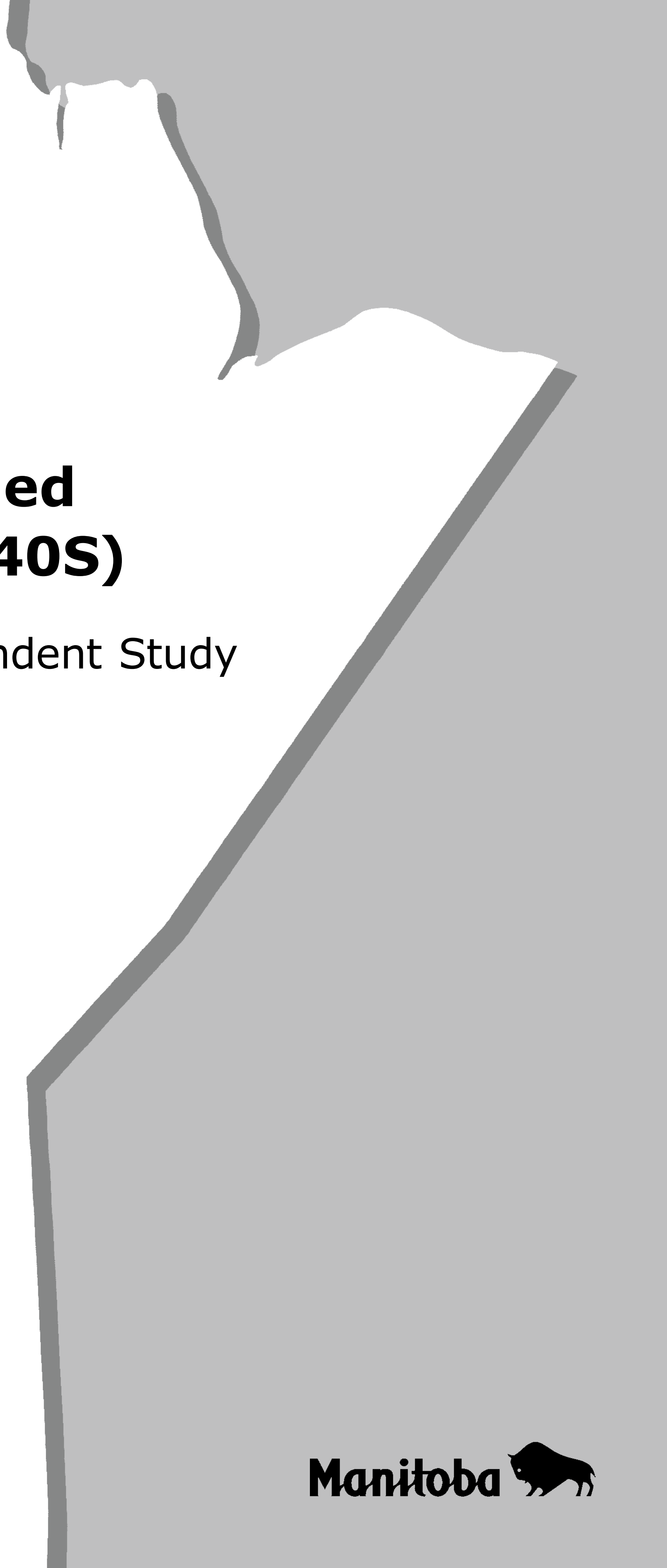
Grade 12 Applied Mathematics (40S)

A Course for Independent Study



Manitoba

Grade 12 Applied Mathematics (40S)

A Course for Independent Study

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for independent study

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Writer	Bonnie Hildebrand	Independent Consultant Winnipeg
Reviewer	Linda Isaak	Independent Consultant Winnipeg
Manitoba Education and Training	Carole Bilyk Coordinator (until August 2016)	Development Unit Instruction, Curriculum and Assessment Branch
	Louise Boissonneault Coordinator	Document Production Services Unit Instruction, Curriculum and Assessment Branch
	Ian Donnelly Project Leader	Early Childhood and Development Unit Instruction, Curriculum and Assessment Branch
	Lynn Harrison Desktop Publisher	Document Production Services Unit Instruction, Curriculum and Assessment Branch
	Nathalie Houle Conseillère pédagogique (from September 2015)	Conseillère pédagogique — Mathématiques 9 à 12 Division du Bureau de l'éducation française Division
	Myrna Klassen Acting Coordinator	Distance Learning Unit Instruction, Curriculum and Assessment Branch
	Gilles Landry Project Manager	Learning Support and Technology Unit Instruction, Curriculum and Assessment Branch
	Philippe Leclercq Conseiller pédagogique (until June 2015)	Conseiller pédagogique — Mathématiques 9 à 12 Division du Bureau de l'éducation française Division
	Grant Moore Publications Editor	Document Production Services Unit Instruction, Curriculum and Assessment Branch

Grade 12 Applied Mathematics (40S)

Introduction

Introduction

Overview

Welcome to Grade 12 Applied Mathematics! This course is a continuation of the concepts you have studied in previous years, as well as an introduction to new topics. It builds upon the applied mathematics topics you were introduced to in Grade 11 Applied Mathematics. You will put to use many of the skills that you have already learned to solve problems and learn new skills along the way. This course helps you develop the skills, ideas, and confidence you will need to continue studying math in the future.

As a student enrolled in an independent study course, you have taken on a dual role—that of a student and a teacher. As a student, you are responsible for mastering the lessons and completing the learning activities and assignments. As a teacher, you are responsible for checking your work carefully, noting areas in which you need to improve, and motivating yourself to succeed.

What Will You Learn in This Course?

In this course, problem solving, communication, reasoning, and mental math are some of the themes you will discover in each module. You will engage in a variety of activities that promote the practical application of symbolic math ideas to the world around you.

There are several areas that you will explore in this course, including linear and non-linear functions, logic, probability, counting techniques, finance, and design and measurement.

How Is This Course Organized?

The Grade 12 Applied Mathematics course consists of the following eight modules:

- Module 1: Functions
- Module 2: Mathematics Research Project
- Module 3: Logical Reasoning
- Module 4: Probability
- Module 5: Financial Mathematics
- Module 6: Techniques of Counting
- Module 7: Sinusoidal Functions
- Module 8: Design and Measurement

Each module in this course consists of several lessons, which contain the following components:

- **Lesson Focus:** The Lesson Focus at the beginning of each lesson identifies one or more specific learning outcomes (SLOs) that are addressed in the lesson. The SLOs identify the knowledge and skills you should have achieved by the end of the lesson.
- **Introduction:** Each lesson begins by outlining what you will be learning in that lesson.
- **Lesson:** The main body of the lesson consists of the content and processes that you need to learn. It contains information, explanations, diagrams, and completed examples.
- **Learning Activities:** Each lesson has a learning activity that focuses on the lesson content. Your responses to the questions in the learning activities will help you to practise or review what you have just learned. Once you have completed a learning activity, check your responses with those provided in the Learning Activity Answer Key found at the end of the applicable module. Do not send your learning activities to the Distance Learning Unit for assessment.
- **Assignments:** Assignments are found throughout each module within this course. At the end of each module, you will mail or electronically submit all your completed assignments from that module to the Distance Learning Unit for assessment. All assignments combined will be worth a total of 55 percent of your final mark in this course.
- **Summary:** Each lesson ends with a brief review of what you just learned.

This course also includes the following appendix:

- **Technology Appendix:** The appendix provides basic information to help you learn how to use certain technology software and applications to complete this course.

What Resources Will You Need for This Course?

You do not need a textbook for this course. All the content is provided directly within the course. You will, however, need access to a variety of resources.

Required Resources

You will require access to an email account if you plan to

- communicate with your tutor/marker by email
- use the learning management system (LMS) to submit your completed assignments

To complete this course, you will require a graphing calculator or access to computer software applications for graphing and financial mathematics operations. To write your examinations, you will need access to the same resources that you used for the modules.

Contact your tutor/marker to make sure that the technology you are using is appropriate for the assignments and the examinations.

Before writing the midterm examination, you will be asked to specify one graphing app or graphing technology and, before writing the final examination, you will be asked to specify one graphing app or graphing technology and one financial app or financial technology. Make sure your tutor/marker has approved your choices prior to writing the examinations.

Optional Resources

It would be helpful if you had access to the following resources:

- **A computer with Internet access:** There are many online resources for this course and references are made to them in the lessons where they would be used. You can choose to use online resources to graph and solve linear and non-linear functions. Many financial mathematics operations can be completed with the help of online sites. You can search for resources online to complete your research project. You may use spreadsheets for design and measurement, the research project, finance, or other modules.

- **A photocopier:** With access to a photocopier/scanner, you could make a copy of your assignments before submitting them so that if your tutor/marker wants to discuss an assignment with you over the phone, each of you will have a copy. It would also allow you to continue studying or to complete further lessons while your original work is with the tutor/marker. Photocopying or scanning your assignments will also ensure that you keep a copy in case the originals are lost.

Who Can Help You with This Course?

Taking an independent study course is different from taking a course in a classroom. Instead of relying on the teacher to tell you to complete a learning activity or an assignment, you must tell yourself to be responsible for your learning and for meeting deadlines. There are, however, two people who can help you be successful in this course: your tutor/marker and your learning partner.

Your Tutor/Marker

Tutor/markers are experienced educators who tutor Independent Study Option (ISO) students and mark assignments and examinations. When you are having difficulty with something in this course, contact your tutor/marker, who is there to help you. Your tutor/marker's name and contact information were sent to you with this course. You can also obtain this information in the learning management system (LMS).

Your Learning Partner

A learning partner is someone **you choose** who will help you learn. It may be someone who knows something about mathematics, but it doesn't have to be. A learning partner could be someone else who is taking this course, a teacher, a parent or guardian, a sibling, a friend, or anybody else who can help you. Most importantly, a learning partner should be someone with whom you feel comfortable and who will support you as you work through this course.

Your learning partner can help you keep on schedule with your coursework, read the course with you, check your work, look at and respond to your learning activities, or help you make sense of assignments. You may even study for your examination(s) with your learning partner. If you and your learning partner are taking the same course, however, your assignment work should not be identical.

One of the best ways that your learning partner can help you is by reviewing your midterm and final practice examinations with you. These are found in the learning management system (LMS), along with their answer keys. Your learning partner can administer your practice examination, check your answers with you, and then help you learn the things that you missed.

How Will You Know How Well You Are Learning?

You will know how well you are learning in this course by how well you complete the learning activities, assignments, and examinations.

Learning Activities

The learning activities in this course will help you to review and practise what you have learned in the lessons. You will not submit the completed learning activities to the Distance Learning Unit. Instead, you will complete the learning activities and compare your responses to those provided in the Learning Activity Answer Key found at the end of each module.

Each learning activity has two parts—Part A has BrainPower questions and Part B has questions related to the content in the lesson.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for you before trying the other questions. Each question should be completed quickly and without using a calculator, and most should be completed without using pencil and paper to write out the steps. Some of the questions will relate directly to content of the course. Some of the questions will review content from previous courses—content that you need to be able to answer efficiently.

Being able to do these questions in a few minutes will be helpful to you as you continue with your studies in mathematics. If you are finding it is taking you longer to do the questions, you can try one of the following:

- work with your learning partner to find more efficient strategies for completing the questions
- ask your tutor/marker for help with the questions
- search online for websites that help you practice the computations so you can become more efficient at completing the questions

None of the assignment questions or examination questions will require you to do the calculations quickly or without a calculator. However, it is for your benefit to complete the questions, as they will help you in the course. Also, being able to complete the BrainPower exercises successfully will help build your confidence in mathematics. BrainPower questions are like a warm-up you would do before competing in a sporting event.

Part B: Course Content Questions

One of the easiest and fastest ways to find out how much you have learned is to complete Part B of the learning activities. These have been designed to let you assess yourself by comparing your answers with the answer keys at the end of each module. There is at least one learning activity in each lesson. You will need a notebook or loose-leaf pages to write your answers.

Make sure you complete the learning activities. Doing so will not only help you to practise what you have learned, but will also prepare you to complete your assignments and the examinations successfully. Many of the questions on the examinations will be similar to the questions in the learning activities. Remember that you **will not submit learning activities to the Distance Learning Unit**.

Assignments

Lesson assignments are located throughout the modules and include questions similar to the questions in the learning activities of previous lessons. The assignments have space provided for you to write your answers on the question sheets. **You need to show all your steps as you work out your solutions, and make sure your answers are clear (include units, where appropriate).**

Once you have completed all the assignments in a module, you will submit them to the Distance Learning Unit for assessment. The assignments are worth a total of 55 percent of your final course mark. You must complete each assignment in order to receive a final mark in this course. **You will mail or electronically submit these assignments to the Distance Learning Unit along with the appropriate cover page once you complete each module.**

The tutor/marker will mark your assignments and return them to you. Remember to keep all marked assignments until you have finished the course so that you can use them to study for your examinations.

Resource Sheet

When you write your midterm and final examinations, you will be allowed to take an Examination Resource Sheet with you into the examination. This sheet will be one letter-sized page, 8½ ″ by 11″, with both sides in your handwriting or typewritten. It is to be submitted with your examination. The Examination Resource Sheet is not worth any marks.

Creating your own resource sheet is an excellent way to review. It also provides you with a convenient reference and quick summary of the important facts of each module. Each student is asked to complete a resource sheet for each module to help with studying and reviewing.

The lesson summaries are written for you to use as a guide, as are the module summaries at the end of each module. Refer to these when you create your own resource sheet. Also, refer to the online glossary found in the learning management system (LMS) to check the information on your resource sheet.

After completing each module's resource sheet, you should summarize the sheets from all of the modules to prepare your Examination Resource Sheet. When preparing your Midterm Examination Resource Sheet, remember that your midterm examination is based on Modules 1 to 4. When preparing your Final Examination Resource Sheet, remember that your final examination is based on Modules 5 to 8.

Midterm and Final Examinations

This course contains a midterm examination and a final examination.

- The **midterm examination** is based on Modules 1 to 4, and is worth 20 percent of your final course mark. You will write the midterm examination when you have completed Module 4.
- The **final examination** is based on Modules 5 to 8 and is worth 25 percent of your final course mark. You will write the final examination when you have completed Module 8.

The two examinations are worth a total of **45 percent** of your final course mark. You will write both examinations under supervision.

In order to do well on the examinations, you should review all of the work that you have completed from Modules 1 to 4 for your midterm examination and Modules 5 to 8 for your final examination, including all learning activities and assignments. You can use your Examination Resource Sheet to bring any formulas and other important information into the examination with you.

You will be required to bring the following supplies when you write both examinations: pens/pencils (2 or 3 of each), metric and imperial rulers, a graphing and/or scientific calculator, and your Examination Resource Sheet. Both examinations are 3 hours in duration.

For the **midterm examination**, graphing technology (either computer software or a graphing calculator) **is required** to complete the examination.

For the **final examination**, graphing and financial applications technology (either computer software or a graphing calculator) **are required** to complete the examination.

Practice Examinations and Answer Keys

To help you succeed in your examinations, you will have an opportunity to complete a Midterm Practice Examination and a Final Practice Examination. These examinations, along with the answer keys, are found in the learning management system (LMS). If you do not have access to the Internet, contact the Distance Learning Unit at 1-800-465-9915 to obtain a copy of the practice examinations.

These practice examinations are similar to the actual examinations you will be writing. The answer keys enable you to check your answers. This will give you the confidence you need to do well on your examinations.

Requesting Your Examinations

You are responsible for making arrangements to have the examinations sent to your proctor from the Distance Learning Unit. Please make arrangements before you finish Module 4 to write the midterm examination. Likewise, you should begin arranging for your final examination before you finish Module 8.

To write your examinations, you need to make the following arrangements:

- **If you are attending school,** your examination will be sent to your school as soon as all the applicable assignments have been submitted. You should make arrangements with your school's ISO school facilitator to determine a date, time, and location to write the examination.

- **If you are not attending school,** check the Examination Request Form for options available to you. Examination Request Forms can be found on the Distance Learning Unit's website, or look for information in the learning management system (LMS). Two weeks before you are ready to write the examination, fill in the Examination Request Form and mail, fax, or email it to

  Distance Learning Unit
  500–555 Main Street
  PO Box 2020
  Winkler MB R6W 4B8
  Fax: 204-325-1719
  Toll-Free Telephone: 1-800-465-9915
  Email: distance.learning@gov.mb.ca

How Much Time Will You Need to Complete This Course?

Learning through independent study has several advantages over learning in the classroom. You are in charge of how you learn and you can choose how quickly you will complete the course. You can read as many lessons as you wish in a single session. You do not have to wait for your teacher or classmates.

From the date of your registration, you have a maximum of **12 months** to complete the course, but the pace at which you proceed is up to you. Read the following suggestions on how to pace yourself.

Chart A: Semester 1

If you want to start this course in September and complete it in January, you can follow the timeline suggested below.

Module	Completion Date
Module 1	Middle of September
Module 2	End of September
Module 3	Middle of October
Module 4	End of October
Midterm Examination	Beginning of November
Module 5	Middle of November
Module 6	End of November
Module 7	Middle of December
Module 8	Middle of January
Final Examination	End of January

Chart B: Semester 2

If you want to start the course in February and complete it in May, you can follow the timeline suggested below.

Module	Completion Date
Module 1	Middle of February
Module 2	End of February
Module 3	Beginning of March
Module 4	Middle of March
Midterm Examination	End of March
Module 5	Beginning of April
Module 6	Middle of April
Module 7	End of April
Module 8	Beginning of May
Final Examination	Middle of May

Chart C: Full School Year (Not Semestered)

If you want to start the course in September and complete it in May, you can follow the timeline suggested below.

Module	Completion Date
Module 1	End of September
Module 2	End of October
Module 3	End of November
Module 4	End of December
Midterm Examination	Middle of January
Module 5	Middle of February
Module 6	Middle of March
Module 7	Beginning of April
Module 8	Beginning of May
Final Examination	Middle of May

Timelines

Do not wait until the last minute to complete your work, since your tutor/marker may not be available to mark it immediately. It may take a few weeks for your tutor/marker to assess your work and return it to you or your school.

If you need this course to graduate this school year, all coursework must be received by the Distance Learning Unit on or before the first Friday in May, and all examinations must be received by the Distance Learning Unit on or before the last Friday in May. Any coursework or examinations received after these deadlines may not be processed in time for a June graduation. Assignments or examinations submitted after these recommended deadlines will be processed and marked as they are received.

When and How Will You Submit Completed Assignments?

When to Submit Assignments

While working on this course, you will submit completed assignments to the Distance Learning Unit nine times. The following chart shows you exactly what assignments you will be submitting.

Submission of Assignments	
Submission	Assignments You Will Submit
1	**Module 1: Functions** Module 1 Cover Sheet Module 1 Cover Assignment: Number Puzzle Assignment 1.1: Polynomial Functions Assignment 1.2: Logarithmic Function
2	**Module 2: Mathematics Research Project** Module 2 Cover Sheet 1 Module 2 Cover Assignment: The Tower of Hanoi Assignment 2.1: Project Proposal
3	**Module 2: Mathematics Research Project** Module 2 Cover Sheet 2 Assignment 2.2: Collecting and Assessing Data Assignment 2.3: Interpreting Data Assignment 2.4: Presentation
4	**Module 3: Logical Reasoning** Module 3 Cover Sheet Module 3 Cover Assignment: Sets Game Assignment 3.1: Sets and Conditional Statements
5	**Module 4: Probability** Module 4 Cover Sheet Module 4 Cover Assignment: Strategies Assignment 4.1: Probability and Odds
6	**Module 5: Financial Mathematics** Module 5 Cover Sheet Module 5 Cover Assignment: Crossing the Canal with Cats Assignment 5.1: Compound Interest Assignment 5.2: Loans and Investments Assignment 5.3: Investment Decisions
7	**Module 6: Techniques of Counting** Module 6 Cover Sheet Module 6 Cover Assignment: Patterns in Numbers Assignment 6.1: Fundamental Counting Principle Assignment 6.2: Applications of Permutations and Combinations
8	**Module 7: Sinusoidal Functions** Module 7 Cover Sheet Module 7 Cover Assignment: Playing Fair Assignment 7.1: Sinusoidal Function Models
9	**Module 8: Design and Measurement** Module 8 Cover Sheet Module 8 Cover Assignment: Container Conundrum Assignment 8.1: Design and Cost Decisions Assignment 8.2: Working within a Project Budget

How to Submit Assignments

In this course, you have the choice of submitting your assignments either by mail or electronically.

- **Mail:** Each time you **mail** something, you must include the print version of the applicable Cover Sheet (found at the end of this Introduction). Complete the information at the top of each Cover Sheet before submitting it along with your assignments.
- **Electronic submission:** You do not need to include a cover sheet when submitting assignments electronically.

Submitting Your Assignments by Mail

If you choose to mail your completed assignments, please photocopy/scan all the materials first so that you will have a copy of your work in case your package goes missing. You will need to place the applicable module Cover Sheet and assignment(s) in an envelope, and address it to

Distance Learning Unit
500–555 Main Street
PO Box 2020
Winkler MB R6W 4B8

Your tutor/marker will mark your work and return it to you by mail

Submitting Your Assignments Electronically

Assignment submission options vary by course. Sometimes assignments can be submitted electronically and sometimes they must be submitted by mail. Specific instructions on how to submit assignments were sent to you with this course. In addition, this information is available in the learning management system (LMS).

If you are submitting assignments electronically, make sure you have saved copies of them before you send them. That way, you can refer to your assignments when you discuss them with your tutor/marker. Also, if the original hand-in assignments are lost, you are able to resubmit them.

Your tutor/marker will mark your work and return it to you electronically.

> The Distance Learning Unit does not provide technical support for hardware-related issues. If troubleshooting is required, consult a professional computer technician.

What Are the Guide Graphics For?

Guide graphics are used throughout this course to identify and guide you in specific tasks. Each graphic has a specific purpose, as described below.

Lesson Introduction: The introduction sets the stage for the lesson. It may draw upon prior knowledge or briefly describe the organization of the lesson. It also lists the learning outcomes for the lesson. Learning outcomes describe what you will learn.

Learning Partner: Ask your learning partner to help you with this task.

Learning Activity: Complete a learning activity. This will help you to review or practise what you have learned and to prepare for an assignment or an examination. You will not submit learning activities to the Distance Learning Unit. Instead, you will compare your responses to those provided in the Learning Activity Answer Key found at the end of the applicable module.

Assignment: Complete an assignment. You will submit your completed assignments to the Distance Learning Unit for assessment at the specified times.

Mail or Electronic Submission: Mail or electronically submit your completed assignments to the Distance Learning Unit for assessment at this time.

Phone Your Tutor/Marker: Telephone your tutor/marker.

Resource Sheet: Indicates material that may be valuable to include on your resource sheet.

Examination: Write your midterm or final examination at this time.

Note: Take note of and remember this important information or reminder.

Getting Started

Take some time right now to skim through the course material, locate your cover sheets, and familiarize yourself with how the course is organized. Get ready to learn!

Remember: If you have questions or need help at any point during this course, contact your tutor/marker or ask your learning partner for help.

Good luck with the course!

Notes

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Module 1: Functions
Cover Sheet

Please complete this sheet and place it on top of your assignments to assist in proper recording of your work. Submit the package to:

Drop-off/Courier Address

Distance Learning Unit
555 Main Street
Winkler MB R6W 1C4

Mailing Address

Distance Learning Unit
500–555 Main Street
PO Box 2020
Winkler MB R6W 4B8

Contact Information

Legal Name: ____________________ Preferred Name: ____________________

Phone: ____________________ Email: ____________________

Mailing Address: ____

City/Town: ______________________________ Postal Code: __________

Attending School: ❑ No ❑ Yes

School Name: ____

Has your contact information changed since you registered for this course? ❑ No ❑ Yes

Note: Please keep a copy of your assignments so that you can refer to them when you discuss them with your tutor/marker.

For Student Use	For Office Use Only	
Module 1 Assignments	**Attempt 1**	**Attempt 2**
Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	__________ Date Received	__________ Date Received
❑ Module 1 Cover Assignment: Number Puzzle	_______ /5	_______ /5
❑ Assignment 1.1: Polynomial Functions	_______ /29	_______ /29
❑ Assignment 1.2: Logarithmic Function Models	_______ /37	_______ /37
	Total: _____ **/71**	**Total:** _____ **/71**

For Tutor/Marker Use
Remarks:

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Module 2: Mathematics Research Project
Cover Sheet 1

Please complete this sheet and place it on top of your assignments to assist in proper recording of your work. Submit the package to:

Drop-off/Courier Address

Distance Learning Unit
555 Main Street
Winkler MB R6W 1C4

Mailing Address

Distance Learning Unit
500–555 Main Street
PO Box 2020
Winkler MB R6W 4B8

Contact Information

Legal Name: ____________________ Preferred Name: ____________________

Phone: ____________________ Email: ____________________

Mailing Address: ____

City/Town: ____________________ Postal Code: __________

Attending School: ❑ No ❑ Yes

School Name: ____

Has your contact information changed since you registered for this course? ❑ No ❑ Yes

Note: Please keep a copy of your assignments so that you can refer to them when you discuss them with your tutor/marker.

For Student Use	For Office Use Only	
Module 2 Cover Assignment and Assignment 2.1	**Attempt 1**	**Attempt 2**
Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	________ Date Received	________ Date Received
❑ Module 2 Cover Assignment: The Tower of Hanoi	______ /10	______ /10
❑ Assignment 2.1: Project Proposal (include rubric)	______ /9	______ /9
	Total: _____ **/19**	**Total:** _____ **/19**

For Tutor/Marker Use
Remarks:

Rubric for Assignment 2.1			
Category	Level 1	Level 2	Level 3
Question	Question is poorly worded, is impractical, or is inappropriate	Question is somewhat vague or too narrow	Question is focused and clearly communicated
Availability of appropriate research	Question can be answered without data or information collection or research, or it is impossible to find data	It is possible but may be difficult to generate primary data or find adequate or appropriate data or information	Meaningful, appropriate, and applicable information or statistics can be generated or are available
Description of proposed primary and/or secondary data collection methods and/or sources to be used	Minimal information is provided regarding how data will be collected or where information will be found	Incomplete information regarding how data will be collected or where information will be found is provided	Intentions regarding primary and/or secondary data collection methods and sources are clearly outlined, including proper bibliographic citation

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Module 2: Mathematics Research Project
Cover Sheet 2

Please complete this sheet and place it on top of your assignments to assist in proper recording of your work. Submit the package to:

Drop-off/Courier Address

Distance Learning Unit
555 Main Street
Winkler MB R6W 1C4

Mailing Address

Distance Learning Unit
500–555 Main Street
PO Box 2020
Winkler MB R6W 4B8

Contact Information

Legal Name: ____________________ Preferred Name: ____________________

Phone: ____________________ Email: ____________________

Mailing Address: ____

City/Town: ____________________ Postal Code: __________

Attending School: ❑ No ❑ Yes

School Name: ____

Has your contact information changed since you registered for this course? ❑ No ❑ Yes

Note: Please keep a copy of your assignments so that you can refer to them when you discuss them with your tutor/marker.

For Student Use	For Office Use Only	
Module 2 Assignments 2.2, 2.3, and 2.4	**Attempt 1**	**Attempt 2**
Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	________ Date Received	________ Date Received
❑ Assignment 2.2: Collecting and Assessing Data (include rubric)	______ /12	______ /12
❑ Assignment 2.3: Interpreting Data (include rubric)	______ /15	______ /15
❑ Assignment 2.4: Presentation (include rubric)	______ /9	______ /9
	Total: _____ **/36**	**Total:** _____ **/36**

For Tutor/Marker Use
Remarks:

Rubric for Assignment 2.2			
Category	Level 1	Level 2	Level 3
Data and information	Inadequate or irrelevant data and information are provided	Data and information provided are incomplete	Complete data and information are provided, including data/information from other sources that contradicts or supports viewpoint
Data collection (primary or secondary)	Poor data collection methods are used, with inappropriate population or sample, or inadequate description Sources are not identified	Data collection methods are not clearly outlined or explained; population and sample are not clearly explained Bibliographic information is incomplete	Collection methods are clearly outlined; population and random, representative sample are identified and used All sources are clearly identified with complete bibliographic information
Assessment	Information and data are not assessed for bias, point of view, accuracy, reliability, and relevance	Assessment of information and data is incomplete	Assessment of the data and information is clearly communicated and includes explanation of bias, point of view, accuracy, reliability, and relevance

Rubric for Assignment 2.3			
Category	Level 1	Level 2	Level 3
Represent data	Representation of data and information is inappropriate or incomplete	Data and information are not clearly represented, or may have errors or omissions	Use of graphs, charts, or equations to appropriately represent data collected is effective and complete
Analysis	Analysis applied to data is inappropriate or irrelevant	Analysis of data and information is incomplete, with limited mathematical or statistical analysis shown	Complete analysis and meaningful application of mathematical, statistical, or regression analysis to model data are included
Interpretation	Incorrectly applied analysis leads to incorrect interpretations	Interpretations are based on weak analysis; analysis may not always be supported by data	Mathematical reasoning or calculations are applied to make sense of analysis, showing a thorough understanding of the issue

Rubric for Assignment 2.4			
Category	Level 1	Level 2	Level 3
Organization	Presentation is disorganized, unclear, or incomplete	Presentation of research is adequate but simplistic	Creative, organized, and purposeful presentation is presented, and ideas are clearly and concisely explained
Process	Process of research is not clearly stated; connection of research to inquiry question is vague or poorly justified	Process of research and methods used to reach a conclusion are not fully stated	Research process is well documented, and controversial issues and multiple viewpoints are identified with supporting data, if applicable
Conclusion	Conclusion is not supported by research	Answer to original question is provided, but only partly justified by research data or information	Logical, conclusive answer to question is presented with justification, based on analysis of data and information

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Module 3: Logical Reasoning
Cover Sheet

Please complete this sheet and place it on top of your assignments to assist in proper recording of your work. Submit the package to:

Drop-off/Courier Address

Distance Learning Unit
555 Main Street
Winkler MB R6W 1C4

Mailing Address

Distance Learning Unit
500–555 Main Street
PO Box 2020
Winkler MB R6W 4B8

Contact Information

Legal Name: ____________________ Preferred Name: ____________________

Phone: ____________________ Email: ____________________

Mailing Address: ____

City/Town: ____________________ Postal Code: __________

Attending School: ❑ No ❑ Yes

School Name: ____

Has your contact information changed since you registered for this course? ❑ No ❑ Yes

Note: Please keep a copy of your assignments so that you can refer to them when you discuss them with your tutor/marker.

For Student Use	For Office Use Only	
Module 3 Assignments	**Attempt 1**	**Attempt 2**
Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	______ Date Received	______ Date Received
❑ Module 3 Cover Assignment: SET® Game	______ /5	______ /5
❑ Assignment 3.1: Sets and Conditional Statements	______ /31	______ /31
	Total: _____ **/36**	**Total:** _____ **/36**

For Tutor/Marker Use
Remarks:

Grade 12 Applied Mathematics (40S)

Module 4: Probability
Cover Sheet

Please complete this sheet and place it on top of your assignments to assist in proper recording of your work. Submit the package to:

Drop-off/Courier Address

Distance Learning Unit
555 Main Street
Winkler MB R6W 1C4

Mailing Address

Distance Learning Unit
500–555 Main Street
PO Box 2020
Winkler MB R6W 4B8

Contact Information

Legal Name: ___________________________ Preferred Name: _________________________

Phone: _________________________________ Email: __________________________________

Mailing Address: ____

City/Town: ____ Postal Code: _______________

Attending School: ❑ No ❑ Yes

School Name: ____

Has your contact information changed since you registered for this course? ❑ No ❑ Yes

Note: Please keep a copy of your assignments so that you can refer to them when you discuss them with your tutor/marker.

For Student Use	For Office Use Only	
Module 4 Assignments	**Attempt 1**	**Attempt 2**
Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	________ Date Received	________ Date Received
❑ Module 4 Cover Assignment: Strategies	_________ /5	_________ /5
❑ Assignment 4.1: Probability and Odds	________ /41	________ /41
	Total: _____ **/46**	**Total:** _____ **/46**

For Tutor/Marker Use
Remarks:

Grade 12 Applied Mathematics (40S)

Module 5: Financial Mathematics
Cover Sheet

Please complete this sheet and place it on top of your assignments to assist in proper recording of your work. Submit the package to:

Drop-off/Courier Address
Distance Learning Unit
555 Main Street
Winkler MB R6W 1C4

Mailing Address
Distance Learning Unit
500–555 Main Street
PO Box 2020
Winkler MB R6W 4B8

Contact Information

Legal Name: ____________________ Preferred Name: ____________________

Phone: ____________________ Email: ____________________

Mailing Address: ____

City/Town: ____________________ Postal Code: ____________

Attending School: ❑ No ❑ Yes

School Name: ____

Has your contact information changed since you registered for this course? ❑ No ❑ Yes

Note: Please keep a copy of your assignments so that you can refer to them when you discuss them with your tutor/marker.

For Student Use	For Office Use Only	
Module 5 Assignments	**Attempt 1**	**Attempt 2**
Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	________ Date Received	________ Date Received
❑ Module 5 Cover Assignment: Crossing the Canal with Cats	________ /5	________ /5
❑ Assignment 5.1: Compound Interest	________ /25	________ /25
❑ Assignment 5.2: Loans and Investments	________ /41	________ /41
❑ Assignment 5.3: Investment Decisions	________ /53	________ /53
	Total: _____ **/124**	**Total:** _____ **/124**

For Tutor/Marker Use
Remarks:

Grade 12 Applied Mathematics (40S)

Module 6: Techniques of Counting
Cover Sheet

Please complete this sheet and place it on top of your assignments to assist in proper recording of your work. Submit the package to:

Drop-off/Courier Address

Distance Learning Unit
555 Main Street
Winkler MB R6W 1C4

Mailing Address

Distance Learning Unit
500–555 Main Street
PO Box 2020
Winkler MB R6W 4B8

Contact Information

Legal Name: ____________________ Preferred Name: ____________________

Phone: ____________________ Email: ____________________

Mailing Address: ____

City/Town: ____________________ Postal Code: ____________

Attending School: ❑ No ❑ Yes

School Name: ____

Has your contact information changed since you registered for this course? ❑ No ❑ Yes

Note: Please keep a copy of your assignments so that you can refer to them when you discuss them with your tutor/marker.

For Student Use	For Office Use Only	
Module 6 Assignments	**Attempt 1**	**Attempt 2**
Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	________ Date Received	________ Date Received
❑ Module 6 Cover Assignment: Patterns in Numbers	________ /5	________ /5
❑ Assignment 6.1: Fundamental Counting Principle	________ /21	________ /21
❑ Assignment 6.2: Applications of Permutations and Combinations	________ /30	________ /30
	Total: _____ /56	**Total: _____ /56**

For Tutor/Marker Use
Remarks:

Grade 12 Applied Mathematics (40S)

Module 7: Sinusoidal Functions
Cover Sheet

Please complete this sheet and place it on top of your assignments to assist in proper recording of your work. Submit the package to:

Drop-off/Courier Address

Distance Learning Unit
555 Main Street
Winkler MB R6W 1C4

Mailing Address

Distance Learning Unit
500–555 Main Street
PO Box 2020
Winkler MB R6W 4B8

Contact Information

Legal Name: ____________________________ Preferred Name: __________________________

Phone: _________________________________ Email: ____________________________________

Mailing Address: ____

City/Town: ____ Postal Code: _______________

Attending School: ❑ No ❑ Yes

School Name: ____

Has your contact information changed since you registered for this course? ❑ No ❑ Yes

Note: Please keep a copy of your assignments so that you can refer to them when you discuss them with your tutor/marker.

For Student Use	For Office Use Only	
Module 7 Assignments	**Attempt 1**	**Attempt 2**
Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	______________ Date Received	______________ Date Received
❑ Module 7 Cover Assignment: Playing Fair	_________ /5	_________ /5
❑ Assignment 7.1: Sinusoidal Function Models	________ /33	________ /33
	Total: _____ **/38**	**Total:** _____ **/38**

For Tutor/Marker Use
Remarks:

Grade 12 Applied Mathematics (40S)

Module 8: Design and Measurement
Cover Sheet

Please complete this sheet and place it on top of your assignments to assist in proper recording of your work. Submit the package to:

Drop-off/Courier Address
Distance Learning Unit
555 Main Street
Winkler MB R6W 1C4

Mailing Address
Distance Learning Unit
500–555 Main Street
PO Box 2020
Winkler MB R6W 4B8

Contact Information

Legal Name: ____________________ Preferred Name: ____________________

Phone: ____________________ Email: ____________________

Mailing Address: ____________________

City/Town: ____________________ Postal Code: __________

Attending School: ❑ No ❑ Yes

School Name: ____________________

Has your contact information changed since you registered for this course? ❑ No ❑ Yes

Note: Please keep a copy of your assignments so that you can refer to them when you discuss them with your tutor/marker.

For Student Use	For Office Use Only	
Module 8 Assignments	**Attempt 1**	**Attempt 2**
Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	__________ Date Received	__________ Date Received
❑ Module 8 Cover Assignment: Container Conundrum	________ /5	________ /5
❑ Assignment 8.1: Design and Cost Decisions	________ /22	________ /22
❑ Assignment 8.2: Working within a Project Budget	________ /10	________ /10
	Total: _____ **/37**	**Total:** _____ **/37**

For Tutor/Marker Use
Remarks:

Released 2019

Printed in Canada
Imprimé au Canada

Grade 12 Applied Mathematics (40S)

Module 1
Functions

Module 1: Functions

Introduction

Welcome to your first module of Grade 12 Applied Mathematics! You will begin with a look back at linear and quadratic functions, and then extend these concepts to include cubic, exponential, and logarithmic functions. You will analyze and describe the characteristics of these functions, and match their graphs to the correct equations. You will determine which function best approximates contextual data and use it to solve problems.

Assignments in Module 1

When you have completed the assignments for Module 1, submit your completed assignments to the Distance Learning Unit either by mail or electronically through the learning management system (LMS). The staff will forward your work to your tutor/marker.

Lesson	Assignment Number	Assignment Title
	Cover Assignment	Number Puzzle
2	Assignment 1.1	Polynomial Functions
6	Assignment 1.2	Logarithmic Function Models

Resource Sheet

When you write your midterm examination, you are encouraged to take a Midterm Examination Resource Sheet with you into the examination. This sheet will be one letter-sized page, 8½ ″ by 11″, with both sides in your handwriting or typewritten. You will submit it with your examination, but you do not receive any marks for it.

Many students have found that preparing a resource sheet is an excellent way to review. It provides you with a summary of the important facts of each module. You should complete a resource sheet for each module to help with your studying and reviewing. Lesson summaries and module summaries are included for you to use as a guide.

You may use the list of instructions provided below to help you with preparing your resource sheet for the material in Module 1. On this sheet, you should record math terms and definitions, formulas, sample questions, or a list of places where you often make mistakes. You should also identify special areas that require extra attention or review by writing the page numbers.

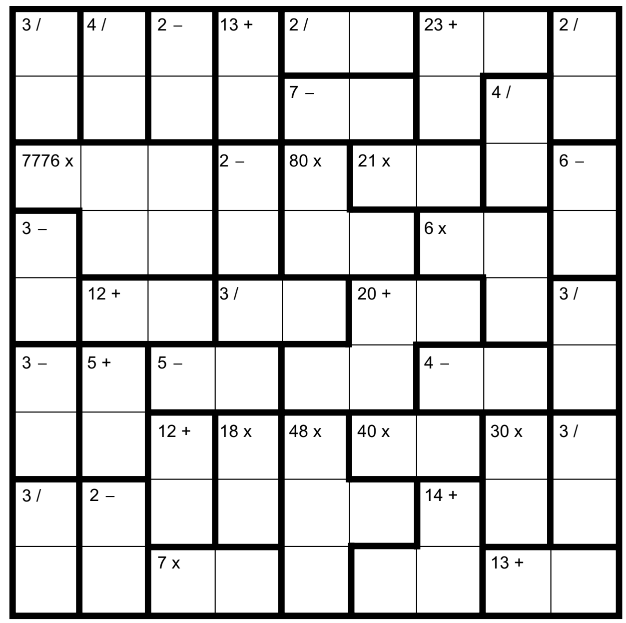
After you have completed each module's resource sheet, you may summarize the sheets from Modules 1, 2, 3, and 4 to prepare your Midterm Examination Resource Sheet. The midterm examination for this course is based on Modules 1 to 4.

Resource Sheet for Module 1

As you go through the lessons of this module, you may want to consider the following suggestions regarding the creation of a resource sheet.

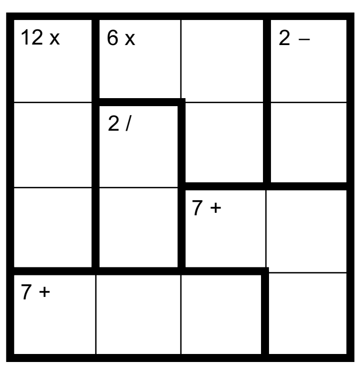
1. List all the important math terms, and define them if necessary.
2. List all the formulas and perhaps a sample problem that shows how the formula is used.
3. If necessary, write the solutions to some problems, showing in detail how you did the calculations.
4. Copy any questions that represent the key points of the lesson, and perhaps include the solutions as well.
5. Identify the problems you found most difficult, and copy the page numbers onto the resource sheet so that you can review them before writing the examination. You may also copy the problems and the solutions onto your resource sheet, and later write them onto your Midterm Examination Resource Sheet.
6. Write any comments, ideas, shortcuts, or other reminders that may be helpful during an examination.

Module 1 Cover Assignment: Number Puzzle



The Sudoku puzzle craze has been called an international math puzzle phenomenon. You may have seen these 9 × 9 grid puzzles in newspapers and online. A variation of the Sudoku is the Inky puzzle created by Jim Bumgardner. They are also known as KenKen® puzzles (created by Tetsuya Miyamoto), Mathdoku, Calcudoku, or Web Kendoku. The rules are similar to Sudoku, where you fill in the grid so each line and column contain the digits 1 through 9. The unique twist to the Inky puzzle is that the "cages" outlined with the heavy lines must contain the numbers (in any order) that produce the given total, using the mathematical operation indicated. Numbers inside the cages may repeat, as long as they are not in the same row or column.

Begin by trying a smaller puzzle, using the digits 1 to 4.



Solution

The only three distinct digits that multiply to 12 are 1 × 3 × 4 in some order, so the bottom left-hand corner must be 2. The only way to arrive at a sum of 7 with three digits in the bottom row is by using 4 + 2 + 1. That leaves 3 for the bottom right corner. A difference of two can be calculated as 4 – 2 or 3 – 1, but since there is a 3 in the last column already, it must be 4 and 2 in some order in that cage, leaving a 1 in the third box in the last column, and another 3 in the third row to complete the 7+ cage. A product of 6 with three digits must be 1 × 2 × 3. Since the third column already has a 3, put it in the second box in the top row. This cage requires a 2 and 1, so the bottom box in the third column must contain the 4. The bottom row is now complete. Place the last 3 in the 12× cage in the only row it can go in, forcing the 1 to go in the top left corner. The rest of the puzzle can now be filled in.

1	3	2	4
3	4	1	2
4	2	3	1
2	1	4	3

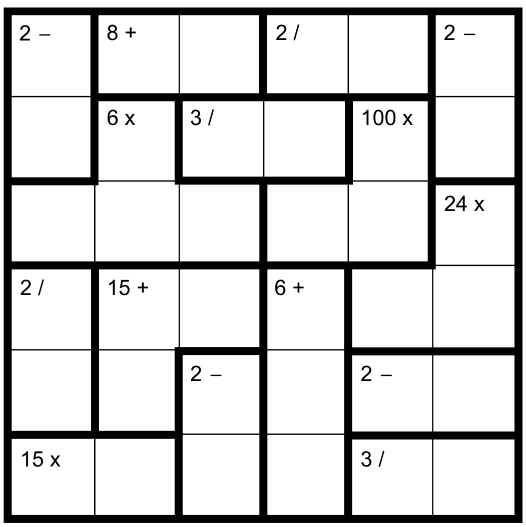
Module 1 Cover Assignment

Number Puzzle

Total: 5 marks

Complete the following 6 × 6 number puzzle and submit it with the rest of your hand-in assignments when you complete Module 1. If you need more practice with smaller puzzles before completing the 6 × 6 puzzle, you can find puzzles online or in puzzle books. You may want to challenge yourself and try a 9 × 9 grid!

Use the digits 1 through 6 to complete each row and column. The digits in the cages must produce, in any order, the given total, using the operation indicated. No number can be used more than once in the same line or column.



Notes

Lesson 1: Characteristics of Polynomial Functions

Lesson Focus

In this lesson, you will

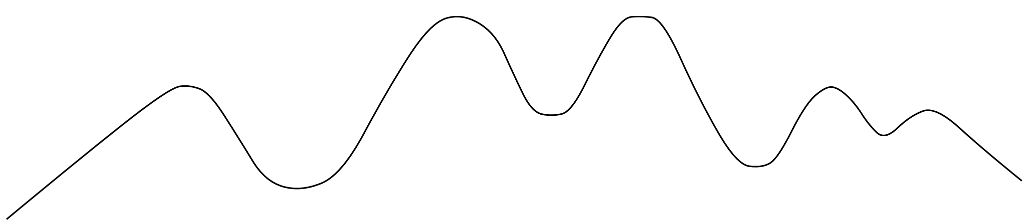
- ❑ describe the characteristics of polynomial functions by analyzing their graphs and equations
- ❑ match polynomial equations in a set to their corresponding graphs

Lesson Introduction

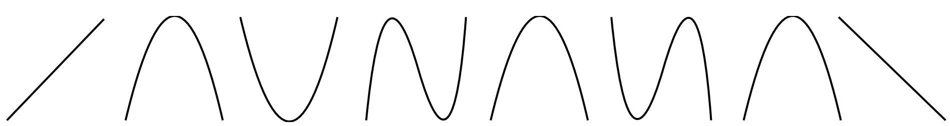
Polynomial functions can be used to describe many real-world applications, including the pathway of a football tossed into the end zone during a Grey Cup game, the revenue generated from the drama club's ticket sales as the price increases, and the relationship between the surface area and volume of an open box.

Polynomial Functions

A rollercoaster at an amusement park has the following track configuration as part of the ride.



Components of the track can be described using polynomial functions such as the following:



Can you identify these graphs as approximating linear, quadratic, or cubic functions?

In previous math courses, you studied linear functions (Grade 10) and quadratic functions (Grade 11). In this lesson, you will extend your knowledge of polynomial functions to include cubic functions.

Recall that a **polynomial function** in one variable is made up of terms consisting of a variable with whole number exponents and real number coefficients, separated by addition or subtraction signs. Terms are written in descending order of power and the polynomial is named by its degree. The degree of a polynomial function is the greatest exponent of the function.

Degree	Name	Example
0	Constant	$f(x) = 2$
1	Linear	$f(x) = x + 2$
2	Quadratic	$f(x) = x^2 - 5x + 6$
3	Cubic	$f(x) = x^3 - 4x^2 + 2x - 6$
4	Quartic	$f(x) = -3x^4 + x$
5	Quintic	$f(x) = 5x^5 + 4x^3 - 6$

Polynomials with a degree higher than 5 are named by their degree. For example, $f(x) = 2x^8 + 6x^7 - x^5 + 3x^4 + 5x^2 - 19$ would be called an eighth degree polynomial. This course will consider polynomial functions with degree of 3 or less.

Technology Requirements

You must have access to graphing technology for this course. There are many possible and appropriate types of software and hardware available, such as graphing calculators, mobile apps, websites, or free graphing programs. Some programs and/or apps can be found by searching or using the urls shown below:

- *WinPlot*
  http://faculty.madisoncollege.edu/alehnen/winptut/Install_Winplot.html
- *Geogebra*
  www.geogebra.org/cms/
- *Graphmatica*
  http://download.cnet.com/Graphmatica/3000-2053_4-10031384.html
- *Meta-Calculator*
  www.meta-calculator.com/online/
  (mobile version is available on the App Store)
- *Desmos*
  https://www.desmos.com/calculator

The technology you choose to use for this module must have the capability to plot data, display graphs, determine the equation of the line or curve of best fit that best describes it, and analyze the data. You may need to try different applications as you complete the assignments in this course. It is your responsibility to learn how to use your chosen technology in order to fulfill the requirements of this course. Read the manual or help file that comes with it or find online tutorials to help you learn how to input equations, adjust the window settings, and view and analyze graphs. When submitting your work for marking, you may need to record your calculator keystrokes, print the screen shots or the graphs you create, or make sketches of the final image in your notes.

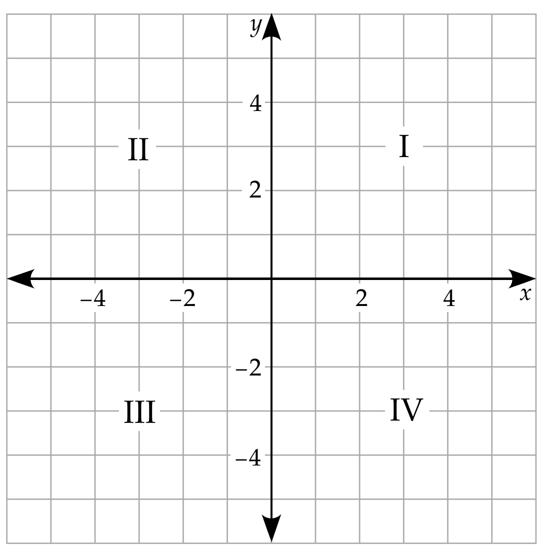
You are required to use technology to complete some parts of this module. Although you may use a variety of apps as you work on the content of this module, you will be asked when submitting assignments and on the day of the midterm examination to specify ***one*** graphing app or graphing technology that you are using for your assessment of the course.

Please see the Technology Appendix for further information on how to use certain technology applications to fulfill the requirements of this course.

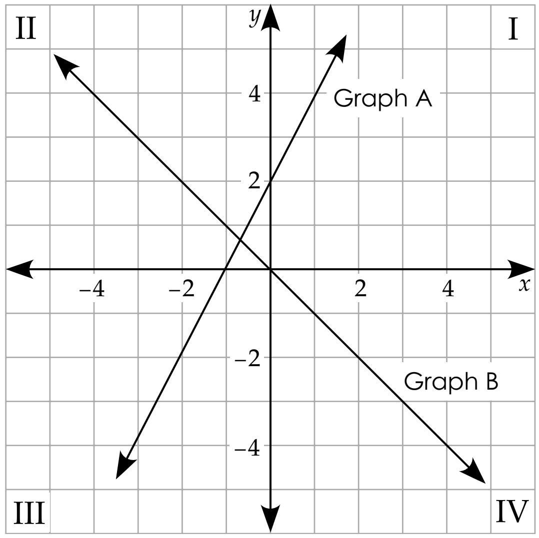
The Graphs of Polynomial Functions

In order to identify and describe the characteristics of polynomial functions, it is helpful to graph them with a continuous curve on a Cartesian grid.

Recall that a Cartesian grid is divided by the x- and y-axes into four quadrants. They are named using roman numerals starting from the top right quadrant and progressing counter-clockwise.



Linear Functions



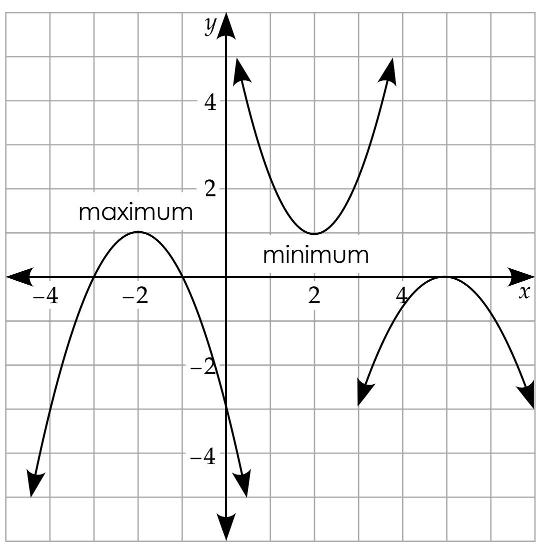
The graph of a linear function is a straight line.

Unless the line is horizontal or vertical, each line has one x-intercept (where $y = 0$) and one y-intercept (where $x = 0$).

The **domain** of each function is $\{x \mid x \in \Re\}$ and the **range** is $\{y \mid y \in \Re\}$.

The **end behaviour** of a function is a description of the shape of the graph, read from left to right, as the value of x continues from negative infinity to positive infinity. The line of Graph A is always increasing and extends from Quadrant III to Quadrant I. The line of Graph B is always decreasing and extends from Quadrant II to Quadrant IV.

Quadratic Functions



The graph of a quadratic function is a parabola. The line may have two, one, or zero x-intercepts. Recall that x-intercepts may also be called **zeros** and they are the roots of a related equation. The line will have one y-intercept.

The end behaviour of the line either extends from Quadrant II to Quadrant I, or from Quadrant III to Quadrant IV, depending on whether the graph opens up (cup shape) or opens down (hill shape).

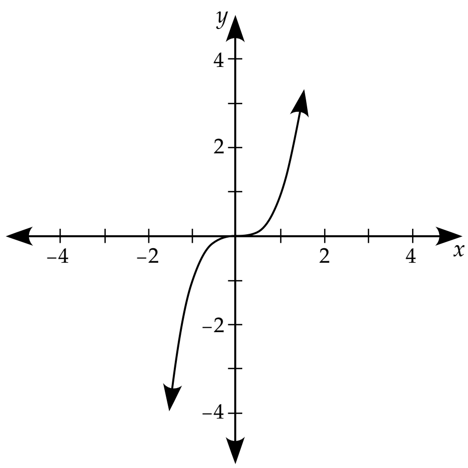
Quadratic functions have either a maximum or a minimum y-value. At this turning point or **vertex**, the line has a smooth curve as the function changes from increasing to decreasing (maximum) or from decreasing to increasing (minimum).

The domain of each function is $\{x \mid x \in \Re\}$ and the range is either $\{y \mid y \leq \text{maximum}, y \in \Re\}$ or $\{y \mid y \geq \text{minimum}, y \in \Re\}$.

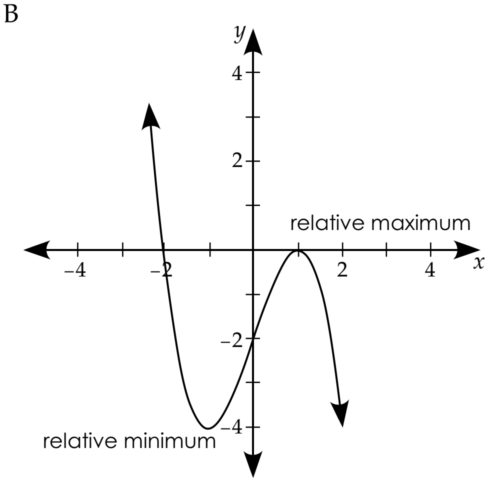
This information may be useful to add to your resource sheet for future reference.

Cubic Functions

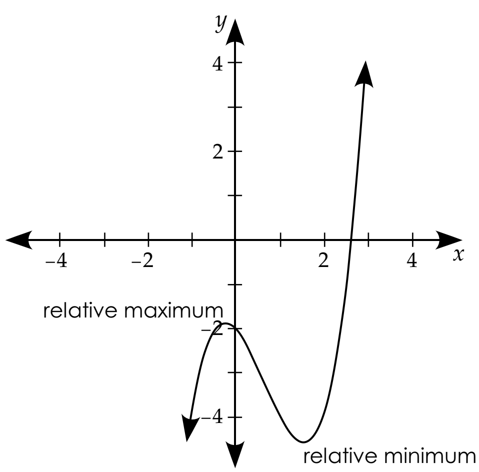
A



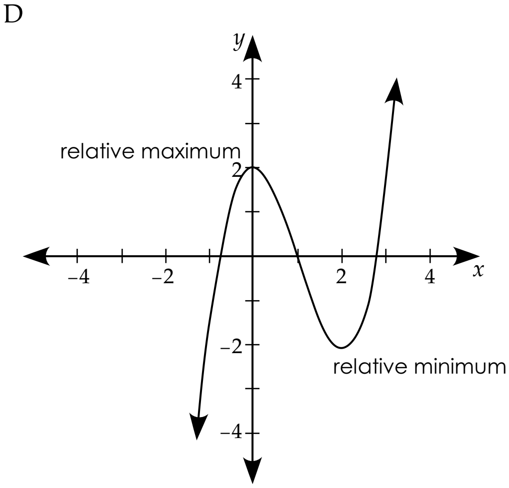
B



C



D



The shape of a cubic function graph is a smooth line with either zero or two turning points (where the function changes between increasing and decreasing). The graph may have one, two, or three x-intercepts and will have one y-intercept.

The end behaviour of the curve is either to extend from Quadrant III to Quadrant I or from Quadrant II to Quadrant IV.

There are no absolute maximum or minimum values in the curve as the line extends to both positive and negative infinity. The domain of each function is $\{x \mid x \in \Re\}$ and the range is $\{y \mid y \in \Re\}$. In a given interval, however, the line may have a peak or valley. These smooth curved turning points are called either a **relative maximum** or a **relative minimum** (that is, maximum or minimum relative to the points on the function in the immediate vicinity).

The first cubic function, graph A above, does not have any turning points. Instead, it flattens out and crosses the x-axis without switching between increasing and decreasing. Graphs B, C, and D have two turning points, so they have both a relative maximum and a relative minimum.

