when $-2 < x < 2$. You can simplify $G(x) = |x^2 - 4|$ to get $G(x) = -x^2 + 4$.

$G(x)$ can be written as the following piece-wise function:

$$G(x) = \begin{cases} x^2 - 4, \leq -2 \\ -x^2 + 4, -2 < x < 2 \\ x^2 - 4, x \geq 2 \end{cases}$$

Learning Activity 8.2

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. If you get paid $11 an hour and you work 30 hours in a week, how much will you get paid (before taxes)?
2. If you travel at a speed of 72 km/h for 6 hours, how far will you travel?
3. State the reciprocal of $\frac{5}{7}$.
4. Calculate 15% of 140.
5. Find the equation of a line with a slope of 3 that crosses through the point (1, 1).
6. Express in exponential form: $\left(\sqrt{5}\right)\left(\sqrt[3]{5}\right)$

continued

Learning Activity 8.2 (continued)

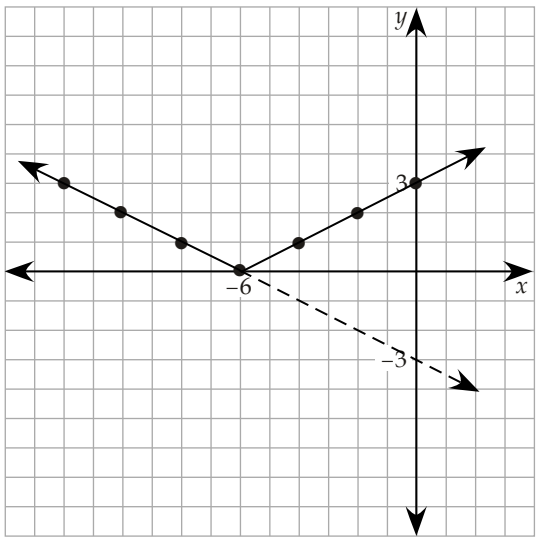
7. Determine the distance between the following two points: (7, −1) and (−8, 15)
8. How many feet are in 24 yards?

Part B: Graphing Absolute Value Functions

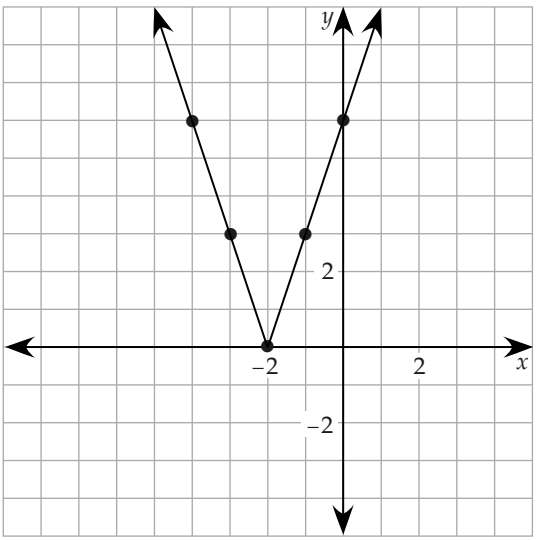
Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Sketch each function and state the values of the intercepts.
   a) $f(x) = |2x - 1|$
   b) $g(x) = |x + 3|$
   c) $h(x) = |x^2 - 1|$
   d) $j(x) = |-x^2 + 2x + 3|$
   e) State the domain and range for each function in (a) to (d).
2. Write each function from Question 1 using piece-wise notation and simplify.
3. The point (2, −5) is on the graph $y = f(x)$. Identify the corresponding point on the graph $y = |f(x)|$.
4. The graph of $y = f(x)$ has an x-intercept of –6 and a y-intercept of −1. What are the x- and y-intercepts on the graph of $y = |f(x)|$?
5. Write the piece-wise function that represents each graph.

a)



b)



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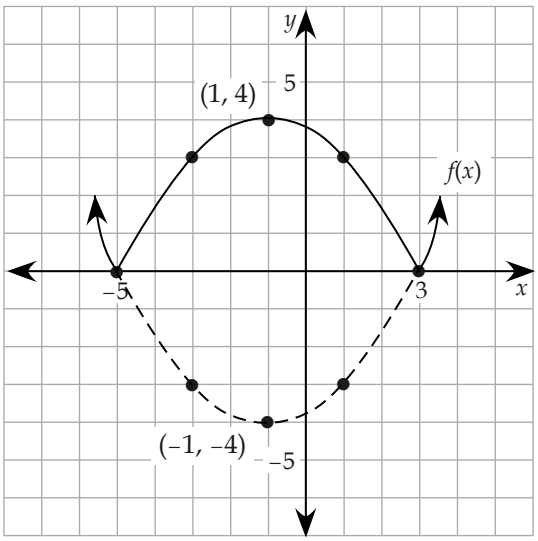
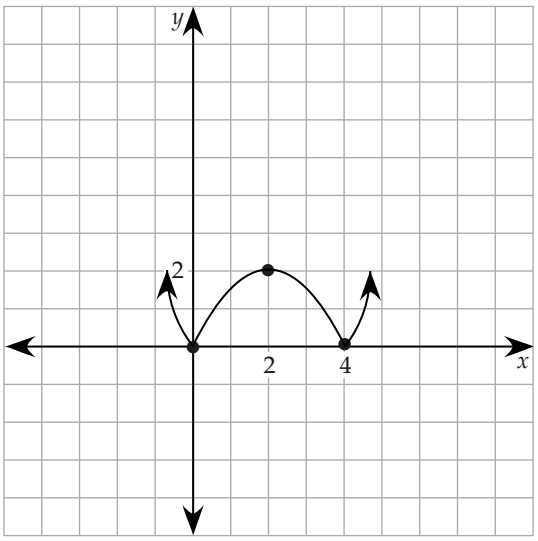
c)



d)



6. Graph $f(x) = |-4x + 3|$ and $g(x) = |4x - 3|$ using a table of values. What do you notice about these two graphs? Explain.

x	$y = -4x + 3$	$f(x) = \lvert -4x + 3 \rvert$	$y = 4x - 3$	$g(x) = \lvert 4x - 3 \rvert$
−3				
−2				
−1				
0				
1				
2				
3				

7. Graph the function $f(x) = |x^2 - 6x + 5|$. Write a different absolute value function having the same graph as $f(x)$.

Lesson Summary

In this lesson, you learned how to graph absolute value functions. You learned how the graphs of absolute value functions are always above or on the x-axis. Any points of the original function that aren't above the x-axis are reflected over the x-axis to obtain positive y-values. You also learned how it was possible for two different functions to have the same absolute value graphs. You will be learning more about this concept when you learn how to solve absolute value equations in the next lesson.

Assignment 8.1

Absolute Value

Total: 42 marks

1. Evaluate. *(1 mark each, for a total of 4 marks)*

   a) $2|\,3(-2) - 4\,| + 1$

   b) $-2|\,-1(3) + 2\,| - 3$

   c) $3|\,2^2 - 4(3) - 3\,| + 1$

   d) $-2|\,(-3)^2 - 11 + 1\,| + 3$

continued

Assignment 8.1: Absolute Value (continued)

2. Determine the distance between each of the following sets of points. *(2 marks each, for a total of 4 marks)*

   a) $(-1, 7)$ and $(-1, -5)$

   b) $(-2, 4)$ and $(-9, 4)$

3. Order the following numbers from least to greatest. *(2 marks)*

$$7.5,\ -8\frac{1}{3},\ \left|-\frac{4}{5}\right|,\ \left|\frac{3}{2}\right|,\ 1.5,\ \left|-\frac{4}{3}\right|$$

continued

Assignment 8.1: Absolute Value (continued)

4. Evaluate the following expressions for $x = -4$. *(2 marks each, for a total of 4 marks)*

   a) $-2|\,-2x + 1\,| + 5$

   b) $2|\,x^2 + 4x + 3\,| - 6$

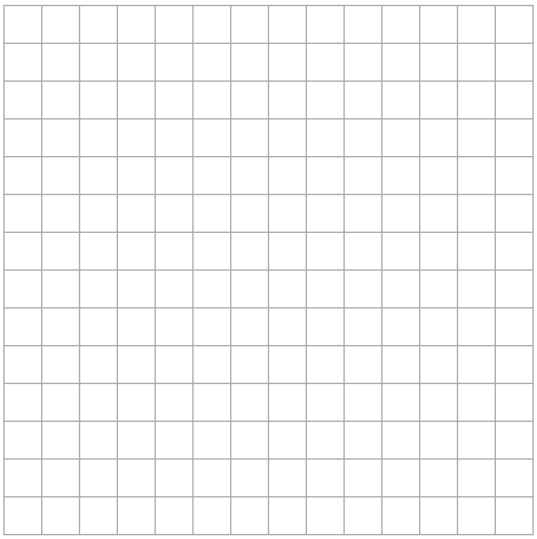
5. On one cold day in winter, the temperature was −42°C with the windchill. By lunchtime, the temperature had climbed to −11°C with no windchill. Determine how many degrees the temperature warmed up during the day. *(2 marks)*

continued

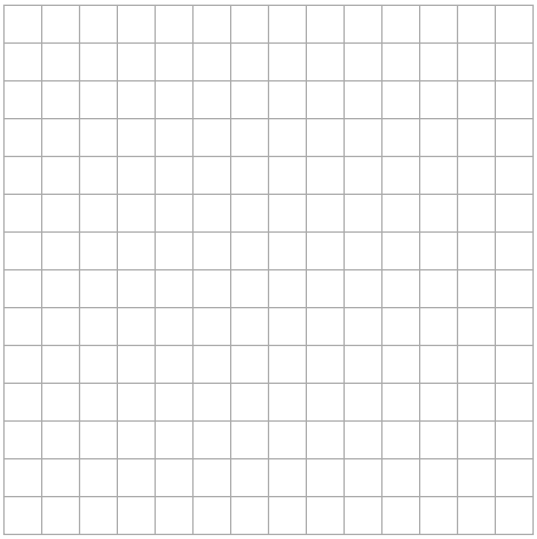
Assignment 8.1: Absolute Value (continued)

6. Sketch each function and state the domain, range, and the values of the intercepts. *(5 marks each, for a total of 10 marks)*

   a) $y = | -3x + 2 |$



   b) $y = | 2(x + 1)^2 - 4 |$



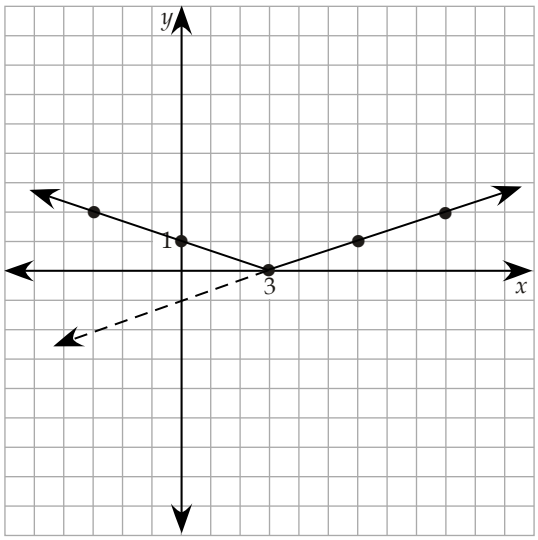
continued

Assignment 8.1: Absolute Value (continued)

7. Write each function from Question 6 using piece-wise notation. *(2.5 marks each, for a total of 5 marks)*

8. Write the piece-wise function that represents each graph. *(4 marks each, for a total of 8 marks)*

a)



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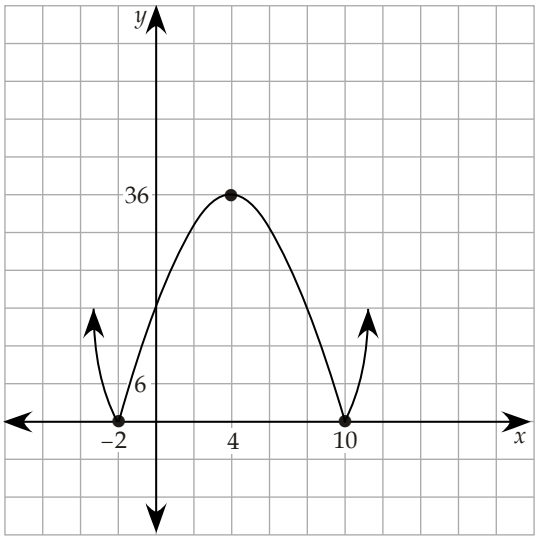
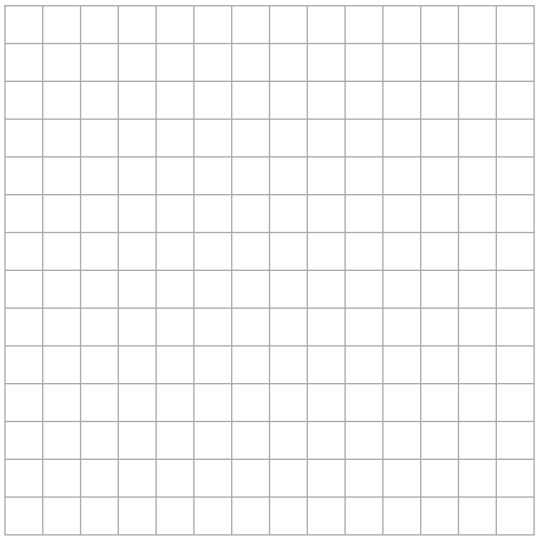
b)



9. Graph $f(x) = |x^2 - 4|$. Write a different absolute value function that would have the same graph as $f(x) = |x^2 - 4|$. *(3 marks)*



Lesson 3: Solving Absolute Value Equations

Lesson Focus

In this lesson, you will

- ❑ learn how to solve absolute value equations algebraically
- ❑ learn how to solve absolute value equations graphically
- ❑ learn how to apply solving absolute value equations to word problems

Lesson Introduction

In the first two lessons, you learned how to find the absolute value of a numerical expression as well as how to graph an absolute value function. These two skills are needed in order to solve absolute value equations. Absolute value equations are useful when you are asked to find the distance from a particular point. Distance is always positive or zero. Therefore, the distance can be either in the negative or positive direction and still be considered a positive distance.

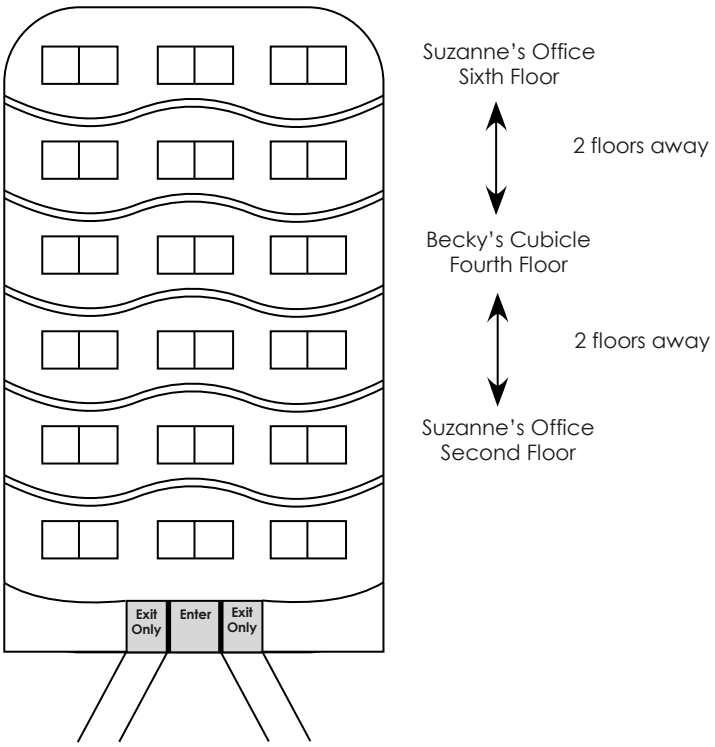
Absolute Value Equations

The absolute value of $\pm a$ is equal to a given $a \geq 0$. Therefore, there are pairs of values on the number line having the same absolute value. This may result in absolute value equations having more than one solution.

Solving Absolute Value Equations Algebraically

If Becky's cubicle is on the fourth floor of her office building and her boss Suzanne's office is two floors away from Becky, what possible floors could Suzanne's office be on?

This expression can be easily solved by adding or subtracting 2 from 4 to get 6 and 2 respectively. Therefore, Suzanne's possible office locations are on the second floor and the sixth floor.



This same type of problem can be expressed in an absolute value equation. Let x represent the floor number for Suzanne's office. You are solving an equation that reads "the distance from x to 4 is 2." As distance is always positive, the first half of the expression will be surrounded by absolute value symbols. The word "is" means "equal to," so your absolute value equation becomes

$$|x - 4| = 2$$

You know the answers to this equation are 6 and 2, but how do you get there?

To solve for x to get a value of $x = 6$, you can simply eliminate the absolute value symbols.

$$x - 4 = 2$$

$$x = 2 + 4$$

$$x = 6$$

How do you solve for x to get a value of $x = 2$? Well, you know that a negative number will have the same absolute value as the positive value of that same number. Therefore, if you make the expression inside the absolute value symbols negative, you should be able to find the value of $x = 2$.

$$-(x - 4) = 2$$
$$-x + 4 = 2$$
$$-x = 2 - 4$$
$$-x = -2$$
$$x = 2$$

In general, to solve an absolute value equation algebraically, solve the equation twice—once when the expression inside the absolute value symbols is positive, and again when the expression inside the absolute value symbols is negative. Consider the following examples.

Example 1

Write the following expressions in words. Using mathematical reasoning, solve for the missing x-values. Then, solve for the missing x-values algebraically.

a) $|x| = 3$

b) $|x - 5| = 2$

Solution

a) $|x| = 3$

This expression can be read as, "The distance from x to 0 is three units." Therefore, the pair of solutions is $x = 3$ or $x = -3$.

Solving algebraically:

Case 1: $x = 3$, when $x \geq 0$

and

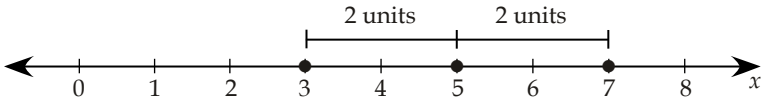
Case 2: $-x = 3$, when $x < 0$

$x = -3$

Therefore, you get the same solutions of $x = 3$ and -3 by considering the x-value as a distance as well as by calculating the answer algebraically.

b) $|x - 5| = 2$

This expression can be read as, "This distance from x to 5 is 2 units." Consider the number line below:



From the number line, you can see that the two values that are 2 units away from 5 are 3 and 7.

Solving algebraically:

Case 1: $x - 5 = 2$, when $x - 5 \geq 0$

$x = 2 + 5$ and $x \geq 5$

$x = 7$ and $x \geq 5$

Case 2: $-(x - 5) = 2$, when $x - 5 < 0$

$-x + 5 = 2$ and $x < 5$

$-x = -3$ and $x < 5$

$x = 3$ and $x < 5$

Thus, the two solutions for x are 3 and 7.

Solving absolute value equations is similar to solving absolute value equations by using the definition of absolute value. The definition of absolute value is:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Consider the following example.

Example 2

Solve the following equations using the definition of absolute value.

a) $|x + 1| = 3$

b) $|3x - 5| = 8$

c) $|2x - 5| = 5 - 3x$

d) $|x^2 + 3x| = 4$

e) $|x^2 - 11x| = -x + 11$

f) $|3x - 15| = 0$

Solutions

a) $|x + 1| = 3$

There are two cases:

Case 1: $x + 1 = 3$, $x + 1 \geq 0$

Case 2: $-(x + 1) = 3$, $x + 1 < 0$

Solve Case 1 and then Case 2:

Case 1: $x + 1 = 3$ and $x + 1 \geq 0$

$x + 1 = 3$ $x + 1 \geq 0$

$x = 2$ $x \geq -1$

Both $x = 2$ and $x \geq -1$ are true.

$\therefore x = 2$ is a solution.

Case 2: $-(x + 1) = 3$ and $x + 1 < 0$

$-(x + 1) = 3$ $x + 1 < 0$

$-x - 1 = 3$ $x < -1$

$-x = 4$

$x = -4$ and $x < -1$

Both $x = -4$ and $x < -1$ are true.

$\therefore x = -4$ is a solution.

The solutions are $x = 2, -4$.

b) $|3x - 5| = 8$

Case 1: $-(3x - 5) = 8$ and $3x - 5 \geq 0$

$3x - 5 = 8$ $3x \geq 5$

$3x = 13$ $x \geq \frac{5}{3}$

$x = \frac{13}{3}$

Since $x = \frac{13}{3}$ is in the interval $x \geq \frac{5}{3}$, then it is a solution.

Case 2: $-(3x - 5) = 8$ and $3x - 5 \geq 0$

$-3x + 5 = 8$ $3x < 5$

$-3x = 3$ $x < \frac{5}{3}$

$x = -1$

Both conditions are satisfied. The solutions are $x = \frac{13}{3}$ or $x = -1$.

c) $|2x - 5| = 5 - 3x$

Case 1: $2x - 5 = 5 - 3x$ and $2x - 5 \geq 0$

$5x = 10$ $2x \geq 5$

$x = 2$ $x \geq \frac{5}{2}$

Since $x = 2$ is not in the interval $x \geq \frac{5}{2}$, then it is not a solution. The value $x = 2$ is called an extraneous root. Also notice that the right side of the equation has a value of -1 when $x = 2$, suggesting $|2x - 5| = -1$. This is false. An absolute value can never have a negative value.

Case 2: $-(2x - 5) = 5 - 3x$ and $2x - 5 < 0$

$-2x + 5 = 5 - 3x$ $2x < 5$

$x = 0$ $x < \frac{5}{2}$

Since $x = 0$ is in the interval $x < \frac{5}{2}$, then $x = 0$ is a solution.

$\therefore$ The only solution is $x = 0$.

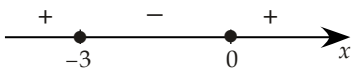
d) $|x^2 + 3x| = 4$

Case 1: $x^2 + 3x = 4$ and $x^2 + 3x \geq 0$

$x^2 + 3x - 4 = 0$ $x(x + 3) \geq 0$

$(x + 4)(x - 1) = 0$ $x = 0, -3$

$x = -4, 1$



Solution: $(-\infty, -3] \cup [0, \infty)$

Both conditions are satisfied because $x = -4$ is in the interval $(-\infty, -3]$ and $x = 1$ is in the interval $[0, \infty)$.

$\therefore$ $x = -4, 1$ are solutions.

Case 2: $-(x^2 + 3x) = 4$ and $x^2 + 3x < 0$

$$-x^2 - 3x = 4$$
$$x^2 + 3x = -4$$
$$x^2 + 3x + 4 = 0$$

discriminant $= b^2 - 4ac = 3^2 - 4(1)(4)$
$= 9 - 16$
$= -7$

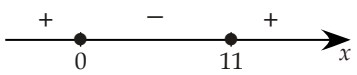
Since the discriminant is negative, there are no real roots.

$\therefore$ The solutions are $x = -4, 1$.

e) $|x^2 - 11x| = -x + 11$

Case 1: $x^2 - 11x = -x + 11$ and $x^2 - 11x \geq 0$

$$x^2 - 11x + x - 11 = 0 \qquad x(x - 11) \geq 0$$
$$x^2 - 10x - 11 = 0 \qquad x = 0, 11$$
$$(x - 11)(x + 1) = 0$$
$$x = 11, -1$$

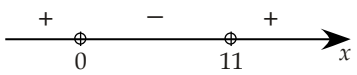


Solution: $(-\infty, 0] \cup [11, \infty)$

Since $x = 11$ is in the interval $[11, \infty)$ and $x = -1$ is in the interval $(-\infty, 0]$, then both are solutions.

Case 2: $-(x^2 - 11x) = -x + 11$ and $x^2 - 11x < 0$

$$-x^2 + 11x + x - 11 = 0 \qquad x(x - 11) < 0$$
$$x^2 - 12x + 11 = 0 \qquad x = 0, 11$$
$$(x - 11)(x - 1) = 0$$
$$x = 11, 1$$



Solution: $(0, 11)$

Since $x = 11$ is not in the interval $(0, 11)$ but $x = 1$ is, then the only solution here is $x = 1$.

$\therefore$ The solutions are $x = 11, -1, 1$.

f) $|3x - 15| = 0$

There are no cases here. There is only one possibility. If $|3x - 15| = 0$, then $3x - 15$ must be zero and $x = 5$ is the answer.

Solving Absolute Value Equations Graphically

To solve an absolute value equation graphically, you can graph the corresponding functions on either side of the equal sign. Once these two functions are graphed, the x-coordinate of the point(s) at which the functions intersect will be the solution(s) to the absolute value equation.

Example 3

Solve the following equations graphically.

a) $|x + 1| = -3$

b) $|x + 2| = 3$

c) $|2x - 5| = 5 - 3x$

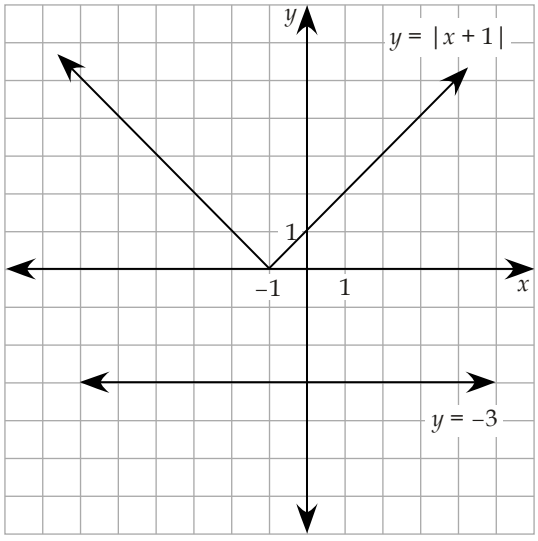
d) $|x^2 + 3x| = 4$

e) $|x^2 - 11x| = -x + 11$

Solutions

a) $|x + 1| = -3$

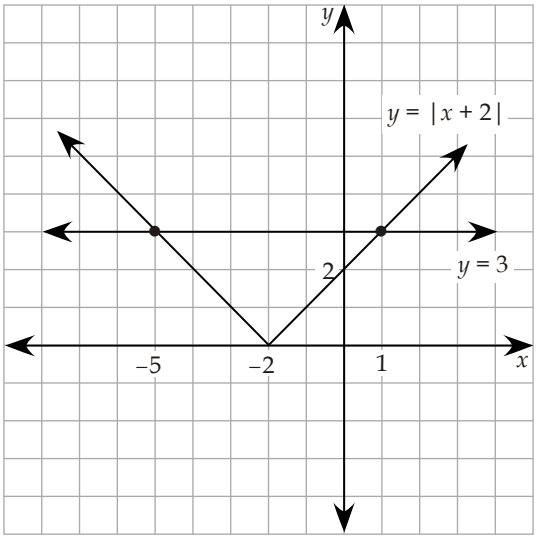
The corresponding functions are $y = |x + 1|$ and $y = -3$.



As the two functions do not intersect, there is no solution. You can also see from the equation itself, there is no solution because an absolute value is always positive or zero, never negative.

b) $|x + 2| = 3$

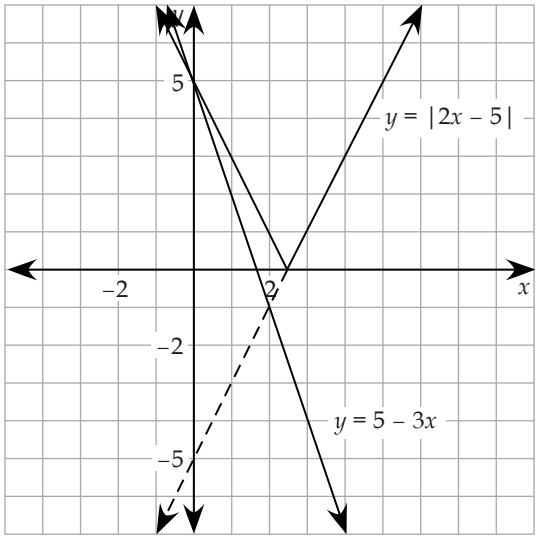
The corresponding functions are $y = x + 2$ and $y = 3$.



From the graph, you can see that there are two points of intersection: one point at (−5, 3) and another point at (1, 3). The x-values of these points are the solutions for x for the absolute value equation. Therefore, the solutions are $x = -5$ and $x = 1$.

c) $|2x - 5| = 5 - 3x$

The corresponding functions are $y = |2x - 5|$ and $y = 5 - 3x$.



From the graph, you can see there is one point of intersection at (0, 5). Therefore, the solution to this absolute value equation is $x = 0$.

d) $|x^2 + 3x| = 4$

The function $y = |x^2 + 3x|$ is found by first graphing $y = x^2 + 3x$. This is a parabola opening up, normal size, and its vertex can be found by completing the square:

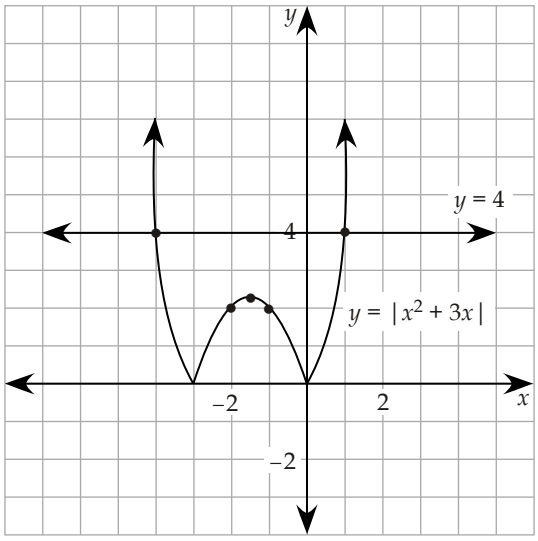
$$y = x^2 + 3x$$

$$y = \left(x^2 + 3x + \frac{9}{4}\right) - \frac{9}{4}$$

$$y = \left(x + \frac{3}{2}\right)^2 - \frac{9}{4}$$

The vertex is $\left(-\frac{3}{2}, \frac{-9}{4}\right)$.

The absolute value transformation reflects any point with a negative y-value over the x-axis. The graph is shown below. The graph of the other function, from the right-hand side of the equation, $y = 4$, is also graphed.



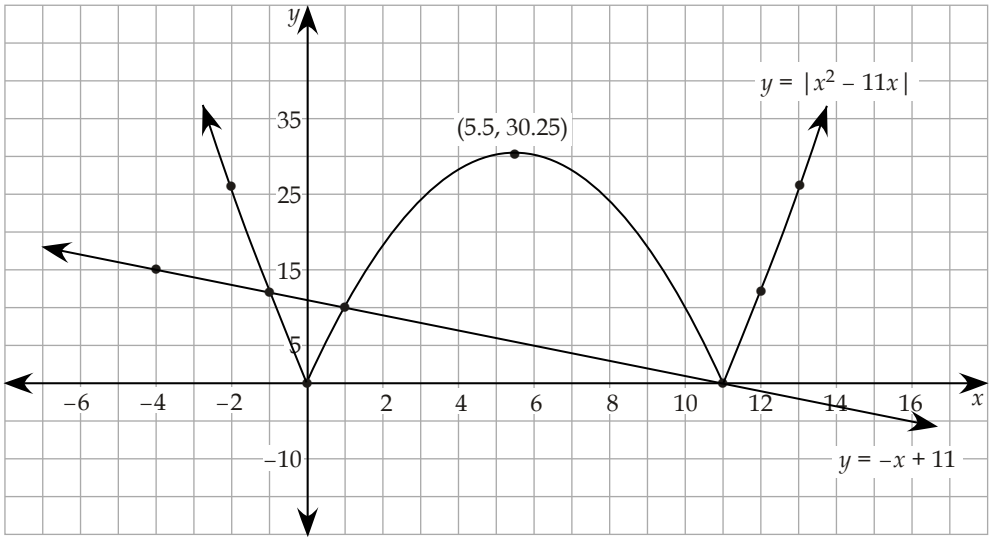
The two solutions are $x = -4$ and $x = 1$, which are the same solutions you arrived at algebraically in Example 2(d) above.

e) $|x^2 - 11x| = -x + 11$

The function corresponding to the left-hand side of the equation is $y = |x^2 - 11x|$ with zeros at $x^2 - 11x = 0$ or $x(x - 11) = 0$. The zeros are 0 and 11. The vertex can be found using the formula $x = -\frac{b}{2a} = \frac{-(-11)}{2(1)} = \frac{11}{2}$.

The y-coordinate of the vertex is $y = \left|\left(\frac{11}{2}\right)^2 - 11\left(\frac{11}{2}\right)\right| = |-30.25| = 30.25$.

The vertex is (5.5, 30.25). The function corresponding to the right-hand side of the equation is $y = -x + 11$. Both functions are shown on the graph below.



From the graph, you can see the two functions intersect at three points: (−1, 12), (1, 10), and (11, 0). The y-coordinate of the first intersection point can be verified by substitution of $x = -1$ into both functions:

$y = |x^2 - 11x| = |(-1)^2 - 11(-1)| = 12$

$y = -x + 11 = -(-1) + 11 = 12$

The three solutions are $x = -1$, $x = 1$, and $x = 11$, which are the same solutions found in Example 2(e) above.

Absolute Value Word Problem

Example 4

A company manufactures screws with a diameter of 1.4 mm. If they differ from this value by more than 0.3 mm, they are rejected.

a) Write an absolute value equation that represents this situation.

b) Solve the absolute value equation.

c) Find the range of acceptable values for the diameter of the screws.

Solution

Let x be an acceptable value for the diameter of the screws measured in mm.

a) Another way of describing this situation is to say "the distance from x to 1.4 is 0.3." This relates to an absolute value equation of $|x - 1.4| = 0.3$.

b) As it may be difficult to see on a graph where these two functions intersect, solving this equation algebraically is the preferable method.

$|x - 1.4| = 0.3$

Case 1: $x - 1.4 = 0.3$ and $x - 1.4 \geq 0$

$x - 1.4 = 0.3$ $x - 1.4 \geq 0$

$x = 1.7$ $x \geq 1.4$

Since $x = 1.7$ is in the interval $x \geq 1.4$, it is a solution.

Case 2: $-(x - 1.4) = 0.3$ and $x - 1.4 < 0$

$-x + 1.4 = 0.3$ $x < 1.4$

$-x = -1.1$

$x = 1.1$

Since $x = 1.1$ is in the interval $x < 1.4$, then it is a solution.

The solutions are $x = 1.7$ mm and $x = 1.1$ mm.

c) The acceptable range of measure for the diameter of the screws is between 1.1 mm and 1.7 mm. You really didn't need all this algebra to answer this question; however, the algebra gives you a process that will work for a much more complicated question.

Before you complete the following learning activity, go back through the lesson and add to your resource sheet. Include notes on how to solve absolute value equations algebraically and graphically. If you found any example difficult, you may find it helpful to include that example on your resource sheet as well.

Learning Activity 8.3

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. You take your best friend out for dinner at a restaurant and the bill comes to $67.32. If you tip the waiter 15%, how much change will you get back if you pay with a $100 bill?
2. Find the equation of a line with a slope of -1 that crosses through the point (5, 2).
3. Evaluate: 3^4
4. Evaluate: $\frac{2}{13} + \frac{6}{7}$
5. If Jacob can read an average of 98 pages a day, how many pages can he read in a week?
6. Express as a radical: $8^{\frac{5}{4}}$. Do not simplify.
7. Three angles in a parallelogram measure 30°, 93°, and 72°. What is the measure of the fourth angle?
8. The equation of a parabola is $y = x^2 + 2x - 15$. What are the x-intercepts?

continued

Learning Activity 8.3 (continued)

Part B: Solving Absolute Value Equations

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Solve the following equations using the definition of absolute value.
   a) $|2x - 3| = 7$
   b) $|3x - 8| = 2 - 4x$
   c) $|x^2 - 4x| = 5$
   d) $|x^2 + 2x| = x + 30$
2. Solve the absolute value equations in Question 1 graphically.
   a) $|2x - 3| = 7$
   b) $|3x - 8| = 2 - 4x$
   c) $|x^2 - 4x| = 4$
   d) $|x^2 + 2x| = x + 30$
3. A manufacturer rejects boxes of cookies when the mass of the box differs from the stated mass of 325 g by more than 4.6 g.
   a) Write an absolute value equation that represents this situation.
   b) Solve the absolute value equation.
   c) State the range of acceptable values for the mass of a box of cookies.
4. Arrange the numbers from 0 to 9 into the boxes so that each absolute value equation is mathematically correct. You can only use each digit once.
   a) $2\left|x + \square\right| = 6$

      $x = 0, -\square$

   b) $\left|2x + 7\right| = \square$

      $x = -\frac{\square}{2}$

   c) $\left|x^2 - \square x\right| = 1$

      $x = \square, 1 + \sqrt{2}, 1 - \sqrt{2}$

   d) $\left|x^2 - 2x - 16\right| = \square$

      $x = -4, -2, \square, 6$

   e) $\left|3x - \square\right| = 24$

      $x = -\square, 11$

continued

Learning Activity 8.3 (continued)

5. Armani and Jennifer are both trying to solve the absolute value equation $|x - 3| = 2x + 1$. Armani's and Jennifer's work is shown below.

   Armani's work:

   Case 1:

   $$|x - 3| = 2x + 1$$
   $$x + 3 = 2x + 1$$
   $$2 = x$$

   Case 2:

   $$-x + 3 = 2x + 1$$
   $$2 = 3x$$
   $$x = \frac{2}{3}$$

   Jennifer's work:

   Case 1:

   $$|x - 3| = 2x + 1$$
   $$x - 3 = 2x + 1$$
   $$-4 = x$$

   Case 2:

   $$-x + 3 = 2x + 1$$
   $$2 = 3x$$
   $$x = \frac{2}{3}$$

   a) Explain whose answer is incorrect.

   b) Explain how a person could have discovered that his or her work was incorrect.

6. Determine whether each statement is always true, sometimes true, or never true. Explain.

   a) The value of $|x + 5|$ is greater than zero.

   b) The value of $|x + a|$, when a is a real number is greater than or equal to zero.

   c) The value of $|5 + a| + a$ is greater than zero.

7. Write an absolute value equation with the indicated solutions or type of solution.

   a) -4

   b) No solution

   c) One solution

   d) Two solutions

Lesson Summary

In this lesson, you learned how to solve absolute value equations. Using what you learned in Lesson 2, you were able to graph the corresponding absolute value functions in order to solve an absolute value equation. You also used the definition of an absolute value in order to solve absolute value equations algebraically.

In the next lesson, you will switch your focus to reciprocal functions. You will learn how to graph reciprocal functions, which are similar to rational functions.

Lesson 4: Graphing Reciprocal Functions: Part 1

Lesson Focus

In this lesson, you will

- ❑ learn how to describe the graph and other properties of the reciprocal of a function
- ❑ learn how to sketch the reciprocal of a given function

Lesson Introduction

The reciprocal of an expression $\frac{a}{b}$ is $\frac{b}{a}$. Finding a reciprocal means interchanging the numerator and the denominator of a fraction. For example, the reciprocal of 2 is $\frac{1}{2}$. You can write 2 as $\frac{2}{1}$. If you interchange the numerator and the denominator of $\frac{2}{1}$, you arrive at an answer of $\frac{1}{2}$.

In this course, you will be learning about reciprocals of linear and quadratic functions. When you have a polynomial $f(x)$, the reciprocal of that polynomial is $\frac{1}{f(x)}$. This is the type of function you will be learning how to graph in the next two lessons.

As mentioned in the introduction to this module, reciprocal relationships occur in many places, including currency exchange rates and various physics applications some of which involve pitch and intensity.

Reciprocal Functions

Reciprocals of linear and quadratic functions are a special type of rational function. You learned about rational functions and equations in Modules 2 and 4.

Recall: A **rational function** is a function of the form $f(x) = \frac{p(x)}{q(x)}$, where $q(x) \neq 0$ and where $p(x)$ and $q(x)$ are polynomials and $q(x)$ contains the variable, x.

Some examples of rational functions are as follows:

$$f(x) = \frac{1}{x-2},\ g(x) = \frac{x+1}{(x+1)(x-2)},\ \text{or}\ h(x) = \frac{x}{x^2+1}$$

Examples of functions that are not rational are

$$t(x) = \frac{1}{4}x + 3,\ j(x) = \frac{1}{\sqrt{x}}$$

You can see that $t(x)$ does not have a variable in the denominator and $j(x)$ has an expression, $\sqrt{x}$, that is not a polynomial in its denominator.

When you graph polynomial functions, such as linear and quadratic functions, the domain is always the set of real numbers. However, rational function domains are often restricted to avoid division by zero.

In the examples of rational functions given above, in $f(x)$, you cannot have $x = 2$ or you will have division by zero. In $g(x)$, you cannot have $x = -1$ or $x = 2$ or you will also have division by zero. In $h(x)$, the domain is the set of all real numbers because there are no real values of x that will make the denominator zero.

In this course, you will only be dealing with reciprocals of linear and quadratic functions. In other words, you will be dealing with rational functions in the form $f(x) = \frac{1}{q(x)}$, where $q(x)$ is either a linear or a quadratic function. To summarize, a reciprocal is obtained by interchanging the numerator and the denominator. Therefore, the reciprocal of a number x is $\frac{1}{x}$, and the reciprocal of a polynomial function $f(x)$ is $\frac{1}{f(x)}$.

It may be helpful for you to include the definition of a reciprocal on your resource sheet.

When sketching the graph of a reciprocal function, you should give special attention to the shape of the graph near x-values that are not in its domain.

Example 1

Sketch the graph of $f(x) = \frac{1}{x}$.

Solution

Because the denominator is zero when $x = 0$, the domain of f is all real numbers as long as $x \neq 0$. To determine the behaviour of f near 0, evaluate $f(x)$ to the left and right of $x = 0$, as shown in the tables below.

As x approaches 0 from the left:

x	-1.0	-0.5	-0.1	-0.01	-0.001	as x approaches 0 from the left,
$f(x)$	-1.0	-2.0	-10	-100	-1000	$f(x)$ approaches $-\infty$

Thus, as x approaches 0 from the left, $f(x)$ becomes a bigger and bigger negative number.

This behaviour of $f(x) = \frac{1}{x}$, written in mathematical notation, is:

$$\text{as } x \longrightarrow 0^-, f(x) \longrightarrow -\infty$$

This is read as follows: "As x approaches 0 from the left, $f(x)$ approaches negative infinity."

Now consider as x approaches 0 from the right.

x	1.0	0.5	0.1	0.01	0.001	as x approaches 0 from the right,
$f(x)$	1.0	2.0	10	100	1000	$f(x)$ approaches $+\infty$

Thus, as x approaches 0 from the right, $f(x)$ increases indefinitely.

The behaviour of $f(x) = \frac{1}{x}$ is denoted by the following:

$$\text{as } x \longrightarrow 0^+, f(x) \longrightarrow +\infty$$

This is read as follows: "As x approaches 0 from the right, $f(x)$ approaches positive infinity."

This behaviour on a graph is represented by a vertical dashed line at $x = 0$.

We call the line $x = 0$ a vertical asymptote of the graph of $f(x)$.

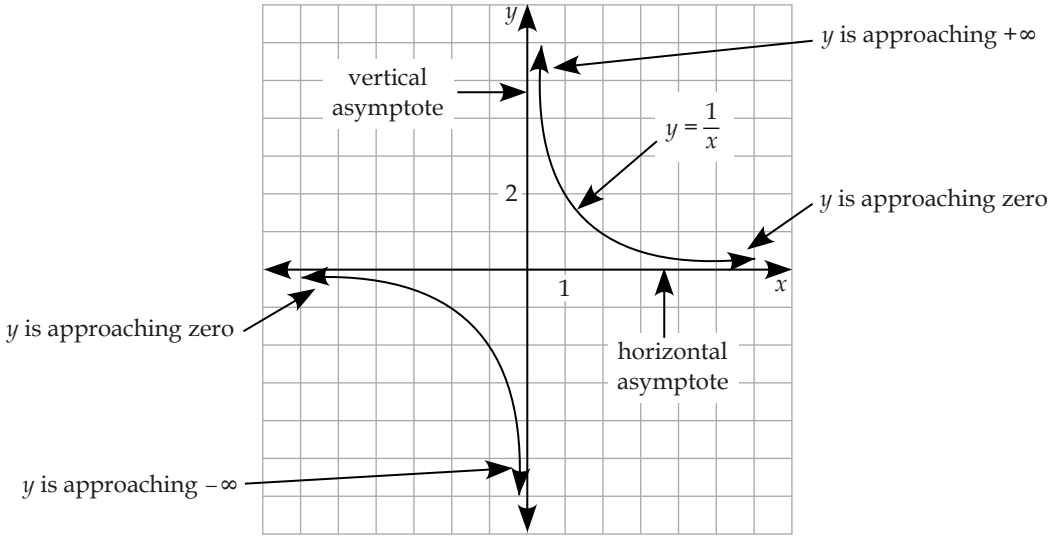
Now let's consider what happens to $f(x)$ as x goes off to infinity to the right or to negative infinity to the left.

x	1	2	10	100	1000	as x approaches $+\infty$,
$f(x)$	1	0.5	0.1	0.01	0.001	$f(x)$ approaches 0 through positive values

x	-1	-2	-10	-100	-1000	as x approaches $-\infty$,
$f(x)$	-1	-0.5	-0.1	-0.01	-0.001	$f(x)$ approaches 0 through negative values

This behaviour on a graph is represented by a horizontal dashed line at $y = 0$. We call the line $y = 0$ a horizontal asymptote of $f(x)$.

Note that the asymptotes are like magnets that draw the curve towards them. However, unlike a magnet, the curve never intersects or touches vertical asymptotes. The horizontal asymptotes describe the behaviour of the function as x goes off to infinity to the left or right.



Vertical Asymptote: The line $x = a$ is a vertical asymptote of the graph of f if $f(x) \to +\infty$ or $f(x) \to -\infty$ as $x \to a$, either from the right or from the left. In other words, if $f(x)$ approaches positive or negative infinity at $x = a$, the line $x = a$ is a vertical asymptote.

Horizontal Asymptote: The line $y = b$ is a horizontal asymptote of the graph of f if $f(x) \to b$ as $x \to +\infty$ or $x \to -\infty$. In other words, if $f(x)$ approaches positive or negative infinity at $y = b$, the line $y = b$ is a horizontal asymptote.

Include these definitions on your resource sheet.

Note that discussing the behaviour of a function near its horizontal asymptote implies you are considering very large positive x-values or very small negative x-values. You are not considering what happens to the function relatively nearer to the origin.

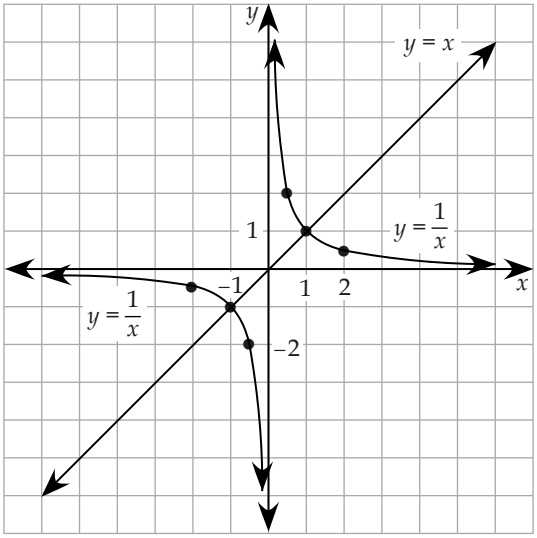
Comparing $y = f(x)$ to $y = \frac{1}{f(x)}$

Consider the table of values showing the function values for $y = x$ and $y = \frac{1}{x}$.

x	$y = x$	$y = \frac{1}{x}$
-2	-2	$-\frac{1}{2}$
-1	-1	-1
$-\frac{1}{2}$	$-\frac{1}{2}$	-2
0	0	undefined
$\frac{1}{2}$	$\frac{1}{2}$	2
1	1	1
2	2	$\frac{1}{2}$

The values for $y = \frac{1}{x}$ are the reciprocals of every y-value of the function $y = x$.

Consider the following coordinate plane, which contains the graphs of $y = x$ and $y = \frac{1}{x}$.



The graph of $y = \frac{1}{x}$ has two distinct pieces that are located on either side of the vertical asymptote, $x = 0$, and the horizontal asymptote, $y = 0$.

Characteristic	$y = x$	$y = \frac{1}{x}$
Domain	$\{x \mid x \in \Re\}$	$\{x \mid x \neq 0, x \in \Re\}$
Range	$\{y \mid y \in \Re\}$	$\{y \mid y \neq 0, x \in \Re\}$
Behaviour as $x \to +\infty$	y approaches $+\infty$	y approaches 0 from above the x-axis
Behaviour as $x \to -\infty$	y approaches $-\infty$	y approaches 0 from below the x-axis
Behaviour at $x = 0$	$y = 0$	y is undefined; vertical asymptote at $x = 0$
Invariant Points	$(-1, -1)$ and $(1, 1)$	

Invariant points are points that do not change in a transformation and thus are the same points on both the graph of $y = f(x)$ and the graph of $y = \frac{1}{f(x)}$.

Graphing Reciprocal Functions

The following guidelines pertain to the asymptotes of reciprocal functions. It may be helpful to include these guidelines on your resource sheet to help you remember how to find vertical and horizontal asymptotes. Guidelines for graphing reciprocal functions are included after Example 2.

Let $f(x) = \dfrac{1}{q(x)}$, where $q(x)$ is a polynomial.

1. The graph of f has a vertical asymptote at each of the zeros of $q(x)$, the denominator.
2. The graph of f has a horizontal asymptote at $y = 0$. (Since the numerator is 1, then $f(x)$ will never be zero.)
3. Any asymptotes that occur other than along the x- or y-axis should be shown on the graph as a dotted line and their equations written on the graph.

Example 2

Find the vertical and horizontal asymptotes of the following:

a) $f(x) = \dfrac{1}{(x+2)^2}$

b) $g(x) = \dfrac{1}{3x-5}$

Solutions

a) The vertical asymptote(s) occur at the zero of the denominator $q(x) = (x + 2)^2$. Since, $x = -2$ is the zero of the denominator, it is also the equation of the vertical asymptote.

As x gets larger (toward ∞ or $-\infty$), the value of $f(x)$ gets smaller but will never be zero. The x-axis is the horizontal asymptote and its equation is $y = 0$.

As you can see, the vertical asymptotes occur at the non-permissible values of the reciprocal function, or the points at which the denominator equals zero.

b) The vertical asymptote(s) occur at the zero of the denominator $q(x) = 3x - 5 = 0$ or $x = \dfrac{5}{3}$. Therefore, $x = \dfrac{5}{3}$ is the zero of the denominator and the equation of the vertical asymptote.

As x gets larger (toward ∞ or $-\infty$), the value of $q(x)$ gets smaller but will never be zero. The x-axis is the horizontal asymptote and the equation of the x-axis is $y = 0$.

Remember that many, but not all, reciprocal functions are discontinuous because of the nature of the asymptotes.

Reciprocal functions can be difficult to graph by merely plotting the reciprocal of several points. Identifying discontinuities, such as vertical and horizontal asymptotes, and the invariant points before you graph can help you find key features of the graphs of reciprocal functions.

Guidelines for Graphing Reciprocal Functions

Let $f(x) = \frac{1}{q(x)}$, where $q(x)$ is a polynomial in factored form.

1. Find and plot the *y-intercept* (if any) by evaluating $f(0)$. Remember the y-intercept is at a point on the y-axis, so the x-coordinate is zero at that point.
2. Sketch the corresponding *vertical asymptotes* by solving the equation of the denominator $q(x) = 0$ to find the zeros of the denominator. Remember the vertical asymptotes are drawn at the location of the zeros of the denominator to indicate the function does not exist there.
3. Sketch the *horizontal asymptote* at $y = 0$. This will always be true when graphing the reciprocal of a polynomial function.
4. Find the *invariant points* of the function. These points remain unchanged when a transformation (such as taking the reciprocal) is applied to a graph. As y-values of 1 and -1 remain unchanged when you take their reciprocal, solving for the denominator equal to 1 and -1 will provide you with the invariant points.
5. Plot the *invariant points* of the function where the y-coordinate is 1 or -1.
6. Use *smooth curves* to complete the graph on each side of the vertical asymptotes.

Example 3

Sketch the graph of $f(x) = \dfrac{1}{x-2}$.

Solution

y-intercept: Let $x = 0$

$$f(0) = \frac{1}{0-2} = -\frac{1}{2}$$

Vertical asymptote: Find the zero of the denominator by letting the expression in the denominator equal 0.

$$x - 2 = 0$$
$$x = 2$$

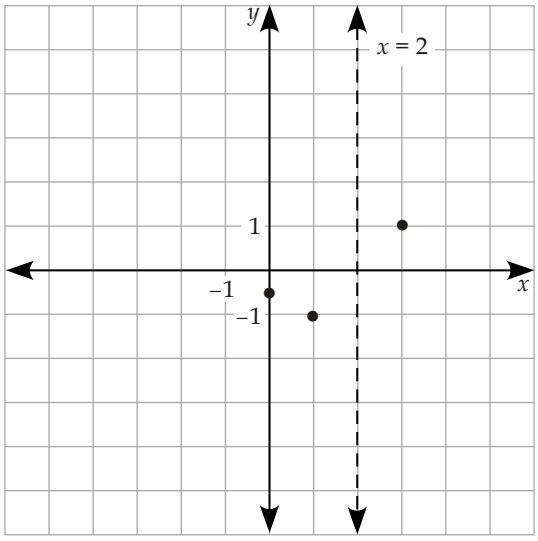
Horizontal asymptote: As x approaches $\pm\infty$, $f(x)$ approaches zero so the horizontal asymptote is the x-axis and its equation is $y = 0$.

Invariant points: To find the invariant points, set the denominator of the function equal to 1 and negative 1 and solve.

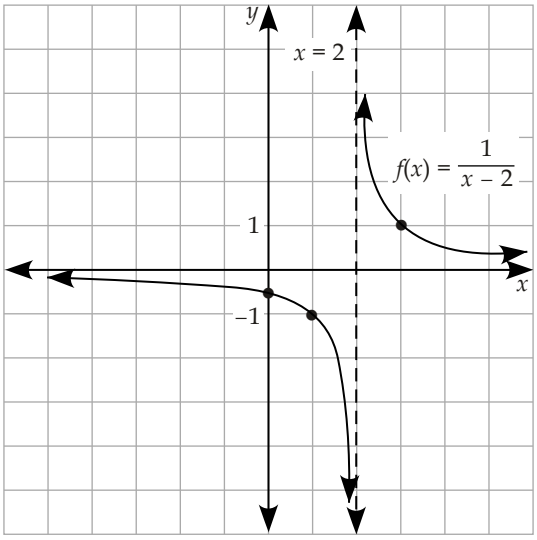
$y = 1$	$y = -1$
$x - 2 = 1$	$x - 2 = -1$
$x = 3$	$x = 1$

Therefore, the two invariant points are (3, 1) and (1, −1).

Draw in the curve: To draw the curve, first plot the two invariant points and the y-intercept. Then, draw in the horizontal and vertical asymptotes.

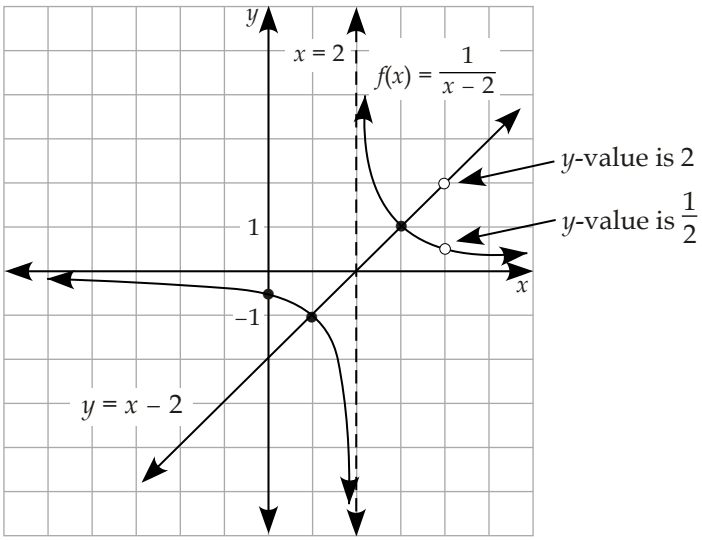


There are going to be two sections of the graph, as there are two different regions separated by the vertical asymptote. To draw the graph, draw a smooth curve through the points on each side as defined by the asymptotes with the ends of the curves approaching, but never touching, the vertical or horizontal asymptotes.



Asymptotes are drawn with a dashed line (unless the asymptote is covered by the x- or y-axis).

To check your work you might find it helpful to sketch both the functions $y = x - 2$ and $y = \frac{1}{x-2}$ on the same set of axes. Then you would be able to confirm the location of invariant points and you could check a few y-values for both functions at the same x-value to be sure the y-values are reciprocals of each other.



Example 4

Sketch the reciprocal of the horizontal line $y = 4$.

Solution

The reciprocal function is defined by $g(x) = \frac{1}{y}$. Then, $g(x) = \frac{1}{4}$.

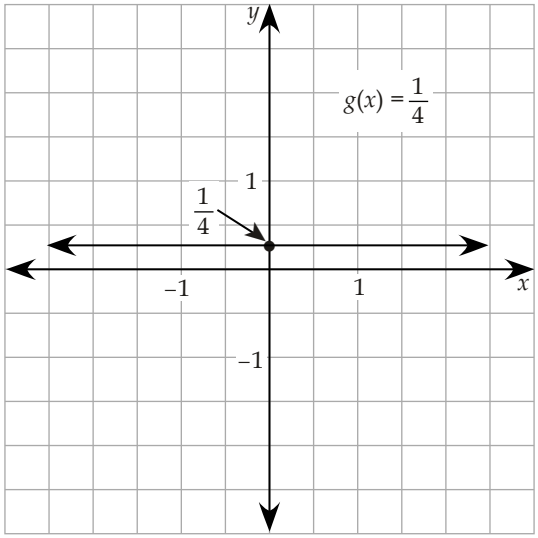
y-intercept: The y-intercept of this function is $\frac{1}{4}$.

Vertical asymptote: There is no vertical asymptote, as the denominator never equals zero.

Horizontal asymptote: As x approaches $\pm\infty$, $g(x)$ remains equal to $\frac{1}{4}$ so there is no horizontal asymptote.

Invariant points: There are no invariant points, as $g(x)$ never equals ± 1.

Draw the curve:



From this example, you can generalize that the reciprocal of a horizontal line is another horizontal line, unless you are considering the line $y = 0$ (where the reciprocal function does not exist). Also, from the definition of a rational function, there needs to be a variable in the denominator of the function. Thus, $g(x) = \frac{1}{4}$ is not considered to be a rational function.

The following examples show graphs of the reciprocal of a quadratic function having either 2, 1, or 0 x-intercepts.

Example 5

Sketch the graph of $f(x) = \frac{1}{x^2 - 4}$.

Solution

y-intercept: Find $f(0)$:

$$f(0) = \frac{1}{0^2 - 4} = \frac{1}{-4} = -\frac{1}{4}$$

Vertical asymptote: Let the denominator, $x^2 - 4$, equal 0, find the zeros, as the places where to draw dotted vertical asymptotes through those x-values.

$$x^2 - 4 = 0$$
$$(x - 2)(x + 2) = 0$$
$$x = 2 \text{ and } x = -2$$

Horizontal asymptote: The horizontal asymptote is the line $y = 0$.

Invariant points: Set the denominator, $x^2 - 4$, equal to 1 and -1.

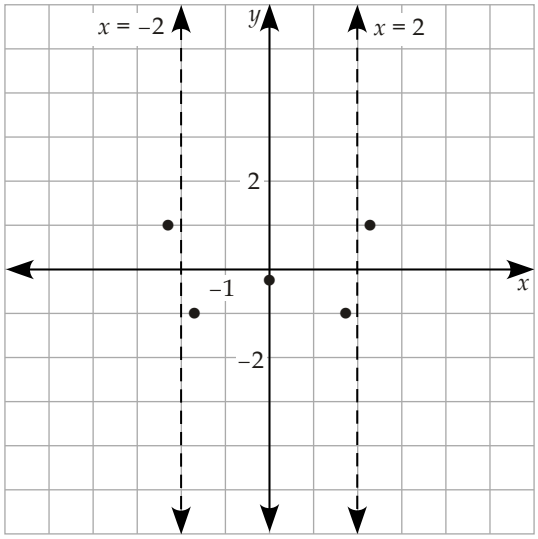
$$y = 1 \qquad\qquad y = -1$$
$$x^2 - 4 = 1 \qquad\qquad x^2 - 4 = -1$$
$$x^2 = 5 \qquad\qquad x^2 = 3$$
$$x = \pm\sqrt{5} \approx \pm 2.2 \qquad\qquad x = \pm\sqrt{3} \approx \pm 1.7$$

Therefore, the four invariant points are at (2.2, 1), (−2.2, 1), (1.7, −1), and (−1.7, −1).

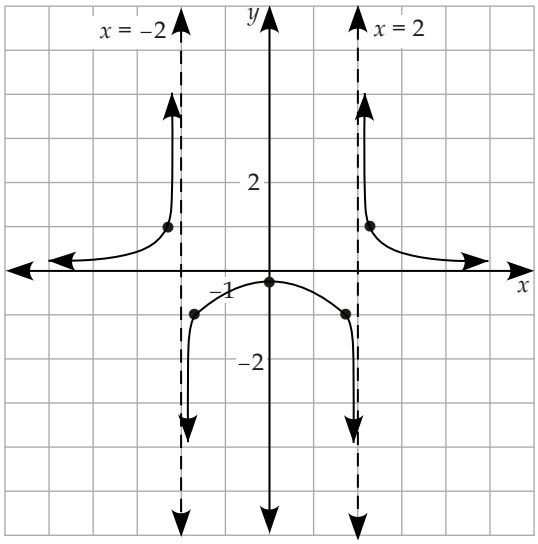
Draw in the curve: First draw the vertical asymptotes and plot the invariant points and the y-intercept. A key point for a quadratic is the vertex. Find the vertex of the original parabola and the corresponding point in the reciprocal function.

V(0, −4) for $y = x^2 - 4$

A key point on the function $f(x) = \frac{1}{x^2 - 4}$ is $\left(0, \frac{-1}{4}\right)$.



Whenever you have a quadratic function in the denominator, if the quadratic has two zeros, you will always have three sections to your graph separated by the two vertical asymptotes. The outer two sections will look similar to the reciprocal of a linear function. However, the inner section will always look like a bounded U-shape with the function approaching two different vertical asymptotes. The final graph is shown below.



Note: To check your work, you might try graphing $y = x^2 - 4$ first. Then determine the reciprocal function directly from the graph of the quadratic by choosing the reciprocal of the y-values at key points such as the zeros, the vertex, and the invariant points.

Characteristic	$y = x^2 - 4$	$y = \frac{1}{x^2 - 4}$
Domain	$\{x \mid x \in \Re\}$	$\{x \mid x \neq -2, 2, x \in \Re\}$
Range	$[-4, \infty)$	$\left\{y \mid y \leq -\frac{1}{4} \text{ or } y > 0, y \in \Re\right\}$
Behaviour as $x \to +\infty$	y approaches $+\infty$	y approaches 0 from above the x-axis
Behaviour as $x \to -\infty$	y approaches $+\infty$	y approaches 0 from above the x-axis
Behaviour at $x = 2$ and -2	$y = 0$	y is undefined; vertical asymptote at $x = 2$ and $x = -2$
Invariant Points	$(\sqrt{5}, 1), (-\sqrt{5}, 1), (\sqrt{3}, -1), (-\sqrt{3}, -1)$	

Look carefully at the range of the quadratic function $y = x^2 - 4$ and the range of its reciprocal. Compare the range to the graph of the functions and confirm that they are consistent.

Example 6

Sketch $f(x) = (x + 3)^2$ and $y = \frac{1}{f(x)}$ on the same set of axes.

Solution

First sketch the parabola $y = (x + 3)^2$ with vertex at $(-3, 0)$ and standard shape. Then sketch. $y = \frac{1}{(x + 3)^2}$.

y-intercept: Let $x = 0$.

$$y = \frac{1}{(0 + 3)^2} = \frac{1}{9}$$

Vertical asymptote: The denominator will equal zero when $x + 3 = 0$. The vertical asymptote occurs at $x = -3$.

Horizontal asymptote: The horizontal asymptote is $y = 0$.

Invariant points: Set the denominator of the function equal to ±1.

$(x + 3)^2 = 1$

$\sqrt{(x + 3)} = \pm\sqrt{1}$

$x + 3 = \pm 1$

$x + 3 = 1$ and $x + 3 = -1$

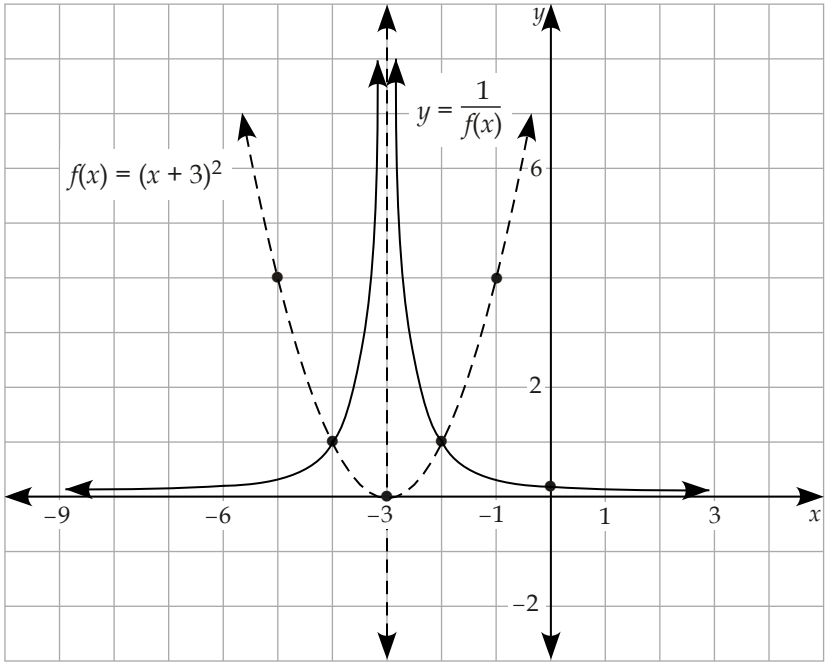
$x = -2, -4$

$(x + 3)^2 = -1$

This equation has no solution as any squared value will never be negative.

The two invariant points are (−2, 1) and (−4, 1).

Draw the curve:



As you can see, graphs of reciprocals of quadratic functions that have only one *x*-intercept (or zero) are different from the graphs of reciprocals of quadratic functions that have two *x*-intercepts. The reciprocal function is split into only two parts.

What happens if the quadratic function has no zeros? Consider the next example.

Example 7

Sketch $g(x) = \dfrac{1}{x^2 + 1}$ and state domain, range, intercepts, and equation(s) of asymptotes.

Solution

Since $x^2 + 1$ is never equal to zero, no vertical asymptotes exist. However, there is a horizontal asymptote at $y = 0$. As x approaches $\pm\infty$, $g(x)$ approaches but never equals zero.

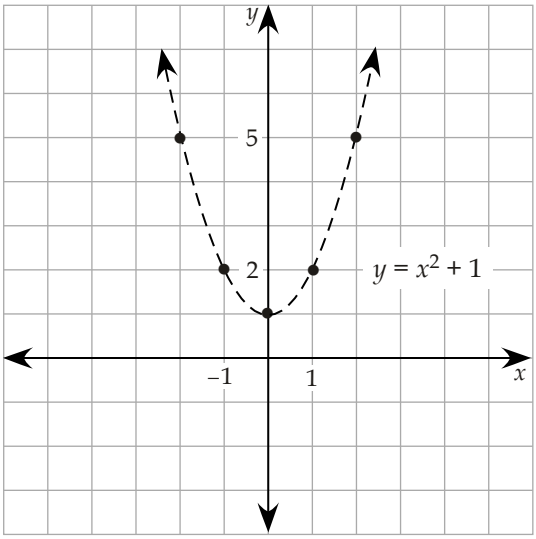
Step 1: Plot the y-intercept

$$g(0) = \frac{1}{0 + 1} = \frac{1}{1} = 1$$

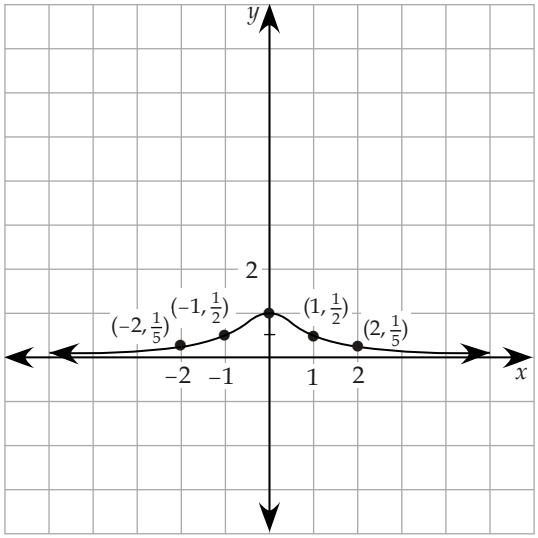
In this situation, the y-intercept is the same as the invariant point.

Step 2: Find additional points on each side of the vertex of the graph of $f(x) = x^2 + 1$. Then find the corresponding reciprocal y-values on the graph of $\dfrac{1}{f(x)}$.

x	$y = x^2 + 1$	$g(x) = \dfrac{1}{x^2 + 1}$
−2	5	$\frac{1}{5}$
−1	2	$\frac{1}{2}$
0	1	1
1	2	$\frac{1}{2}$
2	5	$\frac{1}{5}$



Step 3: Use smooth curves to complete the graph.



The domain, the range, the values of any intercepts that may exist, and the equation of the asymptotes are shown below.

Function	Domain	Range	y-intercept	x-intercepts	Equation of Asymptotes
$g(x) = \frac{1}{x^2 + 1}$	$\Re$	(0, 1]	1	none	$y = 0$

You need to be familiar with the reciprocals of linear and quadratic functions, including quadratic functions having no zeros, one zero, or two zeros. It may be helpful for you to include an example of each different type of reciprocal of quadratic functions on your resource sheet.

Learning Activity 8.4

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. One-third of students in a class of 108 students have never failed a test. How many students have never failed a test?
2. There is a 50% discount on decorations after Halloween. If it costs you $52.44 to buy Halloween decorations before Halloween, how much will it cost you to buy those same decorations the day after Halloween?
3. Find the x-intercepts of the following parabola: $y = x^2 + 6x + 8$
4. Evaluate: $\frac{1}{4} + \frac{2}{7}$
5. State the reciprocal of $\frac{(x+3)(x-4)}{(x+4)(x-1)}$.
6. Express as a power: $\sqrt[2]{12^7}$
7. In which quadrant is an angle of 145° in standard position located?
8. Find the slope of the line defined by the equation $-5x = 2y + 9$.

continued

Learning Activity 8.4 (continued)

Part B: Graphing Reciprocal Functions

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Write the reciprocal of each function.
   a) $y = x - 1$
   b) $y = -3x + 6$
   c) $y = x^2 + 4x - 2$
   d) $y = x^2 - 9$

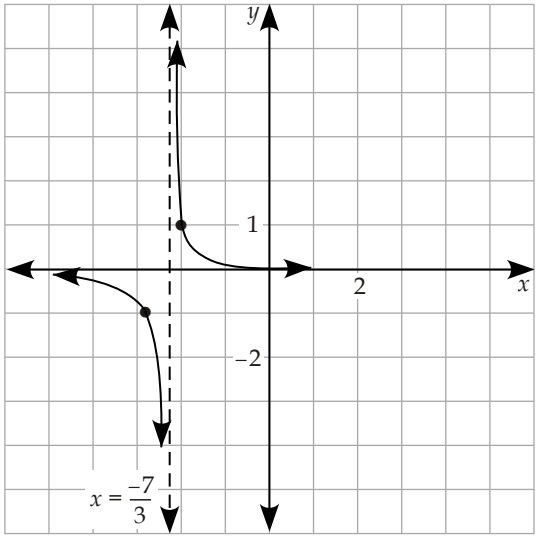
2. For each of the following, identify the (i) x-intercepts of $f(x)$, (ii) the reciprocal function, $y = \frac{1}{f(x)}$, (iii) the non-permissible values of the reciprocal function, and (iv) the equation(s) of the vertical asymptotes of the reciprocal function
   a) $f(x) = 2x + 5$
   b) $f(x) = x^2 - 9$
   c) $f(x) = x^2 + 4x + 3$

3. Match each function with the graph of its reciprocal. Hint: Consider the asymptotes.
   a) $y = x^2 + x - 6$
   b) $y = -2$
   c) $y = x - 5$
   d) $y = x^2 - 2x - 3$
   e) $y = x^2 - 6x + 9$
   f) $y = 3x + 7$
   g) $y = x^2 - 5x + 7$

continued

Learning Activity 8.4 (continued)

i)



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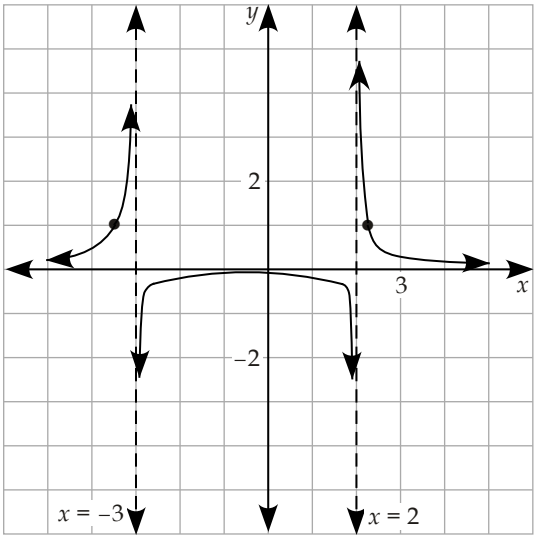

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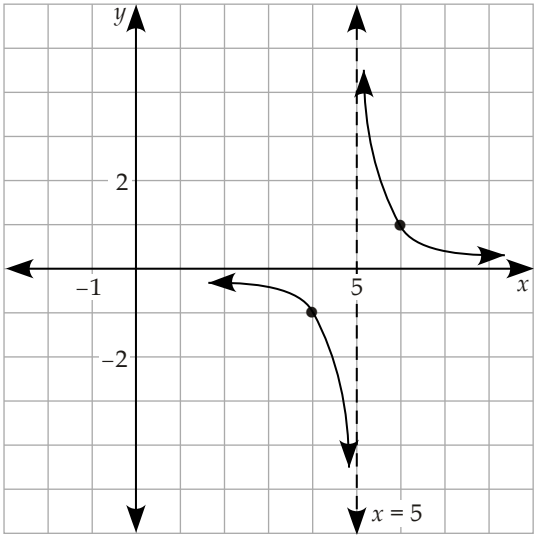

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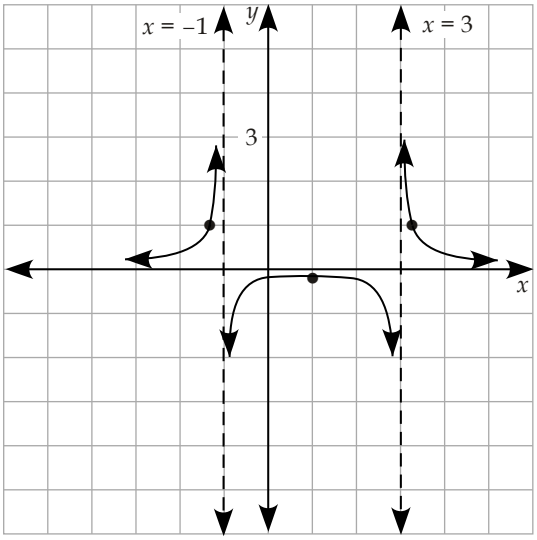

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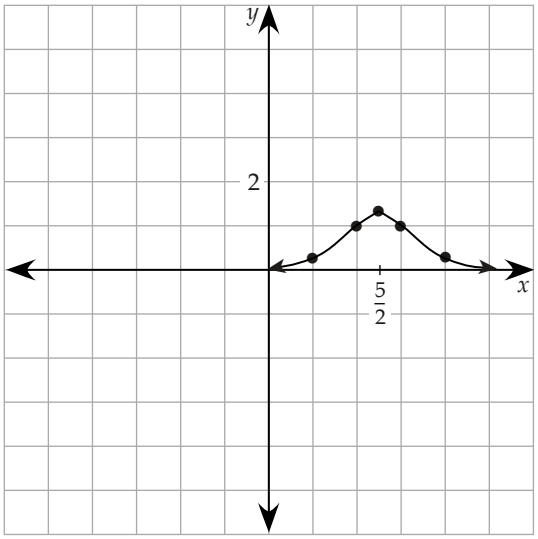

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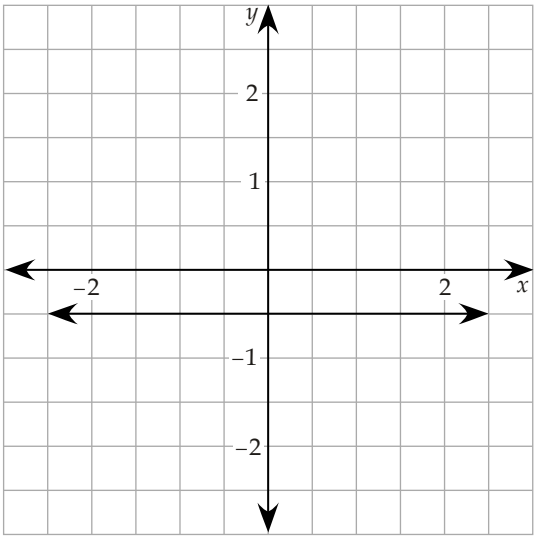
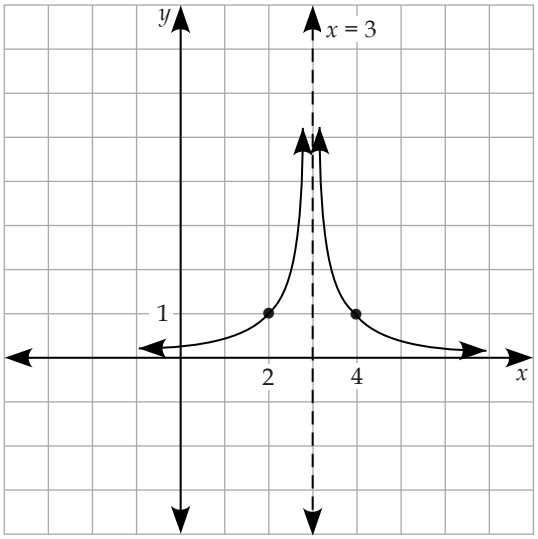


vii)



4. State the equation(s) of the vertical asymptote(s) for each function.

   a) $y = \dfrac{1}{5x - 10}$

   b) $y = \dfrac{1}{x - 4}$

   c) $y = \dfrac{1}{(x + 3)(x - 6)}$

   d) $y = \dfrac{1}{x^2 - x - 20}$

continued

Learning Activity 8.4 (continued)

5. Sketch the graph of each reciprocal function. State the y-intercept, the x-intercept(s), the invariant points, and the equation(s) of the horizontal and vertical asymptote(s), as well as the domain and range.

a) $f(x) = \dfrac{1}{x+3}$

b) $f(x) = \dfrac{1}{2x+8}$

c) $f(x) = \dfrac{1}{x^2+x-2}$

d) $f(x) = \dfrac{1}{x^2-2x-8}$

e) $f(x) = \dfrac{1}{x^2+8x+16}$

f) $f(x) = \dfrac{1}{x^2-2x+6}$

Lesson Summary

In this lesson, you learned what reciprocal functions were. You also learned how to graph these functions by considering their horizontal and vertical asymptotes, as well as their invariant points. In the next lesson, you will expand on this knowledge to learn how to graph a function $y = \dfrac{1}{f(x)}$ when you are given the graph of $y = f(x)$ and also how to find the original function $y = f(x)$ when you are given the graph of $y = \dfrac{1}{f(x)}$.

Before moving onto the next lesson, complete Assignment 8.2.

Assignment 8.2

Absolute Value Equations and Graphing Reciprocal Functions: Part 1

Total: 58 marks

1. Solve the following equations using the definition of absolute value.
   (4 marks each, for a total of 16 marks)

   a) $|3x - 2| = 4$

   b) $|2x + 1| = 13 - 4x$

continued

Assignment 8.2: Absolute Value Equations and Graphing Reciprocal Functions: Part 1

c) $|x^2 - x| = 42$

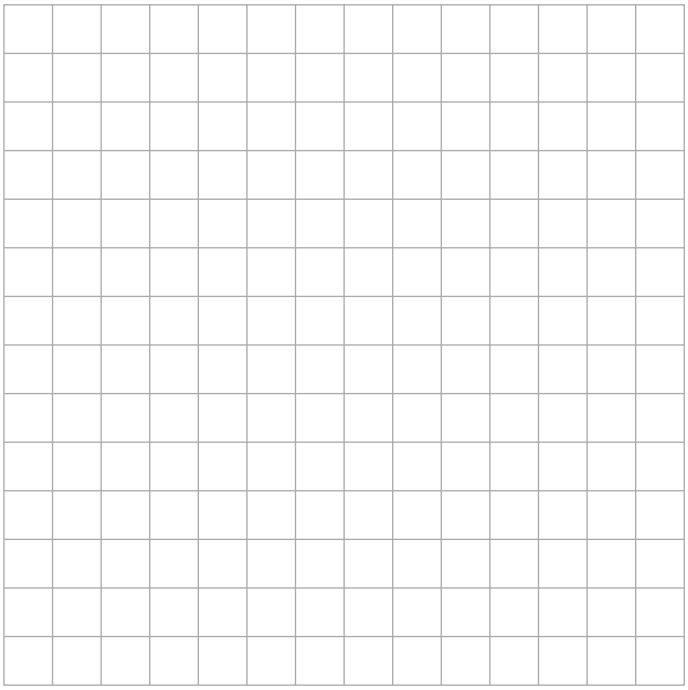
d) $|x^2 + 2x| = x + 6$

continued

Assignment 8.2: Absolute Value Equations and Graphing Reciprocal Functions: Part 1

2. Solve the absolute value equation in Question 1(d) graphically. Show your work. *(4 marks)*

   $| x^2 + 2x | = x + 6$



continued

Assignment 8.2: Absolute Value Equations and Graphing Reciprocal Functions: Part 1

3. A manufacturer rejects 450 g packages of frozen vegetables when the mass of the package differs from the stated mass by more than 15 g.
   a) Write an absolute value equation that represents this situation. *(1 mark)*

   b) Solve the absolute value equation. *(4 marks)*

   c) What are the greatest and lowest acceptable weights for the packages? *(1 mark)*

continued

Assignment 8.2: Absolute Value Equations and Graphing Reciprocal Functions: Part 1

4. For each of the following, write the (i) x-intercepts of $f(x)$, (ii) the reciprocal function of $f(x)$, (iii) the non-permissible values of the reciprocal function, and (iv) the equation(s) of the vertical asymptotes of the reciprocal function.
(4 marks each, for a total of 12 marks)

   a) $f(x) = 3x + 2$

   b) $f(x) = x^2 - 16$

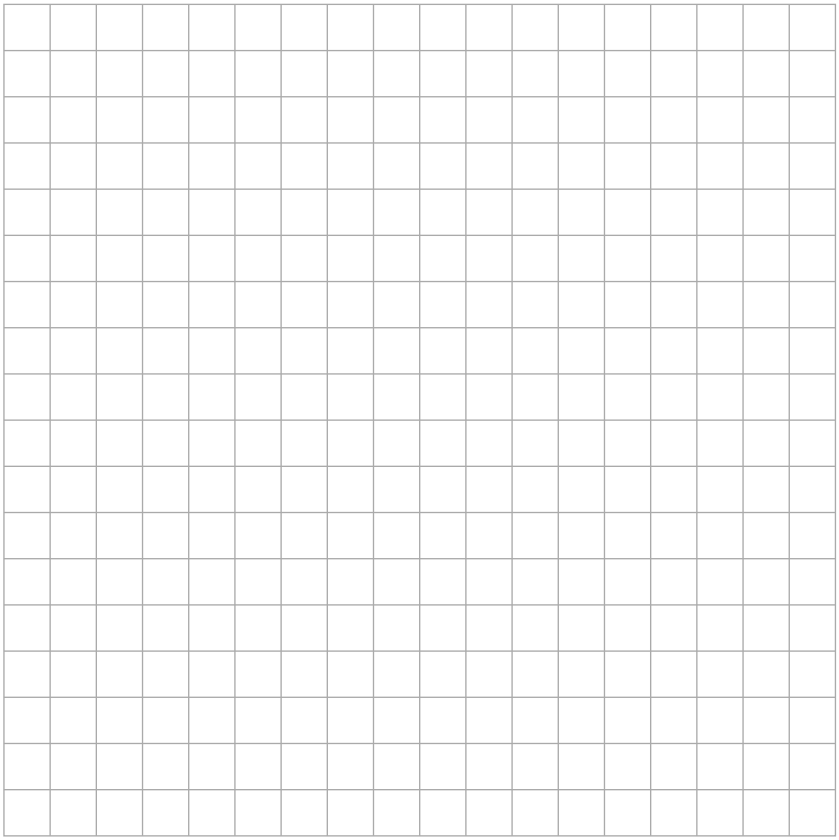
   c) $f(x) = x^2 + 3x - 28$

continued

Assignment 8.2: Absolute Value Equations and Graphing Reciprocal Functions: Part 1

5. Sketch the graph of each reciprocal function. State the y-intercept, the x-intercept(s), the invariant points, and the equation(s) of the horizontal and vertical asymptote(s), as well as the domain and range. *(5 marks each, for a total of 20 marks)*

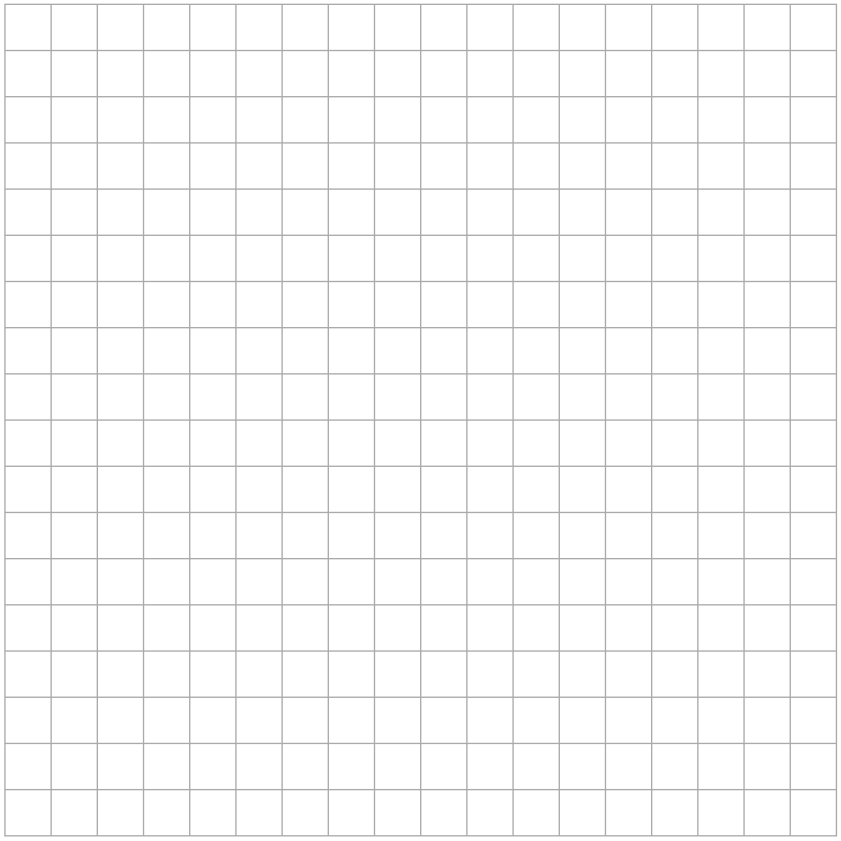
   a) $f(x) = \dfrac{1}{x-8}$



continued

Assignment 8.2: Absolute Value Equations and Graphing Reciprocal Functions: Part 1

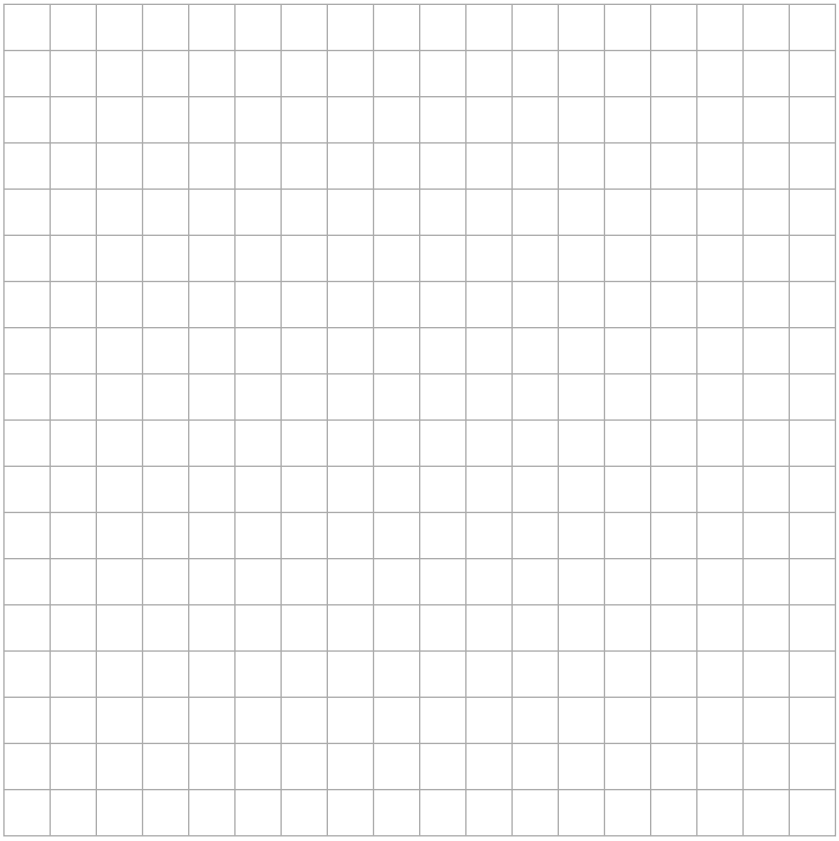
b) $f(x) = \dfrac{1}{x^2 - 3x - 18}$



continued

Assignment 8.2: Absolute Value Equations and Graphing Reciprocal Functions: Part 1

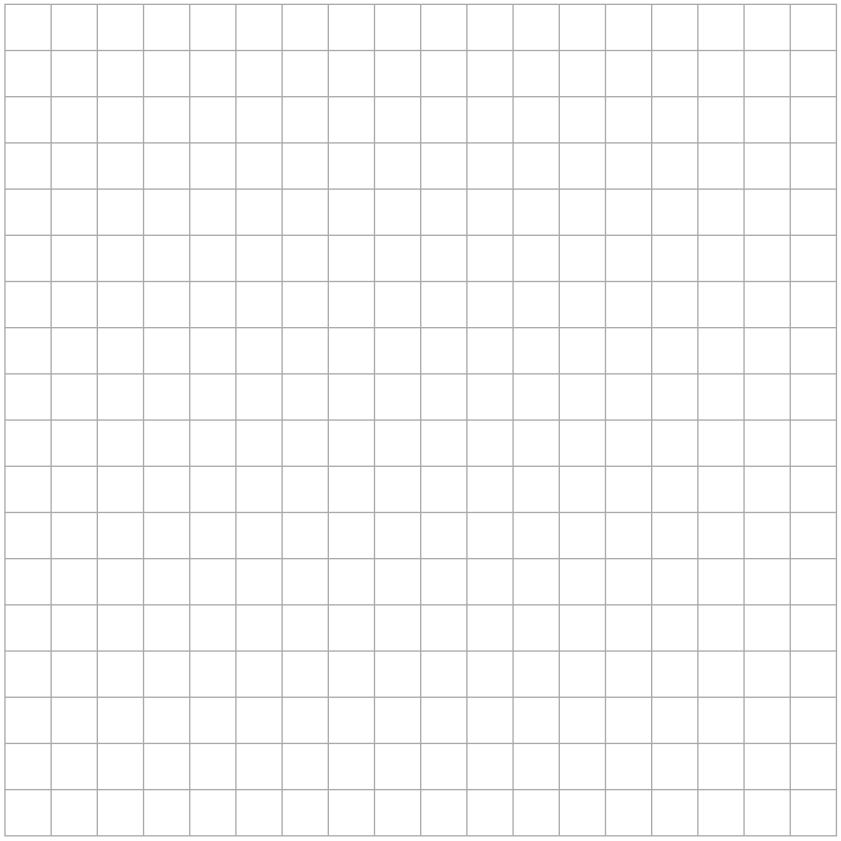
c) $f(x) = \dfrac{1}{x^2 - 4x + 4}$



continued

Assignment 8.2: Absolute Value Equations and Graphing Reciprocal Functions: Part 1

d) $f(x) = \dfrac{1}{x^2 - 2x + 2}$



Notes

Lesson 5: Graphing Reciprocal Functions: Part 2

Lesson Focus

In this lesson, you will

- ❑ learn how to sketch the reciprocal of a function when you are given the graph of the polynomial function
- ❑ learn how to sketch a polynomial function when you are given the graph of its reciprocal function

Lesson Introduction

In the last lesson, you learned how to graph reciprocal functions by discovering their asymptotes and invariant points. In this lesson, you will learn how to graph the reciprocal of a function when you are given the graph of another function. You will also learn how to graph the original function when you are given the graph of the reciprocal of a function. These two different processes will increase your understanding of the relationships between functions and their reciprocals.

Reciprocal Functions

Using what you learned in Lesson 4, you are now able to graph reciprocal functions in various ways. If you have any questions about the content in Lesson 4, make sure you go back and review the material or ask your tutor/marker for an additional explanation.

Graphing $y = \frac{1}{f(x)}$ When You Are Given the Graph of $y = f(x)$

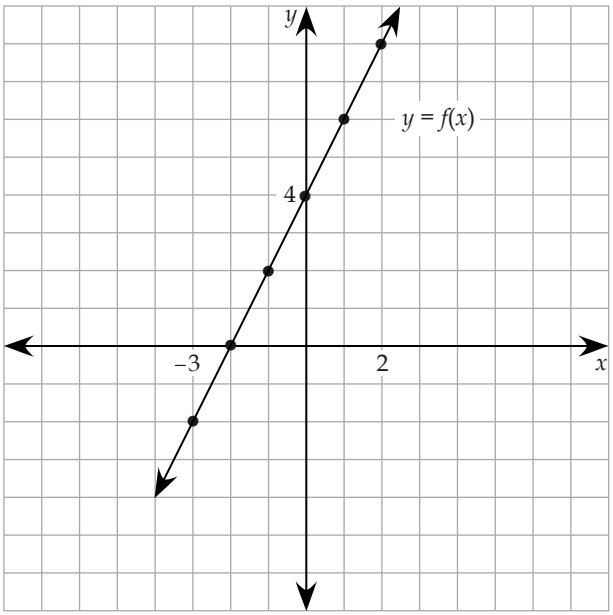
Given the graph of $y = f(x)$, it is possible to sketch the graph of $y = \frac{1}{f(x)}$ in two ways.

Method 1: Algebra Method

You can find the equation of the graph of $f(x)$ and then use the steps explained in Lesson 4 to graph the reciprocal function. However, this can be difficult to do with quadratic functions, as their equations are not always easy to find from points on their graphs.

Example 1

Given the graph of $f(x)$ shown in the graph below, sketch the graph of $y = \frac{1}{f(x)}$.



Solution

To graph the reciprocal of this function using Method 1, you first need to find the equation of this function. The easiest way to do this is to find the slope and the y-intercept.

Slope: 2

y-intercept: 4

Using the slope-intercept equation of a line:

$y = mx + b$

$y = 2x + 4$

Now that you have the equation of the line, you can now graph the reciprocal of this function, $f(x) = \frac{1}{2x + 4}$, using the method you learned in Lesson 4.

y-intercept: Find $f(0)$:

$$f(x) = \frac{1}{2x + 4}$$

$$f(0) = \frac{1}{2(0) + 4} = \frac{1}{4}$$

Vertical asymptote: Let the denominator equal 0, find the zero, and draw the dotted vertical asymptote through this zero.

$$2x + 4 = 0$$
$$2x = -4$$
$$x = -2$$

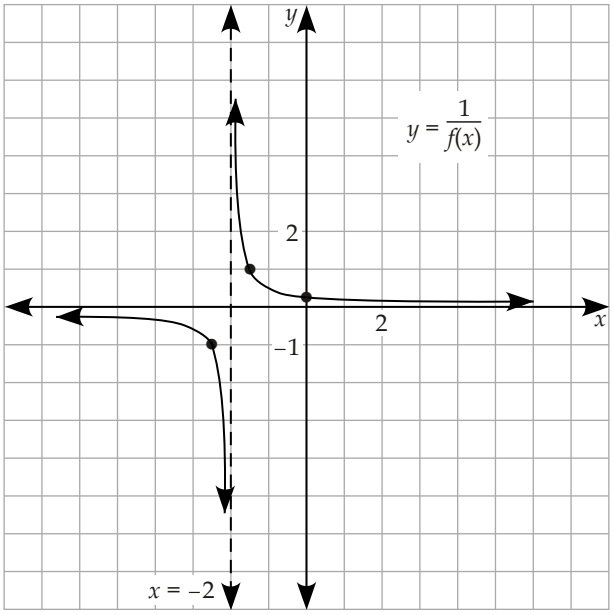
Horizontal asymptote: The horizontal asymptote is the line $y = 0$.

Invariant points: Set the denominator equal to 1 and −1.

$$y = 1 \qquad\qquad y = -1$$
$$2x + 4 = 1 \qquad\qquad 2x + 4 = -1$$
$$2x = -3 \qquad\qquad 2x = -5$$
$$x = -\frac{3}{2} \qquad\qquad x = -\frac{5}{2}$$

Therefore, the two invariant points are $\left(-\frac{3}{2}, 1\right)$ and $\left(-\frac{5}{2}, -1\right)$.

Draw the curve: First, plot the invariant points and the *y*-intercept. Then, draw the asymptote and the curves.

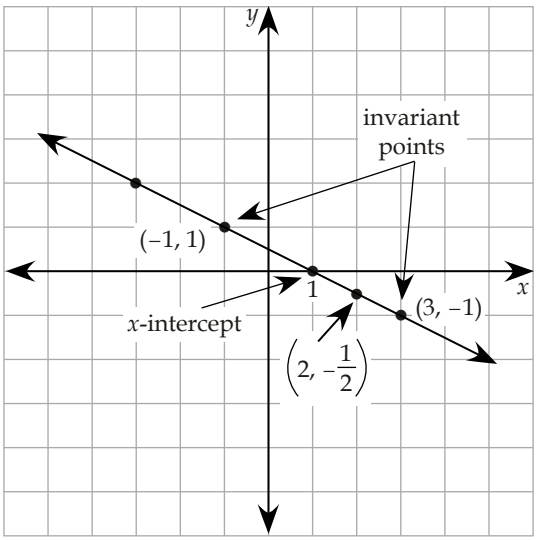


Method 2: Graphing Method

You can find the y-intercept, the equations of the asymptotes, as well as the invariant points from the graph of $y = f(x)$. You can then sketch the graph of $y = \frac{1}{f(x)}$ by using that information.

Example 2

Given the graph of $y = f(x)$ shown in the graph below, sketch the graph of $y = \frac{1}{f(x)}$.



Solution

The y-intercept of $f(x)$ looks like it is $\frac{1}{2}$ so the y-intercept of the reciprocal function is 2.

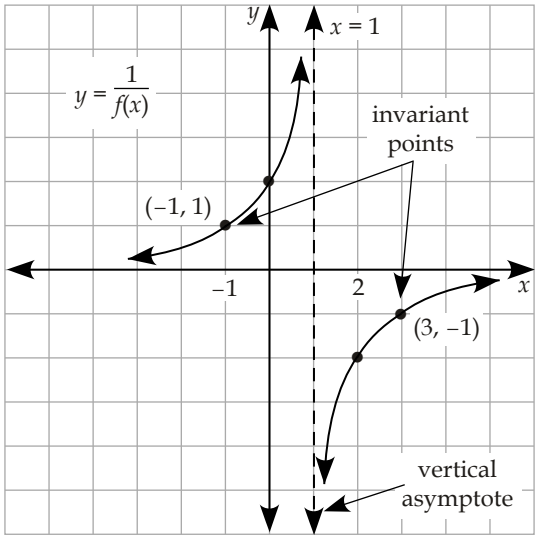
The vertical asymptote(s) occur at the x-intercepts of $f(x)$.

This line crosses the x-axis at the point (1, 0). Therefore, in the reciprocal function, this point is $\left(1, \frac{1}{0}\right)$ and the y-value is not defined here. Since the x-intercept of $f(x)$ is 1, the equation of the vertical asymptote of the reciprocal function would be $x = 1$.

The invariant points occur when the y-coordinate is 1 or −1. These points are (−1, 1) and (3, −1).

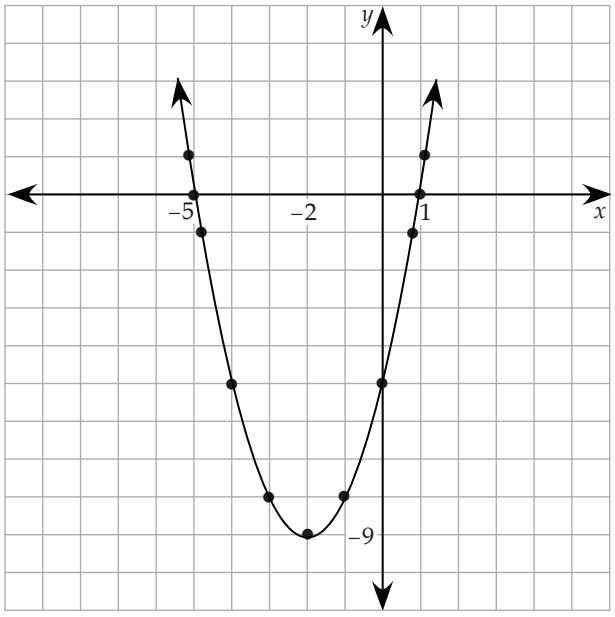
The horizontal asymptote is $y = 0$.

Using this information, you can now graph the reciprocal function.



Example 3

Given the graph of $f(x)$ shown in the graph below, sketch the graph of $y = \frac{1}{f(x)}$.



Solution

The y-intercept of $f(x)$ is -5 so the y-intercept of the reciprocal is $-\frac{1}{5}$.

$f(x)$ crosses the x-axis at -5 and 1 where the y-values are 0. Therefore, the equations of the vertical asymptotes of the reciprocal function would be $x = -5$ and $x = 1$.

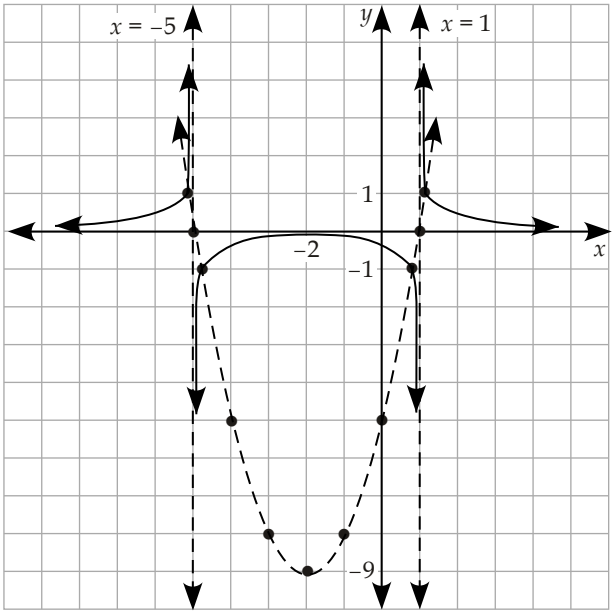
The invariant points occur when the y-coordinate is 1 or -1. These points are roughly (1.2, 1), (−5.2, 1), (0.8, −1), and (−4.8, −1).

The horizontal asymptote is $y = 0$.

The vertex of $f(x)$ is a key point of the parabola and it is at $(-2, -9)$. Its image for the reciprocal function is also a key point and it is at $\left(-2, -\frac{1}{9}\right)$.

Notice that the x-value does not change. The y-value or $f(x)$ value becomes the reciprocal.

Using the y-intercept, the asymptotes, the invariant points, and the key point, you can graph the reciprocal function.



Compare values on the function and its reciprocal. Notice the following:

1. The reciprocal of a positive is still a positive value, and the reciprocal of a negative is a negative.
2. The reciprocal of a large number is a very small number and vice versa.
3. The reciprocal of 0 is undefined. The x-intercepts of the original function give the position of the vertical asymptotes of the reciprocal function.
4. The invariant points occur where the graphs intersect the horizontal lines $y = 1$ and $y = -1$.

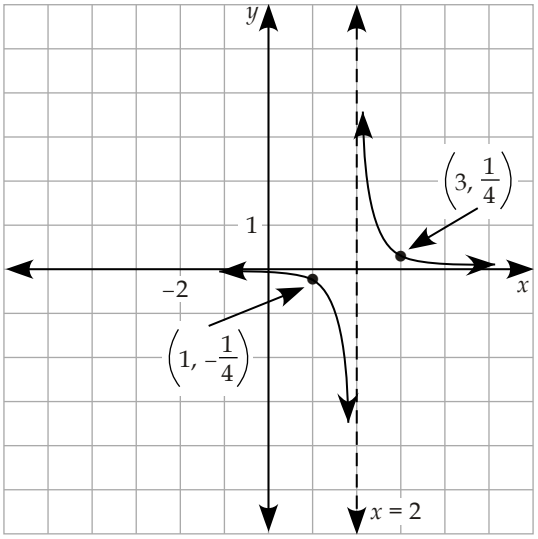
Graphing $y = f(x)$ When You Are Given the Graph of $y = \frac{1}{f(x)}$

When you are given the graph of a reciprocal function, it is possible to "reverse" the reciprocal and sketch the graph of the original function. There are ways to do this for both the reciprocals of linear functions and the reciprocals of quadratic functions.

In the graph of the reciprocal of a function, you can easily see the equation(s) of the vertical asymptote(s). These equations will give you the value(s) of the x-intercept(s) of the original function. Once you find the x-intercepts, you then need to find additional points on the reciprocal graph. The easiest points to use would be invariant points with a y-coordinate of 1 or –1, because these points do not change position in a reciprocal transformation. However, any point on the graph can be used, but you will need to take the reciprocal of the y-value of your point to find the point on the graph of the original function.

Example 4

Given the graph of $y = \frac{1}{f(x)}$ shown on the graph below, sketch the graph of $y = f(x)$.



Solution

The equation of the horizontal asymptote is $y = 0$.

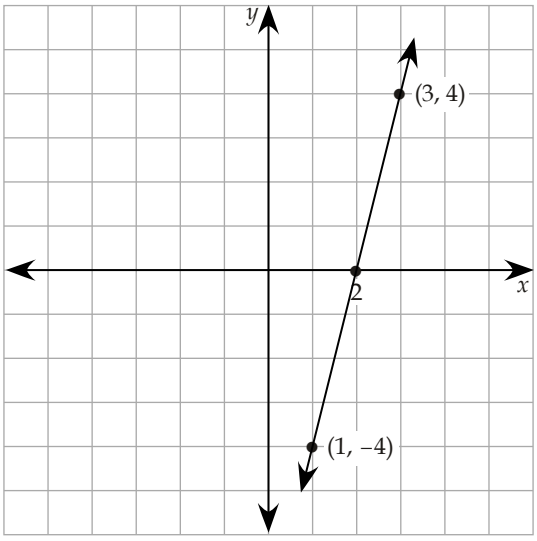
The equation of the vertical asymptote is $x = 2$. From this equation, you know that the graph of $f(x)$ crosses the x-axis, or has an x-intercept at 2. Therefore, one point on the graph of $f(x)$ is (2, 0).

You also have one point, $\left(3, \frac{1}{4}\right)$, on the reciprocal graph. Using this point, you can find the corresponding point on the graph of $f(x)$ by taking the reciprocal of the y-coordinate.

The reciprocal of $\frac{1}{4} = 4$. Therefore, a point on $f(x)$ is (3, 4).

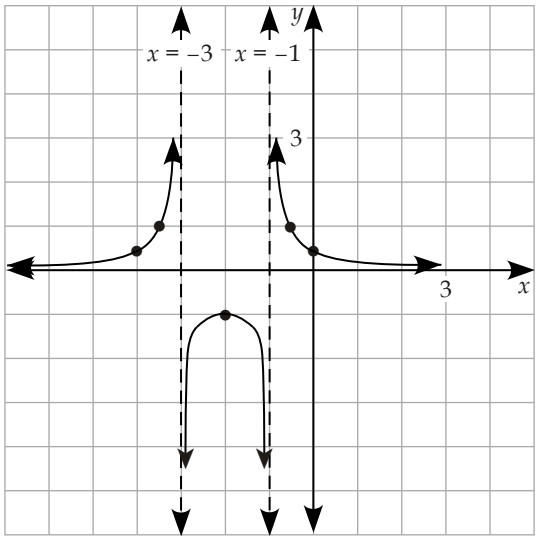
Similarly, the point $\left(1, -\frac{1}{4}\right)$ for the reciprocal function is the point $(1, -4)$ for $f(x)$.

Now that you have the coordinates of the three points, you can sketch the original function.



Example 5

Given the graph of $y = \frac{1}{f(x)}$ shown in the graph below, sketch the graph of $y = f(x)$.



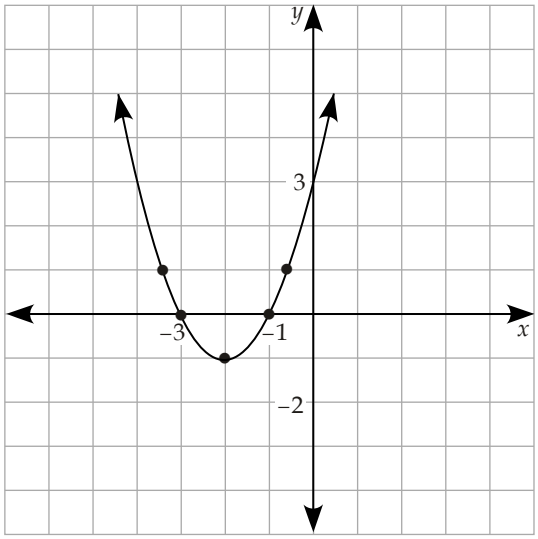
Solution

From the graph, you can see that the two vertical asymptotes occur at $x = -1$ and $x = -3$. Therefore, the graph of $f(x)$ will have zeros at $x = -1$ and $x = -3$. In other words, the function $f(x)$ will be a quadratic of the form, $a(x + 1)(x + 3)$, or $y = a(x + 1)(x + 3)$, where a is some value yet to be determined.

The point $(-2, -1)$ is also given on the reciprocal graph. This corresponds to the same point, $(-2, -1)$, on the graph of $f(x)$, as this is an invariant point. It has a y-coordinate of negative one. The point $(-2, -1)$ will be the vertex of $f(x)$, since $(-2, -1)$ is the maximum value in the reciprocal function in the interval between the asymptotes.

The other invariant points occur where $y = 1$. These coordinates are approximately at $(-3.4, 1)$ and $(-0.6, 1)$.

Using the zeros as well as the invariant points, including the vertex of the quadratic function, you can now graph $f(x)$.



Learning Activity 8.5

Complete the following, and check your answers in the learning activity keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Eric has a big crush on a girl sitting 6 desks away from him. If each desk is 80 cm wide and there are no spaces between them, how far away (in metres) is Eric sitting from his crush?
2. Simplify: $(3^2)^3$
3. What is the slope of the line defined by the equation $3y + 2x - 7 = 0$?
4. If 36% of 1500 is 540, what is 18% of 1500?
5. State the reciprocal of $\dfrac{x^2 + 3x - 4}{x^2 - 2x + 1}$.
6. Express as a power: $\sqrt[3]{16^3}$
7. (45, −72) is on a line with a slope of $\dfrac{20}{17}$. Find another point on this line.
8. Simplify the following fraction: $\dfrac{252}{462}$

continued

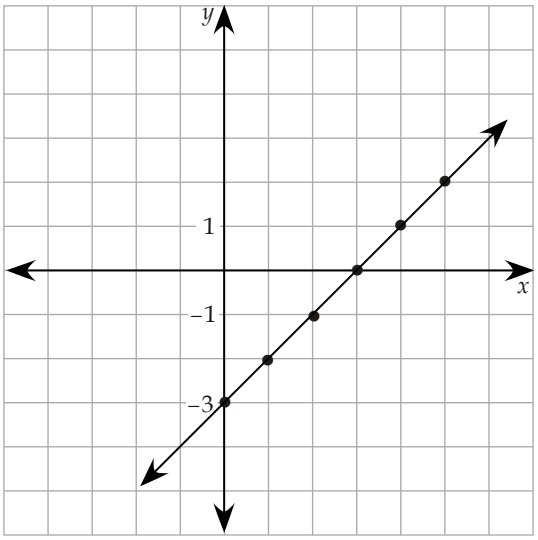
Learning Activity 8.5 (continued)

Part B: Reciprocal Functions

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Given the graph of $y = f(x)$, sketch the graph of $y = \frac{1}{f(x)}$. Describe the method you used.

a)



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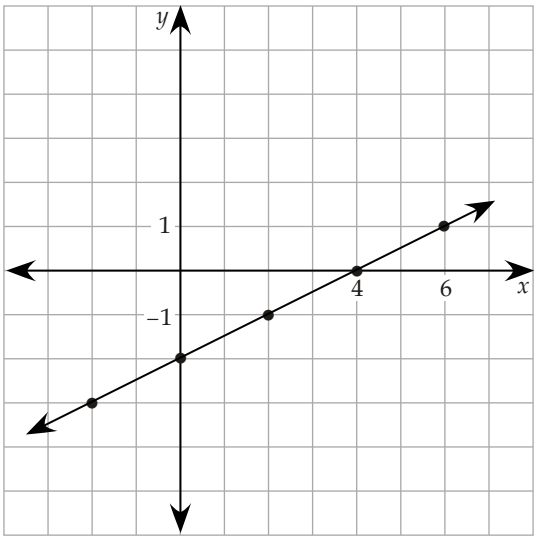

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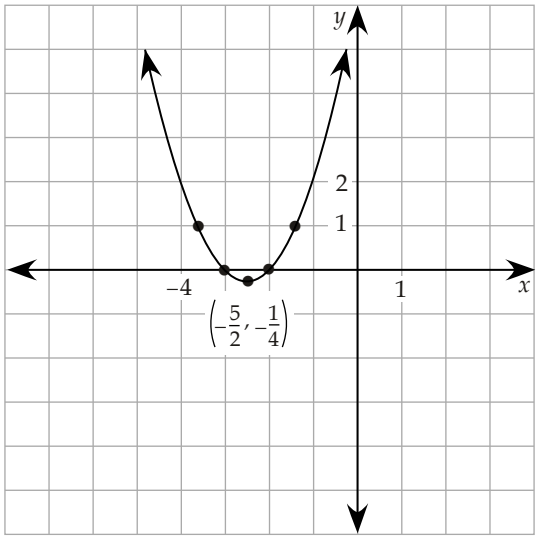
Learning Activity 8.5 (continued)

c)



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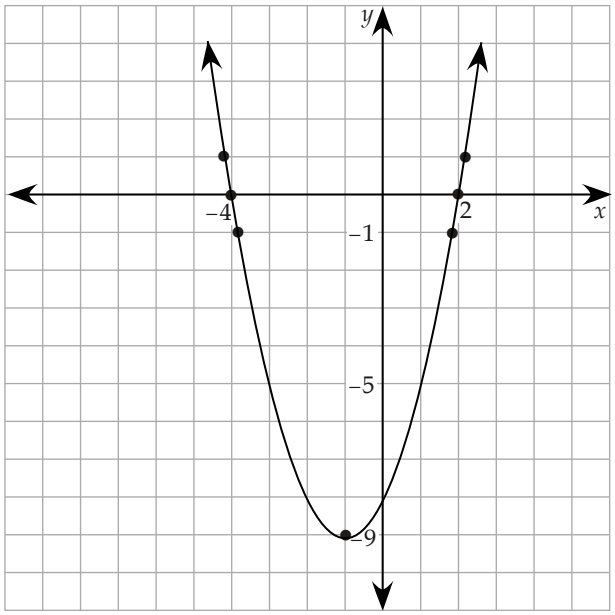

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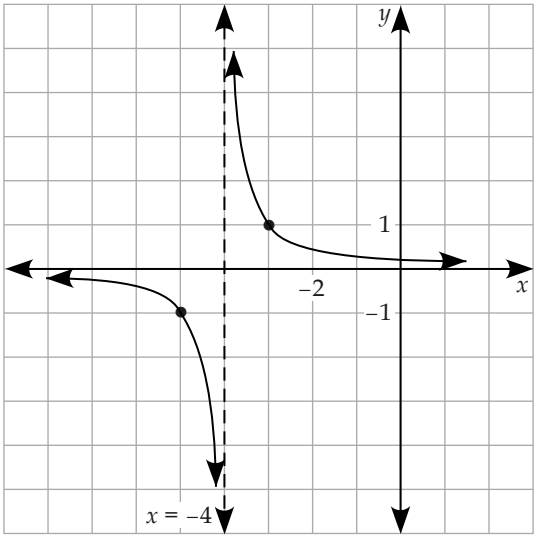
Learning Activity 8.5 (continued)

2. Given the graph of $y = \dfrac{1}{f(x)}$, sketch the graph of $y = f(x)$. Describe the method you used.

a)



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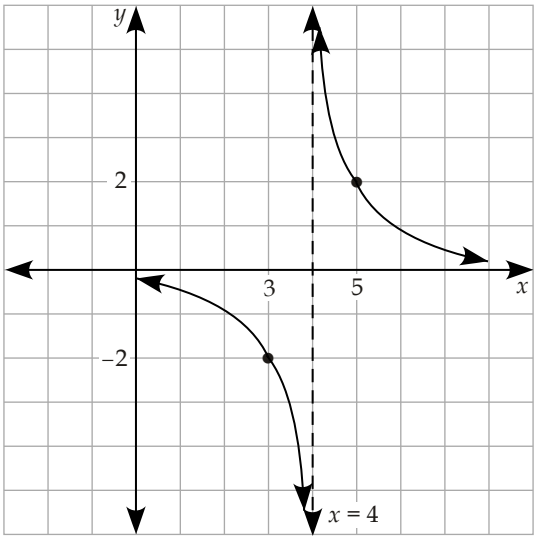

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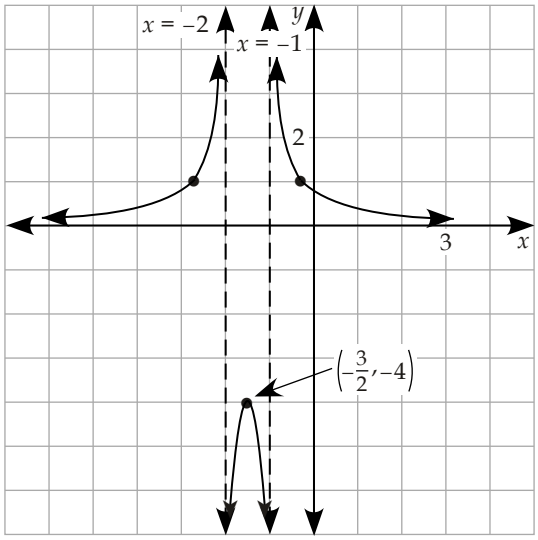
Learning Activity 8.5 (continued)

c)



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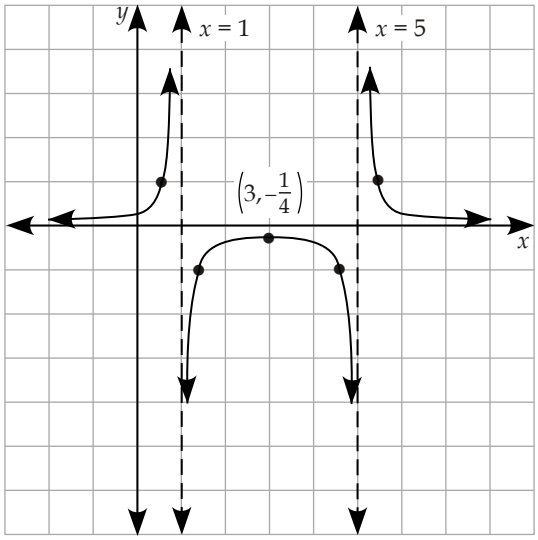

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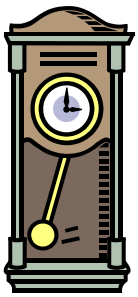
Learning Activity 8.5 (continued)

3. The movement of a pendulum in a grandfather clock can be represented using a reciprocal formula. The frequency, f, is equal to the reciprocal of the period, T. The period is measured in seconds, and measures the time for one cycle, or, in other words, how long it takes the pendulum to swing back and forth once. The frequency is the number of cycles per second.



   a) Write the formula for the frequency of a pendulum.

   b) Sketch the graph of the function in (a).

   c) Using the graph as well as the formula in (a), determine the period of a pendulum when the frequency is 2.1 cycles per second.

Lesson Summary

In this lesson, you learned how to graph reciprocal functions when you were given the graph of a function. You also learned how to graph the original function when you were given the graph of the reciprocal of a function. Being able to graph functions and their reciprocals helps you understand how functions and their reciprocals are related. Make sure you complete the following assignment as it will prepare you for your final exam. Remember, Assignment 8.3 is the last assignment in this course!