Analyzing the Equations of Polynomial Functions

You have seen how the degree of a polynomial function determines the shape of the graph. The characteristics of each graph are connected to the coefficients and the constant in the equation for each function.

The equations of polynomial functions can be written in general form as follows:

Linear: $f(x) = ax + b$, where $a \neq 0$

Quadratic: $f(x) = ax^2 + bx + c$, where $a \neq 0$

Cubic: $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$

The terms are written in descending order of powers. The coefficient of the highest power, a, is called the leading coefficient.

Impact of the Constant and Coefficients in Polynomial Functions

Explore the impact of changing the value of the constant and the sign of the leading coefficient in the equations of polynomial functions.

Example 1

What happens when the value of the constant is changed? Using your choice of technology, graph each set of polynomial functions on a grid, and compare the lines and equations.

Set 1:

$f(x) = x + 0$ or $f(x) = x$

$f(x) = x + 5$

$f(x) = x - 5$

Set 2:

$f(x) = x^2 + 6x$

$f(x) = x^2 + 6x - 5$

$f(x) = x^2 + 6x + 5$

$f(x) = x^2 + 6x + 10$

Set 3:

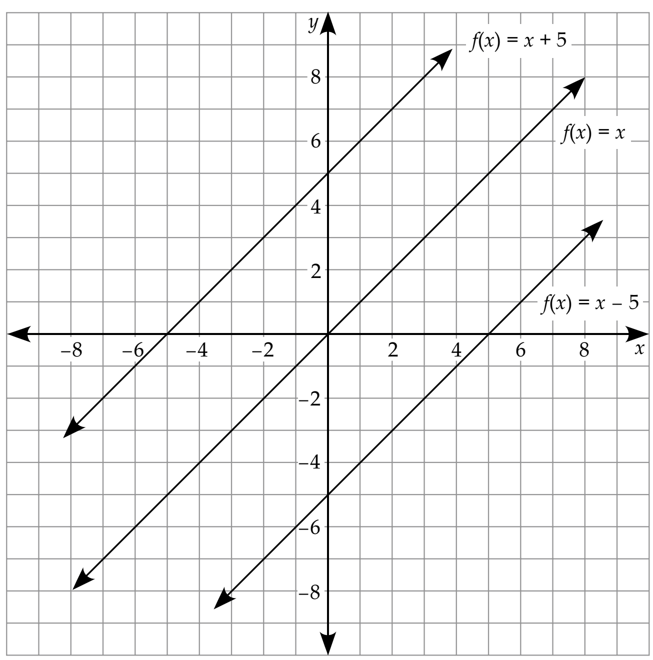
$f(x) = x^3 + 5x^2 + 4x$

$f(x) = x^3 + 5x^2 + 4x + 5$

$f(x) = x^3 + 5x^2 + 4x - 5$

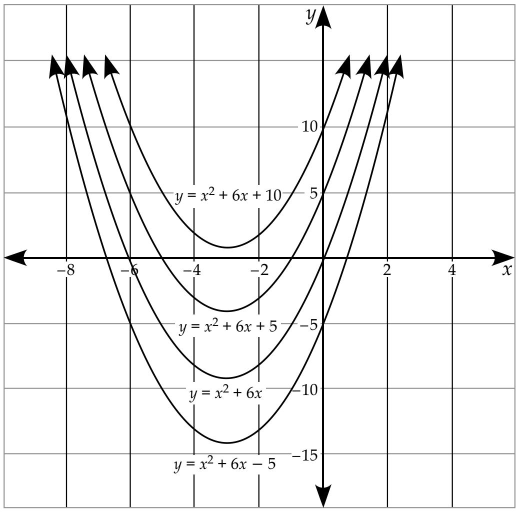
Solution

Set 1:



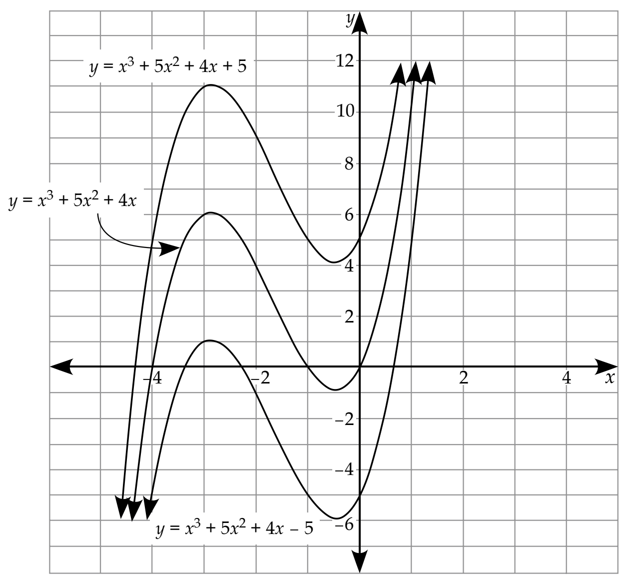
The lines are parallel.

Set 2:



The parabolas are shifted vertically but they are the same shape.

Set 3:



The shape of the curve remains the same, but the place where it crosses the y-axis shifts up if the constant is positive, or down if the constant is negative.

In all three sets, when the value of the constant changed, the y-intercept of the graph changed. The value of the constant in the equation is the same as the y-intercept. Changing the constant shifts the position of the graph vertically but does not affect its shape.

This information may be useful to add to your resource sheet for future reference.

Example 2

What happens when the sign of the leading coefficient is changed? Graph each set of polynomial functions on a grid and compare the lines and equations.

Set 1:

$f(x) = x$

$f(x) = -x$

Set 2:

$f(x) = x^2$

$f(x) = -x^2$

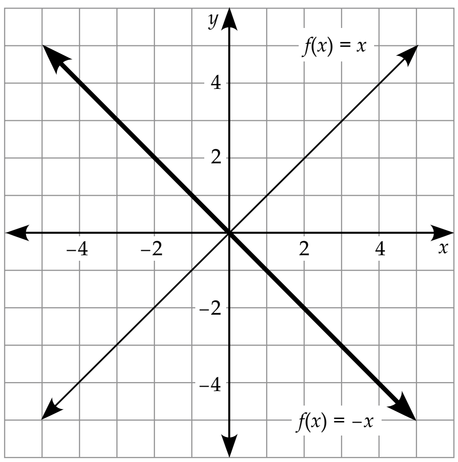
Set 3:

$f(x) = x^3$

$f(x) = -x^3$

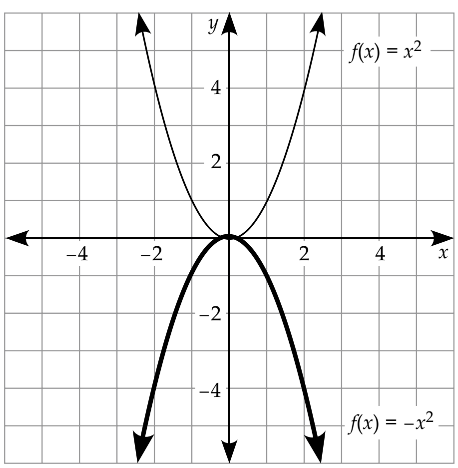
Solution

Set 1:



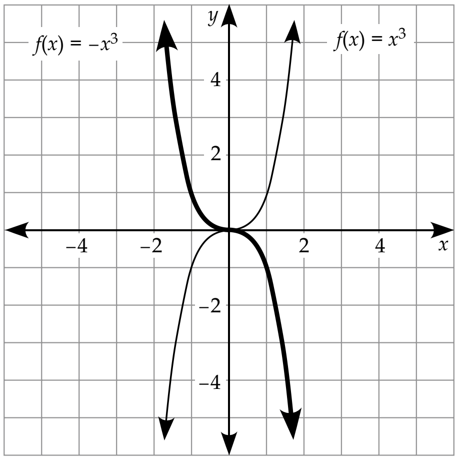
The slope of the line changes from positive to negative.

Set 2:



The direction of opening switches from concave up (cup or valley shaped) to concave down (hill shaped).

Set 3:



Instead of rising to the right, the line falls to the right. The end behaviour is opposite.

In all three sets of graphs, when the sign of the leading coefficient changed, the end behaviour of the line was altered. If the line extended from Quadrant III to Quadrant I, it switched to Quadrant II to Quadrant IV. If the curve extended from Quadrant II to Quadrant I, it switched to Quadrant III to Quadrant IV. When the sign of the leading coefficient changes, the graph is reflected in the x-axis.

This information may be useful to add to your resource sheet for future reference.

Using the Equation to Identify Characteristics of a Polynomial Function

The characteristics of a polynomial function can be described without graphing it by considering the values of the constant and coefficients in the equation.

Example 1

Without graphing the equation, determine the characteristics of the following functions.

a) $f(x) = -3x^2 + 14x + 8$

b) $g(x) = x^3 - 2x^2 - x - 2$

c) $h(x) = (x-1)(x+1)(x-2)$

Solution

a) $f(x) = -3x^2 + 14x + 8$

This is a quadratic polynomial of degree 2, so it will have one turning point. Since the leading coefficient is negative, the parabola will open down. It will have an absolute maximum. The end behaviour of the curve will extend from Quadrant III to Quadrant IV. The y-intercept is at 8 (let $x = 0$). Since this is a positive value, the vertex must be above the x-axis. Therefore, the line will cross the x-axis twice. The domain of the function is $\{x \mid x \in \Re\}$ and the range is $\{y \mid y \leq \text{maximum}, y \in \Re\}$.

b) $g(x) = x^3 - 2x^2 - x - 2$

This is a third degree or cubic function. Just from looking at the general form, it is hard to determine the exact shape. It could have zero or two turning points and one, two, or three x-intercepts. It has a y-intercept at -2, the value of the constant (let $x = 0$). It has a positive leading term so the end behaviour of the line will extend from Quadrant III to Quadrant I. The domain of the function is $\{x \mid x \in \Re\}$ and the range is $\{y \mid y \in \Re\}$, so there is no absolute maximum or minimum, but there would be a relative maximum and minimum if the line has turning points.

c) $h(x) = (x - 1)(x + 1)(x - 2)$

This function is not written in general form, but even in its factored form, it still represents a cubic function of degree 3. If the distributive property were applied and the terms written in descending order of power, it would have a positive leading coefficient (coefficient of each x is +1), so the end behaviour would be from Quadrant III to Quadrant I. It has a y-intercept of +2 (let $x = 0$ and multiply values from each bracket). From the factored form, you can identify the three x-intercepts or zeros of the function as −1, +1, and +2. It must, therefore, have two turning points, a relative maximum, and a relative minimum.

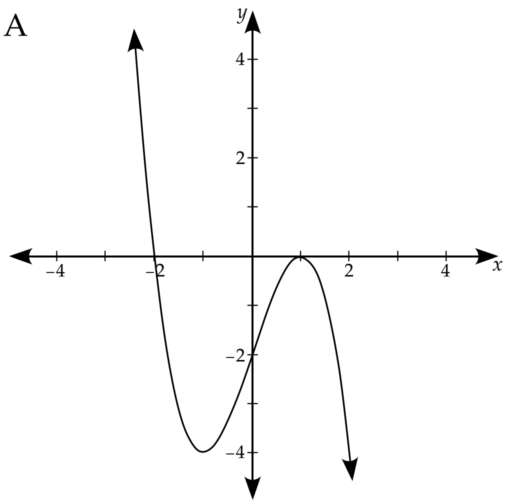
Matching Polynomial Graphs and Equations

Using what you know about polynomial equations and the graphs of the functions, you can match the graphs and equations.

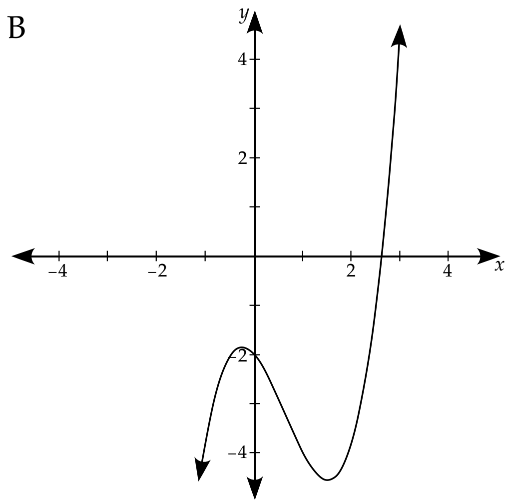
Example 1

Consider the characteristics of the following equations and graphs and match the ones that represent the same function.

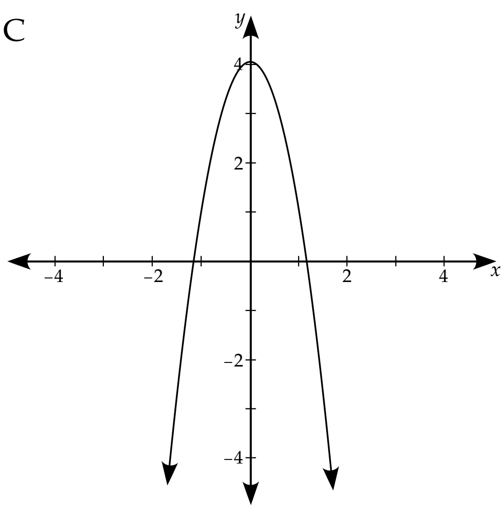
A



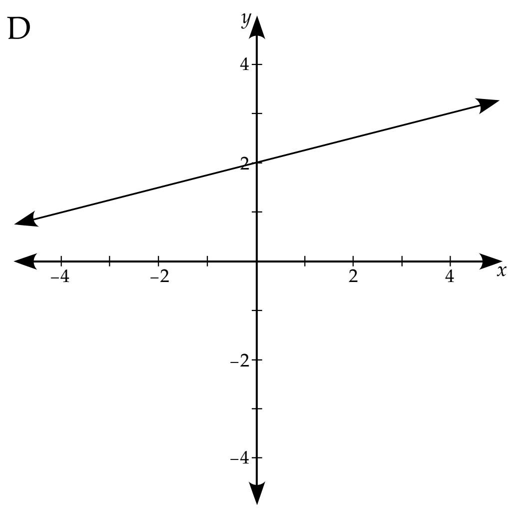
B



C



D



$b(x) = 3x^3 - 3x + 4$

$h(x) = -x^3 + 3x - 2$

$k(x) = \frac{1}{4}x + 2$

$t(x) = -3x^2 + 4$

$m(x) = x^3 - 2x^2 - x - 2$

Solution

Graph A is a cubic function so it must be degree 3. The end behaviour indicates that it has a negative leading coefficient, because it extends from Quadrant II to Quadrant IV. The y-intercept is -2, so the constant must be -2. This graph represents the function $h(x) = -x^3 + 3x - 2$.

Graph B is also a cubic function with a y-intercept of –2, but the leading coefficient must be positive as the line extends from Quadrant III to Quadrant I. This graph represents the function $m(x) = x^3 - 2x^2 - x - 2$.

Graph C is a parabola, so the equation must be degree 2. It opens down, so the leading coefficient must be negative. The y-intercept is 4, so the constant must be 4. This graph represents the function $t(x) = -3x^2 + 4$.

Graph D is linear, a first degree polynomial. The slope and leading coefficient must be positive, and the constant must be 2. This graph represents the function $k(x) = \frac{1}{4}x + 2$.

Example 2

Sketch a graph that matches each of the following sets of characteristics and justify your reasoning.

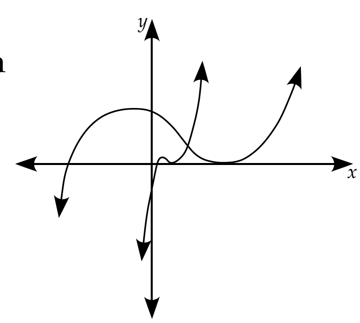
a) Extends from Quadrant III to Quadrant I, two x-intercepts

b) Two turning points, y-intercept at -5, relative minimum in Quadrant III.

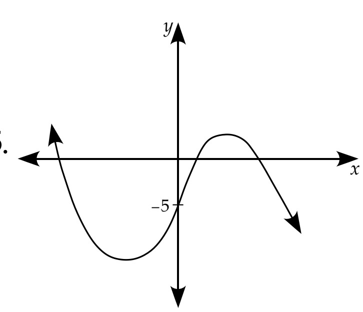
Solution

There are a variety of graphs that would satisfy these conditions.

a) A curve with two x-intercepts could be quadratic or cubic, but for the end behaviour to extend from Quadrant III to Quadrant I, it must be a cubic polynomial with a positive leading coefficient. The curve must cross the x-axis once and be tangent (bounce) once. The y-intercept may be positive or negative. Two possible curves with these characteristics are shown on the left.



b) Two turning points implies a cubic function. A relative minimum in Quadrant III means the end behaviour must extend from Quadrant II to Quadrant IV. The line must cross the y-axis at -5. A possible graph may look similar to the one on the left.



Learning Activity 1.1

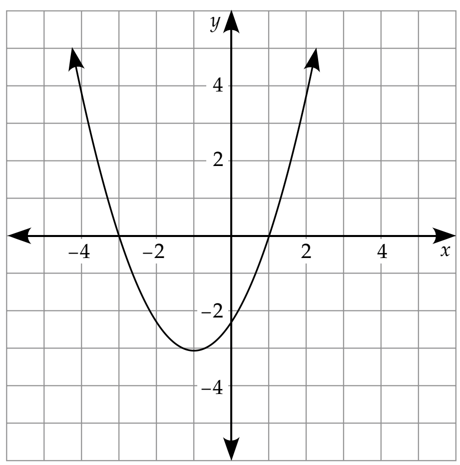
Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Factor: $x^2 - 7x + 10$
2. State the roots of the following quadratic function: $f(x) = x^2 + 5x + 6$
3. The vertex of a quadratic function is at (−5, 31). State the equation of the axis of symmetry.

Complete the following, based on the graph below.



4. State the domain.
5. State the range.
6. State the coordinates of the vertex.
7. State the value of the x-intercepts.
8. State the value of x at the y-intercept.

continued

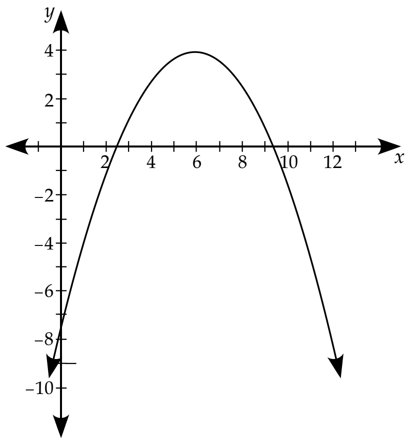
Learning Activity 1.1 (continued)

Part B: Characteristics of Polynomial Functions

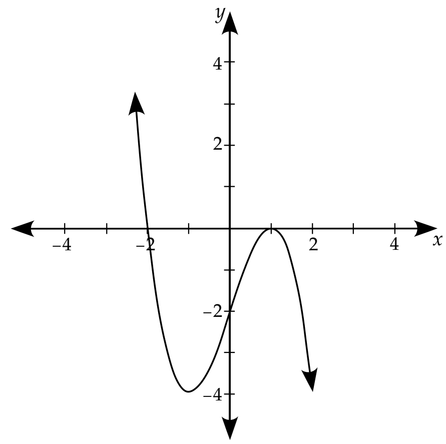
Remember, these questions are similar to the ones that will be on your assignments and midterm examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and midterm examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Given the following graphs of polynomial functions, complete the chart with the required information.

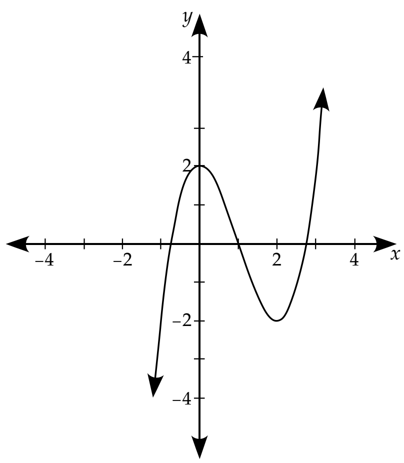
a)



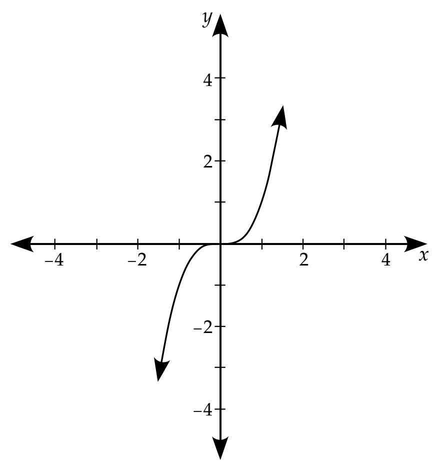
b)



c)



d)



continued

Learning Activity 1.1 (continued)

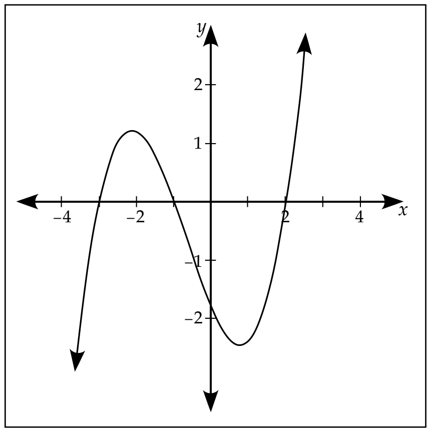
Graph	A	B	C	D
Type of Function				
Degree				
# of x-intercept(s)				
# of y-intercept(s)				
End behaviour				
Absolute or relative maximum or minimum				
Domain				
Range				

2. What is the connection between the degree of a function and the maximum possible number of x-intercepts?
3. What is the connection between the degree of a function and the maximum possible number of turning points?
4. Will a polynomial function always have a y-intercept?
5. How is the end behaviour of a function related to it having an odd or even degree?
6. Use a sketch to show how changing the value of the constant in a cubic equation can affect the number of x-intercepts it has.

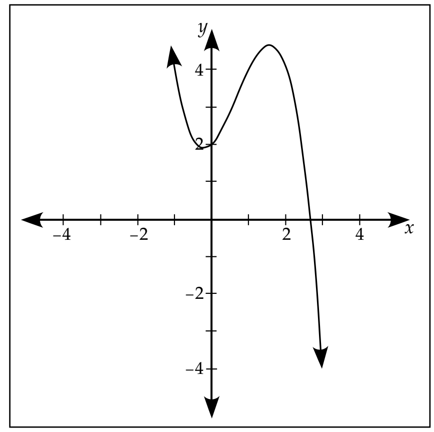
continued

7. Match each of the following graphs with the equation that describes it.

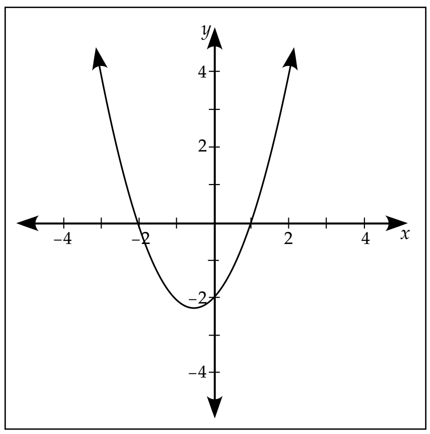
a)



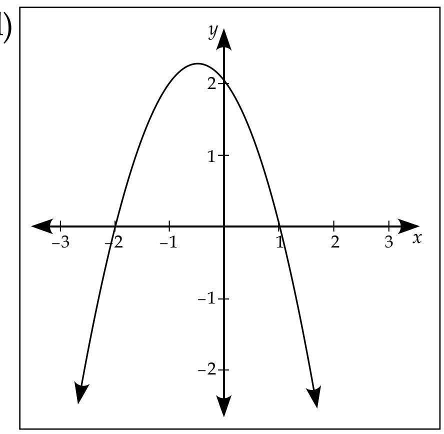
b)



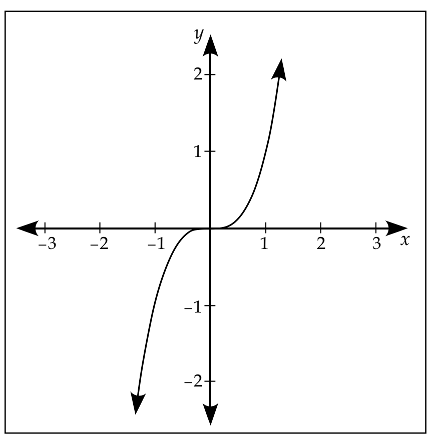
c)



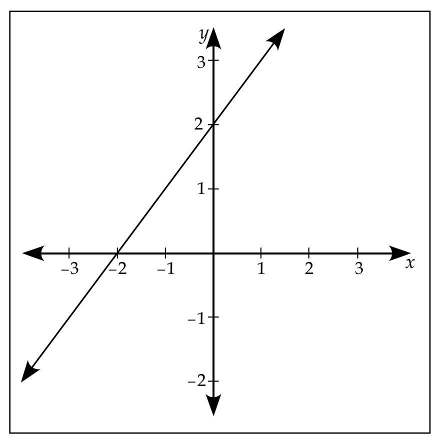
d)



e)



f)



____ $f(x) = x^3$

____ $g(x) = x^2 + x - 2$

____ $h(x) = -x^3 + 2x^2 + x + 2$

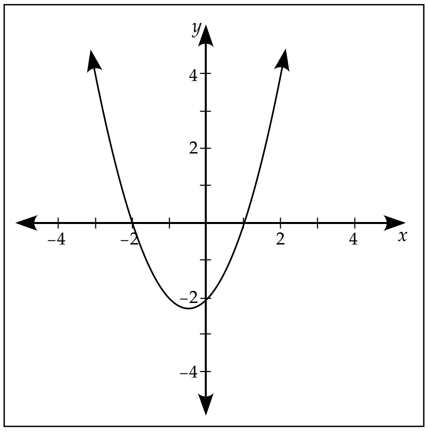
____ $m(x) = 0.3\left(x^3 + 2x^2 - 5x - 6\right)$

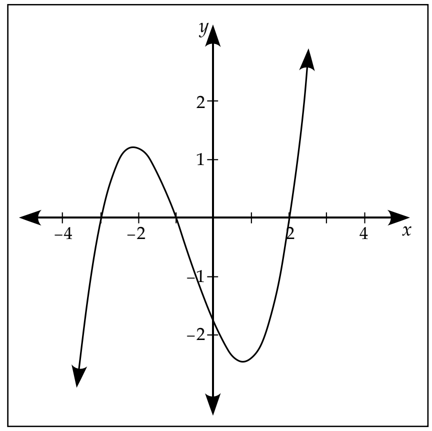
____ $t(x) = x + 2$

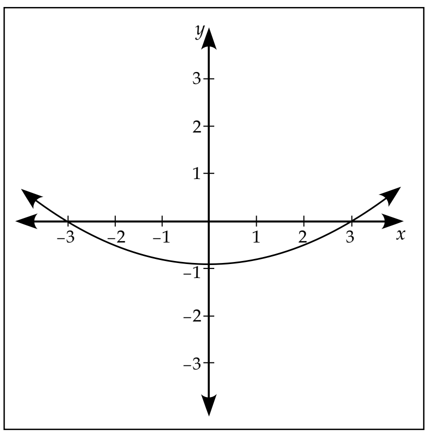
____ $w(x) = -x^2 - x + 2$

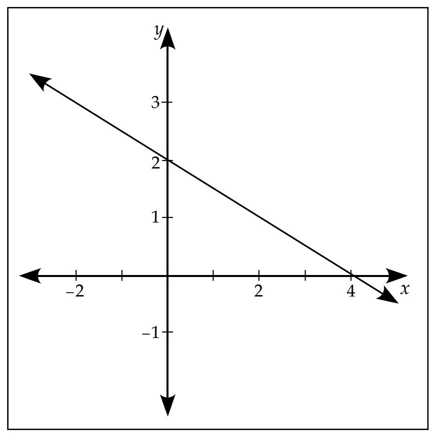
continued

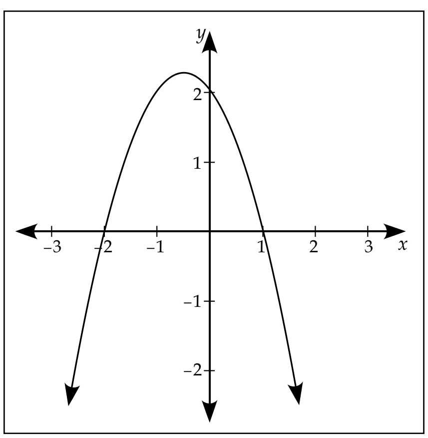
8. Match a polynomial function description to one of the numbered graphs. There are some descriptions that will match with more than one graph, but in the end each graph should be matched to a different description.

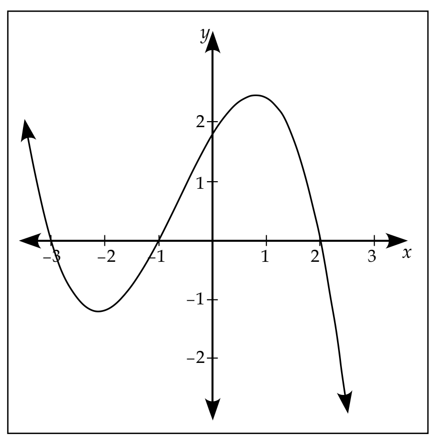
a) 

b) 

c) 

d) 

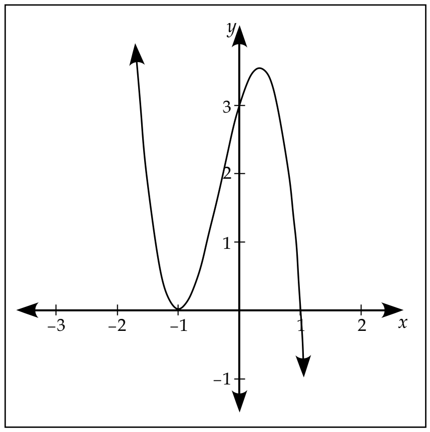
e) 

f) 

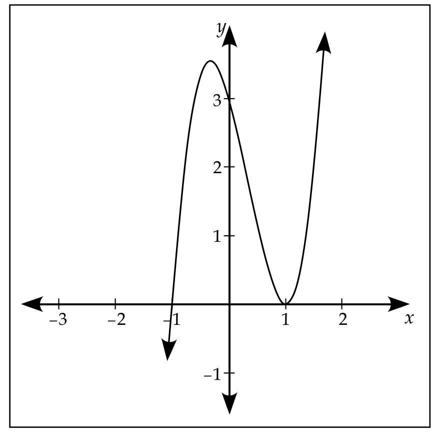
continued

Learning Activity 1.1 (continued)

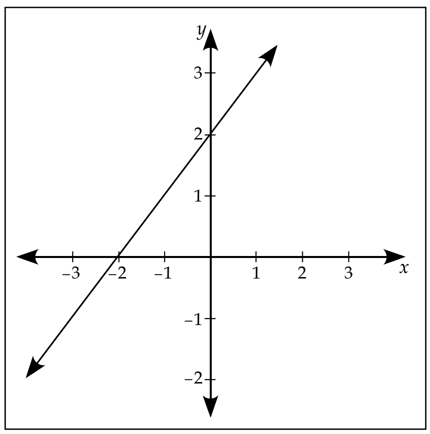
g)



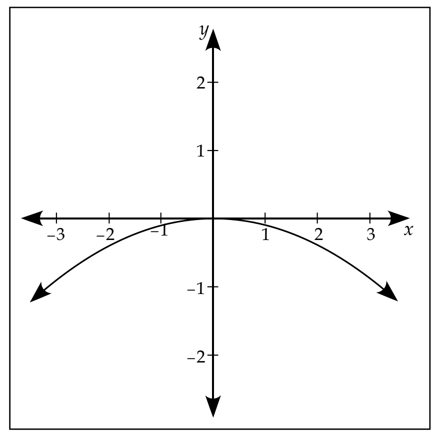
h)



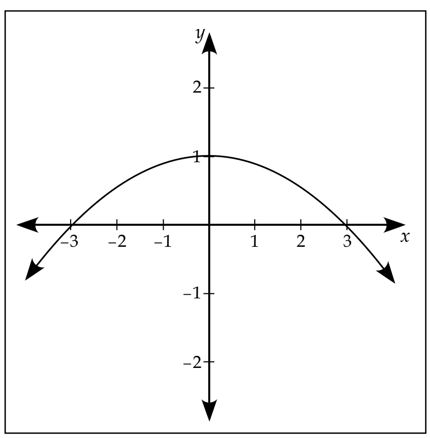
i)



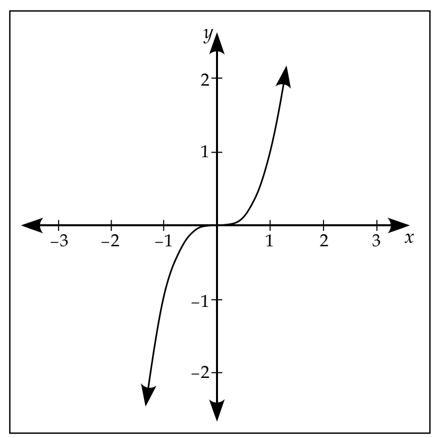
j)



k)



l)



continued

_____ i) This is the graph of a cubic function. The leading coefficient is negative.

_____ ii) This polynomial of degree 2 has two x-intercepts and its axis of symmetry is the y-axis.

_____ iii) This is the graph of a cubic function with x-intercepts at -3, -1, and 2.

_____ iv) This function has roots at 1 and -1.

_____ v) This is the graph of a quadratic function with a minimum function value.

_____ vi) This quadratic function has a maximum of zero.

_____ vii) This quadratic function has a positive leading coefficient much less than one.

_____ viii) This linear function has a positive slope.

_____ ix) The graph of this function has a constant rate of change that is between 0 and -1.

_____ x) The graph of this cubic function has two x-intercepts and a negative leading coefficient.

_____ xi) This is the graph of a parabola with a y-intercept of 2.

_____ xii) This function has no turning points and one x-intercept.

9. Sketch the graph of a possible polynomial function for each of the following sets of characteristics. What can you conclude about the equation of the function with these characteristics?

   a) A relative maximum in Quadrant II, a relative minimum in Quadrant I, a y-intercept of 3.

   b) Range $\{y \mid y \leq -4, y \in \Re\}$, one turning point in Quadrant III.

Lesson Summary

In this lesson, you reviewed, analyzed, and described the characteristics of different types of polynomial graphs and equations. You considered the degree, domain and range, maximum and minimum values, x- and y-intercepts, turning points and end behaviour of graphs, and the constants and leading coefficients in equations. You sketched polynomial graphs and matched them to equations.

Notes

Lesson 2: Applications of Polynomial Functions

Lesson Focus

In this lesson, you will

- ❑ graph data and determine the polynomial function that best approximates the data
- ❑ interpret the graph of a polynomial function that models a situation and explain the reasoning
- ❑ use technology to solve contextual problems that involve data that is represented by graphs of polynomial functions, and explain the reasoning

Lesson Introduction

Trends in real-world data can be modelled using polynomial functions. When data is collected and plotted on a scatterplot graph, you can visualize the relationship between the two variables and describe it using a line or curve of best fit. Using technology, it is usually quite simple to find the equation for the curve that best describes the data and then use it to analyze the data and estimate values or interpolate or extrapolate solutions.

You will now explore how polynomial functions can be used to model data for familiar events, such as throwing a water balloon or calculating the effects of wind chill.

Using Polynomial Functions to Model Data

Equations That Model Data—Projectiles

The height of a projectile as it moves through the air is determined by several things:

- the projectile's vertical velocity, v, in metres per second, at which it begins moving
- the projectile's initial height, s, in metres
- gravity (acceleration due to gravity on Earth is assumed to be a constant, $-9.8\ \text{m/s}^2$)

Its height, h, in metres above the ground can be determined at a given time, t, in seconds using the formula $h = -4.9t^2 + vt + s$.

Example 1

A water balloon is launched from a catapult starting at a height of 1.6 m above the ground with an initial vertical velocity of 17 metres per second.

a) Write the equation that models the height of the water balloon as a function of time, and graph it using technology.

b) What is the maximum height the water balloon reaches?

c) How long is the balloon in the air?

d) When is the water balloon at a height of 10 m?

e) How high is the balloon after 3 seconds?

f) What does the y-intercept of the graph represent?

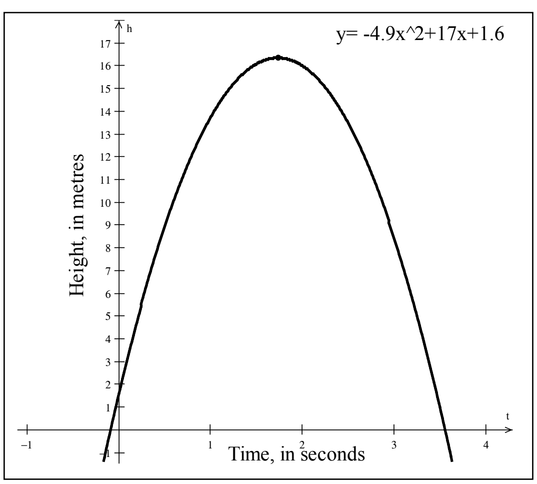
g) State the domain and range for this situation.

Solution

Refer to the Technology Appendix for help in learning how to use certain technology applications to answer the given questions. The appendix provides basic information, some keystrokes or input requirements, and examples of how to graph equations, how to find the coordinates of the vertex, how to find the x-intercepts, how to find the y-intercept, and how to solve for points on the line, as well as intersection points when graphing more than one equation, using a variety of applications. Examples in this lesson show solutions created using *Winplot*. You will get the same answers if you use a different program such as *GeoGeobra* or a TI-83 Plus graphing calculator.

a) Write the equation that models the height of the water balloon as a function of time, and graph it using technology.

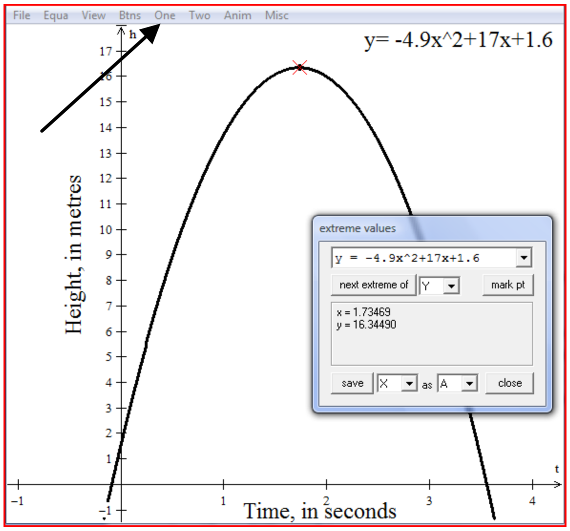
Using initial vertical velocity, $v = 17$, and initial height, $s = 1.6$, the equation that models this data is $h = -4.9t^2 + 17t + 1.6$.



Your graph may appear different from the sample solution provided. Please try to adjust your graph window so it appears similar to the answer provided. Refer to the Technology Appendix for help. In this example, *Winplot* was used to create the graphs.

b) What is the maximum height the water balloon reaches?

To find the maximum height, you need to determine the coordinates of the vertex.

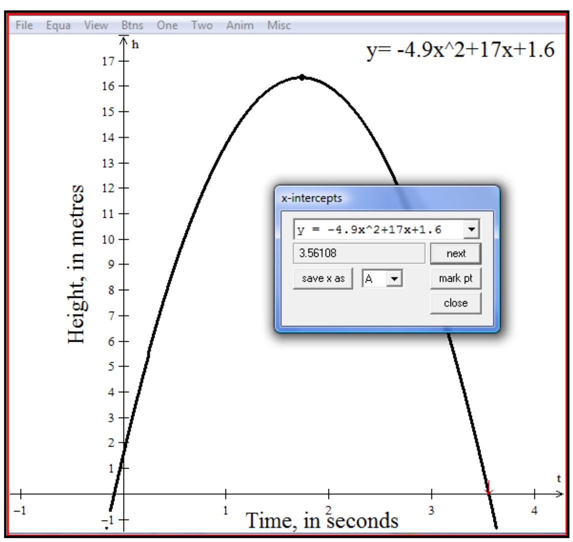


The vertex in the pathway of the water balloon is at approximately (1.7, 16.3). This means the balloon reaches a maximum height of 16.3 m at a time of 1.7 seconds after launch.

c) How long is the balloon in the air?

Ground level is represented by the x-axis, where $h = 0$. Find the x-intercept to determine the time the balloon lands.

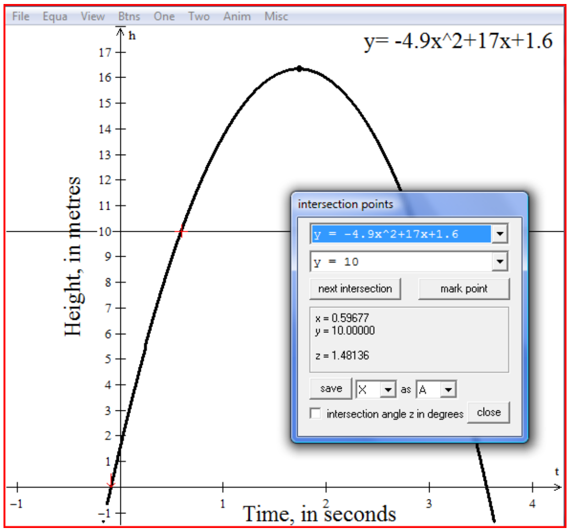
The first intercept is at −0.09169 but this is an irrelevant point in this situation. Negative time has no meaning in this circumstance. The value of the second intercept is given as 3.56108.



The balloon lands about 3.56 seconds after launch.

d) When is the water balloon at a height of 10 m?

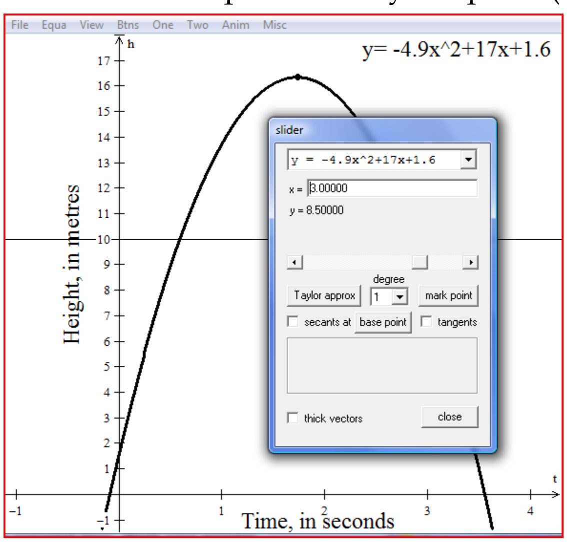
To locate the points where the balloon is at a height of 10 m, you can graph another equation, $h = 10$, and find the points of intersection.



The lines intersect at two points. First at about 0.6 seconds and again at about 2.9 seconds after launch, the balloon is at a height of 10 m.

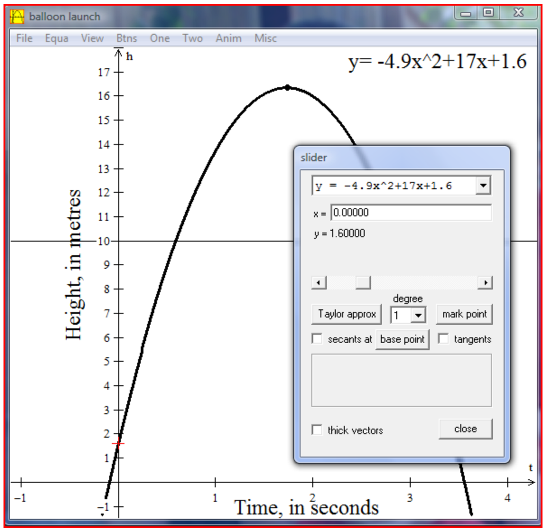
e) How high is the balloon after 3 seconds?

Find the value of h when t is 3. The height of the balloon after 3 seconds is 8.5 m. This is represented by the point (3, 8.5).



f) What does the y-intercept represent?

The y-intercept of the graph is the height of the balloon at the time of launch, when $t = 0$. The point (0, 1.6) is where the graph crosses the y-axis. The balloon is launched from a height of 1.6 m.



g) State the domain and range for this situation.

The domain for this function represents the time the object is in the air and is $\{t \mid 0 \le t \le 3.56, t \in \Re\}$. The range represents the possible height of the object and is $\{h \mid 0 \le h \le 16.3, h \in \Re\}$.

Equations That Model Data—Wind Chill

On hot and humid summer days, the humidity on your skin makes you feel even hotter than the air temperature. This is measured as the humidex. In winter, the cold wind causes the moisture on your skin to evaporate, making it feel colder than the air temperature. This is called the wind chill factor.

Example 1

When the air temperature is at −15 °C, the wind speed in km/h results in the air feeling like the wind chill temperature given in the table below.

Wind Speed (km/h)	Wind Chill (°C)
6	−19
8	−20
15	−23
40	−27
100	−32

a) Using technology, enter the data from the table and create a scatterplot. Fit the window to the data.

b) Describe the relationship and determine what type of polynomial function will best fit the trend of the data.

c) Use technology to determine the quadratic regression equation and graph it on the scatterplot.

d) How well does the line fit the trend of the data points?

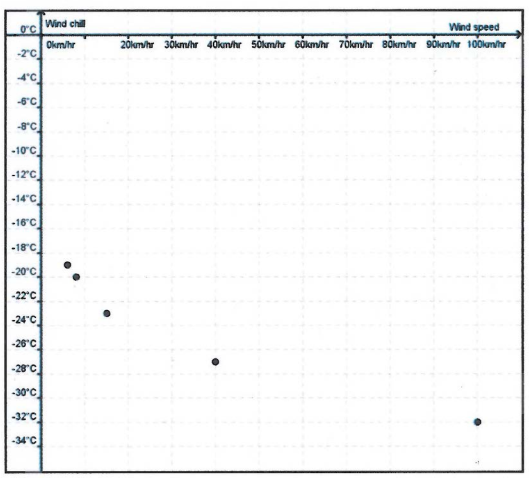
e) Frostbite is a risk when the wind chill factor is lower than −25 °C. If the temperature is −15 °C, what wind speed will make frostbite a risk?

f) When you are driving, your dog likes to hang his head out of the window of the car. If you are cruising in a residential area at 50 km/h, how cold does your dog feel on a day when it is −15 °C?

g) Find the vertex and state its significance and implications on the domain.

Solution

a) Using technology, enter the data from the table and create a scatterplot. Fit the window to the data.



This graph was created using *GeoGebra*. If you use different technology, your graph may appear slightly different. Try to adjust the settings so your graph looks similar to the one above. Refer to the Technology Appendix for help using *GeoGebra* or find alternate assistance if you choose to use other technology applications.

b) Describe the relationship and determine what type of polynomial function will best fit the trend of the data.

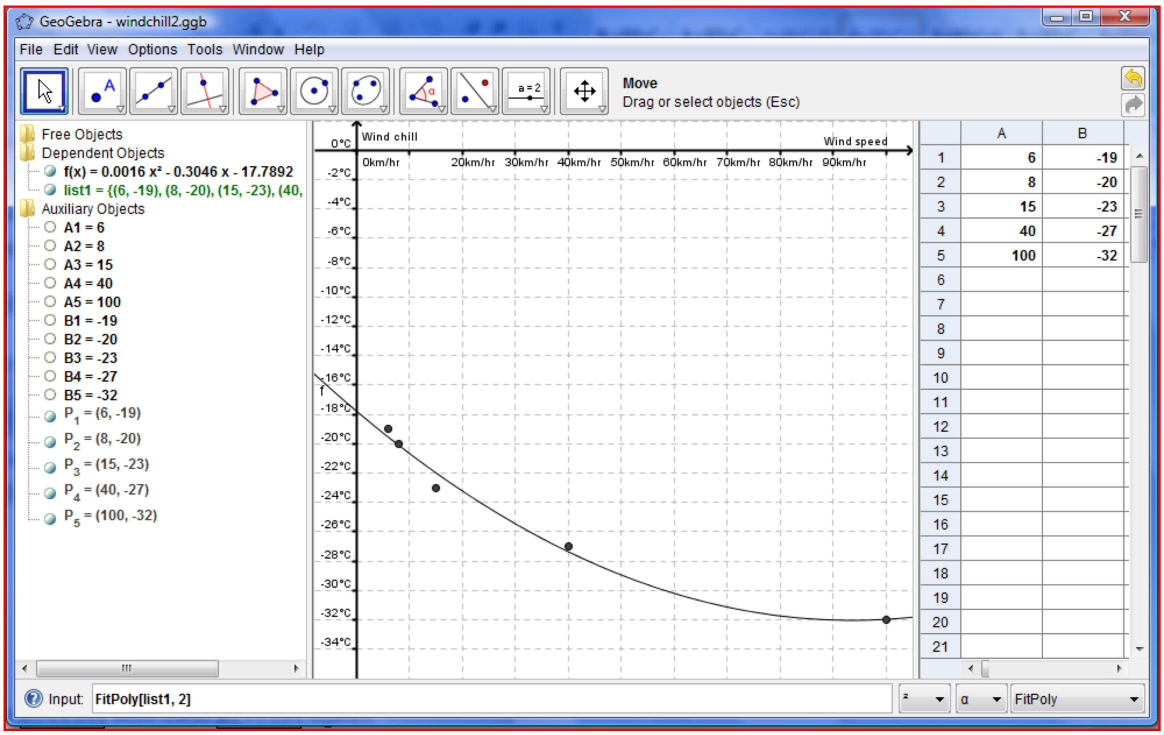
As the wind speed in km/h increases, the wind chill in °C decreases. The points do not line up in a straight line, but rather seem to curve. Part of a parabolic curve will likely follow the trend of these data points. A quadratic function would probably fit this data well.

c) Use technology to determine the quadratic regression equation and graph it on the scatterplot.

The equation of a line or curve of best fit is also called a regression equation. It models or represents the relationship between the variables and can be used to estimate values. You can use technology to find different types of regression equations (quadratic or cubic, for example). Refer to the Technology Appendix for help in determining a regression equation. Note that *Winplot* does not calculate regression equations.

The quadratic regression equation that models the data in this situation is given as $y = 0.0016x^2 - 0.3046x - 17.7892$.

On the *GeoGebra* screen shot below, it is listed on the left pane as the $f(x)$.

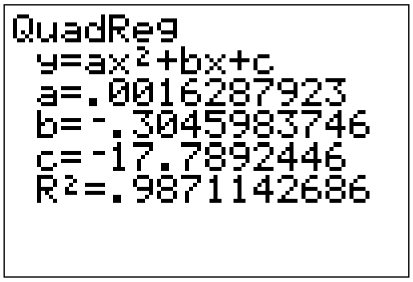


d) How well does the line fit the trend of the data points?

The line matches the trend of the points quite well. The curve does not pass directly through all of the points but is very close to a majority of the points, with some of the points above the curve and some below it.

There is a mathematical value called the **coefficient of determination** that can be used as a diagnostic value. The coefficient of determination, written as R^2, ranges from 0 to 1 and is an indicator of how well the function correlates with or fits the data. If the R^2-value is close to zero, then the data points do not fit the function curve at all. If the R^2-value is close to 1, then the data points fit the function curve very well. An R^2-value of 1 indicates that there is a perfect fit between the data and the function.

The following window shows the result, using the given data, of finding the Quadratic Regression equation with a TI-83 graphing calculator. As shown, the R^2-value for this regression is shown to be 0.987. Since the R^2-value is close to 1, the quadratic equation is a good fit for this data. If you are interested, see the Technology Appendix to learn how to turn on and use this diagnostic tool.

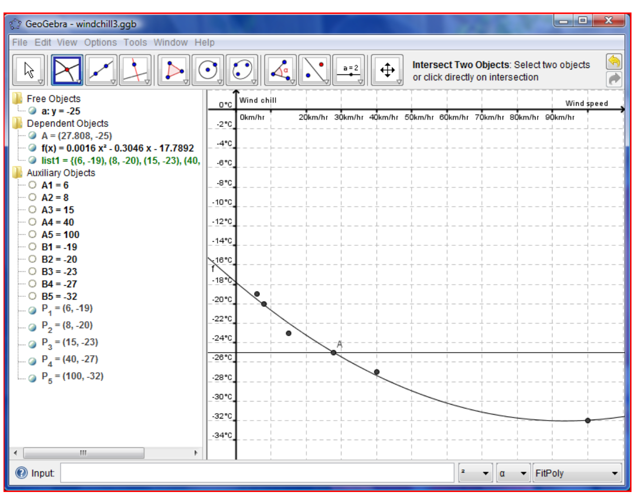


Not all technology applications provide the R^2-value. If you do more courses in statistics in the future, you will learn more about how the coefficient of determination is calculated and used for analysis. For this course, you are not required to use R^2 as part of your analysis. Instead, you can make a qualitative judgment based on a visual inspection of how well a function models the situation, or how closely the function curve follows the trend of the data.

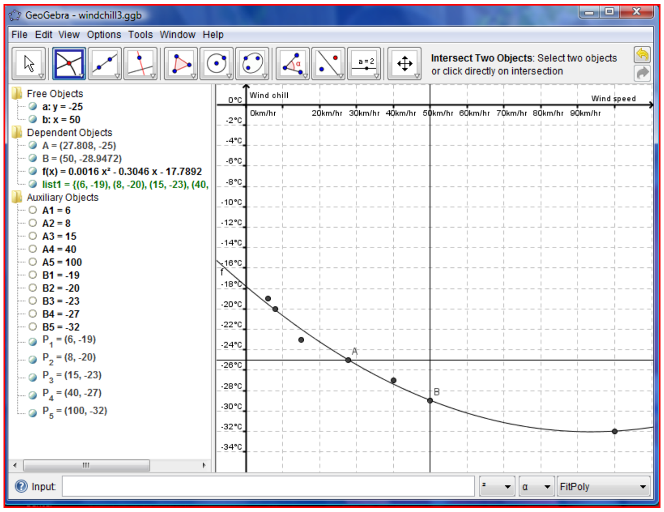
e) Frostbite is a risk when the wind chill factor is lower than −25 °C. If the temperature is −15, what wind speed will make frostbite a risk?

Enter an equation to represent a wind chill of −25 °C and find the point of intersection.

Point A on the screen shot below is the intersection of the line $y = -25$ and the regression equation. The coordinates of Point A are at (27.808, −25). At −15°C, a wind blowing faster than 27.8 km/h will create a risk of frostbite. All points on the line of the regression equation with x-values greater than 27.8 will have a y-value less than −25.



f) When you are driving, your dog likes to hang his head out of the window of the car. If you are cruising in a residential area at 50 km/h, how cold does your dog feel on a day when it is −15 °C?



Solve for the wind chill, y, when the wind speed is $x = 50$ km/h.

The point B in the screen shot above represents the intersection of the line $x = 50$ and the graph of the regression equation. The coordinates of this point are (50, −28.95).

When your dog's head is hanging out of the car window, it feels approximately like −29 °C.

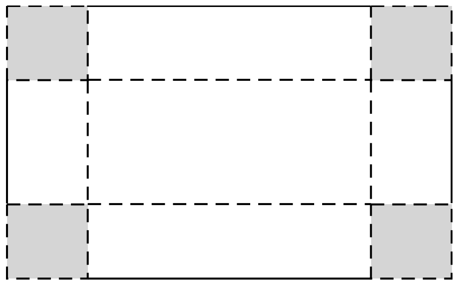
g) Find the vertex and state its significance and implications on the domain.

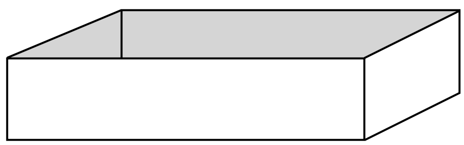
The vertex is at (93.5, −32). A wind blowing at 93.5 km/h will make it feel like −32° C. The characteristic of the quadratic line at this point implies it will now switch from decreasing to increasing, like a parabola. This does not make sense in this situation. As the wind blows faster than 93.5 km/h, the temperature will not feel warmer (unless hypothermia has set in and you really need to seek immediate medical attention). The domain of this function should be limited to values that are reasonable in this application. The effect of the wind is negligible under 5 km/h or above 80 km/h, so realistically the domain should be limited to about $\{x \mid 5 \le x \le 80, x \in \Re\}$.

When using mathematical functions to model real-life situations, you must use some common sense and critical thinking to make sure the function, domain, and range are reasonable. There will be limitations to how well a function describes or models a situation.

Equations That Model Data—Open-Top Box

Open-top boxes in a variety of sizes can be made from flat, rectangular pieces of cardboard by cutting out squares of equal size from each corner and folding up the four sides.

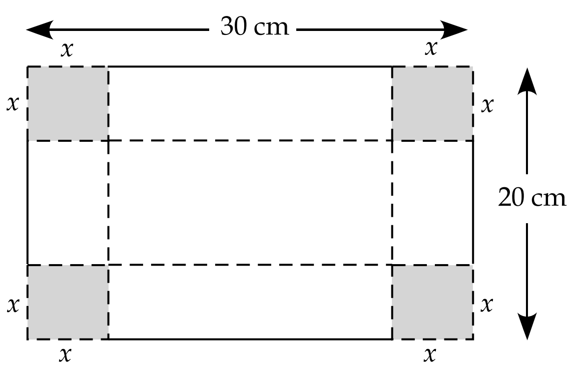




Example 1

You have a 20 cm by 30 cm piece of cardboard. What size squares should be cut out of the corners to produce a box with the maximum volume?

Solution

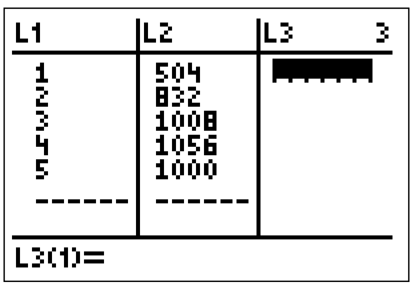


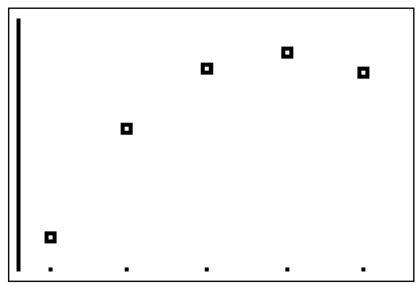
Each cut-out square has sides that are x cm long.

Create a chart with at least five different side lengths for the cut-out squares, and determine the resulting measurements and volume of the box.

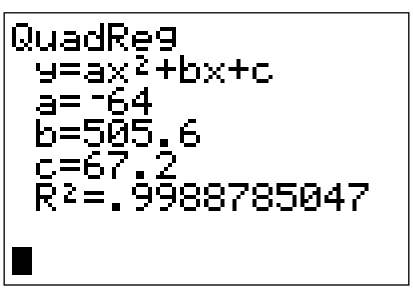
Side Length of Square (cm)	**Resulting Box Length (cm)**	**Resulting Box Width (cm)**	**Volume of Box (cm^3)**
1	28	18	504
2	26	16	832
3	24	14	1008
4	22	12	1056
5	20	10	1000

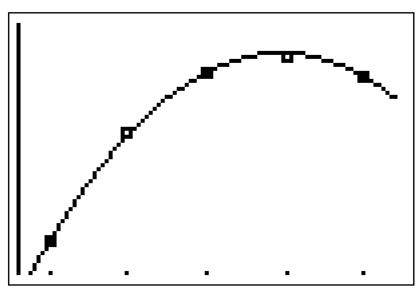
Enter the data for the side lengths of the square and the volume into your graphing program and find the regression equation of best fit. These screen shots are from a Texas Instruments graphing calculator. Your graph may appear different if you used another application. Try to adjust your window settings to make your solution look similar to the screen below.



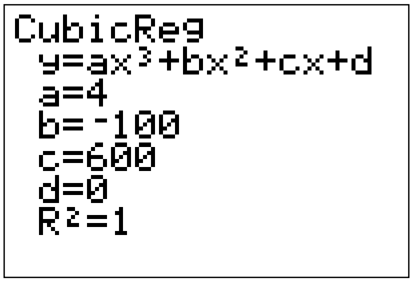


The function is a curve with a maximum value. You could model this data with a quadratic or cubic equation. A quadratic regression equation fits this data very, very closely, but not perfectly (since $R^2 = 0.998$).

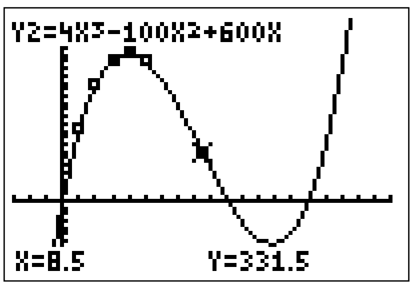




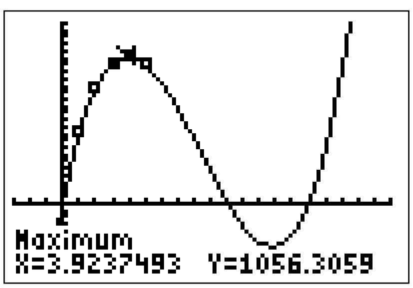
Consider that volume is a three-dimensional calculation. Compare the quadratic regression equation to a cubic regression equation.

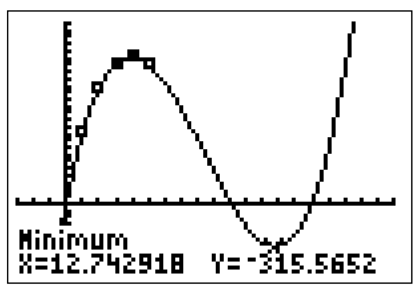


This one is a perfect fit (since $R^2 = 1$)! But even so, you must use common sense in this situation when considering what an appropriate domain and range would be. Since the cardboard is 20 cm wide, the square cut-out must be less than 10 cm (check the diagram at the beginning of this solution).



To see more of the graph, adjust your window settings and press *graph*. Notice that this graph has both a relative maximum and a relative minimum. Find these values.





If you cut out a square that has side lengths of 3.92 cm, your box will have the maximum volume of 1056.31 cm^3. The minimum point is at (12.74, −315.57). However, 12.74 cm is too large of a square to cut out from this piece of cardboard and the resulting volume (*y*-value) would be negative, which has no meaning in this context.

What is the significance of each of the *x*-intercepts?

The *x*-intercepts are found at $x = 0$, $x = 10$, and $x = 15$.

Three zeros are possible for a cubic function. If any one dimension of the box is equal to zero, the volume is zero. A dimension of $x = 0$ makes the height zero, a dimension of $x = 10$ makes the width zero, and a dimension of $x = 15$ makes the length zero. Since the shortest side of the original rectangle is 20 cm, the domain for this function model can only be from 0 to 10, since any x-values outside of this domain would result in negative values for the volume. No box can be made if the height is greater than or equal to 10.

Learning Activity 1.2

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

For questions 1 to 4, simplify the given expression using the exponent laws. Write your answers using positive exponents and the base given.

1. $(2^3)^4$
2. $3^5 \times 3^2$
3. $m^6 \div m^4$
4. $21x^{-5}$
5. Solve: 4^0
6. Solve: h^0
7. Solve for n: $(2^3)^n = 2^{18}$
8. Write as a power: $\sqrt[3]{15x}$

continued

Part B: Polynomial Regression Equations That Model Data

Remember, these questions are similar to the ones that will be on your assignments and midterm examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and midterm examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Using technology, graph each of the following functions. Use technology to find the x-intercept(s), y-intercept, and the coordinates of the relative maximum and minimum (if they exist) of the following polynomial functions. State the domain and range. Include a sketch of the graph with your solutions.
   a) $f(x) = x^3 - 9x^2 + 23x - 15$
   b) $g(x) = x^3 + 3x^2 + 3x + 1$
   c) $h(x) = \frac{1}{3}x^3 - 4x + 3$
   d) $j(x) = -2(x+3)(x-2)(x+1)$

	(a)	(b)	(c)	(d)
Graph				
Zeros				
y-intercept				
Coordinates of relative maximum/ minimum				
Domain				
Range				

continued

2. A toy rocket is fired from a launch pad at a height of 0.3 metres with an initial vertical velocity of 50 metres per second.

   Recall that the height of a projectile is modelled by the function $h = -4.9t^2 + vt + s$.

   a) Will the rocket ever reach a height of 100 metres?
   b) What is the significance of the x- and y-intercepts?

3. A competitive diver springs up from a diving board and then dives into a pool. Her coach uses rapid-fire photography and a stopwatch to record and analyze her dive. He collects the following data.

Time (seconds)	0	0.58	2.1	3.35	4.3
Height above water line (m)	1.3	1.8	0	–1.4	0

   a) Graph the data using technology.
   b) Determine which polynomial function will best model the path of her dive and find the regression equation. Determine how well the line fits the data.
   c) What is the significance of the x-intercepts?
   d) State the domain and range for this situation.

4. Dana helps her mom sell some of her delicious cinnamon buns at various bake sales. They experiment with the price of a dozen buns and come up with the following results.

Price per dozen ($)	3.50	4	4.50	6.25	7.25
Dozens sold (#)	35	33	30	22	17
Revenue ($)					

   a) Complete the chart above by calculating the revenue generated at each price.
   b) Graph the data using price as the independent variable and revenue as the dependent variable.
   c) Determine the quadratic regression equation for the curve of best fit.
   d) Describe how well the curve fits the data.
   e) Determine the price Dana and her mom should charge to maximize their revenue. How many dozen buns could they expect to sell at this price?

continued

5. From a 9" by 12" piece of metal sheeting, small squares are cut out from two of the corners so that the three edges can be bent up and fastened to manufacture a scoop (a rectangular box with no top or front). Find the length of the small squares that are cut out to give the greatest volume. Find the maximum volume.

   In your solution, include a diagram, a chart of values showing possible side lengths of the square, the resulting length and width of the box, and the resulting volume of the scoop. Also include a sketch of a scatterplot with line or curve of best fit, the regression equation calculated using technology, and a statement regarding the fit of the graph to the data.

6. A farmer has a rectangular garden that he wants to enclose with 200 m of fencing.

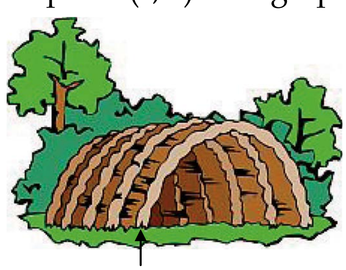
   a) Use the table below to fill in 5 possible pairs of lengths and widths and their corresponding areas.

Length (m)					
Width (m)					
Area (m^2)					

   b) Find the most appropriate regression equation to model the length of the garden and the area enclosed.

   c) What dimensions maximize the area?

7. The entranceway to a tunnel is shown below. The far left of the tunnel opening, at ground level, is said to be the origin. This would be represented by the point (0, 0) on a graph.



   The height of the tunnel at the opening is measured at three places:

   - 3 feet to the right of the origin the height of the tunnel is 4 feet
   - 4.5 feet to the right of the origin the height of the tunnel is 5.5 feet
   - 7 feet to the right of the origin the height of the tunnel is 7.5 feet

continued

a) Find the quadratic regression equation that models the height of the tunnel, y, compared to the distance to the right of the origin, x.

b) How wide is the tunnel at ground level?

c) What is the maximum height of the tunnel?

d) An object that is 8 feet wide and 8 feet tall is to be pulled through the tunnel. Will it fit? Justify your answer.

8. The following table shows the average retail price of gasoline, per litre, for a selection of years in a 30-year period beginning in 1979 (year 0).

Years after 1979	Price of Gas (¢/L)	Years after 1979	Price of Gas (¢/L)
0	21.98	17	58.52
1	26.18	20	59.43
2	35.63	22	70.56
3	43.26	23	70.00
4	45.92	24	74.48
7	45.78	25	82.32
8	47.79	26	92.82
9	47.53	27	97.86
12	57.05	28	102.27
14	54.18	29	115.29

a) Use technology to graph the data as a scatterplot. What polynomial function could be used to model the data? Explain.

b) Determine the cubic regression equation for the data. Use your equation to estimate the average price of gas in 1984 and 1985.

c) Estimate the year in which the average price of gas was 56.0¢/L.

d) Extrapolate the average price of gas in 2012 using this model. Is this reasonable? State any limitations on this model.

Lesson Summary

In this lesson, you considered equations and data from realistic situations. You constructed graphs and used technology to find polynomial regression equations that approximate the data. Based on the correlation coefficient, or visual observation, you determined how well the line or curve of best fit modelled the data. You interpreted the equations and graphs and used them to solve contextual problems.

Assignment 1.1

Polynomial Functions

Total: 29 marks

This is a hand-in assignment. Clearly show the steps in your solutions on the question sheets below and submit these pages when you send in your assignments for marking. If you use a graphing calculator, you may state your keystrokes, include a sketch of what is on your screen, or connect to a computer and print out your screen captures. If you use online graphing tools, state the website address, state what values or equations you used, and sketch or print the graphs. On your midterm examination, you will need to specify ***one*** graphing app or graphing technology that you will use.

Final answers must include units. Answers given without supporting calculations and graphs will not be awarded full marks.

1. Match each of the following polynomial functions with its corresponding graph. Write the number of the correct graph on the line next to the function. *(6 marks)*

____ a) $f(x) = x^3 - x^2 - 6x$

____ b) $g(x) = -x^3 + 4x + 5$

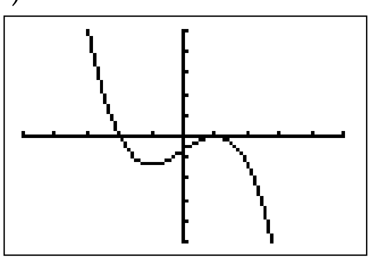
____ c) $k(w) = (x - 4)^3$

____ d) $h(x) = x^3 + 5x^2 + 2x - 8$

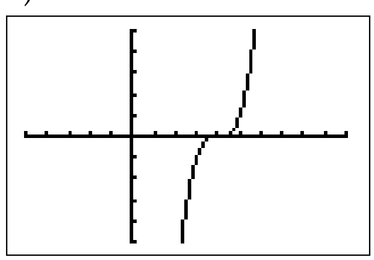
____ e) $b(x) = (x - 4)(x + 2)$

____ f) $t(x) = -\frac{1}{3}x^3 + x - \frac{2}{3}$

i)



ii)



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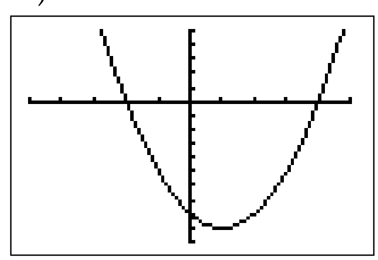
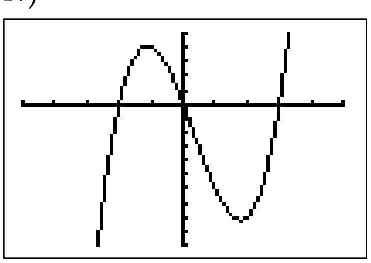
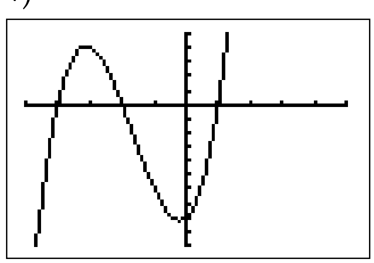


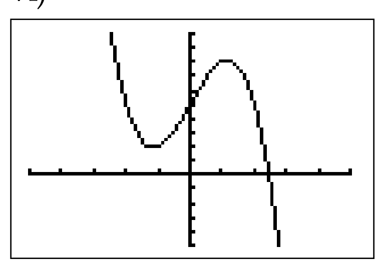
iv)



v)



vi)



Assignment 1.1: Polynomial Functions (continued)

2. Complete the chart below by describing the characteristics of the graphs of the following functions. Examine the sign of the leading coefficient, the constant, and the degree of the polynomial. *(0.5 mark each, for a total of 8 marks)*

Equation	$f(x) = -x^3 + 3x$	$g(x) = x^2 + 7x - 10$	$k(w) = \frac{4}{5}w + \frac{11}{13}$	$h(x) = x^3 - 2x^2 + 5x - 11$
Type of Function				
Degree				
y-intercept				
End Behaviour				

3. A polygon is a closed two-dimensional figure created with three or more straight line segments. A diagonal connects any two non-adjacent vertices of a polygon.
   a) Draw polygons with 4, 5, 6, 7, and 8 sides. Determine how many diagonals each polygon has. Record your results in the chart relating the number of sides to the number of diagonals. *(5 marks)*

# sides					
# diagonals					

b) Graph the data and use technology to find the line or curve of best fit and the regression equation that best models this data. Include a labelled sketch of the graph and state the equation or include a printout showing the scatterplot and regression equation. *(4 marks)*

c) State whether the regression equation is a good fit to the data. Explain. *(2 marks)*

Assignment 1.1: Polynomial Functions (continued)

d) How many diagonals does a polygon with 100 sides have? *(2 marks)*

e) If a polygon has 252 diagonals, how many sides does it have? *(2 marks)*

Lesson 3: Characteristics of Exponential Functions

Lesson Focus

In this lesson, you will

- ❑ describe the characteristics of exponential functions by analyzing their graphs and equations

Lesson Introduction

In Lessons 1 and 2, you used polynomial functions to model real-world data. In some situations, such as investment growth, population trends, cooling curves, and radioactive decay (to mention just a few), the trend of the data points does not follow a pattern similar to the polynomial functions. A different function is needed for a good curve of best fit. These situations are best described by an exponential function, which is the focus of this lesson.

Exponential Functions

An exponential function is defined as $f(x) = a(b)^x$, where $a \neq 0$, $b > 0$, $b \neq 1$, $x \in \Re$.

Recall that a power consists of an exponent and a base. In an exponential function, the variable, x, is in the exponent, it is applied to the base, b, and it is affected by the multiplier, a.

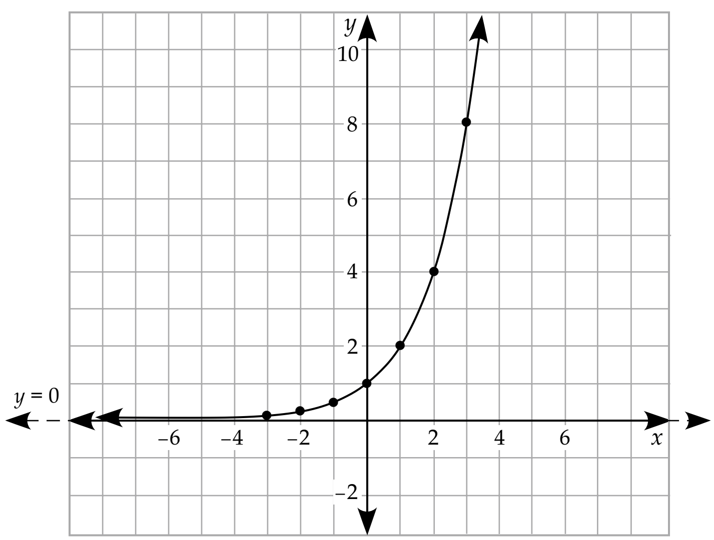
Graphing Exponential Functions

Example 1

Create a table of values and graph the exponential function $f(x) = 2^x$. Describe the characteristics of the graph.

Solution

x	$f(x) = 2^x$
−3	$2^{-3} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$
−2	$2^{-2} = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$
−1	$2^{-1} = \left(\frac{1}{2}\right)^1 = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$
4	$2^4 = 16$
5	$2^5 = 32$



The graph of the function $f(x) = 2^x$ extends from Quadrant II to Quadrant I. It is always increasing, but does so very slowly to the left of the y-intercept, and very quickly to the right of the y-intercept.

The values going towards negative infinity are very, very small, and they get close to the x-axis, but never touch or cross it. The x-axis is an asymptote. An **asymptote** is a line that a function curve gets closer to, but never reaches, as the curve goes off to infinity.

The line has a y-intercept at 1 and no x-intercepts.

The domain is all real numbers, $\{x \mid x \in \Re\}$, and the range is all positive real numbers, but y is never equal to or less than zero, $\{y \mid y > 0, y \in \Re\}$.

Example 2

What happens when the base, b, or the value of a changes? Graph the following exponential functions on the same grid and compare the characteristics of the graphs.

a) $f(x) = 2^x$
$f(x) = 5^x$
$f(x) = 10^x$

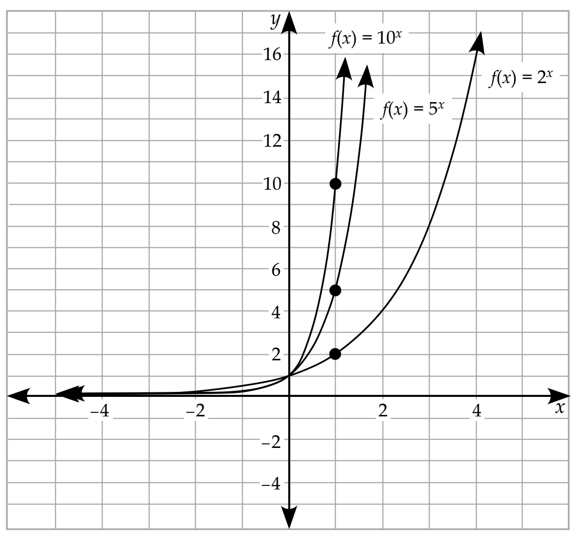
b) $f(x) = 2^x$
$f(x) = \frac{1}{2}^{x}$
$f(x) = \frac{1}{5}^{x}$
$f(x) = \frac{1}{10}^{x}$

c) $f(x) = 2^x$
$f(x) = 5(2)^x$
$f(x) = 10(2)^x$
$f(x) = \frac{1}{2}(2)^x$

d) $f(x) = 2^x$
$f(x) = -1(2)^x$

Solution

a)



When the base of the power increases, the graph rises more quickly as the value of x increases.

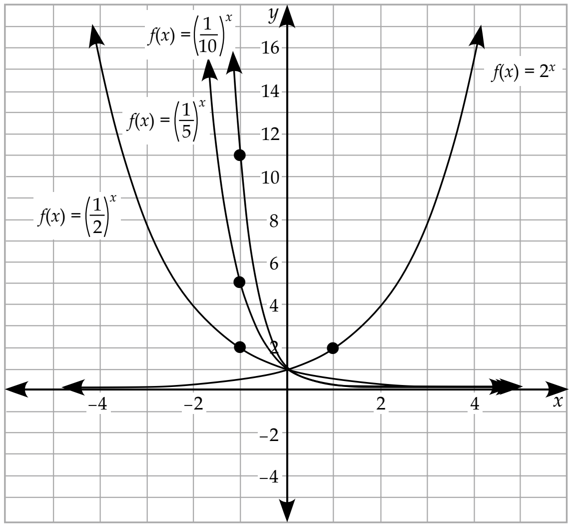
The coordinates for the y-intercept of each function are (0, 1).

The line $y = 0$ is an asymptote.

The end behaviour of each line extends from Quadrant II to Quadrant I.

The domain and range are the same for all three lines, $\{x \mid x \in \Re\}$ and $\{y \mid y > 0, y \in \Re\}$.

b)

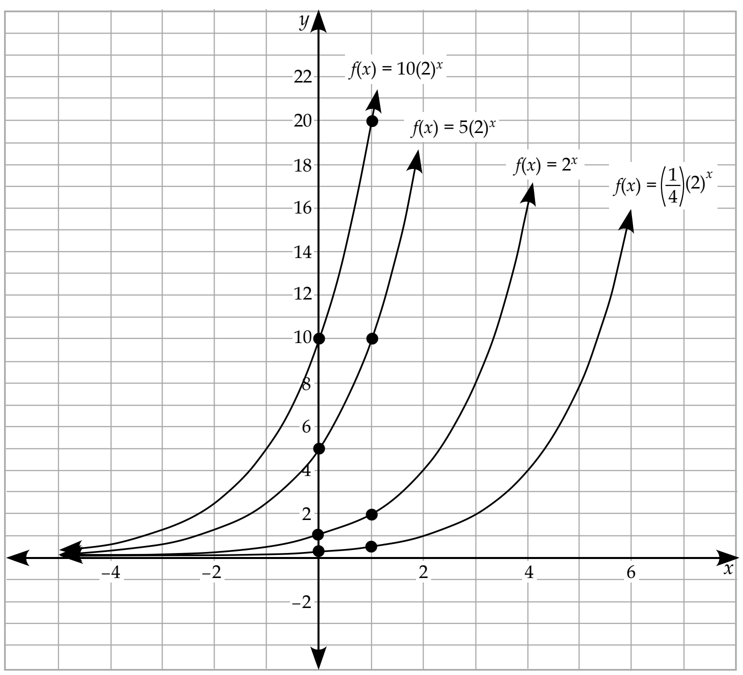


The graph of $f(x) = \left(\frac{1}{2}\right)^x$ is the mirror image of the graph of $f(x) = 2^x$, reflected in the *y*-axis. The end behaviours of all these graphs are the same. The lines extend from Quadrant II to Quadrant I, but if the base is a positive, fractional value between 0 and 1, the line is decreasing.

The equation of the asymptote is $y = 0$.

The *y*-intercept is at (0, 1).

c)

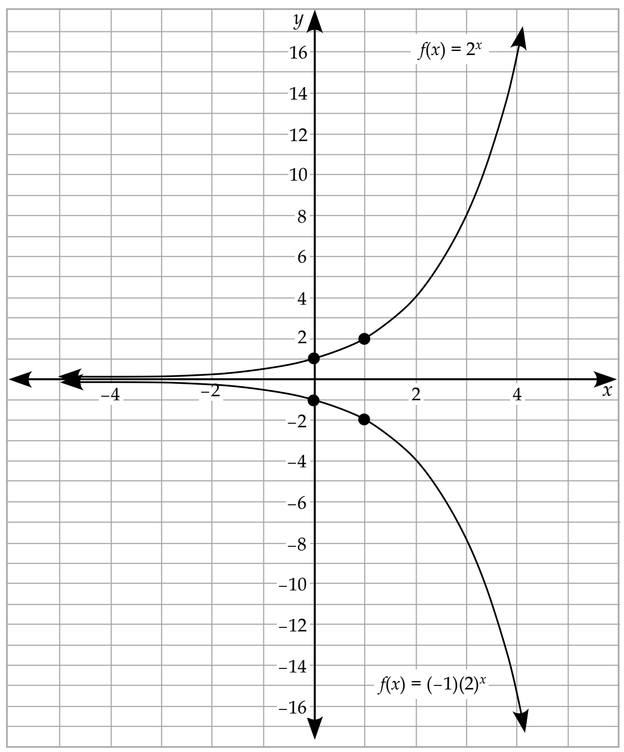


Changing the value of the multiplier, the a-value, either stretches or compresses the graph of $f(x) = 2^x$. If the a-value is greater than 1, it stretches the line vertically and the line increases more quickly. If the value of a is a fraction between 0 and 1, the line is compressed vertically and increases more gradually. Notice the coordinates of the points on each line where $x = 1$. In each graph, where $x = 1$, the y-value is equal to $a \times b$.

The y-intercept for each line is at the value of a.

The lines are increasing. The end behaviours extend from Quadrant II to Quadrant I.

The line $y = 0$ is an asymptote for each function.

d) 

If the power is multiplied by a negative value, the graph of the line is reflected in the x-axis.

The line is decreasing and extends from Quadrant III to Quadrant IV.

The equation of the asymptote is still $y = 0$.

The y-intercept is at the point $(0, -1)$.

The domain is $\{x \mid x \in \Re\}$ but the range of the function, if a is negative, is $\{y \mid y < 0, y \in \Re\}$.

Characteristics of the Graphs of Exponential Functions

You have seen from the above examples that the graph of an exponential function in the form $f(x) = ab^x$ is typically a rapidly increasing function with a horizontal asymptote. The value of the base determines how quickly it increases (if $b > 1$) or decreases (if $0 < b < 1$). The magnitude of the multiplier, a, determines the y-intercept and vertical stretch or compression of the line. The sign of a determines the end behaviour of the line and if it is reflected in the x-axis.

This information may be useful to add to your resource sheet.

The location of an exponential function on a Cartesian grid can be manipulated by introducing other constants into the equation of the function.

Example 1

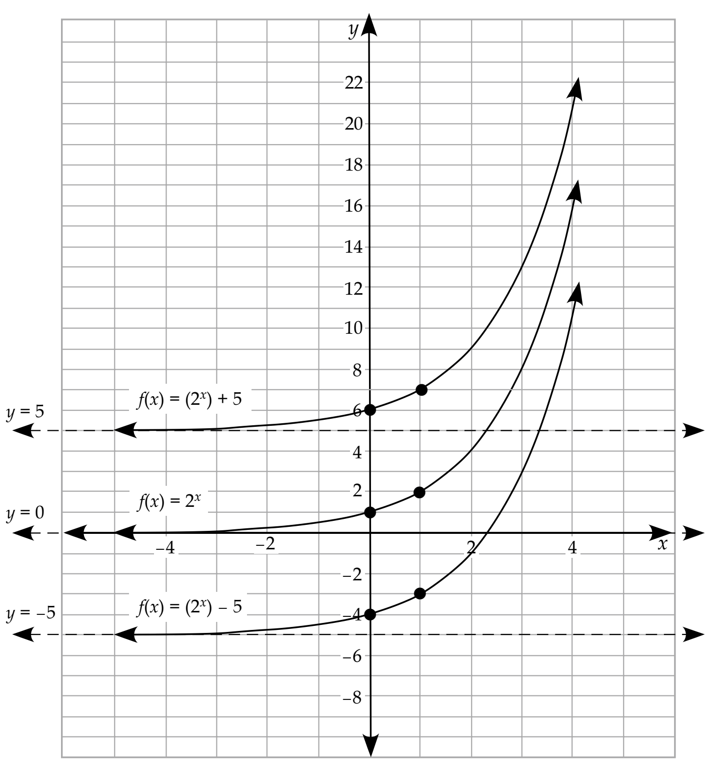
Graph the following exponential functions on the same grid and identify the impact of the constants.

$$f(x) = (2)^x$$

$$f(x) = (2)^x + 5$$

$$f(x) = (2)^x - 5$$

Solution



Adding a constant to the function shifts the graph vertically. Notice the location of the y-intercept, 1, 1 + 5, and 1 − 5.

The horizontal asymptotes are at the value of the constants: $y = 0$, $y = 5$, and $y = -5$.

This information may be useful to add to your resource sheet for future reference.

Matching Exponential Functions and Their Graphs

Using what you know about the characteristics of exponential functions, you can match equations and graphs.

Example 1

Match each of the following graphs with the appropriate equation.

$a(x) = 1.5^x$

$b(x) = 5\left(\frac{1}{125}\right)^x$

$c(x) = 8 \times 4^x - 2$

$d(x) = 5(8)^x$

$e(x) = 2^x$

$f(x) = \left(\frac{1}{3}\right)^x$

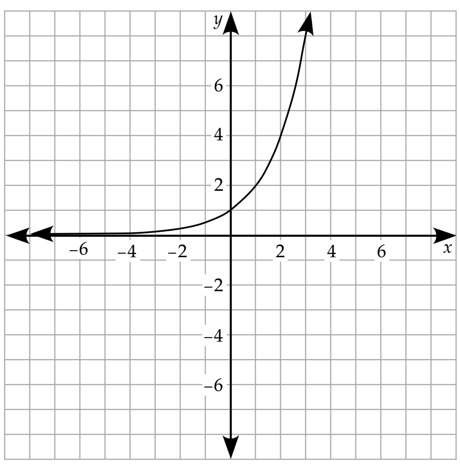
$g(x) = -4(\pi)^x$

$h(x) = \pi^x$

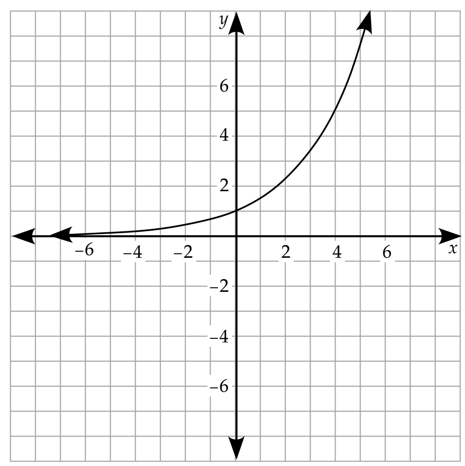
$i(x) = \left(\frac{1}{12}\right)^x$

$j(x) = 5^x + 1$

1.



2.



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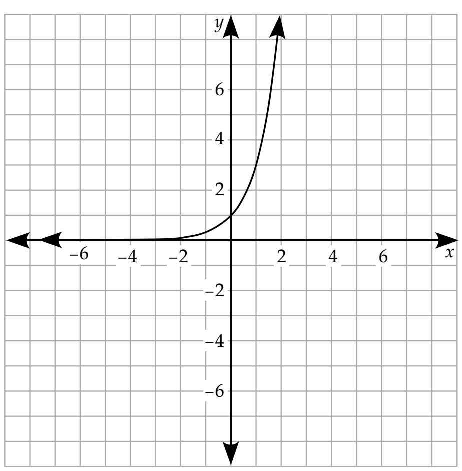

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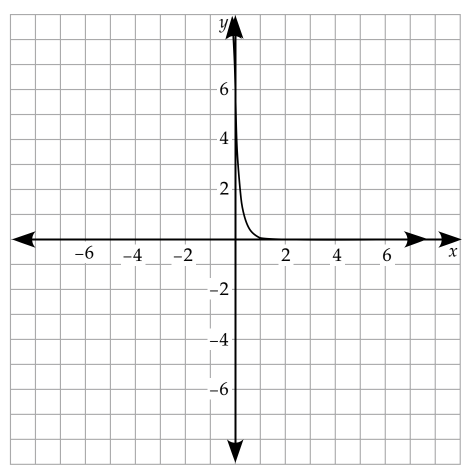

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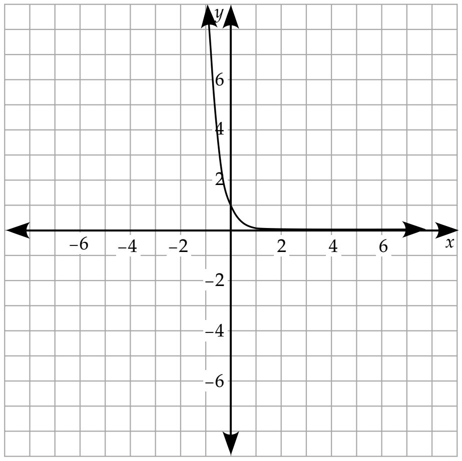
y
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−2
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x
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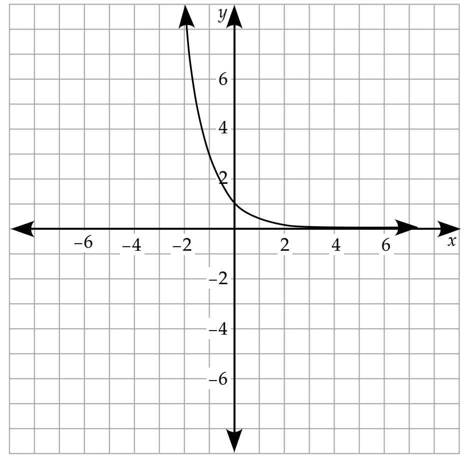
−4
−6

6.


y
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2
4
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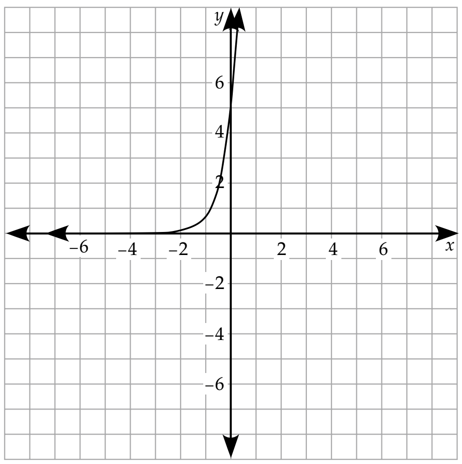
x
−2
−4
−6

7.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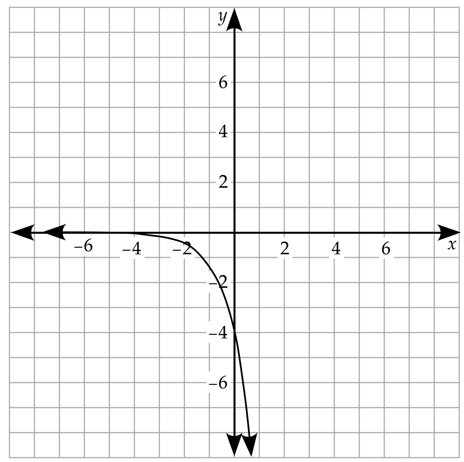
8.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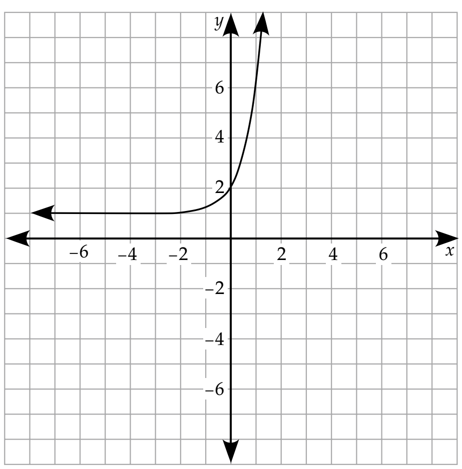
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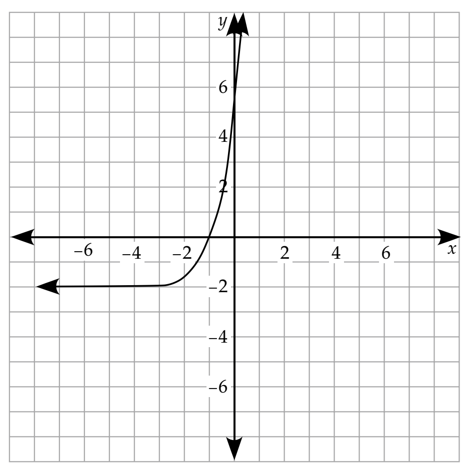
x
−2
−4
−6

10.


y
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−6
−4
−2
2
4
6
x
−2
−4
−6

Solution

Graph 1	$e(x) = 2^x$ (increasing through (1, 2))	Graph 6	$f(x) = \left(\frac{1}{3}\right)^x$ (decreasing, through (−1, 3))
Graph 2	$a(x) = 1.5^x$ (increasing, through (1, 1.5))	Graph 7	$d(x) = 5(8)^x$ (y-intercept at 5, increasing)
Graph 3	$h(x) = (\pi)^x$ (increasing, through (1, π))	Graph 8	$g(x) = -4(\pi)^x$ (y-intercept at −4, increasing)
Graph 4	$b(x) = 5\left(\frac{1}{125}\right)^x$ (y-intercept at 5, decreasing)	Graph 9	$j(x) = (5^x) + 1$ (asymptote at $y = 1$)
Graph 5	$i(x) = \left(\frac{1}{12}\right)^x$ (decreasing, through (−1, 12))	Graph 10	$c(x) = 8 \times (4^x) - 2$ (asymptote at y = −2)

Learning Activity 1.3

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Write 9^5 with a base of 3 and simplify.
2. Expand: $(a + b)^2$
3. Simplify: $25^{\frac{3}{2}}$
4. Solve for x: $6^x = \frac{1}{36}$
5. Write in exponential form: $\sqrt[5]{m^3}$
6. Solve for x: $16^5 = 2^x$
7. A dress costs $100 on Monday. On Tuesday, it is reduced in price by 20% and on Wednesday, it is reduced by 20% again. What is the price of the dress on Wednesday?
8. A boy saves $3 from his weekly allowance the first week he receives it. Each week, he doubles the amount he saves. How much does the boy save from his fourth allowance?

continued

Learning Activity 1.3 (continued)

Part B: Exponential Functions

Remember, these questions are similar to the ones that will be on your assignments and midterm examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and midterm examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Complete the following tables for the given exponential functions. Describe the patterns you see and how they relate to the form of the function $f(x) = a(b)^x$.

x	$f(x) = 2^x$
–1	
0	
1	
2	
3	
4	
5	

x	$f(x) = 3^x$
–1	
0	
1	
2	
3	
4	
5	

x	$f(x) = 10(2^x)$
–1	
0	
1	
2	
3	
4	
5	

x	$f(x) = \left(\frac{1}{2}\right)^x$
–1	
0	
1	
2	
3	
4	
5	

2. Given the function $f(x) = 153^x$, explain why the y-intercept is at (0, 1).

continued

Learning Activity 1.3 (continued)

3. Graph the functions below and complete the chart to describe their characteristics.

	$f(x) = \left(\frac{3}{7}\right)^{x}$	$f(x) = \left(\sqrt{3}\right)^{x} + 5$	$f(x) = -2(10)^{x}$
Graph			
y-intercept			
Equation of Asymptote			
Increasing or Decreasing			
End Behaviour			
Domain			
Range			
Description			

4. Graph the function $f(x) = 1^{x}$ and explain the result.

5. Graph the functions $f(x) = \frac{1}{3}^{x}$ and $f(x) = 3^{-x}$ on the same grid and explain the results using the exponent laws.

6. Given the exponential function $f(x) = 3^{x}$, use the exponent laws to explain why the values to the left of the y-intercept stay small and the values to the right of the y-intercept increase quickly.

Lesson Summary

In this lesson, you explored the characteristics of exponential functions in the form $f(x) = a(b)^x$ by examining their graphs and equations. You described the function in terms of its graph increasing or decreasing, the location of the y-intercept, the equation of its asymptote, the end behaviour, and the domain and range. You analyzed the impact of changing the base, the impact of changing the value of the multiplier a, and the impact of adding a constant to the function and saw how that stretched, compressed, reflected, or shifted the graph of the function. Using these characteristics, you matched equations in a set to their corresponding graphs.

Notes

Lesson 4: Applications of Exponential Functions

Lesson Focus

In this lesson, you will

- ❑ graph data and determine the exponential function that best approximates the data
- ❑ interpret the graph of an exponential function that models a situation and explain the reasoning
- ❑ solve contextual problems using technology

Lesson Introduction

Many real-world situations can be modelled using exponential functions. When you plot data from experiments or observations and observe a rapidly increasing or decreasing trend, an exponential regression equation may be useful as you analyze the data. In this lesson, you will see a variety of applications where an exponential function model is appropriate.

Finding Exponential Regression Equations

Savings Account

When saving your money, what you really want to see is a quick increase in the value of your account. An exponential function that rises quickly is the ideal model for a good savings account.

Example 1

Erik invests $3500.00 into a savings account that has an interest rate of 3%, compounded annually. The following chart models the growth in value of this investment.

Year		Value ($)
0	3500.00	3500.00
1	3500.00 × 1.03	3605.00
2	3605.00 × 1.03	3713.15
3	3713.15 × 1.03	3824.54
4	3824.54 × 1.03	3939.28

a) Graph the results on a scatterplot and find the exponential regression equation.

b) Explain your results.

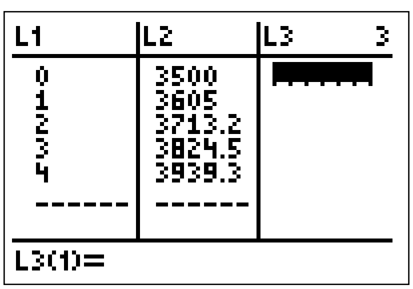
c) How much is the investment worth after 10 years?

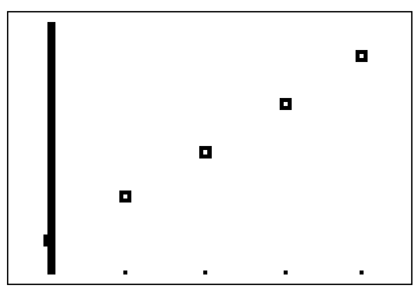
d) If Erik is saving to buy a used car for $5000.00, how long will it take until this investment is worth that amount?

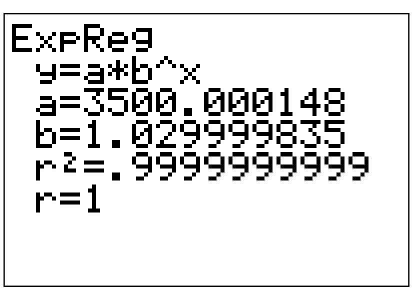
Solution

The following screenshots show the display of a TI-84. Refer to the Technology Appendix or the help files for your chosen technology application for assistance.

a)







Note: Enter all decimal places into the graphing technology of your choice. Always use all decimal places in calculations. The screen shots of the graphing calculator shown above have rounded the values to one decimal place due to space limitations.

The exponential function that models this data is $f(x) = 3500(1.03)^x$.

b) The y-intercept of this line is at $3500.00, the initial amount invested and the a-value in the function. The investment grows in value by a factor of 3% each year. Since the value is increasing and calculated as the investment value plus interest, the factor b is 1.03. The variable, x, is the year.

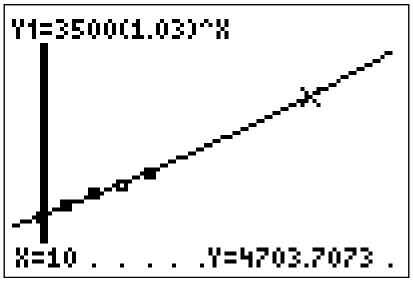
c) Solve for $f(10)$:

$$f(x) = 3500(1.03)^x$$

$$f(10) = 3500(1.03)^{10}$$

$$f(10) = 4703.707328$$

or



You may calculate his investment by using the equation, or you can graph the curve and solve for the y-value where $x = 10$. Adjust your window settings if necessary.

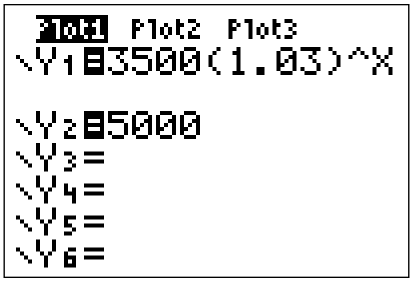
After 10 years this investment would be worth $4703.71.

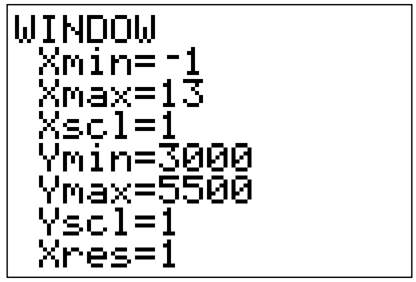
d) Solve for x when $f(x) = 5000$.

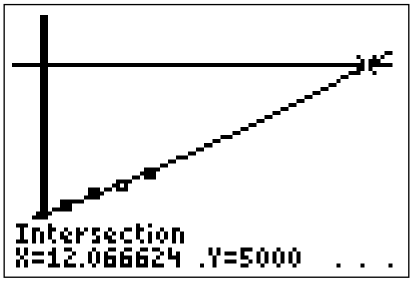
$$f(x) = 3500(1.03)^x$$

$$5000 = 3500(1.03)^x$$

Graph the line $y = 5000$ and find the point of intersection with the line of the function $f(x) = 3500(1.03)^x$. Adjust the window settings as necessary to include the required point.







It will take more than 12 years for Erik's investment to be worth $5000.00.

Paper Folding

Take a piece of paper and fold it in half. Count the number of layers. Repeat as many times as possible. Based on your results, complete the table below.

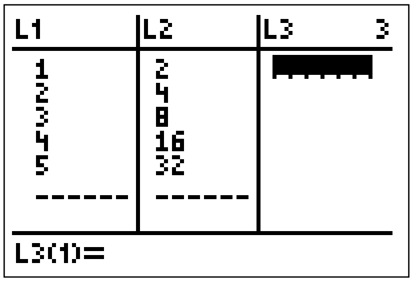
# of folds	# of layers	# of layers written as a power of 2
1		
2		
3		

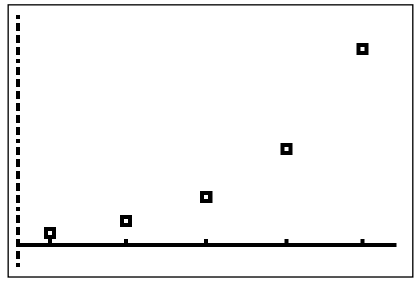
a) Using the data generated, create a scatterplot and find the regression equation for the most appropriate curve of best fit.

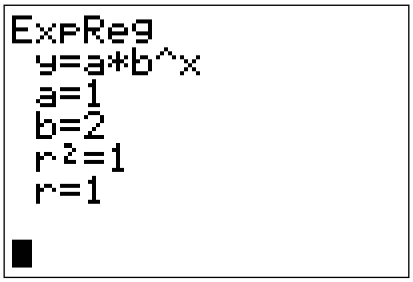
b) Using technology or the regression equation, determine the number of layers created if a piece of paper could be folded 12 times.

Solution

a)







Even though the points could be modelled using a quadratic or cubic function, an exponential regression equation perfectly models the data. Since the number of layers doubles each time, the factor of 2 is represented by the base 2.

b) $f(x) = 1(2)^x$

$f(12) = 1(2)^{12}$

$f(12) = 4096$

Folding a paper 12 times would result in 4096 layers. To find this point on the graph, you would have to solve for the value of y when $x = 12$ and adjust your window setting to include the point (12, 4096).

Battery Life

Exponential functions can be used to model decreasing functions as well. Recall that if b is a fraction between 0 and 1, the line of the function decreases.

Example 1

A cell phone battery is said to lose about 1.75% of its charge capacity every time it is charged. At full charge, a given phone is said to have 200 minutes of talk time.

a) Create a table of values showing the charge capacity in minutes after successive charges.

b) Use technology to find an exponential function or regression equation to model this situation.

c) How much talk time capacity would the battery have after 30 charges?

d) When will the talk time drop below 30 minutes per charge?

Solution

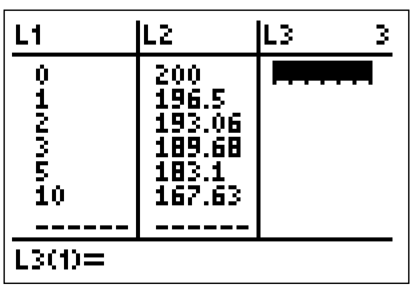
a) Create a table of values showing the charge capacity in minutes after successive charges.

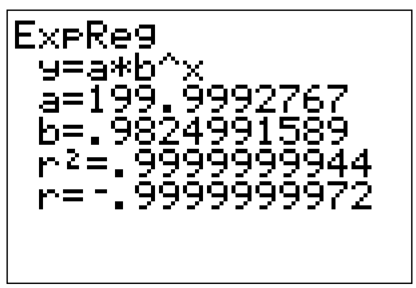
To get a good idea of the trend of the data, a table of values should have at least 5 sets of data.

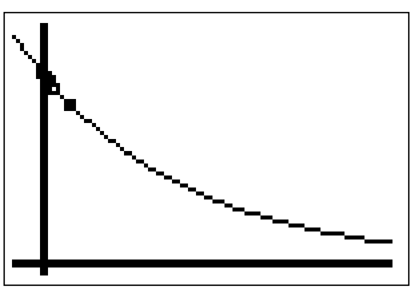
A loss of 1.75% can be thought of as 98.25% of the charge capacity remaining (100% − 1.75% = 98.25%).

Number of Charges			Charge Capacity (minutes)
0	200	200	200
1	(200 × 0.9825) = 196.5	$(200) \times (0.9825)^1 = 196.5$	196.5
2	(196.5 × 0.9825) = 193.06	$(200) \times (0.9825)^2 = 193.06$	193.06
3	(193.06 × 0.9825) = 189.68	$(200) \times (0.9825)^3 = 189.68$	189.68
5		$(200) \times (0.9825)^5 = 183.1$	183.1
10		$(200) \times (0.9825)^{10} = 167.63$	167.63

b) Use technology to find an exponential function or regression equation to model this situation.







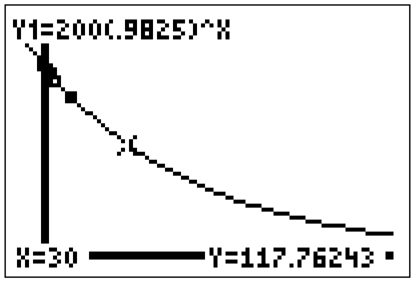
Let T represent the talk time. Let n represent the number of charges.

The amount of talk time left after each charge is $T(n) = 200(0.9825)^n$.

c) How much talk time capacity would the battery have after 30 charges?

You may use technology or the regression equation to determine the capacity when $n = 30$.

After 30 charges, the amount of talk time available would be about 118 minutes.

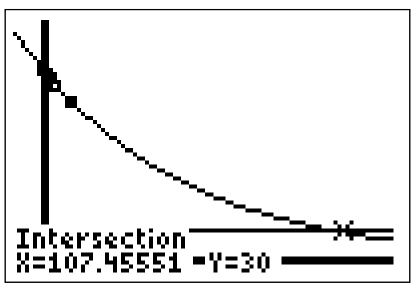


$$T(n) = 200(0.9825)^n$$

$$T(30) = 200(0.9825)^{30}$$

$$T(30) = 117.76$$

d) When will the talk time drop below 30 minutes per charge?



After approximately 107 charges, the talk time would be less than 30 minutes.

Population Growth

The population of a small town is booming due to the oil industry. Experts predict that the population will grow from its population in the year 2012 of 200 and double every three years.

a) Create a table of values showing five doubling periods starting with the year of 2012 as year 0, and sketch the graph.

b) Use technology to find an exponential regression equation that models this data.

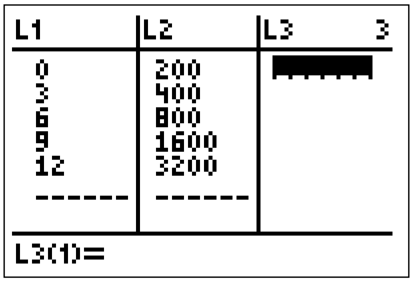
c) Predict how many people will live in the city in 2022.

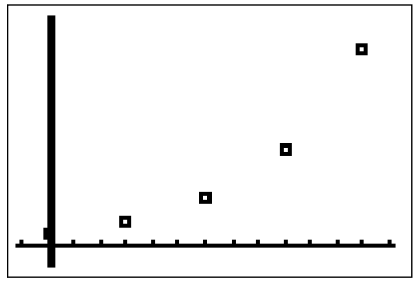
d) Use technology to predict in what year the population will reach 5000.

Solution

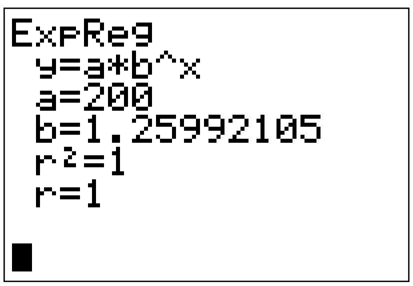
a) Create a table of values showing five doubling periods starting with the year of 2012 as year 0, and sketch the graph.

Year	2012 Year 0	2015 Year 3	2018 Year 6	2021 Year 9	2024 Year 12
Population	200	400	800	1600	3200





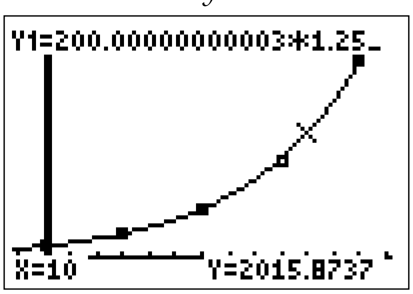
b) Use technology to find an exponential regression equation that models this data.



$y = 200(1.25992105)^x$

c) Predict how many people will live in the city in 2022.

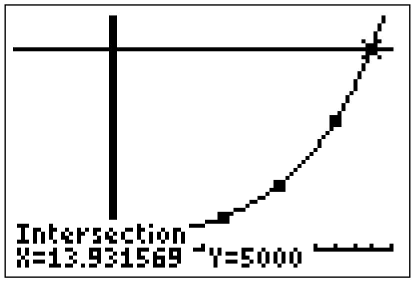
You can solve for y where $x = 10$ using the equation or the graph.



In 2022, the town could have approximately 2015 inhabitants.

d) Use technology to predict in what year the population will reach 5000.

Graph the function and the equation $y = 5000$, and find the point of intersection. You may have to adjust your window settings to include the solution. It will happen around the fourteenth year, or approximately the year 2026.



Continuous Growth or Decay

In the first example in this lesson, you considered the impact of compound interest on an initial investment of \$3500.00. The interest was added to the account annually and subsequent calculations were based on the value of the investment plus the accrued interest. From previous math courses and some of your own life experiences, you may know that the more often the interest is compounded, the more quickly the investment grows in value.

When you look at the compound interest formula now, after exploring exponential functions, it is easy to see how it is a variation of the basic exponential equation $f(x) = a(b)^x$.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where A = total amount accumulated, including interest

P = principal, the initial amount invested or borrowed

r = annual interest rate, as a decimal

n = number of compounding periods per year

t = number of years

This information may be useful to add to your resource sheet for future reference.

Example 1

Compare the final value of an initial investment of $3500 at a rate of 3% after 10 years if it is

a) compounded annually

b) compounded monthly

c) compounded daily

Solution

a) From the example completed earlier in this lesson, the value of the savings account after 10 years with interest compounded annually is about $4703.71.

$$f(x) = 3500(1.03)^x$$

$$f(10) = 3500(1.03)^{10}$$

$$f(10) = 4703.707328$$

b) Use the compound interest formula to determine the value after 10 years, if $n = 12$ (since monthly means 12 times per year).

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$A = 3500\left(1 + \frac{0.03}{12}\right)^{(12)(10)}$$

$$A = 3500(1.0025)^{120}$$

$$A = 4722.737415$$

The value of the savings account when compounded monthly is about $4722.74, approximately $19 higher than the account that is compounded annually.

c) If the account has interest added daily, the value after 10 years is even higher. Daily means $n = 365$ times per year.

$$A = 3500\left(1 + \frac{0.03}{365}\right)^{(365)(10)}$$

$$A = 3500(1.000082192)^{3650}$$

$$A = 4724.451362$$

The account is now worth $4724.45, just a little more ($1.71) than when interest was compounded monthly.

The more often the interest is compounded, the more the investment grows in value. What would happen to the value of the investment if it were compounded twice a day? Every hour? Each minute of the day? Would the bank consider compounding your interest each second? Continuously?

As it turns out, the difference between daily and continuous compounding is not large enough to really make it worthwhile, for you or the financial institution. However, it is possible to calculate the value of continuous compounding, and there are applications where this becomes meaningful.