Notes

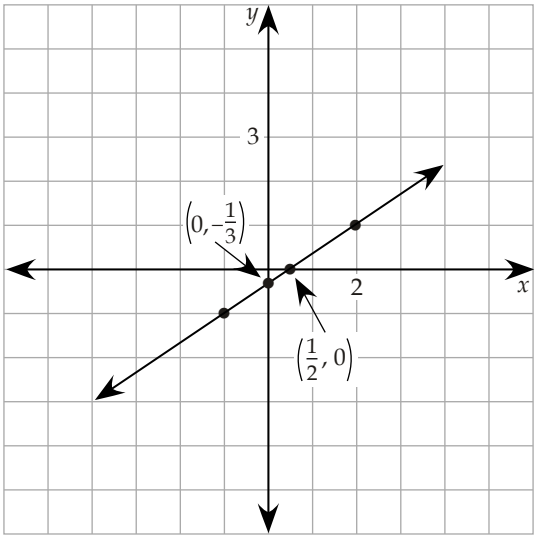
Assignment 8.3

Reciprocal Functions: Part 2

Total: 20 marks

1. Given the graph of $y = f(x)$, sketch the graph of $y = \dfrac{1}{f(x)}$ on the same grid. Describe the method you used. *(5 marks each, for a total of 10 marks)*

   a)



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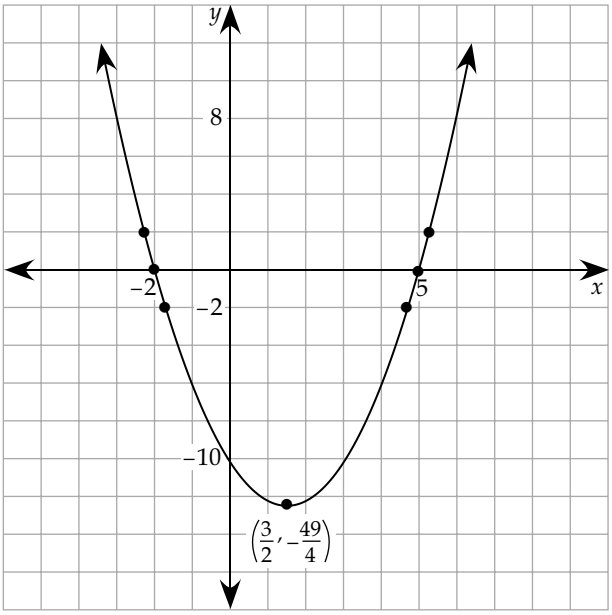
b)



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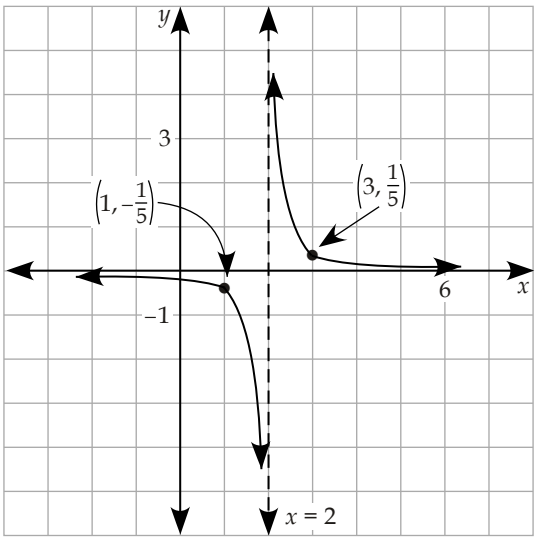
Assignment 8.3: Reciprocal Functions: Part 2 (continued)

2. Given the graph of $y = \frac{1}{f(x)}$, sketch the graph of $y = f(x)$ on the same grid. Describe the method you used. *(5 marks each, for a total of 10 marks)*

a)



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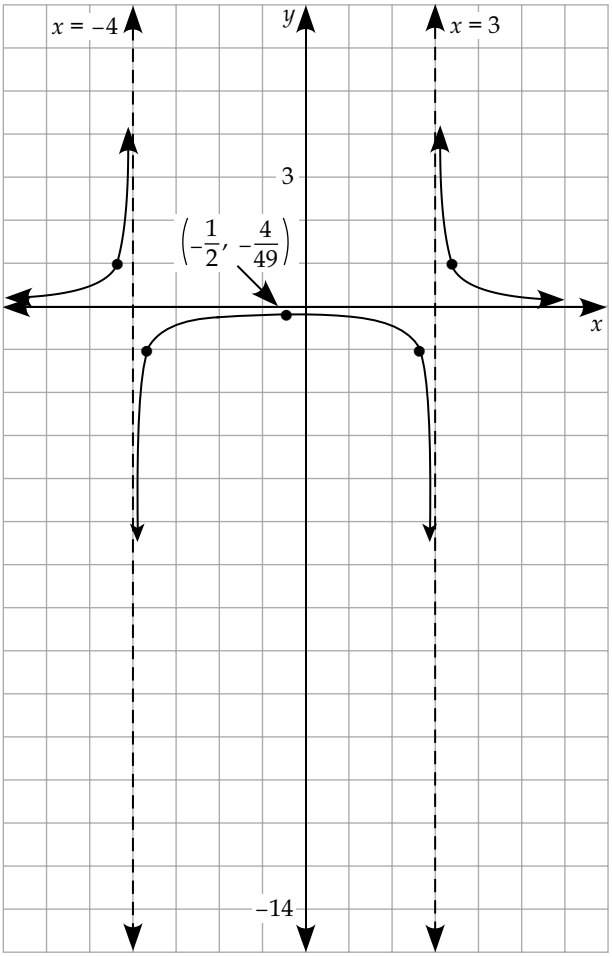


Module 8 Summary

Congratulations! You have completed all of the modules in Grade 11 Pre-Calculus Mathematics! All you have left to do is complete the final examination in this course.

In this module, you learned about two new types of functions: absolute value functions and reciprocal functions. Before you learned how to graph absolute value functions, you learned how to find the absolute value of a number. This was useful when you were asked to solve absolute value equations.

In order to graph reciprocal functions, you related them to rational functions. To do this, you learned how the non-permissible values of rational functions are related to the asymptotes when you are graphing reciprocal functions.

It is now time to take your final examination. The following information will tell you how to apply and prepare for your final exam. It is also a good idea to complete the final practice examination as it will let you know what areas you are struggling with, and perhaps need to study further.

Submitting Your Assignments

It is now time for you to submit Assignments 8.1 to 8.3 to the Distance Learning Unit so that you can receive some feedback on how you are doing in this course. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 8 assignments and organize your material in the following order:

- ❑ Module 8 Cover Sheet (found at the end of the course Introduction)
- ❑ Assignment 81: Absolute Value
- ❑ Assignment 8.2: Absolute Value Equations and Graphing Reciprocal Functions: Part 1
- ❑ Assignment 8.3: Reciprocal Functions: Part 2

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Final Examination

Congratulations, you have finished Module 8 in the course. The final examination is out of 100 marks and worth 25% of your final mark. The examination content will consist of 20% from Modules 1 to 4 and 80% from Modules 5 to 8. In order to do well on this examination, you should review all of your learning activities and assignments from Modules 1 to 8.

You will complete this examination while being supervised by a proctor. You should already have made arrangements to have the examination sent to the proctor from the Distance Learning Unit. If you have not yet made arrangements to write it, then do so now. The instructions for doing so are provided in the Introduction to this module.

You will need to bring the following items to the examination: pens/pencils (2 or 3 of each), blank paper, a ruler, a scientific or graphing calculator, and your Examination Resource Sheet. A maximum of 2.5 hours is available to complete your final examination. When you have completed it, the proctor will then forward it for assessment. Good luck!

At this point you will also have to combine your resource sheets from Modules 1 to 8 onto one 8½" × 11" paper (you may use both sides). Be sure you have all the formulas, definitions, and strategies that you think you will need. This paper can be brought into the examination with you.

Examination Review

You are now ready to begin preparing for your final examination. Please review the content, learning activities, and assignments from Modules 1 to 8.

The final practice examination is also an excellent study aid for reviewing Modules 1 to 8.

You will learn what types of questions will appear on the examination and what material will be assessed. Remember, your mark on the final examination determines 25% of your final mark in this course and you will have 2.5 hours to complete the examination.

Final Practice Examination and Answer Key

To help you succeed in your examination, a practice examination can be found in the learning management system (LMS). The final practice examination is very similar to the actual examination that you will be writing. The answer key is also included so that, when you have finished writing the practice examination, you can check your answers. This will give you the confidence that you need to do well on your examination. If you do not have access to the Internet, contact the Distance Learning Unit at 1-800-465-9915 to get a copy of the practice examination and the answer key.

To get the most out of your final practice examination, follow these steps:

1. Study for the final practice examination as if it were an actual examination.
2. Review those learning activities and assignments from Modules 1 to 8 that you found the most challenging. Reread those lessons carefully and learn the concepts.
3. Contact your learning partner and your tutor/marker if you need help.
4. Review your lessons from Modules 1 to 8, including all of your notes, learning activities, and assignments.
5. Use your module resource sheets to make a draft of your Final Examination Resource Sheet. You can use both sides of an 8½" by 11" piece of paper.
6. Bring the following to the final practice examination: pens/pencils (2 or 3 of each), blank paper, a ruler, a scientific or graphing calculator, and your Final Examination Resource Sheet.
7. Write your final practice examination as if it were an actual examination. In other words, write the entire examination in one sitting, and don't check your answers until you have completed the entire examination. Remember that the time allowed for writing the final examination is 3 hours.
8. Once you have completed the entire practice examination, check your answers against the answer key. Review the questions that you got wrong. For each of those questions, you will need to go back into the course and learn the things that you have missed.
9. Go over your resource sheet. Was anything missing or is there anything that you didn't need to have on it? Make adjustments to your Final Examination Resource Sheet. Once you are happy with it, make a photocopy that you can keep.

Notes

Grade 11 Pre-Calculus Mathematics (30S)

Module 8
Absolute Value and Reciprocal Functions

Learning Activity Answer Keys

MODULE 8:
ABSOLUTE VALUE AND RECIPROCAL FUNCTIONS

Learning Activity 8.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. In the three courses you are taking this semester, you receive final marks of 72%, 83%, and 91%. What is your average?
2. Find the equation of a line with a slope of 2 and a y-intercept of -6.
3. Evaluate and express as a mixed fraction: $\frac{5}{6} + \frac{7}{4}$
4. You have a job at an electronics store. As a bonus for excellent sales this month, your employer offers you either $\frac{1}{20}$ or 7% of your annual salary. Which one is the better choice?
5. Evaluate: 2^6
6. Estimate: $\sqrt{220}$
7. Are these fractions equivalent? $\frac{3}{\sqrt{2}-4}$ and $\frac{3\sqrt{2}+12}{18}$
8. In which quadrant is an angle of 294° located?

Answers:

1. 82% $\left(72 + 83 + 91 = 246; \frac{246}{3} = 82\%\right)$
2. $y = 2x - 6$
3. $2\frac{7}{12}$ $\left(\frac{5}{6} + \frac{7}{4} = \frac{10}{12} + \frac{21}{12} = \frac{31}{12} = 2\frac{7}{12}\right)$
4. 7% is the better choice $\left(\frac{1}{20} = \frac{5}{100}; 7\% = \frac{7}{100}\right)$

5. 64 ($2 \times 2 \times 2 \times 2 \times 2 \times 2$)

6. $\cong 14.8$ (220 is between 196 and 225; therefore, $\sqrt{220}$ is between 14 and 15)

7. No $\left(\dfrac{3}{\sqrt{2}-4} = \dfrac{3\left(\sqrt{2}+4\right)}{\left(\sqrt{2}-4\right)\left(\sqrt{2}+4\right)} = \dfrac{3\sqrt{2}+12}{-14} \neq \dfrac{3\sqrt{2}+12}{18} \right)$

8. Quadrant IV

Part B: Absolute Value

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

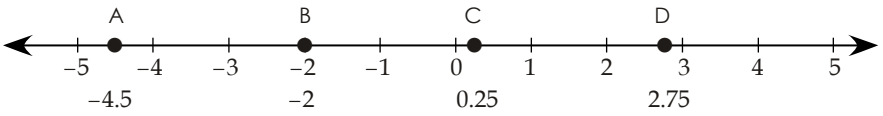
1. Evaluate.
   a) $|-13|$
   b) $|4|$
   c) $|0|$
   d) $|-(52)|$
   e) $2|5-3(2)|+1$
   f) $3|4(3+2)^2-42|-2$

   Answers:

   a) $|-13| = -(-13) = 13$
   b) $|4| = 4$
   c) $|0| = 0$
   d) $|-(52)| = -(-52) = 52$
   e) $2|5-3(2)|+1$
   $= 2|5-6|+1$
   $= 2|-1|+1$
   $= 2[-(-1)]+1$
   $= 2(1)+1$
   $= 2+1$
   $= 3$

f) $3|\,4(3+2)^2 - 42\,| - 2$
$= 3|\,4(5)^2 - 42\,| - 2$
$= 3|\,4(25) - 42\,| - 2$
$= 3|\,100 - 42\,| - 2$
$= 3|\,58\,| - 2$
$= 3(58) - 2$
$= 174 - 2$
$= 172$

2. Use absolute value symbols to write an expression for the distance between each pair of points on the number line. Using this expression, determine the distance between the two points.



a) A and D

Answer:

The distance between A and D can be represented by the absolute value expression $|\,-4.5 - 2.75\,|$ and can be calculated as

$|\,-7.25\,| = -(-7.25) = 7.25.$

b) B and D

Answer:

The distance between B and D can be represented by the absolute value expression $|\,-2 - 2.75\,|$ and can be calculated as

$|\,-4.75\,| = -(-4.75) = 4.75$

c) C and B

Answer:

The distance between C and B can be represented by the absolute value expression $|\,0.25 - (-2)\,|$ and can be calculated as

$|\,0.25 + 2\,| = |\,2.25\,| = 2.25$

3. Describe how distance can be related to absolute value.

Answer:

There are two ways of answering this question. From Question 2, you can see that distances from positive to negative values, as well as distance from negative to positive values, are all positive. This is because distance can be understood as the absolute value of a number. Or, the absolute value of a number can be understood as its distance away from 0.

4. Determine the distance between each of the following sets of points. Include a graph for (b), (c), and (d).

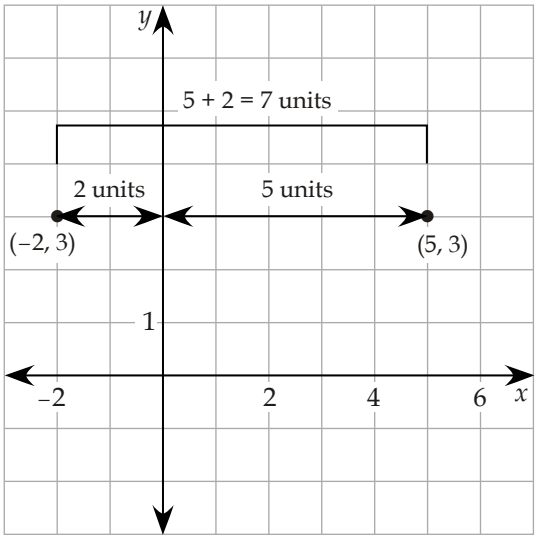
   a) -45 and -23

   Answer:

   $| -45 - (-23) | = | -45 + 23 | = | -22 | = -(-22) = 22$

   b) $(5, 3)$ and $(-2, 3)$

   Answer:

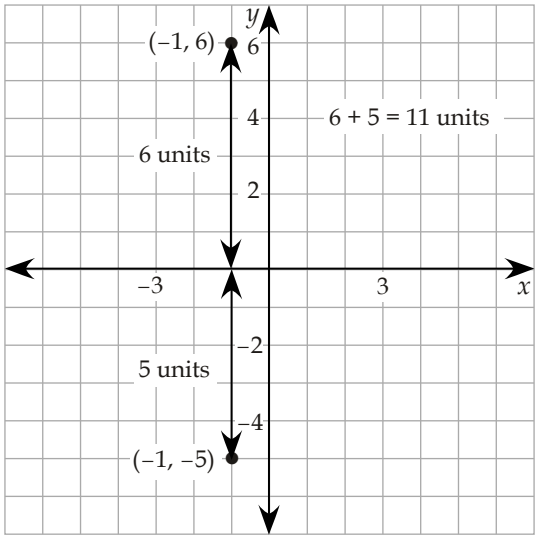


   These points on the coordinate graph have the same y-coordinates and are 7 units apart. This can also be symbolized by subtraction and absolute value:

   $| 5 - (-2) | = | 5 + 2 | = | 7 | = 7$

   c) $(-1, 6)$ and $(-1, -5)$

   Answer:

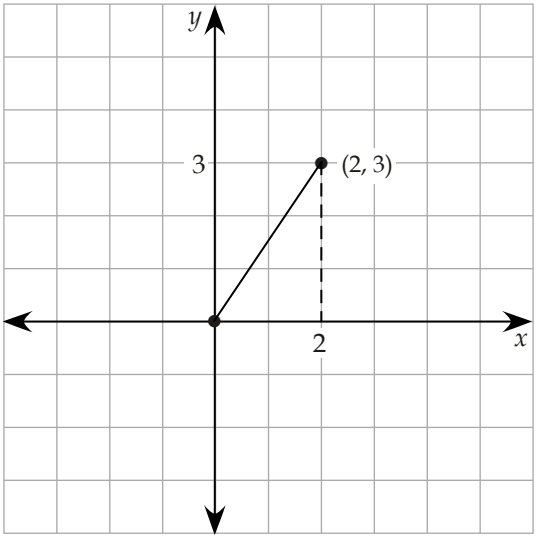


These points on the coordinate graph have the same x-coordinates and are 11 units apart. This can also be symbolized by subtraction and absolute value:

$| 6 - (-5) | = | 6 + 5 | = | 11 | = 11$

d) An extension of finding absolute value on the coordinate graph is to find the distance of a point from the origin—in other words, its absolute value. Find the absolute value for the point (2, 3).

Answer:



The distance can be found by using Pythagorean theorem:

$d = \sqrt{2^2 + 3^2}$

$d = \sqrt{13}$

Since d represents the hypotenuse of a right triangle, its value must be positive and only the principal square root of 13 is considered.

5. Order the following set of absolute values from least to greatest.

$$\left|-\frac{1}{4}\right|,\ \frac{1}{3},\ -3,\ |-3|,\ \left|\frac{2}{9}\right|,\ \left|\frac{3}{2}\right|,\ \frac{1}{2},\ -\frac{7}{3}$$

Answer:

$$\left|-\frac{1}{4}\right| = \frac{1}{4} = 0.25$$

$$\frac{1}{3} = 0.\bar{3}$$

$$-3 = -3$$

$$|-3| = -(-3) = 3$$

$$\left|\frac{2}{9}\right| = \frac{2}{9} = 0.\bar{2}$$

$$\left|\frac{3}{2}\right| = \frac{3}{2} = 1.5$$

$$\frac{1}{2} = 0.5$$

$$-\frac{7}{3} = -2.\bar{3}$$

The order from least to greatest is

$$-3 < -2.\bar{3} < 0.\bar{2} < 0.25 < 0.\bar{3} < 0.5 < 1.5 < 3$$

$$-3 < -\frac{7}{3} < \left|\frac{2}{9}\right| < \left|-\frac{1}{4}\right| < \frac{1}{3} < \frac{1}{2} < \left|\frac{3}{2}\right| < |-3|$$

6. Evaluate the following expressions for $x = -2$.

a) $3|\ -2x - 5\ |\ + 4$

Answer:

$$\begin{aligned}
&3|\ -2x - 5\ |\ + 4 \\
&= 3|\ -2(-2) - 5\ |\ + 4 \\
&= 3|\ 4 - 5\ |\ + 4 \\
&= 3|\ -1\ |\ + 4 \\
&= 3[-1(-1)] + 4 \\
&= 3(1) + 4 \\
&= 3 + 4 \\
&= 7
\end{aligned}$$

b) $3|\, x^2 - 7x + 2\,| - 5$

Answer:

$3|\, x^2 - 7x + 2\,| - 5$
$= 3|\, (-2)^2 - 7(-2) + 2\,| - 5$
$= 3|\, 4 + 14 + 2\,| - 5$
$= 3|\, 20\,| - 5$
$= 3(20) - 5$
$= 60 - 5$
$= 55$

7. Consider the expression $\frac{x}{|x|}$.

a) What is the value of $\frac{x}{|x|}$ when x is positive?

Answer:

Since $x > 0$, the denominator, $|\, x\,|$, will be positive and the numerator will be the same value. The expression becomes $\frac{x}{x}$, which equals 1.

b) What is the value of $\frac{x}{|x|}$ when x is negative?

Answer:

When $x < 0$, the denominator, $|\, x\,|$, will be positive and the numerator will be negative. Therefore, the expression will be negative and will equal -1.

8. In Manitoba, the temperature could go as low as $-40°$ C in the winter. In the summer, the temperature may reach a high of 30° C. Determine the difference in the range of these two temperatures.

Answer:

$|\, -40 - 30\,| = |\, -70\,| = -(-70) = 70$

Therefore, the range in Manitoba's temperatures is 70°C.

9. Charlize works in an office building and uses an electronic key card in order to travel to different floors of the building by elevator. This electronic key card tracks her movements throughout the day. The floors Charlize travelled to throughout one day are displayed in the chart below.

Time	Floor
8 a.m.	1
9:13 a.m.	4
11:41 a.m.	2
12:51 p.m.	7
2:00 p.m.	4
2:10 p.m.	6
2:30 p.m.	4
4 p.m.	1

a) Write and evaluate an absolute value expression to determine the total amount of floor changes throughout Charlize's day.

Answer:

$|4-1|+|2-4|+|7-2|+|4-7|+|6-4|+|4-6|+|1-4|$

$=|3|+|-2|+|5|+|-3|+|2|+|-2|+|-3|$

$=3+2+5+3+2+2+3$

$=20$

b) Write and evaluate an absolute value expression to determine the net change in Charlize's location from the beginning of the day to the end of the day.

Answer:

$|1-1|=|0|=0$

c) Compare the value you calculated in (a) to the value you found in (b). How does the total change compare to the net change of Charlize's location?

Answer:

The net change is the change from the beginning of the day to the end of the day. The total change is the sum of all the changes throughout the day. Therefore, it may look like Charlize didn't change floors at all throughout the day; however, she really travelled through 20 different floors. This is a significant difference.

Learning Activity 8.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. If you get paid $11 an hour and you work 30 hours in a week, how much will you get paid (before taxes)?
2. If you travel at a speed of 72 km/h for 6 hours, how far will you travel?
3. State the reciprocal of $\frac{5}{7}$.
4. Calculate 15% of 140.
5. Find the equation of a line with a slope of 3 that crosses through the point (1, 1).
6. Express in exponential form: $\left(\sqrt{5}\right)\left(\sqrt[3]{5}\right)$
7. Determine the distance between the following two points: (7, −1) and (−8, 15)
8. How many feet are in 24 yards?

Answers:

1. $330 ($30 \times 11$)
2. 432 km ($72 \times 6 = 70 \times 6 + 2 \times 6 = 420 + 12$)
3. $\frac{7}{5}$
4. 21 (10% of 140 = 14; 5% of 140 = 7; 15% of 140 = 14 + 7 = 21)
5. $y = 3x - 2$ ($y = mx + b$; $1 = 3(1) + b$; $1 - 3 = b$; $-2 = b$)
6. $5^{\frac{5}{6}}$ $\left(\left(\sqrt{5}\right)\left(\sqrt[3]{5}\right) = 5^{\frac{1}{2}}5^{\frac{1}{3}} = 5^{\frac{1}{2}+\frac{1}{3}} = 5^{\frac{3}{6}+\frac{2}{6}} = 5^{\frac{5}{6}}\right)$
7. $\sqrt{481}$ $\left(d = \sqrt{(-8-7)^2 + (15-(-1))^2} = \sqrt{(-15)^2 + (16)^2} = \sqrt{225+256}\right)$
8. 72 (There are 3 feet in 1 yard and $3 \times 24 = 72$ feet in 24 yards.)

Part B: Graphing Absolute Value Functions

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Sketch each function and state the values of the intercepts.

   a) $f(x) = |2x - 1|$

   Answer:

   There are two graphs to draw:

   $f(x) = 2x - 1$ when $2x - 1 \geq 0$ or $x \geq \frac{1}{2}$

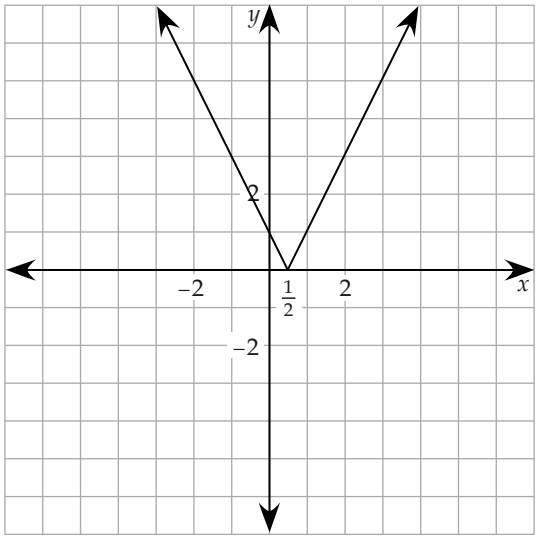
   and $f(x) = -(2x - 1)$ when $2x - 1 < 0$ or $x < \frac{1}{2}$

   The pivot point is the x-intercept.

   $$f(x) = |2x - 1|$$
   $$0 = |2x - 1|$$
   $$0 = 2x - 1$$
   $$\frac{1}{2} = x$$

   The y-intercept is

   $$f(x) = |2x - 1|$$
   $$f(0) = |2(0) - 1|$$
   $$f(0) = |-1|$$
   $$f(0) = 1$$



b) $g(x) = |x + 3|$

Answer:

The piece-wise definition is

$$g(x) = \begin{cases} x + 3, & x + 3 \geq 0 \\ -(x + 3), & x + 3 < 0 \end{cases}$$

Simplify to get

$$g(x) = \begin{cases} x + 3, & x \geq -3 \\ -x - 3, & x < -3 \end{cases}$$

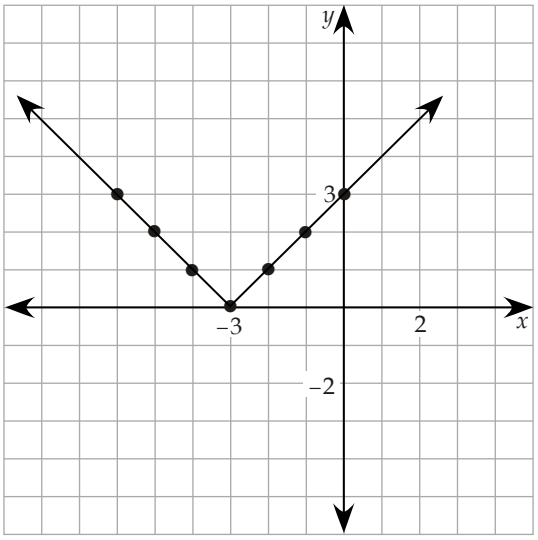
The pivot-point is $x = -3$, which is also the x-intercept.

The y-intercept is

$g(x) = |x + 3|$

$g(0) = |0 + 3|$

$g(0) = 3$



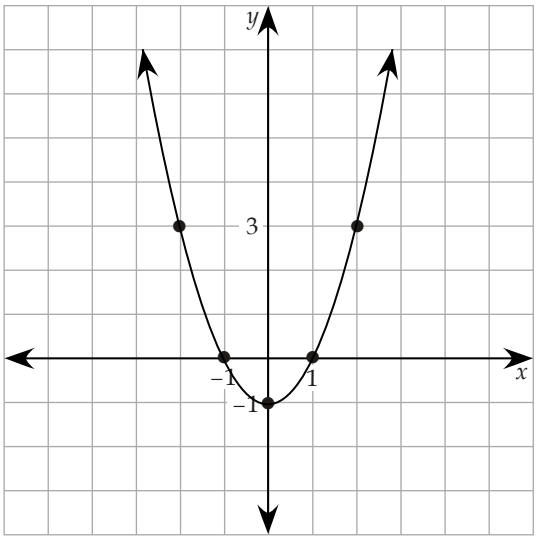
c) $h(x) = |x^2 - 1|$

Answer:

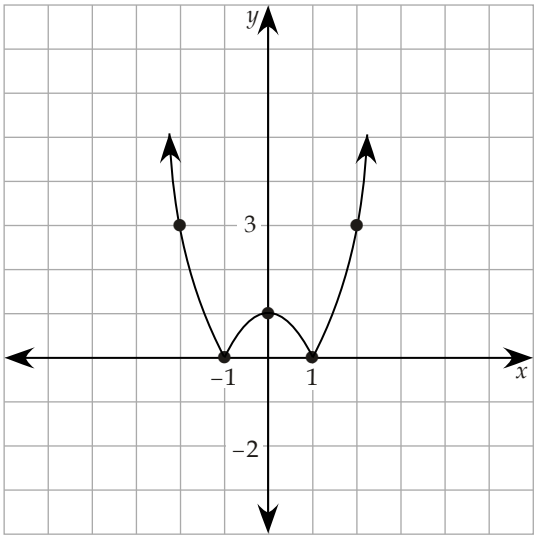
The piece-wise definition is

$$h(x) = \begin{cases} x^2 - 1, & x^2 - 1 \geq 0 \\ -\left(x^2 - 1\right), & x^2 - 1 < 0 \end{cases}$$

Using a graphing transformations method, first graph $y = x^2 - 1$ with vertex (0, 1), opening up and standard size.



Using the absolute value transformation, reflect any points with negative y-values over the x-axis. This would be the portion of the graph in the domain, $-1 < x < 1$.



The x-intercepts are ±1 and the y-intercept is 1.

d) $j(x) = |-x^2 + 2x + 3|$

Answer:

The piece-wise definition is

$$j(x) = \begin{cases} -x^2 + 2x + 3, & -x^2 + 2x + 3 \geq 0 \\ -\left(-x^2 + 2x + 3\right), & -x^2 + 2x + 3 < 0 \end{cases}$$

To sketch the graph of $y = -x^2 + 2x + 3$, first complete the square to find the vertex:

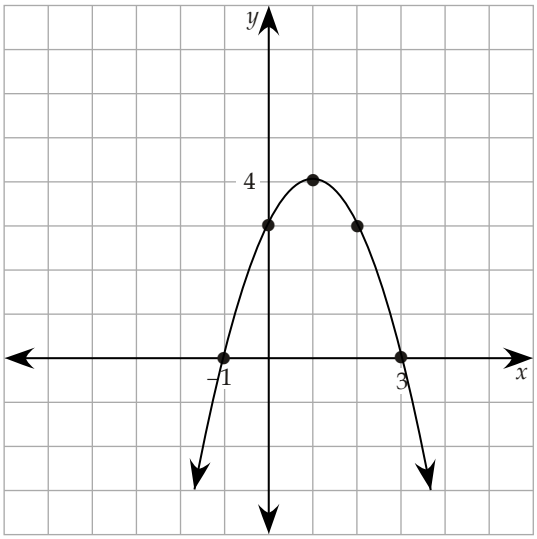
$$y = -x^2 + 2x + 3$$

$$y = -\left(x^2 - 2x \quad\right) + 3$$

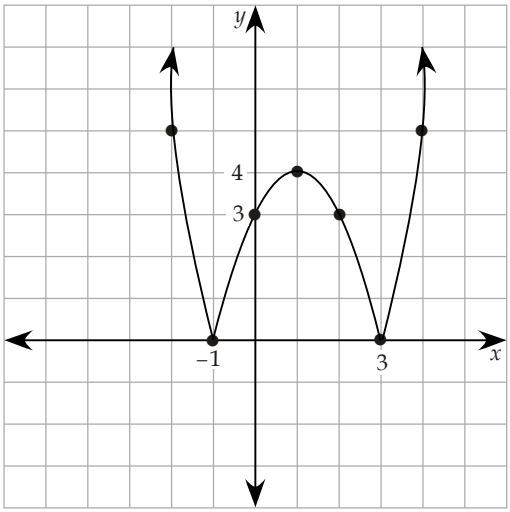
$$y = -\left(x^2 - 2x + 1\right) + 3 - 1(-1)$$

$$y = -(x-1)^2 + 4$$

The parabola opens down and is standard size. The vertex is (1, 4).



The absolute value transformation reflects any points with negative y-values over the x-axis. This occurs in two places in the domain for $x < -1$ and $x > 3$.



The x-intercepts are -1 and 3. The y-intercept is 3.

e) State the domain and range for each function in (a) to (d).

Answer:

Question	Domain	Range
(a)	$\Re$	$[0, \infty)$
(b)	$\Re$	$[0, \infty)$
(c)	$\Re$	$[0, \infty)$
(d)	$\Re$	$[0, \infty)$

2. Write each function from Question 1 using piece-wise notation and simplify.

Answers:

a) $f(x) = |\, 2x - 1 \,|$

The piece-wise function is

$$f(x) = \begin{cases} 2x - 1, & 2x - 1 \geq 0 \\ -(2x - 1), & 2x - 1 < 0 \end{cases}$$

Simplify both the function part and the domain pieces to get:

$$f(x) = \begin{cases} 2x - 1, & x \geq \frac{1}{2} \\ -2x + 1, & x < \frac{1}{2} \end{cases}$$

You can verify your answer from the graph.

b) $g(x) = |x + 3|$

The piece-wise definition is

$$g(x) = \begin{cases} x + 3, & x + 3 \geq 0 \\ -(x + 3), & x + 3 < 0 \end{cases}$$

Simplify the function and domain pieces to get

$$g(x) = \begin{cases} x + 3, & x \geq -3 \\ -x - 3, & x < -3 \end{cases}$$

Check your answer from the graph.

c) $h(x) = |x^2 - 1|$

The piece-wise definition is:

$$h(x) = \begin{cases} x^2 - 1, & x^2 - 1 \geq 0 \\ -\left(x^2 - 1\right), & x^2 - 1 < 0 \end{cases}$$

The function pieces are easily simplified, $h(x) = -(x^2 - 1) = -x^2 + 1$.

The domain pieces can be simplified by using the graph. Separate the pieces into intervals according to the x-intercepts. The simplified version is:

$$h(x) = \begin{cases} x^2 - 1, & x \leq -1 \\ -x^2 + 1, & -1 < x < 1 \\ x^2 - 1, & x \geq 1 \end{cases}$$

Note that the middle piece of $h(x)$ shows the portion where the graph was reflected over the x-axis to get a piece of a parabola opening down with vertex (0, 1).

d) $j(x) = |\ -x^2 + 2x + 3\ |$

The piece-wise definition is

$$j(x) = \begin{cases} -x^2 + 2x + 3, -x^2 + 2x + 3 \geq 0 \\ -\left(-x^2 + 2x + 3\right), -x^2 + 2x + 3 < 0 \end{cases}$$

The function $j(x) = -(-x^2 + 2x + 3)$ simplified is $x^2 - 2x - 3$.

The domain pieces can be simplified from the graph by considering the intervals of the domain according to the x-intercepts.

The simplified version of $j(x)$ becomes

$$j(x) = \begin{cases} x^2 - 2x - 3, x \leq -1 \\ -x^2 + 2x + 3, -1 < x < 3 \\ x^2 - 2x - 3, x \geq 3 \end{cases}$$

Note that the first and last pieces of $j(x)$ indicate the portions of graph of $y = -x^2 + 2x + 3$ that were reflected over the x-axis by the absolute value transformation. They are pieces of a parabola that opens up and has its vertex at $(1, -4)$.

3. The point $(2, -5)$ is on the graph $y = f(x)$. Identify the corresponding point on the graph $y = |\ f(x)\ |$.

   Answer:

   The absolute value transformation can be defined as $(x, y) \rightarrow (x, |\ y\ |)$. The absolute value function turns every negative y-value into a positive y-value. Therefore, the point $(2, -5)$ becomes $(2, 5)$.

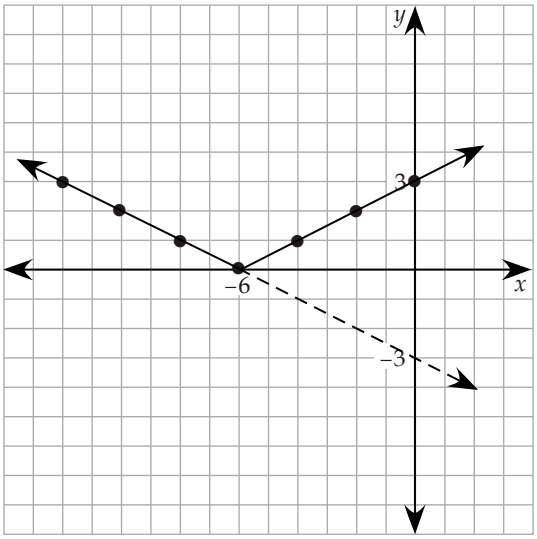
4. The graph of $y = f(x)$ has an x-intercept of -6 and a y-intercept of -1. What are the x- and y-intercepts on the graph of $y = |\ f(x)\ |$?

   Answer:

   The x-intercept of an absolute value function stays the same as the x-intercept is always the invariant point of the absolute value function. Therefore, the x-intercept is still -6. However, an absolute value function does not contain negative y-values. Therefore, this y-intercept would be reflected over the x-axis to result in a y-intercept of 1 in $y = |\ f(x)\ |$.

5. Write the piece-wise function that represents each graph.

a)



Answer:

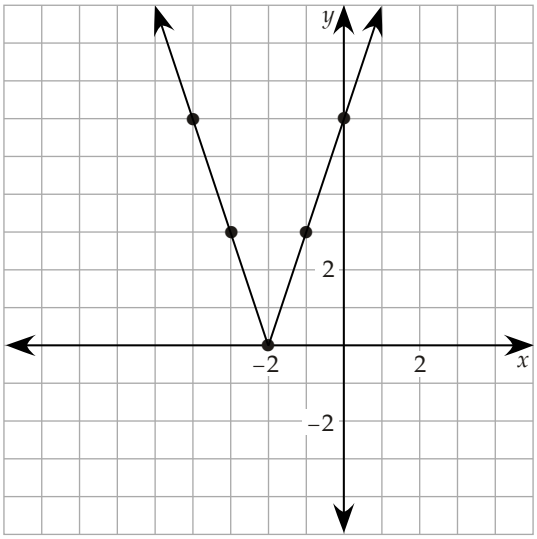
The slope of the line when x is greater than or equal to -6 is $\frac{1}{2}$.

The y-intercept is 3. The slope of the line when x is less than -6 is $-\frac{1}{2}$.

If this line were to continue down below the x-axis, the y-intercept would be -3. Therefore, the piece-wise function for this graph is

$$y = \begin{cases} -\frac{1}{2}x - 3, & x < -6 \\ \frac{1}{2}x + 3, & x \geq -6 \end{cases}$$

b)

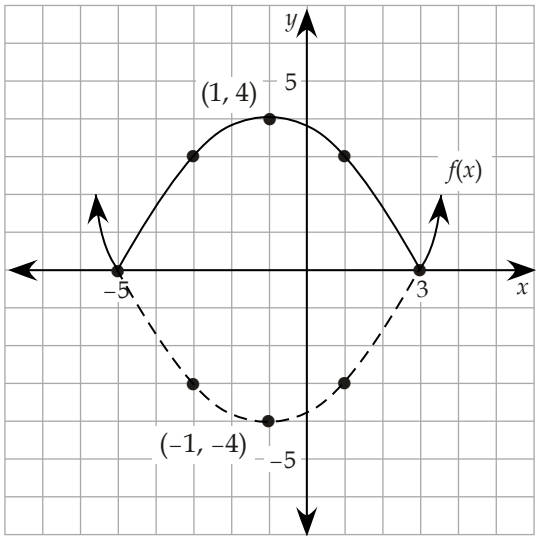


Answer:

The slope of the line when x is greater than or equal to -2 is 3. The y-intercept is 6. Therefore, this piece of the graph has the equation $y = 3x + 6$. When x is less than -2, the line, $y = 3x + 6$, is reflected over the x-axis, resulting in the line $y = -(3x + 6)$ or $y = -3x - 6$. The piece-wise function for this graph is thus

$$y = \begin{cases} -3x - 6, & x < -2 \\ 3x + 6, & x \geq -2 \end{cases}$$

c)



Answer:

This graph represents the absolute of a quadratic function whose vertex was originally at $(-1, -4)$, opening up and y-value reduced by a factor of 4. The pieces of the domain are determined by the x-intercepts. Thus, the piece-wise definition is

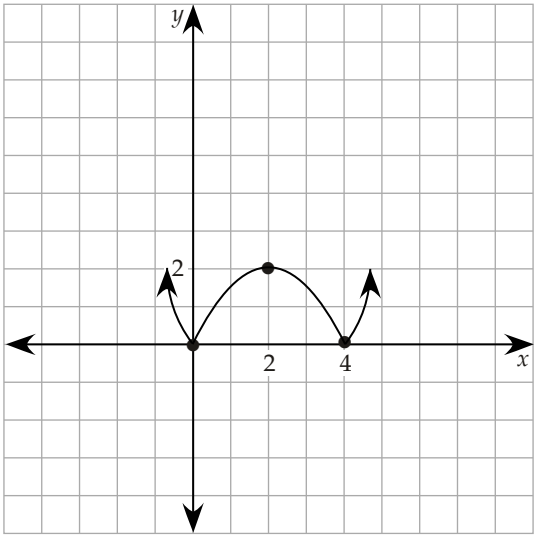
$$f(x) = \begin{cases} \frac{1}{4}(x+1)^2 - 4, & x \leq -5 \\ -\left[\frac{1}{4}(x+1)^2 - 4\right], & -5 < x < 3 \\ \frac{1}{4}(x+1)^2 - 4, & x \geq 3 \end{cases}$$

Note that you might have seen this graph as a parabola opening down with its original vertex at $(-1, 4)$. This possibility is shown in the next solution. Either interpretation will give the same absolute value function.

The simplified version of the function is

$$f(x)=\begin{cases}\frac{1}{4}(x+1)^2-4, & x\le -5\\ -\frac{1}{4}(x+1)^2+4, & -5<x<3\\ \frac{1}{4}(x+1)^2-4, & x\ge 3\end{cases}$$

d)



Answer:

This graph could be the absolute value transformation of a quadratic function with vertex at (2, 2), opening down, and having the y-values reduced by a factor of 2. The original quadratic function would be $y=\frac{-1}{2}(x-2)^2+2$. The tails of the parabola were reflected over the x-axis at pivot points $x = 0$ and $x = 4$ by the absolute value transformation. The piece-wise absolute value function is

$$f(x)=\begin{cases}-\left[\frac{-1}{2}(x-2)^2+2\right], & x<0\\ -\frac{1}{2}(x-2)^2+2, & 0\le x\le 4\\ -\left[\frac{-1}{2}(x-2)^2+2\right], & x>4\end{cases}$$

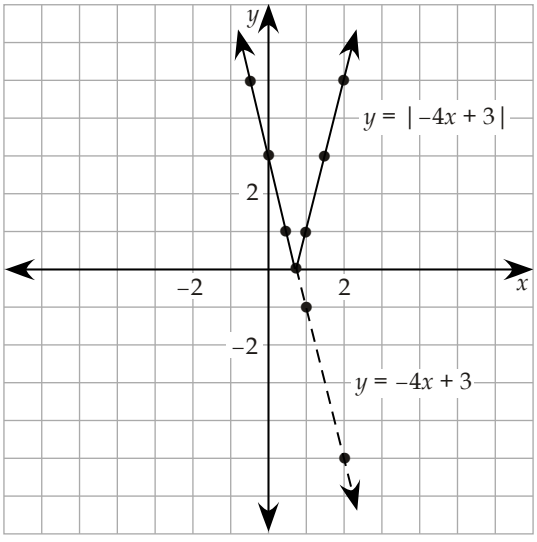
The simplified form:

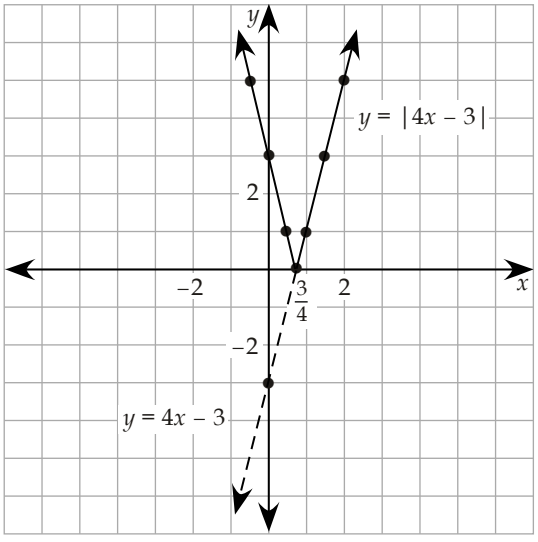
$$f(x) = \begin{cases} \frac{1}{2}(x-2)^2 - 2, & x < 0 \\ -\frac{1}{2}(x-2)^2 + 2, & 0 \le x \le 4 \\ \frac{1}{2}(x-2)^2 - 2, & x > 4 \end{cases}$$

6. Graph $f(x) = |-4x + 3|$ and $g(x) = |4x - 3|$ using a table of values. What do you notice about these two graphs? Explain.

   Answer:

x	$y = -4x + 3$	$f(x) = \lvert -4x + 3 \rvert$	$y = 4x - 3$	$g(x) = \lvert 4x - 3 \rvert$
−3	15	15	−15	15
−2	11	11	−11	11
−1	7	7	−7	7
0	3	3	−3	3
1	−1	1	1	1
2	−5	5	5	5
3	−9	9	9	9





These two absolute value graphs are identical. This is because the expression inside the absolute value symbols can either be positive or negative to yield the same answer. Therefore, $|-4x + 3| = |4x - 3|$.

7. Graph the function $f(x) = |x^2 - 6x + 5|$. Write a different absolute value function having the same graph as $f(x)$.

Answer:

$f(x) = |x^2 - 6x + 5|$

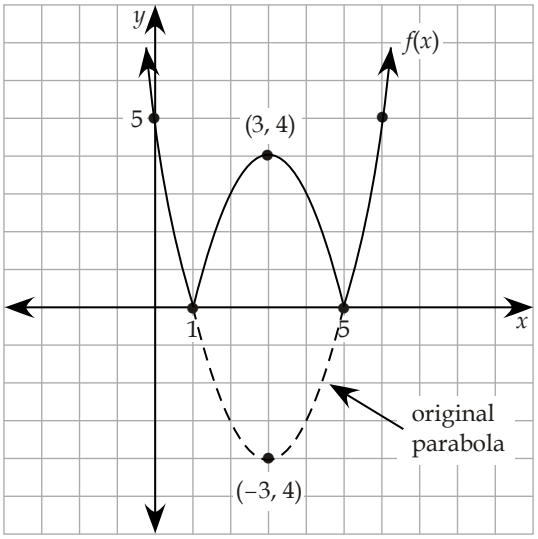
Consider the function $y = x^2 - 6x + 5$. Graph this function and then use the absolute value transformation.

$y = x^2 - 6x + 5$

$y = (x^2 - 6x + 9) + 5 - 9$

$y = (x - 3)^2 - 4$

Vertex $(3, -4)$ opens up with normal size. Reflect the portion of the parabola with negative y-values over the x-axis. The graph of $f(x)$ is shown below.



Another absolute value function having the same graph as $f(x)$ would be the absolute value of the negative of the original quadratic $y = x^2 - 6x + 5$. This function would be

$g(x) = |-(x^2 - 6x + 5)|$

$g(x) = |-x^2 + 6x - 5|$

Learning Activity 8.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. You take your best friend out for dinner at a restaurant and the bill comes to $67.32. If you tip the waiter 15%, how much change will you get back if you pay with a $100 bill?
2. Find the equation of a line with a slope of -1 that crosses through the point (5, 2).
3. Evaluate: 3^4
4. Evaluate: $\frac{2}{13} + \frac{6}{7}$
5. If Jacob can read an average of 98 pages a day, how many pages can he read in a week?
6. Express as a radical: $8^{\frac{5}{4}}$. Do not simplify.
7. Three angles in a parallelogram measure 30°, 93°, and 72°. What is the measure of the fourth angle?
8. The equation of a parabola is $y = x^2 + 2x - 15$. What are the x-intercepts?

Answers:

1. $22.58 ($67.32 × 10% = $6.73; $67.32 × 5% = $3.37; the bill will total $67.32 + $6.73 + $3.37 = $77.42; $100 − $77.42 = $22.58)
2. $y = -x + 7$ ($y = mx + b$; $2 = -1(5) + b$; $2 = -5 + b$; $7 = b$)
3. 81 ($3 \times 3 \times 3 \times 3 = 9 \times 9$)
4. $\frac{92}{91}\left(\frac{14}{91} + \frac{78}{91}\right)$
5. 686 ($90 \times 7 = 630$; $8 \times 7 = 56$; $98 \times 7 = 686$)
6. $\sqrt[4]{8^5}$
7. 165° (The sum of the angles in a parallelogram is 360°. The missing angle is 360° − 30° − 93° − 72° = 165°.)
8. -5 and 3 (Factor the corresponding quadratic equation: $0 = x^2 + 2x - 15 = (x + 5)(x - 3)$.

Part B: Solving Absolute Value Equations

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Solve the following equations using the definition of absolute value.

   a) $|2x - 3| = 7$

   Answer:

   Case 1: $2x - 3 = 7$ and $2x - 3 \geq 0$

   $2x - 3 = 7$ $\qquad$ $2x - 3 \geq 0$

   $2x = 10$ $\qquad$ $2x \geq 3$

   $x = 5$ $\qquad$ $x \geq \frac{3}{2}$

   Since $x = 5$ is in the interval $x \geq \frac{3}{2}$, then $x = 5$ is a solution.

   Case 2: $-(2x - 3) = 7$ and $2x - 3 < 0$

   $-2x + 3 = 7$ $\qquad$ $2x - 3 < 0$

   $-2x = 4$ $\qquad$ $2x < 3$

   $x = -2$ $\qquad$ $x < \frac{3}{2}$

   Since $x = -2$ is in the interval $x < \frac{3}{2}$, then $x = -2$ is a solution.

   $\therefore$ The solutions are $x = 5$ and $x = -2$.

b) $|3x - 8| = 2 - 4x$

Answer:

Case 1: $3x - 8 = 2 - 4x$ and $3x - 8 \geq 0$

$7x = 10$ $3x \geq 8$

$x = \frac{10}{7}$ $x \geq \frac{8}{3}$

Since $x = \frac{10}{7}$ is not in the interval $x \geq \frac{8}{3}$, it is not a solution. It is an extraneous root.

Case 2: $-(3x - 8) = 2 - 4x$ and $3x - 8 < 0$

$-3x + 8 = 2 - 4x$ $3x < 8$

$x = -6$ $x < \frac{8}{3}$

Since $x = -6$ is in the interval $x < \frac{8}{3}$, then it is a solution.

$\therefore$ The only solution is $x = -6$.

c) $|x^2 - 4x| = 4$

Answer:

Case 1: $x^2 - 4x = 4$ and $x^2 - 4x \geq 0$

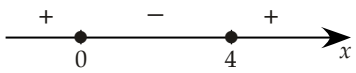
$x^2 - 4x - 4 = 0$ $x(x - 4) \geq 0$

Using the quadratic formula: $x = 0, 4$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, then

$x = 4.828$ or

$x = -0.828$



Solution interval:

$(-\infty, 0] \cup [4, \infty)$

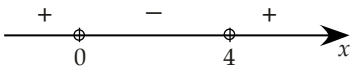
Since $x = 4.828$ is in the interval $[4, \infty)$ and $x = -0.828$ is in the interval $(-\infty, 0]$, then both are solutions.

Case 2: $-\left(x^2 - 4x\right) = 4$ and $x^2 - 4x < 0$

$-x^2 + 4x = 4$

$-x^2 + 4x - 4 = 0$

$x^2 - 4x + 4 = 0$



Solution interval: (0, 4)

Factoring: $(x - 2)(x - 2) = 0$

$x = 2$

Since $x = 2$ is in the interval (0, 4), then $x = 2$ is also a solution.

$\therefore$ The solutions are $x = 4.828$, $x = -0.828$, and $x = 2$.

d) $|x^2 + 2x| = x + 30$

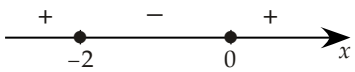
Answer:

Case 1: $x^2 + 2x = x + 30$ and $x^2 + 2x \geq 0$

$x^2 + x - 30 = 0$ $x(x + 2) \geq 0$

$(x + 6)(x - 5) = 0$ $x = 0, -2$

$x = -6, 5$



Solution: $(-\infty, -2] \cup [0, \infty)$

Since $x = -6$ is in the interval $(-\infty, -2]$ and $x = 5$ is in the interval $[0, \infty)$, both are solutions.

Case 2: $-\left(x^2 + 2x\right) = x + 30$ and $x^2 + 2x < 0$

$-x^2 - 2x = x + 30$

$-x^2 - 3x - 30 = 0$

$x^2 + 3x + 30 = 0$

discriminant $= b^2 - 4ac$

$= 3^2 - 4(1)(30)$

$= -111$

There are no real number solutions.

$\therefore$ The only solutions are $x = -6, 5$.

2. Solve the absolute value equations in Question 1 graphically.

   a) $|2x - 3| = 7$

   Answer:

   The corresponding function for the left side of the equation is $y = |2x - 3|$.

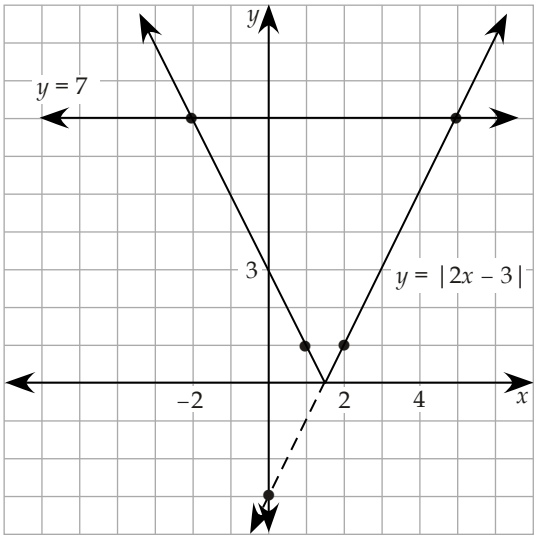
   The line is $y = 2x - 3$ with slope 2 and y-intercept of -3. The pivot point is the x-intercept of

   $$0 = |2x - 3|$$

   $$0 = 2x - 3$$

   $$\frac{3}{2} = x$$

   The corresponding function for the right side of the equation is $y = 7$. The graphs are shown below.



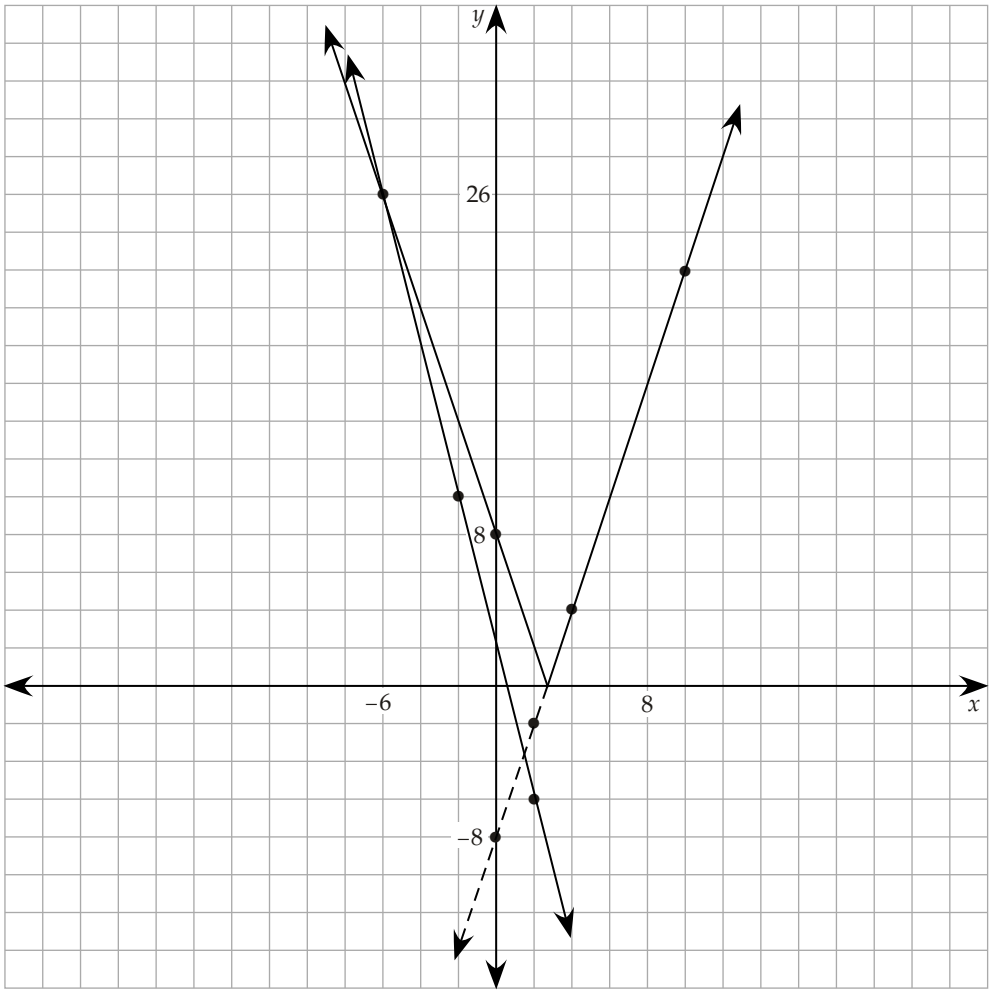
   The two points of intersection are $(-2, 7)$ and $(5, 7)$. The x-coordinates of these points are $x = -2$ and $x = 5$.

   Therefore, as in Question 1(a), the solutions to this absolute value equation are $x = -2$ and $x = 5$.

b) $|3x - 8| = 2 - 4x$

Answer:

Graph $y = |3x - 8|$ and $y = 2 - 4x$ on the same graph. Use a scale of 2 on both axes.



The x-value at which these two functions intersect is -6. As in Question 1(b), this absolute value equation has a solution of $x = -6$.

c) $|x^2 - 4x| = 4$

Answer:

The function for the left side is $y = |x^2 - 4x|$. The parabola $y = x^2 - 4x$ has zeros at

$0 = x^2 - 4x$

$0 = x(x - 4)$

$x = 0, 4$

The vertex for $y = x^2 - 4x$ is

$y = x^2 - 4x$

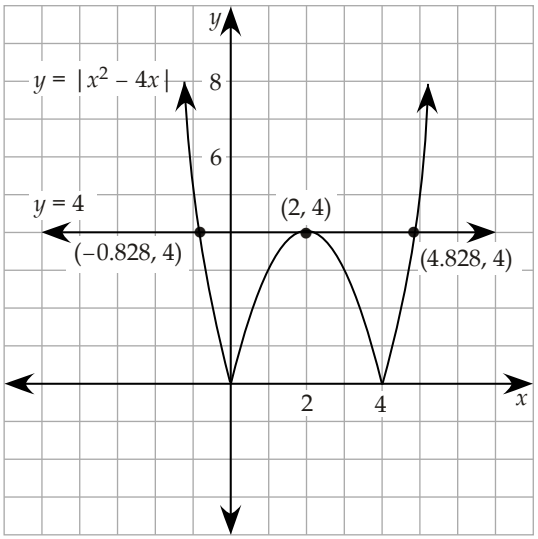
$y = (x^2 - 4x + 4) - 4$

$y = (x - 2)^2 - 4$

Vertex coordinates of $y = x^2 - 4x$ are $(2, -4)$.

Vertex coordinates of $y = |x^2 - 4x|$ are $(2, 4)$.

The graph of both functions, $y = |x^2 - 4x|$ and $y = 4$ are shown below:



To find the x-values for the two outside points, solve $(x^2 - 4x) = 4$ using the quadratic formula $x^2 - 4x - 4 = 0 \Rightarrow x = 4.828$ or $x = -0.828$.

The x-values at which these two functions intersect are $x = -0.828$, $x = 2$, and $x = 4.828$ and are thus the solutions to this absolute value equation, as shown algebraically in Question 1(c).

d) $|x^2 + 2x| = x + 30$

Answer:

The left side function is $y = |x^2 + 2x|$. The vertex of the parabola $y = x^2 + 2x$ is

$y = x^2 + 2x$

$y = (x^2 + 2x + 1) - 1$

$y = (x + 1)^2 - 1$

Vertex $(-1, -1)$

The zeros are

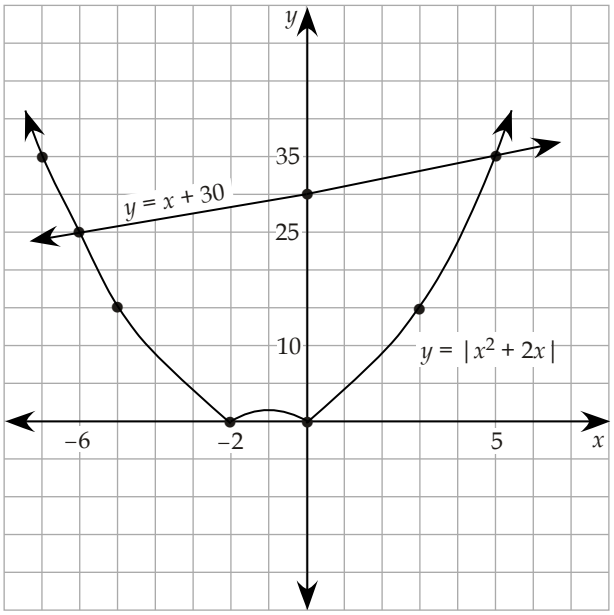
$0 = x^2 + 2x$

$0 = x(x + 2)$

$x = 0 \qquad x + 2 = 0$

$x = -2$

The line $y = x + 30$ on the right side of the equation has a y-intercept of 30 and slope of 1. Use a scale of 5 on the y-axis.