Consider the growth of bacteria in a culture. The model generated in the example above is meaningful, but realistically, the cells do not wait until their "compounding period" or time factor to multiply. They are reproducing continuously. How can the exponential function be manipulated to account for this possibility?

The Natural Exponential Function

You have already seen how the rate and time variables can be manipulated in the formula to model situations where a different compounding period is used, or the time frame for a population to change by a given factor is not equal to one. To model continuous growth, the base must be changed to an irrational number called e.

In the previous lesson, you graphed exponential functions with an irrational value such as π for the base. The function $f(x) = e^x$ is called the **natural exponential function**. The base, e, is an irrational number named after the Swiss mathematician, Leonard Euler (pronounced "oiler", like the team Gretzky played for). The value of e is approximately 2.71828183 . . .

Like π, the number e shows up in many and sometimes surprising applications of mathematics.

There are different applications of the natural exponential function. The basic function is modified to accommodate the parameters of the situation.

Example 1

Compare the value of an investment of $1 after one year, compounded for a variety of compounding periods, given the rate of 100%. (Yes, this is highly unlikely, but for the sake of understanding where the value of e comes from, just go with it!)

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Solution

$$A = 1\left(1 + \frac{1}{n}\right)^{n(1)}$$

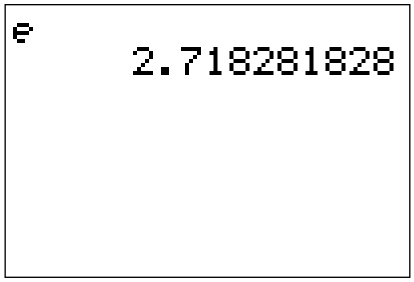
If $P = 1$, $r = 1$, and $t = 1$, then $A = \left(1 + \frac{1}{n}\right)^{n}$.

How Often Compounded	Computation
yearly	$\left(1 + \frac{1}{1}\right)^{1} = 2$
semi-annually	$\left(1 + \frac{1}{2}\right)^{2} = 2.25$
quarterly	$\left(1 + \frac{1}{4}\right)^{4} = 2.44140625$
monthly	$\left(1 + \frac{1}{12}\right)^{12} \approx 2.61303529022\ldots$
weekly	$\left(1 + \frac{1}{52}\right)^{52} \approx 2.69259695444\ldots$
daily	$\left(1 + \frac{1}{365}\right)^{365} \approx 2.71456748202\ldots$
hourly	$\left(1 + \frac{1}{8760}\right)^{8760} \approx 2.71812669063\ldots$
every minute	$\left(1 + \frac{1}{525600}\right)^{525600} \approx 2.7182792154\ldots$
every second	$\left(1 + \frac{1}{31536000}\right)^{31536000} \approx 2.71828247254\ldots$

The value of a $1 investment, compounded every second at 100% for one year, is approximately $2.71828247254 . . .

If the value of n continued to increase to infinity, so that the investment was compounded continually, the value of the function would approach the irrational value e.

Compare the value from the chart with the value of e on your calculator. This function key can be located above the ÷ button on most graphing calculators, and you may have to access it using a 2nd function key.



When using an exponential form of e, or e^x, the function key can be located above the LN button. Check the manual or help file that comes with your device, or find online tutorials to help with this concept.

The compound interest formula, $A = P\left(1 + \frac{r}{n}\right)^{nt}$, if considering continuous compounding, can be rewritten as $A = P(e)^{r(t)}$, since the value of $\left(1 + \frac{1}{n}\right)^{n}$ approaches e as n increases to infinity.

The formula for continuous exponential growth or decay is given as:

$A = Pe^{rt}$

where A = amount, the total accumulated

P = principal, the initial amount

e = 2.718281828 . . .

r = rate per period of time, as a decimal

t = time

Example 2

Use the natural exponential function for continuous compounding to calculate the value of $3500 compounded continuously for ten years at 3%.

Solution

$A = Pe^{rt}$

$A = (3500)e^{(0.03)(10)}$

$A = 4724.505827\ldots$

The value of the investment is approximately $4724.51. Compare this to the value of the investment calculated previously using daily interest. It is only about 6 cents more over the 10 year investment.

Using Base *e* to Model Situations

The biological growth and decay of natural populations is best modelled using the base *e* since the growth or decay is continuous.

Example 1

At the present time, a culture contains 1000 Type A bacteria. A biologist records the growth in the number of bacteria in the culture over time and comes up with the following data and exponential regression equation.

$A = 1000e^{0.025t}$

Time (t) (hours)	Number (A) of Bacteria
0	1000
1	1025
2	1051
3	1078
12	1350
24	1822

a) Use technology to create a graph showing the points and the exponential equation.

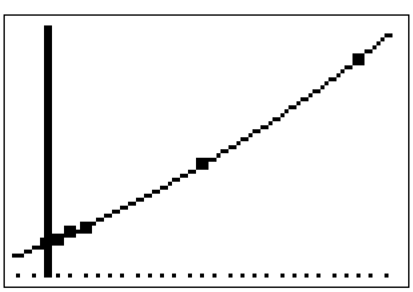
b) Use the equation or the graph to determine the number of bacteria after 36 hours.

c) Use technology to determine approximately how long it would take for the number of bacteria to reach 3000.

d) At what rate do the cells increase?

Solution

a) Use technology to create a graph showing the points and the exponential equation.

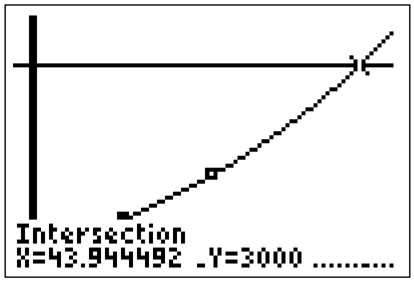


b) Use the equation or the graph to determine the number of bacteria after 36 hours.

$A = 1000e^{0.025t}$

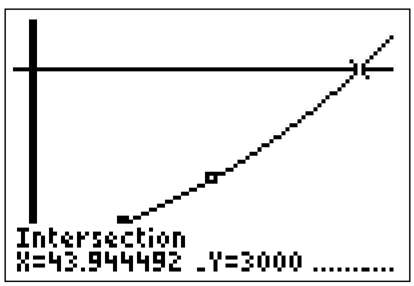
$A = 1000e^{0.025(36)}$

$A = 2459.603111\ldots$



There will be approximately 2460 bacteria after 36 hours.

c) How long would it take for the number of bacteria to reach 3000? Solve for the intersection of $y = 3000$ and the regression equation.



It would take approximately 44 hours until there are 3000 cells in the culture.

d) At what rate do the cells increase?

According to the formula provided, $A = 1000e^{0.025t}$, the rate is 0.025 or 2.5% per hour.

Example 2

A beetle population, A, increases so that the population t days from today is given by the formula $A = 2000e^{0.01t}$.

a) How many beetles are there today?

b) How many beetles will there be tomorrow? In one week?

Solution

a) How many beetles are there today?

Today would be day 0. Substitute $t = 0$ into the equation and solve for A.

$A = 2000e^{0.01(t)}$

$A = 2000e^{0.01(0)}$

$A = 2000e^{0}$

$A = 2000(1)$ Any base raised to the power of zero equals one.

$A = 2000$

There are 2000 beetles today.

b) How many beetles will there be tomorrow? In one week?

Since time is measured in days, tomorrow would be t_1. Let $t = 1$.

$A = 2000e^{0.01(1)}$

$A = 2000e^{0.01}$

$A = 2020$

By tomorrow, the population of beetles will reach 2020.

In one week, $t = 7$.

$A = 2000e^{0.01(7)}$

$A = 2000e^{0.07}$

$A = 2145$

In one week, the population will be 2145 beetles.

Example 3

An 80 g sample of radioactive substance is decaying according to the formula $A = 80e^{-0.2t}$, where A is the amount of material remaining after t years.

Notice that the rate, since it models decay, is stated as a negative value. This means the number of grams is decreasing over time.

a) Given that the initial amount is 80 grams, how much remains after 3 years?

b) Find the half-life of this substance (that is, the time it takes for 80 grams to decay to 40 grams).

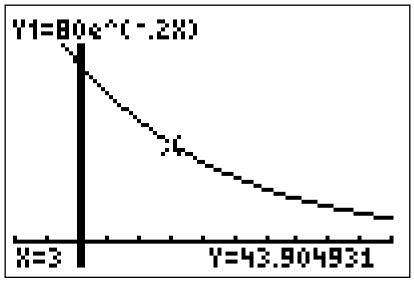
Solution

a) Given that the initial amount is 80 grams, how much remains after 3 years?

$A = 80e^{-0.2t}$

$A = 80e^{-0.2(3)}$

$A = 43.90493089$



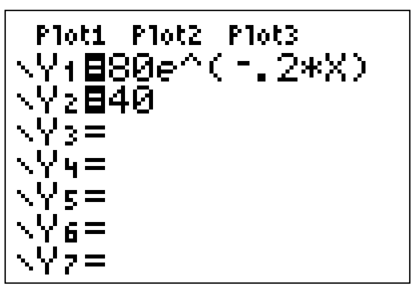
After 3 years, approximately 43.9 grams of the radioactive material remains.

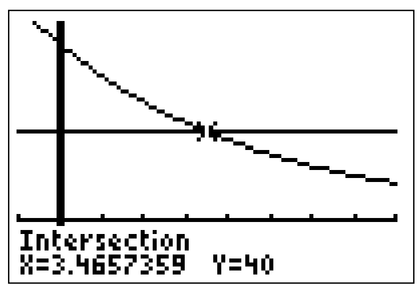
b) Find the half-life of this substance (that is, the time it takes for 80 grams to decay to 40 grams).

The half-life of a substance is the length of time required for the substance to decay to half of its original amount.

$A = 80e^{-0.2t}$

$40 = 80e^{-0.2(t)}$





The half-life of this substance is approximately 3.4657359 years. This can be translated into 3 years plus 170 days (approximately 3 years, 5 months, and 20 days).

You may want to keep track of some of the applications of exponential function models on your resource sheet for future reference.

Learning Activity 1.4

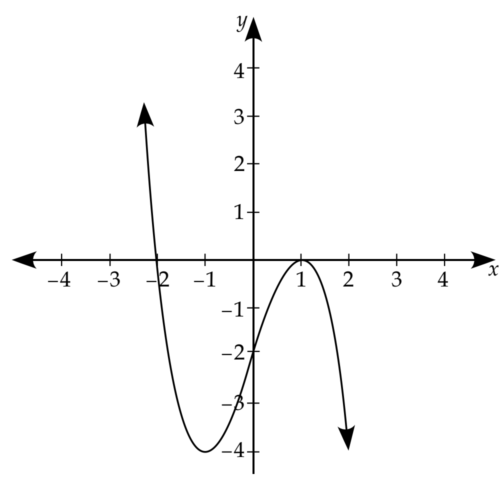
Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. The interest on an investment is compounded bi-weekly. How many times per year is interest added?
2. The interest on an investment is compounded semi-monthly. How many times per year is interest added?

Use the graph of a polynomial function shown here to provide the information requested in questions 3 to 8.



3. The sign of the leading coefficient in the equation
4. The constant in the equation
5. The end behaviour
6. The number of turning points
7. The zeros
8. The degree

continued

Part B: Applications of Exponential Functions

Remember, these questions are similar to the ones that will be on your assignments and midterm examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and midterm examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. The average monthly rent for a 1-bedroom apartment in a major city is $825. Experts predict the rent cost will increase 3.5% each year for the foreseeable future.
   a) Create a table of values showing the average monthly cost to rent a 1-bedroom apartment for the next 5 years. Use the current average monthly cost of $825 as the rent for year 0. Round values to the nearest cent.

Year	0	1	2	3	4	5
Rent ($)	825					

   b) Use technology to graph the points from the table in a scatterplot and find an exponential function that models the situation.
   c) What will the rent for a 1-bedroom apartment be in 10 years?
   d) A man who owns an apartment complex with ten 1-bedroom units is paying $9000 per month for the mortgage on the building. Use technology to determine the number of years until he makes a profit of $5000/month on his apartment complex.

2. The population of a small town is recorded for several years. The data is given below.

Year	2000	2003	2005	2008	2011
Population	250	400	540	860	1375

   a) Use technology to graph the points and find the exponential function that gives the curve of best fit.
   b) By what percent is the population growing each year?
   c) Comment on how well the graph of the regression equation models the data.
   d) Use the equation or graph to estimate the population in 2010.
   e) When the population reaches 8500, a clinic will be built. Use technology to predict when this may happen.

continued

3. The value of a new computer over time from the date of purchase is given in the table below.

# of Months	0	3	6	12	24
Value ($)	1000	875	800	450	400

a) Find the exponential regression equation that best models this data.

b) Comment on the fit of this line.

c) Use technology to determine when the value of the computer will be less than $100.

4. The population of a small town was 750 in the year 2012. The population of the town is estimated to increase by 50% every 5 years.

a) Create a table of values showing the approximate population during 5 growth periods. Use 2012 as year 0.

Year					
Population					

b) Use technology to graph the points and determine an exponential function that models the situation.

c) What will the population be in 2020?

d) When the town reaches a population of 10 000, it will build a community centre. Use technology to determine in what year the community centre may be built.

e) What percent is the population increasing by each year?

5. The value of a particular brand of car is said to decrease by a quarter of its value every 9 months after purchase. The initial value of the car is $32,500.

a) Create a table showing the value of the car over time. Include at least 5 data points.

b) Use technology to determine an exponential function that models the value of the car in this situation.

c) What will the value of the car be after two years?

d) A customer who has purchased this car would like to sell it when it is worth less than $5000. After how many years should the customer sell this vehicle?

continued

6. A 250-gram sample of a certain radioactive element decays continuously. The amount remaining can be modelled by a natural exponential function expressed as $A = 250e^{-0.13t}$.

   a) Complete the following table, using the natural exponential function to show the number of grams remaining.

Time (days)	0	1	2	3	4
Amount (g)					

   b) Use technology to determine the half-life of this substance.

Lesson Summary

In this lesson, you used technology to graph data and found the exponential function that represented the data. Using the regression equation for the curve of best fit, you answered questions about the given situation. You interpreted the equations of increasing and decreasing exponential functions in the forms $y = a(b)^x$ and $A = P\left(1 + \frac{r}{n}\right)^{nt}$. You explored the idea of continuous compounding using $A = Pe^{rt}$ and solved contextual problems using the natural exponential function to model the growth and decay of various populations.

Notes

Lesson 5: Logarithmic Functions

Lesson Focus

In this lesson, you will

- ❑ describe the characteristics of logarithmic functions by analyzing their graphs and equations
- ❑ match logarithmic graphs and their corresponding equations

Lesson Introduction

You can solve an equation such as $x = 10^5$ using what you know about exponents. But how can you solve an equation such as $5 = (10)^x$? Calculating the value of exponents such as this was one of the problems that attracted the interest of John Napier, a mathematician who worked in the early 1600s. Through his research, he defined a new kind of function called a logarithm. The tables of logarithm values that he generated were important for navigation and astronomy. Logarithms (and a scientific calculator) make the solution to an equation such as $5 = 10^x$ very simple. In this lesson, you will learn about this new function type called **logarithms**

Solving Exponential Equations

In previous math courses, you learned how to apply the exponent laws to simplify expressions and solve equations.

Example 1

Solve for x in the following equations by finding a common base.

a) $2^x = 8$

b) $4^x = 8^{1-x}$

Solution

a) $2^x = 8$

$2^x = 2^3$ Express 8 with the same base of 2.

$x = 3$ Exponents are equal.

b) $$\begin{aligned} 4^{x} &= 8^{1-x} \\ \left(2^{2}\right)^{x} &= \left(2^{3}\right)^{1-x} && \text{Rewrite using common base.} \\ 2^{2x} &= 2^{3-3x} && \text{Apply exponent laws.} \\ 2x &= 3 - 3x && \text{Equal bases mean the exponents must be equal.} \\ 5x &= 3 && \text{Solve for } x. \\ x &= \frac{3}{5} \end{aligned}$$

Graphic Solutions

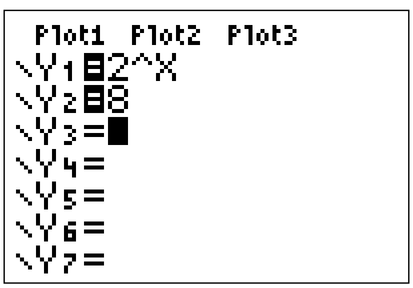
The exponential equations given in the above example could also have been solved using a graphing approach.

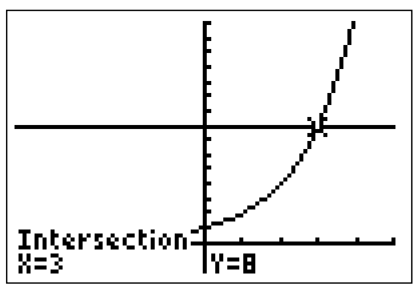
Example 1

Solve $2^{x} = 8$ for x by graphing.

Solution

A graphical approach to solving this problem would involve graphing both sides of the equation and finding the point of intersection, similar to what you did in the previous lesson.

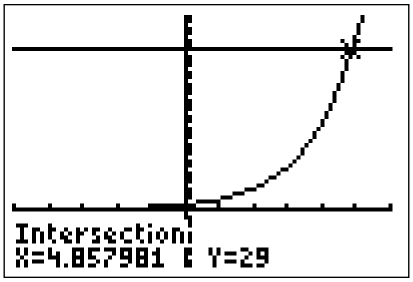




If $x = 3$, then $2^{x} = 8$.

$2^{3} = 8$

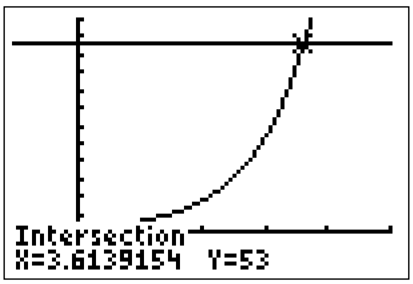
In the above examples, the terms of each equation could easily be written using a common base. This is not always the case. There is no way to solve an exponential equation such as $2^{x} = 29$ using a common base of 2 without some sort of technology. You could use a guess-and-check strategy to find an exponent for the base 2 that is approximately equal to 29. Since $2^{4} = 16$ and $2^{5} = 32$, the value of x would have to be between 4 and 5. With access to technology, you can graph both sides of this equation and find the intersection at an x-value of 4.857981. This means $2^{4.857981} \approx 29$.



Example 2

Solve $3^x = 53$ by graphing.

Solution



This means that 53 can be written as a power of 3, even if the exponent is somewhat awkward: $53 \approx 3^{3.6139154}$.

It is possible to find the exponent required on a given base so that it will equal a certain value, but using a graphing approach may not always be convenient. What you need is some way to solve for x in $53 = 3^x$ without having to graph both sides of the equation. To do this, you need some sort of approach to "undo" the variable in the exponent. You need to find the inverse process that will let you rewrite the equation $y = b^x$ in terms of the exponent, x. This inverse will give you the exponent, x, required on a given base, b, to equal the value, y. A new notation is necessary to write the inverse of a power.

Logarithm Function

The inverse of an exponential function is called the **logarithm function**.

By definition, $a = b^x$ **is equivalent to** $\log_b a = x$.

$y = \log_b x$ is the inverse of $y = b^x$, where $b \neq 1$, $b > 0$, $x > 0$.

The expression $\log_b a = x$ is read as "the log base b of a is x." It means x is the exponent you would put on base b to get a (x is the logarithm, b is the base, and a is called the argument).

The **common base** for logarithms is base 10. If no base is indicated by a subscript in the logarithm, it is assumed that the base is 10. This is the default base your calculator uses and is designated by the LOG key.

Logarithms are just another way of writing an exponent. It is the form you use when you want to find out an unknown exponent on a given base to produce a desired value.

To help you remember where the variables go in the different forms, use the "7 arrow." Draw a 7 as shown in the diagram to keep everything in its right place.

$$\log_b a = x \quad \text{means} \quad b^x = a$$

This information may be useful to add to your resource sheet for future reference.

Example 1

Solve for x:

$10^x = 153$

Solution

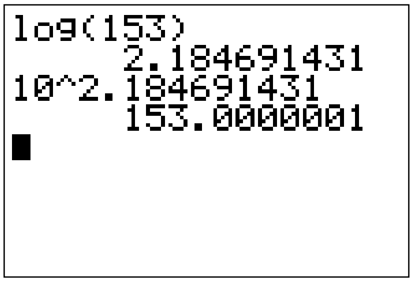
$10^x = 153$ means $\log_{10}(153) = x$ or, since base 10 is the common base, $\log(153) = x$.

Since it is base 10, you can determine the log of 153 using technology.

```
log(153)
     2.184691431
```

This means 2.184691431 is the exponent required on base 10 to produce the value of 153, or $10^{2.184691431} = 153$.

Verify this with technology.



You can see that the answer is not exactly 153. When logarithms involve irrational values, answers are approximate.

Example 2

Express each of the following exponential functions or equations as a common logarithm (base 10).

a) $10^x = y$

b) $10^x = 20$

c) $10^x = 1000$

d) $\sqrt{10} = 10^x$

Solution

a)
$$10^x = y$$
$$\log_{10}(y) = x$$
$$\log(y) = x$$

b)
$$10^x = 20$$
$$\log(20) = x$$

c)
$$10^x = 1000$$
$$\log(1000) = x$$

d)
$$\sqrt{10} = 10^x$$
$$\log(\sqrt{10}) = x$$

Example 3

Solve for x

a) $10^x = 20$

b) $10^x = 1000$

c) $\sqrt{10} = 10^x$

d) $10^x = 200$

Solution

a) $10^x = 20$

$\log(20) = x$

$x = 1.031029996$

b) $10^x = 1000$

$\log(1000) = x$

$x = 3$

c) $\sqrt{10} = 10^x$

$\log\left(\sqrt{10}\right) = x$

$x = 0.5$

d) $10^x = 200$

$\log(200) = x$

$x = 2.301$

Example 4

Rewrite each logarithm in exponential form.

a) $\log(9.5) = x$

b) $\log(1.778) = \frac{1}{4}$

c) $\log(49) = m$

Solution

a) $\log(9.5) = x$

$10^x = 9.5$

b) $\log(1.778) = \frac{1}{4}$

$10^{\frac{1}{4}} = 1.778$

c) $\log(49) = m$

$10^m = 49$

Graphing Logarithmic Functions

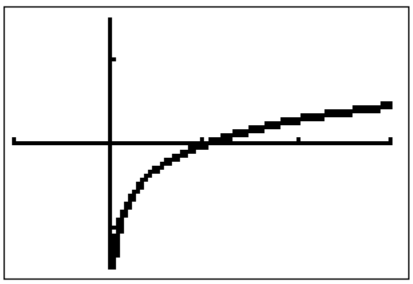
Using technology, you can graph logarithmic functions.

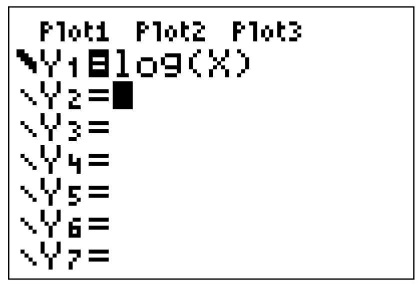
Example 1

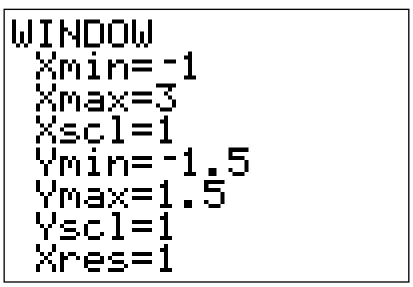
Create the graph of $y = \log_{10} x$.

Solution

Most graphing applications use base 10 as the default so no subscript is needed. Refer to the Technology Appendix or the help files for your choice of graphing technology to determine how to graph logarithms. Adjust the window settings to show the following graph.







The domain is all positive numbers. The range is all real numbers. It is an increasing function. It crosses the x-axis at the point (1, 0) but there is no y-intercept as the y-axis is a vertical asymptote.

This information may be useful to add to your resource sheet for future reference.

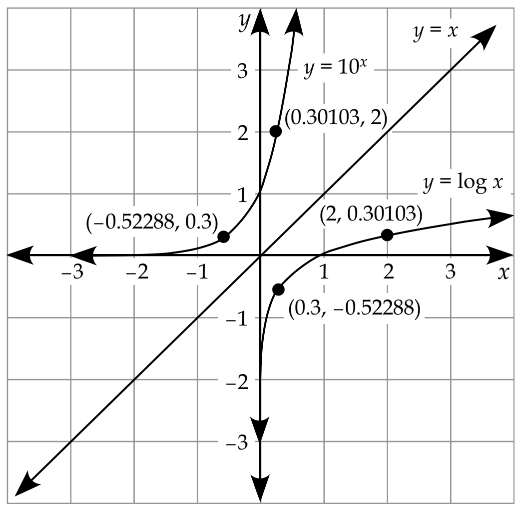
Comparing the Graphs of Logarithm and Exponential Functions

Example 1

Graph the exponential function $y = 10^x$ and its inverse on the same grid and compare the graphs.

Solution

The inverse of $y = 10^x$ is $y = \log_{10} x$.



Adjust the window settings on your choice of graphing technology so the lines on your graph appear similar to the one above.

- The lines are mirror images of each other, reflected in the line $y = x$.
- The exponential graph has a y-intercept at (0, 1) while the log graph has an x-intercept at (1, 0).
- The x- and y-coordinate values of any point in the line $y = 10^x$ are exchanged in the graph of the line $y = \log_{10} x$.
- The x-axis is the asymptote for the exponential graph, while the y-axis is the asymptote for the log graph.
- The exponential graph increases quickly for x-values greater than 0. The log graph increases quickly for values of x between 0 and 1, but the growth seems to taper off as the values of x increase.
- For the function $y = 10^x$, the domain is $\{x \mid x \in \Re\}$ and the range is $\{y \mid y > 0, y \in \Re\}$.
- For the function $y = \log_{10} x$, the domain is $\{x \mid x > 0, x \in \Re\}$ while the range is $\{y \mid y \in \Re\}$. Notice how these are switched.
- The end behaviour for the exponential graph extends from Quadrant II to Quadrant I. The end behaviour of the log graph is Quadrant IV to Quadrant I.

Logarithms of the Natural Exponential Function

As you have noted, when working with logarithmic functions, the default used in graphing technology is base 10. However, logarithms, just like exponential functions, can be used with different values for bases. For example, $7^x = 49$ can be written as $\log_7 (49) = x$, and $3^x = 19$ can be written as $\log_3 (19) = x$, where 7 and 3 are used as bases instead of the default base 10. It is still possible to use technology to calculate the value of x in these examples where the base is not 10, but it takes a little extra work and may involve changing bases, because calculators typically only work with logarithms in base 10 or base e.

Calculators are designed to use base e and base 10 because most applications require one of these two bases.

When you are solving exponential expressions using e (such as e^x), use **natural logarithms** instead of logs. Natural logarithms are denoted by ln, and the key LN is located below the LOG button on TI-83 calculators. The natural logarithm or ln is the inverse of the natural exponential function when the base is e.

$$y = \ln x \text{ is the inverse of } y = e^x$$

$\ln (y) = x$ is pronounced "the ell en of y is equal to x." It means x is the exponent needed on base e to produce the value y. This function is designated by the LN key on the calculator.

The exponential function $y = e^x$ can be rewritten in log form as $y = \log_e x$, but since e is a base that is regularly used, for ease and clarity it is denoted as $y = \ln (x)$.

Calculators are usually programmed to function with logarithms in base e and in base 10.

This information may be useful to add to your resource sheet for future reference.

Example 1

Solve for x:

$e^x = 20.09$

Solution

$e^x = 20.09$

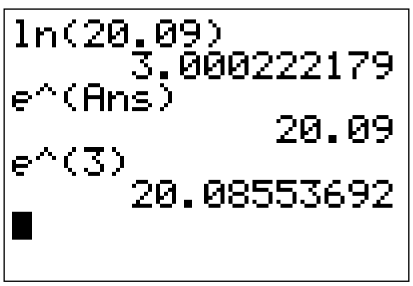
$\log_e(20.09) = x$

or

$\ln(20.09) = x$

$x = 3.000222179$

This means that 3 is approximately the exponent required on base e to produce the value of 20.09. If you use all the decimal places or the "last answer" function on your calculator when verifying your solution, the value will be closer to the original. Remember that e is an irrational value so all answers generated by technology will be approximate.



Example 2

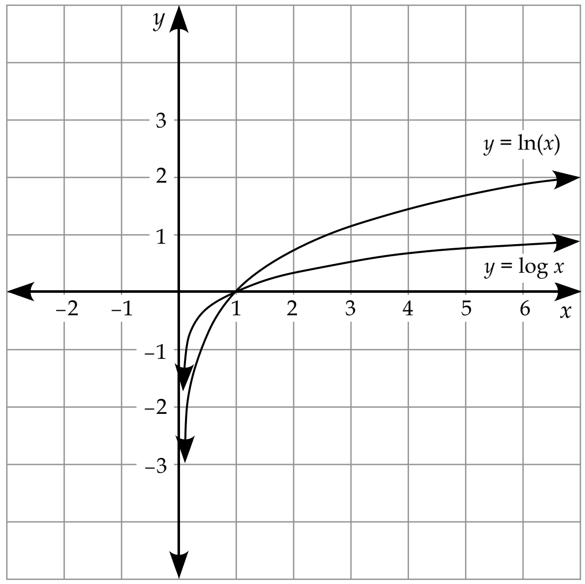
The natural exponential function uses base e, a value of approximately 2.718281828 . . .

Graph the natural logarithm ln (x) and compare it to the graph of the common log with base 10, log (x). Complete the chart describing its characteristics.

Function	Domain	Range	End Behaviour	x-intercept	y-intercept	Equation of Asymptote
$\ln(x) = y$						

Solution

The natural logarithm function is $y = \ln(x)$.



Compared to the logarithm with base 10, the graph of the natural logarithm, with its smaller value for a base, increases more quickly.

Function	Domain	Range	End Behaviour	x-intercept	y-intercept	Equation of Asymptote
$\ln(x) = y$	$\{x \mid x > 0, x \in \Re\}$	$\{y \mid y \in \Re\}$	IV to I	(1, 0)	Does not exist.	$x = 0$

Manipulating a Log Graph

Predict what will happen to the characteristics of the graph of an increasing logarithm function when the value of the multiplier a in the equation $y = a \log(x)$ is changed.

Example 1

Graph each of the following log functions in the form $y = a \log(x)$ on a grid and compare it with the graph of $y = \log(x)$.

a) $y = 5 \cdot \log(x)$

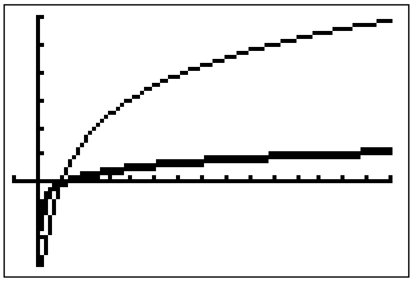
b) $y = \frac{1}{2} \cdot \log(x)$

c) $y = (-2)\log(x)$

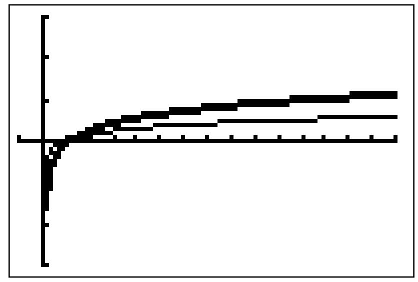
Solution

Since no base is written, each of these log functions is assumed to be base 10.

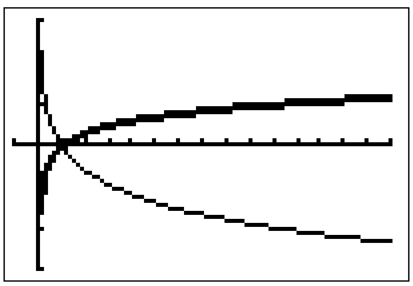
a) $y = 5 \cdot \log(x)$



b) $y = \frac{1}{2} \cdot \log(x)$



c) $y = (-2)\log(x)$



In each of the above graphs, the bold line represents the graph of $y = \log(x)$.

a) Multiplying the log function by a positive value of 5 stretches the graph vertically and it increases more rapidly.

b) Multiplying by a rational a-value between 0 and 1 compresses the graph towards the x-axis and it increases more gradually.

c) If the a-value is negative, the graph decreases and the end behaviour changes to Quadrant I to Quadrant IV.

This information may be useful to add to your resource sheet for future reference.

In the learning activity below, you will have an opportunity to explore what happens when a constant is included in the equation.

Solving Exponential Functions Using Logarithms: Review

You have solved exponential equations by changing the terms to common bases and by graphing both sides of the equation and finding the point of intersection.

Now that you understand that the inverse of an exponential function is a logarithmic function, you can rewrite and use technology to solve exponential equations using base 10 and base *e*.

Example 1: Graphic solution of exponential functions using logarithms

Graph $y = \log(x)$ or $y = \ln(x)$ to solve:

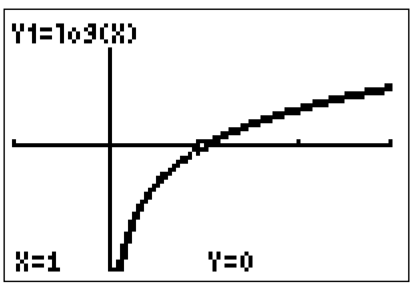
a) $10^x = 2$

b) $e^x = \frac{1}{4}$

Solution

a) $10^x = 2$

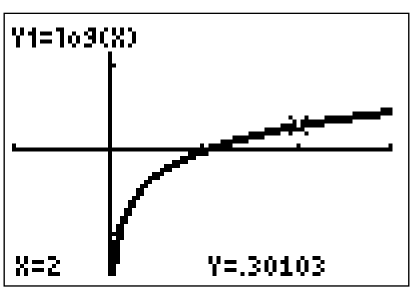
Write as a common logarithm and graph $y = \log(x)$.



In the graph of the base 10 log function $y = \log(x)$, each (x, y) coordinate point on the line tells you what value the exponent, y, on base 10 produces.

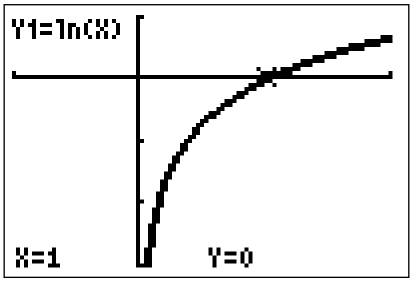
Solve for $x = 2$.

The point (2, 0.30103) means that $10^{0.30103}$ is equal to 2.

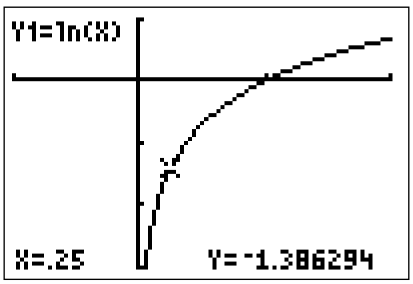


b) $e^x = \frac{1}{4}$

Write as a natural logarithm and graph $y = \ln(x)$.



Solve for $x = \frac{1}{4}$.



(0.25, –1.386294) means $e^{-1.386294} = \frac{1}{4}$.

You can now find the solution to exponential functions without graphing two equations and finding the intersection as you did in the previous lesson.

In fact, you can solve these equations without graphing at all!

Example 2: Calculating the solution of exponential functions using logarithms

Solve:

a) $10^x = 2$

b) $e^x = \frac{1}{4}$

Solution

Use your calculator to determine the exponent required on base 10 to produce the value 2 using logarithms.

a) $10^x = 2$

Write as a logarithm and solve

$\log(2) = x$

$x = 0.3010299957$

```
log(2)
     .3010299957
```

You can verify this by raising the base 10 to the power of x you just found.

```
log(2)
     .3010299957
10^.3010299957
               2
```

$10^{0.3010299957} \approx 2$

b) $e^x = \dfrac{1}{4}$

Write as a natural logarithm and solve.

$\ln(0.25) = x$

$x = -1.386294361$

Verify.

```
ln(.25)
    -1.386294361
e^(-1.386294361)

             .25
```

Remember, the value the calculator finds is not actually an exact answer. It is an irrational number—a non-terminating, non-repeating decimal—but it gives the approximate value.

Matching Logarithmic Graphs and Equations

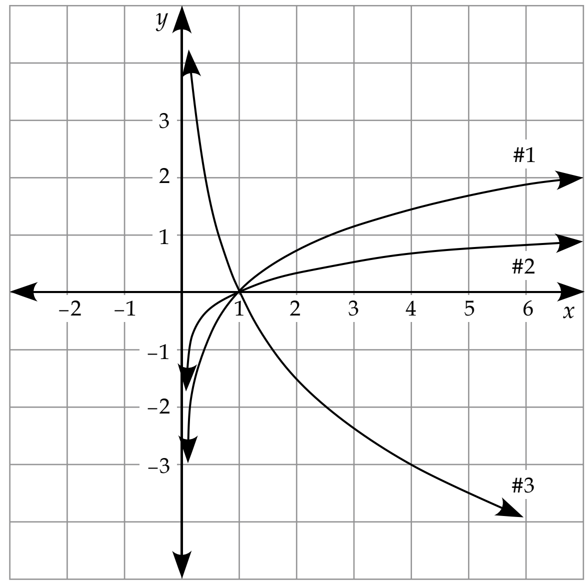
Using what you know about the characteristics of log functions and their corresponding graphs, match each of the following equations with the appropriate graph.

Example 1

$y = -5 \cdot \log(x)$

$y = \ln(x)$

$y = \log(x)$



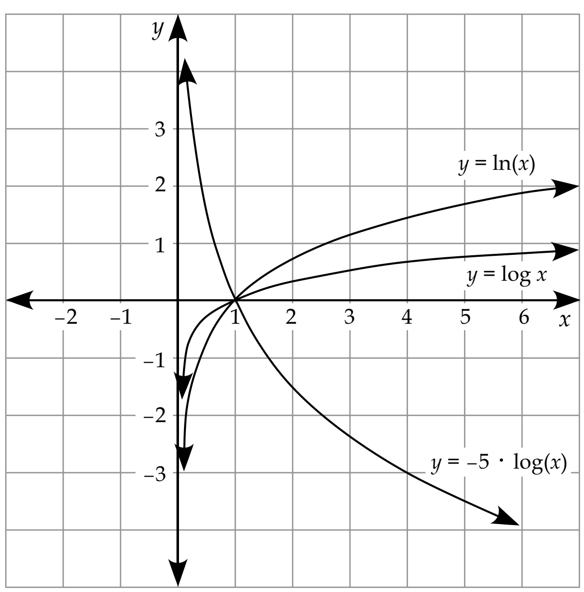
Solution

Graphs #1 and #2 are increasing, while #3 is decreasing.

Graph #3 must have a negative multiplier, so it is the graph of $y = -5 \cdot \log(x)$.

Graph #1 increases more quickly than #2, so it must have a smaller value for a base. $e \approx 2.718281828\ldots$, which is smaller than 10, the base of the common log, so #1 must be $y = \ln(x)$. Also note, when $y = 1$ in a logarithmic function without a multiplier, the value of x is the base.

Graph #2 is $y = \log(x)$.



Learning Activity 1.5

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. The vertex of a parabola with a positive leading coefficient is at (2, 0). How many zeros does the function have?
2. The area of a triangle is 24 m^2. Find the length of the base if the height is 12 m.
3. State the next six numbers in this pattern: 1, 4, 9, 16, . . .
4. An airline overbooks its flights by 10%. Each flight can carry 130 passengers. How many bookings will they accept for a flight?
5. A cube has a volume of 8 $ft.^3$. What are the dimensions of the cube?
6. Convert 30 inches to feet.
7. Calculate a 15% tip on a restaurant bill of $58.
8. Solve for x by finding a common base: $64^x = \sqrt{2}$

Part B: Logarithm Functions

Remember, these questions are similar to the ones that will be on your assignments and midterm examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and midterm examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Express each of the following exponential equations as a logarithm:
   a) $10^x = 5$
   b) $6^3 = 216$
   c) $2^8 = y$
   d) $e^x = 512$

continued

Learning Activity 1.5 (continued)

2. Graph each of the functions and complete the following table.

Function	Domain	Range	End Behaviour	x-intercept	y-intercept	Equation of Asymptote
$y = 10 \log (x)$						
$y = \log (x)$						
$y = -3 \ln (x)$						
$y = \ln (x)$						

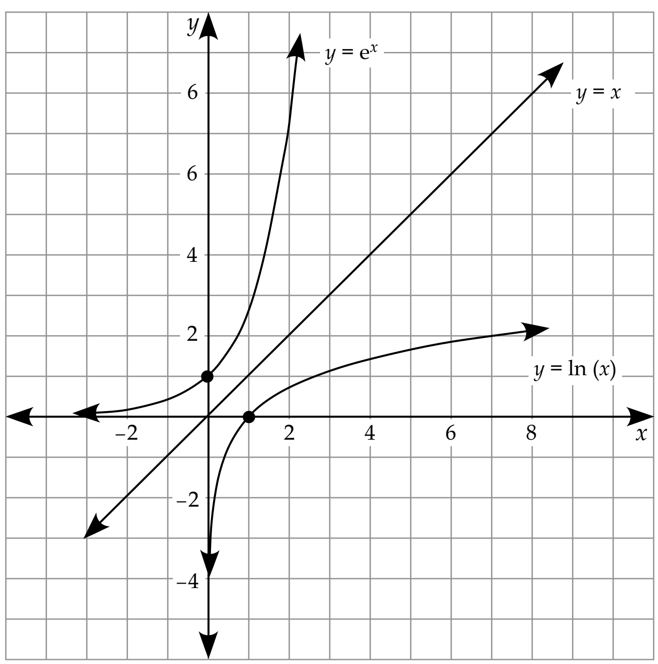
3. Graph each of the following functions. For each, find the value of y when $x = 1$. Summarize the impact the constant has on the characteristics of the logarithm.

   $y = \log (x) + 5$

   $y = \log (x) - 3$

   $y = \ln (x) + 1$

4. Describe the graphs of the functions $y = e^x$ and $y = \ln (x)$ and explain how they are related.



Lesson Summary

In this lesson, you learned about the inverse of an exponential function and used a new notation to express logarithms. You compared the characteristics of exponential and logarithmic functions and graphed them using technology. You learned how to solve common logarithm and natural logarithm equations. You matched equations and graphs in a set.

Notes

Lesson 6: Using Log Functions to Model Data

Lesson Focus

In this lesson, you will

- ❑ graph data and determine the exponential or logarithmic function that best represents it
- ❑ interpret the graph of a function that models a situation and explain your reasoning
- ❑ solve contextual problems using the graphs of exponential and logarithmic functions

Lesson Introduction

There are many situations where logarithmic functions are used to model data. For example, a decibel is a logarithmic unit used to measure sound pressure and voltage ratios, pH is a log scale of acidity, and the Richter scale is a base 10 log scale that describes the amount of energy released during an earthquake.

Logarithmic Models

The basic shape of a logarithm graph suggests that it would approximate a situation that initially has rapid growth, but later increases more slowly.

The logarithmic regression equation used in this course will be the natural logarithm with base e, given as $y = a \cdot \ln(x) + b$, where a is a multiplier and b is a constant. Please take careful note of the form of the equation used by the technology you choose.

Example 1

The average monthly income of a successful business, recorded over a period of 16 years, follows a logarithmic pattern.

Year	2	4	7	9	12	16
Average Monthly Income	\$3000	\$5100	\$6800	\$7600	\$8400	\$9300

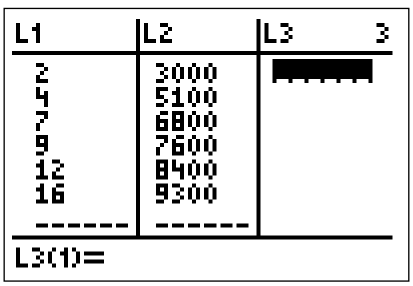
a) Find the natural logarithm regression equation that best models the data.

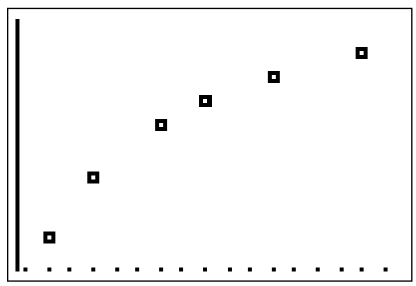
b) Use the equation to find what the average monthly income was in year 1.

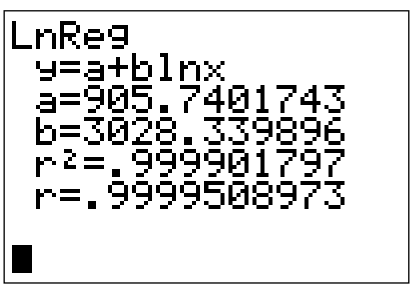
c) After how long can the business expect to earn an average monthly income of \$12,000?

Solution

Refer to the Technology Appendix or the help files associated with your choice of technology if you need help in creating a natural log regression equation.

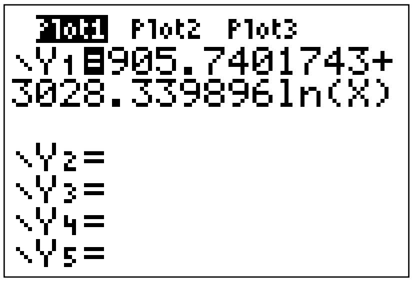


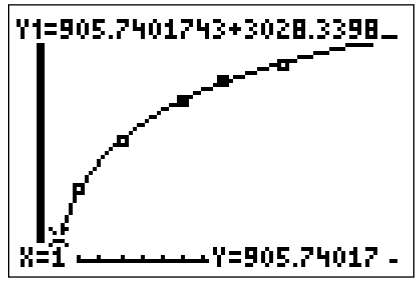




a) The natural logarithm function that models this data is
$y = 3028.339896 \ln (x) + 905.740174$.

b) The average monthly income in year one was $905.74.



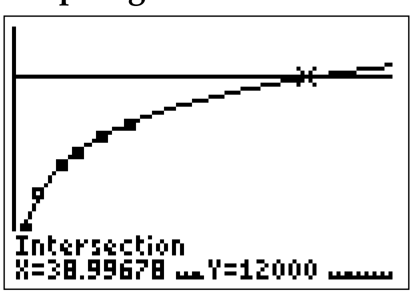


c) The business may expect to earn an average monthly income of $12,000 after about 39 years. You can find this solution using algebra and the regression equation or by graphing the line $y = 12000$ and finding its intersection with the natural log function. You will have to adjust your window settings to include the point.

Algebraic Method:

$$\begin{aligned}
y &= 3028.339896\ \ln(x) + 905.740174 \\
12000 &= 3028.339896\ \ln(x) + 905.740174 \\
12000 - 905.740174 &= 3028.339896\ \ln(x) \\
\frac{11094.25983}{3028.339896} &= \ln(x) \\
3.66347907 &= \ln(x) \quad \leftarrow \text{Write as a natural exponential function.} \\
e^{3.66347907} &= x \\
38.99677967 &= x
\end{aligned}$$

Graphing Method:



Example 2

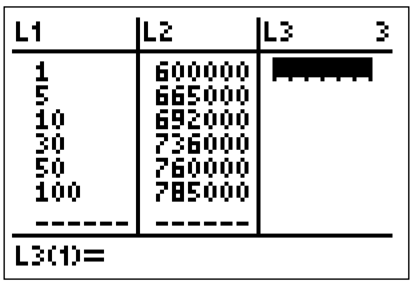
The number of people that have a particular smartphone after it has been released for a certain number of days is given in the table below.

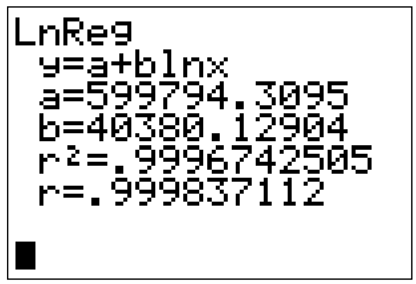
# of days	1	5	10	30	50	100
# of phone users	600 000	665 000	692 000	736 000	760 000	785 000

After how many days will there be 800 000 owners?

Solution

Graph the data to find the natural logarithm function that approximates this data. Use the equation or the graph to find the answer to the question.





Algebraic Method:

$$y = 40380 \ln x + 599794$$

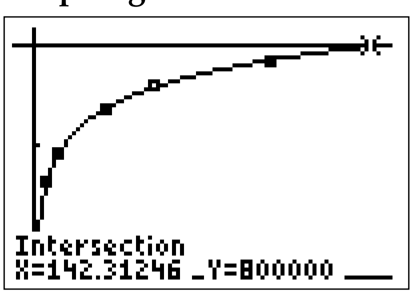
$$800000 = 40380 \ln x + 599794$$

$$\frac{800000 - 599794}{40380} = \ln x$$

$$e^{\left(\frac{800000-599794}{40380}\right)} = x$$

$$x = 142.316$$

Graphing Method:

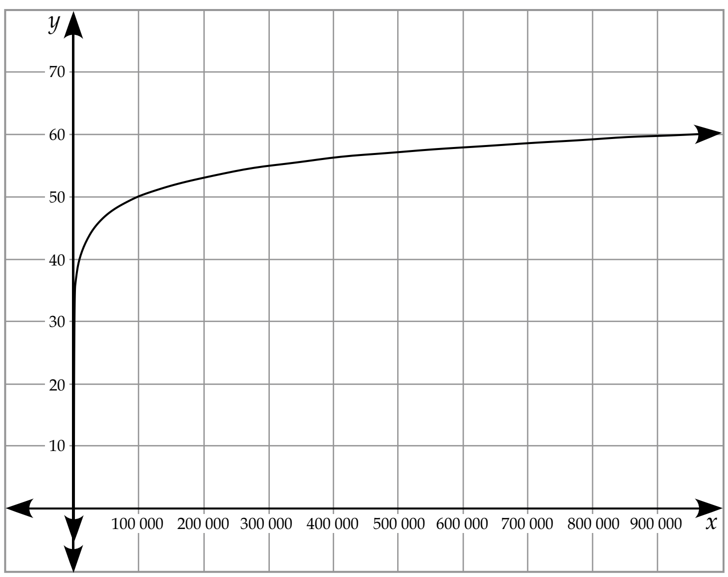


It would take approximately 142 days for the total number of smartphone owners to reach 800 000.

Example 3

The sensation of sound is a common logarithmic function (base 10) measured in bels, a unit named after Alexander Graham Bell. A sound that is 1 bel louder than another sound is actually 10 times more intense. A sound that is 100 times (or 10^2) as intense as another sound is 2 bels louder, and so on. Since this unit is so large, a decibel, (dB) one-tenth of a bel is usually used. The loudness of a sound, L, measured in decibels, is defined by the formula, $L = 10 \log \frac{I}{I_0}$, where I_0 is the minimum sound intensity your ear can detect, given as watts per square centimetre (W/cm^2). For humans, the minimum sound intensity that can be detected is 10^{-16} w/cm^2, so $I_0 = 10^{-16}$. That gives the formula $L = 10 \log \frac{I}{10^{-16}}$.

Let $x = \frac{I}{I_0}$, then graph $y = 10 \log (x)$.



a) Find the number of decibels of a whisper measured at an intensity, I, of 10^{-13} W/cm^2.

b) The loudness of normal conversation is typically around 60 dB. How intense is the power of this sound?

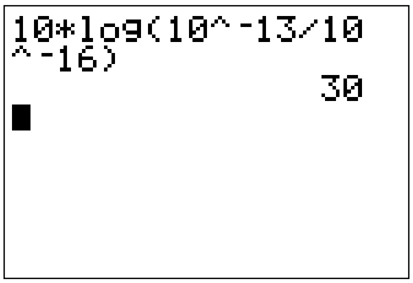
c) A mosquito buzz is said to be about 40 dB. How many times more intense than this is normal conversation?

Solution

a) Find the number of decibels of a whisper measured at an intensity, I, of 10^{-13} W/cm^2.

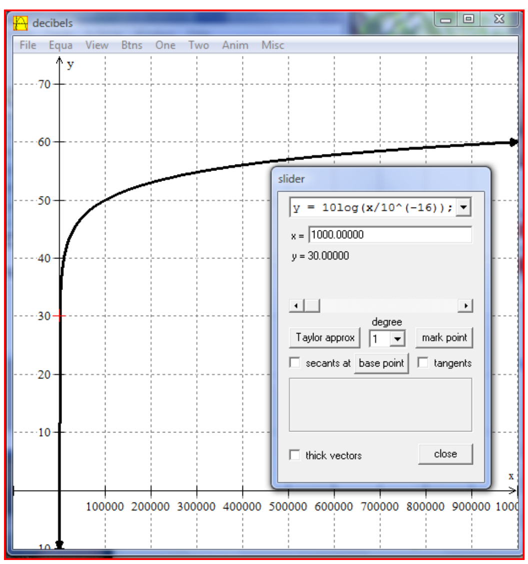
$$L = 10 \log \frac{I}{I_0}$$

$$L = 10 \log \frac{10^{-13}}{10^{-16}}$$



A whisper is 30 dB.

A graphing solution (shown here using Winplot) can be found by solving for $x = \frac{I}{I_0} = \frac{10^{-13}}{10^{-16}}$.



When $x = \frac{10^{-13}}{10^{-16}} = 1000$, then $y = 30$ dB.

b) The loudness of normal conversation is typically around 60 dB. How intense is the power of this sound?

Algebraic Method:

$$L = 10 \log \frac{I}{I_o}$$

$$60 = 10 \log \frac{I}{10^{-16}}$$

$$\frac{60}{10} = \frac{\cancel{10}}{\cancel{10}} \log \frac{I}{10^{-16}}$$

$$6 = \log\left(\frac{I}{10^{-16}}\right)$$

Common log is base 10.
Rewrite log as exponential expression.

$$10^6 = \frac{I}{10^{-16}}$$

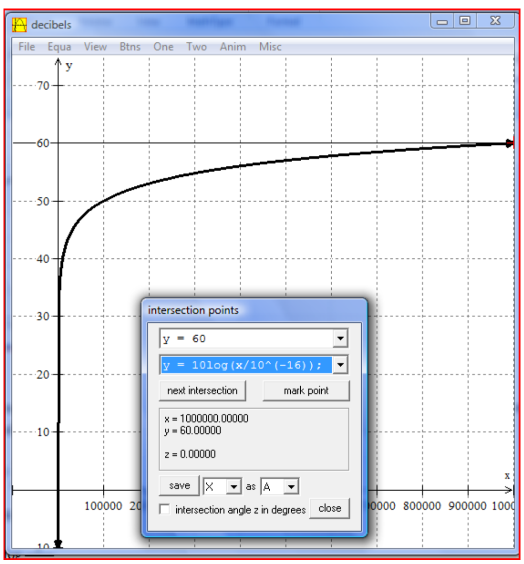
$$10^6 \cdot 10^{-16} = I$$

$$I = 10^{-10}$$

The intensity of normal conversation is about 10^{-10} W/cm^2.

Graphing Method:

On the graph, when $y = 60$, x is 1 000 000.



$$\frac{I}{I_0} = 1\,000\,000$$

$$\frac{I}{10^{-16}} = 10^6$$

$$I = 10^6 \times 10^{-16}$$

$$I = 10^{-10}$$

The intensity is 10^{-10} W/cm^2.

c) A mosquito buzz is said to be about 40 dB. How many times more intense than this is normal conversation?

The intensity of normal conversation is 60 dB, or 2 bels higher than a mosquito buzzing. This means it is 10^2 or 100 times more intense. This can be verified by finding the I-value ratios as well.

$$40 = 10 \log \frac{I}{10^{-16}}$$

$$\frac{40}{10} = \frac{\cancel{10}}{\cancel{10}} \log \frac{I}{10^{-16}}$$

$$4 = \log \left(\frac{I}{10^{-16}}\right)$$

Common log is base 10.
Rewrite log as exponential expression.

$$10^4 = \frac{I}{10^{-16}}$$

$$10^4 \cdot 10^{-16} = I$$

$$I = 10^{-12}$$

Intensity of mosquito buzz.

$$\text{Ratio: } \frac{\text{intensity of conversation}}{\text{intensity of mosquito buzz}} = \frac{10^{-10}}{10^{-12}} = 100$$

Example 4

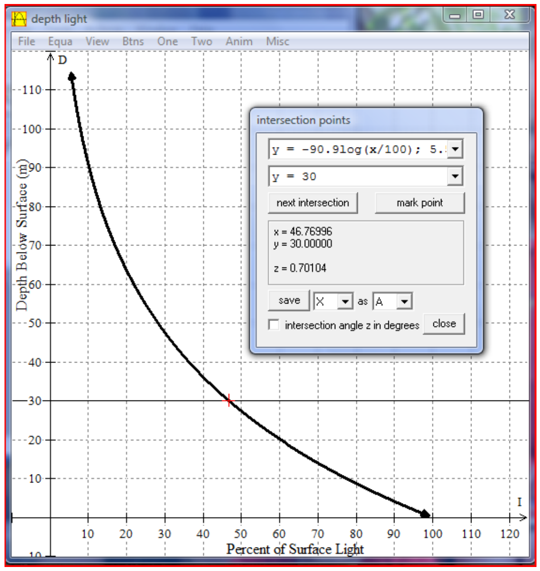
The deeper a diver descends below the surface of the water, the darker it gets. A diver can determine the current depth she is at by measuring the intensity of light, using the formula $D = -90.9 \log \left(\frac{I}{100}\right)$, where D is depth below the surface in metres and I is the percentage of surface light intensity.

a) Graph this function and use it to determine the intensity of the light at a depth of 30 m.

b) How deep is she if there is 25% light intensity?