The x-values at which these two functions intersect are $x = -6$ and $x = 5$ and are thus the solutions to this absolute value equation, as shown algebraically in Question 1(d).

3. A manufacturer rejects boxes of cookies when the mass of the box differs from the stated mass of 325 g by more than 4.6 g.

   a) Write an absolute value equation that represents this situation.

   Answer:

   Let x represent an acceptable weight for a box of cookies, measured in grams.

   $|x - 325| = 4.6$ or $|325 - x| = 4.6$

   b) Solve the absolute value equation.

   Answer:

   $|x - 325| = 4.6$

   Case 1: $x - 325 = 4.5$ and $x - 325 \geq 0$

   $x = 329.6$ $x \geq 325$

   Since $x = 329.6$ is in the interval $x \geq 325$, it is a solution.

   Case 2: $-(x - 325) = 4.6$ and $x - 325 < 0$

   $-x + 325 = 4.6$ $x < 325$

   $-x = -320.4$

   $x = 320.4$

   Since $x = 320.4$ is in the interval $x < 325$, it is a solution.

   The solutions for the absolute value equation are $x = 320.4$ g and $x = 329.6$ g.

   c) State the range of acceptable values for the mass of a box of cookies.

   Answer:

   Thus, the greatest possible acceptable weight of the box of cookies is 329.6 g and the least possible acceptable weight of the box of cookies is 320.4 g.

4. Arrange the numbers from 0 to 9 into the boxes so that each absolute value equation is mathematically correct. You can only use each digit once.

Answer:

a) $2\left|x+\boxed{3}\right|=6$

$x=0,-\boxed{6}$

b) $\left|2x+7\right|=\boxed{0}$

$x=-\frac{\boxed{7}}{2}$

c) $\left|x^2-\boxed{2}x\right|=1$

$x=\boxed{1},\ 1+\sqrt{2},\ 1-\sqrt{2}$

d) $\left|x^2-2x-16\right|=\boxed{8}$

$x=-4,\ -2,\ \boxed{4},\ 6$

e) $\left|3x-\boxed{9}\right|=24$

$x=-\boxed{5},\ 11$

Solutions for each of the absolute value equations:

a) $2\left|x+3\right|=6$

$\left|x+3\right|=\frac{6}{2}$

$\left|x+3\right|=3$

Case 1: $x+3=3$ $\qquad$ $x+3\geq 0$

$x=0$ $\qquad$ $x\geq -3$

Case 2: $-(x+3)=3$ $\qquad$ $x+3<0$

$-x-3=3$ $\qquad$ $x<-3$

$-x=6$

$x=-6$

Solutions are $x=0$ and $x=-6$.

b) $\left|2x+7\right|=0$

$2x+7=0$

$2x=-7$

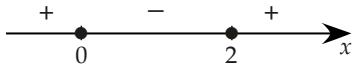
$x=\frac{-7}{2}$

c) $|x^2 - 2x| = 1$

Case 1: $x^2 - 2x = 1$ and $x^2 - 2x \geq 0$

$x^2 - 2x - 1 = 0$ $x(x-2) \geq 0$

$x = \dfrac{2 \pm \sqrt{4 - 4(1)(-1)}}{2(1)}$ $x = 0, 2$



$x = \dfrac{2 \pm \sqrt{8}}{2}$

Solution: $(-\infty, 0] \cup [2, \infty)$

$x = \dfrac{2 \pm 2\sqrt{2}}{2}$

$x = 1 \pm \sqrt{2}$

$x = 1 + \sqrt{2} \approx 2.4$

$x = 1 - \sqrt{2} \approx -0.4$

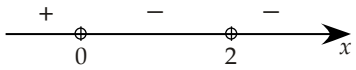
Thus, $x = 1 + \sqrt{2}$ is in the interval $[2, \infty)$ and $x = 1 - \sqrt{2}$ is in the interval $(-\infty, 0]$ and both are solutions.

Case 2: $-(x^2 - 2x) = 1$ and $x^2 - 2x < 0$

$-x^2 + 2x = 1$ $x(x-2) < 0$

$-x^2 + 2x - 1 = 0$ $x = 0, 2$

$x^2 - 2x + 1 = 0$



$(x-1)^2 = 0$

Solution: $(0, 2)$

$x = 1$

Since $x = 1$ is in the interval $(0, 2)$, it is a solution.

$\therefore$ The solutions are 1, $1 + \sqrt{2}$, and $1 - \sqrt{2}$.

d) $|x^2 - 2x - 16| = 8$

Case 1: $x^2 - 2x - 16 = 8$ and $x^2 - 2x - 16 \geq 0$

$x^2 - 2x - 24 = 0$

$(x - 6)(x + 4) = 0$

$x = 6, -4$

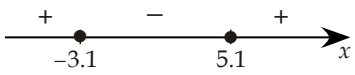
$$x = \frac{2 \pm \sqrt{4 - 4(1)(-16)}}{2(1)}$$

$$x = \frac{2 \pm \sqrt{68}}{2}$$

$$x = \frac{2 \pm 2\sqrt{17}}{2}$$

$x = 1 \pm \sqrt{17}$

$x \approx 5.1$ or -3.1



Approximate solution:

$(-\infty, -3.1] \cup [5.1, \infty)$

Since $x = 6$ and $x = -4$ are in the intervals, they are solutions.

Case 2: $-\left(x^2 - 2x - 16\right) = 8$ and $x^2 - 2x - 16 < 0$

$-x^2 + 2x + 16 = 8$

$-x^2 + 2x + 8 = 0$

$x^2 - 2x - 8 = 0$

$(x - 4)(x + 2) = 0$

$x = 4, -2$

Approximate solution:

$(-3.1, 5.1)$

Since both values $x = 4$ and $x = -2$ are in the interval, they are solutions.

$\therefore$ The solutions are 6, -4, 4, -2.

e) $|3x - 9| = 24$

Case 1: $3x - 9 = 24$ and $3x - 9 \geq 0$

$3x = 33$ $3x \geq 9$

$x = 11$ $x \geq 3$

Since $x = 11$ is in the interval $x \geq 3$, it is a solution.

Case 2: $-(3x - 9) = 24$ and $3x - 9 < 0$

$-3x + 9 = 24$ $3x < 9$

$-3x = 15$ $x < 3$

$x = -5$

Since $x = -5$ is in the interval $x < 3$, it is a solution.

Thus, $x = -5$ and $x = 11$ are the solutions.

5. Armani and Jennifer are both trying to solve the absolute value equation $|x - 3| = 2x + 1$. Armani's and Jennifer's work is shown below.

Armani's work:

Case 1: $|x - 3| = 2x + 1$

$x + 3 = 2x + 1$

$2 = x$

Case 2: $-x + 3 = 2x + 1$

$2 = 3x$

$x = \frac{2}{3}$

Jennifer's work:

Case 1: $|x - 3| = 2x + 1$

$x - 3 = 2x + 1$

$-4 = x$

Case 2: $-x + 3 = 2x + 1$

$2 = 3x$

$x = \frac{2}{3}$

a) Explain whose answer is incorrect.

Answer:

Armani's second equation is correct as she took the negative of the entire expression inside the absolute value symbols. However, the first equation is incorrect. An absolute value equation has possible solutions in two cases, when the expression inside the symbols is positive and when the expression inside the symbols is negative. However, Armani thought that you had to change all the signs to positive inside the absolute value brackets. This is incorrect. Also, she didn't write the interval restrictions for each case.

Jennifer's work is also incorrect. Jennifer solved the equation correctly. However, she forgot to check the intervals of the domain. If Jennifer were to set up her work using the definition of an absolute value, she would have discovered her answer of negative 4 wasn't possible.

Case 1: $x = -4$ as long as $x - 3 \geq 0$ or $x \geq 3$. The solution is not in the interval and is an extraneous solution.

Case 2: $x = \frac{2}{3}$ as long as $x - 3 < 0$ or $x < 3$. This value of x is in the interval $x < 3$ and is a solution.

$\therefore$ Jennifer needed to reject $x = -4$ and keep the only solution, $x = \frac{2}{3}$.

b) Explain how a person could have discovered that his or her work was incorrect.

Answer:

Armani and Jennifer could have discovered their work was incorrect by checking their answers.

Verifying the solutions of $x = 2, \frac{2}{3}$, and -4.

$| x - 3 | = 2x + 1$

Verify $x = 2$.

LHS	RHS
$\| x - 3 \|$	$2x + 1$
$\| 2 - 3 \|$	$2(2) + 1$
1	5

LHS $\neq$ RHS

$x = 2$ is not a solution.

Verify $x = \frac{2}{3}$.

LHS	RHS
$\lvert x - 3 \rvert$	$2x + 1$
$\left\lvert \frac{2}{3} - 3 \right\rvert$	$2\left(\frac{2}{3}\right) + 1$
$\left\lvert -2\frac{1}{3} \right\rvert$	$\frac{4}{3} + 1$
$2\frac{1}{3}$	$2\frac{1}{3}$

LHS = RHS

Verify $x = -4$.

LHS	RHS
$\lvert x - 3 \rvert$	$2x + 1$
$\lvert -4 - 3 \rvert$	$2(-4) + 1$
7	-7

LHS ≠ RHS

$x = -4$ is not a solution.

The only solution of the three is $x = \frac{2}{3}$. By using the domain interval restrictions, extraneous roots can be rejected. By verifying your answers, your calculations may be corrected.

6. Determine whether each statement is always true, sometimes true, or never true. Explain.

   a) The value of $\lvert x + 5 \rvert$ is greater than zero.

   Answer:

   This is sometimes true. The absolute value of any expression is always greater than *or equal to* zero.

b) The value of $|x + a|$, when a is a real number is greater than or equal to zero.

Answer:

This is always true because the absolute value of any expression is always greater than or equal to zero.

c) The value of $|5 + a| + a$ is greater than zero.

Answer:

This is sometimes true. The absolute value of any expression is always greater than or equal to zero. However, if a is a negative number, the expression $|5 + a| + a$ could be less than zero. For example, if $a = -3$, $|5 + (-3)| + (-3) = |2| - 3 = -1$, which is not a positive number.

7. Write an absolute value equation with the indicated solutions or type of solution.

a) -4

Answer:

Answers will vary.

A possible equation is

$$|x - 3| = 7$$

b) No solution

Answer:

Answers will vary. However, in order for an absolute value equation to have no solution, the absolute value expression has to be equal to a negative value. A possible equation is

$$|x + 24| = -3$$

c) One solution

Answer:

Answers will vary. However, answers must have only one solution and not two solutions.

An example is

$$|2x - 6| = 0$$

d) Two solutions

Answer:

Answers will vary. A possible equation is

$$|x + 1| = 5$$

Learning Activity 8.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. One-third of students in a class of 108 students have never failed a test. How many students have never failed a test?
2. There is a 50% discount on decorations after Halloween. If it costs you $52.44 to buy Halloween decorations before Halloween, how much will it cost you to buy those same decorations the day after Halloween?
3. Find the x-intercepts of the following parabola: $y = x^2 + 6x + 8$
4. Evaluate: $\frac{1}{4} + \frac{2}{7}$
5. State the reciprocal of $\frac{(x+3)(x-4)}{(x+4)(x-1)}$.
6. Express as a power: $\sqrt[2]{12^7}$
7. In which quadrant is an angle of 145° in standard position located?
8. Find the slope of the line defined by the equation $-5x = 2y + 9$.

Answers:

1. 36 (One-third of 99 = 33; one-third of 9 = 3; one-third of 108 = 33 + 3 = 36)
2. $26.22 ($52 ÷ 2 = $26 and 0.44 ÷ 2 = 0.22)
3. -2 and -4 ($(x + 2)(x + 4) = 0$)
4. $\frac{15}{28}\left(\frac{7}{28} + \frac{8}{28}\right)$
5. $\frac{(x+4)(x-1)}{(x+3)(x-4)}$
6. $12^{\frac{7}{2}}$
7. Quadrant II
8. $-\frac{5}{2}\left(2y = -5x - 9;\ y = -\frac{5}{2}x - \frac{9}{2}\right)$

Part B: Graphing Reciprocal Functions

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Write the reciprocal of each function.

   a) $y = x - 1$

   Answer:

   $$f(x) = \frac{1}{x - 1}$$

   b) $y = -3x + 6$

   Answer:

   $$f(x) = \frac{1}{-3x + 6}$$

   c) $y = x^2 + 4x - 2$

   Answer:

   $$f(x) = \frac{1}{x^2 + 4x - 2}$$

   d) $y = x^2 - 9$

   Answer:

   $$f(x) = \frac{1}{x^2 - 9}$$

2. For each of the following, identify the (i) x-intercepts of $f(x)$, (ii) the reciprocal function $y = \frac{1}{f(x)}$, (iii) the non-permissible values of the reciprocal function, and (iv) the equation(s) of the vertical asymptotes of the reciprocal function

a) $f(x) = 2x + 5$

Answer:

i) $0 = 2x + 5$

$-5 = 2x$

$x = -\frac{5}{2}$

ii) $y = \frac{1}{2x + 5}$

iii) The non-permissible value is $x = -\frac{5}{2}$.

iv) The equation of the vertical asymptote is $x = -\frac{5}{2}$.

b) $f(x) = x^2 - 9$

Answer:

i) $0 = x^2 - 9$

$0 = (x - 3)(x + 3)$

$x = 3$ or -3

ii) $y = \frac{1}{x^2 - 9}$

iii) The non-permissible values are $x = 3$ and $x = -3$.

iv) The equations of the vertical asymptotes are $x = 3$ and $x = -3$.

c) $f(x) = x^2 + 4x + 3$

Answer:

i) $0 = x^2 + 4x + 3$

$0 = (x + 3)(x + 1)$

$x = -3$ or -1

ii) $y = \frac{1}{x^2 + 4x + 3}$

iii) The non-permissible values are $x = -3$ and $x = -1$.

iv) The equations of the vertical asymptotes are $x = -3$ and $x = -1$.

3. Match each function with the graph of its reciprocal. Hint: Consider the asymptotes.

   a) $y = x^2 + x - 6$

   b) $y = -2$

   c) $y = x - 5$

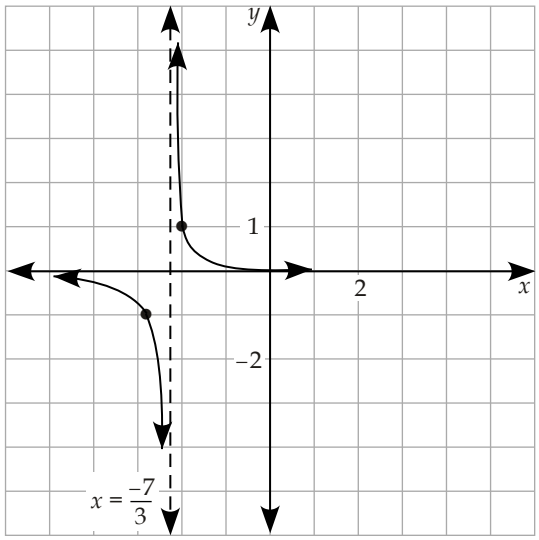
   d) $y = x^2 - 2x - 3$

   e) $y = x^2 - 6x + 9$

   f) $y = 3x + 7$

   g) $y = x^2 - 5x + 7$

i)



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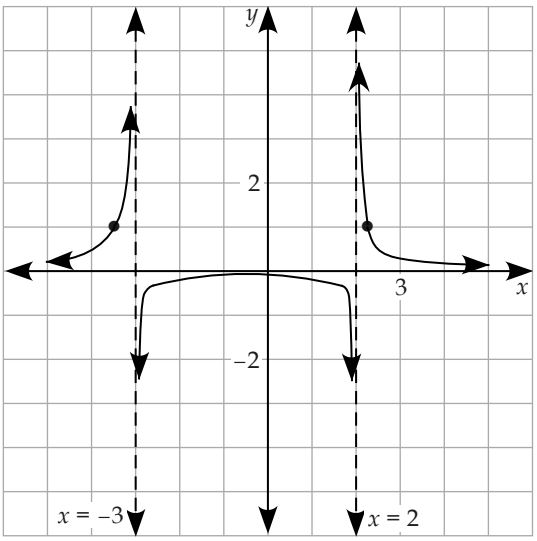

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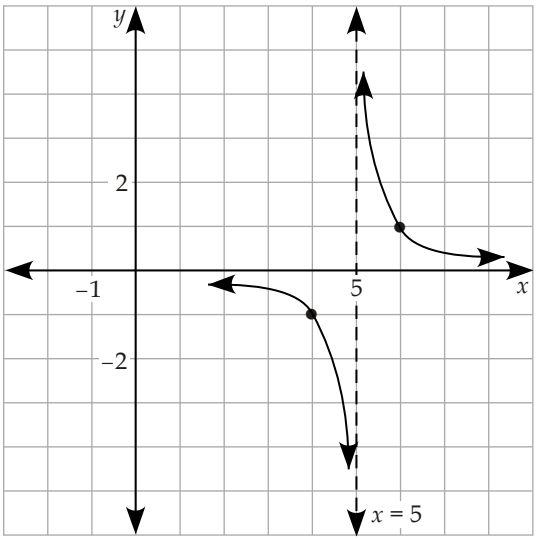

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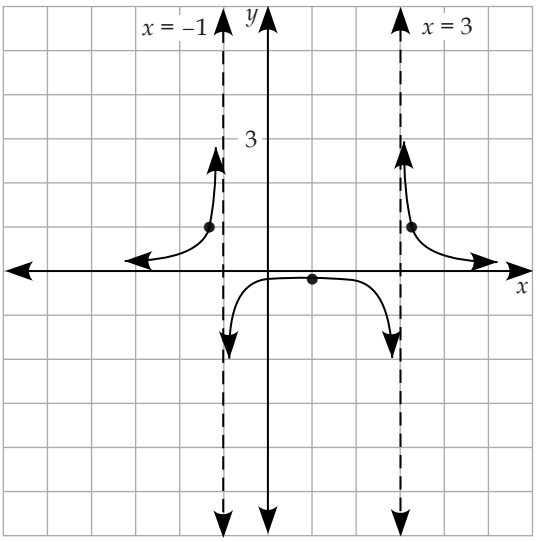

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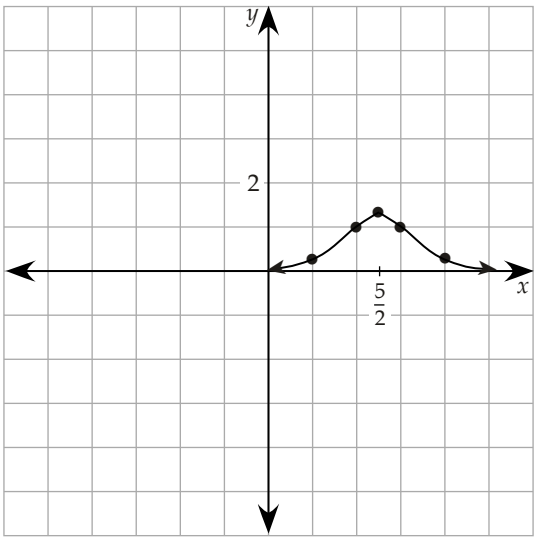

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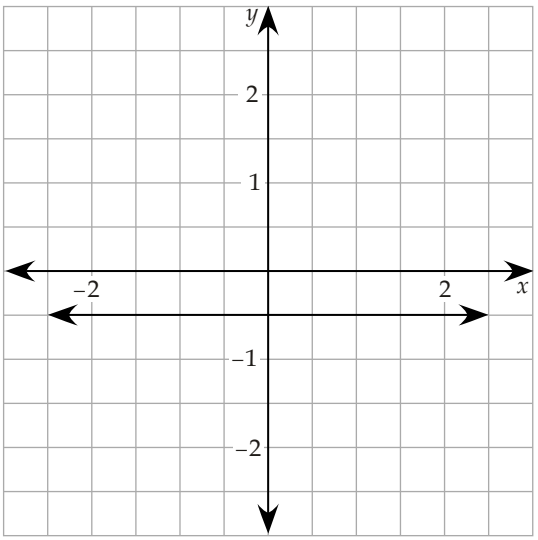
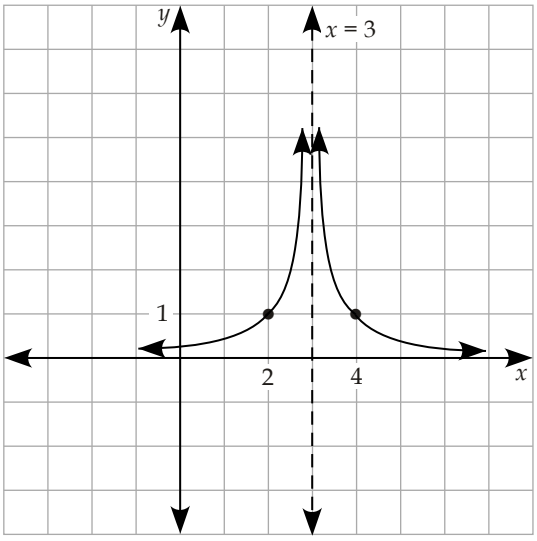


vii)



Answers:

a) (ii) Since $y = (x + 3)(x - 2)$, the vertical asymptotes are at $x = -3$ and $x = 2$.

b) (vi) Since $y = -2$, the reciprocal is a constant, $y = -\frac{1}{2}$.

c) (iii) Since $y = x - 5$, the vertical asymptote is at $x = 5$.

d) (iv) Since $y = (x - 3)(x + 1)$, the vertical asymptotes are at $x = 3$ and $x = -1$.

e) (vii) Since $y = (x - 3)(x - 3)$, only one vertical asymptote at $x = 3$.

f) (i) Since $y = 3x + 7$, the vertical asymptote is at $x = -\frac{7}{3}$.

g) (v) Since $y = x^2 - 5x + 7$ has a discriminate of $b^2 - 4ac = (-5)^2 - 4(1)(7) = -3$, which is negative, then the parabola has no zeros so its reciprocal will have no vertical asymptotes.

4. State the equation(s) of the vertical asymptote(s) for each function.

a) $y = \dfrac{1}{5x - 10}$

Answer:

$5x - 10 = 0$

$5x = 10$

$x = 2$

$\therefore$ Vertical asymptote is $x = 2$.

b) $y = \dfrac{1}{x - 4}$

Answer:

$x - 4 = 0$

$x = 4$

$\therefore$ Vertical asymptote is $x = 4$.

c) $y = \dfrac{1}{(x + 3)(x - 6)}$

Answer:

$(x + 3)(x - 6) = 0$

$x = -3$ or $x = 6$

$\therefore$ Vertical asymptotes are $x = -3$ and $x = 6$.

d) $y = \dfrac{1}{x^2 - x - 20}$

Answer:

$x^2 - x - 20 = 0$

$(x - 5)(x + 4) = 0$

$x = 5$ or $x = -4$

$\therefore$ Vertical asymptotes are $x = 5$ and $x = -4$.

5. Sketch the graph of each reciprocal function. State the y-intercept, the x-intercept(s), the invariant points, and the equation(s) of the horizontal and vertical asymptote(s), as well as the domain and range.

a) $f(x) = \dfrac{1}{x+3}$

b) $f(x) = \dfrac{1}{2x+8}$

c) $f(x) = \dfrac{1}{x^2+x-2}$

d) $f(x) = \dfrac{1}{x^2-2x-8}$

e) $f(x) = \dfrac{1}{x^2+8x+16}$

f) $f(x) = \dfrac{1}{x^2-2x+6}$

Answers:

a) $f(x) = \dfrac{1}{x+3}$

y-intercept: Find $f(0)$:

$$f(0) = \frac{1}{0+3} = \frac{1}{3}$$

Vertical asymptote: Let the denominator, $x + 3$, equal 0, find the zero, and draw the dotted vertical line through that zero.

$$x + 3 = 0$$
$$x = -3$$

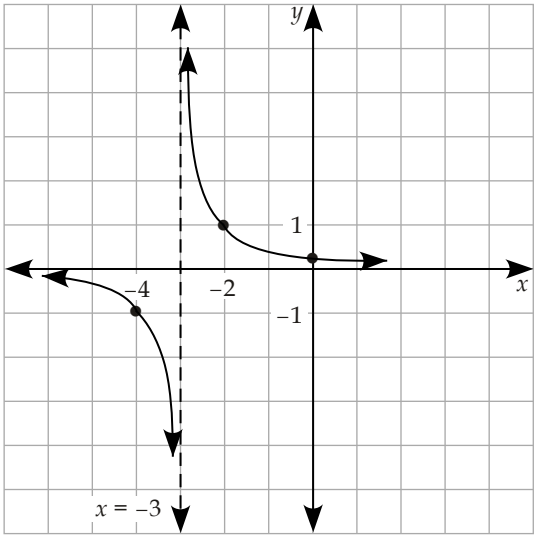
Horizontal asymptote: The horizontal asymptote is the line $y = 0$.

Invariant points: Set the denominator equal to 1 and -1.

$y = 1$	$y = -1$
$x + 3 = 1$	$x + 3 = -1$
$x = -2$	$x = -4$

Therefore, the two invariant points are $(-2, 1)$, and $(-4, -1)$.

Draw the curve: Plot the invariant points and the y-intercept. Draw the asymptotes and draw the curves.



Domain: $\{x \mid x \neq -3, x \in \Re\}$

Range: $\{y \mid y \neq 0, y \in \Re\}$

There is no x-intercept.

b) $f(x) = \dfrac{1}{2x + 8}$

Answer:

y-intercept: Find $f(0)$:

$$f(0) = \frac{1}{2(0) + 8} = \frac{1}{8}$$

Vertical asymptote: Let the denominator, $2x + 8$, equal 0, find the zero, and draw the dotted vertical line through that zero.

$$2x + 8 = 0$$
$$2x = -8$$
$$x = -4$$

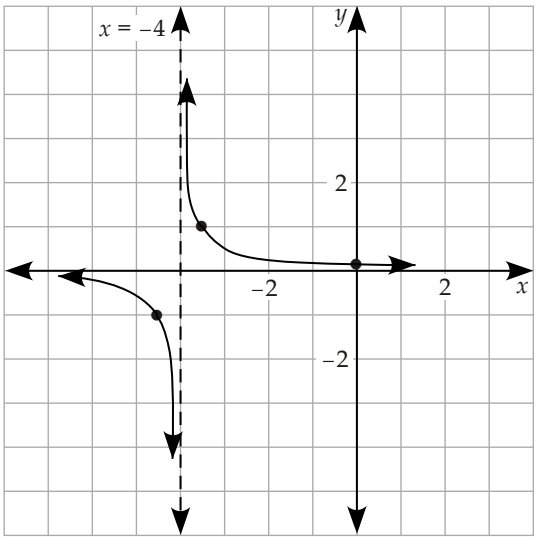
Horizontal asymptote: The horizontal asymptote is the line $y = 0$.

Invariant points: Set the denominator equal to 1 and -1.

$$y = 1 \qquad y = -1$$
$$2x + 8 = 1 \qquad 2x + 8 = -1$$
$$2x = -7 \qquad 2x = -9$$
$$x = -\frac{7}{2} \qquad x = -\frac{9}{2}$$

Therefore, the two invariant points are $\left(-\frac{7}{2}, 1\right)$ and $\left(-\frac{9}{2}, -1\right)$.

Draw the curve: Plot the invariant points and the y-intercept. Draw the asymptotes and draw the curves.



Domain: $\{x \mid x \neq -4, x \in \Re\}$

Range: $\{y \mid y \neq 0, y \in \Re\}$

There is no x-intercept.

c) $f(x) = \dfrac{1}{x^2 + x - 2}$

Answer:

y-intercept: Find $f(0)$:

$$f(0) = \frac{1}{0^2 + 0 - 2} = -\frac{1}{2}$$

Vertical asymptote: Let the denominator equal 0, find the zeros, and draw the dotted vertical asymptotes through those zeros.

$$x^2 + x - 2 = 0$$
$$(x + 2)(x - 1) = 0$$
$$x = -2,\ x = 1$$

Horizontal asymptote: The horizontal asymptote is the line $y = 0$.

Vertex of the original parabola:

$$y = x^2 + x - 2$$
$$y = \left(x^2 + x + \frac{1}{4}\right) - 2 - \frac{1}{4}$$
$$y = \left(x + \frac{1}{2}\right)^2 - \left(-\frac{9}{4}\right)$$

Vertex: $\left(-\dfrac{1}{2}, -\dfrac{9}{4}\right)$

In the reciprocal function, the maximum value in the interval between the vertical asymptotes will occur at the vertex where $x = -\dfrac{1}{2}$ and $y = \dfrac{1}{-\frac{9}{4}} = -\dfrac{4}{9}$.

Invariant points: Set the denominator equal to 1 and −1.

$y = 1$

$x^2 + x - 2 = 1$

$x^2 + x - 3 = 0$

Using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-3)}}{2}$$

$$x = \frac{-1 \pm \sqrt{1^2 + 12}}{2}$$

$$x = \frac{-1 + \sqrt{13}}{2} = 1.3$$

$$x = \frac{-1 - \sqrt{13}}{2} = -2.3$$

$y = -1$

$x^2 + x - 2 = -1$

$x^2 + x - 1 = 0$

Discriminant:

$$b^2 - 4ac = 1 - 4(1)(-1)$$
$$= 5$$

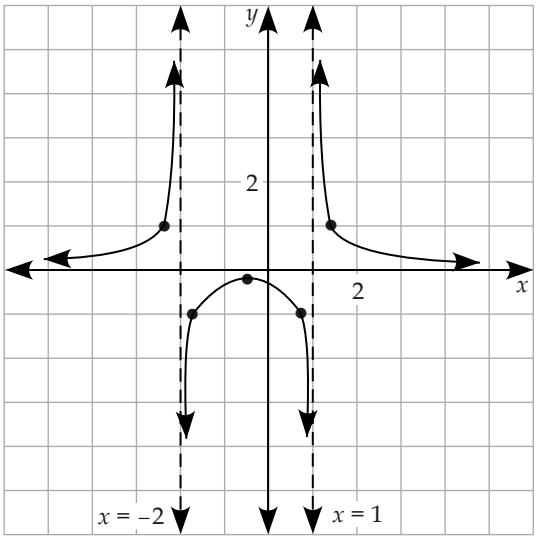
$$x = \frac{-1 \pm \sqrt{5}}{2}$$

$$x = \frac{-1 + \sqrt{5}}{2} = 0.6$$

$$x = \frac{-1 - \sqrt{5}}{2} = -1.6$$

Therefore, the four invariant points are (1.3, 1), (−2.3, 1), (0.6, −1), and (−1.6, −1).

Draw the curve: Plot the invariant points and the *y*-intercept. Draw the asymptotes and draw the curves.



The domain and range can be read from the graph:

Domain: $\{x \mid x \neq -2, 1, x \in \Re\}$

The range is $\left(-\infty, -\frac{4}{9}\right] \cup (0, \infty)$.

There is no x-intercept.

d) $f(x) = \dfrac{1}{x^2 - 2x - 8}$

Answer:

y-intercept: Find $f(0)$:

$$f(0) = \frac{1}{-8} = -\frac{1}{8}$$

Vertical asymptote: Let the denominator equal 0, find the zeros, and draw the dotted vertical asymptotes through those zeros.

$$x^2 - 2x - 8 = 0$$
$$(x - 4)(x + 2) = 0$$
$$x = 4, x = -2$$

Horizontal asymptote: The horizontal asymptote is the line $y = 0$.

Vertex of the parabola:

$$y = x^2 - 2x - 8$$
$$y = \left(x^2 - 2x + 1\right) - 8 - 1$$
$$y = (x - 1)^2 - 9$$

Vertex: $(1, -9)$

In the reciprocal function, the highest point in the interval between the vertical asymptotes is at the reciprocal of the vertex point where $x = 1$ and $y = \dfrac{1}{-9} = -\dfrac{1}{9}$.

Invariant points: Set the denominator equal to 1 and −1.

$y = 1$

$x^2 - 2x - 8 = 1$

$x^2 - 2x - 9 = 0$

Using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-9)}}{2}$$

$$x = \frac{2 \pm \sqrt{4 + 36}}{2}$$

$$x = \frac{2 \pm \sqrt{40}}{2}$$

$$x = \frac{2 \pm 2\sqrt{10}}{2}$$

$$x = 1 \pm \sqrt{10}$$

$y = -1$

$x^2 - 2x - 8 = -1$

$x^2 - 2x - 7 = 0$

Discriminant:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-7)}}{2}$$

$$x = \frac{2 \pm \sqrt{4 + 28}}{2}$$

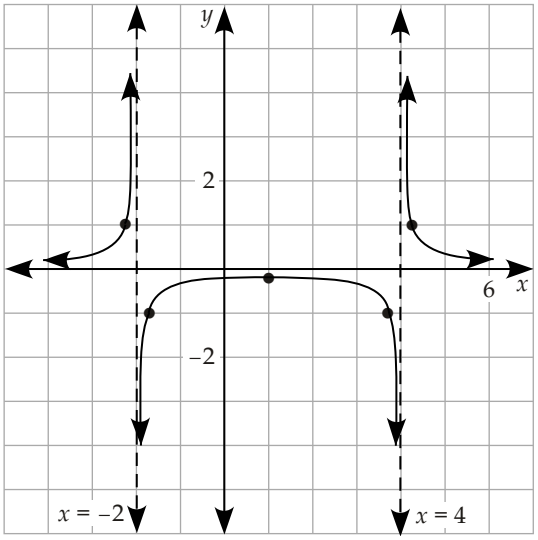
$$x = \frac{2 \pm \sqrt{32}}{2}$$

$$x = \frac{2 \pm 4\sqrt{2}}{2}$$

$$x = 1 \pm 2\sqrt{2}$$

Therefore, the four invariant points are $\left(1 + \sqrt{10}, 1\right)$, $\left(1 - \sqrt{10}, 1\right)$, $\left(1 + 2\sqrt{2}, -1\right)$, and $\left(1 - 2\sqrt{2}, -1\right)$. These points are approximately equal to (4.2, 1), (−2.2, 1), (3.8, −1), and (−1.8, −1).

Draw the curve: Plot the invariant points and the *y*-intercept. Draw the asymptotes and draw the curves.



The domain is $\{x \mid x \neq -2, 4, x \in \Re\}$.

The range is $\left(-\infty, -\frac{1}{9}\right] \cup (0, \infty)$.

There is no x-intercept.

e) $f(x) = \dfrac{1}{x^2 + 8x + 16}$

Answer:

y-intercept: Find $f(0)$:

$$f(0) = \frac{1}{0^2 + 8(0) + 16} = \frac{1}{16}$$

Vertical asymptote: The denominator will equal zero when $x^2 + 8x + 16 = 0$.

$x^2 + 8x + 16 = 0$

$(x + 4)(x + 4) = 0$

The vertical asymptote occurs at $x = -4$.

Horizontal asymptote: The horizontal asymptote is the line $y = 0$.

Vertex of parabola: The parabola is $y = x^2 + 8x + 16 = (x + 4)^2$. Its vertex is $(-4, 0)$. The corresponding point on the reciprocal function is $\left(-4, \frac{1}{0}\right)$, ,which does not exist. The vertical asymptote passes through the vertex of the original parabola since the vertex is on the x-axis.

Invariant points: Set the denominator equal to ± 1.

$y = 1$

$x^2 + 8x + 16 = 1$

$x^2 + 8x + 15 = 0$

$(x + 3)(x + 5) = 0$

$x = -3, -5$

$y = -1$

$x^2 + 8x + 16 = -1$

$x^2 + 8x + 17 = 0$

$b^2 - 4ac = 8^2 - 4(17)$

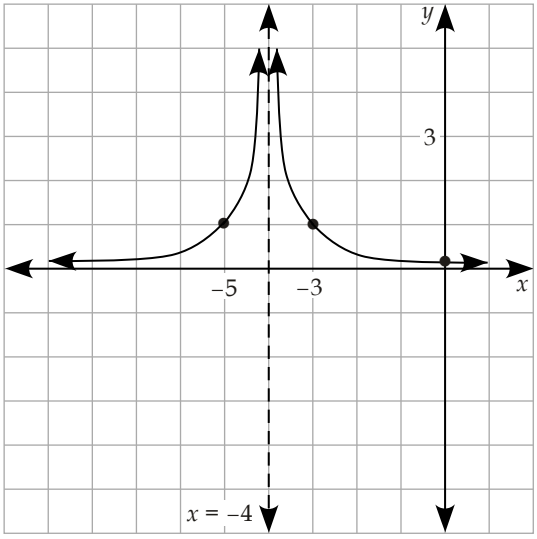
$= 64 - 68$

$= -4$

This equation has no solution as the discriminant is negative.

The two invariant points are $(-3, 1)$ and $(-5, 1)$.

Draw the curve: Plot the invariant points and the y-intercept. Draw the asymptotes and draw the curves.



The domain is $\{x \mid x \neq -4, x \in \Re\}$.

The range is $\{y \mid y > 0, y \in \Re\}$.

There is no x-intercept.

f) $f(x) = \dfrac{1}{x^2 - 2x + 6}$

Answer:

y-intercept: Find $f(0)$:

$$f(0) = \frac{1}{0^2 - 2(0) + 6} = \frac{1}{6}$$

Vertical asymptote: The denominator will equal zero when $x^2 - 2x + 6 = 0$.

$b^2 - 4ac = (-2)^2 - 4(1)(6)$

$= 4 - 24$

$= -20$

This equation has no solution as the discriminant is negative. Therefore, there are no vertical asymptotes.

Horizontal asymptote: The horizontal asymptote is the line $y = 0$.

Invariant points: Set the denominator equal to ±1.

$y = 1$	$y = -1$
$x^2 - 2x + 6 = 1$	$x^2 - 2x + 6 = -1$
$x^2 - 2x + 5 = 0$	$x^2 - 2x + 7 = 0$
$b^2 - 4ac = (-2)^2 - 4(5)$	$b^2 - 4ac = (-2)^2 - 4(7)$
$= 4 - 20$	$= 4 - 28$
$= -16$	$= -24$

Both of these equations have no solution as both discriminants are negative. Thus, there are no invariant points.

To help graph this reciprocal, find the coordinates of the vertex of the quadratic function located in the denominator of $f(x)$. Then, find the reciprocal of that point.

Vertex:

$$y = x^2 - 2x + 6$$

$$x = -\frac{b}{2a} = -\frac{-2}{2} = \frac{2}{2} = 1$$

Once you find the x-coordinate of the vertex, you can substitute that value back into the quadratic function to determine the y-coordinate of the vertex.

$$y = (1)^2 - 2(1) + 6 = 1 - 2 + 6 = 5$$

The vertex is at (1, 5).

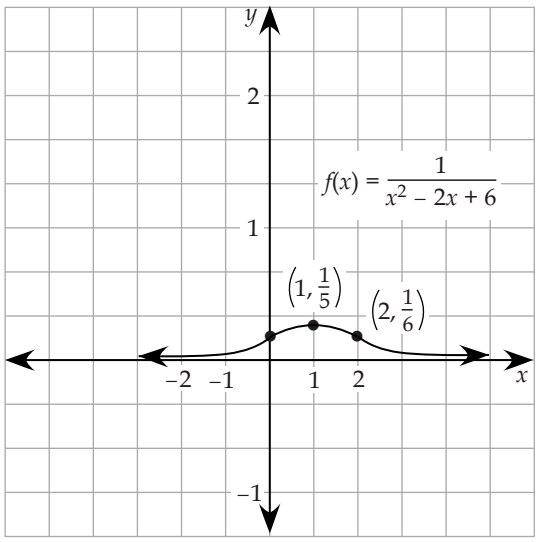
The corresponding point in the reciprocal function is $\left(1, \frac{1}{5}\right)$.

Now find points on each side of the vertex.

x	$y = x^2 - 2x + 6$	$y = \frac{1}{x^2 - 2x + 6}$
0	6	$\frac{1}{6}$
1	5	$\frac{1}{5}$
2	6	$\frac{1}{6}$

Plot the points of the reciprocal function and draw the curve smoothly toward the horizontal asymptote as x goes to infinity to the left and right.

Draw the curve:



The domain is $\{x \mid x \in \Re\}$.

The range is $\left\{y \mid 0 < y \leq \frac{1}{5}, y \in \Re\right\}$.

There is no x-intercept.

Learning Activity 8.5

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Eric has a big crush on a girl sitting 6 desks away from him. If each desk is 80 cm wide and there are no spaces between them, how far away (in metres) is Eric sitting from his crush?
2. Simplify: $(3^2)^3$
3. What is the slope of the line defined by the equation $3y + 2x - 7 = 0$?
4. If 36% of 1500 is 540, what is 18% of 1500?
5. State the reciprocal of. $\dfrac{x^2 + 3x - 4}{x^2 - 2x + 1}$.
6. Express as a power: $\sqrt[3]{16^3}$
7. (45, −72) is on a line with a slope of $\dfrac{20}{17}$. Find another point on this line.
8. Simplify the following fraction: $\dfrac{252}{462}$

Answers:

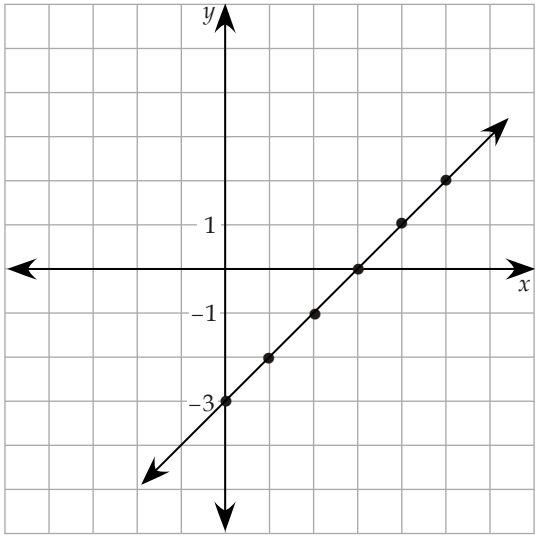
1. 4.8 m ($6 \times 80 = 480$ cm = 4.8 m)
2. $3^6 = 729$ ($9 \times 9 \times 9 = 81 \times 9$; $80 \times 9 = 720 + 1 \times 9 = 9$)
3. $-\dfrac{2}{3}$ $\left(3y = -2x + 7;\ y = -\dfrac{2}{3}x + \dfrac{7}{3}\right)$
4. 270 $\left(\dfrac{540}{2}\right)$
5. $\dfrac{x^2 - 2x + 1}{x^2 + 3x - 4}$
6. 16 $\left(\sqrt[3]{16^3} = 16^{\frac{3}{3}} = 16\right)$
7. (62, −52) (45 + 17, −72 + 20)
8. $\dfrac{6}{11}$ $\left(\dfrac{252}{462} = \dfrac{126}{231} = \dfrac{42}{77} = \dfrac{6}{11}\right)$

Part B: Reciprocal Functions

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Given the graph of $y = f(x)$, sketch the graph of $y = \frac{1}{f(x)}$. Describe the method you used.

a)

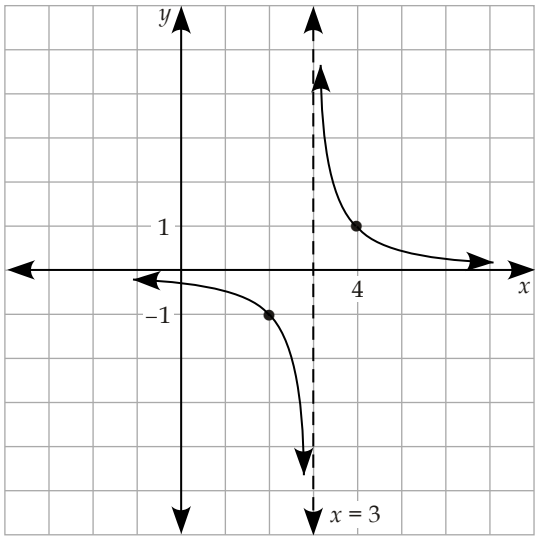


Answer:

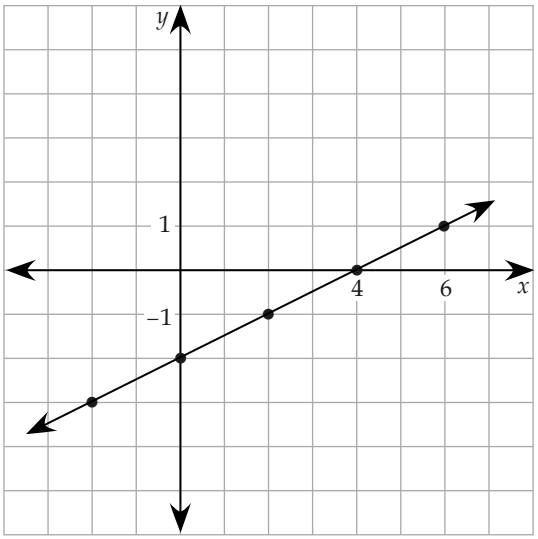
From the graph, you can see that the x-intercept is 3. Therefore, the vertical asymptote in the reciprocal function will be $x = 3$. Also, the two invariant points of this function will be (4, 1) and (2, −1), as these are the points at which the y-coordinate is either 1 or −1.

The horizontal asymptote is $y = 0$.

Using this information, you can graph the reciprocal function.



b)

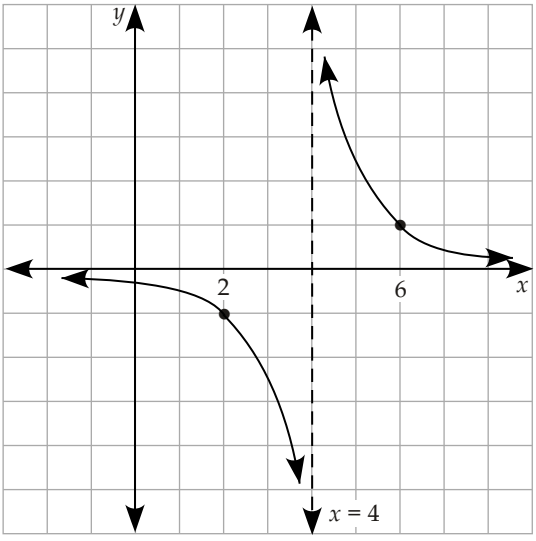


Answer:

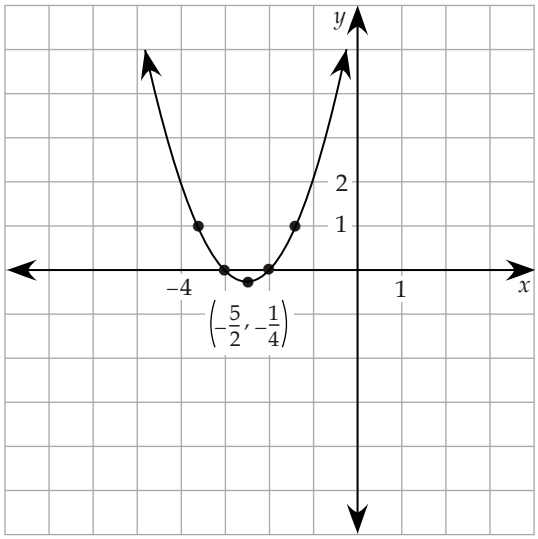
From the graph, you can see that the x-intercept is 4. Therefore, the vertical asymptote in the reciprocal function will be $x = 4$. Also, the two invariant points of this function will be (6, 1) and (2, −1), as these are the
points at which the y-coordinate is either 1 or −1.

The horizontal asymptote is $y = 0$.

Using this information, you can graph the reciprocal function.



c)



Answer:

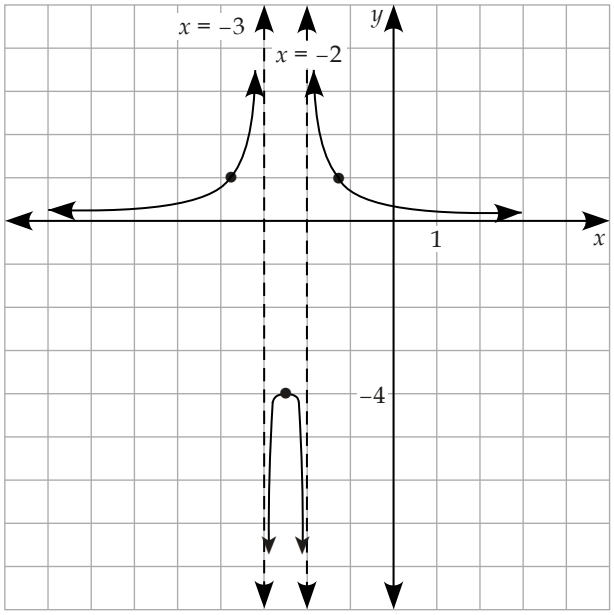
From the graph, you can see that the x-intercepts are -2 and -3. Therefore, the vertical asymptotes in the reciprocal function will have equations of $x = -2$ and $x = -3$.

The invariant points on the function are approximately at (−1.4, 1) and (−3.6, 1).

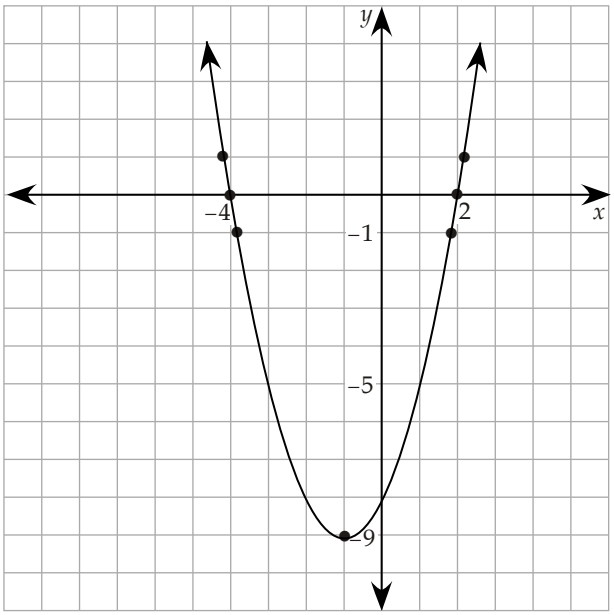
The key point of the parabola is at the vertex at $\left(-\frac{5}{2}, -\frac{1}{4}\right)$.

The corresponding point on the reciprocal function is $\left(-\frac{5}{2}, -4\right)$.

Using this information, you can graph the reciprocal function.



d)



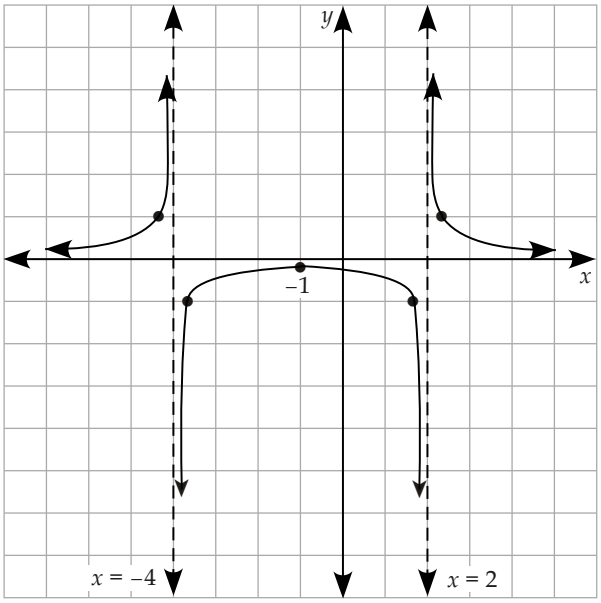
Answer:

From the graph, you can see that the x-intercepts are 2 and -4. Therefore, the vertical asymptotes in the reciprocal function will have equations of $x = 2$ and $x = -4$.

The vertex of the parabola is at $(-1, -9)$. The corresponding point on the reciprocal function is $\left(-1, -\frac{1}{9}\right)$.

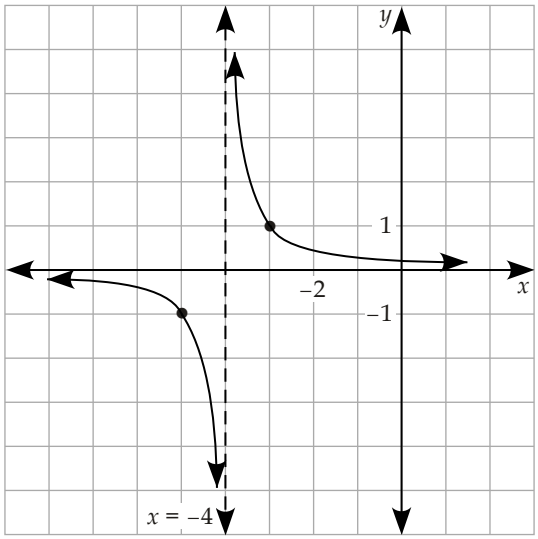
The invariant points are approximately at $(-4.21, 1)$, $(2.2, 1)$ and $(-3.8, -1)$, $(1.8, -1)$.

Using this information, you can graph the reciprocal function.



2. Given the graph of $y = \frac{1}{f(x)}$, sketch the graph of $y = f(x)$. Describe the method you used.

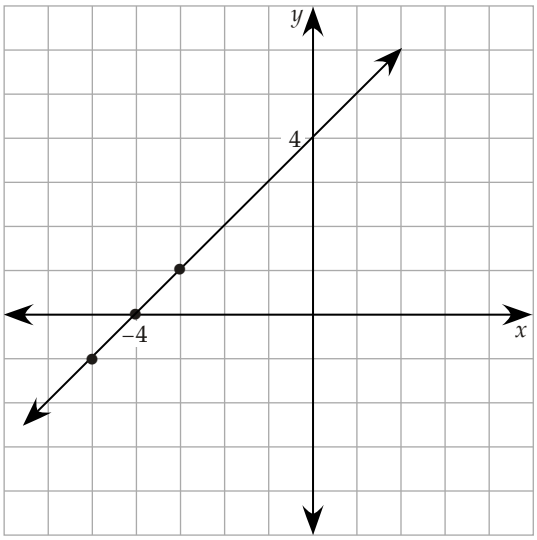
a)



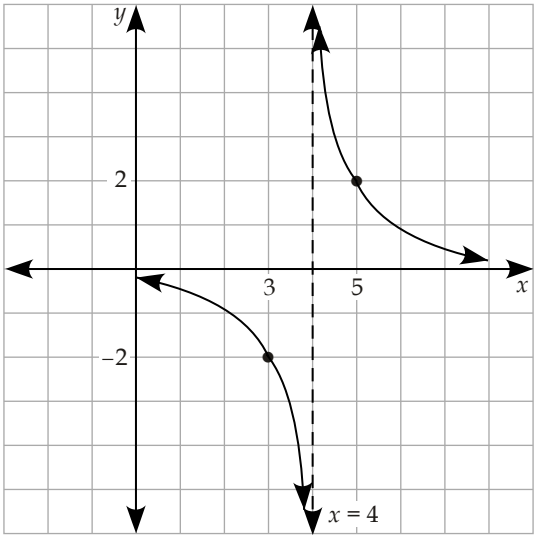
Answer:

The vertical asymptote of this reciprocal graph is at $x = -4$. Therefore, there will be an x-intercept on $f(x)$ as $(-4, 0)$.

There are also two points given: $(-3, 1)$ and $(-5, -1)$. These are invariant points and therefore they stay the same on the graph of $f(x)$. Using these three pieces of information, it is possible to graph $f(x)$.



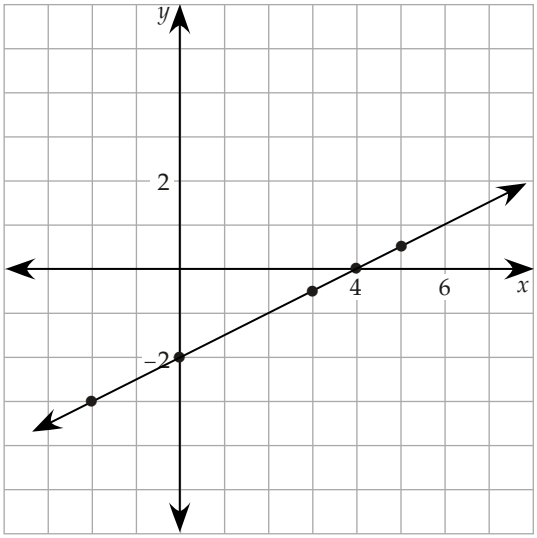
b)



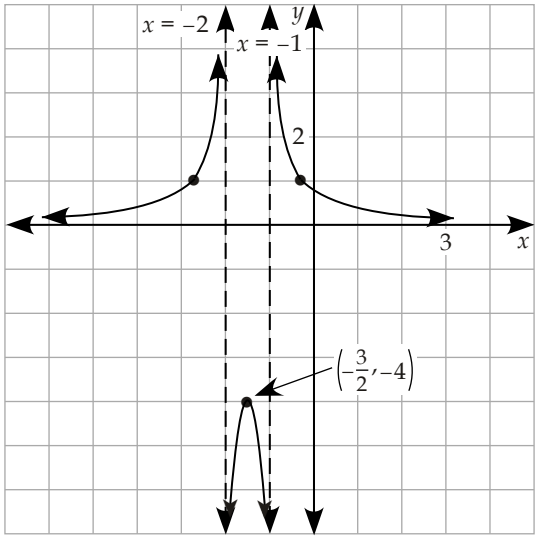
Answer:

The vertical asymptote of this reciprocal graph is at $x = 4$. Therefore, there will be an x-intercept on $f(x)$ as (4, 0).

There are also two points given: (5, 2) and (3, −2). The points on the reciprocal of the function are $\left(5, \frac{1}{2}\right)$ and $\left(3, -\frac{1}{2}\right)$. Using these three pieces of information, it is possible to graph $f(x)$.



c)

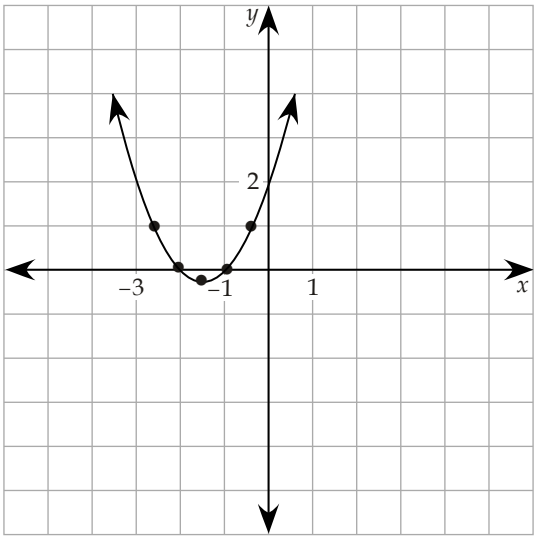


Answer:

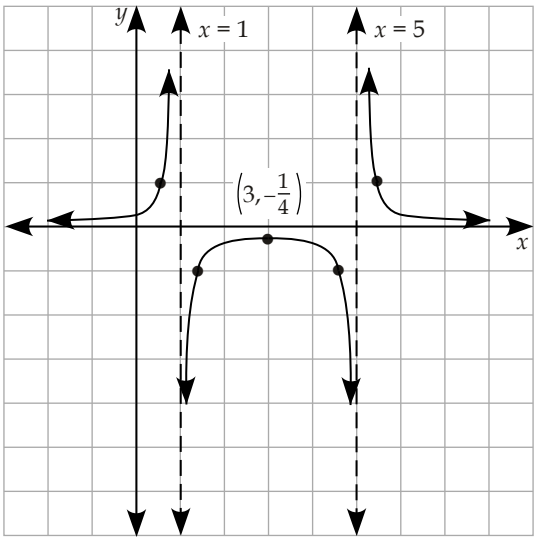
The vertical asymptotes of this reciprocal graph occur at $x = -1$ and $x = -2$ and are thus the x-intercepts in the graph of $f(x)$. In other words, $f(x)$ has the form of $a(x + 1)(x + 2)$.

The one given point is $\left(-\frac{3}{2}, -4\right)$. The corresponding point on the reciprocal of the function is $\left(-\frac{3}{2}, -\frac{1}{4}\right)$ and it will be the vertex of the parabola.

The invariant points are approximately at (−2.6, 1) and (−0.4, 1).



d)



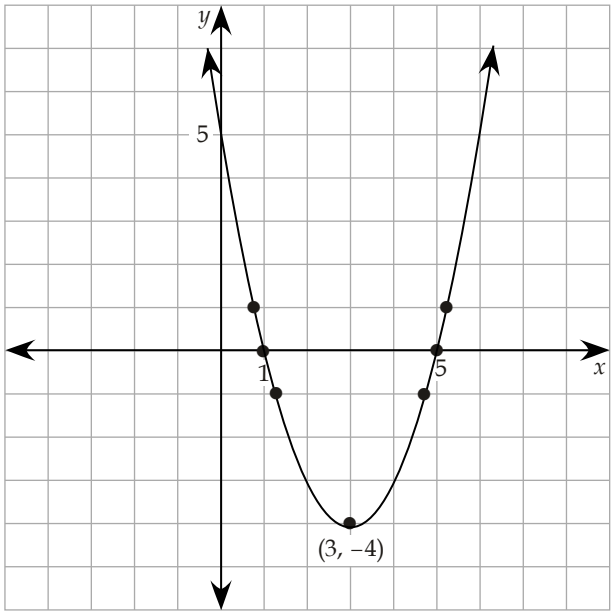
Answer:

The vertical asymptotes of this graph occur at $x = 1$ and $x = 5$ and are thus the x-intercepts in the graph of $f(x)$. In other words, $f(x)$ is of the form $a(x - 1)(x - 5)$ or $f(x) = a(x - 1)(x - 5)$.

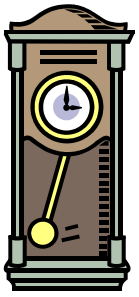
The key point is $\left(3, -\frac{1}{4}\right)$. The corresponding point on $f(x)$ is $(3, -4)$.

It is the vertex of the parabola.

Reading from the graph, the invariant points are approximately at (0.8, 1), (5.2, 1) and (1.3, −1), (4.7, −1).



3. The movement of a pendulum in a grandfather clock can be represented using a reciprocal formula. The frequency, f, is equal to the reciprocal of the period, T. The period is measured in seconds, and measures the time for one cycle, or, in other words, how long it takes the pendulum to swing back and forth once. The frequency is the number of cycles per second.



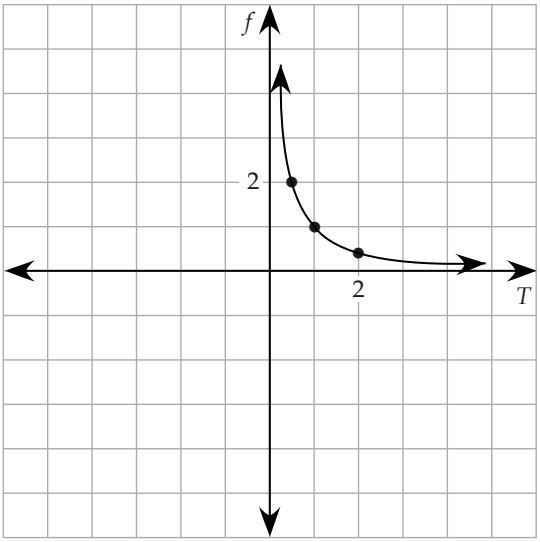
a) Write the formula for the frequency of a pendulum.

Answer:

$$f = \frac{1}{T}$$

b) Sketch the graph of the function in (a).

Answer:



c) Using the graph as well as the formula in (a), determine the period of a pendulum when the frequency is 2.1 cycles per second.

Answer:

Using the graph, the period of a pendulum with a frequency of 2.1 cycles per second is approximately 0.4 s.

Using the formula in (a): $f = \frac{1}{T}$

$$2.1 = \frac{1}{T}$$

$$T = \frac{1}{2.1}$$

$$T = 0.4762 \text{ seconds}$$

Grade 11 Pre-Calculus Mathematics (30S)

Module 8
Absolute Value and Reciprocal Functions

Learning Activity Answer Keys

MODULE 8:
ABSOLUTE VALUE AND RECIPROCAL FUNCTIONS

Learning Activity 8.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. In the three courses you are taking this semester, you receive final marks of 72%, 83%, and 91%. What is your average?
2. Find the equation of a line with a slope of 2 and a y-intercept of -6.
3. Evaluate and express as a mixed fraction: $\frac{5}{6} + \frac{7}{4}$
4. You have a job at an electronics store. As a bonus for excellent sales this month, your employer offers you either $\frac{1}{20}$ or 7% of your annual salary. Which one is the better choice?
5. Evaluate: 2^6
6. Estimate: $\sqrt{220}$
7. Are these fractions equivalent? $\frac{3}{\sqrt{2}-4}$ and $\frac{3\sqrt{2}+12}{18}$
8. In which quadrant is an angle of 294° located?

Answers:

1. 82% $\left(72 + 83 + 91 = 246; \frac{246}{3} = 82\%\right)$
2. $y = 2x - 6$
3. $2\frac{7}{12}$ $\left(\frac{5}{6} + \frac{7}{4} = \frac{10}{12} + \frac{21}{12} = \frac{31}{12} = 2\frac{7}{12}\right)$
4. 7% is the better choice $\left(\frac{1}{20} = \frac{5}{100}; 7\% = \frac{7}{100}\right)$

5. 64 ($2 \times 2 \times 2 \times 2 \times 2 \times 2$)

6. $\cong 14.8$ (220 is between 196 and 225; therefore, $\sqrt{220}$ is between 14 and 15)

7. No $\left(\dfrac{3}{\sqrt{2}-4} = \dfrac{3\left(\sqrt{2}+4\right)}{\left(\sqrt{2}-4\right)\left(\sqrt{2}+4\right)} = \dfrac{3\sqrt{2}+12}{-14} \neq \dfrac{3\sqrt{2}+12}{18} \right)$

8. Quadrant IV

Part B: Absolute Value

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

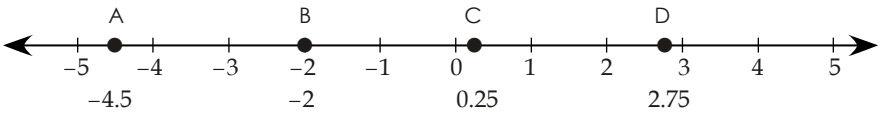
1. Evaluate.
   a) $|-13|$
   b) $|4|$
   c) $|0|$
   d) $|-(52)|$
   e) $2|5-3(2)|+1$
   f) $3|4(3+2)^2-42|-2$

   Answers:

   a) $|-13| = -(-13) = 13$
   b) $|4| = 4$
   c) $|0| = 0$
   d) $|-(52)| = -(-52) = 52$
   e) $2|5-3(2)|+1$
   $= 2|5-6|+1$
   $= 2|-1|+1$
   $= 2[-(-1)]+1$
   $= 2(1)+1$
   $= 2+1$
   $= 3$

f) $3|\,4(3+2)^2 - 42\,| - 2$
$= 3|\,4(5)^2 - 42\,| - 2$
$= 3|\,4(25) - 42\,| - 2$
$= 3|\,100 - 42\,| - 2$
$= 3|\,58\,| - 2$
$= 3(58) - 2$
$= 174 - 2$
$= 172$

2. Use absolute value symbols to write an expression for the distance between each pair of points on the number line. Using this expression, determine the distance between the two points.



a) A and D

Answer:

The distance between A and D can be represented by the absolute value expression $|\,-4.5 - 2.75\,|$ and can be calculated as

$|\,-7.25\,| = -(-7.25) = 7.25.$

b) B and D

Answer:

The distance between B and D can be represented by the absolute value expression $|\,-2 - 2.75\,|$ and can be calculated as

$|\,-4.75\,| = -(-4.75) = 4.75$

c) C and B

Answer:

The distance between C and B can be represented by the absolute value expression $|\,0.25 - (-2)\,|$ and can be calculated as

$|\,0.25 + 2\,| = |\,2.25\,| = 2.25$

3. Describe how distance can be related to absolute value.

Answer:

There are two ways of answering this question. From Question 2, you can see that distances from positive to negative values, as well as distance from negative to positive values, are all positive. This is because distance can be understood as the absolute value of a number. Or, the absolute value of a number can be understood as its distance away from 0.

4. Determine the distance between each of the following sets of points. Include a graph for (b), (c), and (d).

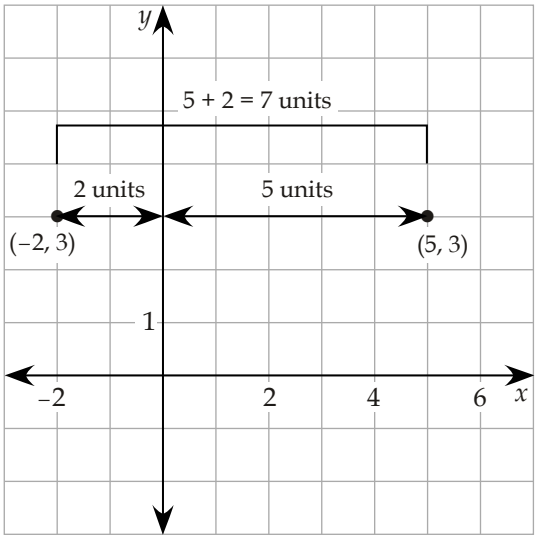
   a) -45 and -23

   Answer:

   $| -45 - (-23) | = | -45 + 23 | = | -22 | = -(-22) = 22$

   b) $(5, 3)$ and $(-2, 3)$

   Answer:

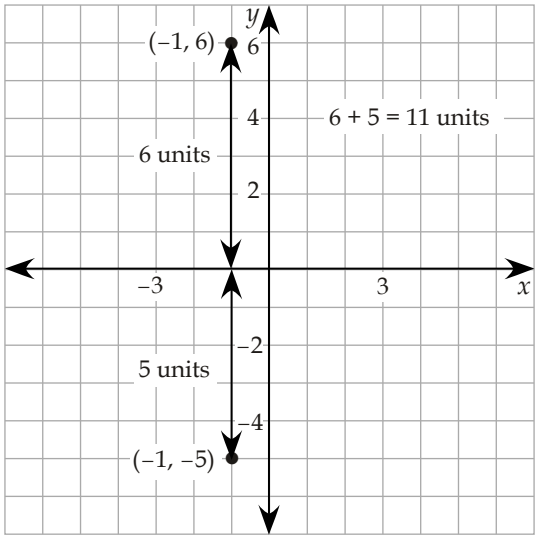


   These points on the coordinate graph have the same y-coordinates and are 7 units apart. This can also be symbolized by subtraction and absolute value:

   $| 5 - (-2) | = | 5 + 2 | = | 7 | = 7$

   c) $(-1, 6)$ and $(-1, -5)$

   Answer:

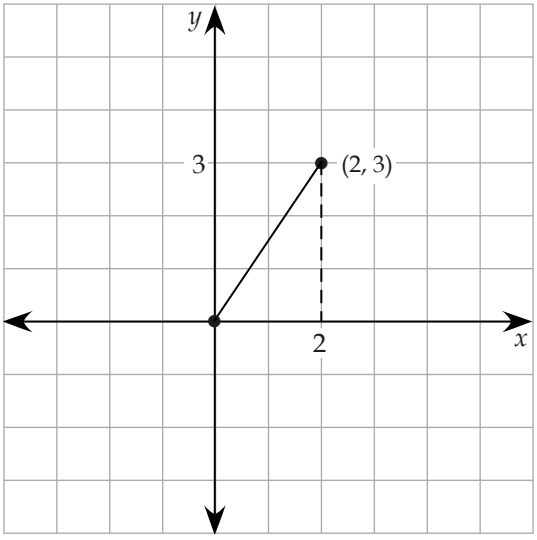


These points on the coordinate graph have the same x-coordinates and are 11 units apart. This can also be symbolized by subtraction and absolute value:

$| 6 - (-5) | = | 6 + 5 | = | 11 | = 11$

d) An extension of finding absolute value on the coordinate graph is to find the distance of a point from the origin—in other words, its absolute value. Find the absolute value for the point (2, 3).

Answer:



The distance can be found by using Pythagorean theorem:

$d = \sqrt{2^2 + 3^2}$

$d = \sqrt{13}$

Since d represents the hypotenuse of a right triangle, its value must be positive and only the principal square root of 13 is considered.

5. Order the following set of absolute values from least to greatest.

$$\left|-\frac{1}{4}\right|,\ \frac{1}{3},\ -3,\ |-3|,\ \left|\frac{2}{9}\right|,\ \left|\frac{3}{2}\right|,\ \frac{1}{2},\ -\frac{7}{3}$$

Answer:

$$\left|-\frac{1}{4}\right| = \frac{1}{4} = 0.25$$

$$\frac{1}{3} = 0.\bar{3}$$

$$-3 = -3$$

$$|-3| = -(-3) = 3$$

$$\left|\frac{2}{9}\right| = \frac{2}{9} = 0.\bar{2}$$

$$\left|\frac{3}{2}\right| = \frac{3}{2} = 1.5$$

$$\frac{1}{2} = 0.5$$

$$-\frac{7}{3} = -2.\bar{3}$$

The order from least to greatest is

$$-3 < -2.\bar{3} < 0.\bar{2} < 0.25 < 0.\bar{3} < 0.5 < 1.5 < 3$$

$$-3 < -\frac{7}{3} < \left|\frac{2}{9}\right| < \left|-\frac{1}{4}\right| < \frac{1}{3} < \frac{1}{2} < \left|\frac{3}{2}\right| < |-3|$$

6. Evaluate the following expressions for $x = -2$.

a) $3|\ -2x - 5\ |\ + 4$

Answer:

$$\begin{aligned}
&3|\ -2x - 5\ |\ + 4 \\
&= 3|\ -2(-2) - 5\ |\ + 4 \\
&= 3|\ 4 - 5\ |\ + 4 \\
&= 3|\ -1\ |\ + 4 \\
&= 3[-1(-1)] + 4 \\
&= 3(1) + 4 \\
&= 3 + 4 \\
&= 7
\end{aligned}$$

b) $3|\, x^2 - 7x + 2\,| - 5$

Answer:

$3|\, x^2 - 7x + 2\,| - 5$
$= 3|\, (-2)^2 - 7(-2) + 2\,| - 5$
$= 3|\, 4 + 14 + 2\,| - 5$
$= 3|\, 20\,| - 5$
$= 3(20) - 5$
$= 60 - 5$
$= 55$

7. Consider the expression $\frac{x}{|x|}$.

a) What is the value of $\frac{x}{|x|}$ when x is positive?

Answer:

Since $x > 0$, the denominator, $|\, x\,|$, will be positive and the numerator will be the same value. The expression becomes $\frac{x}{x}$, which equals 1.

b) What is the value of $\frac{x}{|x|}$ when x is negative?

Answer:

When $x < 0$, the denominator, $|\, x\,|$, will be positive and the numerator will be negative. Therefore, the expression will be negative and will equal -1.

8. In Manitoba, the temperature could go as low as $-40°$ C in the winter. In the summer, the temperature may reach a high of 30° C. Determine the difference in the range of these two temperatures.

Answer:

$|\, -40 - 30\,| = |\, -70\,| = -(-70) = 70$

Therefore, the range in Manitoba's temperatures is 70°C.

9. Charlize works in an office building and uses an electronic key card in order to travel to different floors of the building by elevator. This electronic key card tracks her movements throughout the day. The floors Charlize travelled to throughout one day are displayed in the chart below.

Time	Floor
8 a.m.	1
9:13 a.m.	4
11:41 a.m.	2
12:51 p.m.	7
2:00 p.m.	4
2:10 p.m.	6
2:30 p.m.	4
4 p.m.	1

a) Write and evaluate an absolute value expression to determine the total amount of floor changes throughout Charlize's day.

Answer:

$|4-1|+|2-4|+|7-2|+|4-7|+|6-4|+|4-6|+|1-4|$

$=|3|+|-2|+|5|+|-3|+|2|+|-2|+|-3|$

$=3+2+5+3+2+2+3$

$=20$

b) Write and evaluate an absolute value expression to determine the net change in Charlize's location from the beginning of the day to the end of the day.

Answer:

$|1-1|=|0|=0$

c) Compare the value you calculated in (a) to the value you found in (b). How does the total change compare to the net change of Charlize's location?

Answer:

The net change is the change from the beginning of the day to the end of the day. The total change is the sum of all the changes throughout the day. Therefore, it may look like Charlize didn't change floors at all throughout the day; however, she really travelled through 20 different floors. This is a significant difference.

Learning Activity 8.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. If you get paid $11 an hour and you work 30 hours in a week, how much will you get paid (before taxes)?
2. If you travel at a speed of 72 km/h for 6 hours, how far will you travel?
3. State the reciprocal of $\frac{5}{7}$.
4. Calculate 15% of 140.
5. Find the equation of a line with a slope of 3 that crosses through the point (1, 1).
6. Express in exponential form: $\left(\sqrt{5}\right)\left(\sqrt[3]{5}\right)$
7. Determine the distance between the following two points: (7, −1) and (−8, 15)
8. How many feet are in 24 yards?

Answers:

1. $330 ($30 \times 11$)
2. 432 km ($72 \times 6 = 70 \times 6 + 2 \times 6 = 420 + 12$)
3. $\frac{7}{5}$
4. 21 (10% of 140 = 14; 5% of 140 = 7; 15% of 140 = 14 + 7 = 21)
5. $y = 3x - 2$ ($y = mx + b$; $1 = 3(1) + b$; $1 - 3 = b$; $-2 = b$)
6. $5^{\frac{5}{6}}$ $\left(\left(\sqrt{5}\right)\left(\sqrt[3]{5}\right) = 5^{\frac{1}{2}}5^{\frac{1}{3}} = 5^{\frac{1}{2}+\frac{1}{3}} = 5^{\frac{3}{6}+\frac{2}{6}} = 5^{\frac{5}{6}}\right)$
7. $\sqrt{481}$ $\left(d = \sqrt{(-8-7)^2 + (15-(-1))^2} = \sqrt{(-15)^2 + (16)^2} = \sqrt{225+256}\right)$
8. 72 (There are 3 feet in 1 yard and $3 \times 24 = 72$ feet in 24 yards.)

Part B: Graphing Absolute Value Functions

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Sketch each function and state the values of the intercepts.

   a) $f(x) = |2x - 1|$

   Answer:

   There are two graphs to draw:

   $f(x) = 2x - 1$ when $2x - 1 \geq 0$ or $x \geq \frac{1}{2}$

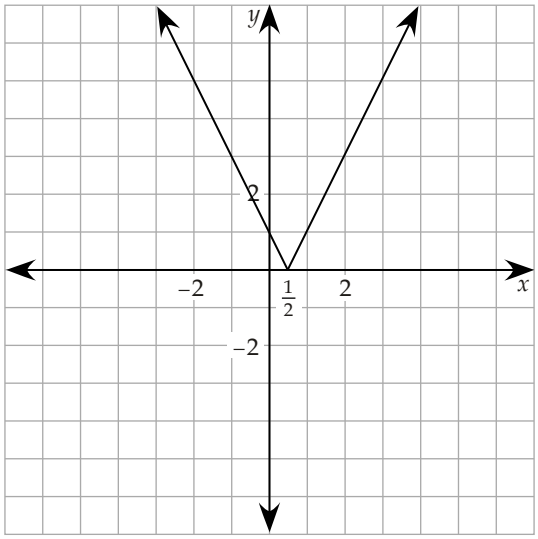
   and $f(x) = -(2x - 1)$ when $2x - 1 < 0$ or $x < \frac{1}{2}$

   The pivot point is the x-intercept.

   $$f(x) = |2x - 1|$$
   $$0 = |2x - 1|$$
   $$0 = 2x - 1$$
   $$\frac{1}{2} = x$$

   The y-intercept is

   $$f(x) = |2x - 1|$$
   $$f(0) = |2(0) - 1|$$
   $$f(0) = |-1|$$
   $$f(0) = 1$$



b) $g(x) = |x + 3|$

Answer:

The piece-wise definition is

$$g(x) = \begin{cases} x + 3, & x + 3 \geq 0 \\ -(x + 3), & x + 3 < 0 \end{cases}$$

Simplify to get

$$g(x) = \begin{cases} x + 3, & x \geq -3 \\ -x - 3, & x < -3 \end{cases}$$

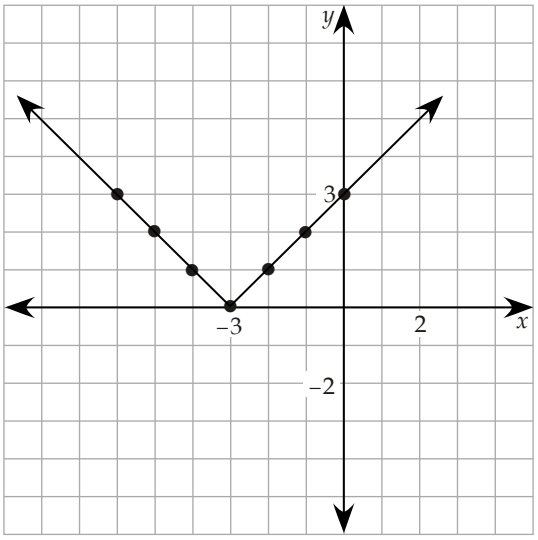
The pivot-point is $x = -3$, which is also the x-intercept.

The y-intercept is

$g(x) = |x + 3|$

$g(0) = |0 + 3|$

$g(0) = 3$



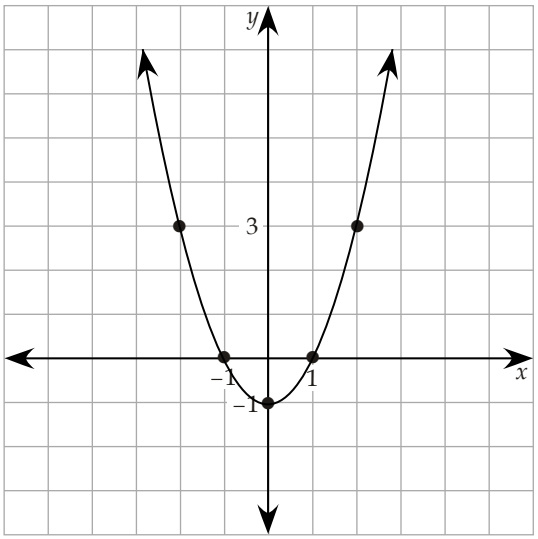
c) $h(x) = |x^2 - 1|$

Answer:

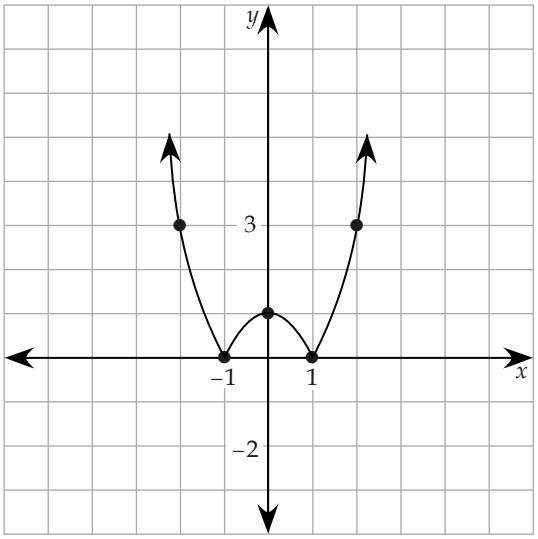
The piece-wise definition is

$$h(x) = \begin{cases} x^2 - 1, & x^2 - 1 \geq 0 \\ -\left(x^2 - 1\right), & x^2 - 1 < 0 \end{cases}$$

Using a graphing transformations method, first graph $y = x^2 - 1$ with vertex (0, 1), opening up and standard size.



Using the absolute value transformation, reflect any points with negative y-values over the x-axis. This would be the portion of the graph in the domain, $-1 < x < 1$.



The x-intercepts are ±1 and the y-intercept is 1.

d) $j(x) = |-x^2 + 2x + 3|$

Answer:

The piece-wise definition is

$$j(x) = \begin{cases} -x^2 + 2x + 3, & -x^2 + 2x + 3 \geq 0 \\ -\left(-x^2 + 2x + 3\right), & -x^2 + 2x + 3 < 0 \end{cases}$$

To sketch the graph of $y = -x^2 + 2x + 3$, first complete the square to find the vertex:

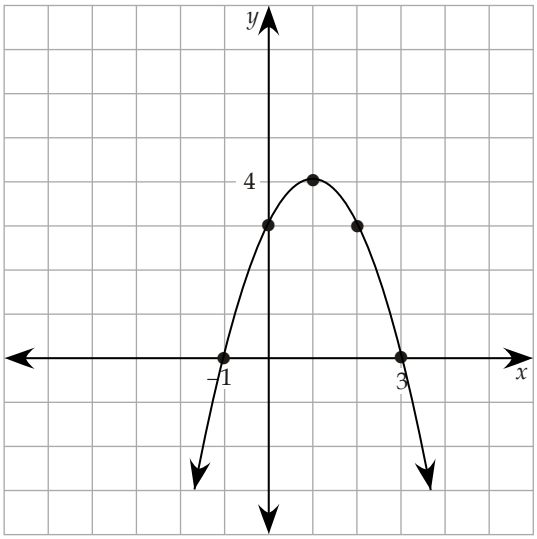
$$y = -x^2 + 2x + 3$$

$$y = -\left(x^2 - 2x \quad\right) + 3$$

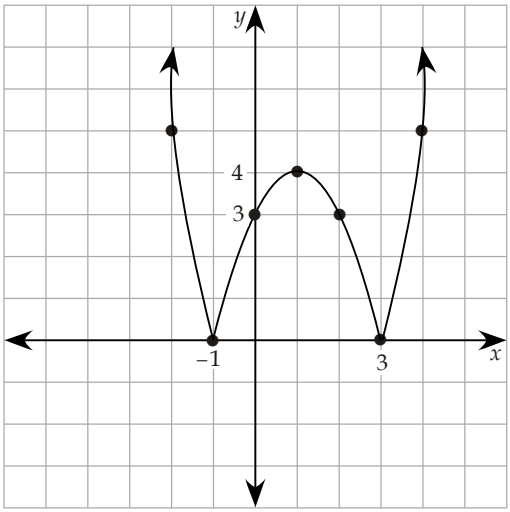
$$y = -\left(x^2 - 2x + 1\right) + 3 - 1(-1)$$

$$y = -(x-1)^2 + 4$$

The parabola opens down and is standard size. The vertex is (1, 4).



The absolute value transformation reflects any points with negative y-values over the x-axis. This occurs in two places in the domain for $x < -1$ and $x > 3$.



The x-intercepts are -1 and 3. The y-intercept is 3.

e) State the domain and range for each function in (a) to (d).

Answer:

Question	Domain	Range
(a)	$\Re$	$[0, \infty)$
(b)	$\Re$	$[0, \infty)$
(c)	$\Re$	$[0, \infty)$
(d)	$\Re$	$[0, \infty)$

2. Write each function from Question 1 using piece-wise notation and simplify.

Answers:

a) $f(x) = |\, 2x - 1 \,|$

The piece-wise function is

$$f(x) = \begin{cases} 2x - 1, & 2x - 1 \geq 0 \\ -(2x - 1), & 2x - 1 < 0 \end{cases}$$

Simplify both the function part and the domain pieces to get:

$$f(x) = \begin{cases} 2x - 1, & x \geq \frac{1}{2} \\ -2x + 1, & x < \frac{1}{2} \end{cases}$$

You can verify your answer from the graph.

b) $g(x) = |x + 3|$

The piece-wise definition is

$$g(x) = \begin{cases} x + 3, & x + 3 \geq 0 \\ -(x + 3), & x + 3 < 0 \end{cases}$$

Simplify the function and domain pieces to get

$$g(x) = \begin{cases} x + 3, & x \geq -3 \\ -x - 3, & x < -3 \end{cases}$$

Check your answer from the graph.

c) $h(x) = |x^2 - 1|$

The piece-wise definition is:

$$h(x) = \begin{cases} x^2 - 1, & x^2 - 1 \geq 0 \\ -\left(x^2 - 1\right), & x^2 - 1 < 0 \end{cases}$$

The function pieces are easily simplified, $h(x) = -(x^2 - 1) = -x^2 + 1$.

The domain pieces can be simplified by using the graph. Separate the pieces into intervals according to the x-intercepts. The simplified version is:

$$h(x) = \begin{cases} x^2 - 1, & x \leq -1 \\ -x^2 + 1, & -1 < x < 1 \\ x^2 - 1, & x \geq 1 \end{cases}$$

Note that the middle piece of $h(x)$ shows the portion where the graph was reflected over the x-axis to get a piece of a parabola opening down with vertex (0, 1).

d) $j(x) = |\ -x^2 + 2x + 3\ |$

The piece-wise definition is

$$j(x) = \begin{cases} -x^2 + 2x + 3, -x^2 + 2x + 3 \geq 0 \\ -\left(-x^2 + 2x + 3\right), -x^2 + 2x + 3 < 0 \end{cases}$$

The function $j(x) = -(-x^2 + 2x + 3)$ simplified is $x^2 - 2x - 3$.

The domain pieces can be simplified from the graph by considering the intervals of the domain according to the x-intercepts.

The simplified version of $j(x)$ becomes

$$j(x) = \begin{cases} x^2 - 2x - 3, x \leq -1 \\ -x^2 + 2x + 3, -1 < x < 3 \\ x^2 - 2x - 3, x \geq 3 \end{cases}$$

Note that the first and last pieces of $j(x)$ indicate the portions of graph of $y = -x^2 + 2x + 3$ that were reflected over the x-axis by the absolute value transformation. They are pieces of a parabola that opens up and has its vertex at $(1, -4)$.

3. The point $(2, -5)$ is on the graph $y = f(x)$. Identify the corresponding point on the graph $y = |\ f(x)\ |$.

   Answer:

   The absolute value transformation can be defined as $(x, y) \rightarrow (x, |\ y\ |)$. The absolute value function turns every negative y-value into a positive y-value. Therefore, the point $(2, -5)$ becomes $(2, 5)$.

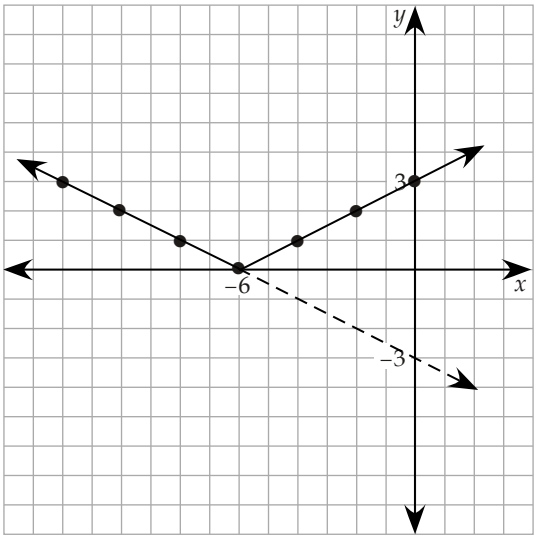
4. The graph of $y = f(x)$ has an x-intercept of -6 and a y-intercept of -1. What are the x- and y-intercepts on the graph of $y = |\ f(x)\ |$?

   Answer:

   The x-intercept of an absolute value function stays the same as the x-intercept is always the invariant point of the absolute value function. Therefore, the x-intercept is still -6. However, an absolute value function does not contain negative y-values. Therefore, this y-intercept would be reflected over the x-axis to result in a y-intercept of 1 in $y = |\ f(x)\ |$.

5. Write the piece-wise function that represents each graph.

a)



Answer:

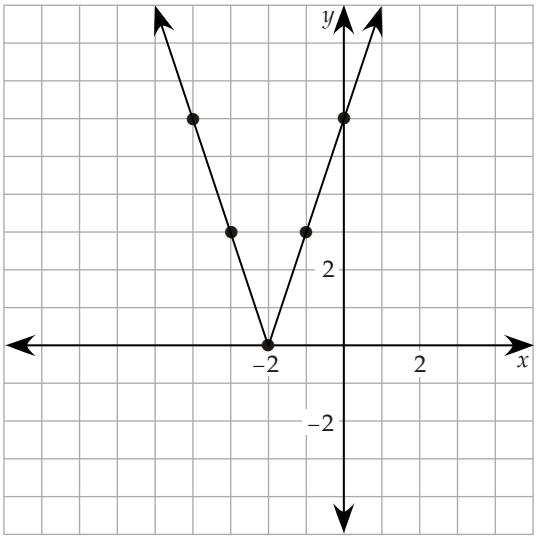
The slope of the line when x is greater than or equal to -6 is $\frac{1}{2}$.

The y-intercept is 3. The slope of the line when x is less than -6 is $-\frac{1}{2}$.

If this line were to continue down below the x-axis, the y-intercept would be -3. Therefore, the piece-wise function for this graph is

$$y = \begin{cases} -\frac{1}{2}x - 3, & x < -6 \\ \frac{1}{2}x + 3, & x \geq -6 \end{cases}$$

b)

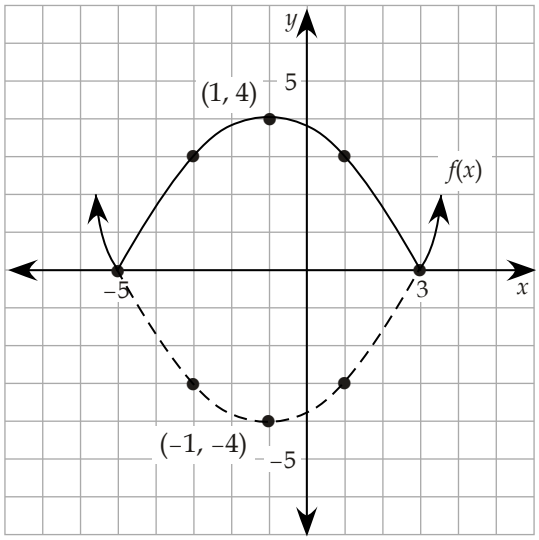


Answer:

The slope of the line when x is greater than or equal to -2 is 3. The y-intercept is 6. Therefore, this piece of the graph has the equation $y = 3x + 6$. When x is less than -2, the line, $y = 3x + 6$, is reflected over the x-axis, resulting in the line $y = -(3x + 6)$ or $y = -3x - 6$. The piece-wise function for this graph is thus

$$y = \begin{cases} -3x - 6, & x < -2 \\ 3x + 6, & x \geq -2 \end{cases}$$

c)



Answer:

This graph represents the absolute of a quadratic function whose vertex was originally at $(-1, -4)$, opening up and y-value reduced by a factor of 4. The pieces of the domain are determined by the x-intercepts. Thus, the piece-wise definition is

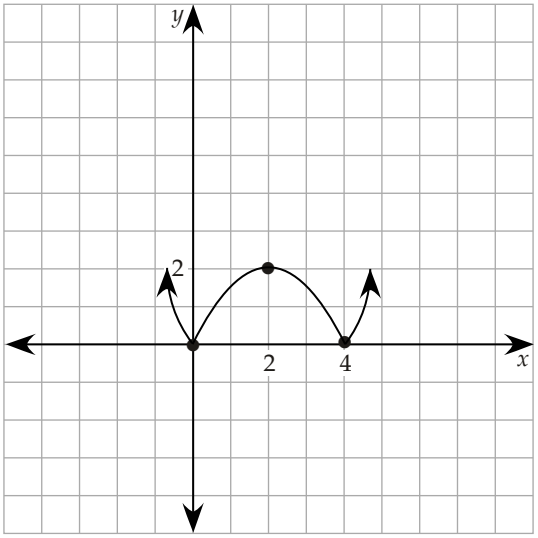
$$f(x) = \begin{cases} \frac{1}{4}(x+1)^2 - 4, & x \leq -5 \\ -\left[\frac{1}{4}(x+1)^2 - 4\right], & -5 < x < 3 \\ \frac{1}{4}(x+1)^2 - 4, & x \geq 3 \end{cases}$$

Note that you might have seen this graph as a parabola opening down with its original vertex at $(-1, 4)$. This possibility is shown in the next solution. Either interpretation will give the same absolute value function.

The simplified version of the function is

$$f(x)=\begin{cases}\frac{1}{4}(x+1)^2-4, & x\le -5\\ -\frac{1}{4}(x+1)^2+4, & -5<x<3\\ \frac{1}{4}(x+1)^2-4, & x\ge 3\end{cases}$$

d)



Answer:

This graph could be the absolute value transformation of a quadratic function with vertex at (2, 2), opening down, and having the y-values reduced by a factor of 2. The original quadratic function would be $y=\frac{-1}{2}(x-2)^2+2$. The tails of the parabola were reflected over the x-axis at pivot points $x = 0$ and $x = 4$ by the absolute value transformation. The piece-wise absolute value function is

$$f(x)=\begin{cases}-\left[\frac{-1}{2}(x-2)^2+2\right], & x<0\\ -\frac{1}{2}(x-2)^2+2, & 0\le x\le 4\\ -\left[\frac{-1}{2}(x-2)^2+2\right], & x>4\end{cases}$$

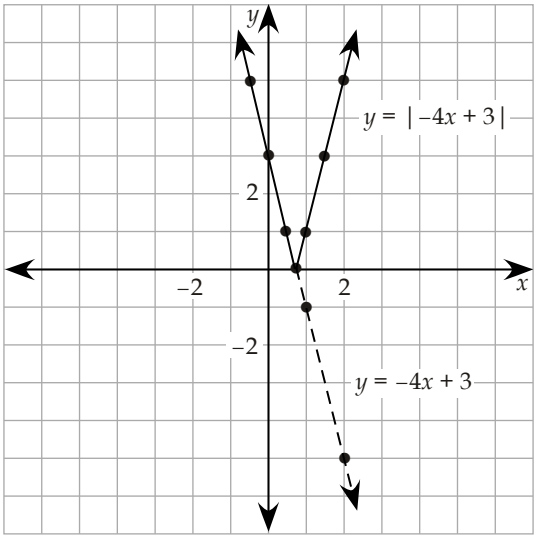
The simplified form:

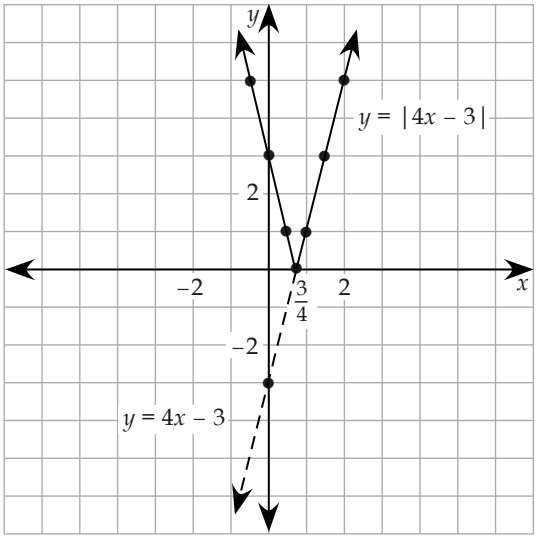
$$f(x) = \begin{cases} \frac{1}{2}(x-2)^2 - 2, & x < 0 \\ -\frac{1}{2}(x-2)^2 + 2, & 0 \le x \le 4 \\ \frac{1}{2}(x-2)^2 - 2, & x > 4 \end{cases}$$

6. Graph $f(x) = |-4x + 3|$ and $g(x) = |4x - 3|$ using a table of values. What do you notice about these two graphs? Explain.

   Answer:

x	$y = -4x + 3$	$f(x) = \lvert -4x + 3 \rvert$	$y = 4x - 3$	$g(x) = \lvert 4x - 3 \rvert$
−3	15	15	−15	15
−2	11	11	−11	11
−1	7	7	−7	7
0	3	3	−3	3
1	−1	1	1	1
2	−5	5	5	5
3	−9	9	9	9





These two absolute value graphs are identical. This is because the expression inside the absolute value symbols can either be positive or negative to yield the same answer. Therefore, $|-4x + 3| = |4x - 3|$.

7. Graph the function $f(x) = |x^2 - 6x + 5|$. Write a different absolute value function having the same graph as $f(x)$.

Answer:

$f(x) = |x^2 - 6x + 5|$

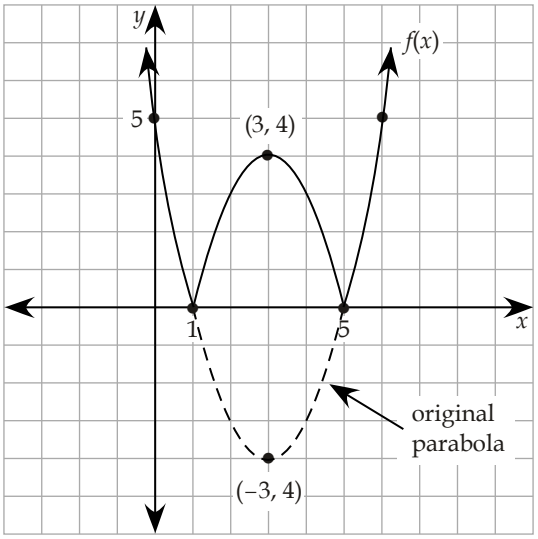
Consider the function $y = x^2 - 6x + 5$. Graph this function and then use the absolute value transformation.

$y = x^2 - 6x + 5$

$y = (x^2 - 6x + 9) + 5 - 9$

$y = (x - 3)^2 - 4$

Vertex $(3, -4)$ opens up with normal size. Reflect the portion of the parabola with negative y-values over the x-axis. The graph of $f(x)$ is shown below.



Another absolute value function having the same graph as $f(x)$ would be the absolute value of the negative of the original quadratic $y = x^2 - 6x + 5$. This function would be

$g(x) = |-(x^2 - 6x + 5)|$

$g(x) = |-x^2 + 6x - 5|$

Learning Activity 8.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. You take your best friend out for dinner at a restaurant and the bill comes to $67.32. If you tip the waiter 15%, how much change will you get back if you pay with a $100 bill?
2. Find the equation of a line with a slope of -1 that crosses through the point (5, 2).
3. Evaluate: 3^4
4. Evaluate: $\frac{2}{13} + \frac{6}{7}$
5. If Jacob can read an average of 98 pages a day, how many pages can he read in a week?
6. Express as a radical: $8^{\frac{5}{4}}$. Do not simplify.
7. Three angles in a parallelogram measure 30°, 93°, and 72°. What is the measure of the fourth angle?
8. The equation of a parabola is $y = x^2 + 2x - 15$. What are the x-intercepts?

Answers:

1. $22.58 ($67.32 × 10% = $6.73; $67.32 × 5% = $3.37; the bill will total $67.32 + $6.73 + $3.37 = $77.42; $100 − $77.42 = $22.58)
2. $y = -x + 7$ ($y = mx + b$; $2 = -1(5) + b$; $2 = -5 + b$; $7 = b$)
3. 81 ($3 \times 3 \times 3 \times 3 = 9 \times 9$)
4. $\frac{92}{91}\left(\frac{14}{91} + \frac{78}{91}\right)$
5. 686 ($90 \times 7 = 630$; $8 \times 7 = 56$; $98 \times 7 = 686$)
6. $\sqrt[4]{8^5}$
7. 165° (The sum of the angles in a parallelogram is 360°. The missing angle is 360° − 30° − 93° − 72° = 165°.)
8. -5 and 3 (Factor the corresponding quadratic equation: $0 = x^2 + 2x - 15 = (x + 5)(x - 3)$.

Part B: Solving Absolute Value Equations

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Solve the following equations using the definition of absolute value.

   a) $|2x - 3| = 7$

   Answer:

   Case 1: $2x - 3 = 7$ and $2x - 3 \geq 0$

   $2x - 3 = 7$ $\qquad$ $2x - 3 \geq 0$

   $2x = 10$ $\qquad$ $2x \geq 3$

   $x = 5$ $\qquad$ $x \geq \frac{3}{2}$

   Since $x = 5$ is in the interval $x \geq \frac{3}{2}$, then $x = 5$ is a solution.

   Case 2: $-(2x - 3) = 7$ and $2x - 3 < 0$

   $-2x + 3 = 7$ $\qquad$ $2x - 3 < 0$

   $-2x = 4$ $\qquad$ $2x < 3$

   $x = -2$ $\qquad$ $x < \frac{3}{2}$

   Since $x = -2$ is in the interval $x < \frac{3}{2}$, then $x = -2$ is a solution.

   $\therefore$ The solutions are $x = 5$ and $x = -2$.

b) $|3x - 8| = 2 - 4x$

Answer:

Case 1: $3x - 8 = 2 - 4x$ and $3x - 8 \geq 0$

$7x = 10$ $3x \geq 8$

$x = \frac{10}{7}$ $x \geq \frac{8}{3}$

Since $x = \frac{10}{7}$ is not in the interval $x \geq \frac{8}{3}$, it is not a solution. It is an extraneous root.

Case 2: $-(3x - 8) = 2 - 4x$ and $3x - 8 < 0$

$-3x + 8 = 2 - 4x$ $3x < 8$

$x = -6$ $x < \frac{8}{3}$

Since $x = -6$ is in the interval $x < \frac{8}{3}$, then it is a solution.

$\therefore$ The only solution is $x = -6$.

c) $|x^2 - 4x| = 4$

Answer:

Case 1: $x^2 - 4x = 4$ and $x^2 - 4x \geq 0$

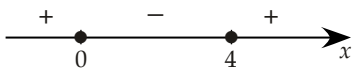
$x^2 - 4x - 4 = 0$ $x(x - 4) \geq 0$

Using the quadratic formula: $x = 0, 4$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, then

$x = 4.828$ or

$x = -0.828$



Solution interval:

$(-\infty, 0] \cup [4, \infty)$

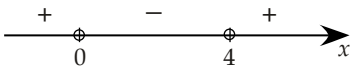
Since $x = 4.828$ is in the interval $[4, \infty)$ and $x = -0.828$ is in the interval $(-\infty, 0]$, then both are solutions.

Case 2: $-\left(x^2 - 4x\right) = 4$ and $x^2 - 4x < 0$

$-x^2 + 4x = 4$

$-x^2 + 4x - 4 = 0$

$x^2 - 4x + 4 = 0$



Solution interval: (0, 4)

Factoring: $(x - 2)(x - 2) = 0$

$x = 2$

Since $x = 2$ is in the interval (0, 4), then $x = 2$ is also a solution.

$\therefore$ The solutions are $x = 4.828$, $x = -0.828$, and $x = 2$.

d) $|x^2 + 2x| = x + 30$

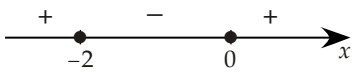
Answer:

Case 1: $x^2 + 2x = x + 30$ and $x^2 + 2x \geq 0$

$x^2 + x - 30 = 0$ $x(x + 2) \geq 0$

$(x + 6)(x - 5) = 0$ $x = 0, -2$

$x = -6, 5$



Solution: $(-\infty, -2] \cup [0, \infty)$

Since $x = -6$ is in the interval $(-\infty, -2]$ and $x = 5$ is in the interval $[0, \infty)$, both are solutions.

Case 2: $-\left(x^2 + 2x\right) = x + 30$ and $x^2 + 2x < 0$

$-x^2 - 2x = x + 30$

$-x^2 - 3x - 30 = 0$

$x^2 + 3x + 30 = 0$

discriminant $= b^2 - 4ac$

$= 3^2 - 4(1)(30)$

$= -111$

There are no real number solutions.

$\therefore$ The only solutions are $x = -6, 5$.

2. Solve the absolute value equations in Question 1 graphically.

   a) $|2x - 3| = 7$

   Answer:

   The corresponding function for the left side of the equation is $y = |2x - 3|$.

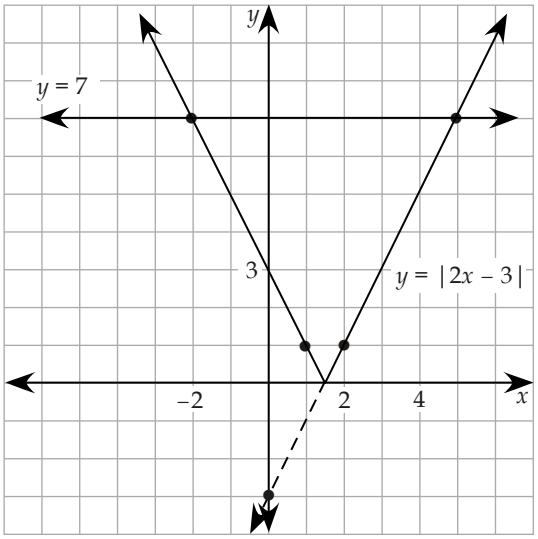
   The line is $y = 2x - 3$ with slope 2 and y-intercept of -3. The pivot point is the x-intercept of

   $$0 = |2x - 3|$$

   $$0 = 2x - 3$$

   $$\frac{3}{2} = x$$

   The corresponding function for the right side of the equation is $y = 7$. The graphs are shown below.



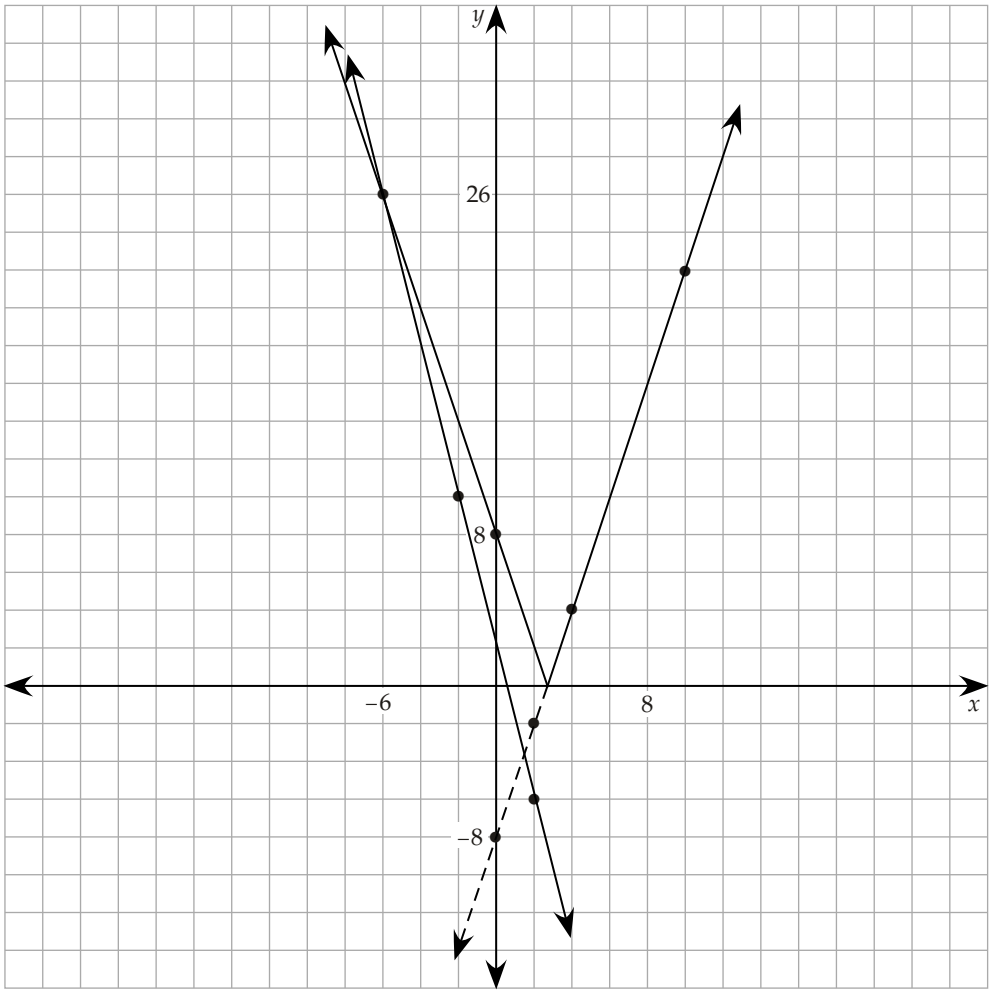
   The two points of intersection are $(-2, 7)$ and $(5, 7)$. The x-coordinates of these points are $x = -2$ and $x = 5$.

   Therefore, as in Question 1(a), the solutions to this absolute value equation are $x = -2$ and $x = 5$.

b) $|3x - 8| = 2 - 4x$

Answer:

Graph $y = |3x - 8|$ and $y = 2 - 4x$ on the same graph. Use a scale of 2 on both axes.



The x-value at which these two functions intersect is -6. As in Question 1(b), this absolute value equation has a solution of $x = -6$.

c) $|x^2 - 4x| = 4$

Answer:

The function for the left side is $y = |x^2 - 4x|$. The parabola $y = x^2 - 4x$ has zeros at

$0 = x^2 - 4x$

$0 = x(x - 4)$

$x = 0, 4$

The vertex for $y = x^2 - 4x$ is

$y = x^2 - 4x$

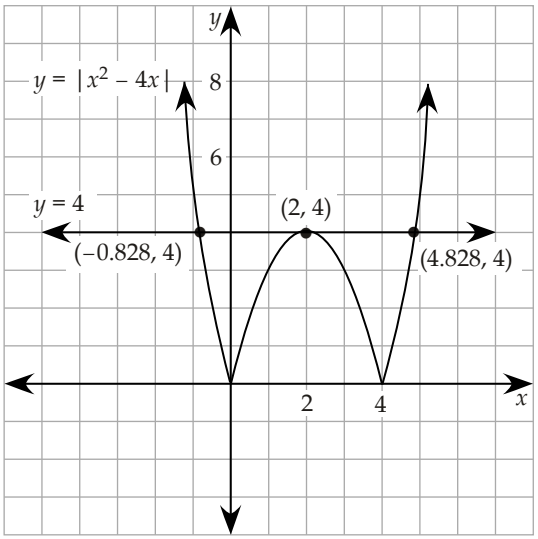
$y = (x^2 - 4x + 4) - 4$

$y = (x - 2)^2 - 4$

Vertex coordinates of $y = x^2 - 4x$ are $(2, -4)$.

Vertex coordinates of $y = |x^2 - 4x|$ are $(2, 4)$.

The graph of both functions, $y = |x^2 - 4x|$ and $y = 4$ are shown below:



To find the x-values for the two outside points, solve $(x^2 - 4x) = 4$ using the quadratic formula $x^2 - 4x - 4 = 0 \Rightarrow x = 4.828$ or $x = -0.828$.

The x-values at which these two functions intersect are $x = -0.828$, $x = 2$, and $x = 4.828$ and are thus the solutions to this absolute value equation, as shown algebraically in Question 1(c).

d) $|x^2 + 2x| = x + 30$

Answer:

The left side function is $y = |x^2 + 2x|$. The vertex of the parabola $y = x^2 + 2x$ is

$y = x^2 + 2x$

$y = (x^2 + 2x + 1) - 1$

$y = (x + 1)^2 - 1$

Vertex $(-1, -1)$

The zeros are

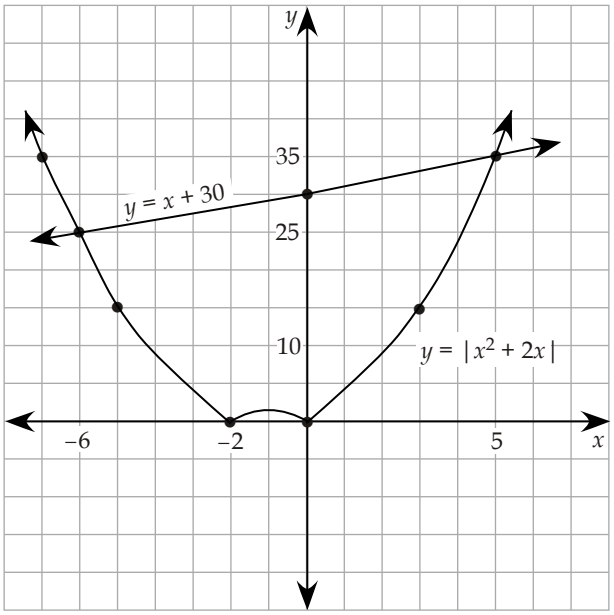
$0 = x^2 + 2x$

$0 = x(x + 2)$

$x = 0 \qquad x + 2 = 0$

$x = -2$

The line $y = x + 30$ on the right side of the equation has a y-intercept of 30 and slope of 1. Use a scale of 5 on the y-axis.



The x-values at which these two functions intersect are $x = -6$ and $x = 5$ and are thus the solutions to this absolute value equation, as shown algebraically in Question 1(d).

3. A manufacturer rejects boxes of cookies when the mass of the box differs from the stated mass of 325 g by more than 4.6 g.

   a) Write an absolute value equation that represents this situation.

   Answer:

   Let x represent an acceptable weight for a box of cookies, measured in grams.

   $|x - 325| = 4.6$ or $|325 - x| = 4.6$

   b) Solve the absolute value equation.

   Answer:

   $|x - 325| = 4.6$

   Case 1: $x - 325 = 4.5$ and $x - 325 \geq 0$

   $x = 329.6$ $x \geq 325$

   Since $x = 329.6$ is in the interval $x \geq 325$, it is a solution.

   Case 2: $-(x - 325) = 4.6$ and $x - 325 < 0$

   $-x + 325 = 4.6$ $x < 325$

   $-x = -320.4$

   $x = 320.4$

   Since $x = 320.4$ is in the interval $x < 325$, it is a solution.

   The solutions for the absolute value equation are $x = 320.4$ g and $x = 329.6$ g.

   c) State the range of acceptable values for the mass of a box of cookies.

   Answer:

   Thus, the greatest possible acceptable weight of the box of cookies is 329.6 g and the least possible acceptable weight of the box of cookies is 320.4 g.

4. Arrange the numbers from 0 to 9 into the boxes so that each absolute value equation is mathematically correct. You can only use each digit once.

Answer:

a) $2\left|x+\boxed{3}\right|=6$

$x=0,-\boxed{6}$

b) $\left|2x+7\right|=\boxed{0}$

$x=-\frac{\boxed{7}}{2}$

c) $\left|x^2-\boxed{2}x\right|=1$

$x=\boxed{1},\ 1+\sqrt{2},\ 1-\sqrt{2}$

d) $\left|x^2-2x-16\right|=\boxed{8}$

$x=-4,\ -2,\ \boxed{4},\ 6$

e) $\left|3x-\boxed{9}\right|=24$

$x=-\boxed{5},\ 11$

Solutions for each of the absolute value equations:

a) $2\left|x+3\right|=6$

$\left|x+3\right|=\frac{6}{2}$

$\left|x+3\right|=3$

Case 1: $x+3=3$ $\qquad$ $x+3\geq 0$

$x=0$ $\qquad$ $x\geq -3$

Case 2: $-(x+3)=3$ $\qquad$ $x+3<0$

$-x-3=3$ $\qquad$ $x<-3$

$-x=6$

$x=-6$

Solutions are $x=0$ and $x=-6$.

b) $\left|2x+7\right|=0$

$2x+7=0$

$2x=-7$

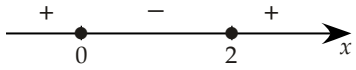
$x=\frac{-7}{2}$

c) $|x^2 - 2x| = 1$

Case 1: $x^2 - 2x = 1$ and $x^2 - 2x \geq 0$

$x^2 - 2x - 1 = 0$ $x(x-2) \geq 0$

$x = \dfrac{2 \pm \sqrt{4 - 4(1)(-1)}}{2(1)}$ $x = 0, 2$



$x = \dfrac{2 \pm \sqrt{8}}{2}$

Solution: $(-\infty, 0] \cup [2, \infty)$

$x = \dfrac{2 \pm 2\sqrt{2}}{2}$

$x = 1 \pm \sqrt{2}$

$x = 1 + \sqrt{2} \approx 2.4$

$x = 1 - \sqrt{2} \approx -0.4$

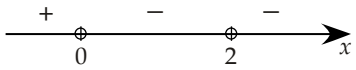
Thus, $x = 1 + \sqrt{2}$ is in the interval $[2, \infty)$ and $x = 1 - \sqrt{2}$ is in the interval $(-\infty, 0]$ and both are solutions.

Case 2: $-(x^2 - 2x) = 1$ and $x^2 - 2x < 0$

$-x^2 + 2x = 1$ $x(x-2) < 0$

$-x^2 + 2x - 1 = 0$ $x = 0, 2$

$x^2 - 2x + 1 = 0$



$(x-1)^2 = 0$

Solution: $(0, 2)$

$x = 1$

Since $x = 1$ is in the interval $(0, 2)$, it is a solution.

$\therefore$ The solutions are 1, $1 + \sqrt{2}$, and $1 - \sqrt{2}$.

d) $|x^2 - 2x - 16| = 8$

Case 1: $x^2 - 2x - 16 = 8$ and $x^2 - 2x - 16 \geq 0$

$x^2 - 2x - 24 = 0$

$(x - 6)(x + 4) = 0$

$x = 6, -4$

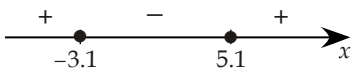
$$x = \frac{2 \pm \sqrt{4 - 4(1)(-16)}}{2(1)}$$

$$x = \frac{2 \pm \sqrt{68}}{2}$$

$$x = \frac{2 \pm 2\sqrt{17}}{2}$$

$x = 1 \pm \sqrt{17}$

$x \approx 5.1$ or -3.1



Approximate solution:

$(-\infty, -3.1] \cup [5.1, \infty)$

Since $x = 6$ and $x = -4$ are in the intervals, they are solutions.

Case 2: $-\left(x^2 - 2x - 16\right) = 8$ and $x^2 - 2x - 16 < 0$

$-x^2 + 2x + 16 = 8$

$-x^2 + 2x + 8 = 0$

$x^2 - 2x - 8 = 0$

$(x - 4)(x + 2) = 0$

$x = 4, -2$

Approximate solution:

$(-3.1, 5.1)$

Since both values $x = 4$ and $x = -2$ are in the interval, they are solutions.

$\therefore$ The solutions are 6, -4, 4, -2.

e) $|3x - 9| = 24$

Case 1: $3x - 9 = 24$ and $3x - 9 \geq 0$

$3x = 33$ $3x \geq 9$

$x = 11$ $x \geq 3$

Since $x = 11$ is in the interval $x \geq 3$, it is a solution.