Solution

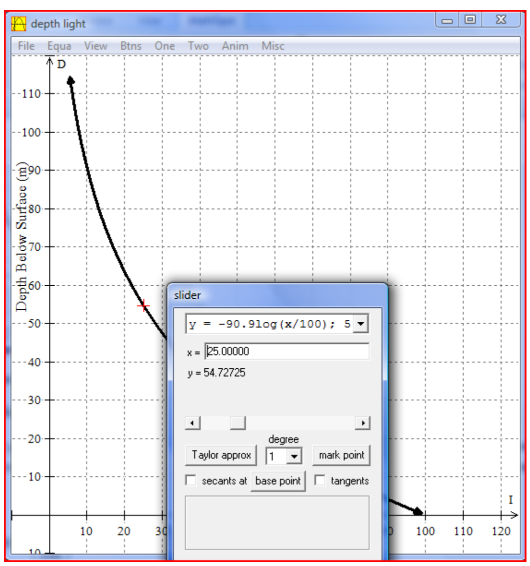
a) Graph this function and use it to determine the intensity of the light at a depth of 30 m.



At 30 m below the surface, there is about 46.8% of surface light.

b) How deep is she if there is 25% light intensity?

From the graph above, it appears that 25% of light remains at a depth of 55 m. You can use the graph to solve for x = 25. Use the formula to verify the result.



$$D = -90.9 \log\left(\frac{I}{100}\right)$$

$$D = -90.9 \log\left(\frac{25}{100}\right)$$

$$D = 54.7$$

The interpolation from the graph was very close to the calculated value.

You may want to write down the types of applications for the logarithm function on your resource sheet for future reference.

Learning Activity 1.6

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Solve for x: $\log(1) = x$
2. Solve: $\ln(e) = x$
3. Solve for x by finding a common base: $100^x = 0.001$
4. Write as a logarithmic function: $10^{1.2} \approx 15.85$
5. Write as a natural exponential function: $\ln(25) \approx 3.22$
6. The population of a town doubles every 5 years. In 2010, the population was 800. What was the population in the year 2000?
7. What is the perimeter of a square with an area of 36 m^2?
8. Estimate the amount of tax you would pay on a purchase of $800 if you must pay 5% GST and 8% PST.

continued

Part B: Applications of Logarithm Functions

Remember, these questions are similar to the ones that will be on your assignments and midterm examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and midterm examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. The growth of Jack's magic bean plant was recorded by government spy agency satellite cameras and its height over time was recorded in the table below.

Hours	2	3	5	8	10
Height (ft.)	952	1510	2212	2858	3165

   a) Find the natural logarithmic regression equation that best models the data.
   b) Use the equation or graph to find how tall the plant was 12 hours after planting.
   c) After how many hours will the plant reach a height of one mile? (5280 ft. = 1 mile)
   d) Find the x-intercept and explain its significance.

2. The number of friends someone has on a social networking website is said to fit a logarithmic model. Consider the data below for a recent new user of the website.

# Days	1	4	9	15	25
Friends	2	19	28	35	41

   a) Find the logarithmic regression equation that best models the data.
   b) Use the equation or graph to find how many friends this person will have after 2 months.
   c) How long will it take until this person has 65 online friends?

continued

3. The pH of a substance, its acidity or alkalinity, is determined by the hydrogen ion concentration, $[H^+]$ in moles per litre, according to the formula $pH = -\log [H^+]$. Water has a neutral pH of 7. Acids have a pH less than 7 and basic (alkaline) solutions have a pH greater than 7.
   a) Orange juice has a hydrogen ion concentration of about 2.8×10^{-4} moles per litre. What is its pH?
   b) The pH of ammonia is 8.9. Determine its hydrogen ion concentration.
4. The Richter Scale expresses the magnitude of an earthquake, R, by measuring the amount of energy E, that is released given in joules. It is expressed by the formula $R = \frac{2}{3} \log \frac{E}{10^{4.8}}$.

   Since the magnitude is based on a log scale of base 10, when the magnitude of an earthquake is increased by 1 unit on the Richter Scale, its energy is increased by approximately 10 times.
   a) Calculate the magnitude, R, of an earthquake that releases 1.778279×10^8 joules of energy.
   b) How much energy is released by a 4.5 magnitude earthquake?
   c) How many times more energy is released by a 4.5 magnitude earthquake compared to a magnitude 2.3?

Lesson Summary

In this lesson, you graphed logarithmic equations, plotted data, and found the natural logarithm regression equation that best modelled the situation. You used the graph and/or equation of the function to interpret the situation and solve contextual problems.

Assignment 1.2

Logarithmic Function Models

Total: 37 marks

This is a hand-in assignment. Clearly show the steps in your solutions on the question sheets below and submit these pages when you send in your assignments for marking. If you use a graphing calculator, you may state your keystrokes, include a sketch of what is on your screen, or connect to a computer and print out your screen captures. If you use online graphing tools, state the website address, state what values or equations you used, and sketch or print the graphs.

Final answers must include units. Answers given without supporting calculations and graphs will not be awarded full marks.

1. The population of a city in Manitoba is recorded for 20 years and the data is provided in the table below, where 1981 is year 1:

Year	1981	1986	1989	1992	1997	2000
Population	69180	65850	64060	63590	62930	62520

   a) The decline in population is said to follow a logarithmic model where $x = 1$ represents the year 1981, $x = 20$ represents the year 2000. Graph the data and find the natural logarithmic regression equation that best models the situation. *(5 marks)*

b) Use the graph or equation to determine what the projected population of this city will be in 2015. *(2 marks)*

2. A 50 g sample of a radioactive substance is said to have a half-life of 18 days.

   a) Complete the table of values showing the number of grams of the substance remaining after each of 5 half-lives. *(3 marks)*

Half-Life	**# of days**	**Grams Remaining**
1		
2		
3		
4		
5		

b) Use technology to find an exponential regression equation that models the number of grams remaining, y, after x number of days. *(3 marks)*

c) Use the graph to determine after how many days there will be less than 5 g left. *(1 mark)*

3. Hot coffee is poured into a cup and left standing on a table. The temperature of the coffee is taken every minute and recorded.

t (in minutes)	1	2	3	4	5	6	7	8	9	10
T (in °C)	92.4	89.6	87.4	84.8	83.1	80.9	78.8	76.6	74.4	73

   a) Graph the data. Include a printout, screenshot, or a sketch of your graph with labels. *(3 marks)*

Assignment 1.2: Logarithmic Function Models (continued)

b) Use technology to determine each of the following: a quadratic, a cubic, an exponential, and a logarithmic regression equation for the data. Include a labelled sketch of each regression equation with a domain of 0 to 60 minutes and a range of 0 to 100 degrees. *(8 marks)*

c) Based on this specific situation presented and the characteristics of the functions studied in this course, indicate which regression equation you believe is the best representation of the data compared to the other functions, and explain your reasoning. Your explanation should include your analysis of how well each graph represents the data, and how appropriate each equation model is to the data represented by the cooling coffee. *(6 marks)*

Assignment 1.2: Logarithmic Function Models (continued)

4. Match the graph with the correct equation. Write the letter of the graph on the line beside the equation. *(6 marks)*

___ $f(x) = \left(\frac{2}{7}\right)^x + 8$

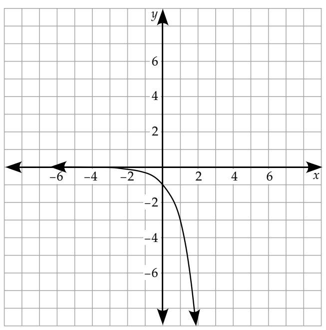
___ $g(x) = 20^x$

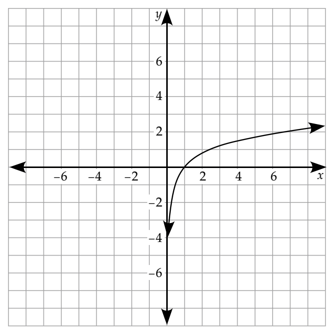
___ $h(x) = -\pi^x$

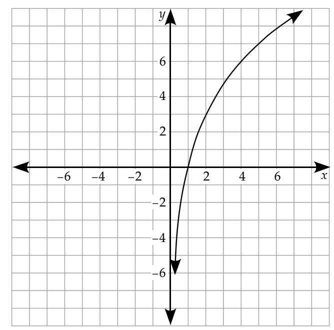
___ $k(x) = 2^x$

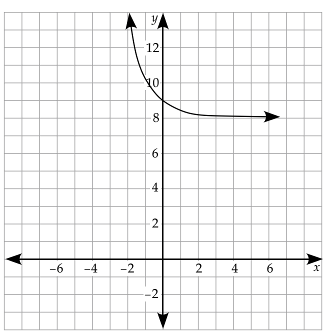
___ $t(x) = \ln x$

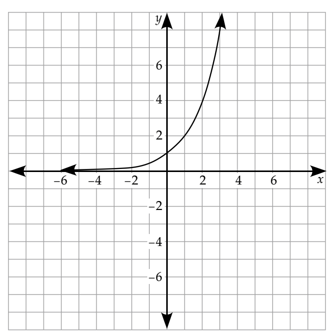
___ $d(x) = 10 \log(x)$

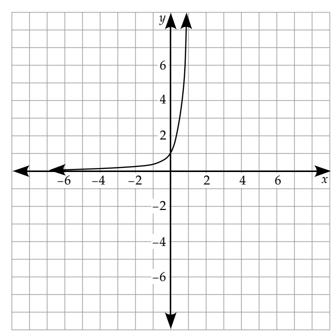
a) 

b) 

c) 

d) 

e) 

f) 

Notes

Module 1 Summary

Congratulations, you have finished the first module in the course! In this module, you learned about the characteristics of polynomial, exponential, and logarithmic functions by analyzing their graphs and equations. Using these characteristics, you matched graphs and equations. You represented data and found the regression equation that best fit the data, and then used that function's graph and/or equation to interpret the situation and solve contextual problems.

In the next module, you will be learning to use mathematics to complete a research project.

Submitting Your Assignments

It is now time for you to submit the Module 1 Cover Assignment and Assignments 1.1 and 1.2 to the Distance Learning Unit so that you can receive some feedback on how you are doing in this course. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 1 assignments and organize your material in the following order:

- ❑ Module 1 Cover Sheet (found at the end of the course Introduction)
- ❑ Cover Assignment: Number Puzzle
- ❑ Assignment 1.1: Polynomial Functions
- ❑ Assignment 1.2: Logarithmic Function Models

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Notes

Grade 12 Applied Mathematics (40S)

Module 1
Functions

Learning Activity Answer Keys

Module 1: Functions

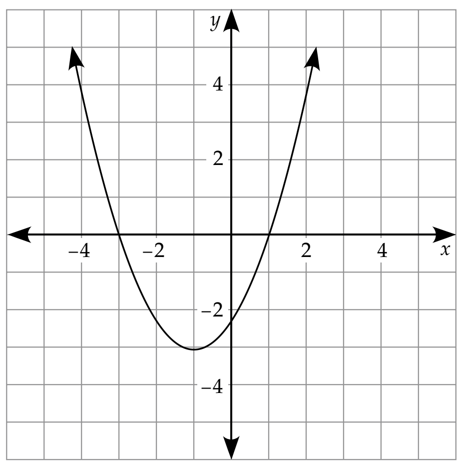
Learning Activity 1.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Factor: $x^2 - 7x + 10$
2. State the roots of the following quadratic function: $f(x) = x^2 + 5x + 6$
3. The vertex of a quadratic function is at (−5, 31). State the equation of the axis of symmetry.

Complete the following, based on the graph below.



4. State the domain.
5. State the range.
6. State the coordinates of the vertex.
7. State the value of the x-intercepts.
8. State the value of x at the y-intercept.

Answers:

1. $(x - 5)(x - 2)$ (Look for two numbers with a product of 10 and a sum of −7; they are −5 and −2.)
2. −3 and −2 (Factored form is $(x + 3)(x + 2)$, so the roots are −3 and −2.)
3. $x = -5$ (The axis of symmetry goes through the x-coordinate of the vertex.)

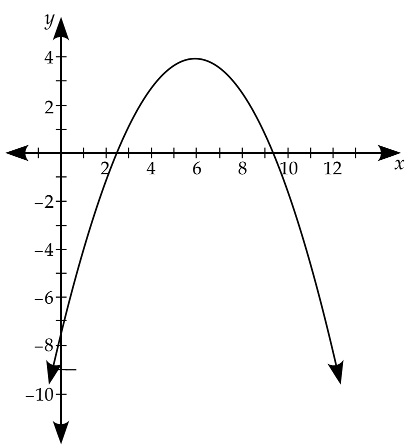
4. $\{x \mid x \in \Re\}$
5. $\{y \mid y \geq -3, y \in \Re\}$
6. $(-1, -3)$
7. The zeros are at -3 and 1.
8. At the y-intercept, the value of $x = 0$.

Part B: Characteristics of Polynomial Functions

Remember, these questions are similar to the ones that will be on your assignments and midterm examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and midterm examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Given the following graphs of polynomial functions, complete the chart with the required information.

a)



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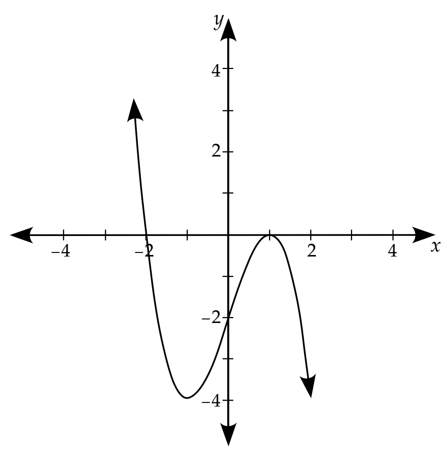
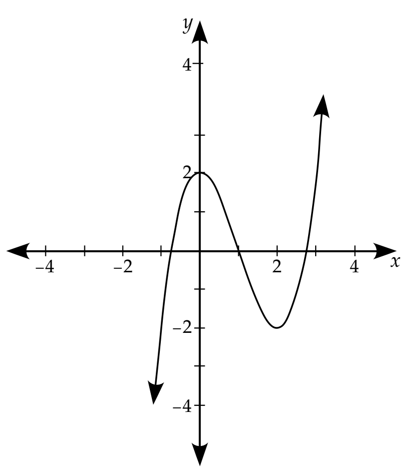


c)



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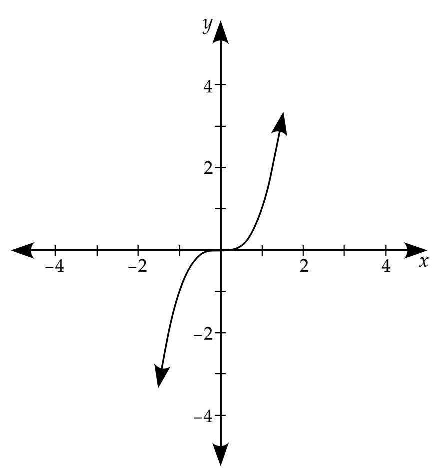


Answer:

Graph	A	B	C	D
Type of Function	Quadratic	Cubic	Cubic	Cubic
Degree	2	3	3	3
# of x-intercept(s)	2	2	3	1
# of y-intercept(s)	1	1	1	1
End behaviour	III to IV	II to IV	III to I	III to I
Absolute or relative maximum or minimum	Absolute maximum	Relative maximum and minimum	Relative maximum and minimum	No absolute or relative maximum or minimum
Domain	$\{x \mid x \in \Re\}$	$\{x \mid x \in \Re\}$	$\{x \mid x \in \Re\}$	$\{x \mid x \in \Re\}$
Range	$\{y \mid y \leq 4, y \in \Re\}$	$\{y \mid y \in \Re\}$	$\{y \mid y \in \Re\}$	$\{y \mid y \in \Re\}$

2. What is the connection between the degree of a function and the maximum possible number of x-intercepts?

   Answer:

   A function of degree n will have at most n real roots. A quadratic function of degree two may have 0, 1, or 2 roots. A cubic function with degree three may have 1, 2, or 3 roots.

3. What is the connection between the degree of a function and the maximum possible number of turning points?

   Answer:

   A function of degree n will have at most $(n - 1)$ turning points. A linear function, degree 1, has no turning points; a quadratic function, degree 2, has one turning point; and a cubic function, degree 3, may have zero or two turning points.

4. Will a polynomial function always have a y-intercept?

   Answer:

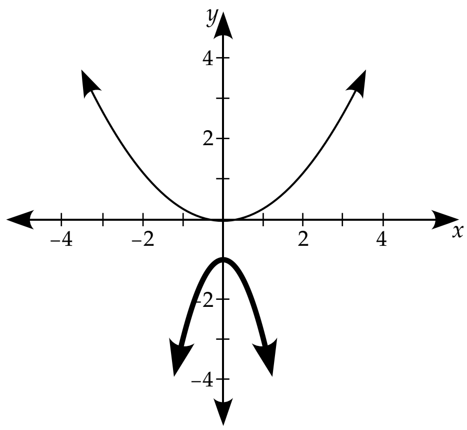
   Yes. If the graph of the function is drawn with a continuous line that extends towards negative and positive infinity, it will at some point cross the y-axis. However, depending on the function, it may or may not cross the x-axis.

5. How is the end behaviour of a function related to it having an odd or even degree?

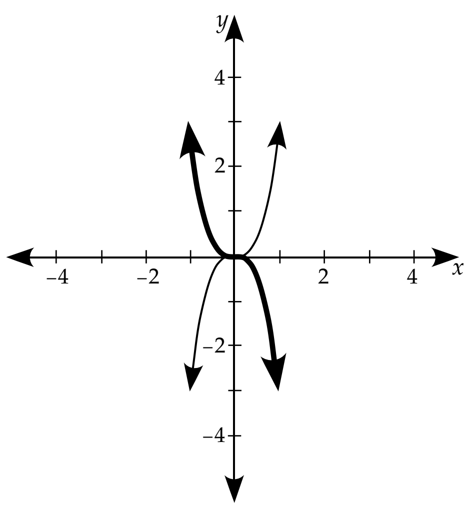
Answer:

Quadratic functions are degree 2. If the degree is even and the leading term has a positive sign, the line will extend from Quadrant II to Quadrant I. That is, it will have similar behaviour at the ends. It will open up. It will rise to the left and right.

If the degree is even and the leading term has a negative sign, the line will extend from Quadrant III to Quadrant IV. That is, it will open down or fall to the left and the right.



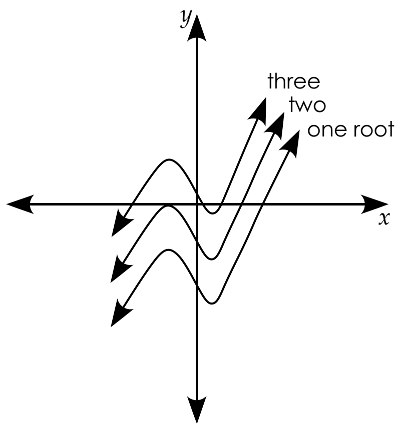
Linear functions are degree 1 and cubic functions are degree 3. If the degree is odd and the leading term has a positive sign, the line will extend from Quadrant III to Quadrant I. That is, it will have opposite behaviour at the ends. It will fall to the left and rise to the right. If the degree is odd and the leading term has a negative sign, the end behaviour of the line will be from Quadrant II to Quadrant IV. It will rise to the left and fall to the right.



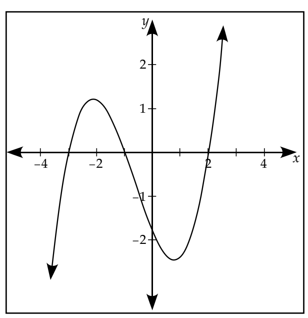
6. Use a sketch to show how changing the value of the constant in a cubic equation can affect the number of x-intercepts it has.

   Answer:

   Changing the constant in a polynomial equation shifts the graph vertically and changes the y-intercept of the graph. Moving the line up or down will change where the relative maximum and minimum are located and, as a result, will affect where the line crosses the x-axis, and it may also affect how many times the line crosses the x-axis.



7. Match each of the following graphs with the equation that describes it.

a)


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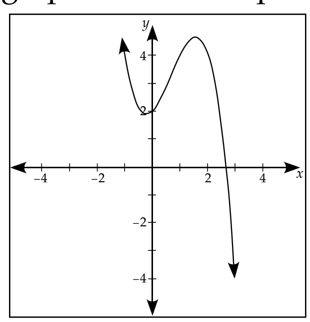

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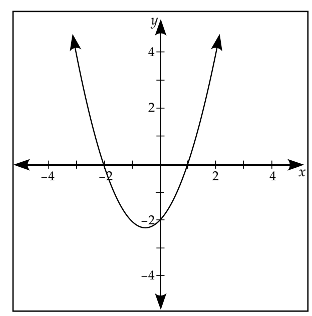

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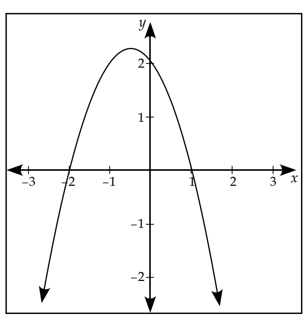

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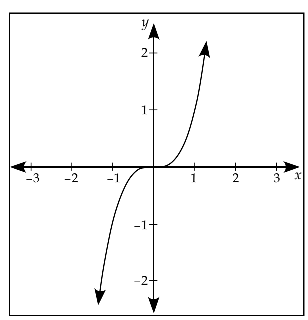

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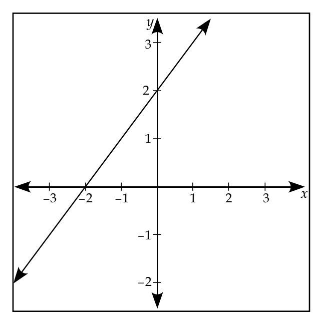


(e) $f(x) = x^3$

(c) $g(x) = x^2 + x - 2$

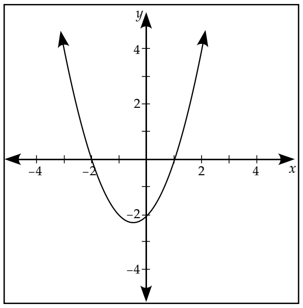
(b) $h(x) = -x^3 + 2x^2 + x + 2$

(a) $m(x) = 0.3\left(x^3 + 2x^2 - 5x - 6\right)$

(f) $t(x) = x + 2$

(d) $w(x) = -x^2 - x + 2$

8. Match a polynomial function description to one of the numbered graphs. There are some descriptions that will match with more than one graph, but in the end each graph should be matched to a different description.

a)


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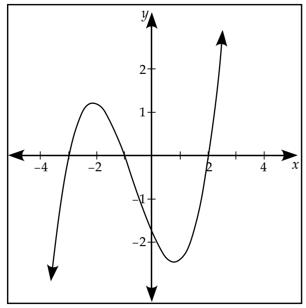
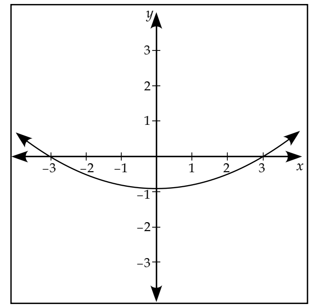


c)


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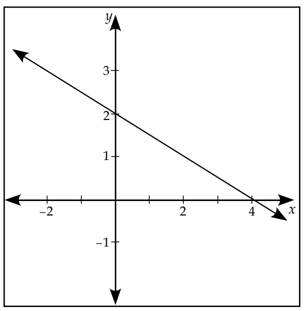

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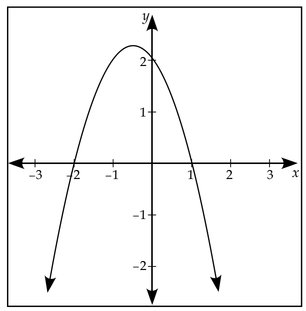

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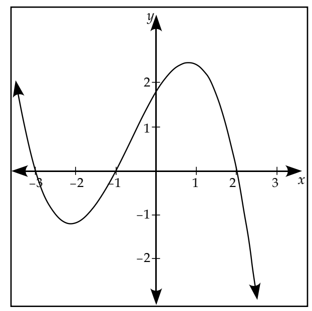

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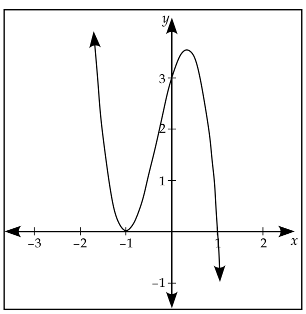
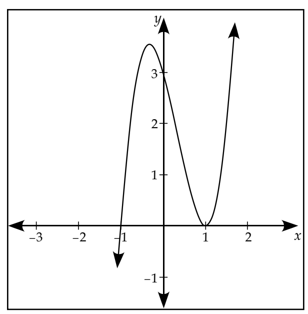
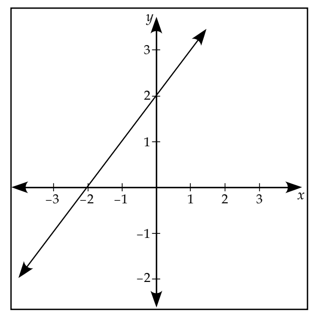


h)


i)


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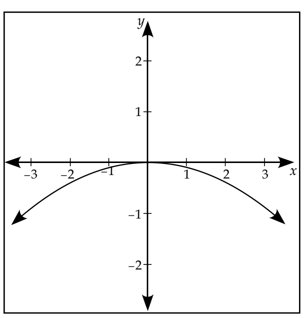

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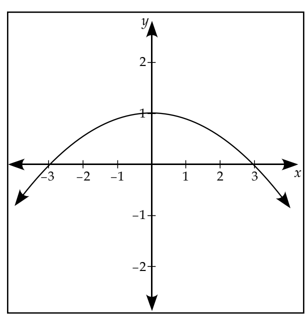

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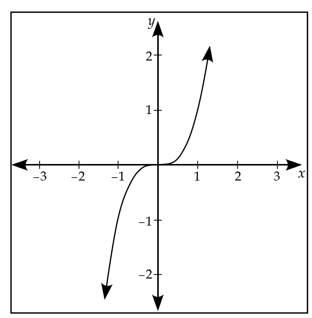


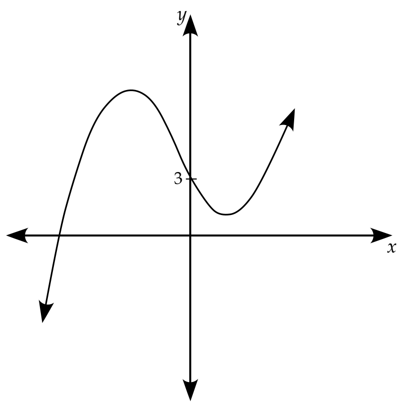
Answer:

(f)	i)	This is the graph of a cubic function. The leading coefficient is negative.
(k)	ii)	This polynomial of degree 2 has two x-intercepts and the axis of symmetry is the y-axis.
(b)	iii)	This is the graph of a cubic function with x-intercepts at −3, −1, and 2.
(h)	iv)	This function has roots at 1 and −1.
(a)	v)	This is the graph of a quadratic function with a minimum function value.
(j)	vi)	This quadratic function has a maximum of zero.
(c)	vii)	This quadratic function has a positive leading coefficient much less than one.
(i)	viii)	This linear function has a positive slope.
(d)	ix)	The graph of this function has a constant rate of change that is between 0 and −1.
(g)	x)	The graph of this cubic function has two x-intercepts and a negative leading coefficient.
(e)	xi)	This is the graph of a parabola with a y-intercept of 2.
(l)	xii)	This function has no turning points and one x-intercept.

9. Sketch the graph of a possible polynomial function for each of the following sets of characteristics. What can you conclude about the equation of the function with these characteristics?

   a) A relative maximum in Quadrant II, a relative minimum in Quadrant I, a y-intercept of 3.

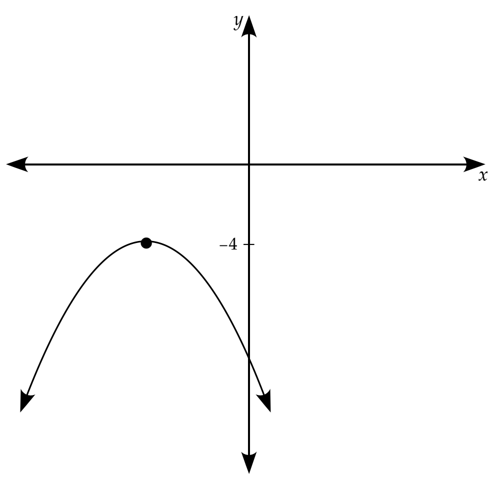
   Answer:



   In order to have two turning points, you can conclude the equation must be degree three or cubic. It must have a positive leading coefficient in order to have the stated end behaviour. The constant must be 3 to have a y-intercept of 3. Other solutions are possible.

   b) Range $\{y \mid y \leq -4, y \in \Re\}$, one turning point in Quadrant III.

   Answer:



   A quadratic function has one turning point. If the range is less than or equal to −4, it must have a maximum value of −4 somewhere in Quadrant III. This means it has a negative leading coefficient. The end behaviour will be from Quadrant III to Quadrant IV. The constant will be a negative number. Other solutions are possible.

Learning Activity 1.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

For questions 1 to 4, simplify the given expression using the exponent laws. Write your answers using positive exponents and the base given.

1. $(2^3)^4$
2. $3^5 \times 3^2$
3. $m^6 \div m^4$
4. $21x^{-5}$
5. Solve: 4^0
6. Solve: h^0
7. Solve for n: $(2^3)^n = 2^{18}$
8. Write as a power: $\sqrt[3]{15x}$

Answers:

1. $(2^3)^4 = 2^{12}$ (When you raise a power to a power, multiply the exponents.)
2. $3^5 \times 3^2 = 3^7$ (When you multiply like bases, add the exponents.)
3. $m^6 \div m^4 = m^2$ (When you divide like bases, subtract the exponents.)
4. $21x^{-5} = \dfrac{21}{x^5}$ (Take the reciprocal of a base with a negative exponent and write it with a positive exponent.)
5. $4^0 = 1$ (Any base raised to the power of zero is equal to one.)
6. $h^0 = 1$ (Any base raised to the power of zero is equal to one.)
7. $\left(2^3\right)^n = 2^{18}$ (When the bases are equal, the exponents are equal.)

   $\left(2^3\right)^n = \left(2^3\right)^6$

   $n = 6$
8. $\sqrt[3]{15x} = (15x)^{\frac{1}{3}} \quad \left(x^{\frac{1}{n}} \text{ means the } n^{\text{th}} \text{ root of } x\right)$

Part B: Polynomial Regression Equations That Model Data

Remember, these questions are similar to the ones that will be on your assignments and midterm examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and midterm examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Using technology, graph each of the following functions. Use technology to find the x-intercept(s), y-intercept, and the coordinates of the relative maximum and minimum (if they exist) of the following polynomial functions. State the domain and range. Include a sketch of the graph with your solutions.

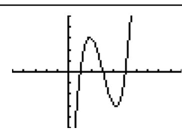
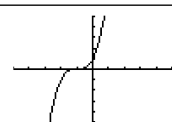
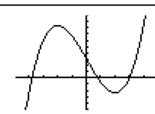
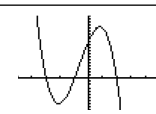
   a) $f(x) = x^3 - 9x^2 + 23x - 15$

   b) $g(x) = x^3 + 3x^2 + 3x + 1$

   c) $h(x) = \frac{1}{3}x^3 - 4x + 3$

   d) $j(x) = -2(x+3)(x-2)(x+1)$

   Answer:

	(a)	(b)	(c)	(d)
Graph				
Zeros	$x = 1$, $x = 3$, $x = 5$	$x = -1$	$x = -3.79$, $x = 0.79$, $x = 3$	$x = -3$, $x = -1$, $x = 2$
y-intercept	$y = -15$	$y = 1$	$y = 3$	$y = 12$
Coordinates of relative maximum/ minimum	Relative maximum at (1.845, 3.079) and relative minimum at (4.155, –3.079)	None	Relative maximum at (−2, 8.33) and relative minimum at (2, −2.33)	Relative maximum at (0.786, 16.418) and relative minimum at (−2.12, −8.12)
Domain	$\{x \mid x \in \Re\}$	$\{x \mid x \in \Re\}$	$\{x \mid x \in \Re\}$	$\{x \mid x \in \Re\}$
Range	$\{y \mid y \in \Re\}$	$\{y \mid y \in \Re\}$	$\{y \mid y \in \Re\}$	$\{y \mid y \in \Re\}$

2. A toy rocket is fired from a launch pad at a height of 0.3 metres with an initial vertical velocity of 50 metres per second.

   Recall that the height of a projectile is modelled by the function $h = -4.9t^2 + vt + s$.

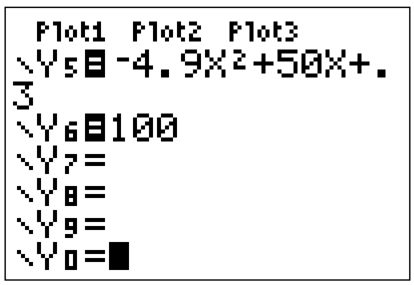
   a) Will the rocket ever reach a height of 100 metres?

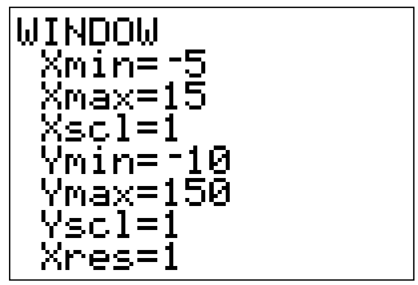
      Answer:

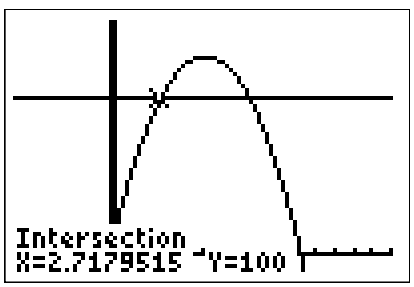
      Method 1:

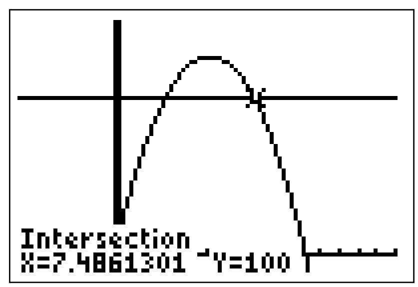
      The function that models this situation is $h = -4.9t^2 + 50t + 0.3$.

      The graph of the line $h = 100$ intersects the graph of the function at $t = 2.7$ and $t = 7.5$, so yes, the rocket reaches a height of 100 metres on the way up and again on the way down, at about 2.7 seconds and 7.5 seconds into the flight.



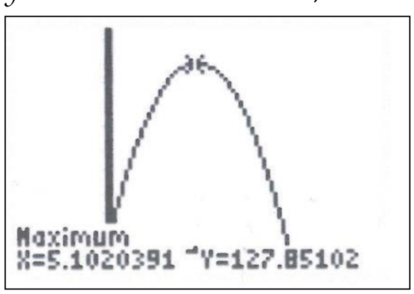






      Method 2:

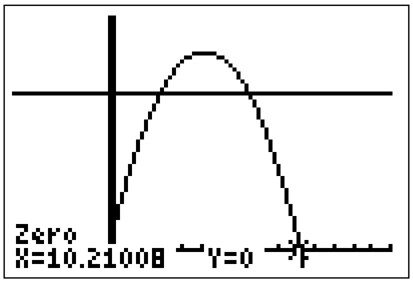
      Or, the coordinates of the maximum are (5.1020391, 127.85102). Since the y-value exceeds 100 m, the rocket reaches and exceeds this height.



b) What is the significance of the *x*- and *y*-intercepts?

Answer:

The *y*-intercept gives the height (h = 0.3 metres) of the rocket on the launch pad at t = 0. The *x*-intercept on the left is meaningless in this situation because it represents negative time. The *x*-intercept on the right represents the time the rocket touches down on the ground after the flight—about 10.2 seconds after takeoff.

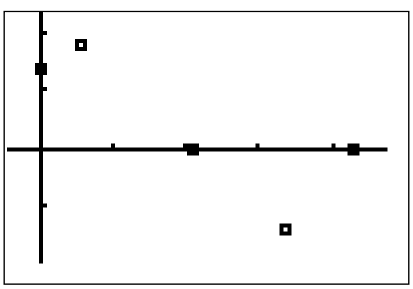


3. A competitive diver springs up from a diving board and then dives into a pool. Her coach uses rapid-fire photography and a stopwatch to record and analyze her dive. He collects the following data.

Time (seconds)	0	0.58	2.1	3.35	4.3
Height above water line (m)	1.3	1.8	0	–1.4	0

a) Graph the data using technology.

Answer:

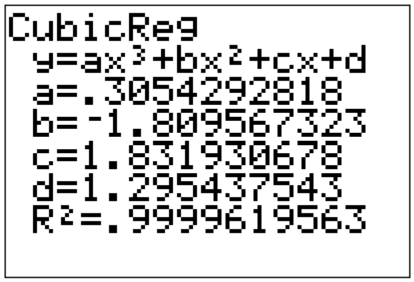


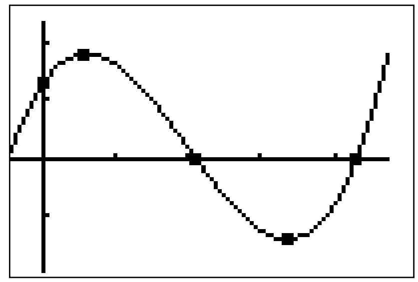
Your graph may appear different. Try to adjust the window settings so it resembles the sample solution provided.

b) Determine which polynomial function will best model the path of her dive and find the regression equation. Determine how well the line fits the data.

Answer:

The line needs to have two turning points so a cubic regression equation would model this situation best. The polynomial function is approximately $y = 0.3x^3 - 1.8x^2 + 1.8x + 1.3$.





You do not need to use technology that shows the coefficient of determination (R^2); you can inspect the line visually and make a judgment of how well the line follows the trend of the data. It passes through all the points and the shape of the graph models the pathway of the dive, so it is a good representation of the data.

c) What is the significance of the x-intercepts?

Answer:

At 2.1 seconds, the diver enters the water and then swims below the surface until 4.3 seconds, when she comes back up to the surface.

d) State the domain and range for this situation.

Answer:

According to the polynomial function that models this situation, and applying common sense, the domain is $\{x \mid 0 \leq x \leq 4.3, x \in \Re\}$. This represents the time, in seconds, of her dive. The range would be $\{y \mid -1.39 \leq y \leq 1.8, y \in \Re\}$. This represents her depth below the water surface and the height she achieves when springing up off the board to begin her dive.

4. Dana helps her mom sell some of her delicious cinnamon buns at various bake sales. They experiment with the price of a dozen buns and come up with the following results.

Price per dozen ($)	3.50	4	4.50	6.25	7.25
Dozens sold (#)	35	33	30	22	17
Revenue ($)					

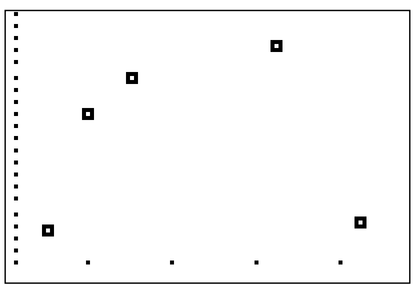
a) Complete the chart above by calculating the revenue generated at each price.

Answer:

Price per dozen ($)	3.50	4	4.50	6.25	7.25
Dozens sold (#)	35	33	30	22	17
Revenue ($)	122.50	132.00	135.00	137.50	123.25

b) Graph the data using price as the independent variable and revenue as the dependent variable.

Answer:



c) Determine the quadratic regression equation for the curve of best fit.

Answer:

$y = -4.90x^2 + 52.94x - 2.52$

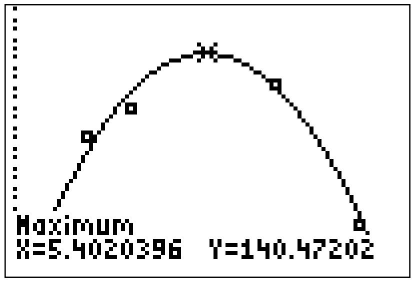
d) Describe how well the curve fits the data.

Answer:

The graph of the regression equation fits the data points quite well. It is a very good approximation of the data, although, realistically, many factors contribute to the sale of cinnamon buns so the line doesn't go through each point perfectly.

e) Determine the price Dana and her mom should charge to maximize their revenue. How many dozen buns could they expect to sell at this price?

Answer:

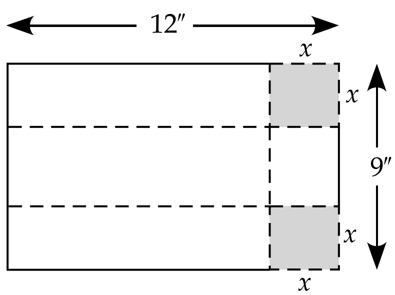


The vertex is at (5.40, 140). According to this model, Dana and her mom would maximize their revenue if they sold their cinnamon buns at $5.40 per dozen. This would generate revenue of approximately $140. To do so, they would expect to sell about 26 dozen buns.

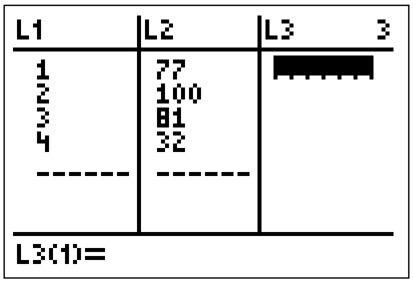
5. From a 9″ by 12″ piece of metal sheeting, small squares are cut out from two of the corners so that the three edges can be bent up and fastened to manufacture a scoop (a rectangular box with no top or front). Find the length of the small squares that are cut out to give the greatest volume. Find the maximum volume.

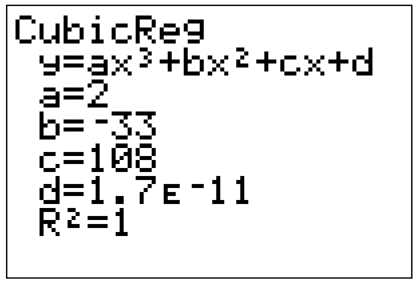
   In your solution, include a diagram, a chart of values showing possible side lengths of the square, the resulting length and width of the box, and the resulting volume of the scoop. Also include a sketch of a scatterplot with line or curve of best fit, the regression equation calculated using technology, and a statement regarding the fit of the graph to the data.

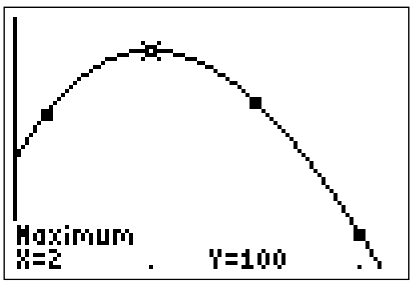
   Answer:



Side Length of Square	Resulting Box Length	Resulting Box Width	Volume of Box
1″	11″	7″	77 in.^{3}
2″	10″	5″	100 in.^{3}
3″	9″	3″	81 in.^{3}
4″	8″	1″	32 in.^{3}







Since volume is a three-dimensional calculation, a cubic polynomial is the best model. The equation for the curve of best fit is $y = 2x^3 - 33x^2 + 108x$. The regression equation is a perfect fit. The graph follows the trend of the data perfectly and goes through each point. **Note:** The coefficient for *d* given above by the calculator is shown in scientific notation (meaning 1.7×10^{-11}), which is effectively 0. You can confirm that the *y*-intercept is actually exactly zero but is approximated by a very, very small number in the display due to the software algorithm used by the graphing calculator.

The relative maximum of this function is at (2, 100). The maximum volume of the scoop is 100 cubic inches when the square cut out of the corners is 2 inches along each side.

6. A farmer has a rectangular garden that he wants to enclose with 200 m of fencing.

   a) Use the table below to fill in 5 possible pairs of lengths and widths and their corresponding areas.

   Answer:

Length (m)	10	20	30	40	50
Width (m)	90	80	70	60	50
Area (m^2)	900	1600	2100	2400	2500

   Other dimensions are possible. However, the length plus width must be 100.

   b) Find the most appropriate regression equation to model the length of the garden and the area enclosed.

   Answer:

   $y = -x^2 + 100x$

   $R^2 = 1$

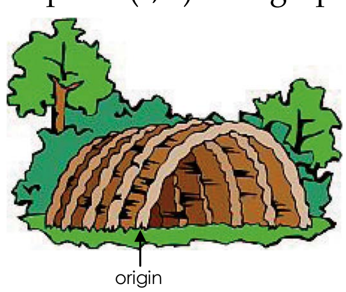
   Area is a two-dimensional calculation, so a quadratic regression equation is the most appropriate.

   c) What dimensions maximize the area?

   Answer:

   The vertex of this function is at (50, 2500). A garden that measures 50 m by 50 m would have the maximum area of 2500 m^2.

7. The entranceway to a tunnel is shown below. The far left of the tunnel opening, at ground level, is said to be the origin. This would be represented by the point (0, 0) on a graph.



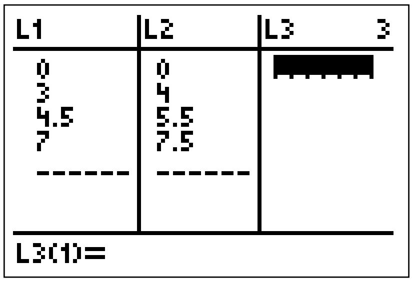
The height of the tunnel at the opening is measured at three places:

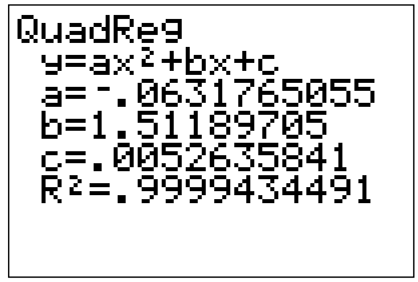
- 3 feet to the right of the origin the height of the tunnel is 4 feet
- 4.5 feet to the right of the origin the height of the tunnel is 5.5 feet
- 7 feet to the right of the origin the height of the tunnel is 7.5 feet

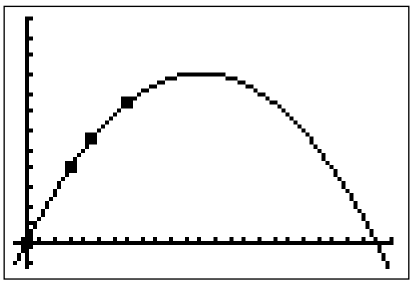
a) Find the quadratic regression equation that models the height of the tunnel, y, compared to the distance to the right of the origin, x.

Answer:

Create a table of values and use the data to calculate a quadratic regression equation.





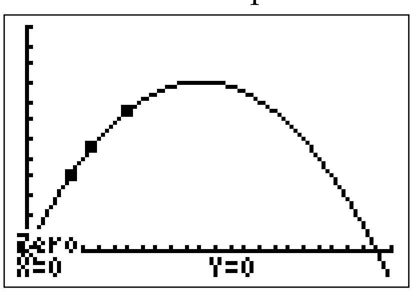


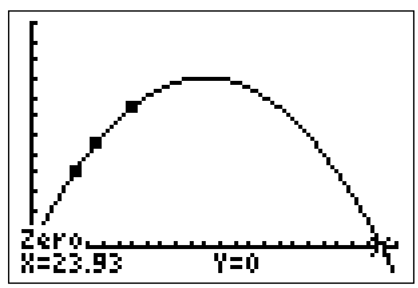
$y = -0.0631765x^2 + 1.511897x + 0.00526358$

b) How wide is the tunnel at ground level?

Answer:

Find the x-intercepts.

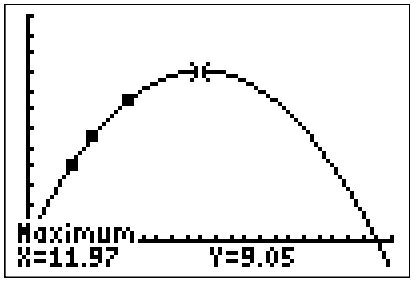




The tunnel is 23.93 feet across the bottom.

c) What is the maximum height of the tunnel?

Answer:

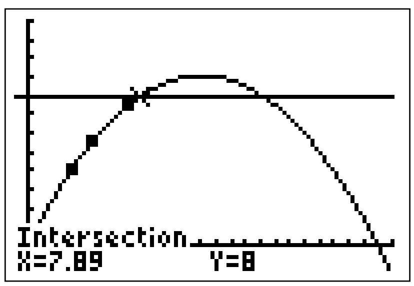


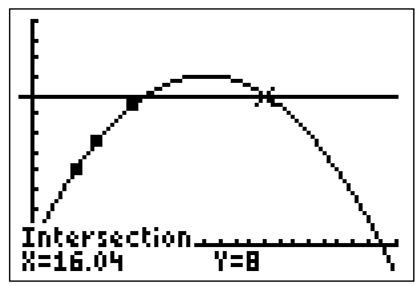
The maximum height of the tunnel is 9.05 feet at a point 11.97 feet horizontally from the origin.

d) An object that is 8 feet wide and 8 feet tall is to be pulled through the tunnel. Will it fit? Justify your answer.

Answer:

Graph a line $y = 8$ and find the distance between the intersection points with the regression equation. Between these two intersection points, the tunnel is higher than 8 feet.





The distance between the two intersection points is 16.04 − 7.89 = 8.15.

There is a 8.15 foot (or 8′1.8″) wide portion of the tunnel that is higher than 8 feet. If the object is pulled through the exact centre of the tunnel, it will fit.

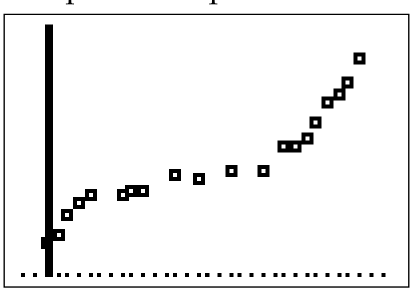
8. The following table shows the average retail price of gasoline, per litre, for a selection of years in a 30-year period beginning in 1979 (year 0).

Years after 1979	Price of Gas (¢/L)	Years after 1979	Price of Gas (¢/L)
0	21.98	17	58.52
1	26.18	20	59.43
2	35.63	22	70.56
3	43.26	23	70.00
4	45.92	24	74.48
7	45.78	25	82.32
8	47.79	26	92.82
9	47.53	27	97.86
12	57.05	28	102.27
14	54.18	29	115.29

a) Use technology to graph the data as a scatterplot. What polynomial function could be used to model the data? Explain.

Answer:

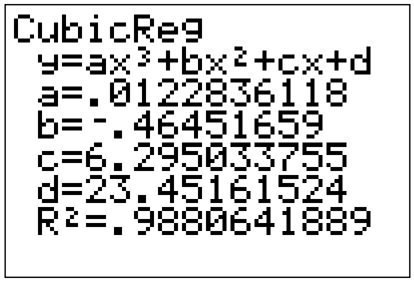
The price is dependent on the number of years after 1979.



The points are continually increasing, but do not lie in a straight line. They extend from Quadrant III to Quadrant I, so a cubic function could be used to describe this data. There do not appear to be any turning points, but the graph flattens out and then continues to increase, similar to the characteristics of the graph $f(x) = x^3$.

b) Determine the cubic regression equation for the data. Use your equation to estimate the average price of gas in 1984 and 1985.

Answer:



The cubic function that models this data is
$y = 0.0123x^3 - 0.465x^2 + 6.295x + 23.452$.

The year 1984 is the fifth year after 1979, so solve for when $x = 5$, using the graph of the line and technology or substitute $x = 5$ into the equation and solve.

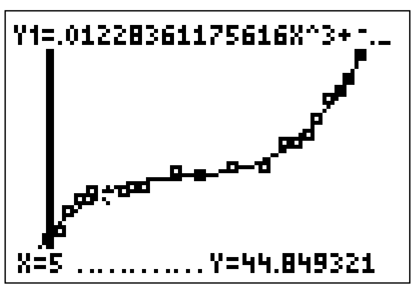
$$y = 0.0123x^3 - 0.465x^2 + 6.295x + 23.452$$

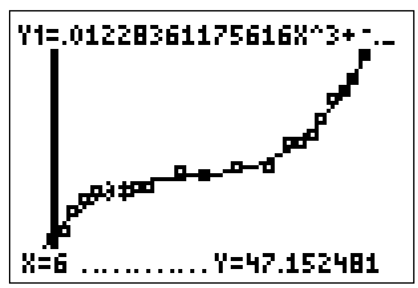
$$y = 0.0123(5)^3 - 0.465(5)^2 + 6.295(5) + 23.452$$

$$y = 1.5375 - 11.625 + 31.475 + 23.452$$

$$y = 44.8$$

The average price of gas in 1984 was about 44.8¢/L.



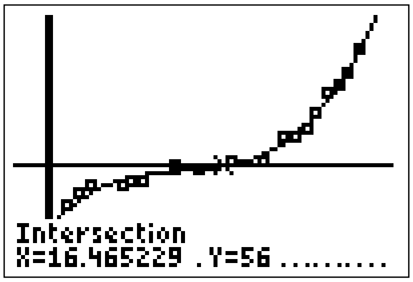


The average price of gas in 1985 can be found using a similar method, where $x = 6$. It was approximately 47.2¢/L.

c) Estimate the year in which the average price of gas was 56.0¢/L.

Answer:

The year when the average price of gas was 56.0¢/L can be found by graphing the equation $y = 56.0$ and finding the point of intersection of the lines.



The average retail price of gas was 56.0¢/L during the year represented by 16.5. This would correspond to about midway through 1995.

d) Extrapolate the average price of gas in 2012 using this model. Is this reasonable? State any limitations on this model.

Answer:

$$y = 0.0123x^3 - 0.465x^2 + 6.295x + 23.452$$

$$y = 0.0123(33)^3 - 0.465(33)^2 + 6.295(33) + 23.452$$

$$y = 166.8$$

Or you may solve for the value of $x = 33$, using the regression equation graph. You may have to adjust the window settings to include the value of $x = 33$.

According to this model, the average price of gas in 2012 was 166.8¢/L. This value is not reasonable. The model assumes that the price of gas increases consistently and quickly from 2008 onwards, when in fact the price of gas varies. A cubic function accurately describes the trend of the prices in the interval of the data, but extending it further is not appropriate. The price of gas likely has many turning points in a longer interval and a different regression equation would have to be used to predict prices outside of the given data points.

Learning Activity 1.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Write 9^5 with a base of 3 and simplify.
2. Expand: $(a + b)^2$
3. Simplify: $25^{\frac{3}{2}}$
4. Solve for x: $6^x = \frac{1}{36}$
5. Write in exponential form: $\sqrt[5]{m^3}$
6. Solve for x: $16^5 = 2^x$
7. A dress costs \$100 on Monday. On Tuesday, it is reduced in price by 20% and on Wednesday, it is reduced by 20% again. What is the price of the dress on Wednesday?
8. A boy saves \$3 from his weekly allowance the first week he receives it. Each week, he doubles the amount he saves. How much does the boy save from his fourth allowance?

Answers:

1. 3^{10} $(9^5 = (3^2)^5 = 3^{10})$
2. $a^2 + 2ab + b^2$ $$\left(\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a(a+b) + b(a+b)\\ &= a^2 + 2ab + b^2\end{aligned}\right)$$

   (Write as the product of two binomials and apply the distributive property.)
3. 125 $\left(25^{\frac{3}{2}} = \left(\sqrt{25}\right)^3 = 5^3 = 125\right)$

4. $x = -2 \left(\begin{aligned} 6^x &= \frac{1}{36} \\ 6^x &= \frac{1}{6^2} \\ 6^x &= 6^{-2} \\ x &= -2 \end{aligned} \right)$

5. $\sqrt[5]{m^3} = m^{\frac{3}{5}}$

6. $x = 20 \left(\begin{aligned} 16^5 &= 2^x \\ \left(2^4\right)^5 &= 2^x \\ 2^{20} &= 2^x \end{aligned} \right)$

7. \$64 (Monday—\$100; Tuesday—0.8 × 100 = \$80; Wednesday—0.8 × 80 = \$64)

8. \$24 (Week 1: \$3; Week 2: \$6; Week 3: \$12; Week 4: \$24)

Part B: Exponential Functions

Remember, these questions are similar to the ones that will be on your assignments and midterm examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and midterm examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Complete the following tables for the given exponential functions. Describe the patterns you see and how they relate to the form of the function $f(x) = a(b)^x$.

x	$f(x) = 2^x$
–1	0.5
0	1
1	2
2	4
3	8
4	16
5	32

x	$f(x) = 3^x$
–1	0.3333
0	1
1	3
2	9
3	27
4	81
5	243

x	$f(x) = 10(2^x)$
–1	5
0	10
1	20
2	40
3	80
4	160
5	320

x	$f(x) = \left(\frac{1}{2}\right)^x$
–1	2
0	1
1	0.5
2	0.25
3	0.125
4	0.0625
5	0.03125

The value of the base, b, is the factor by which the function increases for an increase of 1 in x. If $b = 2$, the values double. If $b = 3$, the values triple.