Case 2: $-(3x - 9) = 24$ and $3x - 9 < 0$

$-3x + 9 = 24$ $3x < 9$

$-3x = 15$ $x < 3$

$x = -5$

Since $x = -5$ is in the interval $x < 3$, it is a solution.

Thus, $x = -5$ and $x = 11$ are the solutions.

5. Armani and Jennifer are both trying to solve the absolute value equation $|x - 3| = 2x + 1$. Armani's and Jennifer's work is shown below.

Armani's work:

Case 1: $|x - 3| = 2x + 1$

$x + 3 = 2x + 1$

$2 = x$

Case 2: $-x + 3 = 2x + 1$

$2 = 3x$

$x = \frac{2}{3}$

Jennifer's work:

Case 1: $|x - 3| = 2x + 1$

$x - 3 = 2x + 1$

$-4 = x$

Case 2: $-x + 3 = 2x + 1$

$2 = 3x$

$x = \frac{2}{3}$

a) Explain whose answer is incorrect.

Answer:

Armani's second equation is correct as she took the negative of the entire expression inside the absolute value symbols. However, the first equation is incorrect. An absolute value equation has possible solutions in two cases, when the expression inside the symbols is positive and when the expression inside the symbols is negative. However, Armani thought that you had to change all the signs to positive inside the absolute value brackets. This is incorrect. Also, she didn't write the interval restrictions for each case.

Jennifer's work is also incorrect. Jennifer solved the equation correctly. However, she forgot to check the intervals of the domain. If Jennifer were to set up her work using the definition of an absolute value, she would have discovered her answer of negative 4 wasn't possible.

Case 1: $x = -4$ as long as $x - 3 \geq 0$ or $x \geq 3$. The solution is not in the interval and is an extraneous solution.

Case 2: $x = \frac{2}{3}$ as long as $x - 3 < 0$ or $x < 3$. This value of x is in the interval $x < 3$ and is a solution.

$\therefore$ Jennifer needed to reject $x = -4$ and keep the only solution, $x = \frac{2}{3}$.

b) Explain how a person could have discovered that his or her work was incorrect.

Answer:

Armani and Jennifer could have discovered their work was incorrect by checking their answers.

Verifying the solutions of $x = 2, \frac{2}{3}$, and -4.

$| x - 3 | = 2x + 1$

Verify $x = 2$.

LHS	RHS
$\| x - 3 \|$	$2x + 1$
$\| 2 - 3 \|$	$2(2) + 1$
1	5

LHS $\neq$ RHS

$x = 2$ is not a solution.

Verify $x = \frac{2}{3}$.

LHS	RHS
$\lvert x - 3 \rvert$	$2x + 1$
$\left\lvert \frac{2}{3} - 3 \right\rvert$	$2\left(\frac{2}{3}\right) + 1$
$\left\lvert -2\frac{1}{3} \right\rvert$	$\frac{4}{3} + 1$
$2\frac{1}{3}$	$2\frac{1}{3}$

LHS = RHS

Verify $x = -4$.

LHS	RHS
$\lvert x - 3 \rvert$	$2x + 1$
$\lvert -4 - 3 \rvert$	$2(-4) + 1$
7	-7

LHS ≠ RHS

$x = -4$ is not a solution.

The only solution of the three is $x = \frac{2}{3}$. By using the domain interval restrictions, extraneous roots can be rejected. By verifying your answers, your calculations may be corrected.

6. Determine whether each statement is always true, sometimes true, or never true. Explain.

   a) The value of $\lvert x + 5 \rvert$ is greater than zero.

   Answer:

   This is sometimes true. The absolute value of any expression is always greater than *or equal to* zero.

b) The value of $|x + a|$, when a is a real number is greater than or equal to zero.

Answer:

This is always true because the absolute value of any expression is always greater than or equal to zero.

c) The value of $|5 + a| + a$ is greater than zero.

Answer:

This is sometimes true. The absolute value of any expression is always greater than or equal to zero. However, if a is a negative number, the expression $|5 + a| + a$ could be less than zero. For example, if $a = -3$, $|5 + (-3)| + (-3) = |2| - 3 = -1$, which is not a positive number.

7. Write an absolute value equation with the indicated solutions or type of solution.

a) -4

Answer:

Answers will vary.

A possible equation is

$$|x - 3| = 7$$

b) No solution

Answer:

Answers will vary. However, in order for an absolute value equation to have no solution, the absolute value expression has to be equal to a negative value. A possible equation is

$$|x + 24| = -3$$

c) One solution

Answer:

Answers will vary. However, answers must have only one solution and not two solutions.

An example is

$$|2x - 6| = 0$$

d) Two solutions

Answer:

Answers will vary. A possible equation is

$$|x + 1| = 5$$

Learning Activity 8.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. One-third of students in a class of 108 students have never failed a test. How many students have never failed a test?
2. There is a 50% discount on decorations after Halloween. If it costs you $52.44 to buy Halloween decorations before Halloween, how much will it cost you to buy those same decorations the day after Halloween?
3. Find the x-intercepts of the following parabola: $y = x^2 + 6x + 8$
4. Evaluate: $\frac{1}{4} + \frac{2}{7}$
5. State the reciprocal of $\frac{(x+3)(x-4)}{(x+4)(x-1)}$.
6. Express as a power: $\sqrt[2]{12^7}$
7. In which quadrant is an angle of 145° in standard position located?
8. Find the slope of the line defined by the equation $-5x = 2y + 9$.

Answers:

1. 36 (One-third of 99 = 33; one-third of 9 = 3; one-third of 108 = 33 + 3 = 36)
2. $26.22 ($52 ÷ 2 = $26 and 0.44 ÷ 2 = 0.22)
3. -2 and -4 ($(x + 2)(x + 4) = 0$)
4. $\frac{15}{28}\left(\frac{7}{28} + \frac{8}{28}\right)$
5. $\frac{(x+4)(x-1)}{(x+3)(x-4)}$
6. $12^{\frac{7}{2}}$
7. Quadrant II
8. $-\frac{5}{2}\left(2y = -5x - 9;\ y = -\frac{5}{2}x - \frac{9}{2}\right)$

Part B: Graphing Reciprocal Functions

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Write the reciprocal of each function.

   a) $y = x - 1$

   Answer:

   $$f(x) = \frac{1}{x - 1}$$

   b) $y = -3x + 6$

   Answer:

   $$f(x) = \frac{1}{-3x + 6}$$

   c) $y = x^2 + 4x - 2$

   Answer:

   $$f(x) = \frac{1}{x^2 + 4x - 2}$$

   d) $y = x^2 - 9$

   Answer:

   $$f(x) = \frac{1}{x^2 - 9}$$

2. For each of the following, identify the (i) x-intercepts of $f(x)$, (ii) the reciprocal function $y = \frac{1}{f(x)}$, (iii) the non-permissible values of the reciprocal function, and (iv) the equation(s) of the vertical asymptotes of the reciprocal function

a) $f(x) = 2x + 5$

Answer:

i) $0 = 2x + 5$

$-5 = 2x$

$x = -\frac{5}{2}$

ii) $y = \frac{1}{2x + 5}$

iii) The non-permissible value is $x = -\frac{5}{2}$.

iv) The equation of the vertical asymptote is $x = -\frac{5}{2}$.

b) $f(x) = x^2 - 9$

Answer:

i) $0 = x^2 - 9$

$0 = (x - 3)(x + 3)$

$x = 3$ or -3

ii) $y = \frac{1}{x^2 - 9}$

iii) The non-permissible values are $x = 3$ and $x = -3$.

iv) The equations of the vertical asymptotes are $x = 3$ and $x = -3$.

c) $f(x) = x^2 + 4x + 3$

Answer:

i) $0 = x^2 + 4x + 3$

$0 = (x + 3)(x + 1)$

$x = -3$ or -1

ii) $y = \frac{1}{x^2 + 4x + 3}$

iii) The non-permissible values are $x = -3$ and $x = -1$.

iv) The equations of the vertical asymptotes are $x = -3$ and $x = -1$.

3. Match each function with the graph of its reciprocal. Hint: Consider the asymptotes.

   a) $y = x^2 + x - 6$

   b) $y = -2$

   c) $y = x - 5$

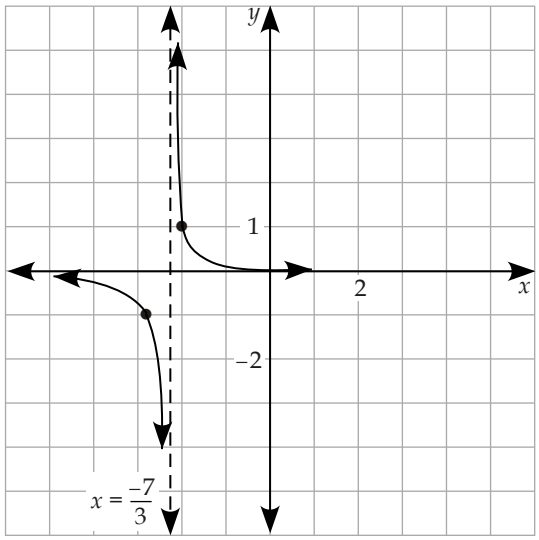
   d) $y = x^2 - 2x - 3$

   e) $y = x^2 - 6x + 9$

   f) $y = 3x + 7$

   g) $y = x^2 - 5x + 7$

i)



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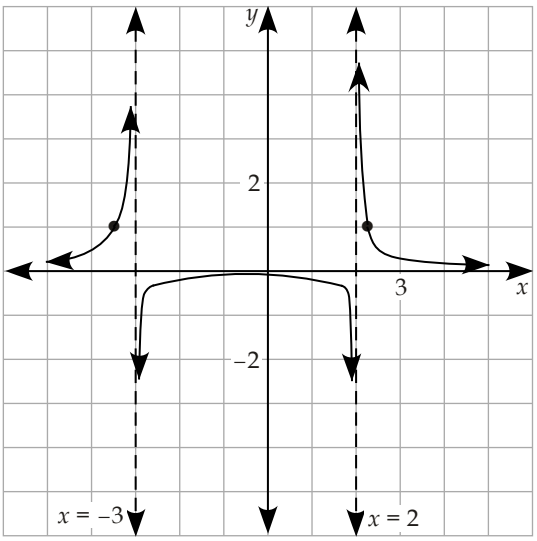

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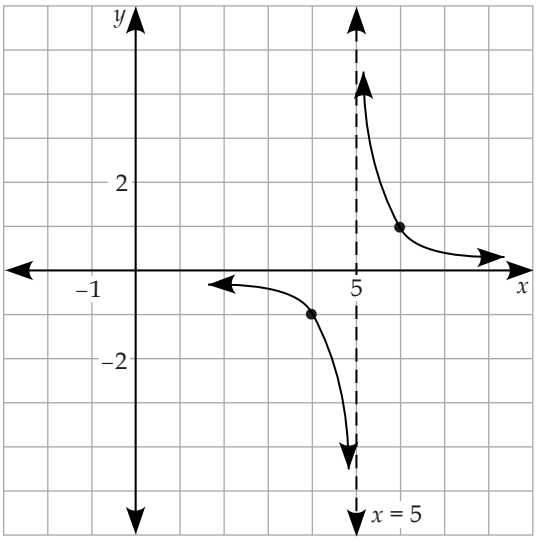

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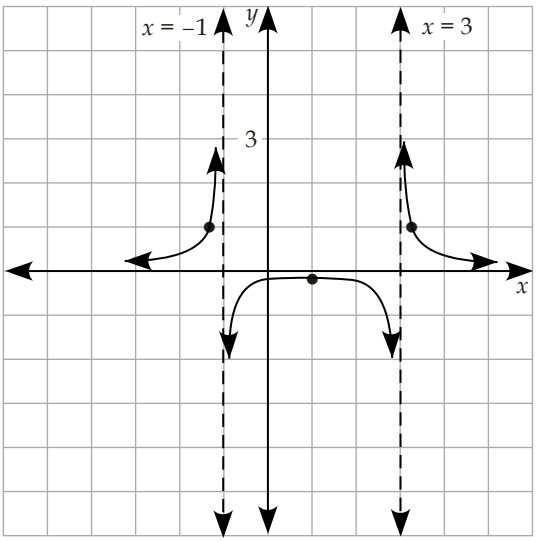
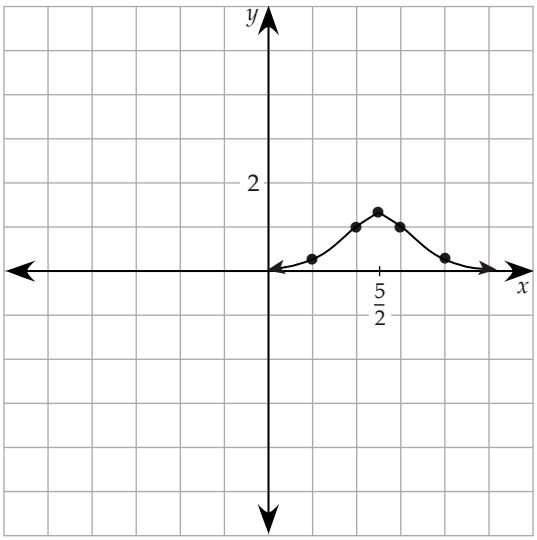
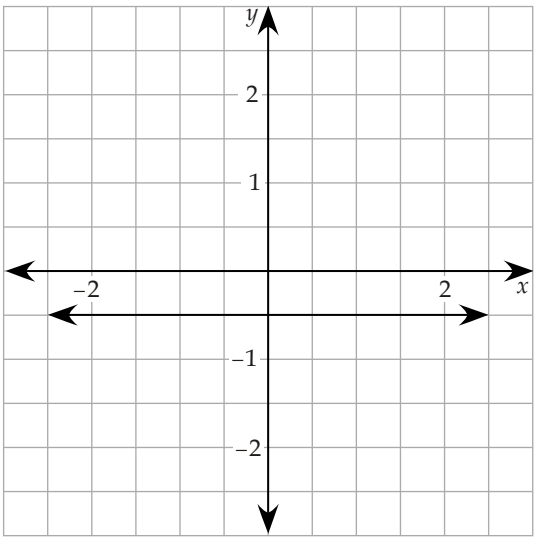


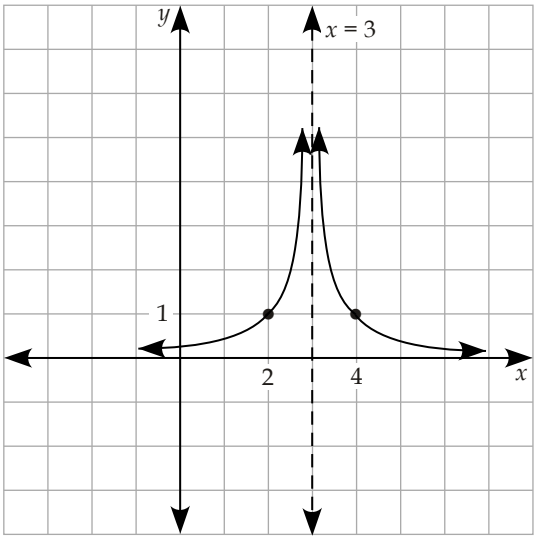
v)



vi)



vii)



Answers:

a) (ii) Since $y = (x + 3)(x - 2)$, the vertical asymptotes are at $x = -3$ and $x = 2$.

b) (vi) Since $y = -2$, the reciprocal is a constant, $y = -\frac{1}{2}$.

c) (iii) Since $y = x - 5$, the vertical asymptote is at $x = 5$.

d) (iv) Since $y = (x - 3)(x + 1)$, the vertical asymptotes are at $x = 3$ and $x = -1$.

e) (vii) Since $y = (x - 3)(x - 3)$, only one vertical asymptote at $x = 3$.

f) (i) Since $y = 3x + 7$, the vertical asymptote is at $x = -\frac{7}{3}$.

g) (v) Since $y = x^2 - 5x + 7$ has a discriminate of $b^2 - 4ac = (-5)^2 - 4(1)(7) = -3$, which is negative, then the parabola has no zeros so its reciprocal will have no vertical asymptotes.

4. State the equation(s) of the vertical asymptote(s) for each function.

a) $y = \dfrac{1}{5x - 10}$

Answer:

$5x - 10 = 0$

$5x = 10$

$x = 2$

$\therefore$ Vertical asymptote is $x = 2$.

b) $y = \dfrac{1}{x - 4}$

Answer:

$x - 4 = 0$

$x = 4$

$\therefore$ Vertical asymptote is $x = 4$.

c) $y = \dfrac{1}{(x + 3)(x - 6)}$

Answer:

$(x + 3)(x - 6) = 0$

$x = -3$ or $x = 6$

$\therefore$ Vertical asymptotes are $x = -3$ and $x = 6$.

d) $y = \dfrac{1}{x^2 - x - 20}$

Answer:

$x^2 - x - 20 = 0$

$(x - 5)(x + 4) = 0$

$x = 5$ or $x = -4$

$\therefore$ Vertical asymptotes are $x = 5$ and $x = -4$.

5. Sketch the graph of each reciprocal function. State the y-intercept, the x-intercept(s), the invariant points, and the equation(s) of the horizontal and vertical asymptote(s), as well as the domain and range.

a) $f(x) = \dfrac{1}{x+3}$

b) $f(x) = \dfrac{1}{2x+8}$

c) $f(x) = \dfrac{1}{x^2+x-2}$

d) $f(x) = \dfrac{1}{x^2-2x-8}$

e) $f(x) = \dfrac{1}{x^2+8x+16}$

f) $f(x) = \dfrac{1}{x^2-2x+6}$

Answers:

a) $f(x) = \dfrac{1}{x+3}$

y-intercept: Find $f(0)$:

$$f(0) = \frac{1}{0+3} = \frac{1}{3}$$

Vertical asymptote: Let the denominator, $x + 3$, equal 0, find the zero, and draw the dotted vertical line through that zero.

$$x + 3 = 0$$
$$x = -3$$

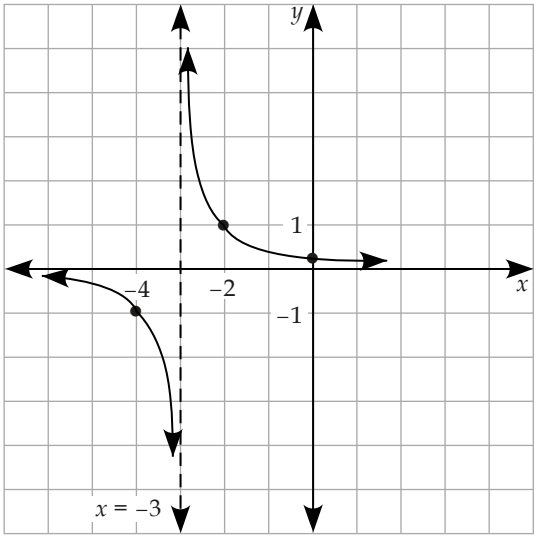
Horizontal asymptote: The horizontal asymptote is the line $y = 0$.

Invariant points: Set the denominator equal to 1 and -1.

$y = 1$	$y = -1$
$x + 3 = 1$	$x + 3 = -1$
$x = -2$	$x = -4$

Therefore, the two invariant points are $(-2, 1)$, and $(-4, -1)$.

Draw the curve: Plot the invariant points and the y-intercept. Draw the asymptotes and draw the curves.



Domain: $\{x \mid x \neq -3, x \in \Re\}$

Range: $\{y \mid y \neq 0, y \in \Re\}$

There is no x-intercept.

b) $f(x) = \dfrac{1}{2x + 8}$

Answer:

y-intercept: Find $f(0)$:

$$f(0) = \frac{1}{2(0) + 8} = \frac{1}{8}$$

Vertical asymptote: Let the denominator, $2x + 8$, equal 0, find the zero, and draw the dotted vertical line through that zero.

$$2x + 8 = 0$$
$$2x = -8$$
$$x = -4$$

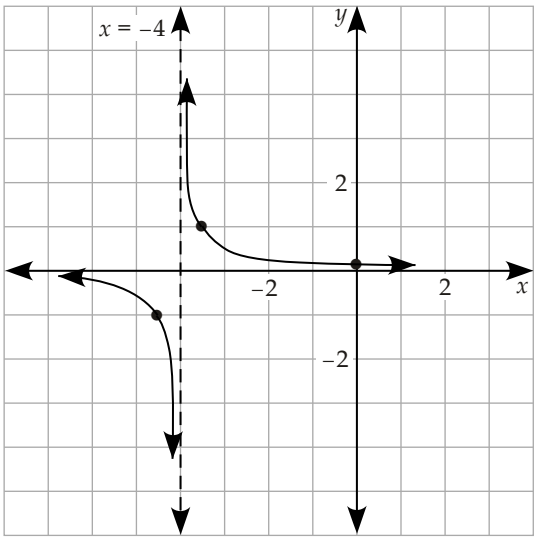
Horizontal asymptote: The horizontal asymptote is the line $y = 0$.

Invariant points: Set the denominator equal to 1 and -1.

$$y = 1 \qquad y = -1$$
$$2x + 8 = 1 \qquad 2x + 8 = -1$$
$$2x = -7 \qquad 2x = -9$$
$$x = -\frac{7}{2} \qquad x = -\frac{9}{2}$$

Therefore, the two invariant points are $\left(-\frac{7}{2}, 1\right)$ and $\left(-\frac{9}{2}, -1\right)$.

Draw the curve: Plot the invariant points and the y-intercept. Draw the asymptotes and draw the curves.



Domain: $\{x \mid x \neq -4, x \in \Re\}$

Range: $\{y \mid y \neq 0, y \in \Re\}$

There is no x-intercept.

c) $f(x) = \dfrac{1}{x^2 + x - 2}$

Answer:

y-intercept: Find $f(0)$:

$$f(0) = \frac{1}{0^2 + 0 - 2} = -\frac{1}{2}$$

Vertical asymptote: Let the denominator equal 0, find the zeros, and draw the dotted vertical asymptotes through those zeros.

$$x^2 + x - 2 = 0$$
$$(x + 2)(x - 1) = 0$$
$$x = -2,\ x = 1$$

Horizontal asymptote: The horizontal asymptote is the line $y = 0$.

Vertex of the original parabola:

$$y = x^2 + x - 2$$
$$y = \left(x^2 + x + \frac{1}{4}\right) - 2 - \frac{1}{4}$$
$$y = \left(x + \frac{1}{2}\right)^2 - \left(-\frac{9}{4}\right)$$

Vertex: $\left(-\dfrac{1}{2}, -\dfrac{9}{4}\right)$

In the reciprocal function, the maximum value in the interval between the vertical asymptotes will occur at the vertex where $x = -\dfrac{1}{2}$ and $y = \dfrac{1}{-\frac{9}{4}} = -\dfrac{4}{9}$.

Invariant points: Set the denominator equal to 1 and −1.

$y = 1$

$x^2 + x - 2 = 1$

$x^2 + x - 3 = 0$

Using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-3)}}{2}$$

$$x = \frac{-1 \pm \sqrt{1^2 + 12}}{2}$$

$$x = \frac{-1 + \sqrt{13}}{2} = 1.3$$

$$x = \frac{-1 - \sqrt{13}}{2} = -2.3$$

$y = -1$

$x^2 + x - 2 = -1$

$x^2 + x - 1 = 0$

Discriminant:

$$b^2 - 4ac = 1 - 4(1)(-1)$$
$$= 5$$

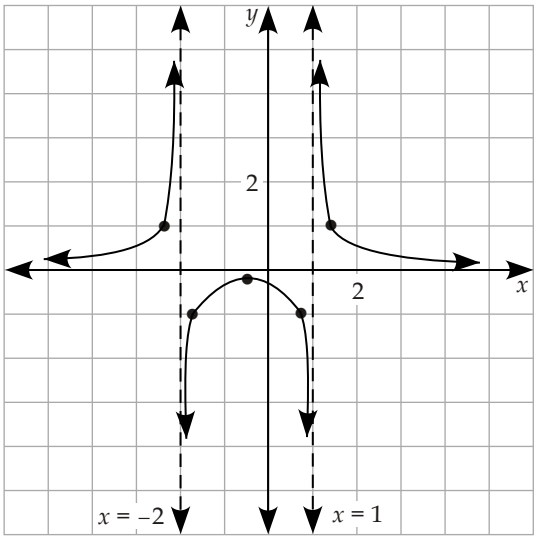
$$x = \frac{-1 \pm \sqrt{5}}{2}$$

$$x = \frac{-1 + \sqrt{5}}{2} = 0.6$$

$$x = \frac{-1 - \sqrt{5}}{2} = -1.6$$

Therefore, the four invariant points are (1.3, 1), (−2.3, 1), (0.6, −1), and (−1.6, −1).

Draw the curve: Plot the invariant points and the *y*-intercept. Draw the asymptotes and draw the curves.



The domain and range can be read from the graph:

Domain: $\{x \mid x \neq -2, 1, x \in \Re\}$

The range is $\left(-\infty, -\frac{4}{9}\right] \cup (0, \infty)$.

There is no x-intercept.

d) $f(x) = \dfrac{1}{x^2 - 2x - 8}$

Answer:

y-intercept: Find $f(0)$:

$$f(0) = \frac{1}{-8} = -\frac{1}{8}$$

Vertical asymptote: Let the denominator equal 0, find the zeros, and draw the dotted vertical asymptotes through those zeros.

$$x^2 - 2x - 8 = 0$$
$$(x - 4)(x + 2) = 0$$
$$x = 4, x = -2$$

Horizontal asymptote: The horizontal asymptote is the line $y = 0$.

Vertex of the parabola:

$$y = x^2 - 2x - 8$$
$$y = \left(x^2 - 2x + 1\right) - 8 - 1$$
$$y = (x - 1)^2 - 9$$

Vertex: $(1, -9)$

In the reciprocal function, the highest point in the interval between the vertical asymptotes is at the reciprocal of the vertex point where $x = 1$ and $y = \dfrac{1}{-9} = -\dfrac{1}{9}$.

Invariant points: Set the denominator equal to 1 and −1.

$y = 1$

$x^2 - 2x - 8 = 1$

$x^2 - 2x - 9 = 0$

Using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-9)}}{2}$$

$$x = \frac{2 \pm \sqrt{4 + 36}}{2}$$

$$x = \frac{2 \pm \sqrt{40}}{2}$$

$$x = \frac{2 \pm 2\sqrt{10}}{2}$$

$$x = 1 \pm \sqrt{10}$$

$y = -1$

$x^2 - 2x - 8 = -1$

$x^2 - 2x - 7 = 0$

Discriminant:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-7)}}{2}$$

$$x = \frac{2 \pm \sqrt{4 + 28}}{2}$$

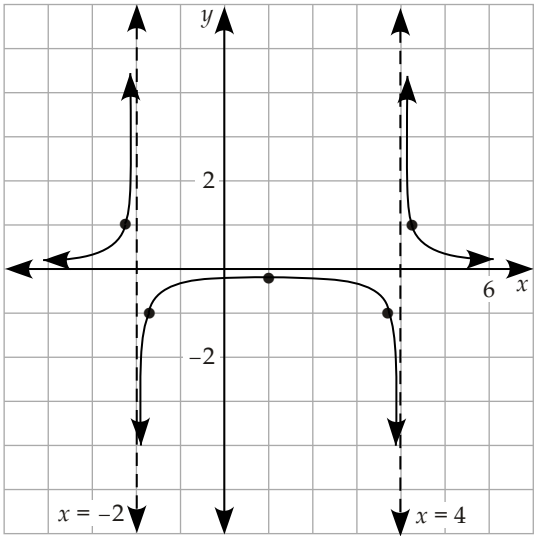
$$x = \frac{2 \pm \sqrt{32}}{2}$$

$$x = \frac{2 \pm 4\sqrt{2}}{2}$$

$$x = 1 \pm 2\sqrt{2}$$

Therefore, the four invariant points are $\left(1 + \sqrt{10}, 1\right)$, $\left(1 - \sqrt{10}, 1\right)$, $\left(1 + 2\sqrt{2}, -1\right)$, and $\left(1 - 2\sqrt{2}, -1\right)$. These points are approximately equal to (4.2, 1), (−2.2, 1), (3.8, −1), and (−1.8, −1).

Draw the curve: Plot the invariant points and the *y*-intercept. Draw the asymptotes and draw the curves.



The domain is $\{x \mid x \neq -2, 4, x \in \Re\}$.

The range is $\left(-\infty, -\frac{1}{9}\right] \cup (0, \infty)$.

There is no x-intercept.

e) $f(x) = \dfrac{1}{x^2 + 8x + 16}$

Answer:

y-intercept: Find $f(0)$:

$$f(0) = \frac{1}{0^2 + 8(0) + 16} = \frac{1}{16}$$

Vertical asymptote: The denominator will equal zero when $x^2 + 8x + 16 = 0$.

$x^2 + 8x + 16 = 0$

$(x + 4)(x + 4) = 0$

The vertical asymptote occurs at $x = -4$.

Horizontal asymptote: The horizontal asymptote is the line $y = 0$.

Vertex of parabola: The parabola is $y = x^2 + 8x + 16 = (x + 4)^2$. Its vertex is $(-4, 0)$. The corresponding point on the reciprocal function is $\left(-4, \frac{1}{0}\right)$, ,which does not exist. The vertical asymptote passes through the vertex of the original parabola since the vertex is on the x-axis.

Invariant points: Set the denominator equal to ± 1.

$y = 1$

$x^2 + 8x + 16 = 1$

$x^2 + 8x + 15 = 0$

$(x + 3)(x + 5) = 0$

$x = -3, -5$

$y = -1$

$x^2 + 8x + 16 = -1$

$x^2 + 8x + 17 = 0$

$b^2 - 4ac = 8^2 - 4(17)$

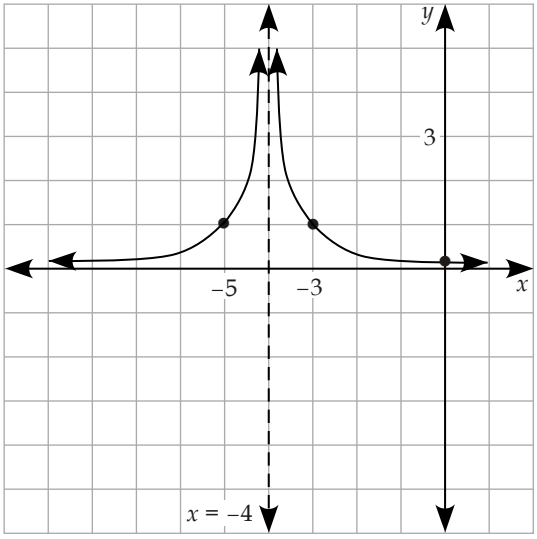
$= 64 - 68$

$= -4$

This equation has no solution as the discriminant is negative.

The two invariant points are $(-3, 1)$ and $(-5, 1)$.

Draw the curve: Plot the invariant points and the y-intercept. Draw the asymptotes and draw the curves.



The domain is $\{x \mid x \neq -4, x \in \Re\}$.

The range is $\{y \mid y > 0, y \in \Re\}$.

There is no x-intercept.

f) $f(x) = \dfrac{1}{x^2 - 2x + 6}$

Answer:

y-intercept: Find $f(0)$:

$$f(0) = \frac{1}{0^2 - 2(0) + 6} = \frac{1}{6}$$

Vertical asymptote: The denominator will equal zero when $x^2 - 2x + 6 = 0$.

$b^2 - 4ac = (-2)^2 - 4(1)(6)$

$= 4 - 24$

$= -20$

This equation has no solution as the discriminant is negative. Therefore, there are no vertical asymptotes.

Horizontal asymptote: The horizontal asymptote is the line $y = 0$.

Invariant points: Set the denominator equal to ±1.

$y = 1$	$y = -1$
$x^2 - 2x + 6 = 1$	$x^2 - 2x + 6 = -1$
$x^2 - 2x + 5 = 0$	$x^2 - 2x + 7 = 0$
$b^2 - 4ac = (-2)^2 - 4(5)$	$b^2 - 4ac = (-2)^2 - 4(7)$
$= 4 - 20$	$= 4 - 28$
$= -16$	$= -24$

Both of these equations have no solution as both discriminants are negative. Thus, there are no invariant points.

To help graph this reciprocal, find the coordinates of the vertex of the quadratic function located in the denominator of $f(x)$. Then, find the reciprocal of that point.

Vertex:

$$y = x^2 - 2x + 6$$

$$x = -\frac{b}{2a} = -\frac{-2}{2} = \frac{2}{2} = 1$$

Once you find the x-coordinate of the vertex, you can substitute that value back into the quadratic function to determine the y-coordinate of the vertex.

$$y = (1)^2 - 2(1) + 6 = 1 - 2 + 6 = 5$$

The vertex is at (1, 5).

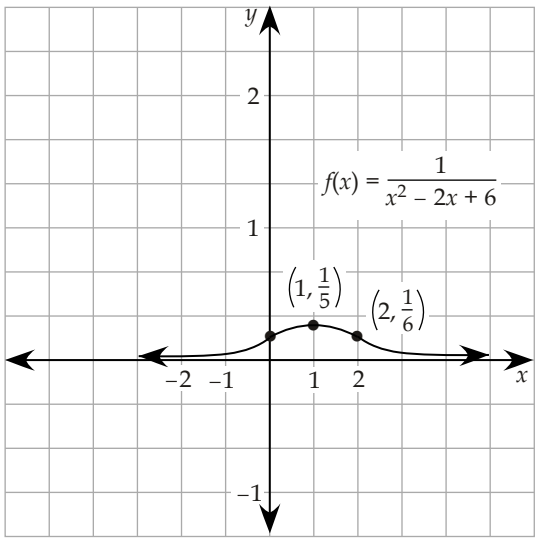
The corresponding point in the reciprocal function is $\left(1, \frac{1}{5}\right)$.

Now find points on each side of the vertex.

x	$y = x^2 - 2x + 6$	$y = \frac{1}{x^2 - 2x + 6}$
0	6	$\frac{1}{6}$
1	5	$\frac{1}{5}$
2	6	$\frac{1}{6}$

Plot the points of the reciprocal function and draw the curve smoothly toward the horizontal asymptote as x goes to infinity to the left and right.

Draw the curve:



The domain is $\{x \mid x \in \Re\}$.

The range is $\left\{y \mid 0 < y \leq \frac{1}{5}, y \in \Re\right\}$.

There is no x-intercept.

Learning Activity 8.5

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing out many steps on paper.

1. Eric has a big crush on a girl sitting 6 desks away from him. If each desk is 80 cm wide and there are no spaces between them, how far away (in metres) is Eric sitting from his crush?
2. Simplify: $(3^2)^3$
3. What is the slope of the line defined by the equation $3y + 2x - 7 = 0$?
4. If 36% of 1500 is 540, what is 18% of 1500?
5. State the reciprocal of. $\dfrac{x^2 + 3x - 4}{x^2 - 2x + 1}$.
6. Express as a power: $\sqrt[3]{16^3}$
7. (45, −72) is on a line with a slope of $\dfrac{20}{17}$. Find another point on this line.
8. Simplify the following fraction: $\dfrac{252}{462}$

Answers:

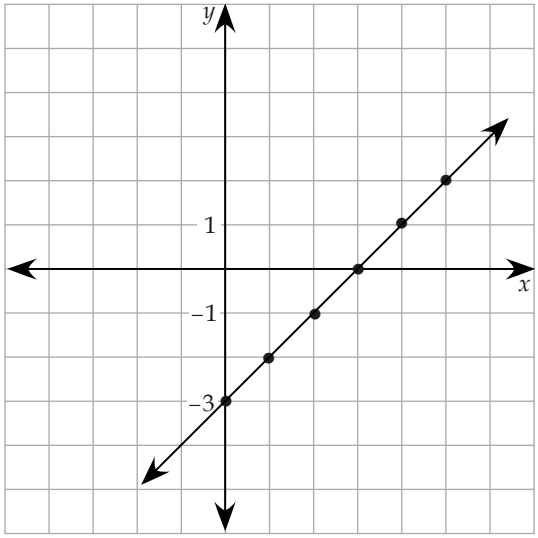
1. 4.8 m ($6 \times 80 = 480$ cm = 4.8 m)
2. $3^6 = 729$ ($9 \times 9 \times 9 = 81 \times 9$; $80 \times 9 = 720 + 1 \times 9 = 9$)
3. $-\dfrac{2}{3}$ $\left(3y = -2x + 7;\ y = -\dfrac{2}{3}x + \dfrac{7}{3}\right)$
4. 270 $\left(\dfrac{540}{2}\right)$
5. $\dfrac{x^2 - 2x + 1}{x^2 + 3x - 4}$
6. 16 $\left(\sqrt[3]{16^3} = 16^{\frac{3}{3}} = 16\right)$
7. (62, −52) (45 + 17, −72 + 20)
8. $\dfrac{6}{11}$ $\left(\dfrac{252}{462} = \dfrac{126}{231} = \dfrac{42}{77} = \dfrac{6}{11}\right)$

Part B: Reciprocal Functions

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn them.

1. Given the graph of $y = f(x)$, sketch the graph of $y = \frac{1}{f(x)}$. Describe the method you used.

a)

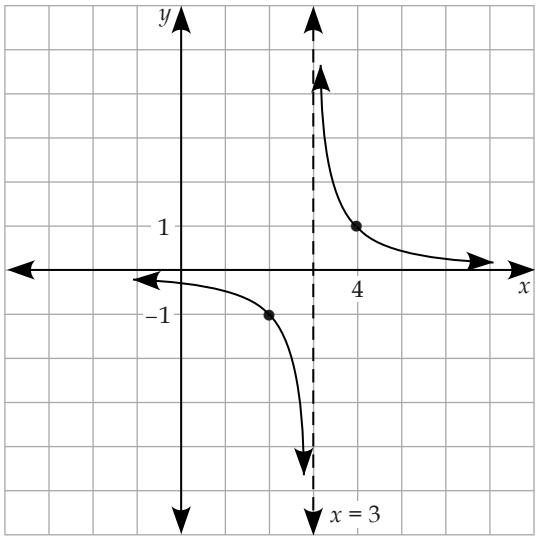


Answer:

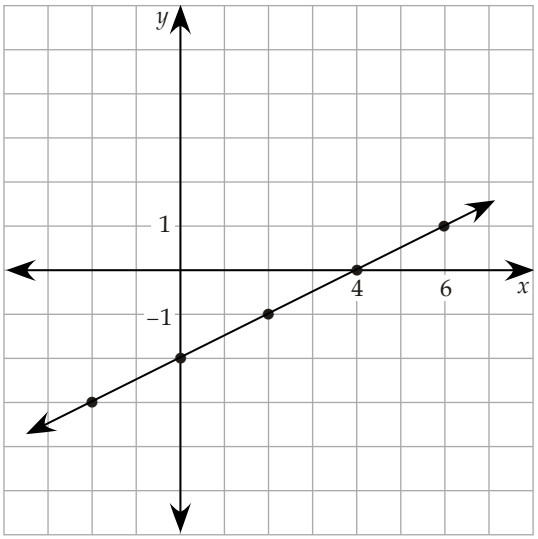
From the graph, you can see that the x-intercept is 3. Therefore, the vertical asymptote in the reciprocal function will be $x = 3$. Also, the two invariant points of this function will be (4, 1) and (2, −1), as these are the points at which the y-coordinate is either 1 or −1.

The horizontal asymptote is $y = 0$.

Using this information, you can graph the reciprocal function.



b)

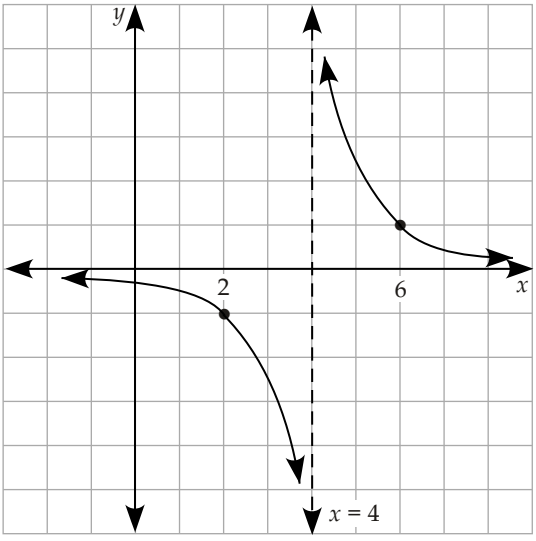


Answer:

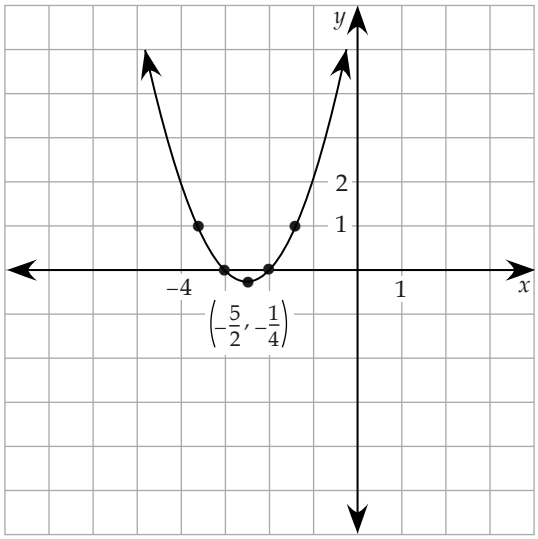
From the graph, you can see that the x-intercept is 4. Therefore, the vertical asymptote in the reciprocal function will be $x = 4$. Also, the two invariant points of this function will be (6, 1) and (2, −1), as these are the
points at which the y-coordinate is either 1 or −1.

The horizontal asymptote is $y = 0$.

Using this information, you can graph the reciprocal function.



c)



Answer:

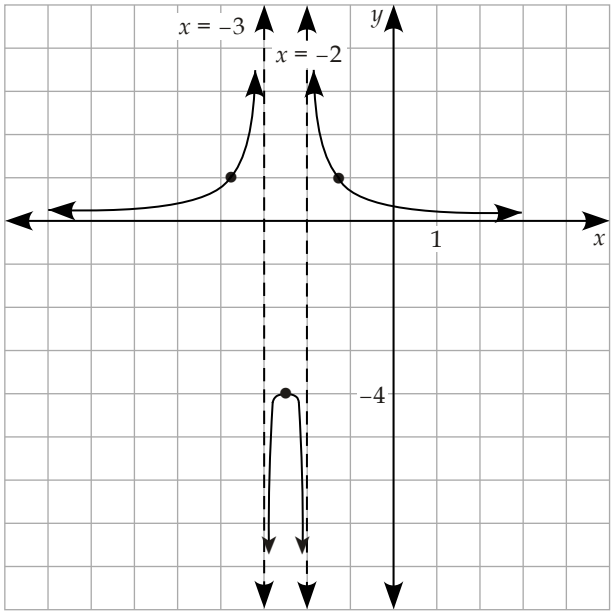
From the graph, you can see that the x-intercepts are -2 and -3. Therefore, the vertical asymptotes in the reciprocal function will have equations of $x = -2$ and $x = -3$.

The invariant points on the function are approximately at (−1.4, 1) and (−3.6, 1).

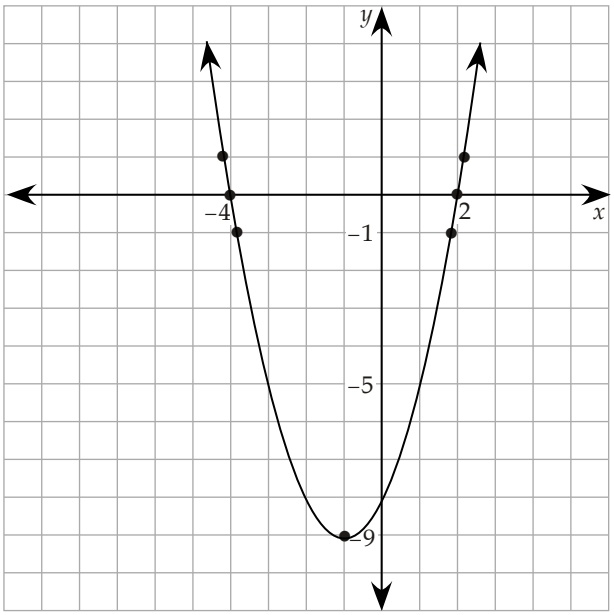
The key point of the parabola is at the vertex at $\left(-\frac{5}{2}, -\frac{1}{4}\right)$.

The corresponding point on the reciprocal function is $\left(-\frac{5}{2}, -4\right)$.

Using this information, you can graph the reciprocal function.



d)



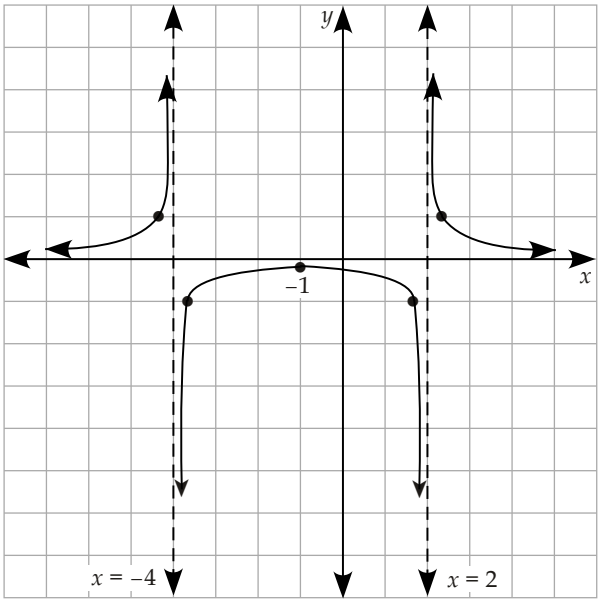
Answer:

From the graph, you can see that the x-intercepts are 2 and -4. Therefore, the vertical asymptotes in the reciprocal function will have equations of $x = 2$ and $x = -4$.

The vertex of the parabola is at $(-1, -9)$. The corresponding point on the reciprocal function is $\left(-1, -\frac{1}{9}\right)$.

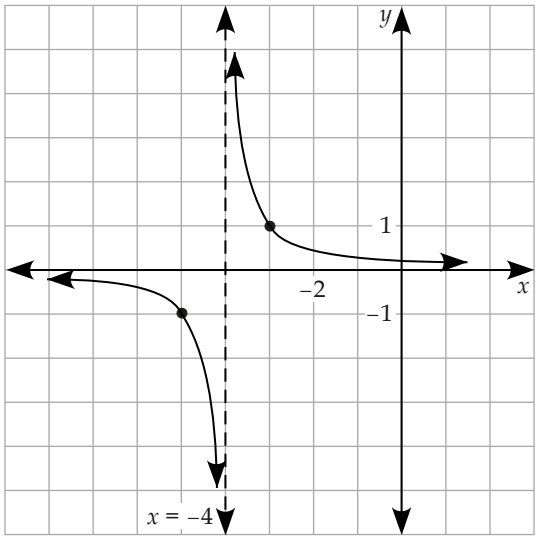
The invariant points are approximately at $(-4.21, 1)$, $(2.2, 1)$ and $(-3.8, -1)$, $(1.8, -1)$.

Using this information, you can graph the reciprocal function.



2. Given the graph of $y = \frac{1}{f(x)}$, sketch the graph of $y = f(x)$. Describe the method you used.

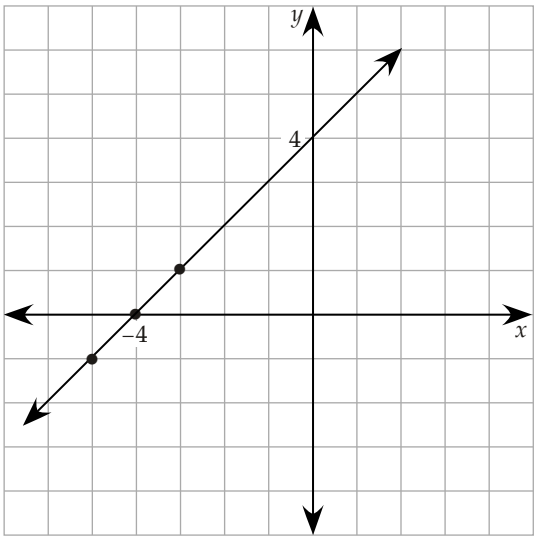
a)



Answer:

The vertical asymptote of this reciprocal graph is at $x = -4$. Therefore, there will be an x-intercept on $f(x)$ as $(-4, 0)$.

There are also two points given: $(-3, 1)$ and $(-5, -1)$. These are invariant points and therefore they stay the same on the graph of $f(x)$. Using these three pieces of information, it is possible to graph $f(x)$.



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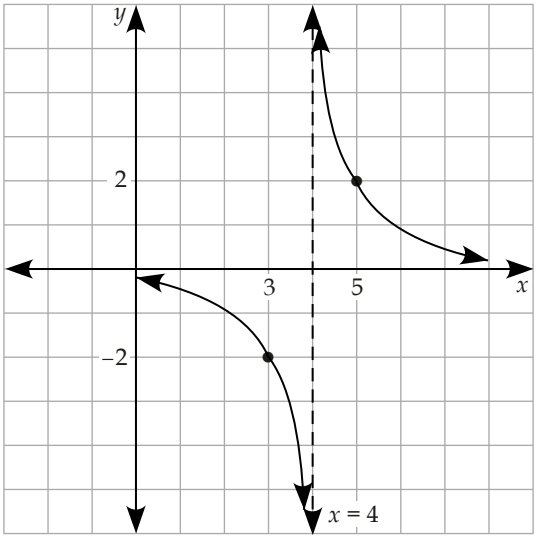
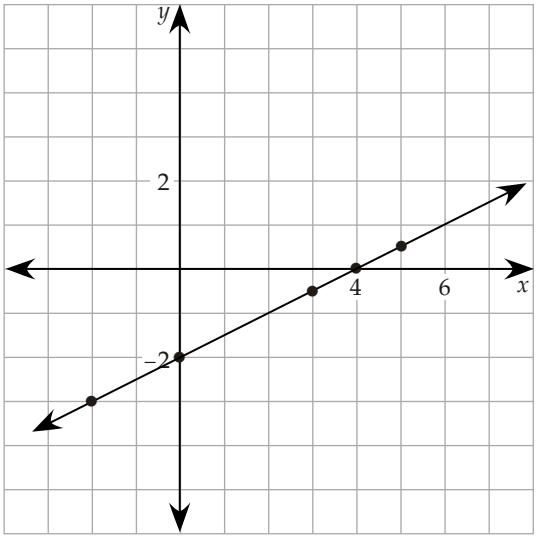


Answer:

The vertical asymptote of this reciprocal graph is at $x = 4$. Therefore, there will be an x-intercept on $f(x)$ as (4, 0).

There are also two points given: (5, 2) and (3, −2). The points on the reciprocal of the function are $\left(5, \frac{1}{2}\right)$ and $\left(3, -\frac{1}{2}\right)$. Using these three pieces of information, it is possible to graph $f(x)$.



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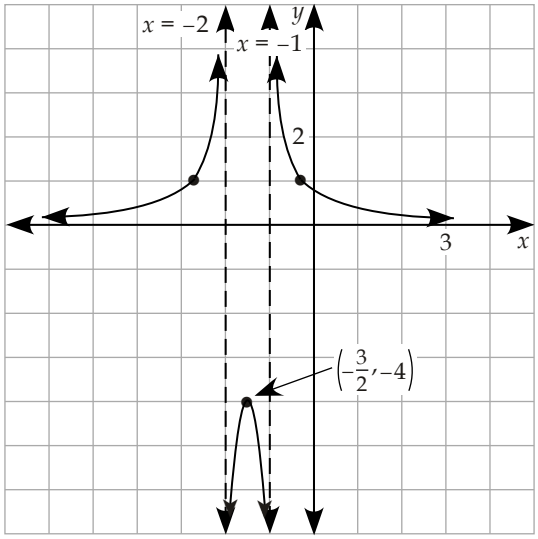
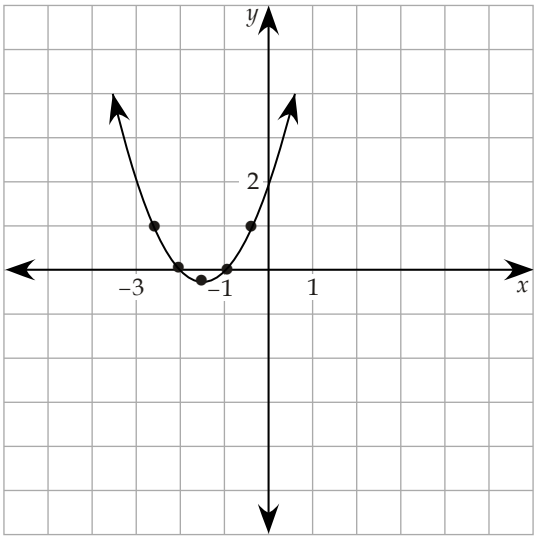


Answer:

The vertical asymptotes of this reciprocal graph occur at $x = -1$ and $x = -2$ and are thus the x-intercepts in the graph of $f(x)$. In other words, $f(x)$ has the form of $a(x + 1)(x + 2)$.

The one given point is $\left(-\frac{3}{2}, -4\right)$. The corresponding point on the reciprocal of the function is $\left(-\frac{3}{2}, -\frac{1}{4}\right)$ and it will be the vertex of the parabola.

The invariant points are approximately at (−2.6, 1) and (−0.4, 1).



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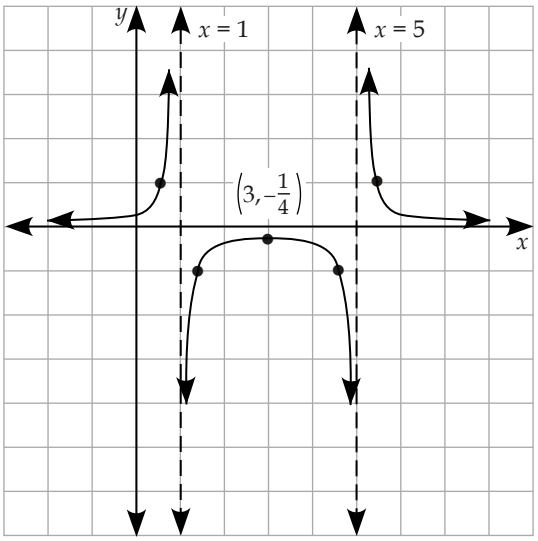


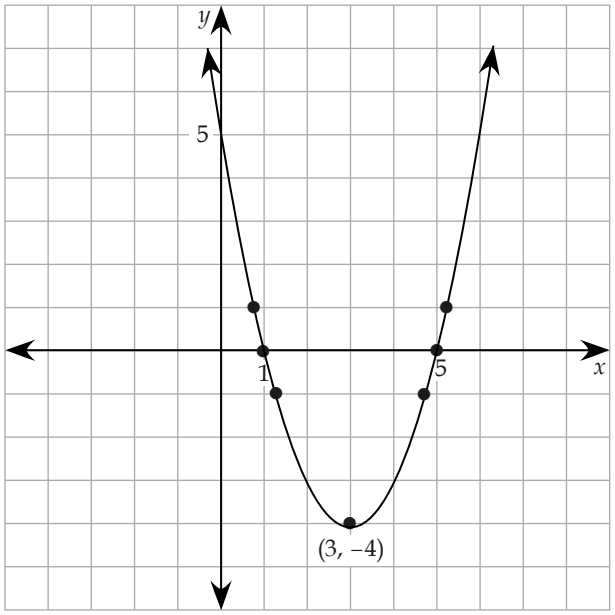
Answer:

The vertical asymptotes of this graph occur at $x = 1$ and $x = 5$ and are thus the x-intercepts in the graph of $f(x)$. In other words, $f(x)$ is of the form $a(x - 1)(x - 5)$ or $f(x) = a(x - 1)(x - 5)$.

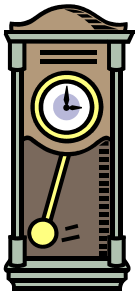
The key point is $\left(3, -\frac{1}{4}\right)$. The corresponding point on $f(x)$ is $(3, -4)$.

It is the vertex of the parabola.

Reading from the graph, the invariant points are approximately at (0.8, 1), (5.2, 1) and (1.3, −1), (4.7, −1).



3. The movement of a pendulum in a grandfather clock can be represented using a reciprocal formula. The frequency, f, is equal to the reciprocal of the period, T. The period is measured in seconds, and measures the time for one cycle, or, in other words, how long it takes the pendulum to swing back and forth once. The frequency is the number of cycles per second.



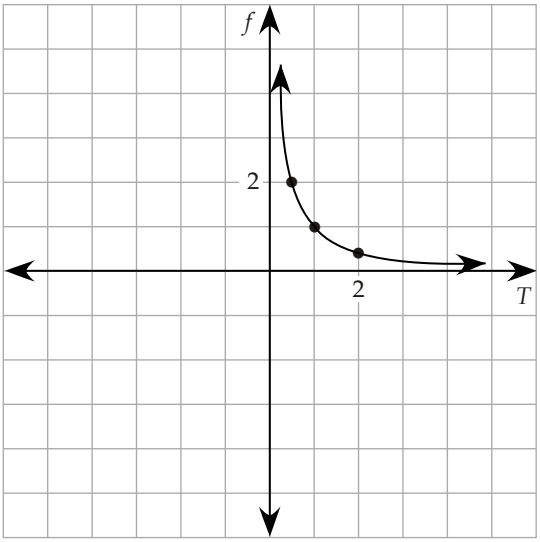
a) Write the formula for the frequency of a pendulum.

Answer:

$$f = \frac{1}{T}$$

b) Sketch the graph of the function in (a).

Answer:



c) Using the graph as well as the formula in (a), determine the period of a pendulum when the frequency is 2.1 cycles per second.

Answer:

Using the graph, the period of a pendulum with a frequency of 2.1 cycles per second is approximately 0.4 s.

Using the formula in (a): $f = \frac{1}{T}$

$$2.1 = \frac{1}{T}$$

$$T = \frac{1}{2.1}$$

$$T = 0.4762 \text{ seconds}$$

Grade 11 Pre-Calculus Mathematics (30S)

Glossary

Appendix A: Glossary

absolute value of an expression
A positive number of zero that is denoted by vertical bars surrounding an expression (it represents the distance a number is from the origin); the absolute value of an expression is defined as:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

absolute value equation
An equation that involves the absolute value of a variable.

absolute value function

$$y = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

acute angle
An angle that is less than 90° and more than 0°.

ambiguous case
A situation that may arise when solving oblique triangles in which there may be zero, one, or two solutions.

arithmetic sequence
A list of numbers in which each succeeding term is found by adding the same value to the previous term; it is also the range of a linear function whose domain is the natural numbers.

arithmetic sequence formula for the general term
$t_n = t_1 + (n - 1)d$

arithmetic series
The sum of consecutive terms in an arithmetic sequence.

asymptote
A line that a function approaches, but never touches or crosses as its domain values or its range values approach ∞ or $-\infty$.

axis of symmetry
A vertical line passing through the vertex of a quadratic function which divides the parabola into two symmetrical parts (*see* symmetrical).

boundary line
A line or curve of the graph of an inequality that divides the Cartesian plane into two regions; this line will be solid if the inequality involves the signs $\geq$ or $\leq$. This line will be broken or dashed if the inequality involves the signs > or <.

coefficient
The number multiplying the variable(s) in a term.

Example

7 is the coefficient of $7x^3$

conjugate of a binomial
The conjugate of $a + b$ is $a - b$ and the conjugate of $a - b$ is $a + b$; there is only one sign change.

Example
the conjugate of $2 + \sqrt{3}$ is $2 - \sqrt{3}$

common difference
The value that is added to each term in an arithmetic sequence in order to get the next term.

common ratio
The value that is multiplied to each term in a geometric sequence in order to get the next term.

completing the square
The process of changing a quadratic function written in general form $f(x) = ax^2 + bx + c$ into vertex form $f(x) = a(x - p)^2 + q$.

constant term
An expression that has no variables. It is either a number or a symbol that represents a number, such as π.

convergent series
A series with an infinite number of terms in which its sequence of partial sums approaches a fixed value.

Cosine Law
Defined as $c^2 = a^2 + b^2 - 2ab \cos C$, where a, b, and c are sides of ΔABC, and is used to solve oblique triangles.

difference of squares
A binomial where the terms are subtracted and each term is a perfect square.

discriminant
The expression $b^2 - 4ac$ is called the discriminant. It enables you to determine the nature of the roots of a quadratic equation without actually finding the roots.

divergent series
A series with an infinite number of terms in which the sequence of partial sums does not approach a fixed value.

domain
The set of all x-values for which a function is defined.

double root/double zero
A value that appears as a root or a zero of a quadratic equation twice.

entire radical
An equivalent form of a mixed radical expression.

Example
the mixed radical, $2\sqrt{3}$, is equivalent to $\sqrt{12}$, which is an entire radical

exact value
An answer that does not contain approximated decimal values.

exponential function
A function in the form of $y = ab^x$.

extraneous root
A number obtained when solving an equation that does not satisfy the initial restrictions on the variable, or is not a root of the original equation.

factor
To write an expression as a product.

factored form of a quadratic function
$y = a(x - r_1)(x - r_2)$, where a is the leading coefficient and r_1 and r_2 are the x-intercepts.

finite series
A series that terminates or ends.

function
A relation such that for each x-value, there is exactly one value of y.

geometric sequence
A sequence in which the ratio of any term and its preceding term is a fixed number.

geometric sequence formula for the general term
$t_n = t_1 r^{n-1}$

geometric series
The sum of the terms of a geometric sequence.

greatest common factor (GCF)

The greatest number or term that is a factor of two or more numbers or terms.