If b is a whole number greater than 1, the function increases. If the value of b is a fraction between zero and one, the function decreases.

The value of the multiplier, a, is the y-intercept, since $b^0 = 1$ so $f(x) = a$, when $x = 0$.

The values are all positive and get closer to zero (without ever being equal to zero) for large negative x-values in the first three tables, and large positive x-values in the last table. So the x-axis, $y = 0$, is a horizontal asymptote.

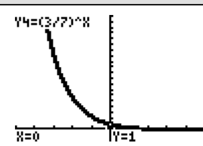
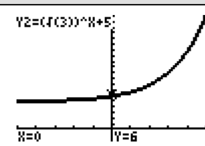
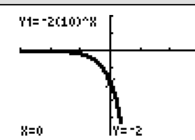
2. Given the function $f(x) = 153^x$, explain why the y-intercept is at (0, 1).

   Answer:

   At the y-intercept, $x = 0$. Any base, raised to the power 0 is equal to 1 (that is, $153^0 = 1$), so the y-intercept will be at (0, 1).

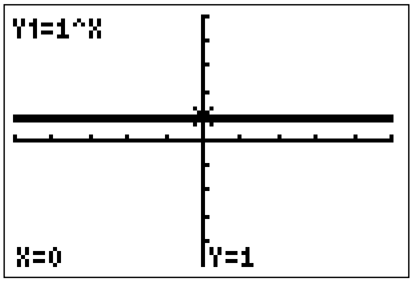
3. Graph the functions below and complete the chart to describe their characteristics.

   Answer:

	$f(x) = \left(\frac{3}{7}\right)^x$	$f(x) = \left(\sqrt{3}\right)^x + 5$	$f(x) = -2(10)^x$
Graph			
y-intercept	1	6	−2
Equation of Asymptote	$y = 0$	$y = 5$	$y = 0$
Increasing or Decreasing	Decreasing	Increasing	Decreasing
End Behaviour	II to I	II to I	III to IV
Domain	$\{x \mid x \in \Re\}$	$\{x \mid x \in \Re\}$	$\{x \mid x \in \Re\}$
Range	$\{y \mid y > 0, y \in \Re\}$	$\{y \mid y > 5, y \in \Re\}$	$\{y \mid y < 0, y \in \Re\}$
Description	Since the base is a fraction between 0 and 1, the graph is decreasing.	$\sqrt{3} \approx 1.73$, so the line increases a bit more gradually than the graph of $f(x) = 2^x$. It is shifted vertically, up 5 units.	Since it is multiplied by a negative 2, the graph is reflected in the y-axis and is decreasing as a result.

4. Graph the function $f(x) = 1^x$ and explain the result.

   Answer:

   The power with a base of 1 is a constant function and its graph is a horizontal line at $y = 1$. The value of 1, raised to any power, will still equal 1.

   $$1^{-12} = \frac{1}{1^{12}} = \frac{1}{1} = 1$$

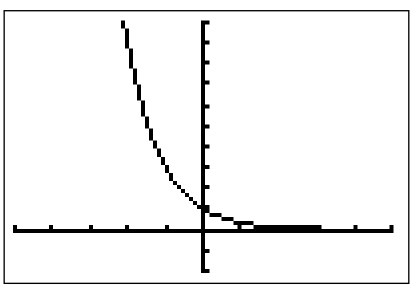
   $$1^0 = 1$$

   $$1^1 = 1$$

   $$1^{1379146} = 1$$

5. Graph the functions $f(x) = \frac{1}{3}^{x}$ and $f(x) = 3^{-x}$ on the same grid and explain the results using the exponent laws.

   Answer:

X	Y4	Y5
-3	27	27
-2	9	9
-1	3	3
0	1	1
1	.33333	.33333
2	.11111	.11111
3	.03704	.03704

   X= -3

   The graphs of these two functions result in the exact same curve. This is seen in the table of values for these two functions. To simplify an expression with a negative exponent, the law $a^{-m} = \left(\frac{1}{a}\right)^m$ implies that you take the reciprocal of the base and write it with a positive exponent.

   To write $f(x) = 3^{-x}$ with a positive exponent, you must take the reciprocal of the base. The reciprocal of 3 is $\frac{1}{3}$. Therefore, $3^{-x} = \left(\frac{1}{3}\right)^x$ and these two functions are equivalent, resulting in the same curve on a graph.

6. Given the exponential function $f(x) = 3^x$, use the exponent laws to explain why the values to the left of the y-intercept stay small and the values to the right of the y-intercept increase quickly.

   Answer:

   To the right of the y-intercept, the x-values are positive. The exponent in a power indicates the number of times the base is multiplied by itself. The base is first squared, then cubed, then multiplied by itself repeatedly 4, 5, 6 . . . times. This results in the function increasing in value very, very quickly.

   To the left of the y-intercept, the values for x are negative. To simplify a negative exponent, you can take the reciprocal of the base and write it with a positive exponent. If the numerator is equal to one and the denominator gets increasingly larger, the value of the rational expression gets smaller and smaller. However, when a positive base, such as 3, is raised to an exponent, the result will never be negative and will never be zero. So, as you move left, the curve approaches a horizontal asymptote at $y = 0$.

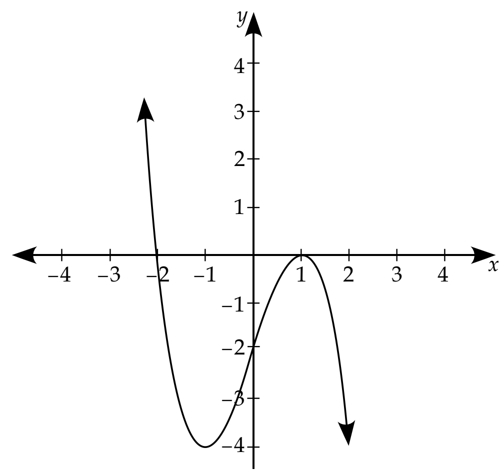
Learning Activity 1.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. The interest on an investment is compounded bi-weekly. How many times per year is interest added?
2. The interest on an investment is compounded semi-monthly. How many times per year is interest added?

Use the graph of a polynomial function shown here to provide the information requested in questions 3 to 8.



3. The sign of the leading coefficient in the equation
4. The constant in the equation
5. The end behaviour
6. The number of turning points
7. The zeros
8. The degree

Answers:

1. 26 times per year (bi-weekly means every two weeks, 52 ÷ 2 = 26)
2. 24 times per year (semi-monthly means two times per month, 12 × 2 = 24)
3. Negative (The graph decreases as it moves to the right; the graph moves from Quadrant II to Quadrant IV.)
4. –2 (the value of the y-intercept)
5. Quadrant II to Quadrant IV
6. Two (The graph changes from decreasing to increasing and then back to decreasing.)
7. –2, 1 (The x-intercepts or zeros refer to the value of the x-coordinate where the graph crosses the x-axis, or where $y = 0$.)
8. Three (illustrates characteristics of the graph of a cubic function)

Part B: Applications of Exponential Functions

Remember, these questions are similar to the ones that will be on your assignments and midterm examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and midterm examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. The average monthly rent for a 1-bedroom apartment in a major city is $825. Experts predict the rent cost will increase 3.5% each year for the foreseeable future.

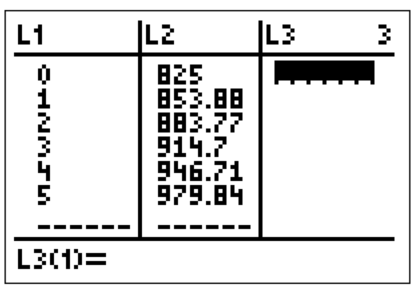
   a) Create a table of values showing the average monthly cost to rent a 1-bedroom apartment for the next 5 years. Use the current average monthly cost of $825 as the rent for year 0. Round values to the nearest cent.

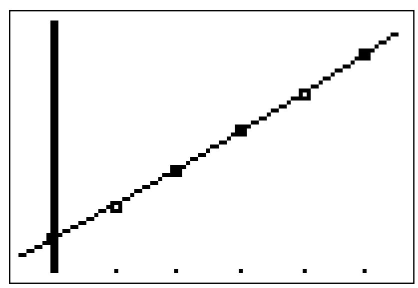
   Answer:

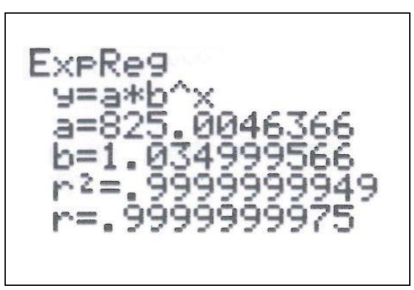
Year	0	1	2	3	4	5
Rent ($)	825	853.88	883.77	914.70	946.71	979.84

   b) Use technology to graph the points from the table in a scatterplot and find an exponential function that models the situation.

   Answer:







The exponential regression equation that models the monthly cost of rent over the next few years is given as $y = 825(1.035)^x$.

c) What will the rent for a 1-bedroom apartment be in 10 years?

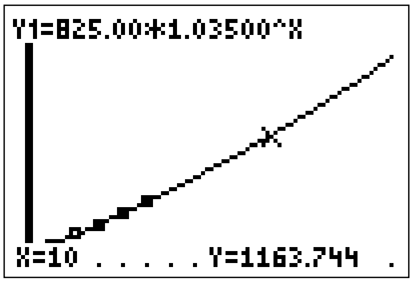
Answer:

You can use the equation or the graph to solve for y when $x = 10$.

$$y = 825(1.035)^x$$

$$y = 825(1.035)^{10}$$

$$y = 1163.743978$$



In 10 years, the cost to rent a 1-bedroom apartment will be about $1163.74 per month.

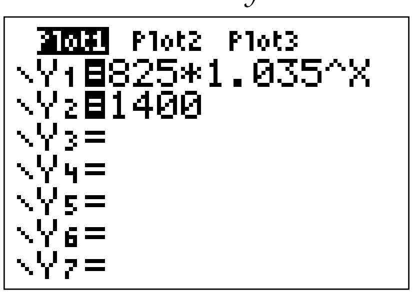
d) A man who owns an apartment complex with ten 1-bedroom units is paying $9000 per month for the mortgage on the building. Use technology to determine the number of years until he makes a profit of $5000/month on his apartment complex.

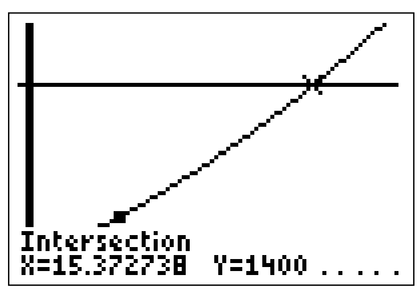
Answer:

Currently, the owner is paying $9000 per month on his 10 apartments. That means the owner is paying $900 on each apartment per month. You can assume this will stay the same. If the owner wants an additional $500 profit on each apartment ($5000/10), he will want a total rent of:

$900 + $500 = $1400 per apartment

Solve for x when $y = 1400$.





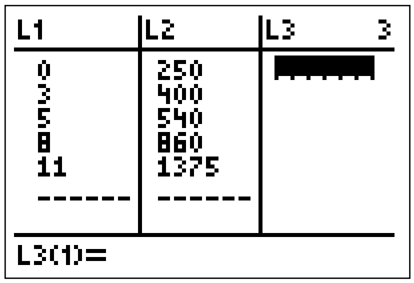
The landlord will be making a total of $5000 profit monthly after 15 years (assuming the rate increase is 3.5% per year).

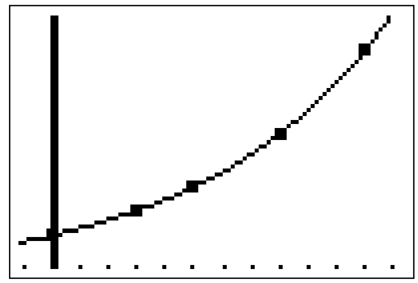
2. The population of a small town is recorded for several years. The data is given below.

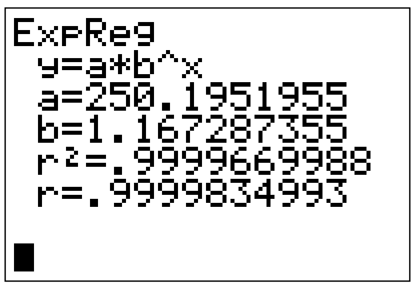
Year	2000	2003	2005	2008	2011
Population	250	400	540	860	1375

a) Use technology to graph the points and find the exponential function that gives the curve of best fit.

Answer:







The exponential function that approximates this data is $y(x) = 250(1.167)^x$ where x is the number of years from the year 2000.

b) By what percent is the population growing each year?

Answer:

From the exponential regression equation the growth factor is about 16.7% each year (1.167287 − 1).

c) Comment on how well the graph of the regression equation models the data.

Answer:

The coefficient of determination for this equation is 0.99997, which is very close to 1. The line passes through all the points and follows the trend of the data. It is a very good fit for the data.

d) Use the equation or graph to estimate the population in 2010.

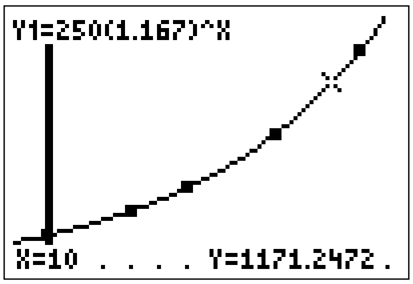
Answer:

Solve for $x = 10$.

$$y(x) = 250(1.167)^x$$

$$y(10) = 250(1.167)^{10}$$

$$y(10) = 1171.25$$



In 10 years, the population will be about 1171 inhabitants.

e) When the population reaches 8500, a clinic will be built. Use technology to predict when this may happen.

Answer:

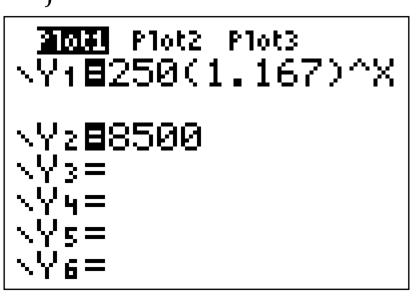
Solve for when $y(x) = 8500$.

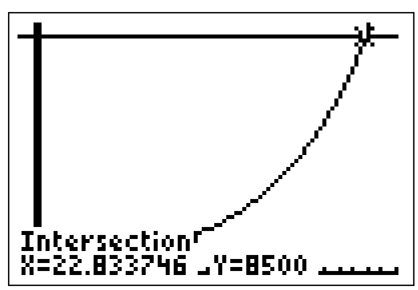
$$y(x) = 250(1.167)^x$$

$$8500 = 250(1.167)^x$$

Graph both sides of the equation and solve for the point of intersection.

Adjust the window as necessary.





The population will reach 8500 in about 22.8 years.

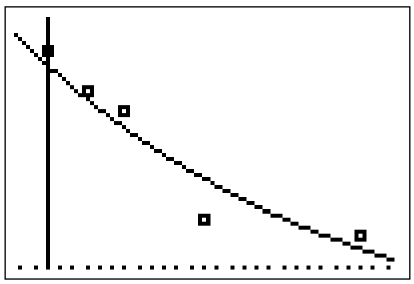
3. The value of a new computer over time from the date of purchase is given in the table below.

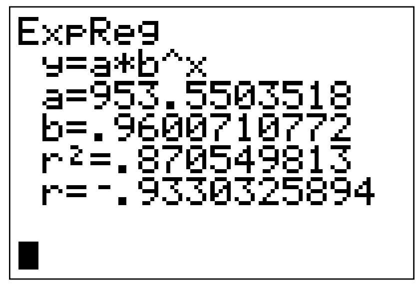
# of Months	0	3	6	12	24
Value ($)	1000	875	800	450	400

a) Find the exponential regression equation that best models this data.

Answer:

The function $y(x) = 953.55(0.96)^x$ approximates this data. Notice that the base, b, is less than one. This results in a graph that illustrates exponential decay or decreasing values over time.





b) Comment on the fit of this line.

Answer:

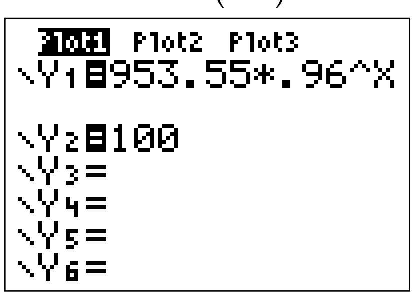
The fit is not perfect, but the line is reasonably close to four of the data points (some points are above the line and some are below the line), and the trend does seem to follow an exponential decay model. Many factors influence the price of technology over time, but this function remains a reasonably good estimate of its value.

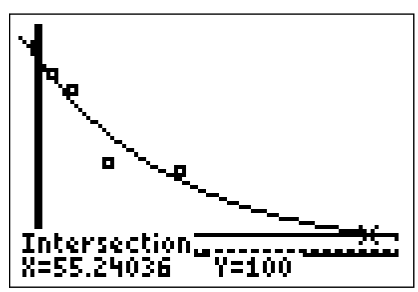
c) Use technology to determine when the value of the computer will be less than $100.

Answer:

This point can be found where $y = 100$. Solve by graphing both sides of the following equation. Adjust your window to view the point of intersection.

$$100 = 953.55(0.96)^x$$





The computer will be worth less than $100 after about 55 months (55 ÷ 12 = 4.583 years or 4 years and 7 months).

4. The population of a small town was 750 in the year 2012. The population of the town is estimated to increase by 50% every 5 years.

a) Create a table of values showing the approximate population during 5 growth periods. Use 2012 as year 0.

Answer:

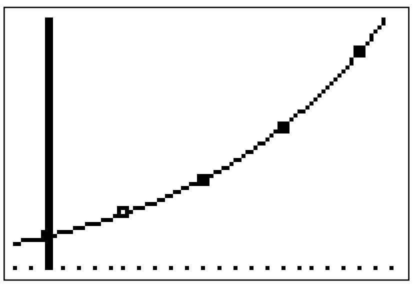
Year	2012– year 0	2017– year 5	2022– year 10	2027– year 15	2032– year 20
Population	750	1125	1687.5	2531.25	3796.875

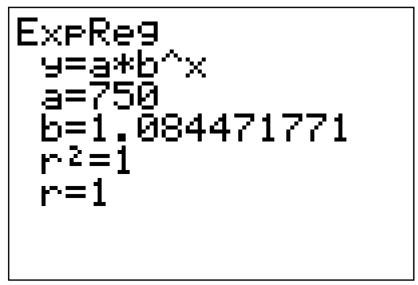
$750 \times 1.5 = 1125$

$1125 \times 1.5 = 1687.5$, etc.

b) Use technology to graph the points and determine an exponential function that models the situation.

Answer:



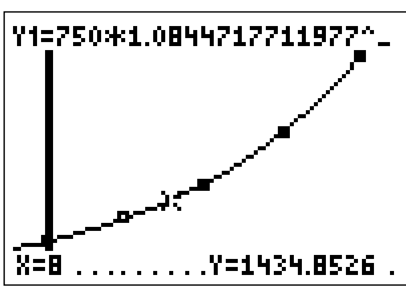


$x = 750 \times 1.084471771^x$

c) What will the population be in 2020?

Answer:

2020 to 2012 = 8 years. The year 2020 is represented as year 8 in this function. Solve for $x = 8$ using the equation or technology.



$y = 750 \times 1.084471771^x$

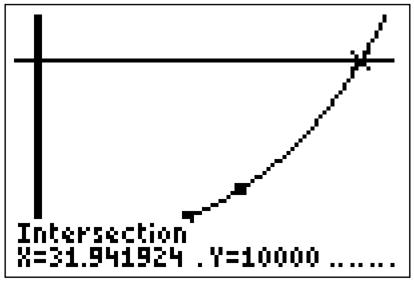
$y = 750 \times 1.084471771^8$

$y = 1434.852561$

In 2020, the population will be about 1435 inhabitants.

d) When the town reaches a population of 10 000, it will build a community centre. Use technology to determine in what year the community centre may be built.

Answer:



The intersection of the line y = 10 000 and the function is at (31.941924, 10 000). This means that the community centre may be built in year 32, according to this function. This represents 2012 + 32 = 2044, or the year 2044.

e) What percent is the population increasing by each year?

Answer:

According to the regression equation $y = 750 \times 1.084471771^x$, the growth factor is 1.084471771. The population is increasing at a rate of 8.447% per year. According to the information provided initially, the population increases by 50% or 1.5 times every 5 years. The rate per year could also be determined using $\sqrt[5]{1.5} = 1.084471771$.

5. The value of a particular brand of car is said to decrease by a quarter of its value every 9 months after purchase. The initial value of the car is $32,500.

a) Create a table showing the value of the car over time. Include at least 5 data points.

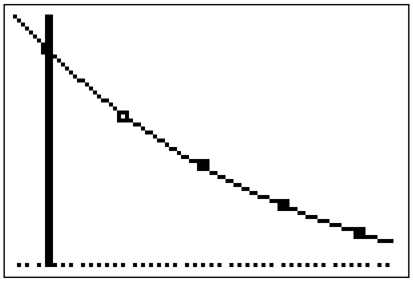
Answer:

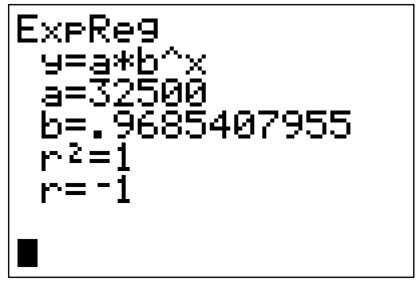
Time (months)	0	9	18	27	36
Value ($)	32 500	24 375	18 281.25	13 710.9375	10 283.20313

32 500 × 0.75 = 24 375

b) Use technology to determine an exponential function that models the value of the car in this situation.

Answer:



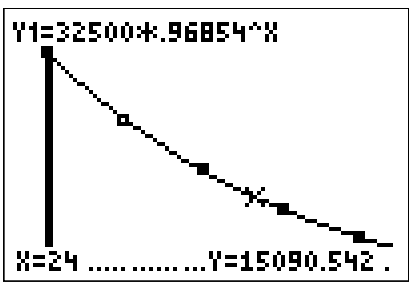


$y = 32500 \times 0.96854^x$

c) What will the value of the car be after two years?

Answer:

Two years is 24 months. Solve for y when $x = 24$, using the graph or the equation.



$$y = 32500 \times 0.96854^x$$

$$y = 32500 \times 0.96854^{24}$$

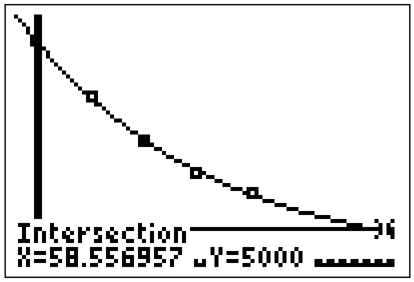
$$y = 15090.54193$$

Two years after purchase, the vehicle will be worth about $15 090.54.

d) A customer who has purchased this car would like to sell it when it is worth less than $5000. After how many years should the customer sell this vehicle?

Answer:

Use technology to solve for x when $y = 5000$. Graph both sides of the equation $5000 = 32500 \times 0.96854^x$ and find the point of intersection. Adjust the window as necessary.



The point of intersection is at (58.556957, 5000).

The owner should sell the car sometime during the 58th month.

$58 \div 12 = 4.83$

After driving the car for four years and 10 months, the owner should sell it.

6. A 250-gram sample of a certain radioactive element decays continuously. The amount remaining can be modelled by a natural exponential function expressed as $A = 250e^{-0.13t}$.

   a) Complete the following table, using the natural exponential function to show the number of grams remaining.

   Answer:

Time (days)	0	1	2	3	4
Amount (g)	250	219.5238577	192.7628965	169.2642186	148.630137

   b) Use technology to determine the half-life of this substance.

   Answer:

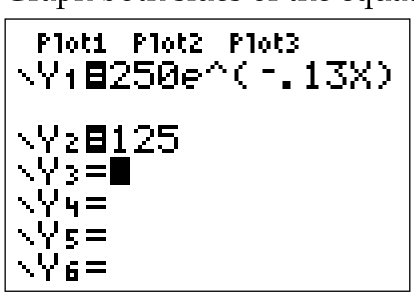
   The half-life of this substance is the time it takes to decay to half of the original amount.

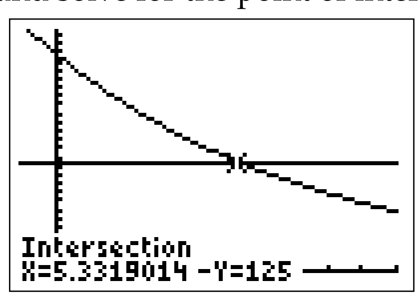
   $250 \div 2 = 125$ g is half of the original amount.

   $A = 250e^{-0.13(t)}$

   $125 = 250e^{-0.13(t)}$

   Graph both sides of the equation and solve for the point of intersection.





The half-life of this substance is approximately 5.33 days.

Learning Activity 1.5

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. The vertex of a parabola with a positive leading coefficient is at (2, 0). How many zeros does the function have?
2. The area of a triangle is 24 m^2. Find the length of the base if the height is 12 m.
3. State the next six numbers in this pattern: 1, 4, 9, 16, . . .
4. An airline overbooks its flights by 10%. Each flight can carry 130 passengers. How many bookings will they accept for a flight?
5. A cube has a volume of 8 $ft.^3$. What are the dimensions of the cube?
6. Convert 30 inches to feet.
7. Calculate a 15% tip on a restaurant bill of $58.
8. Solve for x by finding a common base: $64^x = \sqrt{2}$

Answers:

1. One (The function will open up. It touches the x-axis at 2, so there will be one root or x-intercept (also called a zero) at (2, 0).)
2. The height of the triangle is 4 m.

$$\left(\begin{aligned} A &= \frac{bh}{2} \\ 24 &= \frac{12(h)}{2} \\ h &= 4 \end{aligned} \right)$$

3. . . . 25, 36, 49, 64, 81, 100 . . . (perfect square numbers)
4. 143 (10% of 130 is 13; 130 + 13 = 143)

5. The cube is 2 ft. long by 2 ft. wide by 2 ft. deep.

$$\begin{pmatrix} V = (\text{side length})^3 \\ 8 = s^3 \\ 2^3 = s^3 \\ s = 2 \end{pmatrix}$$

6. 2.5 (There are 12 inches in one foot; 12 × 2 = 24; 30 – 24 = 6; 6 inches = $\frac{1}{2}$ foot, so there are 2.5 feet in 30 inches; or, 30 ÷ 12 = 2.5)
7. $8.70 (10% would be $5.80; half of 10% is 5%, so 5% would be $2.90; the sum of these values will give you 15%; 5.80 + 2.90 = 8.70; a 15% tip would be $8.70)
8. $\frac{1}{12}$

$64^x = \sqrt{2}$	Recall that the square root of 2 can be written as $2^{\frac{1}{2}}$.
$(2^6)^x = 2^{\frac{1}{2}}$	Rewrite the terms with a common base and apply the exponent laws.
$6x = \frac{1}{2}$	Equate the exponents if the bases are equal.
$x = \frac{1}{12}$	Solve for x.

Part B: Logarithm Functions

Remember, these questions are similar to the ones that will be on your assignments and midterm examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and midterm examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

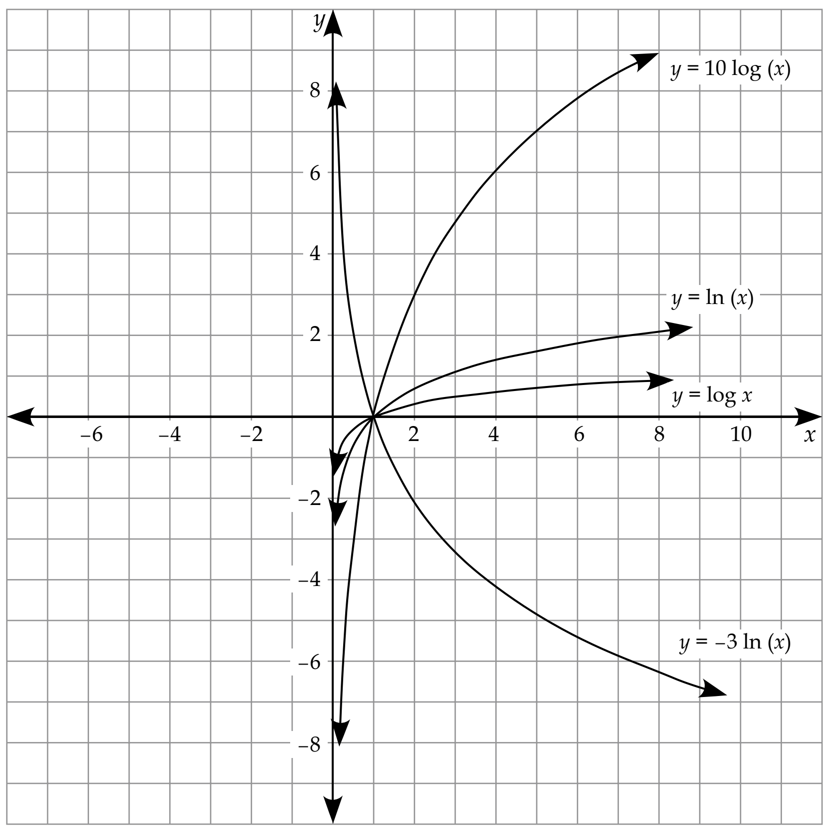
1. Express each of the following exponential equations as a logarithm:

	Answers:
a) $10^x = 5$	$\log_{10} 5 = x$ or $x = \log(5)$
b) $6^3 = 216$	$\log_6 216 = 3$
c) $2^8 = y$	$\log_2 y = 8$
d) $e^x = 512$	$\log_e 512 = x$ or $x = \ln(512)$

2. Graph each of the functions and complete the following table.

Answer:

Function	Domain	Range	End Behaviour	x-intercept	y-intercept	Equation of Asymptote
$y = 10 \log (x)$	$\{x \mid x > 0, x \in \Re\}$	$\{y \mid y \in \Re\}$	IV to I	(1, 0)	Does not exist	$x = 0$
$y = \log (x)$	$\{x \mid x > 0, x \in \Re\}$	$\{y \mid y \in \Re\}$	IV to I	(1, 0)	Does not exist	$x = 0$
$y = -3 \ln (x)$	$\{x \mid x > 0, x \in \Re\}$	$\{y \mid y \in \Re\}$	I to IV	(1, 0)	Does not exist	$x = 0$
$y = \ln (x)$	$\{x \mid x > 0, x \in \Re\}$	$\{y \mid y \in \Re\}$	IV to I	(1, 0)	Does not exist	$x = 0$



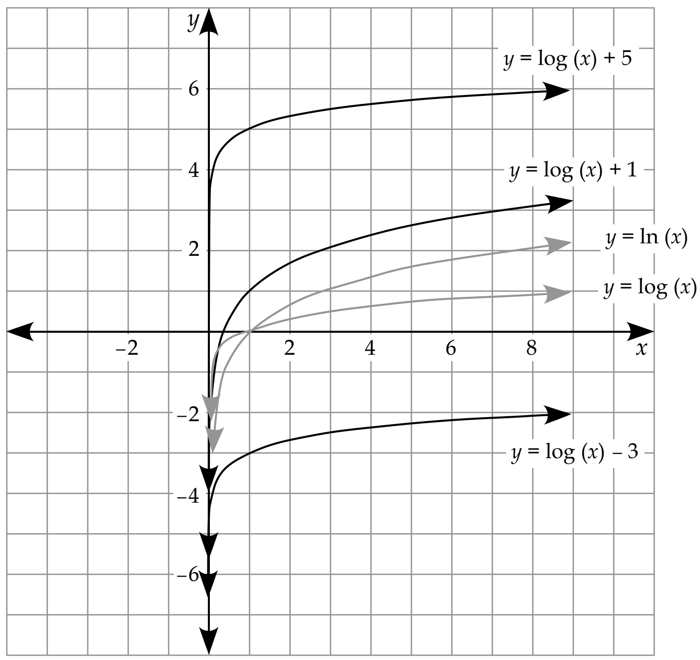
3. Graph each of the following functions. For each, find the value of y when $x = 1$. Summarize the impact the constant has on the characteristics of the logarithm.

   $y = \log(x) + 5$

   $y = \log(x) - 3$

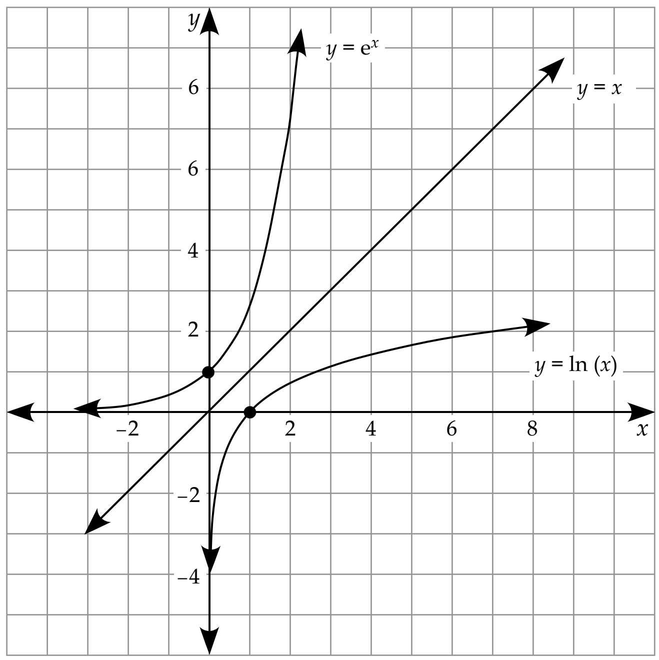
   $y = \ln(x) + 1$

   Answer:



The constant shifts the graph vertically. The point (1, 0) is the x-intercept of the graph of $y = \ln x$ and $y = \log x$. In the graph of a common or natural log with a constant, the point where $x = 1$ is shifted up if the constant is positive and down if the constant is negative. As a result, the x-intercept is changed. The shape of the line, the end behaviour, and domain and range are not affected by the constant.

4. Describe the graphs of the functions $y = e^x$ and $y = \ln(x)$ and explain how they are related.



Answer:

These functions are inverses of each other. The graphs are reflections in the line $y = x$.

The natural exponential graph $y = e^x$ is an increasing graph. If x is negative, the value of y is close to zero. If x is positive, the value of y increases very quickly. The line extends from Quadrant II to Quadrant I. It has a y-intercept at 1 and $y = 0$ is a horizontal asymptote. The domain is all real numbers, and the range is all values of y greater than 0.

The natural logarithm graph $y = \ln(x)$ is also an increasing graph. For values of x between 0 and 1, the value of y is negative. It has an x–intercept at 1. If x is positive and greater than 1, the values of y increase gradually. The line $x = 0$ is a vertical asymptote. The end behaviour of the line extends from Quadrant IV to Quadrant I. The domain is all x-values greater than 0. The range is all real numbers.

Learning Activity 1.6

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Solve for x: log (1) = x
2. Solve: ln (e) = x
3. Solve for x by finding a common base: $100^x = 0.001$
4. Write as a logarithmic function: $10^{1.2} \approx 15.85$
5. Write as a natural exponential function: ln (25) ≈ 3.22
6. The population of a town doubles every 5 years. In 2010, the population was 800. What was the population in the year 2000?
7. What is the perimeter of a square with an area of 36 m^2?
8. Estimate the amount of tax you would pay on a purchase of $800 if you must pay 5% GST and 8% PST.

Answers:

1. 0 (Write in exponential form: $10^x = 1$; since any base raised to the power 0 equals 1, $10^0 = 1$ and $x = 0$)
2. 1 (write in exponential form: $e^x = e$; $x = 1$)
3. $x = \frac{-3}{2}$

$$\left(\begin{aligned} 100^x &= 0.001 \\ \left(10^2\right)^x &= 10^{-3} \\ 2x &= -3 \\ x &= \frac{-3}{2} \end{aligned}\right)$$

4. log (15.85) ≈ 1.2
5. $e^{3.22} \approx 25$
6. 200 (The inverse of doubling is halving; if there are 800 in 2010, there were 400 in 2005 and 200 in 2000.)

7. 24 m

   $A = s^2$

   $s = \sqrt{A}$

   $s = \sqrt{36}$

   $s = 6$

   $P = 4s$

   $P = 4(6)$

   $P = 24\text{ m}$

8. Taxes would be just over \$100. (5 + 8 = 13; half of 25% is 12.5%, so this is a good estimate for the amount of taxes you will pay; 50% of 800 is 400, 25% of 800 is \$200, so 12.5% is \$100; Or, 13% of \$100 = \$13; there are 8 hundreds, so 8 × 13 = \$104)

Part B: Applications of Logarithm Functions

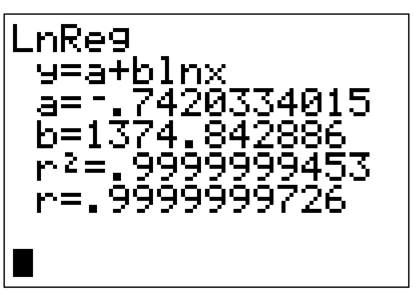
Remember, these questions are similar to the ones that will be on your assignments and midterm examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and midterm examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. The growth of Jack's magic bean plant was recorded by government spy agency satellite cameras and its height over time was recorded in the table below.

Hours	2	3	5	8	10
Height (ft.)	952	1510	2212	2858	3165

   a) Find the natural logarithmic regression equation that best models the data.

   Answer:

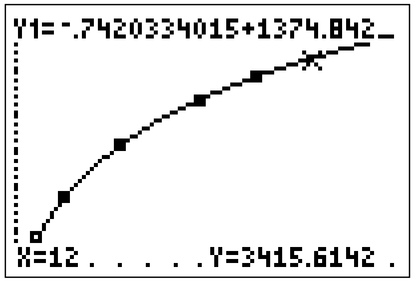


   The equation that models this data is $y = 1374.842886\,(\ln)\,(x) - 0.7420334015$.

b) Use the equation or graph to find how tall the plant was 12 hours after planting.

Answer:

Use a graph.



Adjust the window as necessary and solve for y when $x = 12$.

The plant would be about 3415.6 feet tall.

Or, use the equation.

$$y = 1374.842886(\ln)(x) - 0.7420334015$$

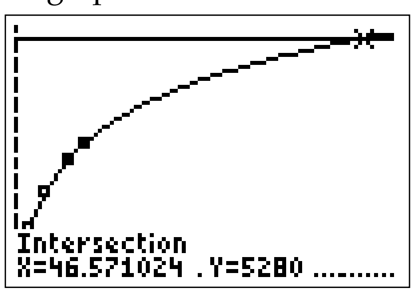
$$y = 1374.842886(\ln)(12) - 0.7420334015$$

$$y = 3415.614196$$

c) After how many hours will the plant reach a height of one mile? (5280 ft. = 1 mile)

Answer:

Use a graph.



Or, use the equation.

$$y = 1374.842886(\ln)(x) - 0.7420334015$$

$$5280 = 1374.842886(\ln)(x) - 0.7420334015$$

$$\frac{5280 + 0.7420334015}{1374.842886} = \ln x$$

$$e^{\left(\frac{5280.7420334015}{1374.842886}\right)} = x$$

$$e^{3.840978549} = x$$

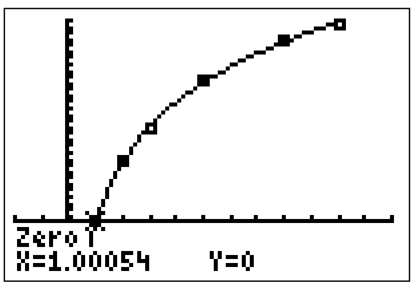
$$x = 46.57102418$$

It will take about 46 and a half hours to reach a height of one mile.

d) Find the x-intercept and explain its significance.

Answer:

Use the graph.



Or, use the equation.

$$y = 1374.842886(\ln)(x) - 0.7420334015$$

$$0 = 1374.842886(\ln)(x) - 0.7420334015$$

$$\frac{0.7420334015}{1374.842886} = \ln x$$

$$e^{\left(\frac{0.7420334015}{1374.842886}\right)} = x$$

$$e^{3.840978549} = x$$

$$x = 1.000539868$$

The x-intercept is at approximately (1, 0). This means that the plant broke the surface of the ground approximately 1 hour after being planted.

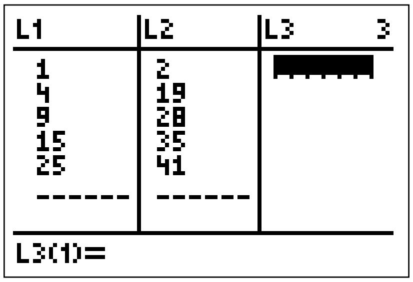
2. The number of friends someone has on a social networking website is said to fit a logarithmic model. Consider the data below for a recent new user of the website.

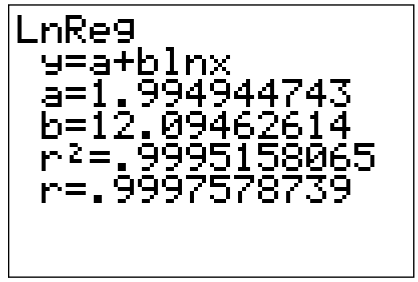
# Days	1	4	9	15	25
Friends	2	19	28	35	41

a) Find the logarithmic regression equation that best models the data.

Answer:

$y = 12 \ln x + 2$





b) Use the equation or graph to find how many friends this person will have after 2 months.

Answer:

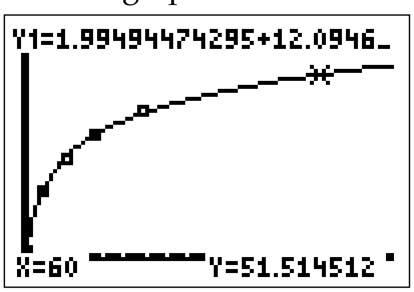
Use the equation.

$$y = 12.094626141626 \ln x + 1.99494474295$$

$$y = 12.094626141626 \ln(60) + 1.99494474295$$

$$y \approx 51.51451152$$

Or, use the graph.



This person will have about 51 friends after 2 months, assuming there are about 30 days in a month.

c) How long will it take until this person has 65 online friends?

Answer:

Use the equation.

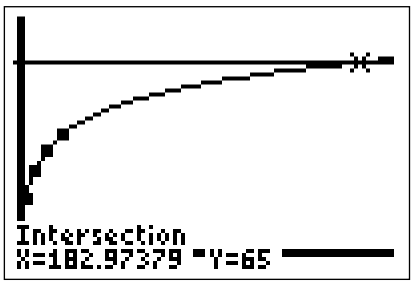
$$y = 12.094626141626 \ln x + 1.99494474295$$

$$65 = 12.094626141626 \ln x + 1.99494474295$$

$$e^{\frac{63.00505526}{12.094626141626}} = x$$

$$x = 182.9737932$$

Or, use the graph.



It will take about 183 days.

3. The pH of a substance, its acidity or alkalinity, is determined by the hydrogen ion concentration, $[H^+]$ in moles per litre, according to the formula $pH = -\log [H^+]$. Water has a neutral pH of 7. Acids have a pH less than 7 and basic (alkaline) solutions have a pH greater than 7.

   a) Orange juice has a hydrogen ion concentration of about 2.8×10^{-4} moles per litre. What is its pH?

   Answer:

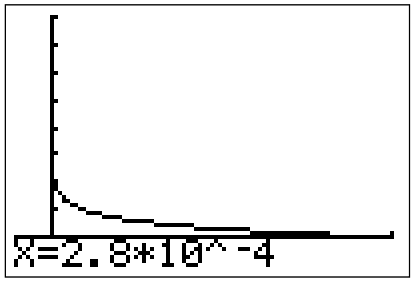
   Use the equation.

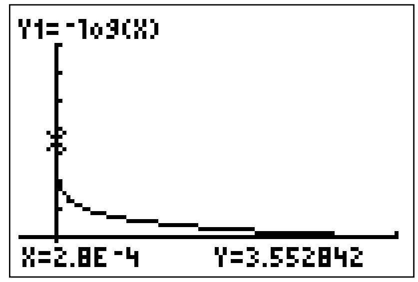
$$pH = -\log \left[H^+\right]$$

$$pH = -\log \left[2.8 \times 10^{-4}\right]$$

$$pH = 3.552841969$$

Or, use the graph.





The pH of orange juice is acidic. It is approximately 3.6 on the pH scale, which is less than 7.

b) The pH of ammonia is 8.9. Determine its hydrogen ion concentration.

Answer:

$pH = -\log\left[H^+\right]$

$8.9 = -\log\left[H^+\right]$

$-8.9 = \log\left[H^+\right]$

$10^{-8.9} = \left[H^+\right]$

$\left[H^+\right] = 1.26 \times 10^{-9}$

The hydrogen ion concentration of the alkaline solution is about 1.26×10^{-9} moles per litre.

4. The Richter Scale expresses the magnitude of an earthquake, R, by measuring the amount of energy E, that is released given in joules. It is expressed by the formula $R = \frac{2}{3}\log\frac{E}{10^{4.8}}$.

Since the magnitude is based on a log scale of base 10, when the magnitude of an earthquake is increased by 1 unit on the Richter Scale, its energy is increased by approximately 10 times.

a) Calculate the magnitude, R, of an earthquake that releases 1.778279×10^8 joules of energy.

Answer:

Note on solution strategies: Logarithmic functions often use values that are very large and very small, so using the equation rather than the graph is sometimes more convenient than trying to set an appropriate window to view the necessary values. Either strategy—using the graph or the equation—is acceptable.

$$R = \frac{2}{3} \log \frac{E}{10^{4.8}}$$

$$R = \frac{2}{3} \log \frac{1.778279 \times 10^8}{10^{4.8}}$$

$$R = 2.3$$

It would be a magnitude 2.3 earthquake.

b) How much energy is released by a 4.5 magnitude earthquake?

Answer:

$$R = \frac{2}{3} \log \frac{E}{10^{4.8}}$$

$$4.5 = \frac{2}{3} \log \frac{E}{10^{4.8}}$$

$$4.5 \times \frac{3}{2} = \log \frac{E}{10^{4.8}}$$

$$6.75 = \log \frac{E}{10^{4.8}}$$

$$10^{6.75} = \frac{E}{10^{4.8}}$$

$$E = 3.548133892 \times 10^{11} \text{ joules}$$

c) How many times more energy is released by a 4.5 magnitude earthquake compared to a magnitude 2.3?

Answer:

$$3.548133892 \times 10^{11} \div 1.778279 \times 10^8 = 1995.262775$$

The 4.5 magnitude earthquake releases almost 2000 times more energy than a magnitude 2.3 earthquake.

Grade 12 Applied Mathematics (40S)

Module 2
Mathematics Research Project

Module 2: Mathematics Research Project

Introduction

Mathematics is many things: a way to count, measure, and quantify; a pattern used to order and organize; a language to describe; and a method to predict or find a value. It is a tool used to manipulate, create, design, and help explain and gather information. Mathematics is useful for science, psychology, medicine, linguistics, sports, cooking, flying, exploring, and communicating. And so much more!

In this module, you will select a current event or area of interest and use mathematics to help you research and collect quantitative data in order to answer a question, analyze and interpret your findings, and create a presentation to showcase your work.

Assignments in Module 2

When you have completed the assignments for Module 2, submit your completed assignments to the Distance Learning Unit either by mail or electronically through the learning management system (LMS). The staff will forward your work to your tutor/marker.

There are two assignment submissions for Module 2. You will find **two** module cover sheets for Module 2 at the end of the course Introduction.

- **Submission 1:** Assignment 2.1: Project Proposal **must** be submitted, along with the Module 2 Cover Assignment: The Tower of Hanoi. You must **respond** to the feedback from your tutor/marker for Assignment 2.1 before you start to work on Assignments 2.2 to 2.4.
- **Submission 2:** Assignments 2.2 to 2.4 must be submitted together as a group once you have completed the module.

Lesson	Assignment Number	Assignment Title
	Cover Assignment	The Tower of Hanoi
1	Assignment 2.1	Project Proposal
2	Assignment 2.2	Collecting and Assessing Data
3	Assignment 2.3	Interpreting Data
4	Assignment 2.4	Presentation

Resource Sheet

When you write your midterm examination, you are encouraged to take a Midterm Examination Resource Sheet with you into the examination. This sheet will be one letter-sized page, 8½ ″ by 11″, with both sides in your handwriting or typewritten. You will submit it with your examination, but you do not receive any marks for it.

Many students have found that preparing a resource sheet is an excellent way to review. It provides you with a summary of the important facts of each module. You should complete a resource sheet for each module to help with your studying and reviewing. Lesson summaries and module summaries are included for you to use as a guide.

You may use the list of instructions provided below to help you with preparing your resource sheet for the material in Module 2. On this sheet, you should record math terms and definitions, formulas, sample questions, or a list of places where you often make mistakes. You should also identify special areas that require extra attention or review by writing the page numbers.

After you have completed each module's resource sheet, you may summarize the sheets from Modules 1, 2, 3, and 4 to prepare your Midterm Examination Resource Sheet. The midterm examination for this course is based on Modules 1 to 4.

Resource Sheet for Module 2

As you go through the lessons of this module, you may want to consider the following suggestions regarding the creation of a resource sheet.

1. List all the important math terms, and define them if necessary.
2. List all the formulas and perhaps a sample problem that shows how the formula is used.
3. If necessary, write the solutions to some problems, showing in detail how you did the calculations.
4. Copy any questions that represent the key points of the lesson, and perhaps include the solutions as well.
5. Identify the problems you found most difficult, and copy the page numbers onto the resource sheet so that you can review them before writing the examination. You may also copy the problems and the solutions onto your resource sheet, and later write them onto your Midterm Examination Resource Sheet.
6. Write any comments, ideas, shortcuts, or other reminders that may be helpful during an examination.

Overview

Read through this entire module before beginning to work on it so that you have a complete understanding of the expectations of this project. There are **four** steps to this project. You need to complete the assignment for the first step, the project proposal, and submit it electronically to the Distance Learning Unit for feedback from your tutor/marker. Do not proceed with the other three project steps until you have received this feedback and responded to your tutor/marker feedback. The four steps in this project are

- Step 1: Project Proposal—topic and question selection and data source identification (Assignment 2.1)
- Step 2: Collecting and Assessing Data (Assignment 2.2)
- Step 3: Interpreting Data—analyzing and interpreting the data using mathematical methods (Assignment 2.3)
- Step 4: Presentation (Assignment 2.4)

You *must* submit Assignment 2.1: Project Proposal electronically to the Distance Learning Unit and respond to feedback from your tutor/marker before continuing with Assignments 2.2, 2.3, and 2.4. Refer to How to Submit Assignments found in the course Introduction for instructions on how to submit assignments electronically. Submitting your work in this way will ensure that your assignments reach your tutor/marker quickly, and that you receive your feedback much faster.

Although it will take longer, if it is not possible to submit Assignment 2.1 electronically to the Distance Learning Unit, it can be sent and the feedback received by regular mail. **Remember, you must respond to the feedback from your tutor/marker whether you use regular mail or submit your work electronically.**

Remember that there are two separate assignment submissions for Module 2. When you submit Assignment 2.1: Project Proposal to the Distance Learning Unit for feedback, you must also include the Module 2 Cover Assignment and the appropriate Module 2 Cover Sheet for these two assignments (Module 2, Cover Sheet 1).

While you are waiting for the feedback from your tutor/marker regarding your project proposal, you might want to begin Module 3. Do not continue with the Module 2 assignments until you have responded to the feedback from your tutor/marker regarding the Project Proposal, Assignment 2.1.

Evaluation

This module is handled differently from the other modules in the course. In this module, you are asked to submit Assignment 2.1: Project Proposal (the topic and data source identification) separately in order to receive feedback from your tutor/marker before proceeding to the next lesson and assignment in the module project. This will help to ensure that your final presentation is on track and fulfills the requirements and expectations of this module. Each of the four steps in this project will be assessed according to the rubrics provided in this module and with the cover sheets. As you complete each section, you should assess your own work according to the corresponding rubric to see how well you have met the expectations.

It is critical that your topic and question for research are appropriate before committing time and energy into research and analysis. After responding to the feedback you receive for your work on Assignment 2.1, you should complete and submit Assignments 2.2, 2.3, and 2.4 together to the Distance Learning Unit for assessment. A rubric is provided for each step of the project (or each assignment) and can be found at the end of each assignment in the course and with the cover sheets found at the end of the course Introduction. Use each rubric to do a self-evaluation for each step.

If you require assistance with either the Cover Assignment or with Assignments 2.2, 2.3, or 2.4, you should contact your tutor/marker.

It is acceptable to work on Module 3 and on the individual project assignments in Module 2 concurrently. Working on the next module while you are waiting for and responding to feedback from your tutor/marker may be a good time management strategy. During the process of conducting your research and completing the analysis, you may find that you need to redefine your original question, find alternate data sources, or refine your testing methods. Feedback from your tutor/marker as you work through this process is a very valuable resource.

When you submit Assignment 2.1 by mail, you need to include the appropriate cover sheet (Module 2, Cover Sheet 1). Similarly, when you submit Assignments 2.2 to 2.4, you need to include the appropriate cover sheet (Module 2, Cover Sheet 2). Remember to contact your tutor/marker or check your work submission guidelines found in the course Introduction for instructions on how to submit work electronically.

Module 2 Cover Assignment: The Tower of Hanoi

The Tower of Hanoi is a puzzle that was invented by a French mathematician in 1883. It involves disks of different sizes that are moved between three pegs, with specific constraints on how and where they can be stacked.

To begin, the disks are stacked on a single peg in order by size, with the largest at the bottom. The object of the game is to move the entire stack to another peg, ending with the same arrangement by size.

You can only move one disk at a time, taking it from the top of one stack and placing it on the top of another stack. A larger disk may never be placed on top of a smaller disk.

The goal is to use the fewest number of moves possible.

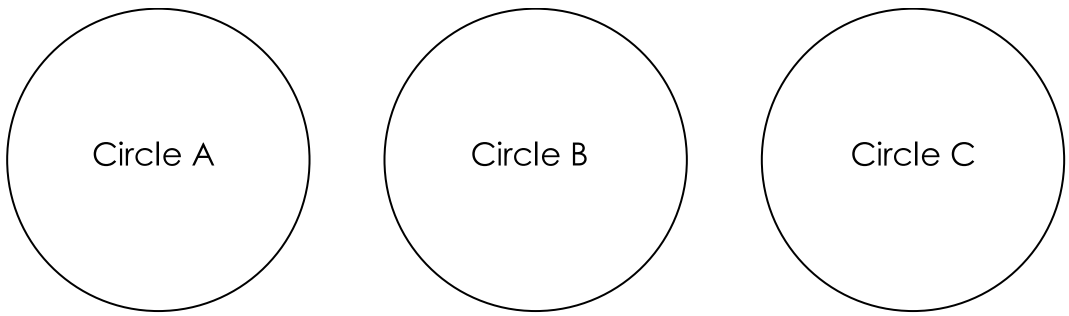
Notes

Module 2 Cover Assignment

The Tower of Hanoi

Total: 10 marks

Create your own version of this puzzle using coins and the circles below.



Stack a loonie, a quarter, a nickel, a penny, and a dime in order by size (largest on the bottom) on Circle A. Moving one coin at a time from one circle to another, move all the coins to Circle C so that the coins end up in the original order with the largest on the bottom. Make sure to move only the top coin from any stack and that a larger coin is never placed on top of a smaller coin. Coins can be moved back and forth among stacks placed on Circle A, Circle B, and Circle C.

Note: If you do not have any of one or more types of coins, you can use any five objects that are each different in size. Examples may include lids from plastic containers, buttons, or circular game pieces. The important characteristics of the five objects you use are:

- they stack easily
- all five are different sizes

a) Determine the fewest number of moves required to transfer a stack of five coins from Circle A to Circle C so that they are in the original order. *(1 mark)*

b) Describe the strategy and/or patterns you used to solve this puzzle. *(3 marks)*

c) Identify the pattern in the number of moves required. **Hint:** You may have to determine how many moves would be required if you started with two coins, with three coins, etc. *(3 marks)*

d) If a stack of 64 coins were to be moved at a rate of one move per second, it would take about 585 billion years to complete the puzzle. Verify this based on the pattern you found. *(3 marks)*

Lesson 1: Project Proposal: Topic and Question Selection and Data Source Identification

Lesson Focus

In this lesson, you will

- ❑ identify a current event or area of interest and determine a question to motivate research
- ❑ identify primary or secondary data sources pertinent to your question that you intend to use

Lesson Introduction

The first critical step in this research project is the project proposal in which you choose an appropriate topic and question. One aspect to consider is your interest in and motivation for learning more about the topic. You should begin by identifying several current events or areas of interest that you would like to know more about. You may want to look in newspapers, visit online sites, and watch the news, keeping in mind your personal interests (sports, arts, environment, health, nutrition, science, etc.).

Topic and Question Selection

At this point, you should brainstorm about specific topics related to each of your areas of interest, which could motivate your search for data and information. You may find that creating a word web or mind map of your thoughts is a helpful strategy to organize and generate your creative ideas.

An important aspect of creating a project proposal is to narrow your options and choose the one question that most interests you and lends itself best to the requirements of this project. This question must not be something that a quick visit to a Wikipedia site will answer, although that site may be one of the sources you use to research information. The question must be some sort of an inquiry-based issue for which you will need to find or collect quantitative (numerical) data and information, analyze it, and then come to your own conclusion. Your final decision regarding the question you plan to research and answer should be based on available resources and your ability to find an answer given the timeline.

Your subject, topic, and question possibilities are endless. You are not in any way limited to the concepts covered in this course, although you will need to engage in some mathematical reasoning and analysis of the data you use or provide information regarding a topic that involves mathematics.

Note: It is important to remember that your proposal must outline how you will use Grade 12 skills to analyze your data, and that it must reflect a significant effort at this level. Each module in this course requires many hours to complete, and the research project is expected to demonstrate a similar time commitment.

Data Source Identification

Along with your proposed question, you need to submit a list of potential data sources or methods of data collection you intend to use. Before progressing too far into this project, it is important to determine whether the resources exist for you to find an answer to your inquiry question.

With respect to your chosen topic involving a current event or an area of interest, there are two categories of data that you may use for this project—primary and secondary.

Primary Data

Primary data consist of information or statistics that you personally collect. It may be the results of a survey or an experiment you conduct where you record results. Consider what type of data you can measure, what credible population/sample sources you can tap into, and what equipment you have available to conduct experiments.

Whenever possible, use primary data or sources that publish their own primary data. It is always possible that slight errors are incorporated into the analysis of data and information by secondary sources and, if republished, quoted, and copied by multiple parties, the data may become less and less reliable.

Secondary Data

Secondary data are statistics or information that someone else has collected, recorded, and published in some form that is accessible to you. It may be found in science textbooks, psychology journals, sports or technology magazines, or online at the Statistics Canada or Environment Canada websites. Results may be published in articles covering topics such as literature, art, politics, music, gaming, forensics, and architecture. You may find data in the financial markets sections of a newspaper. The value of using second party data compared to collecting your own depends on the nature and focus of your topic and question.

As noted above, try to use sources that publish their own primary data, rather than reprints of information with a vague source and history. Keep in mind that you also need to be very aware of how the data was collected and by whom, in order to identify potential biases and to assess the reliability, relevance, and accuracy of the data. You may have to sift through a lot of secondary data to find the information pertinent to your question. As well, you may find data from multiple sources that either support or contradict each other.

Qualitative vs. Quantitative Data

The type of data and information that you should collect and analyze is quantitative data. Quantitative data is statistical data that can be measured or counted using numerical scales or units. It can be categorized, recorded in tables, and graphed. Qualitative data, which describes the observable characteristics of a population, is important in some cases, but for this project you should be concerned with data that can be measured numerically.

Avoiding Plagiarism

In this project, you will likely be using data and information that is not your own. If you submit your project in such a way that another person's intellectual property appears to be your own, you are plagiarizing. This means you are taking credit for someone else's work. In the collaborative world of the Internet and with the wide accessibility of information, it only makes sense to share and learn together with, and from, others. It becomes very important, therefore, to give credit where credit is due. You are encouraged to use the ideas of others but you must record and submit a bibliography citing all sources of information, ideas, and data you used. If you include word-for-word quotations, use quotation marks and endnotes or footnotes to cite the source in the text body, and include the source in the bibliography. If you paraphrase content or use someone else's ideas, reference the source in the text and include it in the bibliography.

Appropriate ways to cite sources and write a bibliography in MLA and APA styles can be found online or in reference materials available in the library. Typically, include as much of the following information as possible for each source you reference and use in your report:

Last name, First name. *Book Title*. City: Publisher, Year of publication.

Last name, First name. "Article Title." *Journal Title Series*, Volume. Issue (Year published): page(s).

Last name, First name. "Article Title." *Website Title*. Publisher of Website, Day Month Year article was published. Day Month Year article was accessed. <URL>.

Some examples:

Rothstein, Edward. "Opening the Doors to the Life of Pi." *The New York Times*. The New York Times Company, 13 December 2012. Accessed on 30 March 2013. http://www.nytimes.com/2012/12/14/arts/design/museum-of-mathematics-at-madison-square-park.html.

"Waste disposal by source, province and territory." *Statistics Canada*. Government of Canada, 22 December 2010. Accessed on 30 March 2013. http://www.statcan.gc.ca/tables-tableaux/sum-som/l01/cst01/envir25a-eng.htm.

Wikipedia contributors. "Fibonacci." *Wikipedia, The Free Encyclopedia*. Wikipedia, The Free Encyclopedia, 26 March 2013. Retrieved 30 March 2013 from http://en.wikipedia.org/w/index.php?title=Fibonacci&oldid=547019561.

A bibliography of all sources used is required in the final presentation of this project, so begin now by keeping a detailed record of the all sites and resources you explore. You may want to print pages from sites you use for your own records. The Internet is often a temporary publishing site and there are no guarantees that sites and links that are active today will still be alive tomorrow.

Lesson Summary

In this lesson, you began the process to create your mathematics research project by developing the project proposal, which includes a good question based on a current event or area of interest. You learned about primary and secondary data, you learned the difference between quantitative and qualitative data, and you identified the sources and methods you intend to use to collect or generate data to answer your question. You began to record the bibliographic information of all sources you intend to use in this project.

Assignment 2.1

Project Proposal

Total: 9 marks

This is a hand-in assignment. You must submit this assignment in order to receive feedback from your tutor/marker regarding the question and data sources or methods you intend to use in your research. You need to act on the feedback you receive **before** continuing with Lessons 2 to 4.

Attach the cover sheet, indicating that you are submitting the Module 2 Cover Assignment and Assignment 2.1. In order to receive a more prompt response, electronically submit your assignment, if possible, rather than using regular mail.

Topic and question selection and data source identification. *(9 marks)*

1. List at least three different current events or areas of interest that would be suitable for a research project based on quantitative data. Indicate what types of data could be collected for each of your selected topics.
2. For each of these topics, define a question that could be a specific focus of your research, based on the types of data you identified above.
3. From these topics, select your choice of a focused question to motivate your pursuit and analysis of research material. Explain how your topic is inquiry-based and how it is appropriate for the collection and analysis of quantitative data and information. Discuss how you will use Grade 12 skills to analyze your data. Remember that the completed project is expected to reflect a significant investment of time and effort.
4. Identify sources and methods to be used if you are generating primary data, and a minimum of four resources to be used for secondary data and information. Include detailed bibliographic information, using appropriate format, for all sources.

Your tutor/marker will indicate either “complete” or “incomplete” when this assignment is returned. You must receive a “complete” designation on this assignment before moving on to any other lessons or assignments in Module 2. You may, however, proceed to work on Module 3 during this process.

You will receive a “complete” designation when you earn a minimum of 6 out of 9 points.

The following rubric will be used when marking the assignment.

Assignment 2.1: Project Proposal (continued)

Rubric for Assignment 2.1			
Category	Level 1	Level 2	Level 3
Question	Question is poorly worded, is impractical, or is inappropriate	Question is somewhat vague or too narrow	Question is focused and clearly communicated
Availability of appropriate research	Question can be answered without data or information collection or research, or it is impossible to find data	It is possible but may be difficult to generate primary data or find adequate or appropriate data or information	Meaningful, appropriate, and applicable information or statistics can be generated or are available
Description of proposed primary and/or secondary data collection methods and/or sources to be used	Minimal information is provided regarding how data will be collected or where information will be found	Incomplete information regarding how data will be collected or where information will be found is provided	Intentions regarding primary and/or secondary data collection methods and sources are clearly outlined, including proper bibliographic citation

Feedback is required on Assignment 2.1 before continuing with Assignments 2.2 to 2.4.

Submitting Your Assignments

It is now time for you to submit the Module 2 Cover Assignment and Assignment 2.1 to the Distance Learning Unit so that you can receive some feedback on how you are doing with the project proposal. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 2 Cover Assignment and Assignment 2.1 and organize your material in the following order:

- ❑ Module 2 Cover Sheet 1 (found at the end of the course Introduction)
- ❑ Cover Assignment: The Tower of Hanoi
- ❑ Assignment 2.1: Project Proposal

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Notes

Lesson 2: Collecting and Assessing the Data

Lesson Focus

In this lesson, you will

- ❑ collect primary or secondary data related to your research question
- ❑ assess the accuracy, reliability, and relevance of the data by identifying bias, analyzing the collection methods, and determining if the data is relevant and consistent

Lesson Introduction

After carefully choosing a project question and getting approval for your project proposal, you are ready to proceed with the collection and analysis of data. The data may involve a numerical survey or the data may be information that is gathered from a number of sources regarding your topic of choice. In this lesson, you will learn some important points to consider regarding the quality of the data you collect.

Collecting Data

Once you have identified the sources and collection methods for the data you intend to use and have received and responded to feedback from your tutor/marker, you can engage in gathering, recording, and assessing your statistical and/or informational data. Keep accurate records of everything you do and use. As you read and work with the data, be vigilant and critical in your analysis of it.

Assessing Data

The fact that information has been published does not mean that the data is accurate, reliable, or relevant. Often, books that end up in a school library have had to pass the scrutiny of librarians and teachers before they are put on the shelves. The Internet has no such regulatory safety net. Qualified professionals and "weirdo quacks" alike can publish their findings and analyses. It is up to you—the researcher—to judge the reliability of what you read and use. Here are some points to consider when assessing the accuracy, reliability, and relevance of data and information.

Reliability

Ask the following questions when assessing your research:

- Is what you are reading dependable?
- Would another test or experiment produce the same results?
- Are the results quoted on a reputable site or in a scholarly article, showing that qualified experts agree with them?
- Is the data being published by a primary or secondary source?

Try to find the same type of information or results published either online on another site, or in print in a book or magazine, and compare them. Reliable data is consistent. It should be taken from a sample that was an appropriately selected random and representative cross-section of the population.

Accuracy

Accurate data will be close to the true or actual values. It should be free from errors and carefully collected and recorded. Examine the bibliography provided on the site. Determine if the sources used by the author are trustworthy and adhere to high standards, or appear to be opinionated and far-fetched. Look for when the data was published or last updated. Check to ensure that the site includes the latest research and information available.

Relevance

Try to determine the purpose of the data and information to make sure it is in line with what your interest is.

- Why was the information published?
- Is the site trying to sell something, to inform others, to teach, to persuade?
- Who is the intended audience?

Author

Find out who actually wrote the information you are considering and what their credentials are. Using information posted on a personal blog may not have the same degree of reliability as using statistical data published by a governmental agency.

- Does the author speak with authority?
- Is the author working for a reputable organization?
- Is the author's name and contact information given?
- Who holds the copyright to the published material?

Bias

Depending on who wrote the information and why it is published, bias may be inherent in the results. Advertising or sponsorship on the site may indicate the need to be suspicious of the content of that page. Look for biased wording or examples of opinion rather than facts being used. Determine if underlying assumptions are affecting the reliability of the data.

Lesson Summary

In this lesson, you collected and assessed primary and/or secondary data for accuracy, reliability, and relevance.

Notes

Assignment 2.2

Collecting and Assessing Data

Total: 12 marks

Work on this assignment only **after** you have received and responded to the feedback from your tutor/marker and received a "complete" designation for Assignment 2.1: Project Proposal. This is a hand-in assignment.

Assignments 2.2, 2.3, and 2.4 should be submitted together, so hand in all three as a package when they are complete. However, you may ask for and receive feedback and advice from your tutor/marker as you are working on each individual assignment. When you have completed Assignments 2.2, 2.3, and 2.4, attach the appropriate cover sheet (Module 2, Cover Sheet 2) to your work and submit them for marking. Remember to include the rubrics for Assignments 2.2, 2.3, and 2.4, which are included with the cover sheet, found at the end of the course Introduction.

Collecting and assessing the data. *(12 marks)*

(**Note:** You may have both, but only one type of data is required—either primary or secondary.)

Primary data

1. State your focus question, and explain the collection method used.
2. Identify the population and describe the process used for selecting a random, representative sample.
3. Outline the test, experiment, or survey method used.
4. Assess the accuracy, reliability, and relevance of the data. Identify the point of view and/or biases in the test or collection method and explain how that affects the data. Identify data or information from other sources that supports or contradicts your data. Be sure to refer to the information in Lesson 2 regarding data assessment.
5. Include a copy of all raw data or information generated or collected and all pertinent bibliographic information.

Assignment 2.2: Collecting and Assessing Data (continued)

Secondary data

1. State your focus question, and identify the sources of your data.
2. Explain the collection methods used by the authors. Identify the population and the sample.
3. Assess the accuracy, reliability, and relevance of the data. Identify the point of view and/or biases in the test or collection method and explain how that affects the data. Identify data or information from other sources that supports or contradicts your data. Be sure to refer to the information in Lesson 2 regarding data assessment.
4. Include a copy of all raw data or information used and all pertinent bibliographic information.

The following rubric will be used when marking the assignment. Your tutor/marker will use the rubric included with the module cover sheet to provide feedback.

Rubric for Assignment 2.2			
Category	Level 1	Level 2	Level 3
Data and information	Inadequate or irrelevant data and information are provided	Data and information provided are incomplete	Complete data and information are provided, including data/information from other sources that contradicts or supports viewpoint
Data collection (primary or secondary)	Poor data collection methods are used, with inappropriate population or sample, or inadequate description Sources are not identified	Data collection methods are not clearly outlined or explained; population and sample are not clearly explained Bibliographic information is incomplete	Collection methods are clearly outlined; population and random, representative sample are identified and used All sources are clearly identified with complete bibliographic information
Assessment	Information and data are not assessed for bias, point of view, accuracy, reliability, and relevance	Assessment of information and data is incomplete	Assessment of the data and information is clearly communicated and includes explanation of bias, point of view, accuracy, reliability, and relevance

Lesson 3: Interpreting the Data

Lesson Focus

In this lesson, you will

- ❑ use mathematical methods (as appropriate) to represent, analyze, and interpret your data

Lesson Introduction

To make sense of the research information, facts, and figures you have compiled, you now need to analyze and interpret them in a way that is meaningful and appropriate. By applying mathematical reasoning and statistical and regression analysis, you can explore the potential answers to your question.

Representing, Analyzing, and Interpreting the Data

First, consider how to best represent the data and information you have collected. This may be done in many different ways, including charts, graphs, lists, tables, diagrams, paragraph structure, and so forth. The most appropriate display will depend on the nature and quantity of your data. You can choose how to best represent your data.

The type of data and information you have will affect how you analyze it. In previous math courses, you found the mean, median, and mode of data. You used linear equations to approximate data and model patterns in contextual situations. In the first module of this course, you explored the characteristics of polynomial, exponential, and logarithmic functions and used them to model data and represent trends. You used graphs to interpolate, extrapolate, and calculate values based on the graph and equation representing the information. Other statistical measures such as z-scores, standard deviation, normal distribution, confidence intervals, and margin of error (covered in Grade 11 Applied Mathematics) may be appropriate ways to describe and evaluate your data.

In this step of the project, you need to analyze your data and information in a meaningful and appropriate way in order to come to an interpretation and conclusion. This part of the project may require you to look back at previous course notes or investigate online for help in using statistical measures. Ongoing dialogue with your tutor/marker will help guide you towards appropriate analysis options.

Your interpretation of the results should lead to a logical conclusion regarding the initial question you researched. You may end up with an interpretation that surprises you. It may support or refute what you initially set out to prove. Base your explanation on the data and information you collected, and support it as specifically as possible.

Lesson Summary

In this lesson, you represented your data/information and applied mathematical reasoning, statistical methods, and/or regression analysis to examine it. You interpreted the results to come to a conclusion regarding the initial research question.

Assignment 2.3

Interpreting Data

Total: 15 marks

Work on this assignment only **after** you have received and responded to the feedback from your tutor/marker and received a “complete” designation for Assignment 2.1: Project Proposal. This is a hand-in assignment.

Assignments 2.2, 2.3, and 2.4 should be submitted together, so hand in all three as a package when they are complete. However, you may ask for and receive feedback and advice from your tutor/marker as you are working on each individual assignment. When you have completed Assignments 2.2, 2.3, and 2.4, attach the appropriate cover sheet (Module 2, Cover Sheet 2) to your work and submit them for marking. Remember to include the rubrics for Assignments 2.2, 2.3, and 2.4, which are included with the cover sheet, found at the end of the course Introduction.

Analyzing and interpreting the data using mathematical methods. *(15 marks)*

1. Represent your data/information in a way that clearly communicates the results of your research.
2. Analyze the data or information you collected or found using appropriate mathematical reasoning, statistical methods, or regression analysis.
3. Interpret the results of your analysis of the data. Use this to answer the initial research question. Justify your conclusions based on the research and analysis.

The following rubric will be used when marking the assignment. Your tutor/marker will use the rubric included with the module cover sheet to provide feedback.

Assignment 2.3: Interpreting Data (continued)

Rubric for Assignment 2.3			
Category	Level 1	Level 2	Level 3
Represent data	Representation of data and information is inappropriate or incomplete	Data and information are not clearly represented, or may have errors or omissions	Use of graphs, charts, or equations to appropriately represent data collected is effective and complete
Analysis	Analysis applied to data is inappropriate or irrelevant	Analysis of data and information is incomplete, with limited mathematical or statistical analysis shown	Complete analysis and meaningful application of mathematical, statistical, or regression analysis to model data are included
Interpretation	Incorrectly applied analysis leads to incorrect interpretations	Interpretations are based on weak analysis; analysis may not always be supported by data	Mathematical reasoning or calculations are applied to make sense of analysis, showing a thorough understanding of the issue

Lesson 4: Presentation

Lesson Focus

In this lesson, you will

- ❑ create a final presentation of your research project

Lesson Introduction

The presentation of your question, research, analysis, and the conclusions you reached to answer your question may be done with or without technology. You may be creative in your organization and presentation of the data, process, and conclusions, but it should be focused, clear, concise, and engaging. If applicable, identify how your results may be controversial. Represent multiple viewpoints or sides of the issue that are possible, based on the contradictory or supporting evidence you collected.

Lesson Summary

In this lesson, you tied together everything you have completed in this module and presented your research question, data, process, and conclusions.

Notes

Assignment 2.4

Presentation

Total: 9 marks

Work on this assignment only **after** you have received and responded to the feedback from your tutor/marker and received a “complete” designation for Assignment 2.1: Project Proposal. This is a hand-in assignment.

Assignments 2.2, 2.3, and 2.4 should be submitted together, so hand in all three as a package when they are complete. However, you may ask for and receive feedback and advice from your tutor/marker as you are working on each individual assignment. When you have completed Assignments 2.2, 2.3, and 2.4, attach the appropriate cover sheet (Module 2, Cover Sheet 2) to your work and submit them for marking. Remember to include the rubrics for Assignments 2.2, 2.3, and 2.4, which are included with the cover sheet, found at the end of the course Introduction.

Create a final presentation of your research project. *(9 marks)*

1. Effectively and clearly organize and communicate the question, data/information, process of analysis, and conclusion of your research project.
2. Include graphs, charts, or models to help communicate your research.

Note: Your presentation can include aspects of work from Assignments 2.2 and 2.3, but **should not** be a resubmission of your previous work.

The following rubric will be used when marking the assignment. Your tutor/marker will use the rubric included with the module cover sheet to provide feedback.

Assignment 2.4: Presentation (continued)

Rubric for Assignment 2.4			
Category	Level 1	Level 2	Level 3
Organization	Presentation is disorganized, unclear, or incomplete	Presentation of research is adequate but simplistic	Creative, organized, and purposeful presentation is presented, and ideas are clearly and concisely explained
Process	Process of research is not clearly stated; connection of research to inquiry question is vague or poorly justified	Process of research and methods used to reach a conclusion are not fully stated	Research process is well documented, and controversial issues and multiple viewpoints are identified with supporting data, if applicable
Conclusion	Conclusion is not supported by research	Answer to original question is provided, but only partly justified by research data or information	Logical, conclusive answer to question is presented with justification, based on analysis of data and information

Module 2 Summary

Congratulations, you have finished Module 2! In this module, you collected primary and/or secondary data and information related to a current event or area of interest. You assessed the data, interpreted it using mathematical reasoning, statistical method, or regression analysis, and presented your results in a final project.

In the next module, you will be learning about logic, set theory, and conditional statements.

Submitting Your Assignments

It is now time for you to submit the Module 2 Assignments 2.2 to 2.4 to the Distance Learning Unit so that you can receive some feedback on how you are doing in this course. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 2 assignments and organize your material in the following order:

- ❑ Module 2 Cover Sheet 2 (found at the end of the course Introduction)
- ❑ Assignment 2.2: Collecting and Assessing Data
- ❑ Assignment 2.3: Interpreting Data
- ❑ Assignment 2.4: Presentation

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Notes

Grade 12 Applied Mathematics (40S)

Module 3
Logical Reasoning

Module 3:
Logical Reasoning

Introduction

Welcome to Module 3. In this module, you will be introduced to set theory, look at some of its applications, and solve problems involving conditional statements. This module is part of the logical reasoning topic found in applied mathematics. It is designed to help you develop your ability to think critically and creatively as you solve problems. The skills you use in this module will be applied in many other lessons in this course, and you will find applications for these skills in your daily life as well.

Assignments in Module 3

When you have completed the assignments for Module 3, submit your completed assignments to the Distance Learning Unit either by mail or electronically through the learning management system (LMS). The staff will forward your work to your tutor/marker.

Lesson	Assignment Number	Assignment Title
	Cover Assignment	SET® Game
3	Assignment 3.1	Sets and Conditional Statements

Resource Sheet

When you write your midterm examination, you are encouraged to take a Midterm Examination Resource Sheet with you into the examination. This sheet will be one letter-sized page, 8½ ″ by 11″, with both sides in your handwriting or typewritten. You will submit it with your examination, but you do not receive any marks for it.

Many students have found that preparing a resource sheet is an excellent way to review. It provides you with a summary of the important facts of each module. You should complete a resource sheet for each module to help with your studying and reviewing. Lesson summaries and module summaries are included for you to use as a guide.

You may use the list of instructions provided below to help you with preparing your resource sheet for the material in Module 3. On this sheet, you should record math terms and definitions, formulas, sample questions, or a list of places where you often make mistakes. You should also identify special areas that require extra attention or review by writing the page numbers.

After you have completed each module's resource sheet, you may summarize the sheets from Modules 1, 2, 3, and 4 to prepare your Midterm Examination Resource Sheet. The midterm examination for this course is based on Modules 1 to 4.

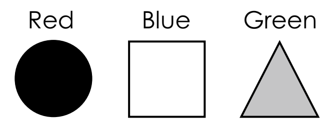
Resource Sheet for Module 3

As you go through the lessons of this module, you may want to consider the following suggestions regarding the creation of a resource sheet.

1. List all the important math terms, and define them if necessary.
2. List all the formulas and perhaps a sample problem that shows how the formula is used.
3. If necessary, write the solutions to some problems, showing in detail how you did the calculations.
4. Copy any questions that represent the key points of the lesson, and perhaps include the solutions as well.
5. Identify the problems you found most difficult, and copy the page numbers onto the resource sheet so that you can review them before writing the examination. You may also copy the problems and the solutions onto your resource sheet, and later write them onto your Midterm Examination Resource Sheet.
6. Write any comments, ideas, shortcuts, or other reminders that may be helpful during an examination.

The goal in the game of SET® is to identify a set of three pictures, based on the colour, shape, number of symbols, and shading depicted.

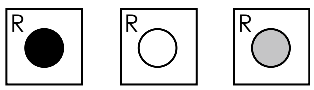
The pictures have a combination of features:



- colour—red, blue, green
- shape—circle, square, triangle
- number—one, two, or three symbols
- shading—solid, outline, shaded

A **set** is three pictures in which each feature is either all the same or all different. Consider the following examples of sets:

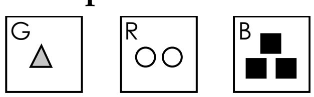
Example 1



Colour is indicated by the first letter in Red, Blue, or Green. You may colour the pictures to make them easier to identify.

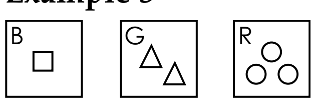
- colour—all the same: all red
- shape—all the same: all circles
- number—all the same: one of each
- shading—all different: solid, outline, shaded

Example 2



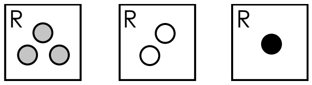
- colour—all different
- shape—all different
- number—all different
- shading—all different

Example 3



- colour—all different
- shape—all different
- number—all different
- shading—all same

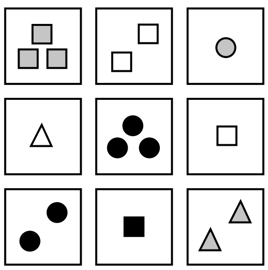
Example 4



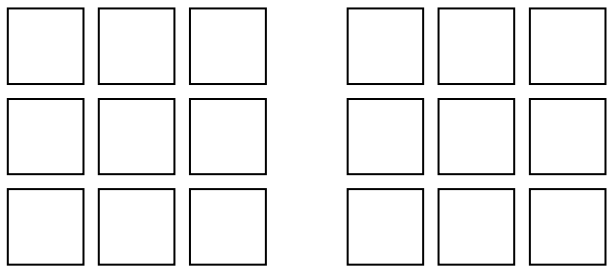
- colour—all same
- shape—all same
- number—all different
- shading—all different

To learn how to play, begin with a simpler game using only three of the four features: shape, number, and shading.

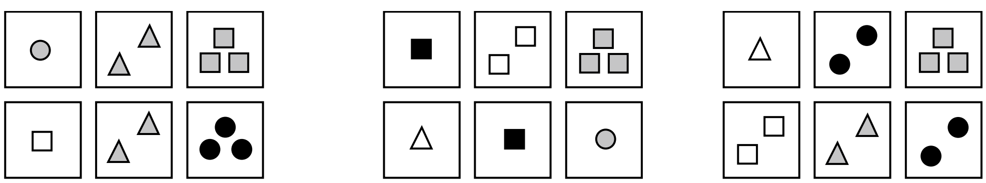
Find all the possible sets of three given the following nine pictures. Draw the solution sets in the frames below.



Possible sets:



Solution



A solution set can be found by answering **yes** to the following four questions:

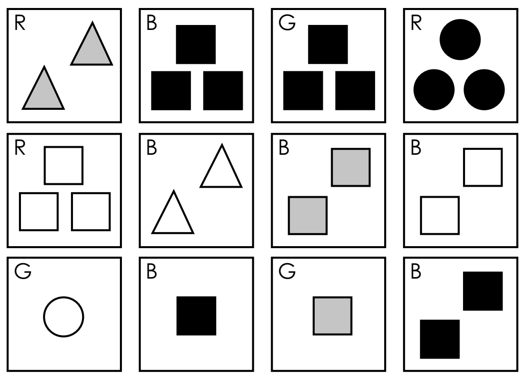
- Are the colours all the same or all different in the pictures in each set?
- Are the shapes all the same or all different in the pictures in each set?
- Are the numbers of shapes all the same or all different in the pictures in each set?
- Are the shapes shaded all the same or all different in the pictures in each set?

Module 3 Cover Assignment

SET® Game

Total: 5 marks

Find five out of six possible sets of three in the following 12 pictures. Draw your solutions neatly in the boxes provided. Colour the diagrams to make it easier to complete.



Notes

Lesson 1: Set Theory

Lesson Focus

In this lesson, you will

- ❑ learn how to solve problems that involve the application of set theory

Lesson Introduction

When you use a search engine such as Google, you are actually using knowledge of sets that you may not be aware that you are using. You may search for the set of movies that have won the Oscar for best picture. You may refine your search by looking for a subset of the best picture winners that includes only the animated films. Alternatively, you may choose to expand the set of movies to include the best picture winners and the movies that received the Oscar for best actors. Set theory is a large branch of mathematics that you will be introduced to in this lesson.

Sets and Subsets

A **set** is a defined group of distinct objects (that is, each object is different). Each object in the group is called an **element** or a **member** of the set.

To designate a set, name it with a capital letter and list the elements inside braces. For example $A = \{1, 8, 27, 64\}$ can be read as "A is the set whose members are 1, 8, 27 and 64." Since these elements are all perfect cube numbers, you could say "A is the set of perfect cube numbers between 0 and 100."

The epsilon symbol, $\in$, is used to denote that an object is a member or element of a set. For example, $27 \in A$ means "27 is an element of set A." The symbol, $\notin$, denotes that an object is not an element of a set. For example, $30 \notin A$ means "30 is not an element of A."

Sets that have the same elements are **equal sets**. For example, $B = \{1, 10, 100\}$ and $C = \{100, 1, 10\}$ are equal sets, as all their members are identical.

Equivalent sets have the same number of elements. For example, $F = \{a, b, c, d, e\}$ and $G = \{v, w, x, y, z\}$ are equivalent sets since they both have five elements. The number of elements in set F is written as $n(F) = 5$, since there are five elements in set F.

A set containing no elements is an **empty set** or a **null set**. It is denoted by empty braces { } or ∅.

If it is possible to count the number of elements in a set, the set is said to be finite. If there is a mama bear, papa bear, and baby bear at the zoo, the set may be Z = {m, p, b} and $n(Z) = 3$. Some sets are uncountable, like the number of integers, $I = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. This set is considered to be **infinite**.

The set of all elements considered part of a specified situation is called a **universal set**. The universal set of digits is $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

A **subset** of a set contains some, all, or none of the members of another set. Some of the possible subsets of set *U* given above could be

- the set of even numbers, $E = \{0, 2, 4, 6, 8\}$, since each element in set *E* is also a member of set *U*.
- the subset of irrational numbers, $I = \{\ \}$, an empty set, since there are no irrational numbers in the universal set of digits, *U*.

A **proper subset** contains some, but not all, of the related set. A subset of the universal set of digits *U*, set $O = \{1, 3, 5, 7, 9\}$, containing the odd numbers is a proper subset of *U*. The notation $O \subset U$ or $U \supset O$ indicates that "*O* is a proper subset of *U*" or "*O* is properly contained in *U*." This also means that *U* contains everything in set *O* plus more.

The set of elements in the universal set that are not part of a subset are the **complement** of that subset. For example, O', called *O* prime, is the set of all elements in the universal set *U* that are not in subset *O* ($O' = \{0, 2, 4, 6, 8\}$). **Note:** You may also see the complement of set *O* denoted using a bar, $\overline{O}$, or with a superscript *c*, O^c.)

You can see that set $O' = \{0, 2, 4, 6, 8\}$ and set $E = \{0, 2, 4, 6, 8\}$ have common elements, while sets $E = \{0, 2, 4, 6, 8\}$ and $O = \{1, 3, 5, 7, 9\}$ have no elements in common. Sets that have no elements in common are called **disjoint sets**.

It may be useful to add these definitions, symbols, and examples to your resource sheet.

Example 1

a) Define universal set *F* as the numbers in the Fibonacci sequence between 0 and 100. **Note:** In the Fibonacci sequence, each number is the sum of the two previous numbers, {1, 2, 3, 5, 8, 13 . . .}.

b) Define subset *A* as the odd numbers in set *F*.

c) Define the complement of set *A*, denoted as A'.

d) Define a subset of F, named set B, so it is equivalent to subset A and explain your reasoning.

e) Is subset $G = \{2, 3, 5, 13, 89\}$ a proper subset of F? Explain your reasoning.

f) Define a subset of F, named D, where D and G are disjoint sets.

g) Define a subset of F, named E, containing the negative numbers in the Fibonacci sequence between 0 and 100.

Solution

a) Define universal set F as the numbers in the Fibonacci sequence between 0 and 100. **Note:** In the Fibonacci sequence, each number is the sum of the two previous numbers, {1, 2, 3, 5, 8, 13 . . .}.

$F = \{1, 2, 3, 5, 8, 13, 21, 34, 55, 89\}$

b) Define subset A as the odd numbers in set F.

$A = \{1, 3, 5, 13, 21, 55, 89\}$

c) Define the complement of set A, denoted as A'.

$A' = \{2, 8, 34\}$

d) Define a subset of F, named set B, so it is equivalent to subset A, and explain your reasoning.

$B = \{1, 2, 3, 5, 8, 55, 89\}$.

Since $n(A) = 7$ and $n(B) = 7$ these sets are equivalent. You may choose any 7 elements from the universal set F to create a set equivalent to subset A.

e) Is subset $G = \{2, 3, 5, 13, 89\}$ a proper subset of F? Explain your reasoning.

$G \subset F$

$2 \in F, 3 \in F, 5 \in F, 13 \in F, 89 \in F$

Yes, G is a proper subset since it is properly contained in F. That is, each element in set G is a member of set F, and F contains additional elements.

f) Define a subset of F, named D, where D and G are disjoint sets.

$D = \{8, 21, 34\}$

Since set D has no elements in common with set G, they are disjoint. You may choose any combination of elements from the universal set that are not elements of G. D is a proper subset of F since it does not contain all the elements of F.

g) Define a subset of F, named E, containing the negative numbers in the Fibonacci sequence between 0 and 100.

$E = \{\}$ or $\varnothing$

Example 2

a) List all the possible subsets of set $S = \{0, 1, 2\}$.

b) Set $S = \{0, 1, 2\}$ has three elements, set $R = \{a, b\}$ has two elements, and set $Q = \{@\}$ has 1 element. List all possible subsets for S, R, and Q. Determine the relationship between the number of elements in the set and the possible number of subsets.

Solution

a) $A = \{0\}$ $B = \{1\}$ $C = \{2\}$ $D = \{0, 1\}$
$E = \{0, 2\}$ $F = \{1, 2\}$ $G = \{0, 1, 2\}$ $H = \{\ \}$

There are 8 possible subsets.

b) $S = \{0, 1, 2\}$

The eight possible subsets are:

$A = \{0\}$ $B = \{1\}$ $C = \{2\}$ $D = \{0, 1\}$
$E = \{0, 2\}$ $F = \{1, 2\}$ $G = \{0, 1, 2\}$ $H = \{\ \}$

$R = \{a, b\}$

The four possible subsets are:

$I = \{a\}$ $J = \{b\}$ $K = \{a, b\}$ $L = \{\ \}$

$Q = \{@\}$

The two possible subsets are

$M = \{@\}$ $N = \{\ \}$

A set with 1 element has 2 possible subsets.

A set with 2 elements has 4 possible subsets.

A set with 3 elements has 8 possible subsets.

A set with n elements would have 2^n subsets.

Connecting Words and Notation

You have seen how elements in a universal set can be common to two or more subsets, or belong to one subset but not another, while some subsets are disjoint.

Example 1

Given: Set $A = \{1, 2, 3, 4, 5, 6, 7\}$; Set $B = \{2, 4, 6, 8\}$; Set $C = \{4, 8, 12\}$

a) List the elements that belong to Set *B* AND Set *C*.

b) List the elements that belong to Set *A* AND Set *C*.

c) List the elements that belong to Set *B* OR Set *C*.

d) List the elements that belong to Set *A* but NOT Set *B*.

Solution

a) The set of elements that are common to two sets is referred to as the **intersection** of the sets. This is denoted by the symbol $\cap$. $B \cap C = \{4, 8\}$.

b) The intersection of A and C is denoted as $A \cap C = \{4\}$.

c) The set of elements that belong to at least one of set *B* or *C* is read as the **union** of *B* and *C*. It is denoted as $B \cup C = \{2, 4, 6, 8, 12\}$.

d) $\{1, 3, 5, 7\}$

This can be denoted as the intersection of *A* and the complement of *B*, or $A \cap B'$.

It may be helpful for you to include these definitions, notations, and examples on your resource sheet.

Number Lines

Sets of numbers may be organized graphically using number lines. You may remember using set notation to write the solutions to some answers in a previous math course.

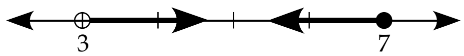
Example 1

Draw a number line to illustrate the solution set for the following:

a) $n > 3$ and $n \leq 7$

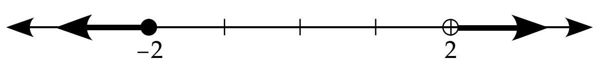
b) $a > 2$ or $a \leq -2$

Solution

a) 

Since the number, n, must be larger than 3, draw a hollow dot at 3 to indicate the values go up to but do not include that value. Shade the line towards the right, indicating that all values larger than 3 are part of the solution set. Draw a solid dot at seven, since the number, n, must be less than or equal to 7, indicating the value of 7 is included in the solution set. Shade the line towards the left as the solution is all values less than or equal to 7. Since the solution set is the intersection of these sets (indicated by "and"), the solution is the part of the line where the arrows overlap.

In set notation, the solution can be written as $\{n \mid 3 < n \leq 7, n \in \Re\}$, which is read, "the set of of n-values such that 3 is less than n and n is less than or equal to 7, where n is a real number."

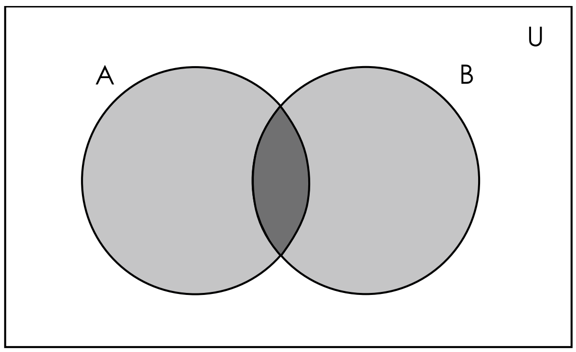
b) 

Draw a solid dot at −2 and shade to the left of the dot for "less than or equal to." Draw a hollow dot at 2 and shade to the right for "greater than." Since this question asks for the union (indicated by the word "or"), the solution is both portions of the shaded line. The solution can be written $\{a \mid a \leq -2 \cup a > 2, a \in \Re\}$ and is read as "the set of a values, such that a is less than or equal to −2 or a is greater than 2, where a is a real number."

Venn Diagrams

Another way to graphically organize set information is to use a Venn diagram. Venn diagrams can be used to illustrate the logical relationships between sets of data or numbers with specific properties.

In a Venn diagram, a rectangle represents the universal set and closed curves represent subsets. The darker shaded overlapping area in the diagram below illustrates the intersection of the sets $A \cap B$, and would contain the elements common to both A and B. The union of sets $A \cup B$ would include the entire shaded region of both A and B, while the white region represents the elements not in A or B, the complement of $A \cup B$.



Example 1

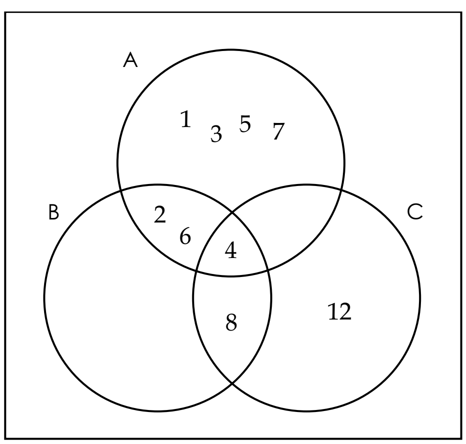
Create a Venn diagram to represent the following sets:

$A = \{1, 2, 3, 4, 5, 6, 7\}$

$B = \{2, 4, 6, 8\}$

$C = \{4, 8, 12\}$

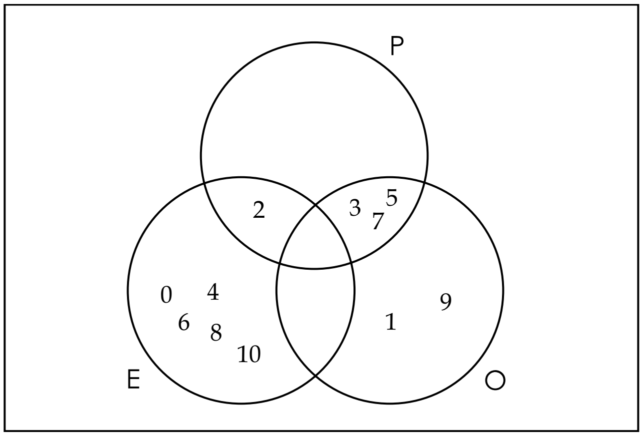
Solution



Example 2

a) Create a Venn diagram to illustrate the universal set, U = {whole numbers from 0 to 10 (inclusive)}, organized into three subsets: O = {odd numbers}, E = {even numbers}, and P = {prime numbers}.

Solution



b) Are the following statements true or false? Explain.

i) $P \subset U$ (remember this can be read as "P is contained in U")

ii) $O' = E$

iii) $E \cap P = \{0, 2, 4, 6, 8, 10\}$

iv) $O \cup P = \{1, 2, 3, 5, 7, 9\}$

v) $E \cap O = \varnothing$

vi) $1 \in P$

Solution

i) True. P is a proper subset of U.

ii) True. The complement of O is all the numbers that are NOT odd. All non-odd numbers are even and are part of subset E.

iii) False. The intersection of subsets E and P is the elements they have in common, $E \cap P = \{2\}$.

iv) True. The union of sets O and P includes elements in either O, P, or both.

v) True. These sets are disjoint.

vi) False. The number 1 is not considered a prime number.

The numbers in the real number system, $\Re$, are classified according to common characteristics and are named as follows:

- $N = \{1, 2, 3, \ldots\}$
- $W = \{0, 1, 2, 3, \ldots\}$
- $I = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- Q = Any rational number expressed in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$; that is, any number that can be written as a fraction (or ratio). In decimal form, the digits of a rational number either terminate, such as 0.25, or repeat, such as 0.33333 . . .
- Q' = An irrational number cannot be expressed as a fraction. In decimal form, the digits of an irrational number never terminate or repeat, such as π.

Example 3

a) Create a Venn diagram to represent some examples from the real number system.

b) Identify:

i) $N \cup W$

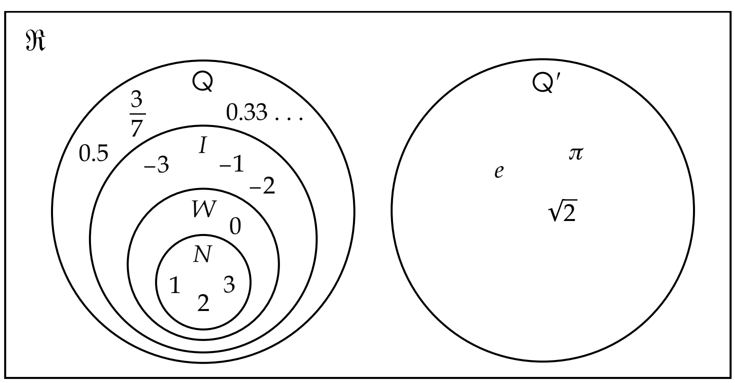
ii) $I \cap W$

iii) $Q \cap W'$

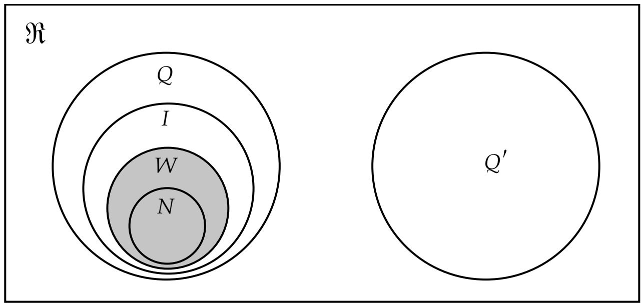
iv) $Q \cap Q'$

Solution

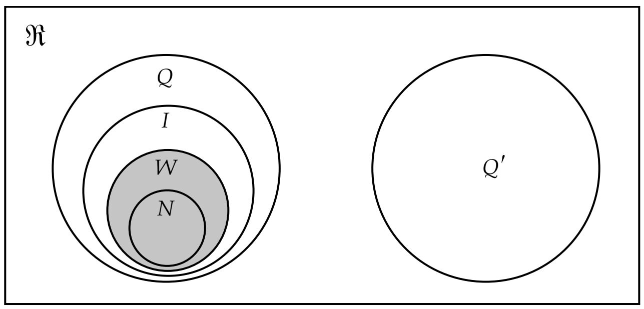
a)



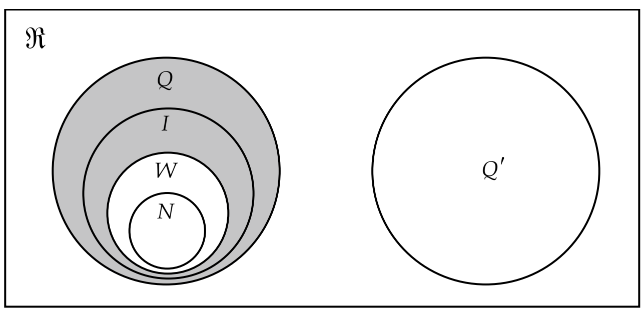
b) i) $N \cup W$



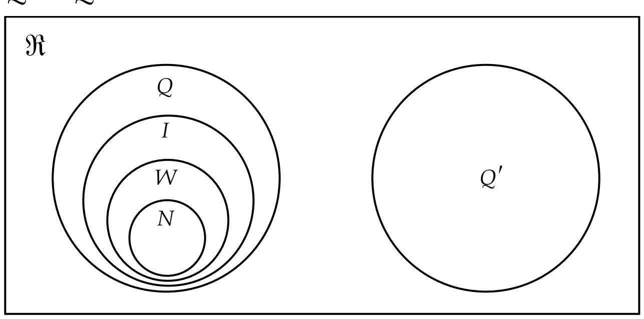
ii) $I \cap W$



iii) $Q \cap W'$



iv) $Q \cap Q'$



Learning Activity 3.1

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Write the symbol that denotes the following terminology.

   and

   or

   element

   null

   intersection

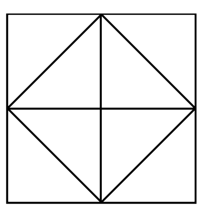
   union

   proper subset

   complement

2. On Monday, a friend borrows $10 and tells you he will pay it back in 30 days. What day of the week will you get your money back?

3. How many rectangles are in the figure below?

4. How many degrees are required to rotate all around a circle?

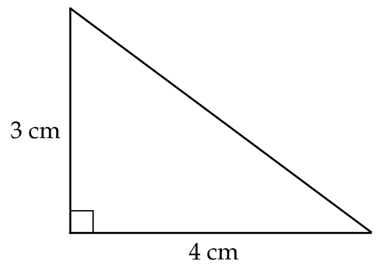
5. How many degrees are in a triangle?

6. How many degrees are in a right angle?

continued

Learning Activity 3.1 (continued)

7. What is the length of the hypotenuse in this right triangle?



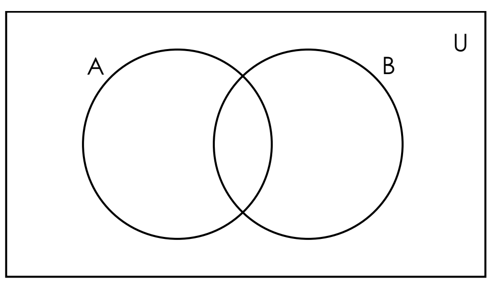
8. What is the degree of a quadratic equation?

Part B: Sets and Venn Diagrams

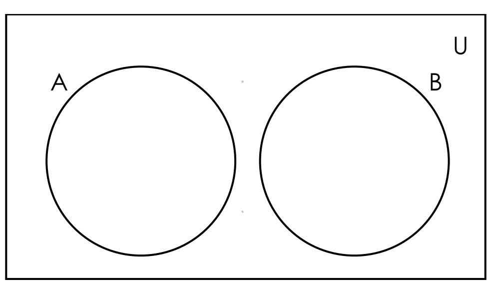
Remember, these questions are similar to the ones that will be on your assignments and midterm examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and midterm examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Represent the following by shading the correct region on the Venn diagram provided and explain what it means.

   a) $A \cup B$



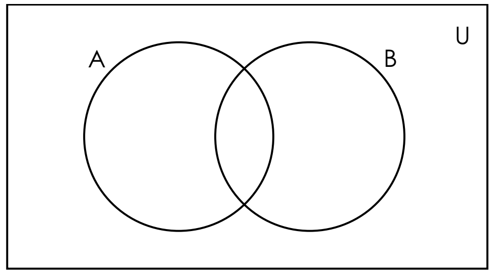
   b) $A \cup B$



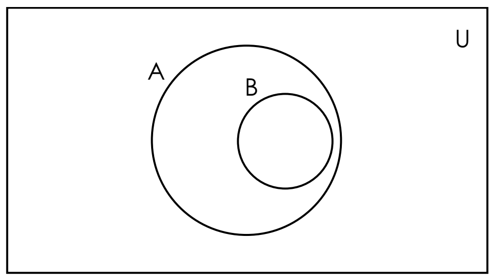
continued

Learning Activity 3.1 (continued)

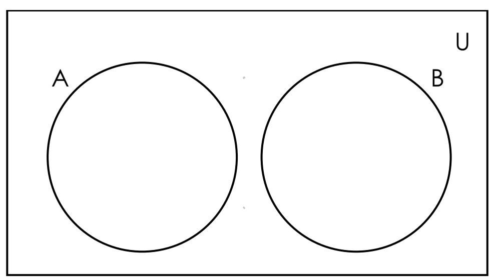
c) $A \cap B$



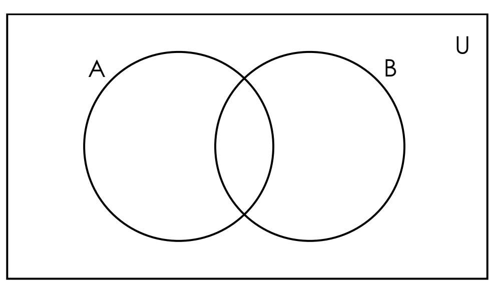
d) $A \cap B$



e) $A \cap B$



f) A'



continued

Learning Activity 3.1 (continued)

2. Write the following sets by listing the elements within braces:
   a) the two-digit even numbers between 81 and 99
   b) the Prairie provinces in Canada
   c) the months in the calendar that start with a vowel

3. Set S is all two-digit multiples of 6 and set F is all two-digit multiples of 5.
   a) Write these sets by listing their elements in braces.
   b) Are the following statements true or false? Explain.
      i) $84 \in S$
      ii) $50 \notin F$
      iii) Set S and set F are equal sets
      iv) Set S and set F are equivalent sets
      v) Set F is a finite set
   c) Create set P so that $P \subset F$.
   d) Find $S \cap F$.

4. If $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$
   $X = \{1, 2, 3, 4, 5\}$
   $Y = \{1, 2, 3\}$
   $Z = \{4, 6, 8\}$
   a) Create a Venn diagram, showing sets X, Y, and Z.
   b) Find the following:
      i) X and Y
      ii) X or Z
      iii) $Y \cap Z$
      iv) $Y \cup Z$
      v) X'
      vi) $(Y \cup Z)'$
      vii) not X and not Z

continued

Learning Activity 3.1 (continued)

5. The following sets of real numbers are identified.

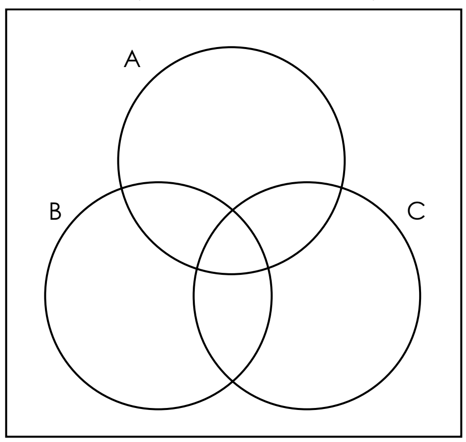
   $A = \{x | -5 \leq x < 22, x \in \Re\}$

   $B = \{x | x \leq -5, x \in \Re\}$

   $C = \{x | x \geq 17, x \in \Re\}$

   a) Illustrate these sets on the same number line.
   b) State the solution to the following:
      i) $A \cup B$
      ii) $A \cap B$
      iii) $B \cap C$
      iv) A'

6. a) If you wanted to colour each region of a Venn diagram a different colour, how many colours would you need to colour the following diagram?



   b) If the subsets are *A*, *B*, and *C*, use logic notation to define each region that would be represented by a different colour.

continued

Learning Activity 3.1 (continued)

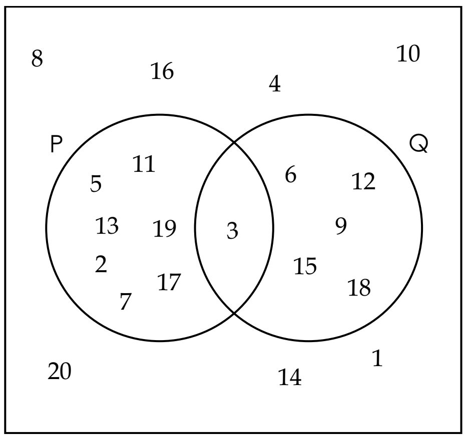
7. Consider the following sets:

   $U = \{1, 2, 3, \ldots 20\}$

   $P = \{$all prime numbers between 1 and 20$\}$

   $Q = \{$all multiples of 3 between 1 and 20$\}$

   In order to determine $(P \cup Q)'$, Alex drew a Venn diagram to illustrate these sets as follows:

   He incorrectly concluded that $(P \cup Q)' = \{3\}$. Find Alex's error and correct it.

Lesson Summary

In this lesson, you were introduced to set theory and explored sets and the elements in subsets, including disjoint, proper, universal, and empty sets. You defined the intersection, union, and complement of sets and illustrated these using regions on a Venn diagram. You organized information and solved problems using Venn diagrams.

Lesson 2: Conditional Statements

Lesson Focus

In this lesson, you will

- ❑ analyze an "if–then" statement, make a conclusion, and explain the reasoning
- ❑ determine the converse, inverse, and contrapositive of a conditional statement, determine its truth, and, if it is false, provide a counter-example
- ❑ identify and describe contexts in which a biconditional statement can be justified
- ❑ learn that the truth of any statement does not imply the truth of its converse, inverse, or contrapositive
- ❑ use truth tables to summarize the possible results of logical arguments that involve biconditional, converse, inverse, or contrapositive statements

Lesson Introduction

Have you ever had the unfortunate experience of getting into an argument with someone who takes all your words and twists them around in such a manner that those same words now imply the opposite of what you originally meant? Has someone ever tried to prove something to you using legitimate statements but then drew incorrect conclusions from them? What do arguments and mathematics have in common?

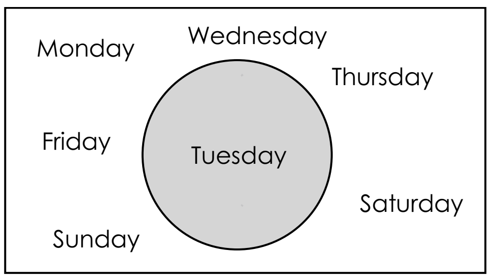
Mathematics gives us a language to express abstract concepts. To use and interpret mathematics, you need to be able to test the validity of statements. Using logic, statements can be manipulated and the relationships between concepts can be identified. Throw in some creative experimentation and who knows what interesting mathematical developments will result! If the new ideas are consistent and logical, they can be shown to be valid and, as a result, will help you to win an argument. Understanding logic is, therefore, an important part of the mathematical process.

Making a Statement

A statement is different from an expression of opinion. A statement is either true or false. It cannot be both. For example, an opinion would be "Monday is the worst day of the week," whereas a statement would be "Today is Tuesday."

A Venn diagram illustrating this statement would have a subset containing Tuesday in the universal set Days of the Week. The complement of the subset consists of every day that is **not** Tuesday.

If today really **is** Tuesday, then the statement "Today is Tuesday" is true and is represented by the element "Tuesday" inside the subset (shaded area). If today **is not** Tuesday, then the statement "Today is Tuesday" is false and is represented by the area of the Venn diagram outside the shaded area.



Negating Statements

If a statement is true, the negation of that statement is false.

Example 1

A given statement, denoted by the letter "p" states: "A right angle is 90°." This is true.

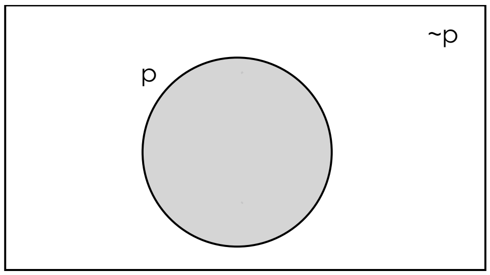
The negation of that statement would be "not p." This can be denoted as $\sim p$ or $\neg p$: "A right angle is **not** 90°." This is false.

In a table, these results are recorded as follows.

p	$\sim p$
T	F

If p is true, then $\sim p$ is false (read "$\sim p$" as "not p").

As a Venn diagram, p is true within the shaded area and false (not p) outside the shaded area.



If a statement is false, its negation is true.

Example 2

Statement, p: The moon is made of green cheese. This is false.

Negation, $\sim p$: The moon is **not** made of green cheese. This is true.

Add a row to the above table to include this result.

p	$\sim p$	
T	F	← if p is true, then not p is false
F	T	← if p is false, then not p is true

Compound Statements

If you combine multiple statements with the connecting words from the previous lesson, AND or OR, you have a **compound statement**.

It is raining. Toni drives to work.

It is raining and Toni drives to work. (conjunction, AND)

It is raining or Toni drives to work. (disjunction, OR)

Conditional Statements

Using the form "If p, then q," a compound statement can be structured to become a **conditional statement**. A conditional statement is an "if . . . then . . ." statement made up of a hypothesis, p, (if . . .) and a conclusion, q, (then . . .).

If it is raining then Toni drives to work. (if p, then q)

Hypothesis: It is raining

Conclusion: Toni drives to work.

A conditional statement can be written as "If p, then q" or denoted as $p \Rightarrow q$. This is read as "p implies q."

The converse of "If it is raining, then Toni drives to work."

The converse of a conditional statement is created by switching the order of the statements. In other words, q becomes the hypothesis and p is the conclusion.