Examples:

- the greatest common factor of 12 and 18 is 6
- the greatest common factor of $2xy$ and $2yz$ is $2y$

horizontal asymptote

The line $y = b$ is a horizontal asymptote of the graph of f if $f(x) \to b$ as $x \to \infty$ or $x \to -\infty$.

horizontal shift

The graph of $f(x) = (x - p)^2$ is said to be obtained from the graph of $y = x^2$ by a horizontal shift of p units left if $p < 0$ and p units right if $p > 0$.

improper fraction

A fraction that is larger than one where the numerator is greater than the denominator; some texts also include 1 as being an improper fraction.

index of a radical

The number n in the expression, $\sqrt[n]{a}$.

Example

the index of the radical, $\sqrt[3]{8}$, is 3

infinite geometric series

A geometric series that has an unlimited number of terms.

infinite series

Any series that has an unlimited number of terms.

integer

A number that can be written without a fractional or decimal part and may be written with a negative sign; it is a number in the set $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

invariant points

Points on a graph that remain in the same location on both the graph of $y = f(x)$ and the graph of $y = \frac{1}{f(x)}$, or, in general, an equivalent point on $f(x)$ and its transformed graph; their x-coordinates are equal and their y-coordinates are equal.

inverse operation's property for radicals

$\sqrt[n]{a^n} = a$, where $a \geq 0$, $n \geq 2$, $n \in N$

irrational number

A number that cannot be written in the form, $\frac{a}{b}$, where $a \in \mathrm{N}$, $b \in \mathrm{N}$; an irrational number can be thought of as a decimal number with an infinite number of digits that never repeat.

leading coefficient

Coefficient of the highest power of a polynomial expression or equation.

Example

4 is the leading coefficient of $4x^3 - 2x + 3$.

least common multiple (LCM)

LCM of two or more polynomials is the simplest polynomial that is a multiple of each of the original polynomials.

linear function

A function whose graph is an oblique or horizontal line; its form is $f(x) = mx + b$ or $y - y_1 = m(x - x_1)$ or $y = b$.

lowest common denominator (LCD)

LCD of two or more fractions is the least common multiple (LCM) of the denominators of the fractions.

maximum value
The largest value that a function attains over its domain.

Example

the function $f(x) = -x^2 + 2$ has a maximum value of 2.

minimum value
The smallest value that a function attains over its domain.

Example

the function $f(x) = x^2 + 5$ has a minimum value of 5

mixed radical
The product of a real number and a radical.

monomial
A polynomial with only one term, where the variable has a whole number exponent.

multiple root
The root of an equation that occurs more than once.

Example

In the cubic equation $(x - 3)^2(x - 1) = 0$, the root of 3 occurs twice so 3 is a multiple root.

natural number
Any number in the set of numbers, N = {1, 2, 3, . . .}.

nature of the roots
Describes how many and what type of roots a quadratic equation has.

non-permissible values
The numbers excluded from the domain of a rational expression and which make the value of the denominator equal zero.

oblique triangles
Triangles that do not have a right angle.

parabola
The graph of a quadratic function, which has a U-shape.

perfect square trinomial
A quadratic expression that is the square of a binomial.

Example

$x^2 + 6x + 9$ is the square of $(x + 3)$.

piece-wise function
Function defined by different formulas in different parts of its domain.

point of intersection
The location on a plane that two functions have in common, written as an ordered pair, (x, y).

polynomial
A mathematical expression with one or more terms; an addition, subtraction, or multiplication of monomials.

power of a power law for exponents
When a power is raised to a power, you can simplify the expression by multiplying the exponents.

Example

$(2^3)^4 = 2^{3\times4} = 2^{12}$

principal square root
The positive square root of a number.

product property of radicals

$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$

where: $a \geq 0$

$b \geq 0$

n is a positive integer ≥ 2

If n is an odd number, $a, b \in \Re$.

proper fraction
A fraction whose numerator is less than its denominator.

quadrant

One of four sections of a coordinate grid separated by horizontal and vertical axes; quadrants are numbered counter-clockwise, beginning with the top right-hand quadrant, as 1st, 2nd, 3rd, and 4th (or I, II, III, and IV). The points on the axes are not included in any quadrant.

quadrantal angle

An angle that has as its initial side a ray from the origin along the positive x-axis and its terminal side along one of the axes.

quadratic equation

An equation that can be written in the general form $ax^2 + bx + c = 0$, where a, b, and c are real numbers, and $a \neq 0$.

quadratic formula

Quadratic formula is used to find the roots of a quadratic equation written in the form $ax^2 + bx + c = 0$; the formula is $x = \dfrac{-b \pm \sqrt{b^2 + 4ac}}{2a}$

quadratic function

A polynomial function of degree 2

Example

$y = x^2 - 2x + 1$ or $y = (x - 3)(x - 2)$

quotient property of radicals

$\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

where: $a \geq 0$

$b > 0$

n is a positive integer ≥ 2

If n is an odd number, then $a, b \in \Re$ and $b \neq 0$.

radical

Another name for a root, such as a square root or cube root.

radical symbol

$\sqrt{\ }$

radicand

Everything included under the radical symbol in a radical.

range

The set of all y-values for which a function is defined.

rational exponent

The exponent of an expression that is a rational number; an expression with a rational exponent can be written in radical form.

Example

$64^{\frac{2}{3}}$ can be written as $\left(\sqrt[3]{64}\right)^2$ or as $\sqrt[3]{64^2}$

rational expression

$\dfrac{p}{q}$ is a rational expression if p and q are polynomials, $q \neq 0$ and q contains a variable.

rational number

$\dfrac{a}{b}$ is a rational number if a and b are integers and $b \neq 0$.

rationalized denominator

A denominator that does not contain radicals.

real number

A number which is either a rational number or an irrational number.

reciprocal

Obtained by interchanging the numerator and the denominator.

Example

the reciprocal of x is $\dfrac{1}{x}$ and the reciprocal of a polynomial $f(x)$ is $\dfrac{1}{f(x)}$

recursive definition/recursive formula

When, after the first term or the first few terms of a sequence, the remaining terms are defined in terms of the value(s) of the preceding term(s).

reference angle

The positive acute angle formed between the terminal arm of θ and the nearer portion of the x-axis, denoted by θ_r.

relation

A set of ordered pairs, (x, y).

restrictions for the value of the radicand

Values of a variable that ensure the radicand is not negative.

Example

the restrictions for the radical expression $\sqrt{x-5}$ are $x \geq 5$

roots

The solutions of an equation; also called the zeros of the function or the x-intercepts of the corresponding graph.

sigma notation

The symbol $\sum_{i=1}^{n} t_i$; it represents the sum $t_1 + t_2 + t_3 + \ldots + t_n$.

Sine Law

Used to solve oblique ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \text{ or}$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

slope formula

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

solution region of an inequality

The set of all ordered pairs that make the inequality true.

solution set

The set of values or the interval of values that makes the equation or inequality true.

square of a binomial

For $(a + b)^2$ is $a^2 + 2ab + b^2$ and is a perfect square trinomial.

standard position of an angle

- its first side, called the initial side, is a ray from the origin along the positive x-axis
- its second side, called the terminal side, is any ray from the origin
- the angle is measured from the initial to the terminal side in a counter-clockwise direction and has a positive value

sum of an arithmetic series

For any arithmetic series $t_1 + t_2 + t_3 + \ldots + t_n$ its sum is

$$S_n = \frac{n}{2}\left(2t_1 + (n-1)d\right) \text{ or } S_n = \left(\frac{t_1 + t_n}{2}\right)(n).$$

sum of infinite geometric series formula

$$S_\infty = \frac{t_1}{1-r}, |r| < 1$$

sum of geometric series formula

$$S_n = \frac{t_1\left(1 - r^n\right)}{1-r}$$

symmetrical

Having symmetry; two sides of a shape are balanced about a line or point; two sides of a shape are mirror images; a shape may have one or more lines of symmetry.

symmetry

The property of having the same size and shape across a dividing line or around a point.

term

A number or the product of a number and a variable(s).

Example

the expression $7x^2 + 2x + 5$ has three terms.

trinomial

A polynomial in the form $ax^2 + bx + c$, where a, b, and c are non-zero real numbers.

vertex

Turning point of a parabola.

vertex form of a quadratic function

$y = f(x)$ is $f(x) = a(x - p)^2 + q$, where a, p, and q are real numbers and the vertex is (p, q).

vertical asymptote

The line $x = a$ is a vertical asymptote of the graph of f if $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a$, either from the right or from the left side of a.

vertical compression

If $0 < a < 1$ and a is a real number, the graph of $f(x) = ax^2$ can be obtained from the graph of $y = x^2$ by multiplying each of the y-values by a; the resulting parabola will be wider than the parabola $y = x^2$.

vertical shift

The graph of $f(x) = ax^2 + q$ is obtained from the graph of $y = ax^2$ by moving it up q units if $q > 0$ and down q units if $q < 0$.

vertical stretch

If $a > 1$, the graph of $f(x) = ax^2$ can be obtained from the graph of $y = x^2$ by multiplying the y-values by a; the resulting parabola will be narrower than the parabola $y = x^2$.

x-intercept

The x-coordinate of the point where the graph of the function intersects the x-axis; the y-coordinate of this point is zero.

y-intercept

The y-coordinate of the point where the graph of the function intersects the y-axis; the x-coordinate of this point is zero.

zero product property

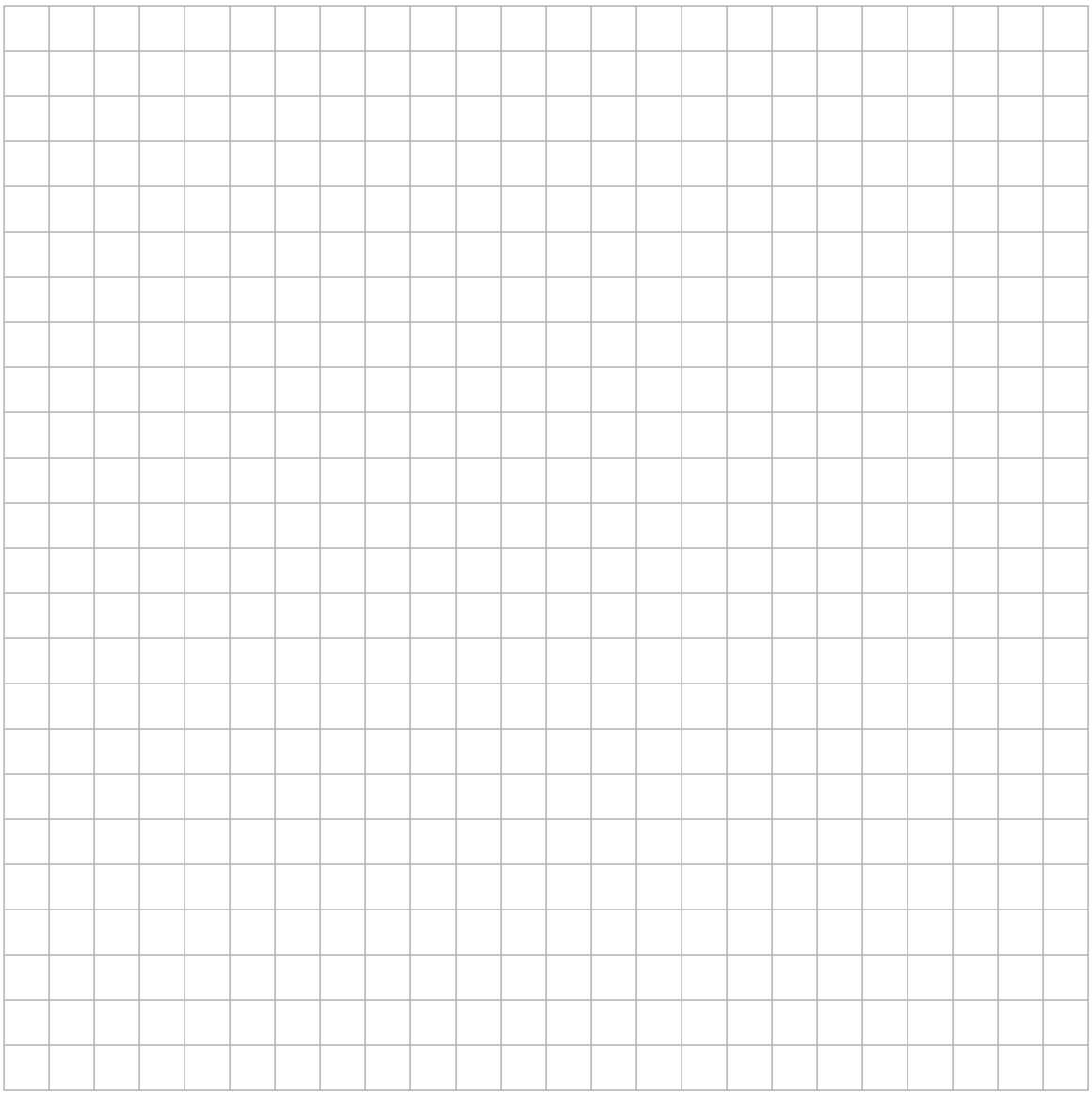
If $ab = 0$, then either $a = 0$ or $b = 0$ must be true; in other words, for the property to apply, the equation must be equal to zero and the expression must be a product obtained by factoring.

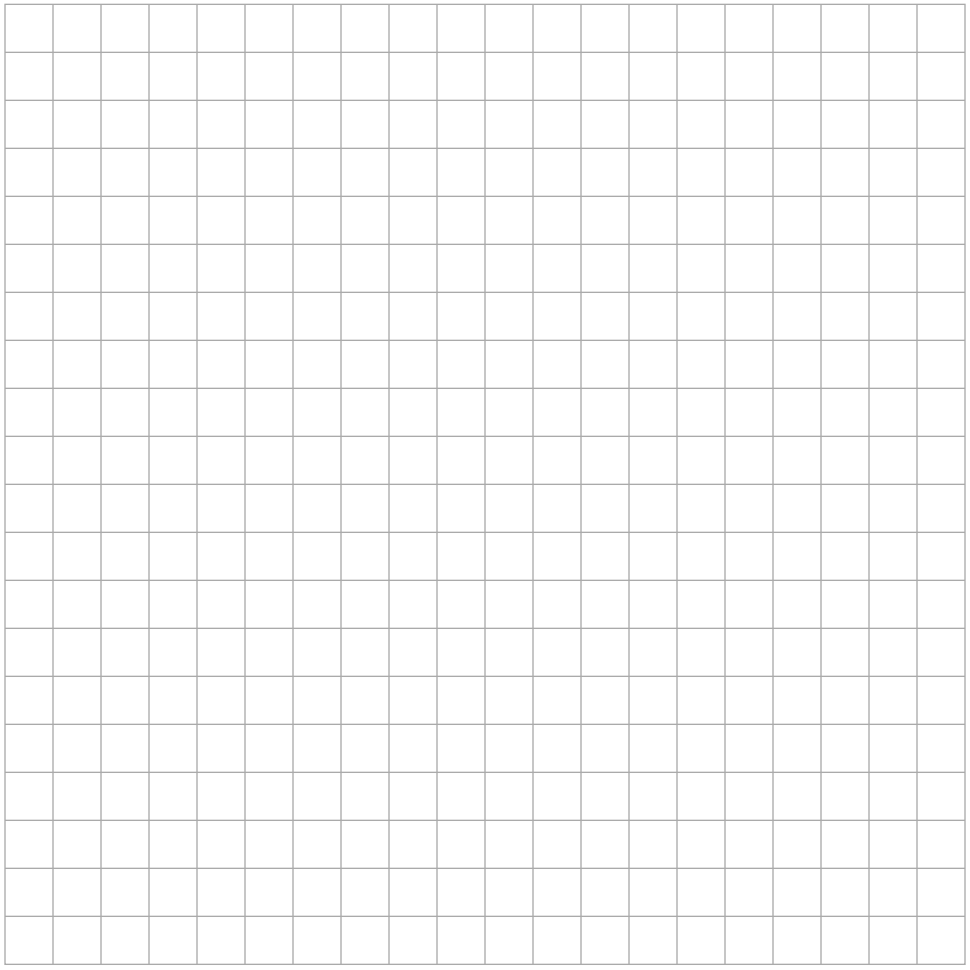
zeros

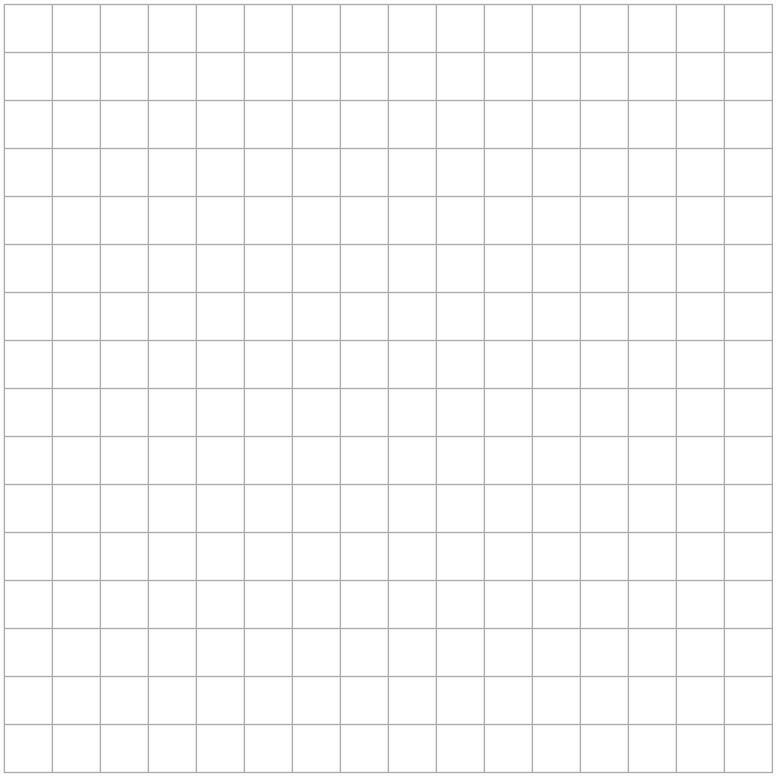
The value(s) of x which make the function zero; these values of x also are called the x-intercepts of the graph and occur where the graph crosses or touches the x-axis.

Grade 11 Pre-Calculus Mathematics (30S)

Graph Paper







Grade 11 Pre-Calculus Mathematics (30S)

Midterm Practice Examination

Grade 11 Pre-Calculus Mathematics

Midterm Practice Examination

Name: _____

Student Number: _____

Attending ❑ Non-Attending ❑

Phone Number: _____

Address: _____

_____

_____

For Marker's Use Only

Date: _____

Final Mark: _____ /100 = _____ %

Comments:

Instructions

The midterm examination will be weighted as follows:

Modules 1–4	100%

The format of the examination will be as follows:

Module 1: Sequences and Series	24 marks
Module 2: Factoring and Rational Expressions	23 marks
Module 3: Quadratic Functions	25 marks
Module 4: Solving Rational and Quadratic Equations	28 marks

Time allowed: 2.5 hours

Note: You are allowed to bring the following to the examination: pencils (2 or 3 of each), blank paper, a ruler, a scientific or graphing calculator, and your Midterm Exam Resource Sheet. Your Midterm Exam Resource Sheet must be handed in with the examination. You will receive your Midterm Exam Resource Sheet back from your tutor/marker with the next module work that is submitted for marking.

Show all calculations and formulas used. Include units where appropriate. Clearly state your final answer.

Name: _____

Answer all questions to the best of your ability. Show all your work.

Module 1: Sequences and Series *(24 marks)*

1. For each sequence below, indicate whether it is geometric, arithmetic, or neither. If it is geometric, state the value of r (the common ratio). If it is arithmetic, state the value of d (the common difference). *(1 mark each, for a total of 3 marks)*

   a) $81, 9, 1, \frac{1}{9}, \ldots$

   b) $1, 6, 9, 10, \ldots$

   c) $1, 4, 7, 10, \ldots$

2. Explain the relationship between arithmetic sequences and linear functions. *(2 marks)*

3. Write the defining linear function of the following arithmetic sequence. *(2 marks)*

   3, 8, 13, . . .

4. Which term of the arithmetic sequence, $-4, 2, 8, \ldots$, is the number 170? *(3 marks)*

Name: _____

5. Find the sum of the first 125 terms defined by $t_1 = 4$, $t_n = t_{n-1} + 7$. *(3 marks)*

6. Find the total distance travelled by a ball in coming to rest, if it is dropped from a height of 12 m and it rebounds $\frac{2}{3}$ of its previous height every time it hits the ground *(4 marks)*

7. Consider the geometric sequence $-1, 2, -4, \ldots$

   a) Write the function for the geometric sequence. *(2 marks)*

   b) Using this function, find t_{14} of the sequence. *(1 mark)*

   c) Which term of the geometric sequence is the number -4096? *(2 marks)*

Name: _____

8. If possible, find the value of $\sum_{k=1}^{\infty} 64\left(\frac{3}{4}\right)^{k-1}$. *(2 marks)*

Module 2: Factoring and Rational Expressions *(23 marks)*

1. Factor the following expressions completely.

   a) $x^2y - 7xy + 10y$ *(2 marks)*

   b) $4x^2 + 12x + 9$ *(2 marks)*

   c) $36x^2 - 64y^2$ *(1 mark)*

   d) $25(x - 4)^2 - 49(y - 5)^2$ *(3 marks)*

Name: _____

2. Without factoring, determine whether the binomial $(x - 1)$ is a factor of the following polynomial. Explain. *(2 marks)*

   $2x^2 - x - 1$

3. Create an equivalent rational expression for the following rational expression. Explain how you know the rational expression you created is equivalent to the original rational expression. State the non-permissible values of your rational expression. Do not simplify your answer. *(2 marks)*

   $$\frac{x + 2}{x^2}$$

4. Perform the indicated operation and simplify your answer. State the non-permissible values.

a) $\dfrac{x+1}{x^2+x-6}+\dfrac{x^2}{x^2-x-2}$ *(4 marks)*

b) $\dfrac{x^2+x-2}{x^2-x-6}\times\dfrac{x^2+x-12}{2x^2-2x-40}$ *(3 marks)*

Name: _____

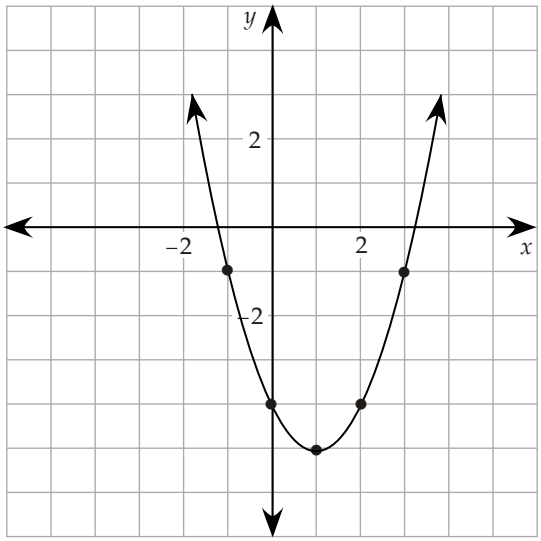
c) $\dfrac{2x^2-8}{4x^2} \div \dfrac{x^2-x-2}{x^3+x^2} \times \dfrac{8x}{x^2-x-6}$ *(3 marks)*

5. Create a rational expression with the following non-permissible values. Do not simplify your answer. *(1 mark)*

   -3 and -5

Module 3: Quadratic Functions *(25 marks)*

1. Given the graph below, answer the following questions.



a) What are the coordinates of the vertex? *(1 mark)*

b) What is the range? *(1 mark)*

c) What is the equation of the axis of symmetry? *(1 mark)*

d) State whether the graph has a maximum or a minimum value and what that value is. *(1 mark)*

e) Write a quadratic function in vertex form to represent this graph. *(2 marks)*

Name: _____

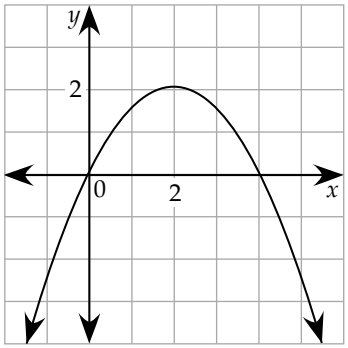
2. Match each equation to its corresponding graph. Place (a), (b), (c), and (d) next to its corresponding graph below. *(1 mark each, for a total of 4 marks)*

   a) $y = 2(x - 2)^2 - 2$

   b) $y = -\frac{1}{2}(x - 2)^2 + 2$

   c) $y = -\frac{1}{2}(x + 2)^2 + 2$

   d) $y = 2(x + 2)^2 - 2$



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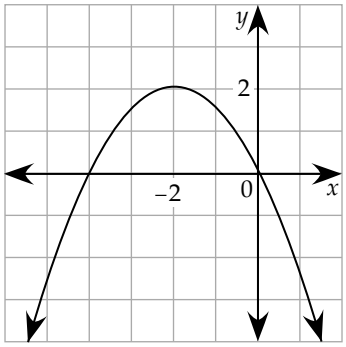

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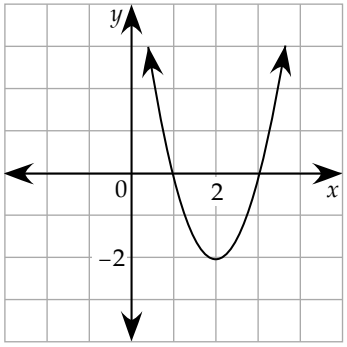

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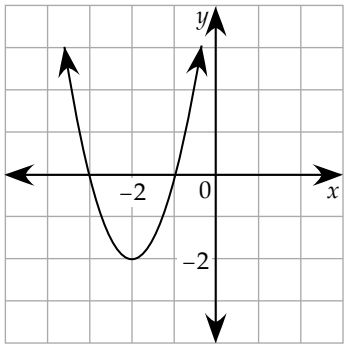

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3. For what value of k is the expression $x^2 - 11x + k$ a perfect square trinomial? *(1 mark)*

4. Consider the parabola $y = 3x^2 + 12x - 36$.
   a) Write the function in vertex form by completing the square. *(2 marks)*

   b) Find the coordinates of the vertex. *(1 mark)*

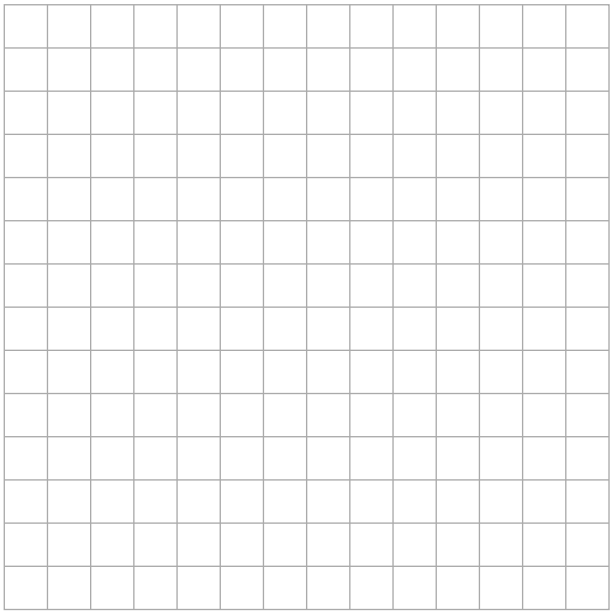
   c) Find the y-intercept. *(1 mark)*

   d) State the domain. *(1 mark)*

   e) Write an equation for the axis of symmetry. *(1 mark)*

Name: _____

f) Sketch the graph. *(2 marks)*



g) Find the x-intercepts from the graph. *(1 mark)*

5. A projectile is shot straight up from a height of 40 m with an initial velocity of 30 m/s. Its height in metres after t seconds is given by $h(t) = 40 + 30t - 5t^2$. *(1.5 marks each, for a total of 3 marks)*

   a) After how many seconds does the projectile reach its maximum height?

   b) Find the maximum height above the ground that the ball reaches.

6. Consider the following quadratic functions. Determine how many x-intercepts the corresponding graph has by considering the values of a and q. *(1 mark each, for a total of 2 marks)*

   a) $y = -2(x + 3)^2 - 15$

   b) $y = 9(x + 2)^2$

Name: _____

Module 4: Solving Rational and Quadratic Equations *(28 marks)*

1. Find the roots of the following equations. Explain which method you used to solve each equation and why.

   a) $x^2 - 9x + 18 = 0$ *(2 marks)*

   b) $x^2 - 5x - 1 = 0$ *(3 marks)*

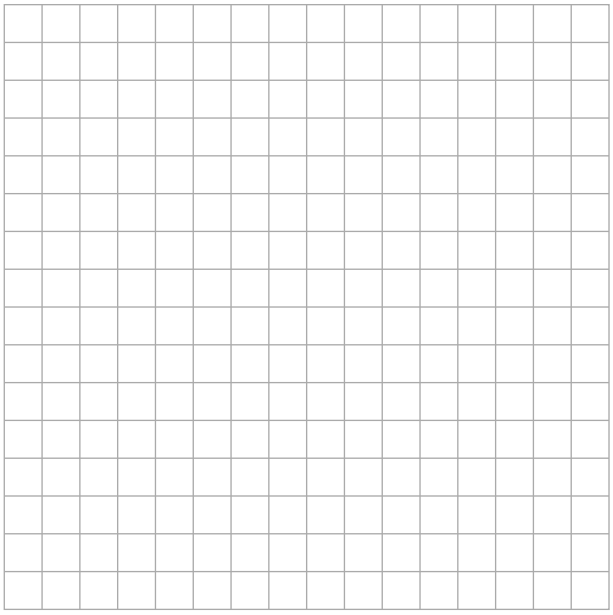
2. Use the quadratic formula to find the roots of $2x^2 + 6x - 1 = 0$. *(2 marks)*

3. Solve $x^2 - x - 3 = 0$ by completing the square. *(3 marks)*

Name: _____

4. Solve $(x - 7)^2 = 36$ by taking square roots. *(2 marks)*

5. Solve $(x - 2)^2 - 1 = 0$ by graphing. *(2 marks)*



6. Solve for x. State the non-permissible values of the rational equation. *(4 marks)*

$$\frac{x-2}{x+3} - 3 = \frac{4(x+3)}{x-2}$$

Name: _____

7. The discriminant of the quadratic equation $4x^2 + 2x - k = 0$ is 84. *(1 mark each, for a total of 2 marks)*

   a) Find the value of k.

   b) State the nature of the roots without solving for the roots.

8. Tia can ride her bicycle 1 km/h faster than Denzel. They begin at the same spot and at the same time, travelling in opposite directions. After travelling for the same length of time, they stop and realize that Tia has travelled 21 km and Denzel has travelled 19 km. How fast was Denzel travelling? *(4 marks)*

   Recall: Distance = Rate $\times$ Time, or Time = $\frac{\text{Distance}}{\text{Rate}}$

Name: _____

9. Andrea can mow a lawn by herself in 4 hours. Samantha can mow the same lawn in 3 hours. If the girls work together, how long will it take to mow this lawn? *(4 marks)*

Grade 11 Pre-Calculus Mathematics (30S)

Midterm Practice Examination
Answer Key

Grade 11 Pre-Calculus Mathematics

Midterm Practice Examination
Answer Key

Name: _____

Student Number: _____

Attending ❑ Non-Attending ❑

Phone Number: _____

Address: _____

_____

_____

For Marker's Use Only

Date: _____

Final Mark: _____ / 100 = _____ %

Comments:

Answer Key

Instructions

The midterm examination will be weighted as follows:

Modules 1–4	100%

The format of the examination will be as follows:

Module 1: Sequences and Series	24 marks
Module 2: Factoring and Rational Expressions	23 marks
Module 3: Quadratic Functions	25 marks
Module 4: Solving Rational and Quadratic Equations	28 marks

Time allowed: 2.5 hours

Note: You are allowed to bring the following to the examination: pencils (2 or 3 of each), blank paper, a ruler, a scientific or graphing calculator, and your Midterm Exam Resource Sheet. Your Midterm Exam Resource Sheet must be handed in with the examination. You will receive your Midterm Exam Resource Sheet back from your tutor/marker with the next module work that is submitted for marking.

Show all calculations and formulas used. Include units where appropriate. Clearly state your final answer.

Name: _____

Answer all questions to the best of your ability. Show all your work.

Module 1: Sequences and Series *(24 marks)*

1. For each sequence below, indicate whether it is geometric, arithmetic, or neither. If it is geometric, state the value of *r* (the common ratio). If it is arithmetic, state the value of *d* (the common difference). *(1 mark each, for a total of 3 marks)* (Lesson 3)

   a) $81, 9, 1, \frac{1}{9}, \ldots$

   Answer:

   Geometric sequence with a common ratio of $\frac{1}{9}$.

   b) $1, 6, 9, 10, \ldots$

   Answer:

   Neither.

   c) $1, 4, 7, 10, \ldots$

   Answer:

   Arithmetic sequence with a common difference of 3.

2. Explain the relationship between arithmetic sequences and linear functions. *(2 marks)* (Lesson 1)

   Answer:

   If the domain of a linear function is the set of natural numbers (or positive integers), the range is called an arithmetic sequence. Also, the common difference of an arithmetic sequence is the same as the slope in the corresponding linear function.

3. Write the defining linear function of the following arithmetic sequence. *(2 marks)* (Lesson 1)

   3, 8, 13, . . .

   Answer:

   One point on line: (1, 3)

   Slope: $d = 8 - 3 = 5$

   Defining linear function: $y - y_1 = m(x - x_1)$

   $y - 3 = 5(x - 1)$

   $y = 5x - 5 + 3$

   $y = 5x - 2$

4. Which term of the arithmetic sequence, $-4, 2, 8, \ldots$, is the number 170? *(3 marks)* (Lesson 1)

   Answer:

   Method 1: Using the general term formula $t_n = t_1 + (n - 1)d$:

   $170 = -4 + (n - 1)(6)$

   $170 + 4 = 6n - 6$

   $174 + 6 = 6n$

   $180 = 6n$

   $30 = n$

   Therefore, the 30th term is 170.

   Method 2: Using the defining linear equation:

   $m = 6$ and points $(1, -4)$ and $(n, 170)$

   $y - y_1 = m(x - x_1)$

   $170 - (-4) = 6(n - 1)$

   $174 = 6(n - 1)$

   $29 = n - 1$

   $30 = n$

   Therefore, the 30th term is 170.

Name: _____

5. Find the sum of the first 125 terms defined by $t_1 = 4$, $t_n = t_{n-1} + 7$. *(3 marks)* (Lesson 2)

   Answer:

   The arithmetic sequence is 4, 11, 18, . . .

   The common difference = 7

   To find t_{125}, use the formula $t_n = t_1 + (n - 1)d$

   $t_{125} = 4 + (125 - 1)7$

   $t_{125} = 4 + 875 - 7$

   $t_{125} = 872$

   Sum of the series: $S_{125} = \frac{4 + 872}{2}(125) = 54\,750$

   Or $S_n = \frac{n}{2}\left[2t_1 + (n-1)d\right]$

   $S_{125} = \frac{125}{2}\left[2(4) + (125-1)(7)\right]$

   $S_{125} = \frac{125}{2}\left[8 + (124)(7)\right]$

   $S_{125} = 54\,750$

6. Find the total distance travelled by a ball in coming to rest, if it is dropped from a height of 12 m and it rebounds $\frac{2}{3}$ of its previous height every time it hits the ground *(4 marks)* (Lesson 5)

   Answer:

   down series: $12 + \frac{2}{3}(12) + \left(\frac{2}{3}\right)^2(12) + \ldots$

   down distance: $\frac{12}{1 - \frac{2}{3}} = 36$ m

   up series: $\frac{2}{3}(12) + \left(\frac{2}{3}\right)^2(12) + \left(\frac{2}{3}\right)^3(12) + \ldots$

   up distance = down distance − 12 = 24 m

   Total distance travelled = up distance + down distance = 24 m + 36 m = 60 m.

7. Consider the geometric sequence $-1, 2, -4, \ldots$ (Lesson 3)

a) Write the function for the geometric sequence. *(2 marks)*

Answer:

$t_1 = -1$

$r = \frac{2}{-1} = -2$

$t_n = t_1 r^{n-1}$

$t_n = -1(-2)^{n-1}$

b) Using this function, find t_{14} of the sequence. *(1 mark)*

Answer:

$t_{14} = -1(-2)^{14-1}$

$t_{14} = -1(-2)^{13}$

$t_{14} = -1(-8192)$

$t_{14} = 8192$

c) Which term of the geometric sequence is the number -4096? *(2 marks)*

Answer:

$t_n = -1(-2)^{n-1}$

$-4096 = -1(-2)^{n-1}$

$4096 = (-2)^{n-1}$

$(-2)^{12} = (-2)^{n-1}$

$12 = n - 1$

$13 = n$

Therefore, -4096 is the 13th term in this geometric sequence.

Name: _____

8. If possible, find the value of $\sum_{k=1}^{\infty} 64\left(\frac{3}{4}\right)^{k-1}$. *(2 marks)* (Lesson 5)

Answer:

$r = \frac{3}{4}$ and r satisfies the condition: $|r| < 1$

$t_1 = 64$

$S_\infty = \frac{t_1}{1-r}$ if $|r| < 1$

$S_\infty = \frac{64}{1-\frac{3}{4}} = 256$

Module 2: Factoring and Rational Expressions *(23 marks)*

1. Factor the following expressions completely. (Lessons 1 and 2)

   a) $x^2y - 7xy + 10y$ *(2 marks)*

   Answer:

   $= y(x^2 - 7x + 10)$

   $= y(x - 2)(x - 5)$

   b) $4x^2 + 12x + 9$ *(2 marks)*

   Answer:

   $= 4x^2 + 6x + 6x + 9$

   $= 2x(2x + 3) + 3(2x + 3)$

   $= (2x + 3)(2x + 3)$

   $= (2x + 3)^2$

   c) $36x^2 - 64y^2$ *(1 mark)*

   Answer:

   $= 4(9x^2 - 16y^2)$

   $= 4(3x - 4y)(3x + 4y)$

   d) $25(x - 4)^2 - 49(y - 5)^2$ *(3 marks)*

   Answer:

   $25(x - 4)^2 - 49(y - 5)^2$

   Let $w = x - 4$

   $t = y - 5$

   $25w^2 - 49t^2$

   $(5w - 7t)(5w + 7t)$

   $[5(x - 4) - 7(y - 5)][5(x - 4) + 7(y - 5)]$

   $[5x - 20 - 7y + 35][5x - 20 + 7y - 35]$

   $(5x - 7y + 15)(5x + 7y - 55)$

Name: _____

2. Without factoring, determine whether the binomial $(x - 1)$ is a factor of the following polynomial. Explain. *(2 marks)* (Lesson 2)

   $2x^2 - x - 1$

   Answer:

   If $(x - 1)$ were a factor of this polynomial, then when $x = 1$ is substituted into the polynomial, the polynomial would equal zero.

   $2(1)^2 - 1 - 1 = 2 - 1 - 1 = 1 - 1 = 0$

   This polynomial equals zero when $x = 1$ is substituted in for x. Therefore, $(x - 1)$ is a factor of this polynomial.

3. Create an equivalent rational expression for the following rational expression. Explain how you know the rational expression you created is equivalent to the original rational expression. State the non-permissible values of your rational expression. Do not simplify your answer. *(2 marks)* (Lesson 3)

   $$\frac{x+2}{x^2}$$

   Answer:

   Answers may vary. A sample answer is:

   $$\frac{x+2}{x^2} = \frac{(x+2)(x-2)}{x^2(x-2)}$$

   The numerator and denominator are both multiplied by the same non-zero polynomial. Because any polynomial divided by itself is one, you are essentially multiplying the expression by 1. Anything multiplied by 1 is equal to itself.

   The non-permissible values are $x = 0$ and $x = 2$.

4. Perform the indicated operation and simplify your answer. State the non-permissible values.

a) $\dfrac{x+1}{x^2+x-6}+\dfrac{x^2}{x^2-x-2}$ *(4 marks)* (Lesson 5)

Answer:

$$\frac{x+1}{x^2+x-6}+\frac{x^2}{x^2-x-2}$$

$$=\frac{x+1}{(x-2)(x+3)}+\frac{x^2}{(x-2)(x+1)}$$

$$\text{LCD}=(x-2)(x+3)(x+1)$$

$$=\frac{(x+1)(x+1)}{(x-2)(x+3)(x+1)}+\frac{x^2(x+3)}{(x-2)(x+1)(x+3)}$$

$$=\frac{x^2+2x+1+x^3+3x^2}{(x-2)(x+3)(x+1)}$$

$$=\frac{x^3+4x^2+2x+1}{(x-2)(x+3)(x+1)}$$

Non-permissible values: $x = 2, -3, -1$

b) $\dfrac{x^2+x-2}{x^2-x-6}\times\dfrac{x^2+x-12}{2x^2-2x-40}$ *(3 marks)* (Lesson 4)

Answer:

$$\frac{x^2+x-2}{x^2-x-6}\times\frac{x^2+x-12}{2x^2-2x-40}$$

$$=\frac{(x+2)(x-1)}{(x-3)(x+2)}\times\frac{(x+4)(x-3)}{2\left(x^2-x-20\right)}$$

$$=\frac{(x+2)(x-1)}{(x-3)(x+2)}\times\frac{(x+4)(x-3)}{2(x-5)(x+4)}$$

$$=\frac{x-1}{2(x-5)}$$

Non-permissible values: $x = 3, -2, 5, -4$

Name: _____

c) $\frac{2x^2-8}{4x^2} \div \frac{x^2-x-2}{x^3+x^2} \times \frac{8x}{x^2-x-6}$ *(3 marks)* (Lesson 4)

Answer:

$$\frac{2x^2-8}{4x^2} \div \frac{x^2-x-2}{x^3+x^2} \times \frac{8x}{x^2-x-6}$$

$$= \frac{2\left(x^2-4\right)}{4x^2} \times \frac{x^3+x^2}{x^2-x-2} \times \frac{8x}{x^2-x-6}$$

$$= \frac{2(x-2)(x+2)}{4x^2} \times \frac{x^2(x+1)}{(x-2)(x+1)} \times \frac{8x}{(x-3)(x+2)}$$

$$= \frac{4x}{x-3}$$

Non-permissible values: $x = 0, -1, 2, 3, -2$

5. Create a rational expression with the following non-permissible values. Do not simplify your answer. *(1 mark)* (Lesson 3)

   –3 and –5

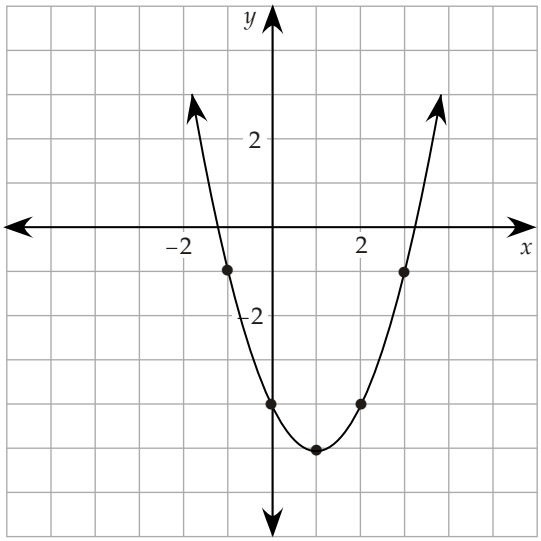
   Answer:

   If -3 and -5 are non-permissible values, then $(x + 3)$ and $(x + 5)$ are factors of the denominator of the rational expression. Answers may vary. However, $(x + 3)$ and $(x + 5)$ have to be factors of the denominator. A sample answer is:

$$\frac{1}{(x+3)(x+5)}$$

Module 3: Quadratic Functions *(25 marks)*

1. Given the graph below, answer the following questions. (Lessons 1 to 4)



a) What are the coordinates of the vertex? *(1 mark)*

Answer:

$(1, -5)$

b) What is the range? *(1 mark)*

Answer:

R: $\{y \mid y \geq -5\}$ or $[-5, \infty)$ or $\{y \mid y \geq -5, y \in \Re\}$

c) What is the equation of the axis of symmetry? *(1 mark)*

Answer:

$x = 1$

d) State whether the graph has a maximum or a minimum value and what that value is. *(1 mark)*

Answer:

It has a minimum value when $y = -5$.

e) Write a quadratic function in vertex form to represent this graph. *(2 marks)*

Answer:

Since the graph is normal width, the value of a is 1.

$y = (x - 1)^2 - 5$

Name: _____

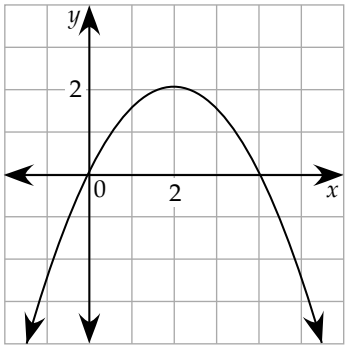
2. Match each equation to its corresponding graph. Place (a), (b), (c), and (d) next to its corresponding graph below. *(1 mark each, for a total of 4 marks)* (Lessons 1 to 4)

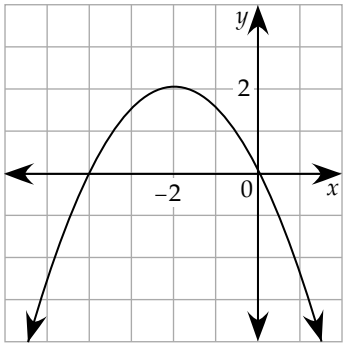
   a) $y = 2(x - 2)^2 - 2$

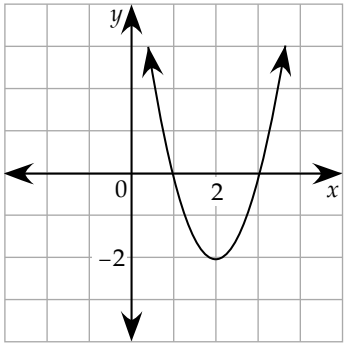
   b) $y = -\frac{1}{2}(x-2)^2 + 2$

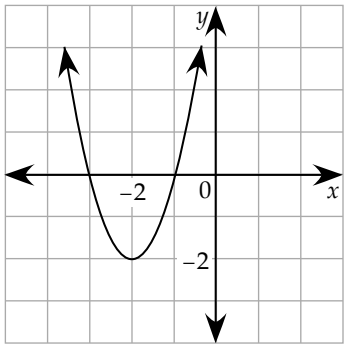
   c) $y = -\frac{1}{2}(x + 2)^2 + 2$

   d) $y = 2(x + 2)^2 - 2$

(b) 

(c) 

(a) 

(d) 

3. For what value of k is the expression $x^2 - 11x + k$ a perfect square trinomial? *(1 mark)* (Lesson 5)

   Answer:

$$\left(\frac{-11}{2}\right)^2 = \frac{121}{4}$$

4. Consider the parabola $y = 3x^2 + 12x - 36$. (Lesson 5)

   a) Write the function in vertex form by completing the square. *(2 marks)*

   Answer:

   $y = 3(x^2 + 4x) - 36$

   $y = 3(x^2 + 4x + 4) - 36 - 4(3)$

   $y = 3(x + 2)^2 - 48$

   b) Find the coordinates of the vertex. *(1 mark)*

   Answer:

   $(-2, -48)$

   c) Find the y-intercept. *(1 mark)*

   Answer:

   Let $x = 0$

   $y = 3x^2 + 12x - 36$

   $y = 3(0)^2 + 12(0) - 36$

   $y = -36$

   d) State the domain. *(1 mark)*

   Answer:

   D: $\{x \mid x \in \Re\}$ or $(-\infty, \infty)$

   e) Write an equation for the axis of symmetry. *(1 mark)*

   Answer:

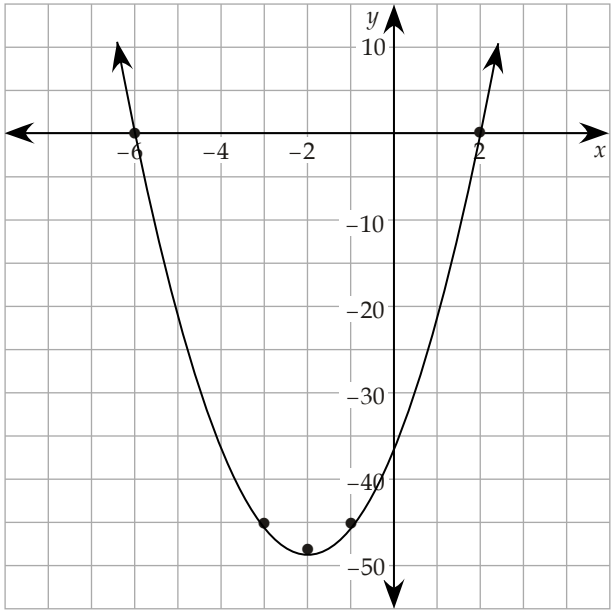
   $x = -2$

Name: _____

f) Sketch the graph. *(2 marks)*

Answer:

The vertex is $(-2, -48)$. The curve opens up and has a narrow width. The y-values are three times the normal ones.



g) Find the x-intercepts from the graph. *(1 mark)*

Answer:

The x-intercepts are located at $x = -6$ and $x = 2$.

5. A projectile is shot straight up from a height of 40 m with an initial velocity of 30 m/s. Its height in metres after t seconds is given by $h(t) = 40 + 30t - 5t^2$. *(1.5 marks each, for a total of 3 marks)* (Lesson 7)

   a) After how many seconds does the projectile reach its maximum height?

   Answer:

   $h(t) = -5t^2 + 30t + 40$

   Time at maximum height $= -\frac{b}{2a} = -\frac{30}{2(-5)} = 3$ seconds

   Note: This question could also be solved by completing the square to find the vertex at (3, 85), where 3 seconds is the answer to (a) and 85 metres is the answer to (b).

   b) Find the maximum height above the ground that the ball reaches.

   Answer:

   Maximum height $= h(3) = -5(3)^2 + 30(3) + 40$

   $= 85$ metres

6. Consider the following quadratic functions. Determine how many x-intercepts the corresponding graph has by considering the values of a and q. *(1 mark each, for a total of 2 marks)* (Lesson 6)

   a) $y = -2(x + 3)^2 - 15$

   Answer:

   In this equation, $a = -2 < 0$, and $q = -15 < 0$. Therefore, as both a and q are less than zero, the corresponding graph has no x-intercepts.

   Or

   The graph opens down and the vertex is below the x-axis. There are no x-intercepts.

   b) $y = 9(x + 2)^2$

   Answer:

   In this equation, $a = 9 > 0$, and $q = 0$. Therefore, the corresponding graph has one (1) x-intercept, as the vertex is on the x-axis.

   Or

   The graph opens up and the vertex is on the x-axis. There is one x-intercept.

Name: _____

Module 4: Solving Rational and Quadratic Equations *(28 marks)*

1. Find the roots of the following equations. Explain which method you used to solve each equation and why. (Lessons 1 to 4)

   a) $x^2 - 9x + 18 = 0$ *(2 marks)*

   Answer:

   This equation is easily factorable.

   $(x - 6)(x - 3) = 0$

   $x = 6, 3$

   b) $x^2 - 5x - 1 = 0$ *(3 marks)*

   Answer:

   This equation is not easily factorable. Also, if you were to complete the square, there would be fractions. Therefore, the quadratic formula should be used.

   Note: If you use the completing-the-square method, you should still arrive at the correct answer.

   $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

   $$x^2 - 5x - 1 = 0$$

   $$a = 1, b = -5, c = -1$$

   $$x = \frac{5 \pm \sqrt{(-5)^2 - 4(1)(-1)}}{2(1)}$$

   $$x = \frac{5 \pm \sqrt{25 + 4}}{2}$$

   $$x = \frac{5 \pm \sqrt{29}}{2}$$

   $$x = \frac{5 + \sqrt{29}}{2}, \frac{5 - \sqrt{29}}{2}$$

   Note: Unless specifically mentioned, you may use any method to solve quadratic equations.

2. Use the quadratic formula to find the roots of $2x^2 + 6x - 1 = 0$. *(2 marks)* (Lesson 4)

Answer:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 2,\ b = 6,\ c = -1$$

$$x = \frac{-6 \pm \sqrt{6^2 - 4(2)(-1)}}{2(2)}$$

$$x = \frac{-6 \pm \sqrt{36 + 8}}{4}$$

$$x = \frac{-6 \pm \sqrt{44}}{4}$$

$$x = \frac{-6 \pm \sqrt{4}\sqrt{11}}{4}$$

$$x = \frac{-6 \pm 2\sqrt{11}}{4}$$

$$x = \frac{-3 \pm \sqrt{11}}{2}$$

$$x = \frac{-3 + \sqrt{11}}{2}, \frac{-3 - \sqrt{11}}{2}$$

Name: _____

3. Solve $x^2 - x - 3 = 0$ by completing the square. *(3 marks)* (Lesson 4)

   Answer:

$$x^2 - x - 3 = 0$$
$$x^2 - x = 3$$
$$x^2 - x + \frac{1}{4} = 3 + \frac{1}{4}$$
$$\left(x - \frac{1}{2}\right)^2 = \frac{13}{4}$$
$$x - \frac{1}{2} = \pm\sqrt{\frac{13}{4}}$$
$$x = \frac{1}{2} \pm \frac{\sqrt{13}}{2}$$
$$x = \frac{1 + \sqrt{13}}{2}, \frac{1 - \sqrt{13}}{2}$$

4. Solve $(x - 7)^2 = 36$ by taking square roots. *(2 marks)* (Lesson 2)

   Answer:

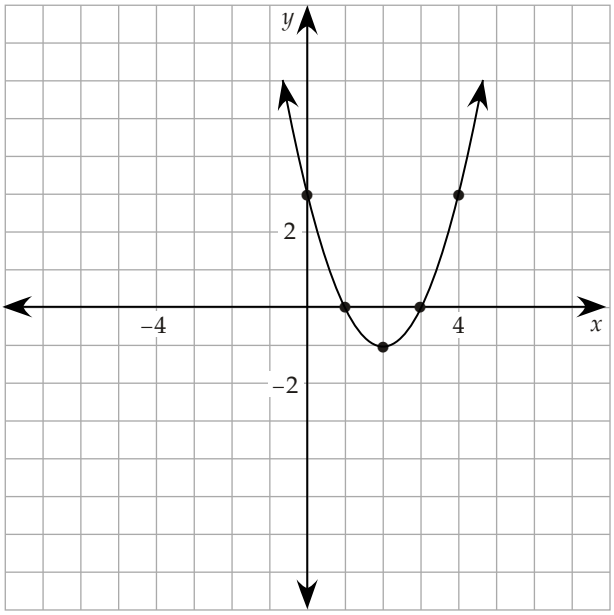
$$(x - 7)^2 = 36$$
$$x - 7 = \pm 6$$
$$x = 7 \pm 6$$
$$x = 7 + 6, 7 - 6$$
$$x = 13, 1$$

5. Solve $(x - 2)^2 - 1 = 0$ by graphing. *(2 marks)* (Lesson 1)

Answer:

Let $y = (x - 2)^2 - 1$

Vertex is $(2, -1)$; the curve opens up and its width is normal.



The x-intercepts (or zeros, or roots) of this quadratic equation are 3 and 1.

Name: _____

6. Solve for x. State the non-permissible values of the rational equation. *(4 marks)* (Lesson 6)

$$\frac{x-2}{x+3} - 3 = \frac{4(x+3)}{x-2}$$

Answer:

$$\frac{x-2}{x+3} - 3 = \frac{4(x+3)}{x-2}$$

$\text{LCD} = (x+3)(x-2)$

Non-permissible values are $x = -3, 2$.

$$\frac{x-2}{x+3}(x+3)(x-2) - 3(x+3)(x-2) = \frac{4(x+3)}{x-2}(x+3)(x-2)$$

$$(x-2)(x-2) - 3\left(x^2 - 2x + 3x - 6\right) = 4(x+3)(x+3)$$

$$\left(x^2 - 4x + 4\right) - 3\left(x^2 + x - 6\right) = 4\left(x^2 + 6x + 9\right)$$

$$x^2 - 4x + 4 - 3x^2 - 3x + 18 = 4x^2 + 24x + 36$$

$$-2x^2 - 7x + 22 = 4x^2 + 24x + 36$$

$$0 = 6x^2 + 31x + 14$$

$$0 = 6x^2 + 3x + 28x + 14$$

$$0 = 3x(2x+1) + 14(2x+1)$$

$$0 = (3x+14)(2x+1)$$

$$x = \frac{-14}{3}, \frac{-1}{2}$$

7. The discriminant of the quadratic equation $4x^2 + 2x - k = 0$ is 84. *(1 mark each, for a total of 2 marks)* (Lesson 5)

   a) Find the value of k.

   Answer:

   $a = 4$, $b = 2$, $c = -k$

   The discriminant $= b^2 - 4ac$

   $84 = (2)^2 - 4(4)(-k)$

   $84 = 4 + 16k$

   $80 = 16k$

   $5 = k$

   b) State the nature of the roots without solving for the roots.

   Answer:

   As the discriminant is positive and not a perfect square, there are 2 real, irrational, unequal roots.

Name: _____

8. Tia can ride her bicycle 1 km/h faster than Denzel. They begin at the same spot and at the same time, travelling in opposite directions. After travelling for the same length of time, they stop and realize that Tia has travelled 21 km and Denzel has travelled 19 km. How fast was Denzel travelling? *(4 marks)* (Lesson 6)

   Recall: Distance = Rate × Time, or Time = $\frac{\text{Distance}}{\text{Rate}}$

   Answer:

   Let r = Denzel's rate.

	Rate ×	Time =	Distance
Tia	$r + 1$	$\frac{21}{r+1}$	21
Denzel	r	$\frac{19}{r}$	19

   Tia's time of travel = Denzel's time of travel

$$\frac{21}{r+1} = \frac{19}{r}$$

$$\text{LCD} = (r+1)(r)$$

$$\frac{21}{r+1}(r+1)(r) = \frac{19}{r}(r+1)(r)$$

$$21r = 19(r+1)$$

$$21r = 19r + 19$$

$$2r = 19$$

$$r = 9.5$$

   Therefore, Denzel was travelling at 9.5 km/h.

9. Andrea can mow a lawn by herself in 4 hours. Samantha can mow the same lawn in 3 hours. If the girls work together, how long will it take to mow this lawn? *(4 marks)* (Lesson 6)

Answer:

The following equation can be used to solve this problem:

(per hour rate) × (hours worked) = portion of lawn mowed

In one hour, the Andrea can mow $\frac{1}{4}$ of the lawn. Therefore, Andrea has a per-hour rate of $\frac{1}{4}$.

In one hour, Samantha can mow $\frac{1}{3}$ of the lawn. Therefore, Samantha has a per-hour rate of $\frac{1}{3}$.

	Per Hour Rate	**Hours Worked**	**Portion of Lawn Mowed**
Andrea	$\frac{1}{4}$	x	$\frac{x}{4}$
Samantha	$\frac{1}{3}$	x	$\frac{x}{3}$

When both girls are working together, the portion of the lawn mowed in x hours is the whole lawn, and is represented in the equation as 1. The equation becomes:

$$\frac{x}{4} + \frac{x}{3} = 1$$

LCD = 12

$$\frac{3x}{12} + \frac{4x}{12} = 1$$

$$\frac{7x}{12} = 1$$

$$7x = 12$$

$$x = \frac{12}{7} \approx 1.7 \text{ hours}$$

Therefore, both of these girls working together can mow the lawn in approximately 1.7 hours.

Grade 11 Pre-Calculus Mathematics (30S)

Final Practice Examination

Grade 11 Pre-Calculus Mathematics

Final Practice Examination

Name: _____

Student Number: _____

Attending ❑ Non-Attending ❑

Phone Number: _____

Address: _____

_____

_____

For Marker's Use Only

Date: _____

Final Mark: _____ /100 = _____ %

Comments:

Instructions

The final examination will be weighted as follows:

Modules 1–8	100%

The format of the examination will be as follows:

Module 1:	5 marks	Module 5:	19 marks
Module 2:	5 marks	Module 6:	19 marks
Module 3:	5 marks	Module 7:	23 marks
Module 4:	5 marks	Module 8:	19 marks

Time allowed: 2.5 hours

Note: You are allowed to bring the following to the examination: pencils (2 or 3 of each), blank paper, a ruler, a scientific or graphing calculator, and your Final Exam Resource Sheet. Your Final Exam Resource Sheet must be handed in with the examination.

Show all calculations and formulas used. Include units where appropriate. Clearly state your final answer. **Diagrams may not be drawn to scale.**

Name: _____

Answer all questions to the best of your ability. Show all your work.

Module 1: Sequences and Series *(5 marks)*

1. Write the defining linear function of the following arithmetic sequence. *(2 marks)*

   99, 103, 107, . . .

2. Use a formula to find the value of $\sum_{k=2}^{8} 81\left(\frac{1}{3}\right)^{k-1}$. *(3 marks)*

Module 2: Factoring and Rational Expressions *(5 marks)*

1. Simplify. Identify all non-permissible values. *(5 marks)*

$$\frac{5x^2 + 14x - 19}{2x^2 + 5x + 3} \div \left(\frac{3x - 12}{x^2 - 2x - 8} + \frac{2x - 2}{x^2 + 4x - 5} \right)$$

Name: _____

Module 3: Quadratic Functions *(5 marks)*

1. Given the following parabola in vertex form, complete the following questions. *(1 mark each, for a total of 5 marks)*

$$y = \frac{1}{2}(x-3)^2 + 2$$

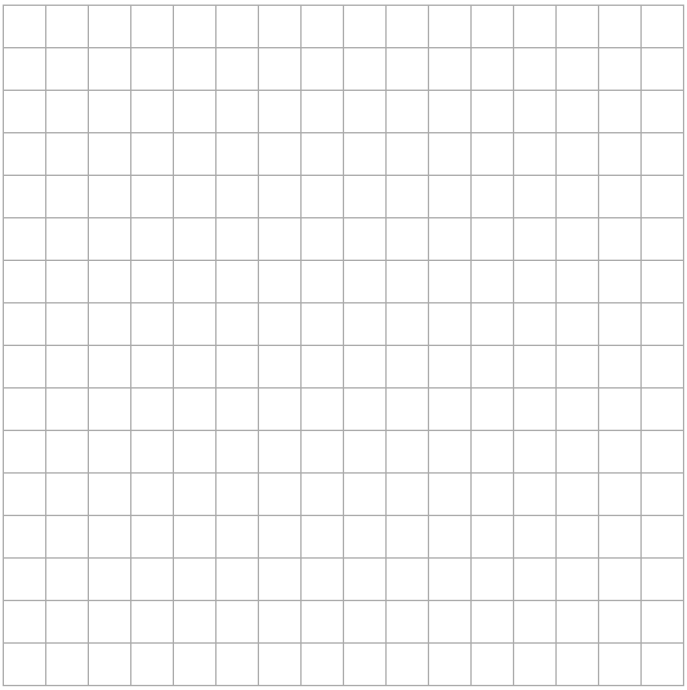
a) Identify the range.

b) Identify the direction of the opening.

c) Identify the axis of symmetry.

d) Identify the y-intercept.

e) Sketch the graph of the parabola.



Name: _____

Module 4: Solving Rational and Quadratic Equations *(5 marks)*

1. Solve the following quadratic equation using any method you wish. Explain why you chose the method you used. *(2 marks)*

   $2x^2 + 25 = 15x$

2. Solve the following rational equation. Identify any non-permissible values. *(3 marks)*

$$\frac{3x+1}{x} = -\frac{x}{x+1}$$

Name: _____

Module 5: Radicals *(19 marks)*

1. Order the following radical expressions from least to greatest. Do not use a calculator. *(2 marks)*

   $6\sqrt{3}$, $11\sqrt{7}$, $9\sqrt{5}$, $2\sqrt{15}$

2. Simplify each of the following. All answers must have rationalized denominators. Assume all variables are non-negative.

   a) $3\sqrt{19} + 2\sqrt{76}$ *(1 mark)*

b) $\left(\sqrt{8xy} - \sqrt{x^3}\right)\left(\sqrt{4y^2}\right)$ *(2 marks)*

c) $\dfrac{5\sqrt{8} - 2\sqrt{5}}{3 - 2\sqrt{3}}$ *(3 marks)*

Name: _____

3. Identify the values for each of the variables for which each radical expression is defined. *(1 mark each, for a total of 2 marks)*

   a) $\sqrt{6 - 3x}$

   b) $\sqrt{32x^2}$

4. Find the solutions for each of the following equations. Check your solutions for extraneous roots. Determine any restrictions on the variable.

   a) $3 - x = \sqrt{2x - 3}$ *(4 marks)*

Name: _____

b) $\sqrt{2x + 4} + 3 = 9$ *(3 marks)*

5. The period, P, measured in seconds, of a pendulum is the time it takes to complete one full swing. The period can be found using the formula $P = 2\pi\sqrt{\frac{L}{9.8}}$, where L measures the length of the pendulum in metres. How long should a pendulum be to complete one full swing in 4.3 seconds? Round your answer to four decimal places. Check your answer for extraneous solutions. *(2 marks)*

Name: _____

Module 6: Systems of Equations and Inequalities *(19 marks)*

1. Solve the following system of equations algebraically and graphically. *(6 marks)*

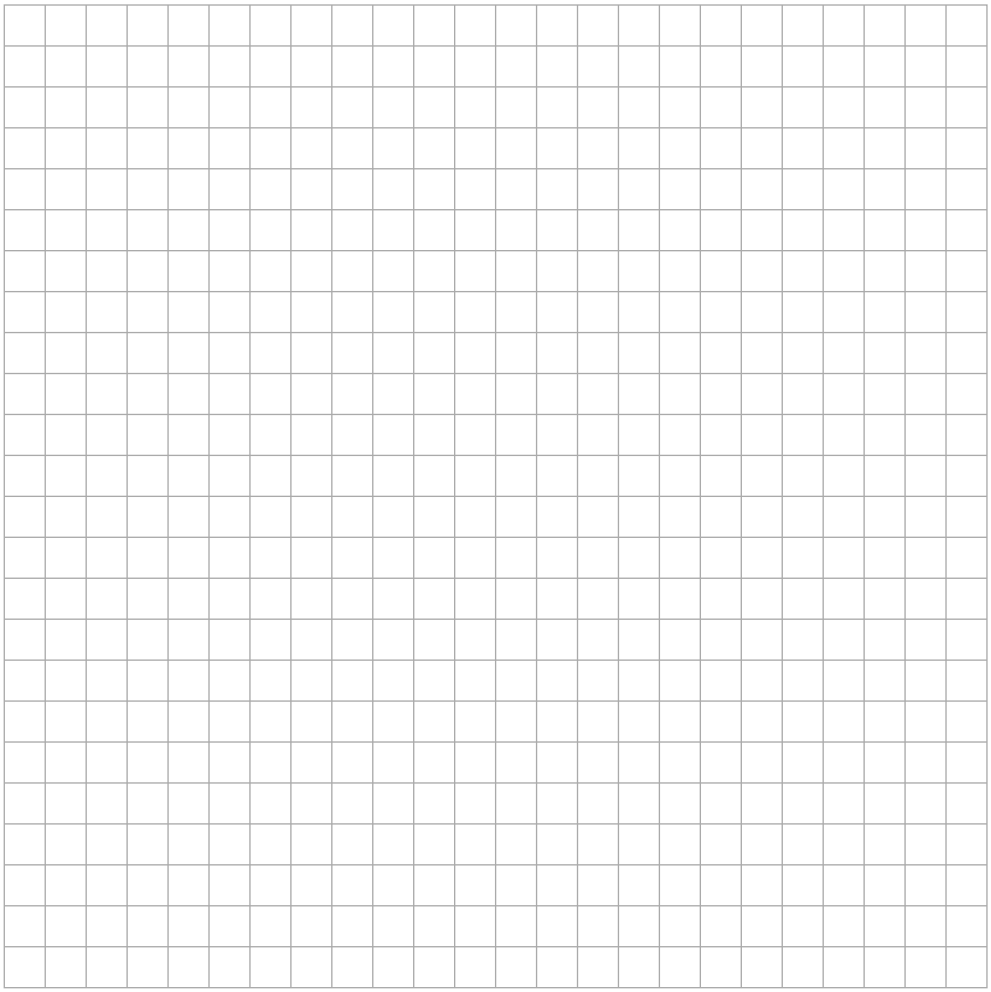
   $y = (x - 3)^2$

   $y = -3(x + 1)^2 + 12$

   Answer:

   Algebraic Solution

Graphing Solution



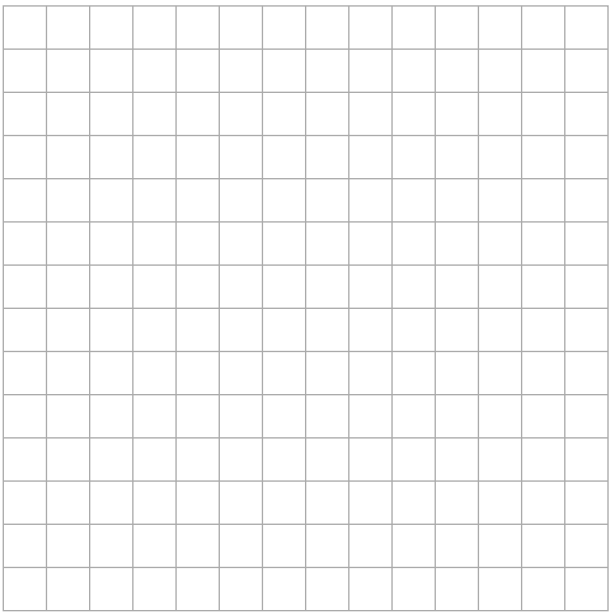
Name: _____

2. The sum of two numbers is 22. Their product is 117.
   a) Write a system of equations to represent this problem. *(1 mark)*

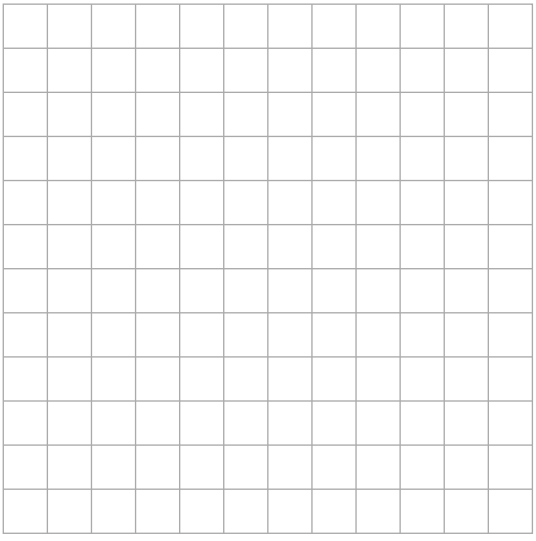
   b) Solve the system of equations to find the two numbers. *(3 marks)*

3. Solve the following inequalities by graphing. *(2 marks each, for a total of 6 marks)*

a) $y < 3x - 8$

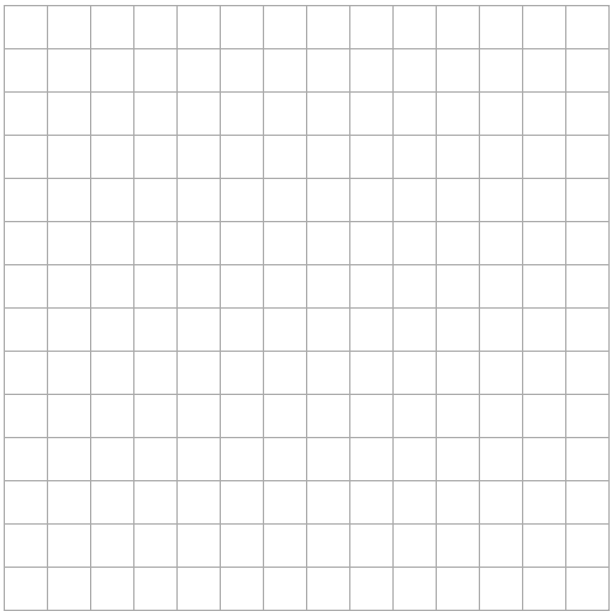


b) $0 \leq x^2 - 4$



Name: _____

c) $y > (x + 1)^2 - 3$



4. Use the sign analysis method to solve the following quadratic inequality. *(3 marks)*

$0 \leq 2x^2 - 5x - 3$

Name: _____

Module 7: Trigonometry *(23 marks)*

1. P(2, −3) is a point on the terminal side of angle θ in standard position. Determine $\sin\theta$, $\cos\theta$, and $\tan\theta$ for the following point. Also, determine the distance from the origin to the point P(2, −3). Rationalize any denominators. *(5 marks)*

2. Given the following angles in standard position, determine the exact values of the sine, cosine, and tangent ratios. Show how you determined the exact values. *(2 marks each, for a total of 4 marks)*

   a) 240°

   b) 45°

Name: _____

3. Sketch the angle 342° in standard position and find its reference angle. Determine the other angles that have the same reference angle as the given angle for θ in the interval [0°, 360°]. *(3 marks)*

4. Determine the solution set for each of the following trigonometric equations over the interval [0°, 360°]. Round to the nearest degree where necessary. *(2 marks each, for a total of 4 marks)*

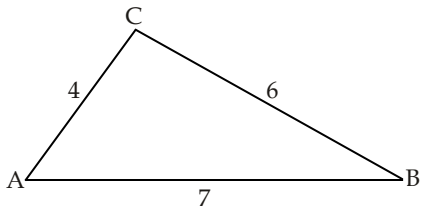
   a) $\cos\theta = \frac{1}{5}$

   b) $\sin\theta - 1 = 0$

Name: _____

5. In ΔABC, $\angle C = 41°$, $c = 6$, and $a = 8$. Find all possible values for b and $\angle B$. Draw a diagram and round off answers to two decimal places. *(5 marks)*

6. Find the measure of $\angle A$ in ΔABC (to the nearest degree). *(2 marks)*



Name: _____

Module 8: Absolute Value and Reciprocal Functions *(19 marks)*

1. Evaluate. *(1 mark each, for a total of 2 marks)*

   a) $4\,|\,3(-7) + 4\,|\,-3$

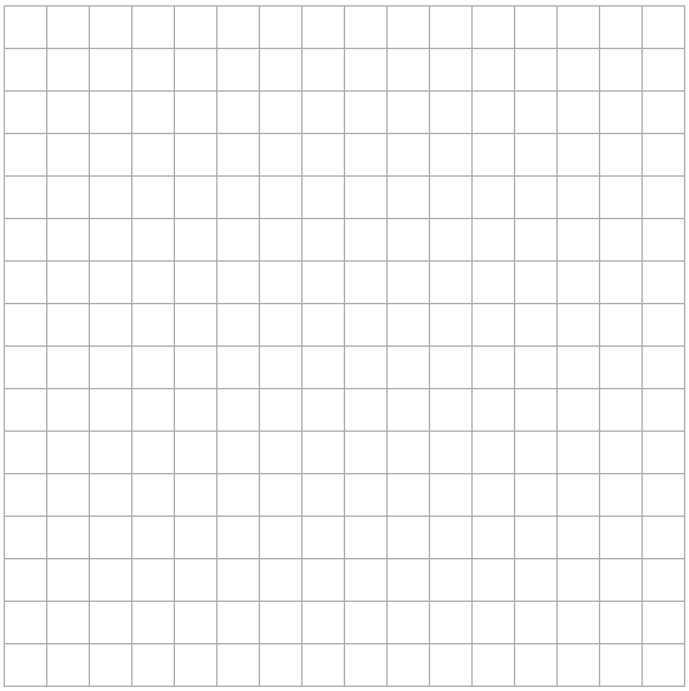
   b) $-2\,|\,-2(3) + 1\,|\,+5$

2. Solve the following absolute value equation algebraically. *(4 marks)*

   $|\,x^2 - x\,| = 6$

3. Solve the following absolute value equation graphically. *(4 marks)*

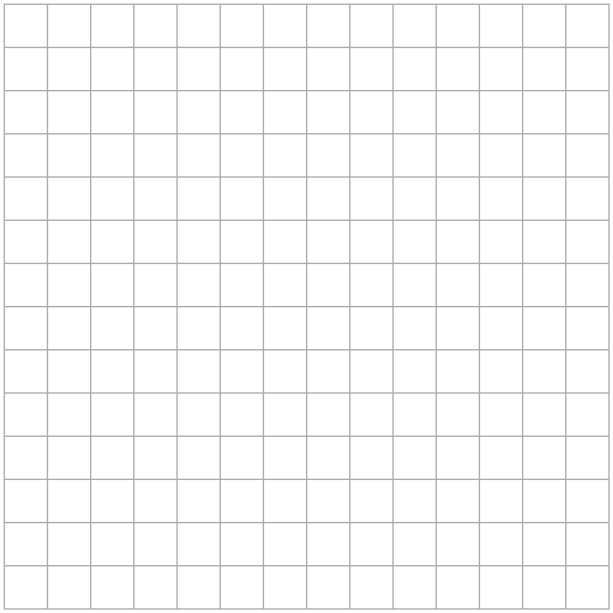
$| x + 4 | = 5$



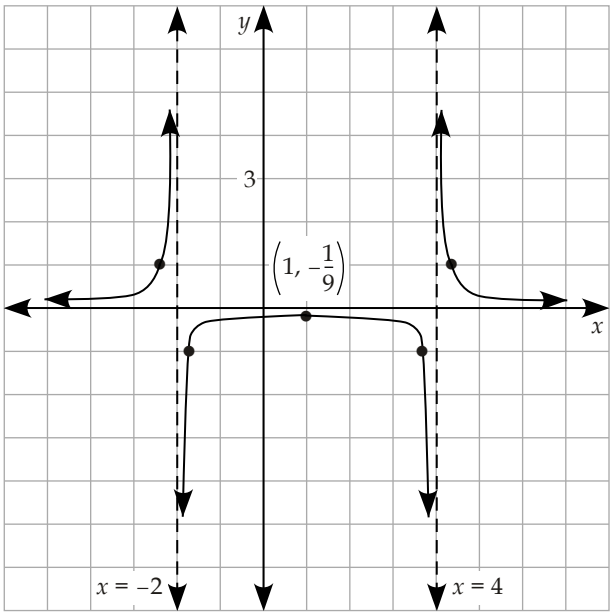
Name: _____

4. Sketch the graph of the reciprocal of the given function. State the equation(s) of the horizontal and vertical asymptote(s). *(4 marks)*

   $y = 2x - 6$



5. Given the graph of $y = \frac{1}{f(x)}$, sketch the graph of $y = f(x)$. *(5 marks)*



Grade 11 Pre-Calculus Mathematics (30S)

Final Practice Examination
Answer Key

Grade 11 Pre-Calculus Mathematics

Final Practice Examination
Answer Key

Name: _____

Student Number: _____

Attending ❑ Non-Attending ❑

Phone Number: _____

Address: _____

_____

_____

For Marker's Use Only

Date: _____

Final Mark: _____ / 100 = _____ %

Comments:

Answer Key

Instructions

The final examination will be weighted as follows:

Modules 1–8	100%

The format of the examination will be as follows:

Module 1:	5 marks	Module 5:	19 marks
Module 2:	5 marks	Module 6:	19 marks
Module 3:	5 marks	Module 7:	23 marks
Module 4:	5 marks	Module 8:	19 marks

Time allowed: 2.5 hours

Note: You are allowed to bring the following to the examination: pencils (2 or 3 of each), blank paper, a ruler, a scientific or graphing calculator, and your Final Exam Resource Sheet. Your Final Exam Resource Sheet must be handed in with the examination.

Show all calculations and formulas used. Include units where appropriate. Clearly state your final answer. **Diagrams may not be drawn to scale.**

Name: _____

Answer all questions to the best of your ability. Show all your work.

Module 1: Sequences and Series *(5 marks)*

1. Write the defining linear function of the following arithmetic sequence. *(2 marks)* (Lesson 1)

   99, 103, 107, . . .

   Answer:

   Points on line: (1, 99), (2, 103), (3, 107)

   Slope: $d = 4$

   Defining Linear Function:

   $$y - y_1 = m(x - x_1)$$
   $$y - 99 = 4(x - 1)$$
   $$y = 4x - 4 + 99$$
   $$y = 4x + 95$$

2. Use a formula to find the value of $\sum_{k=2}^{8} 81\left(\frac{1}{3}\right)^{k-1}$. *(3 marks)* (Lesson 4)

   Answer:

   Notice k's initial value is 2. Thus, n is only 7, not 8.

   The first few terms are $81\left(\frac{1}{3}\right)^1$, $81\left(\frac{1}{3}\right)^2$, $81\left(\frac{1}{3}\right)^3$, ...

   Thus, $t_1 = 27$ and $r = \frac{1}{3}$.

   $$S_n = \frac{t_1\left(1 - r^n\right)}{1 - r}$$

   $$S_7 = \frac{27\left(1 - \left(\frac{1}{3}\right)^7\right)}{1 - \frac{1}{3}} = \frac{27\left(\frac{2186}{2187}\right)}{\frac{2}{3}} = 27\left(\frac{2186}{2187}\right)\left(\frac{2}{3}\right) = 40\frac{13}{27}$$

Module 2: Factoring and Rational Expressions *(5 marks)*

1. Simplify. Identify all non-permissible values. *(5 marks)* (Lessons 4 and 5)

$$\frac{5x^2+14x-19}{2x^2+5x+3} \div \left(\frac{3x-12}{x^2-2x-8}+\frac{2x-2}{x^2+4x-5}\right)$$

Answer:

$$\frac{5x^2+14x-19}{2x^2+5x+3} \div \left(\frac{3x-12}{x^2-2x-8}+\frac{2x-2}{x^2+4x-5}\right)$$

$$\frac{(5x+19)(x-1)}{(2x+3)(x+1)} \div \left(\frac{3(x-4)}{(x-4)(x+2)}+\frac{2(x-1)}{(x+5)(x-1)}\right)$$

$$= \frac{(5x+19)(x-1)}{(2x+3)(x+1)} \div \left(\frac{3}{(x+2)}+\frac{2}{(x+5)}\right)$$

$$= \frac{(5x+19)(x-1)}{(2x+3)(x+1)} \div \left(\frac{3(x+5)}{(x+2)(x+5)}+\frac{2(x+2)}{(x+5)(x+2)}\right)$$

$$= \frac{(5x+19)(x-1)}{(2x+3)(x+1)} \div \left(\frac{3x+15}{(x+2)(x+5)}+\frac{2x+4}{(x+5)(x+2)}\right)$$

$$= \frac{(5x+19)(x-1)}{(2x+3)(x+1)} \div \left(\frac{5x+19}{(x+2)(x+5)}\right)$$

$$= \frac{(5x+19)(x-1)}{(2x+3)(x+1)} \cdot \left(\frac{(x+2)(x+5)}{5x+19}\right)$$

$$= \frac{(x-1)}{(2x+3)(x+1)} \cdot \left(\frac{(x+2)(x+5)}{1}\right)$$

$$= \frac{(x+2)(x+5)(x-1)}{(2x+3)(x+1)}$$

$$x \neq \frac{-3}{2}, -1, 4, -2, -5, 1, \frac{-19}{5}$$

Name: _____

Module 3: Quadratic Functions *(5 marks)*

1. Given the following parabola in vertex form, complete the following questions. *(1 mark each, for a total of 5 marks)*

   $y = \frac{1}{2}(x-3)^2 + 2$ (Lessons 1 to 4)

   a) Identify the range.

   Answer:

   R: $\{y \mid y \geq 2, y \in \Re\}$

   b) Identify the direction of the opening.

   Answer:

   The parabola opens upward

   c) Identify the axis of symmetry.

   Answer:

   $x = 3$

   d) Identify the y-intercept.

   Answer:

   $y = \frac{1}{2}(x-3)^2 + 2$

   let $x = 0$

   $y = \frac{1}{2}(0-3)^2 + 2$

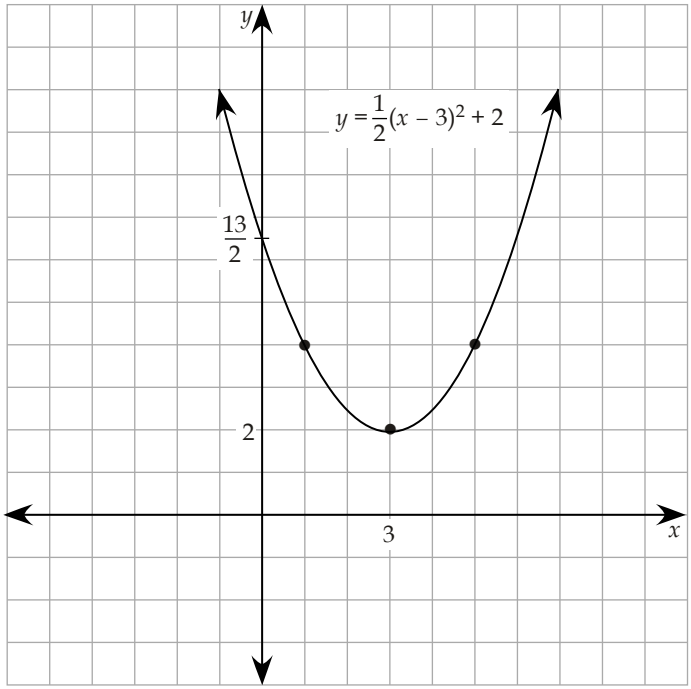
   $y = \frac{1}{2}(9) + 2$

   $y = \frac{13}{2}$

e) Sketch the graph of the parabola.

Answer:

The vertex is (3, 2), the curve opens up, and its size is wide. The y-values are $\frac{1}{2}$ of the normal y-values.



Name: _____

Module 4: Solving Rational and Quadratic Equations *(5 marks)*

1. Solve the following quadratic equation using any method you wish. Explain why you chose the method you used. *(2 marks)* (Lessons 1 to 4)

   $2x^2 + 25 = 15x$

   Answer:

   $$2x^2 + 25 = 15x$$
   $$2x^2 - 15x + 25 = 0$$
   $$2x^2 - 10x - 5x + 25 = 0$$
   $$2x(x-5) - 5(x-5) = 0$$
   $$(2x-5)(x-5) = 0$$
   $$x = \frac{5}{2} \text{ or } x = 5$$

   You can use any method. As long as you complete the question correctly and explain why you chose the method you did, you will get full marks.

   A possible explanation is:

   I chose to solve this quadratic equation by factoring because it is easily factorable.

2. Solve the following rational equation. Identify any non-permissible values. *(3 marks)*

$$\frac{3x+1}{x} = -\frac{x}{x+1}$$ (Lesson 6)

Answer:

Non-permissible values: $x = 0$ or $x = -1$

LCD = $x(x+1)$

$$\frac{3x+1}{x}(x)(x+1) = \frac{-x}{x+1}(x)(x+1)$$

$$(3x+1)(x+1) = -x(x)$$

$$3x^2 + 4x + 1 = -x^2$$

$$4x^2 + 4x + 1 = 0$$

$$4x^2 + 2x + 2x + 1 = 0$$

$$2x(2x+1) + 1(2x+1) = 0$$

$$(2x+1)^2 = 0$$

$$x = -\frac{1}{2}$$

Name: _____

Module 5: Radicals *(19 marks)*

1. Order the following radical expressions from least to greatest. Do not use a calculator. *(2 marks)* (Lesson 1)

   $6\sqrt{3}$, $11\sqrt{7}$, $9\sqrt{5}$, $2\sqrt{15}$

   Answer:

   $6\sqrt{3} = \sqrt{36}\sqrt{3} = \sqrt{108}$

   $11\sqrt{7} = \sqrt{121}\sqrt{7} = \sqrt{847}$

   $9\sqrt{5} = \sqrt{81}\sqrt{5} = \sqrt{405}$

   $2\sqrt{15} = \sqrt{4}\sqrt{15} = \sqrt{60}$

   Thus, $\sqrt{60} < \sqrt{108} < \sqrt{405} < \sqrt{847}$ and $2\sqrt{15} < 6\sqrt{3} < 9\sqrt{5} < 11\sqrt{7}$.

2. Simplify each of the following. All answers must have rationalized denominators. Assume all variables are non-negative. (Lesson 2)

   a) $3\sqrt{19} + 2\sqrt{76}$ *(1 mark)*

   Answer:

   $3\sqrt{19} + 2\sqrt{76}$

   $= 3\sqrt{19} + 2\sqrt{4}\sqrt{19}$

   $= 3\sqrt{19} + 2(2)\sqrt{19}$

   $= 3\sqrt{19} + 4\sqrt{19}$

   $= 7\sqrt{19}$

b) $\left(\sqrt{8xy}-\sqrt{x^3}\right)\left(\sqrt{4y^2}\right)$ *(2 marks)* (Lesson 3)

Answer:

$$\left(\sqrt{8xy}-\sqrt{x^3}\right)\left(\sqrt{4y^2}\right)$$
$$=\sqrt{4}\sqrt{2}\sqrt{x}\sqrt{y}\sqrt{4}\sqrt{y}\sqrt{y}-\sqrt{x}\sqrt{x}\sqrt{x}\sqrt{4}\sqrt{y}\sqrt{y}$$
$$=4y\sqrt{2xy}-2xy\sqrt{x}$$

c) $\dfrac{5\sqrt{8}-2\sqrt{5}}{3-2\sqrt{3}}$ *(3 marks)* (Lesson 4)

Answer:

$$\frac{5\sqrt{8}-2\sqrt{5}}{3-2\sqrt{3}}$$
$$=\frac{5\sqrt{8}-2\sqrt{5}}{3-2\sqrt{3}}\cdot\frac{3+2\sqrt{3}}{3+2\sqrt{3}}$$
$$=\frac{15\sqrt{8}+10\sqrt{24}-6\sqrt{5}-4\sqrt{15}}{9+6\sqrt{3}-6\sqrt{3}-4\sqrt{9}}$$
$$=\frac{15\sqrt{4}\sqrt{2}+10\sqrt{4}\sqrt{6}-6\sqrt{5}-4\sqrt{15}}{9-4(3)}$$
$$=\frac{30\sqrt{2}+20\sqrt{6}-6\sqrt{5}-4\sqrt{15}}{-3}$$
$$=\frac{-30\sqrt{2}-20\sqrt{6}+6\sqrt{5}+4\sqrt{15}}{3}$$

Name: _____

3. Identify the values for each of the variables for which each radical expression is defined. *(1 mark each, for a total of 2 marks)*

   a) $\sqrt{6-3x}$ (Lesson 1)

   Answer:

   In order for this radical expression to be defined, the radicand has to be greater than or equal to zero.

   $6 - 3x \geq 0$

   $6 \geq 3x$

   $2 \geq x$

   $x \leq 2$

   Thus, $x \leq 2$ in order for this radical expression to be defined.

   b) $\sqrt{32x^2}$ (Lesson 1)

   Answer:

   Because the index is two, the radicand has to be greater than or equal to zero for this expression to be defined. As any value squared will always be greater than or equal to zero, $x \in \Re$ for this expression to be defined.

4. Find the solutions for each of the following equations. Check your solutions for extraneous roots. Determine any restrictions on the variable.

a) $3 - x = \sqrt{2x - 3}$ *(4 marks)* (Lesson 5)

Answer:

Restrictions on the variable:

$2x - 3 \geq 0$

$2x \geq 3$

$x \geq \frac{3}{2}$

$(3 - x)^2 = 2x - 3$

$9 - 6x + x^2 = 2x - 3$

$x^2 - 8x + 12 = 0$

$(x - 6)(x - 2) = 0$

$x = 6$ or $x = 2$

Check $x = 2$

LHS	RHS
$3 - x$	$\sqrt{2x - 3}$
$3 - 2$	$\sqrt{2(2) - 3}$
1	$\sqrt{1}$
	1

LHS = RHS

Check $x = 6$

LHS	RHS
$3 - x$	$\sqrt{2x - 3}$
$3 - 6$	$\sqrt{2(6) - 3}$
-3	$\sqrt{9}$
	3

LHS ≠ RHS

Therefore, the only solution is $x = 2$.

Name: _____

b) $\sqrt{2x+4}+3=9$ *(3 marks)* (Lesson 5)

Answer:

Restrictions on the variable:

$2x+4 \geq 0$

$2x \geq -4$

$x \geq -2$

$\sqrt{2x+4}+3=9$

$\sqrt{2x+4}=6$

$2x+4=6^2$

$2x+4=36$

$2x=32$

$x=16$

Check $x = 16$

LHS	RHS
$\sqrt{2x+4}+3$	9
$\sqrt{2(16)+4}+3$	
$6+3$	
9	

LHS = RHS

$\therefore$ The solution is $x = 16$.

5. The period, P, measured in seconds, of a pendulum is the time it takes to complete one full swing. The period can be found using the formula $P = 2\pi\sqrt{\frac{L}{9.8}}$, where L measures the length of the pendulum in metres. How long should a pendulum be to complete one full swing in 4.3 seconds? Round your answer to four decimal places. Check your answer for extraneous solutions. *(2 marks)* (Lesson 5)

Answer:

$$P = 2\pi\sqrt{\frac{L}{9.8}}$$

Restrictions: $L \geq 0$ and $P \geq 0$

Note: P is positive or zero because it is equal to a positive number, 2π, multiplied by the principal square root. Also, it would not make sense for either variable to be negative.

$$P = 2\pi\sqrt{\frac{L}{9.8}}$$

$$4.3 = 2\pi\sqrt{\frac{L}{9.8}}$$

$$4.3^2 = (2\pi)^2\left(\frac{L}{9.8}\right)$$

$$18.49 = \frac{4\pi^2 L}{9.8}$$

$$L = \frac{18.49 \times 9.8}{4\pi^2}$$

$$4.5899 = L$$

Check: $L \geq 0$

LHS	RHS
P	$2\pi\sqrt{\frac{L}{9.8}}$
4.3	$2\pi\left(\sqrt{\frac{4.5899}{9.8}}\right)$
	4.3

LHS = RHS

The pendulum should be 4.5899 m long.

Name: _____

Module 6: Systems of Equations and Inequalities *(19 marks)*

1. Solve the following system of equations algebraically and graphically. *(6 marks)*

$$y = (x-3)^2$$
$$y = -3(x+1)^2 + 12$$

(Lesson 2)

Answer:

Algebraic Solution

$y = (x-3)^2$ $\qquad$ $y = -3(x+1)^2 + 12$

$y = x^2 - 6x + 9$ $\qquad$ $y = -3\left(x^2 + 2x + 1\right) + 12$

$y = -3x^2 - 6x + 9$

Substitute the expression for y in Equation (1) into the y-variable of Equation (2).

$$x^2 - 6x + 9 = -3x^2 - 6x + 9$$
$$4x^2 = 0$$
$$x = 0$$

Solve for y:

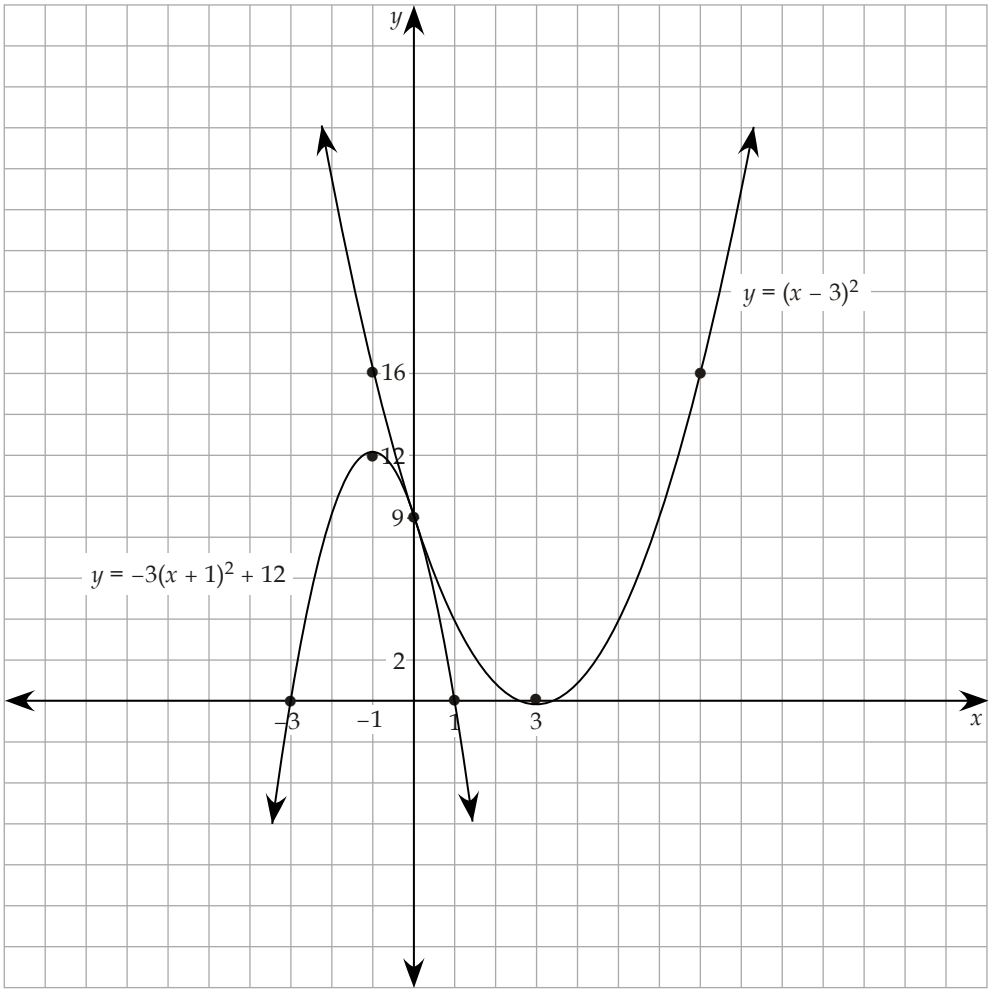
$$y = (x-3)^2$$
$$y = (0-3)^2$$
$$y = 9$$

Thus, the solution to this system of equations is (0, 9).

Graphing Solution

The parabola, $y = (x - 3)^2$, has its vertex at (3, 0), it opens up, and has normal shape.

The parabola, $y = -3(x + 1)^2 + 12$, has its vertex at $(-1, 12)$, opens down, and its shape is narrower. The y-coordinates are multiplied by 3.



The solution to this system of equations is the point at which the two functions intersect, or (0, 9).

Name: _____

2. The sum of two numbers is 22. Their product is 117.

   a) Write a system of equations to represent this problem. *(1 mark)* (Lesson 1)

   Answer:

   Let the first number be represented by m and the second number be represented by n. The system of equations is:

   $$\begin{cases} m + n = 22 \\ mn = 117 \end{cases}$$

   b) Solve the system of equations to find the two numbers. *(3 marks)*

   Answer:

   Substitute $m = 22 - n$ into $nm = 117$.

   $$(22 - n)(n) = 117$$
   $$22n - n^2 - 117 = 0$$
   $$n^2 - 22n + 117 = 0$$
   $$(n - 13)(n - 9) = 0$$
   $$n = 13 \text{ or } n = 9$$

   Substituting into the equation $m = 22 - n$, you get:

$n = 13$	$n = 9$
$m = 22 - \text{n}$	$m = 22 - \text{n}$
$m = 22 - 13$	$m = 22 - 9$
$m = 9$	$m = 13$

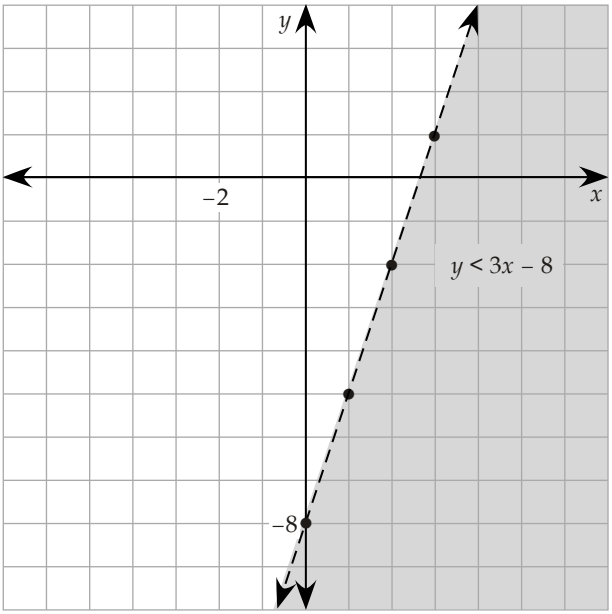
   Therefore, the two numbers are 9 and 13.

3. Solve the following inequalities by graphing. *(2 marks each for a total of 6 marks)*

a) $y < 3x - 8$ (Lesson 3)

Answer:

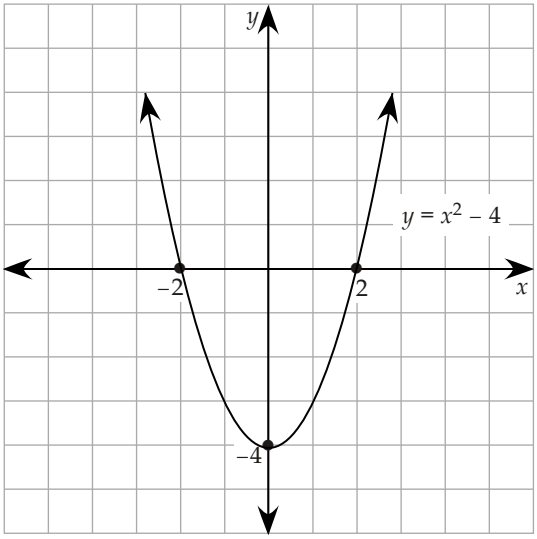
The slope of the line is 3. The y-intercept is -8. The boundary line is dotted and the shading is below the line.



b) $0 \leq x^2 - 4$ (Lesson 4)

Answer:

You are interested in the domain where the expression is greater than or equal to 0. Write $x^2 - 4 \geq 0$ and write the corresponding function as $y = x^2 - 4$. The parabola has its vertex at $(0, -4)$, opens up, and has normal shape.



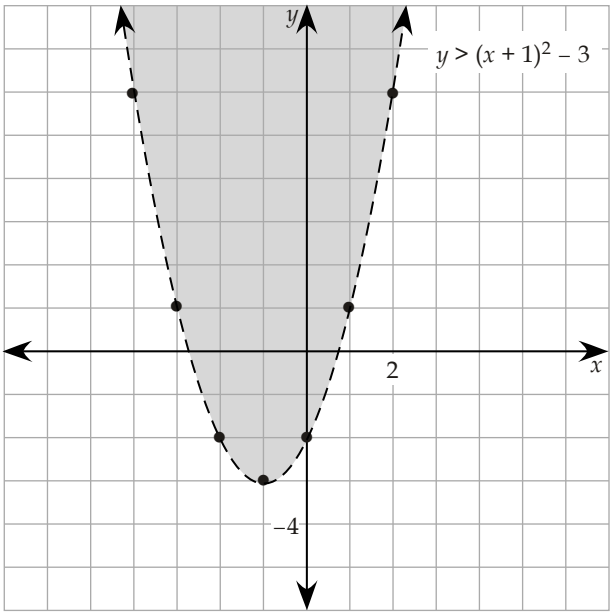
The parabola crosses the x-axis at -2 and 2. The parabola lies on or above the x-axis when $x \leq -2$ and when $x \geq 2$. Therefore, the solution set is $\{x \mid x \leq -2 \text{ or } x \geq 2, x \in \Re\}$.

Name: _____

c) $y > (x + 1)^2 - 3$ (Lesson 5)

Answer:

The parabola has its vertex at $(-1, -3)$, opens up, and has normal shape. The boundary line is dotted and the region above the parabola is shaded.



4. Use the sign analysis method to solve the following quadratic inequality. *(3 marks)*

$0 \le 2x^2 - 5x - 3$ (Lesson 4)

Answer:

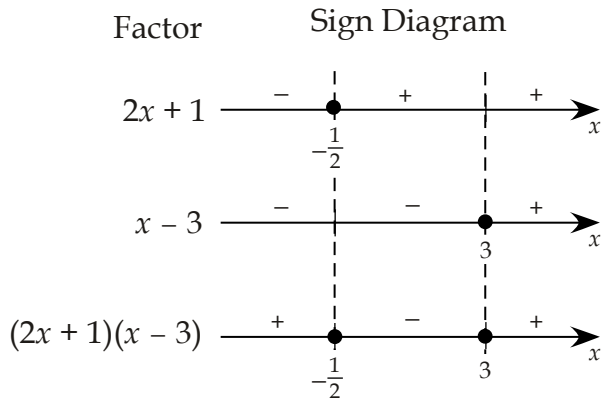
Write as $2x^2 - 5x - 3 \ge 0$.

You want to consider all values of the expression that are greater than or equal to zero.

Factoring the corresponding quadratic equation:

$$2x^2 - 5x - 3 = 0$$
$$2x^2 - 6x + x - 3 = 0$$
$$2x(x - 3) + 1(x - 3) = 0$$
$$(2x + 1)(x - 3) = 0$$
$$2x + 1 = 0, \; x - 3 = 0$$
$$x = -\frac{1}{2} \text{ or } x = 3$$

The critical points are $-\frac{1}{2}$ and 3.



The solution set contains the points where $2x^2 - 5x - 3 \ge 0$ and is $\left\{x \mid x \le -\frac{1}{2} \text{ or } x \ge 3, x \in \Re\right\}$.

Name: _____

Module 7: Trigonometry *(23 marks)*

1. P(2, −3) is a point on the terminal side of angle θ in standard position. Determine sin θ, cos θ, and tan θ for the following point. Also, determine the distance from the origin to the point P(2, −3). Rationalize any denominators. *(5 marks)* (Lesson 1)

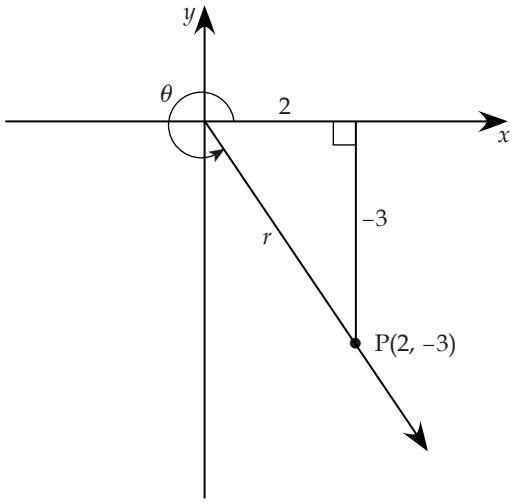
   Answer:

   $r = \sqrt{2^2 + (-3)^2} = \sqrt{13}$

   $\sin\theta = \frac{y}{r} = \frac{-3}{\sqrt{13}} = \frac{-3\sqrt{13}}{13}$

   $\cos\theta = \frac{x}{r} = \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13}$

   $\tan\theta = \frac{y}{x} = \frac{-3}{2} = -\frac{3}{2}$

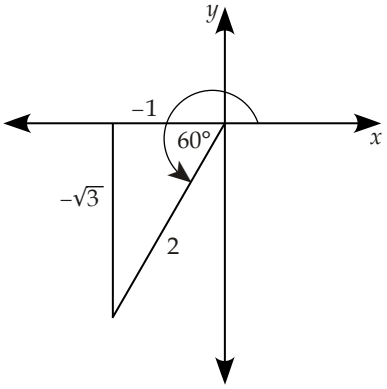


2. Given the following angles in standard position, determine the exact values of the sine, cosine, and tangent ratios. Show how you determined the exact values. *(2 marks each, for a total of 4 marks)* (Lesson 3)

a) 240°

Answer:

$\theta_r = 240° - 180° = 60°$



$$\sin 240° = \frac{-\sqrt{3}}{2}$$

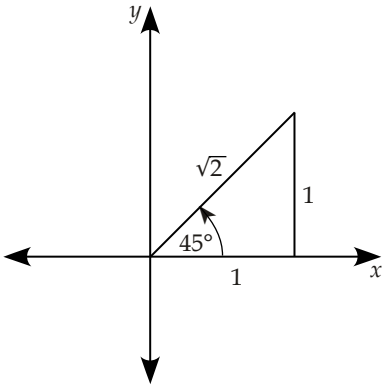
$$\cos 240° = \frac{-1}{2}$$

$$\tan 240° = \frac{-\sqrt{3}}{-1} = \sqrt{3}$$

b) 45°

Answer:

$\theta_r = 45°$



$$\sin 45° = \frac{1}{\sqrt{2}}$$

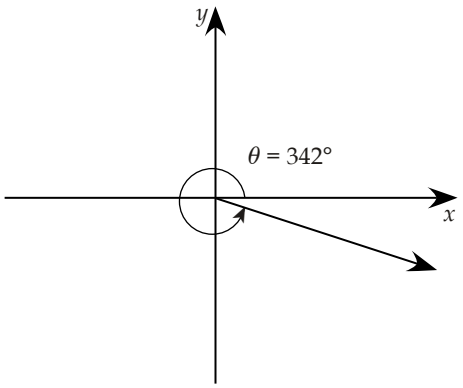
$$\cos 45° = \frac{1}{\sqrt{2}}$$

$$\tan 45° = 1$$

Name: _____

3. Sketch the angle 342° in standard position and find its reference angle. Determine the other angles that have the same reference angle as the given angle for θ in the interval [0°, 360°]. *(3 marks)* (Lesson 2)

   Answer:



The reference angle for 342° is 360° − 342° = 18°.

The other angles that have this same reference angle are:

18°	in Quadrant I
180° − 18° = 162°	in Quadrant II
180° + 18° = 198°	in Quadrant III

4. Determine the solution set for each of the following trigonometric equations over the interval [0°, 360°]. Round to the nearest degree where necessary. *(2 marks each, for a total of 4 marks)* (Lesson 2)

   a) $\cos\theta = \frac{1}{5}$

   Answer:

   $\theta_r = \cos^{-1}\left(\frac{1}{5}\right)$

   $\theta_r = 78°$

   As $\cos\theta > 0$, θ must be in Quadrants I or IV.

   In Quadrant I, $\theta = 78°$.

   In Quadrant IV, $\theta = 360° - 78° = 282°$.

   b) $\sin\theta - 1 = 0$

   Answer:

   $\sin\theta = 1$

   This is a quadrantal angle through the point (0, 1) and only occurs when θ is 90°.

Name: _____

5. In ΔABC, ∠C = 41°, c = 6, and a = 8. Find all possible values for b and ∠B. Draw a diagram and round off answers to two decimal places. *(5 marks)* (Lessons 4 and 6)

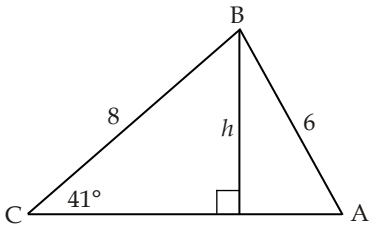
Answer:

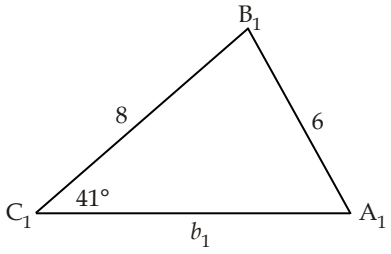
Since AB < BC, find h:

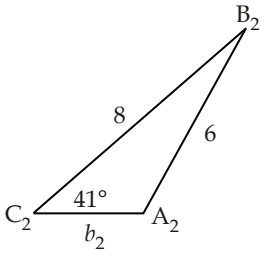
$h = 8 \sin 41° = 5.25$

Since h < AB, there are two solutions.

Draw two triangles and solve.







$$\frac{6}{\sin 41°} = \frac{8}{\sin A}$$

$$6 \sin A = 8 \sin 41°$$

$$\sin A = \frac{8 \sin 41°}{6}$$

$$\angle A = \sin^{-1}\left(\frac{8 \sin 41°}{6}\right)$$

$\angle A_{related} = 61.01°$, with ∠A in Quadrants I and II.

$\angle A_1 = 61.01°$

$\angle B_1 = 180° - (41° + 61.01°)$

$= 77.99°$

$$\frac{6}{\sin 41°} = \frac{b_1}{\sin 77.99°}$$

$$b_1 = \frac{6 \sin 77.99°}{\sin 41°}$$

$b_1 = 8.95$ units

$\angle A_2 = 180° - 61.01°$

$= 118.99°$

$\angle B_2 = 180° - (41° + 118.99°)$

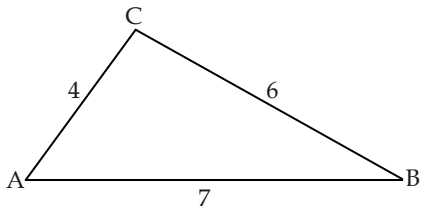
$= 20.01°$

$$\frac{6}{\sin 41°} = \frac{b_2}{\sin 20.01°}$$

$$b_2 = \frac{6 \sin 20.01°}{\sin 41°}$$

$b_2 = 3.13$ units or 3.14 units

6. Find the measure of $\angle A$ in ΔABC (to the nearest degree). *(2 marks)* (Lesson 5)



Answer:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$6^2 = 4^2 + 7^2 - 2(4)(7) \cos A$$

$$6^2 - 4^2 - 7^2 = -2(4)(7) \cos A$$

$$-29 = -56 \cos A$$

$$0.517857 = \cos A$$

$$A = \cos^{-1}(0.517857)$$

$$\angle A = 59°$$

Name: _____

Module 8: Absolute Value and Reciprocal Functions *(19 marks)*

1. Evaluate. *(1 mark each, for a total of 2 marks)* (Lesson 1)

   a) $4\,|\,3(-7) + 4\,|\,-3$

   Answer:

   $4\,|\,3(-7) + 4\,|\,-3$

   $= 4\,|\,-17\,|\,-3$

   $= 4(17) - 3$

   $= 65$

   b) $-2\,|\,-2(3) + 1\,|\,+5$

   Answer:

   $-2\,|\,-2(3) + 1\,|\,+5$

   $= -2\,|\,-5\,|\,+5$

   $= -2(5) + 5$

   $= -5$

2. Solve the following absolute value equation algebraically. *(4 marks)* (Lesson 3)

   $|\,x^2 - x\,| = 6$

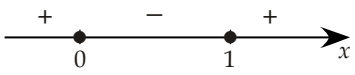
   Answer:

   Case 1: $x^2 - x = 6$ and $x^2 - x \geq 0$

   $x^2 - x - 6 = 0$ $x(x - 1) = 0$

   $(x - 3)(x + 2) = 0$ $x = 0, 1$

   $x = 3, -2$

   $x \in (-\infty, 0] \cup [1, \infty)$

   Since $x = 3$ is in the interval $[1, \infty)$ and $x = -2$ is in the interval $(-\infty, 0]$, both are solutions.

   Case 2: $-(x^2 - x) = 6$ and $x^2 - x < 0$

   $-x^2 + x = 6$

   $-x^2 + x - 6 = 0$

   $x^2 - x + 6 = 0$

   $b^2 - 4ac = (-1)^2 - 4(1)(6)$

   $= 1 - 24$

   $= -23$

   There is no solution.

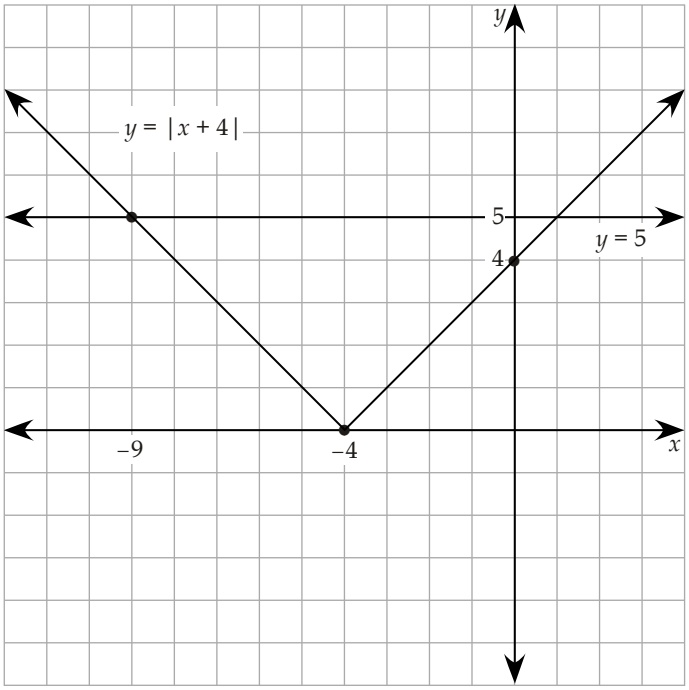
Therefore, the solutions to this absolute value equation are $x = 3$ and $x = -2$

3. Solve the following absolute value equation graphically. *(4 marks)*

   $|x + 4| = 5$ (Lesson 3)

   Answer:

   From the left side of the equation, graph $y = |x + 4|$. From the right side, graph $y = 5$. The two graphs intersect at the points $(-9, 5)$ and $(1, 5)$. The solutions for the equation are $x = -9$ and $x = 1$.



Name: _____

4. Sketch the graph of the reciprocal of the given function. State the equation(s) of the horizontal and vertical asymptote(s). *(4 marks)* (Lesson 4)

   $y = 2x - 6$

   Answer:

   The reciprocal function is: $f(x) = \dfrac{1}{2x - 6}$

   y-intercept: Find *f*(0):

   $$f(0) = \frac{1}{2(0) - 6} = \frac{1}{-6} = -\frac{1}{6}$$

   Vertical asymptote:

   $$\begin{aligned} 2x - 6 &= 0 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

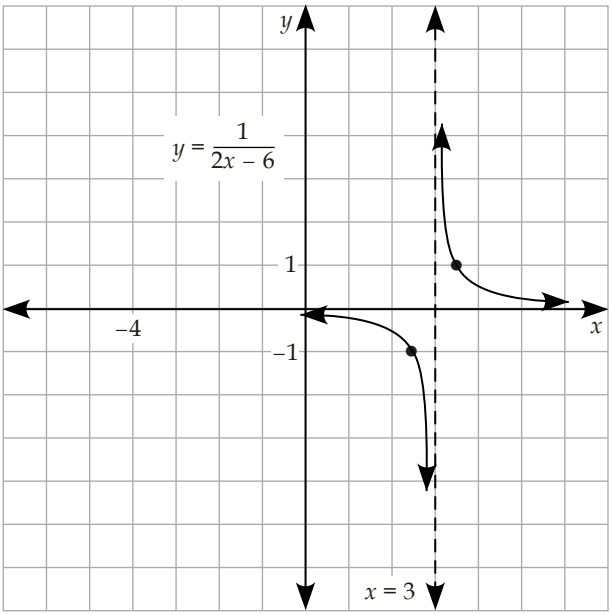
   Horizontal asymptote: The horizontal asymptote is the line $y = 0$.

   Invariant points:

   $$\begin{array}{lll} \text{Let } 2x - 6 = 1 & \text{or} & 2x - 6 = -1 \\ 2x = 7 & & 2x = 5 \\ x = \dfrac{7}{2} & & x = \dfrac{5}{2} \end{array}$$

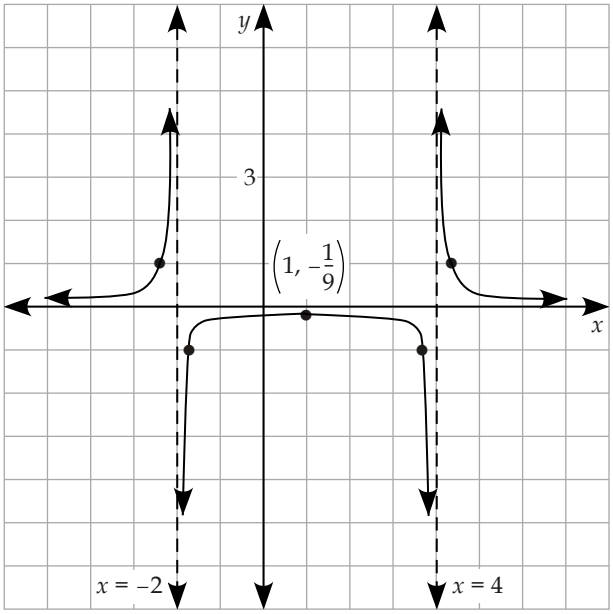
   The invariant points are $\left(\frac{7}{2}, 1\right)$ and $\left(\frac{5}{2}, -1\right)$.

Draw the curve:



Name: _____

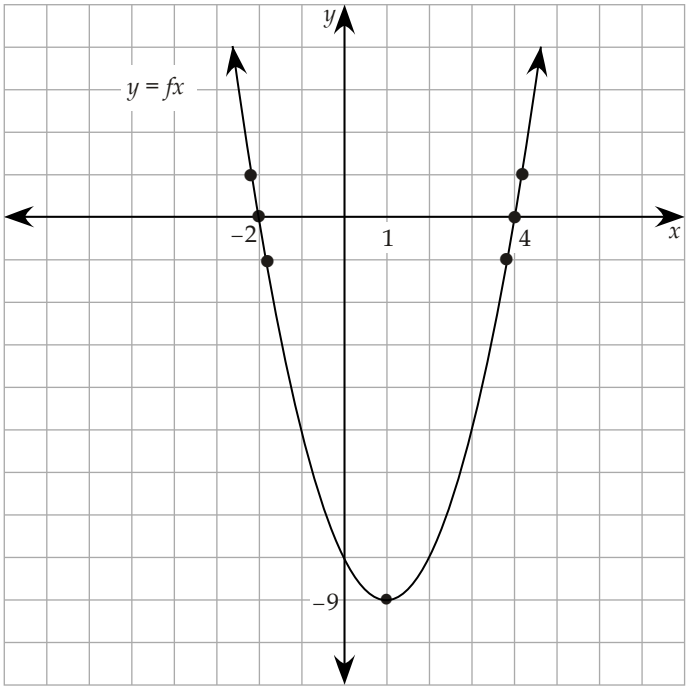
5. Given the graph of $y = \dfrac{1}{f(x)}$, sketch the graph of $y = f(x)$. *(5 marks)* (Lesson 5)



Answer:

The vertical asymptotes of this reciprocal graph are at $x = -2$ and $x = 4$, and are thus the x-intercepts in the graph of $f(x)$. In other words, $f(x)$ is of the form $f(x) = a(x + 2)(x - 4)$.

The key point is $\left(1, -\dfrac{1}{9}\right)$. The corresponding point in the graph of the function, $f(x)$, is $(1, -9)$.

The invariant points can be read from the graph and are approximately $(-2.2, 1)$, $(4.2, 1)$, $(-1.8, -1)$, and $(3.8, -1)$.

The parabola has its vertex at (1, −9) and opens up.