If Toni drives to work, then it is raining.

A converse statement can be written as "If q, then p" or in logic notation as $q \Rightarrow p$.

The inverse of "If it is raining, then Toni drives to work."

If you negate the hypothesis, p, and negate the conclusion, q, of a compound statement, you have its inverse.

If it is not raining, then Toni does not drive to work.

In logic notation "if not p, then not q" is written as $\sim p \Rightarrow \sim q$. As seen above, the tilde symbol "~" is one way to denote negation. The symbol "¬" or the word "not" may also be used.

The contrapositive of "If it is raining, then Toni drives to work."

If you negate both the hypothesis and the conclusion of the converse of a conditional statement, you have what is called the **contrapositive**. It is a combination of the converse and the inverse. Recall that in the converse, q is the hypothesis and p is the conclusion. Now they are also negated.

If Toni does not drive to work, then it is not raining.

The contrapositive in logic notation is written as $\sim q \Rightarrow \sim p$.

It may be helpful for you to include a summary of these terms and notations on your resource sheet.

Example 3

Given the following conditional statements, write the converse, inverse, and contrapositive.

a) If a polygon has three sides, then it is a triangle.

b) If $\angle A = 70°$, then $\angle A$ is acute.

Solution

a)

Conditional statement, $p \Rightarrow q$:	If a polygon has three sides, then it is a triangle.
Converse, $q \Rightarrow p$:	If a polygon is a triangle, then it has three sides.
Inverse, $\sim p \Rightarrow \sim q$:	If a polygon does not have three sides, then it is not a triangle.
Contrapositive, $\sim q \Rightarrow \sim p$:	If a polygon is not a triangle, then it does not have three sides.

Note that the hypothesis can be either p, q, $\sim p$, or $\sim q$. The conclusion can be either p, q, $\sim p$, or $\sim q$.

b)

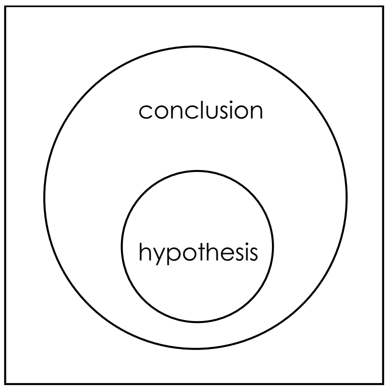
Conditional statement:	If $\angle A = 70°$, then $\angle A$ is acute.
Converse:	If $\angle A$ is acute, then $\angle A = 70°$.
Inverse:	If $\angle A \neq 70°$, then $\angle A$ is not acute.
Contrapositive:	If $\angle A$ is not acute, then $\angle A \neq 70°$.

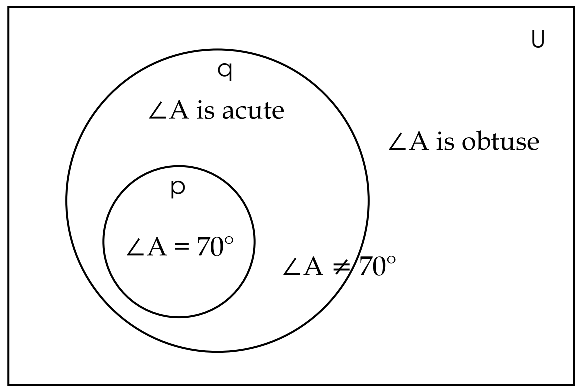
Example 4

Illustrate the conditional statement "If ∠A = 70°, then ∠A is acute." using a Venn diagram.

Solution

Draw a large circle for the conclusion and a small circle contained within it for the hypothesis.





If the hypothesis is a subset logically contained in the set of possible conclusions, then the statement is true.

Truth Tables

Truth tables are another way to analyze the results of conditional, inverse, converse, and contrapositive statements, and indicate if they are true or false.

The truth of a conditional statement depends on whether or not the hypothesis and conclusion are true or false. There are four cases that need to be considered. If the hypothesis, p, is true, the conclusion, q, may be either true or false. If the hypothesis is false, the conclusion may be either true or false. The four cases may be organized graphically in a chart, called a truth table.

p hypothesis	q conclusion	
T	T	← Case 1: The hypothesis, p, is true, and the conclusion, q, is true.
T	F	← Case 2: The hypothesis, p, is true, and the conclusion, q, is false.
F	T	← Case 3: The hypothesis, p, is false, and the conclusion, q, is true.
F	F	← Case 4: The hypothesis, p, is false, and the conclusion, q, is false.

Example 1

Given the conditional statement "If I wake up early, then I go for a run.", verify when the statement is true or false. Consider all four possible cases.

Solution

A conditional statement is denoted as $p \Rightarrow q$, where p is the hypothesis (I wake up early), and q is the conclusion (I go for a run). These statements may be either true or false. Consider the four cases.

Case 1: Hypothesis, p, is true. Conclusion, q, is true.

Hypothesis: I wake up early. For Case 1, assume this to be true.

Conclusion: I go for a run. For Case 1, assume this to be true.

You did what you said you would do, so the statement is true.

Case 2: Hypothesis, p, is true. Conclusion, q, is false.

Hypothesis: I wake up early. For Case 2, assume this to be true.

Conclusion: I go for a run. For Case 2, assume this to be false.

You woke up early but did not do what you said you would do, so the statement is false.

Case 3: Hypothesis, p, is false. Conclusion, q, is true.

Hypothesis: I wake up early. False.

Conclusion: I go for a run. True.

You overslept and did not wake up early so you are not obligated to go for a run. Even though you did go, it does not violate the hypothesis so the statement is still true.

Case 4: Hypothesis, p, is false. Conclusion, q, is false.

Hypothesis: I wake up early. False.

Conclusion: I go for a run. False.

Since you did not wake up early, you are not obligated to go for a run. You could run or stay in bed but, either way, the hypothesis is not violated so the conditional statement is still true.

The results could be summarized in the table.

p hypothesis	q conclusion	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The only time a conditional statement is false is when the hypothesis is true, but the conclusion is not. In all other cases, the statement is true.

You may want to include this chart and an explanation or example on your resource sheet.

Example 2

Given the conditional statement "If I chew gum, then I get a toothache.", write the converse and create a truth table to verify when the converse of this statement is true or false.

Solution

The converse statement is "If I get a toothache, then I chew gum." This is denoted as $q \Rightarrow p$, where q is now the hypothesis (I get a toothache) and p is the conclusion (I chew gum). These statements may be either true or false. Consider the four cases, keeping in mind that p and q are switched for the converse.

p conclusion	q hypothesis	$q \Rightarrow p$	
T	T	T	Case 1
T	F		Case 2
F	T		Case 3
F	F		Case 4

Case 1: Hypothesis, q, is true. Conclusion, p, is true.

Hypothesis: I get a toothache. Assume this to be true for Case 1.

Conclusion: I chew gum. Again, assume this to be true for Case 1.

The conclusion agrees with the hypothesis, so the converse statement is true.

p conclusion	q hypothesis	$q \Rightarrow p$
T	T	T
T	F	T
F	T	
F	F	

Case 2: Hypothesis, q, is false. Conclusion, p, is true.

Hypothesis: I get a toothache. False.

Conclusion: I chew gum. True.

You did not get a toothache, but you still chewed gum. This does not violate the converse statement, so it is still true.

p conclusion	*q* hypothesis	$q \Rightarrow p$
T	T	T
T	F	T
F	T	F
F	F	

Case 3: Hypothesis, *q*, is true. Conclusion, *p*, is false.

Hypothesis: I get a toothache. True.

Conclusion: I chew gum. False.

You get a toothache but you did not chew gum. The converse statement is false when the hypothesis is true, so the conclusion is false.

Case 4: Hypothesis, *q*, is false. Conclusion, *p*, is false.

Hypothesis: I get a toothache. False.

Conclusion: I chew gum. False.

You did not get a toothache and so are not obligated to chew gum. The converse statement is still true.

The results could be summarized in the truth table.

p	*q*	$q \Rightarrow p$
T	T	T
T	F	T
F	T	F
F	F	T

Similar to the conditional statement, a converse statement is only **false when the hypothesis (in this statement it is *q*) is true, but the conclusion (in a converse statement, this is *p*) is not**. In all other cases, the statement is true.

You may want to include this chart and an explanation or example on your resource sheet.

Similar truth tables can be constructed for inverse and contrapositive statements. Identify when the hypothesis is **true** but the conclusion is **false**. In this case, the statement will be false. In all other cases, the statement is true.

It may be difficult to understand how a false hypothesis, with a false conclusion, can result in a true statement. Consider the statement, "If pigs fly, then I am an elephant." In this statement, *p* and *q* are obviously false—pigs do not fly and I am not an elephant. However, the statement as given is still true.

p	q	$\sim p$ negation	$\sim q$ negation	$p \Rightarrow q$ conditional	$q \Rightarrow p$ converse	$\sim p \Rightarrow \sim q$ inverse	$\sim q \Rightarrow \sim p$ contrapositive
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Take a look at the results in each column and see if you can determine why each element is true or false based on *p* and *q* or $\sim p$ and $\sim q$.

You may want to include this table on your resource sheet.

Example 3

Explain how to complete the truth table column for the contrapositive statement, $\sim q \Rightarrow \sim p$, using the hypothesis $\sim q$ and the conclusion $\sim p$.

Solution

The hypothesis and conclusion for the contrapositive use the negations of *p* and *q*. You saw earlier in this lesson that if "*q*" is true, then "not *q*" is false. The same goes for *p* and $\sim p$.

p	q	$\sim p$ conclusion	$\sim q$ hypothesis	$\sim q \Rightarrow \sim p$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Locate the case in which the hypothesis, ~*q*, is true but the conclusion, ~*p*, is false. In this case, the contrapositive statement will be false. All other statements will be true.

The Truth of Inverse, Converse, and Contrapositive Statements

You have seen that a conditional statement, its converse, inverse, and contrapositive may or may not be true. If the hypothesis is true and the conclusion is false, the statement is false. If the hypothesis and the conclusion are both true, or the hypothesis is false, the statement is true.

> If a conditional statement is true, then the contrapositive statement will also be true, while the converse and inverse may not necessarily be true.

Example 4

Given the true conditional statement, "If triangles are congruent, then the areas are equal", state the contrapositive, converse, and inverse, and indicate if they are true or false.

Solution

Contrapositive: If areas are not equal, then triangles are not congruent. True.

Converse: If the areas are equal, then triangles are congruent. False. Two triangles may have the same area yet be a different shape and size.

Inverse: If triangles are not congruent, then the areas are not equal. False.
It is possible for two non-congruent triangles to have the same area.

Notice that the original conditional statement is true and so is the contrapositive. In this case, the converse and inverse statements are not true.

Biconditional Statements

If a conditional statement is true and its converse is also true, the statement can be written as a biconditional statement. A biconditional statement is used when the hypothesis and conclusion must either both be true or both be false. This expression is written as "p if and only if q" and is denoted by a bi-directional arrow, $\Leftrightarrow$, or $p \Leftrightarrow q$.

Example 5

Given the conditional statement, "If a polygon has three sides, then it is a triangle", write the converse statement and determine if it is true. If the conditional statement and the converse are both true, write the statement as a biconditional statement.

Solution

The conditional statement is true. A three-sided polygon is a triangle. The converse would be, "If a polygon is a triangle, then it has three sides." This is also true. The conditional statement can be written as a biconditional statement:

A polygon has three sides if and only if it is a triangle.

Counter-Examples

A statement may be thought to be true as long as no example exists to disprove it. However, only one counter-example is needed to disprove a statement. Consider this converse statement.

"If $\angle A$ is acute, then $\angle A = 70°$."

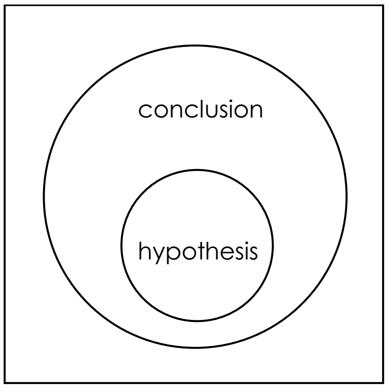
$\angle A$ may be any measure between 0° and 90°. For example, it may be acute and equal to 55°, so the conclusion is false. The angle is still acute but this counter-example shows that it is not necessarily equal to 70°, so the statement is false.

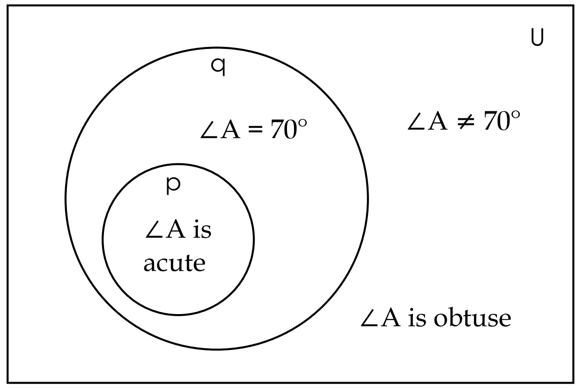
Example 6

Illustrate the statement "If ∠A is acute, then ∠A = 70°", using a Venn diagram, and show why it is false.

Solution

Draw a large circle for the conclusion and a small circle contained within it for the hypothesis.





If the hypothesis is a subset logically contained in the set of possible conclusions, then the statement is true. In this situation, it is possible for an acute angle other than 70° to be in *p*, so the statement is false.

Learning Activity 3.2

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Is 903 528 divisible by 2?
2. Is 903 528 divisible by 3?
3. Is 903 528 divisible by 4?
4. Is 903 528 divisible by 5?
5. Is 903 528 divisible by 6?
6. Is 903 528 divisible by 8?
7. Is 903 528 divisible by 9?
8. Is 903 528 divisible by 12?

continued

Part B: Statements and Truth Tables

Remember, these questions are similar to the ones that will be on your assignments and midterm examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and midterm examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Consider the following conditional statements:
   a) "If I am swimming off the coast of B.C., then I am swimming in salt water."
      i) Write the hypothesis and conclusion.
      ii) Is the conditional statement true? If it is false, provide a counter-example.
      iii) Write the converse, inverse, and contrapositive and determine their truth. Give a counter-example for any statement that is false.
      iv) Can the original statement be written as a biconditional statement? Explain.
   b) "If the batter gets three strikes, then the batter is out."
      i) Write the hypothesis and conclusion.
      ii) Is the conditional statement true? If it is false, provide a counter-example.
      iii) Write the converse, inverse, and contrapositive, and determine their truth. Give a counter-example for any statement that is false.
2. In what case is a conditional statement false?
3. You are given the conditional statement, "If there is a severe blizzard, then schools are closed."
   a) Consider the four cases (TT, TF, FT, FF) and determine if they are true or false.
   b) Complete a truth table to summarize the results.
4. You are given the conditional statement, "If a number is a multiple of four, then it is a multiple of two."
   a) Construct a truth table for the contrapositive of this statement.
   b) Compare these truth values to the values in the truth table for a conditional statement.

continued

Learning Activity 3.2 (continued)

5. You are given the conditional statement, "If a number has a repeating or terminating decimal, then it can be written as a fraction."
   a) Construct a truth table for the inverse statement.
   b) Compare these truth values to the values in the truth table for a converse statement.

6. Complete the following truth table. Describe any significant patterns or relationships you notice.

p	q	$\sim p$	$\sim q$	$p \Rightarrow q$	$q \Rightarrow p$	$\sim p \Rightarrow \sim q$	$\sim q \Rightarrow \sim p$
T	T						
T	F						
F	T						
F	F						

7. A given conditional statement is false.
   a) What can you conclude about its hypothesis and conclusion?
   b) What can you conclude about the truth of the contrapositive of the given conditional statement?
   c) What can you conclude about the truth of the converse and inverse of the given conditional statement?
   d) What can you conclude if a conditional statement and its converse are both true?

8. Given the statement, "If you are Canadian, then you are from North America."
   a) Write the conditional statement, its inverse, converse, and contrapositive.
   b) Determine if the conditional statement, its inverse, converse, and contrapositive are true or false. Give a counter-example if the statement is false.
   c) Is this a biconditional statement? Explain.

9. Given the statement, "If a triangle is isosceles, then it has two congruent sides."
   a) Write the conditional statement, its inverse, converse, and contrapositive.
   b) Determine if this can be written as a biconditional statement.

Lesson Summary

In this lesson, you analyzed "if–then" statements, determined the converse, inverse, and contrapositive of a conditional statement, and determined its truth. To demonstrate that a conditional statement was false, you provided a counter-example. You identified when a biconditional statement is justified. As well, you summarized and analyzed the truth value of statements in a truth table.

Notes

Lesson 3: Using Logic

Lesson Focus

In this lesson, you will

- ❑ use set theory in applications such as Internet searches, games, and puzzles
- ❑ arrange information using graphic organizers
- ❑ make and justify a decision using "what if?" questions in contexts such as probability, finance, sports, games, and puzzles, with or without technology

Lesson Introduction

Beyond arithmetic and geometry, you have used mathematics as a tool in science, you have seen it demonstrated in the patterns in nature, and you have manipulated the language of math to explain and describe things in your daily life, such as time, technology, sports, music, and so much more. As well, you have seen how mathematics and logic are connected. This lesson gives you the opportunity to see and use some of these connections.

Applications of Logic

Internet Searches

It is quite likely that as you work on your research project for this course, you will use the resources available to you on the Internet. Unless someone recommends a specific site and gives you the URL (webpage address) for that site, chances are you will make use of a search engine such as Google, Yahoo, or Bing to find websites of interest.

On a given day, a simple search for "jets" using Google returned about 130 million results in 0.25 seconds. The exact same search on another day may return slightly different search results, as the Internet changes regularly. There is no way you can access all those sites, and chances are that many of them are not pertinent to your query. So how do you limit the number of websites returned so it only lists the sites that have the information you really want? You use logic.

Typically, a search engine will match the terms you provide and list the pages that include them. The more precise you are in expressing what you are searching for, the better chance you have that the search engine will find relevant pages. Sometimes words have multiple meanings. If you are only interested in a specific use or application of a word, you need to clearly express that or the search engine will include websites that are of no interest to you. On the other hand, sometimes there are two or more words that refer to a given idea, and you may want to search for pages with either one or the other term.

Of the 130,000,000 sites listed in response to the search for "jets," some will include information about the Winnipeg Jets, others will be about the New York Jets (American football team), while others will have information about military jets, private planes for sale, music groups with the name Jets, and on and on.

You can give the search engine specific instructions about how it should conduct your search by using mathematical operators, words, or symbols with the terms you give it. The words AND and OR, quotation marks, and the "−" sign are the most commonly used ones.

The "−" sign before a word will exclude any site that has that word. The search for "jets−football" will eliminate sites referring to the New York Jets football team, but Google still returns 214 million possible sites.

If you are looking for sites that include the words Winnipeg OR Jets, Google returns about 211,000,000 results. These sites would include those that use the word *Winnipeg*, the word *Jets*, or both words.

The search query "Winnipeg AND Jets," limits the results to sites that have both of those terms. With this operator included in the search, Google returns about 31,400,000 results, significantly fewer than the original search.

If you want the search engine to find pages that have the exact phrase "Winnipeg Jets," you can use quotation marks. Using this query, Google only returns 9,000,000 sites.

You can use combinations of these operators to further refine your search.

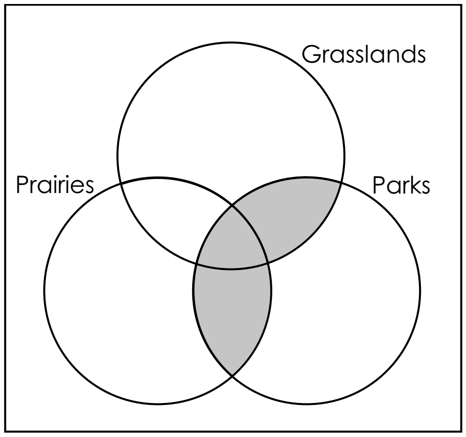
Using the query "Winnipeg Jets AND 1996 AND roster AND schedule," still results in over 3 million sites, but will limit the sites to those that could help you find who was playing in the final game before the original team left Winnipeg to become the Phoenix Coyotes.

Example 1

Use a Venn diagram to illustrate the results of the following search engine query:

(grasslands OR prairies) AND parks

Solution

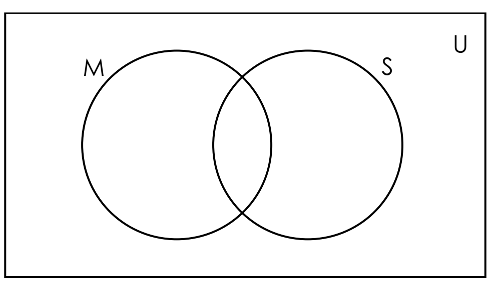


Venn Diagrams

Venn diagrams may also be used in practical, problem-solving applications.

Example 2

Of the 435 Grade 12 students in a particular high school, 275 are taking mathematics, 235 are taking science, and 189 are taking both mathematics and science. Use a Venn diagram to determine how many students are taking neither mathematics nor science.



Solution

Create a Venn diagram to represent the universal set U = {grade 12 students} and the subsets M = {students taking math} and S = {students taking science}.

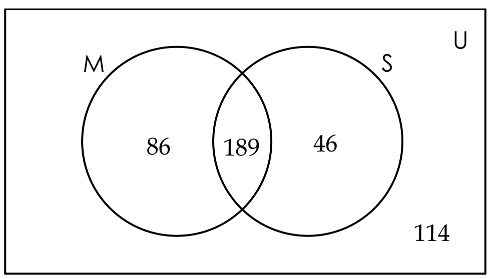
Fill in the Venn diagram, beginning with the number with the most restrictions. Place 189 in the intersection of M and S, the region $M \cap S$.

Calculate how many students are taking only mathematics and place that number in the correct region on the diagram (275 − 189 = 86).

Calculate and record how many students are taking only science (235 − 189 = 46).

The number taking neither mathematics nor science will be the complement of $M \cup S$ (435 − (189 + 86 + 46) = 114).

The number of students taking neither mathematics nor science is 114.



Puzzles

Logic puzzles make use of AND, OR, and NOT terminology in clues that are designed to give you just enough information to deduce the solution to the puzzle. A popular type of puzzle is the grid puzzle. You are given categories with different options, each of which can be used only once. A grid allows you to cross-reference each possible option. Mark an X in a box that is a false match or an impossible combination, and a O in the box that corresponds to a true match or related pair. Using simple logic, you can solve for the one correct solution to the puzzle.

Example 3

Five friends were born the same year, but each friend's birthday falls in a different month of the year and on a different day of the week. Using the clues below and the chart provided, determine the month of the year and day of the week on which each friend's birthday falls.

The names of the friends are Abe, Ben, Mark, Paul, and Tom. The possible birthday months are January, April, May, August, and October. The possible days of the week are Sunday, Tuesday, Thursday, Friday, and Saturday.

1. Paul was born in April but not on Saturday. Abe's birthday was not on Friday or Thursday.
2. The boy whose birthday is on Tuesday was born earlier in the year than Ben and Mark.
3. Tom was not born in January and his birthday was on the weekend.
4. Mark was not born in October nor was his birthday on a weekday. The friend whose birthday was in May was born on Sunday.
5. Tom was born before Ben, whose birthday was not on Friday. Mark was not born in August.

	January	April	May	August	October	Sunday	Tuesday	Thursday	Friday	Saturday
Abe										
Ben										
Mark										
Paul										
Tom										
Sunday										
Tuesday										
Thursday										
Friday										
Saturday										

Solution

Clue 1 indicates that Paul was born in April, so put an O at the intersection of Paul and April, and an X in each other square in that row and column. The intersections of Saturday and Paul, and Abe and Friday or Thursday, can have an X as well.

	January	April	May	August	October	Sunday	Tuesday	Thursday	Friday	Saturday
Abe		X						X	X	
Ben		X								
Mark		X								
Paul	X	O	X	X	X					X
Tom		X								
Sunday										
Tuesday										
Thursday										
Friday										
Saturday										

From Clue 2, you know that Ben and Mark were not born on Tuesday. At least one boy was born earlier in the year than them, so neither Ben nor Mark was born in January. If the Tuesday birthday was before Ben and Mark, Tuesday cannot be in the last two months, so put an X in Tuesday and August and October in the bottom portion of the grid.

	January	April	May	August	October	Sunday	Tuesday	Thursday	Friday	Saturday
Abe		X						X	X	
Ben	X	X					X			
Mark	X	X					X			
Paul	X	O	X	X	X					X
Tom		X								
Sunday										
Tuesday				X	X					
Thursday										
Friday										
Saturday										

Clue 3 tells you that Tom was not born in January, so that means Abe was born in January. Fill in an O for Abe and January and place an X for the other months. Going back to Clue 1, you now know that Abe's January birthday was not on Thursday or Friday, so now you can X those intersections in the bottom grid. Tom's birthday is on a weekend, so put an X where Tom and Tuesday, Thursday, or Friday intersect.

	January	April	May	August	October	Sunday	Tuesday	Thursday	Friday	Saturday
Abe	O	X	X	X	X			X	X	
Ben	X	X					X			
Mark	X	X					X			
Paul	X	O	X	X	X					X
Tom	X	X					X	X	X	
Sunday										
Tuesday				X	X					
Thursday	X									
Friday	X									
Saturday										

From Clue 4, you know to put an X where Mark and October intersect, as well as Mark and the weekdays Thursday or Friday. You are told that Sunday and May are a true match, so put an O there and an X elsewhere in that row and column.

Now make connections between this true match and matches you previously confirmed. Abe and January are a true match. Using logic, if Sunday AND May are a match, Sunday and January are NOT a match. Therefore Abe's birthday in January cannot be on Sunday. The same applies to Paul. If Paul's birthday is in April and the Sunday birthday is in May, Paul's birthday is NOT on Sunday.

	January	April	May	August	October	Sunday	Tuesday	Thursday	Friday	Saturday
Abe	O	X	X	X	X	X		X	X	
Ben	X	X					X			
Mark	X	X			X		X	X	X	
Paul	X	O	X	X	X	X				X
Tom	X	X					X	X	X	
Sunday	X	X	O	X	X					
Tuesday			X	X	X					
Thursday	X		X							
Friday	X		X							
Saturday			X							

Now consider Clue 5. Ben and Tom's birthdays could only be in May, August, or October. Ben cannot have been born in May, as Tom's birthday comes before his. Tom could not have been born in October, as Ben's birthday is after his. Place an X in these two impossible intersections. Put an X in the intersection of Ben and Friday, and one to show Mark and August are a false match. October now has only one possible true match so Ben was born in October and Mark in May, so Tom must be born in August. Paul's birthday is on Friday, as that is the only possible match in the Friday column. If Paul matches Friday, and Paul matches April, then Friday and April must match. When the false matches are crossed out, it becomes clear that Abe's birthday is on Tuesday, and therefore January matches Tuesday. Now any row or column that has two matches in it (in the May column, both Mark and Sunday have an O) can be included in the appropriate row (place an O in the Mark row under Sunday and X out all false matches in that grid). Using similar logic, the rest of the grid can be completed.

	January	April	May	August	October	Sunday	Tuesday	Thursday	Friday	Saturday
Abe	O	X	X	X	X	X	O	X	X	X
Ben	X	X	X	X	O	X	X	O	X	X
Mark	X	X	O	X	X	O	X	X	X	X
Paul	X	O	X	X	X	X	X	X	O	X
Tom	X	X	X	O	X	X	X	X	X	O
Sunday	X	X	O	X	X					
Tuesday	O	X	X	X	X					
Thursday	X	X	X	X	O					
Friday	X	O	X	X	X					
Saturday	X	X	X	O	X					

The solution to this puzzle is the following:

- Abe, January, Tuesday
- Ben, October, Thursday
- Mark, May, Sunday
- Paul, April, Friday
- Tom, August, Saturday

You can confirm this solution is correct by re-reading the five clues with this solution birthday information in mind. The clues should always yield a true statement.

If–Then

"If" is a little but powerful word in mathematics! It is the driving force behind interesting discoveries and creative solutions. With the word "if," you can imagine whatever you want and try to figure out what would happen if it were true.

Imagine how you might answer the following:

- If I were to get a better mortgage rate from my bank, how much less would my monthly payments be and how much less interest would I pay?
- What if there was another line, the four-point line, on a basketball court floor and shots from behind that line scored 4 points? How would that change the game and the outcome?
- If I had a thousand loonies and lined them up along the TransCanada highway, how far would I get? How much would a stack of loonies the height of the CN tower in Toronto be worth?
- If I opened each of the 130,000,000 web pages returned by Google in a query for "jets," how long would it take?

You may choose to use technology to answer these types of questions, base your conclusion on known formulas, or strike off in a new direction and come up with your own novel, creative solution. It is in the asking and answering of "what–if" questions that winning strategies for games are found, solutions for puzzles become clear, and an understanding of how mathematics and the world are related becomes evident. Answers to these questions could lead to new questions, interesting connections, and diverse applications. If the answer appears impossible to find, it is not a failure—it is another opportunity to ask "Well then, what–if . . .?"

Learning Activity 3.3

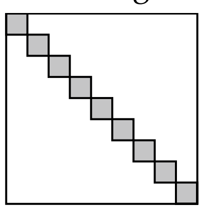
Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. What is 11% of $1800?

2. Each shaded square along the diagonal is 1 unit^2. What is the area of the unshaded region?



3. At a hockey game, U represents everyone in the arena building. If S represents the people watching the game, what does S' (the complement of set S) represent?

4. Of 100 people surveyed, half of them prefer chewing gum brand A. Half of the rest prefer brand B. How many prefer something else?

5. How many prime numbers are there between 30 and 40?

6. Write in exponential form: $\log 100 = n$

7. What is the y-intercept of $y = e^x$?

8. Solve for x: $25^3 = 5^x$

continued

Part B: Using Logic

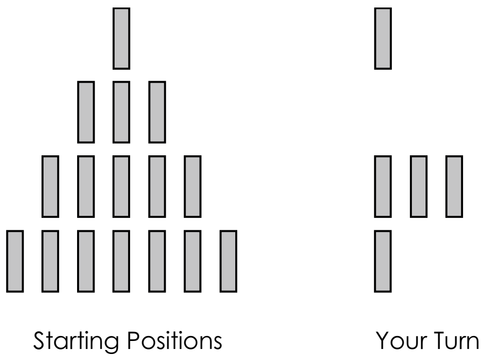
Remember, these questions are similar to the ones that will be on your assignments and midterm examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and midterm examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Abe, Bert, Carla, and Diana each choose different natural numbers between 1 and 30. Bert's number is $\frac{1}{7}$th of Abe's number. Abe's number is the product of Bert's and Carla's numbers. Diana's number is the smallest of all the numbers chosen and is the difference between Carla's and Bert's numbers. What number did each person choose?
2. In a class of 20 students, 10 play hockey, 14 play football, and 6 play both. How many students play neither hockey nor football? Draw a Venn diagram to illustrate your solution.
3. A survey asked people to identify if they learned about news and information from newspapers (N), the radio (R), or the Internet (I).
   - 49% get their news from the newspaper
   - 62% get their news from the Internet
   - 17% get their news from the newspaper and radio
   - 21% get their news from the newspaper and Internet
   - 8% get their news from the newspaper, radio, and Internet
   - 35% get their news from the Internet but not the newspaper or radio
   - 78% get their news from the Internet or radio

   a) Create a Venn diagram to illustrate the above information.
   b) What percentage of the population surveyed did not get their news and information from any of the three sources listed above?
4. In a survey of 46 students, 20 said they listened to pop music and 14 said they listened to classical music. However, the survey allowed the students to select more than one type of music, so some students selected both pop and classical music. Eighteen students said they did not listen to either. How many students like to listen to pop but not classical? How many listen to classical but not pop music? Use a Venn diagram to illustrate your answer.

continued

5. In the game of Nim, 16 markers are arranged in four rows, as shown on the left in the diagram below. Two players take turns removing any number of markers from any one row. The player who is forced to remove the last marker loses the game. After a number of turns, the remaining tiles are shown in the diagram on the right.



If it is your turn and the remaining markers are arranged as shown on the right above, which marker(s) should you remove to win the game?

6. What if a Venn diagram had four subsets and you were to colour each possible region a different colour? How would it look and how many colours would be needed if each region was a unique colour? Draw and colour a Venn diagram with four regions.

Bonus Puzzle (optional):

Two girls (Amanda and Preesha) and three boys (Alvin, Jon, and Cameron) spent a day at the zoo. Each of the five children had a different favourite animal that they wanted to see—chimpanzee, aardvark, baboon, leopard, or gazelle. They each enjoyed a different snack—peanuts, chips, pretzels, nacho chips with cheese, or ice cream. As well, they each purchased a different novelty—a stuffed bunny, a T-shirt, a keychain, a rubber chicken, or a teddy bear.

Use the clues to match each child with his or her favourite animal, snack, and souvenir.

- Preesha didn't care to see the aardvarks. The person who wanted to see them got so excited that they dumped a packet of peanuts.
- One of the girls enjoyed the baboons the most.
- The teddy bear got some nacho cheese on it when its owner got distracted.
- After visiting his favourite animal at the zoo, Jon was wearing a new T-shirt with matching spots.
- Both Amanda and Cameron enjoyed some type of chips for snack.

continued

Learning Activity 3.3 (continued)

- The chimpanzee almost got some nachos, since the person who wanted to see that exhibit considered sharing that snack with the animals there.
- The stuffed animals were not purchased by boys.
- The boy who bought the T-shirt got pretzel crumbs all over it.
- The aardvark keychain was purchased by the person who wanted to see that animal.

	aardvark	baboon	chimpanzee	gazelle	leopard	rubber chicken	t-shirt	teddy bear	stuffed bunny	key chain	chips	ice cream	pretzels	nachos	peanuts
Alvin															
Amanda															
Jon															
Cameron															
Preesha															
chips															
ice cream															
pretzels															
nachos															
peanuts															
rubber chicken															
t-shirt															
teddy bear															
stuffed bunny															
key chain															

Fill the chart with your final answer.

Name	Favourite Animal	Snack	Souvenir
Alvin			
Amanda			
Jon			
Cameron			
Preesha			

Lesson Summary

In this lesson, you used logical reasoning to solve problems involving set theory and conditional statements in applications such as Internet queries, logic puzzles, and games. You considered "if–then" statements and used graphic organizers to analyze and summarize the results of statements.

Assignment 3.1

Sets and Conditional Statements

Total: 31 marks

This is a hand-in assignment. Clearly show the steps in your solutions on the question sheets below, and submit these pages when you submit your assignments for marking. Final answers must include units. Answers given without supporting calculations will not be awarded full marks.

1. Use the set of natural numbers from 1 to 100 (inclusive) as the universal set.

   a) State the following subsets by listing their elements inside the braces. *(2 marks)*

   A = {perfect square numbers}

   B = {perfect cube numbers}

   b) Determine if subsets A and B are disjoint sets and explain your reasoning. *(2 marks)*

   c) Set $C = \{16, 36, 64\}$. Is $C \subset A$? Explain. *(1 mark)*

2. If the universal set U = {natural numbers less than 20}, consider the following subsets:

   P = {prime numbers}

   F = {factors of 18}

   H = {perfect square numbers}

   a) Create a Venn diagram. *(3 marks)*

   b) Determine the following: *(5 marks)*

      i) $P \cup H =$

      ii) $F \cap H =$

      iii) $F \cap H' =$

      iv) $P \cap H =$

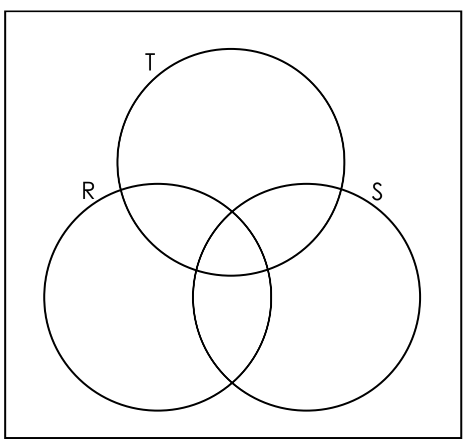
      v) $(P \cup H)' =$

3. In a gathering of 18 people, it was discovered that 8 could speak French and 11 could speak English. There were 4 people who could speak both languages. How many could not speak either language? *(2 marks)*

4. Each member of a sports club plays at least one of the following sports: soccer, rugby, or tennis. The following information is given:

   - 163 play tennis; 36 play tennis and rugby; 13 play tennis and soccer
   - 6 play all three sports; 11 play soccer and rugby; 208 play rugby or tennis
   - 98 play soccer or rugby

   Use this information to complete the following Venn diagram and determine the number of members in the club. *(3 marks)*



5. Given the conditional statement: "If two numbers are prime, then their sum is an even number."
   a) Determine if this conditional statement is true. If it is false, provide a counter-example. *(1 mark)*

   b) Write the contrapositive. *(1 mark)*

6. Given the statement: "If you invest in the stock market, then you will get rich."
   a) State the hypothesis and conclusion of the statement. *(1 mark)*

   b) Write the inverse of the conditional statement. *(1 mark)*

7. Consider the conditional statement, "If lines are perpendicular, then they meet at 90°." Can this statement be written as a biconditional statement? Explain your reasoning. *(2 marks)*

Assignment 3.1: Sets and Conditional Statements (continued)

8. Given the following conditional statement: "If I eat an orange, then I am getting some vitamin C."

   a) Write the inverse, converse, and contrapositive of this statement. *(3 marks)*

   b) Determine if the conditional statement, its inverse, converse, and contrapositive are true or false. Give a counter-example if the statement is false. *(4 marks)*

Notes

Module 3 Summary

Congratulations, you have finished Module 3! In this module, you learned to solve problems involving the application of set theory and conditional statements. You used logical notation to denote sets, the elements in subsets, and the union and intersection of sets. You organized data in Venn diagrams. You considered "if–then" statements, counter-examples, and the truth of a statement and its converse, inverse, and contrapositive. You completed truth tables to summarize the results of the four possible cases in logical arguments. You solved puzzles by using logic to complete grids.

As you complete Module 3, make sure your resource sheet is up to date and keep it in a safe place so you can refer back to it when it comes time to prepare for your midterm examination.

In the next module, you will be learning about odds, probability, and their applications.

Submitting Your Assignments

It is now time for you to submit the Module 3 Cover Assignment and Assignment 3.1 to the Distance Learning Unit so that you can receive some feedback on how you are doing in this course. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 3 assignments and organize your material in the following order:

- ❑ Module 3 Cover Sheet (found at the end of the course Introduction)
- ❑ Cover Assignment: SET® Game
- ❑ Assignment 3.1: Sets and Conditional Statements

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Notes

Grade 12 Applied Mathematics (40S)

Module 3
Logical Reasoning

Learning Activity Answer Keys

MODULE 3:
LOGICAL REASONING

Learning Activity 3.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Write the symbol that denotes the following terminology.

   and

   or

   element

   null

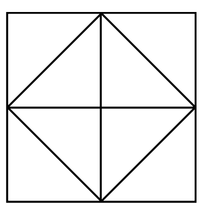
   intersection

   union

   proper subset

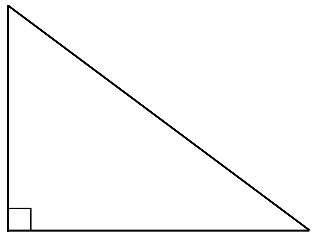
   complement

2. On Monday, a friend borrows $10 and tells you he will pay it back in 30 days. What day of the week will you get your money back?
3. How many rectangles are in the figure below?



4. How many degrees are required to rotate all around a circle
5. How many degrees are in a triangle?
6. How many degrees are in a right angle?

7. What is the length of the hypotenuse in this right triangle?



8. What is the degree of a quadratic equation?

Answers:

1. Write the symbol that denotes the following terminology.

and	$\cap$
or	$\cup$
element	$\in$
null	$\varnothing$ or { }
intersection	$\cap$
union	$\cup$
proper subset	$\subset$
complement	$'$

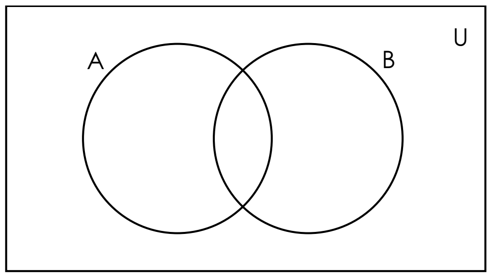
2. Wednesday (A week has 7 days and $4 \times 7 = 28$; 4 weeks plus two days would be 30 days, and that takes you to a Wednesday.)
3. 10 (6 squares and 4 other rectangles)
4. 360°
5. 180°
6. 90°
7. 5 cm $\left(\sqrt{3^2 + 4^2} = \sqrt{25} = 5\right)$
8. 2

Part B: Sets and Venn Diagrams

Remember, these questions are similar to the ones that will be on your assignments and midterm examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and midterm examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

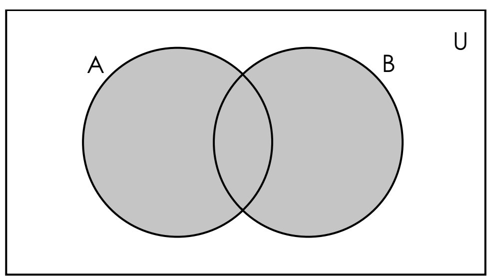
1. Represent the following by shading the correct region on the Venn diagram provided and explain what it means.

   a) $A \cup B$

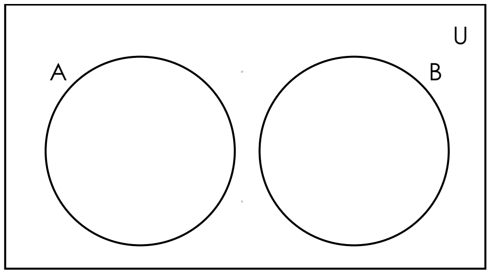


Answer:

This is the union of sets *A* and *B*. It includes all elements that belong to at least one of the sets, *A* or *B*. The entire region of these sets is shaded.

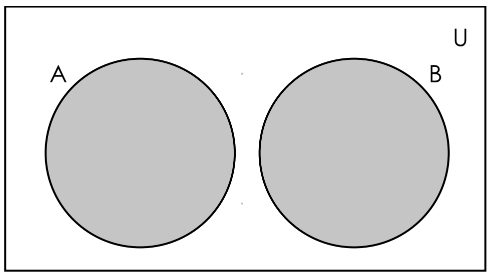


b) $A \cup B$

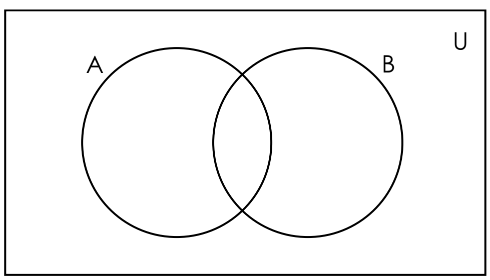


Answer:

These sets are disjoint. Both regions are shaded since the union of sets includes elements in *A* or *B*.

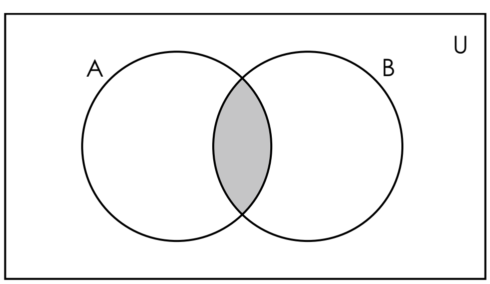


c) $A \cap B$

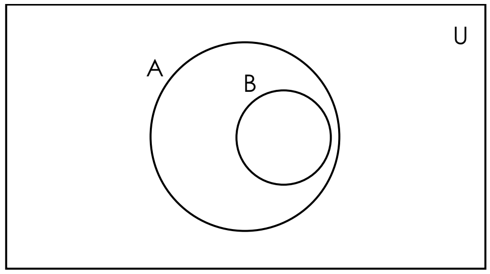


Answer:

The intersection of sets includes the elements that are common to both *A* and *B*. The region of overlap should be shaded on the diagram.

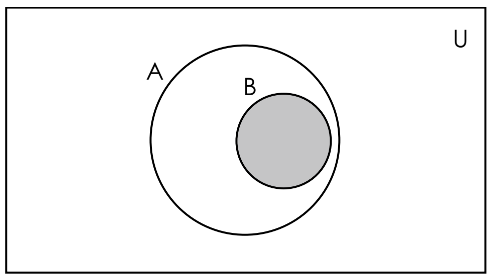


d) $A \cap B$

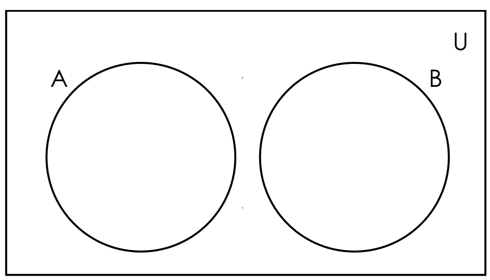


Answer:

$B \subset A$ so the intersection of these sets is the set B, as all elements in B are common to both sets.

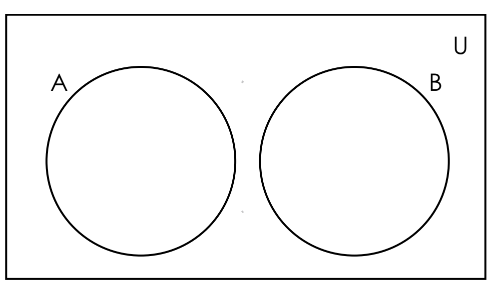


e) $A \cap B$

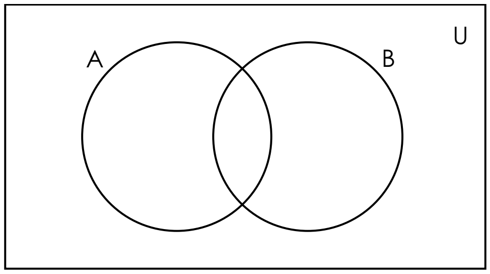


Answer:

If there are no common elements between sets the intersection is the empty set. Nothing is shaded.

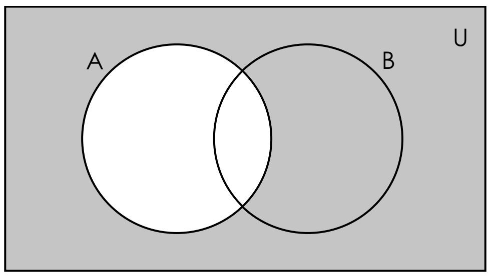


f) A'



Answer:

The complement of A is everything not in A.



2. Write the following sets by listing the elements within braces:
   a) the two-digit even numbers between 81 and 99
   b) the Prairie provinces in Canada
   c) the months in the calendar that start with a vowel

   Answers:

   a) $E = \{82, 84, 86, 88, 90, 92, 94, 96, 98\}$
   b) P = {Alberta, Saskatchewan, Manitoba}
   c) M = {April, August, October}

3. Set S is all two-digit multiples of 6 and set F is all two-digit multiples of 5.
   a) Write these sets by listing their elements in braces.

      Answer:

      $S = \{12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 78, 84, 90, 96\}$

      $F = \{10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95\}$

b) Are the following statements true or false? Explain.

i) $84 \in S$

ii) $50 \notin F$

iii) Set *S* and set *F* are equal sets

iv) Set *S* and set *F* are equivalent sets

v) Set *F* is a finite set

Answers:

i) True. 84 is an element in set *S*.

ii) False. 50 is a member of set *F*.

iii) False. The elements in these sets are not identical.

iv) False. $n(S) = 15$ while $n(F) = 18$.

v) True. The elements in this set are countable.

c) Create set *P* so that $P \subset F$.

Answer:

You may create any proper subset of *F*. It must contain some but not all of the elements in *F*. A possible solution could be $P = \{10, 20, 30, 40\}$.

d) Find $S \cap F$.

Answer:

$\{30, 60, 90\}$

4. If $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$

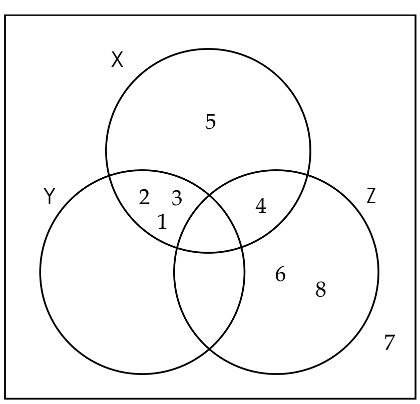
$X = \{1, 2, 3, 4, 5\}$

$Y = \{1, 2, 3\}$

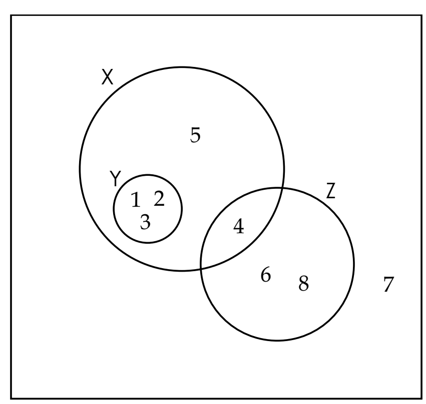
$Z = \{4, 6, 8\}$

a) Create a Venn diagram, showing sets *X*, *Y*, and *Z*.

Answer:



or



b) Find the following:

i) X and Y

ii) X or Z

iii) $Y \cap Z$

iv) $Y \cup Z$

v) X'

vi) $(Y \cup Z)'$

vii) not X and not Z

Answers:

i) X and $Y = \{1, 2, 3\}$

ii) X or $Z = \{1, 2, 3, 4, 5, 6, 8\}$

iii) $Y \cap Z = \{\,\}$ or $\varnothing$

iv) $Y \cup Z = \{1, 2, 3, 4, 6, 8\}$

v) $X' = \{6, 7, 8\}$

vi) $(Y \cup Z)' = \{5, 7\}$

vii) not X and not $Z = \{7\}$

5. The following sets of real numbers are identified.

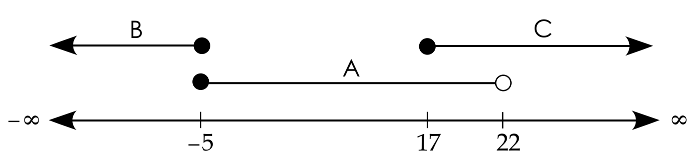
$A = \{x \mid -5 \leq x < 22,\ x \in \Re\}$

$B = \{x \mid x \leq -5,\ x \in \Re\}$

$C = \{x \mid x \geq 17,\ x \in \Re\}$

a) Illustrate these sets on the same number line.

Answer:



The solid dot represents a value that is included, symbolized by ≤ or ≥, while the open dot is used to represent a point on the line that is less than or greater than the value but is not included, symbolized by < or >. The arrows indicate the line continues to include all values as they approach positive and/or negative infinity.

b) State the solution to the following:

i) $A \cup B$

ii) $A \cap B$

iii) $B \cap C$

iv) A'

Answers:

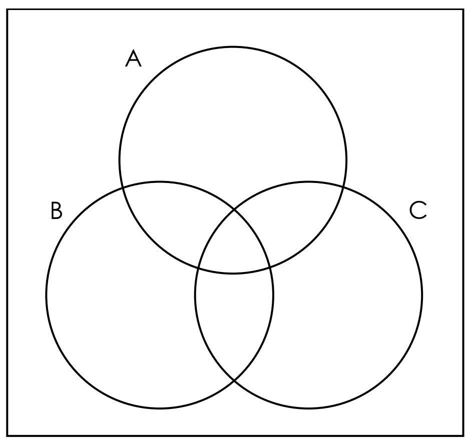
i) $\{x \mid x < 22, x \in \Re\}$

ii) $\{-5\}$

iii) { } or $\varnothing$

iv) $\{x \mid x < 5 \cup x \geq 22, x \in \Re\}$

6. a) If you wanted to colour each region of a Venn diagram a different colour, how many colours would you need to colour the following diagram?



Answer:

You would need 8 colours. There are 2^n or $2^3 = 8$ subsets possible.

b) If the subsets are A, B, and C, use logic notation to define each region that would be represented by a different colour.

Answer:

Colour #1 $A \cap B \cap C$ (centre region consisting of 3 intersecting circles)

Colour #2 $A \cap B$ (where only A and B intersect)

Colour #3 $A \cap C$

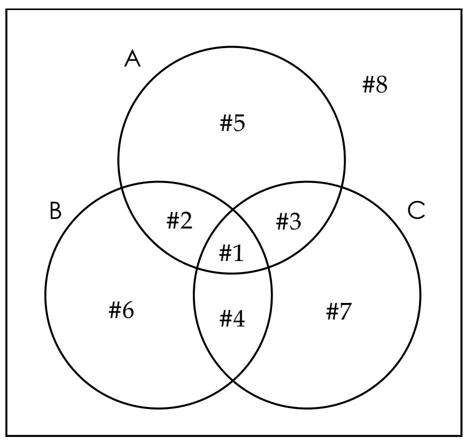
Colour #4 $B \cap C$

Colour #5 $A \cap (B \cup C)'$ (The part of circle A that does not intersect B or C)

Colour #6 $B \cap (A \cup C)'$

Colour #7 $C \cap (A \cup B)'$

Colour #8 $(A \cup B \cup C)'$ (The area inside the universal set rectangle that is not a part of any circle)



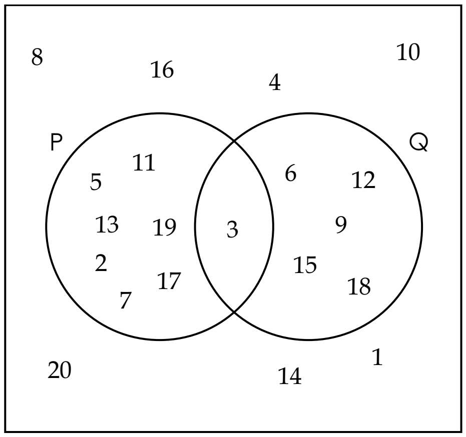
7. Consider the following sets:

   $U = \{1, 2, 3, \ldots 20\}$

   P = {all prime numbers between 1 and 20}

   Q = {all multiples of 3 between 1 and 20}

   In order to determine $(P \cup Q)'$, Alex drew a Venn diagram to illustrate these sets as follows:



   He incorrectly concluded that $(P \cup Q)' = \{3\}$. Find Alex's error and correct it.

   Answer:

   Alex found the intersection of $P \cap Q$ not the complement of $P \cup Q$.

   The only common element in sets P and Q is 3. The complement of $P \cup Q$ would be all the numbers not in either P or Q, or
   $(P \cup Q)' = \{1, 4, 8, 10, 14, 16, 20\}$.

Learning Activity 3.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Is 903 528 divisible by 2?
2. Is 903 528 divisible by 3?
3. Is 903 528 divisible by 4?
4. Is 903 528 divisible by 5?
5. Is 903 528 divisible by 6?
6. Is 903 528 divisible by 8?
7. Is 903 528 divisible by 9?
8. Is 903 528 divisible by 12?

Answers:

1. Yes (All even numbers are divisible by 2.)
2. Yes (Add the digits in the number. If that sum is divisible by 3, then the entire number is: 9 + 0 + 3 + 5 + 2 + 8 = 27; 27 is divisible by 3, so 903 528 is divisible by 3.)
3. Yes (Look at the last two digits of the number, in this case 28. If that number is divisible by 4, then the entire number is divisible by 4.)
4. No (The last digit must be zero or five to be divisible by 5.)
5. Yes (If a number is divisible by 2 and 3, it is divisible by 6. See divisibility rules above.)
6. Yes (Look at the last three digits of the number, in this case 528. Since $528 \div 8 = 66$, it is divisible by 8, then the entire number is divisible by 8.)
7. Yes (Find the sum of all the digits in the number. If the sum is divisible by 9, the entire number is divisible by 9: 9 + 0 + 3 + 5 + 2 + 8 = 27; 27 is divisible by 9, so 903 528 is divisible by 9.)
8. Yes (If the sum of the digits is divisible by 3 and the last two digits are divisible by 4, the entire number is divisible by 12.)

Part B: Statements and Truth Tables

Remember, these questions are similar to the ones that will be on your assignments and midterm examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and midterm examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Consider the following conditional statements:
   a) "If I am swimming off the coast of B.C., then I am swimming in salt water."
      i) Write the hypothesis and conclusion.

         Answer:

         Hypothesis (p): "I am swimming off the coast of B.C."

         Conclusion (q): "I am swimming in salt water."

      ii) Is the conditional statement true? If it is false, provide a counter-example.

         Answer:

         It is true. The salty Pacific Ocean is off the coast of B.C.

      iii) Write the converse, inverse, and contrapositive and determine their truth. Give a counter-example for any statement that is false.

         Answer:

         Converse (if q then p): "If I am swimming in salt water, then I am swimming off the coast of B.C."

            False. I could be swimming in the Caribbean, or in a salt water pool at a hotel.

         Inverse (if not p then not q): "If I am not swimming off the coast of B.C., then I am not swimming in salt water."

            False. I could be swimming in Newfoundland in the salt water off the eastern coast.

         Contrapositive (if not q then not p): "If I am not swimming in salt water, then I am not swimming off the coast of B.C."

            True. The contrapositive of a true statement is also true.

      iv) Can the original statement be written as a biconditional statement? Explain.

         Answer:

         No, a conditional statement and its converse must both be true in order to be a valid "if and only if" statement.

b) "If the batter gets three strikes, then the batter is out."

i) Write the hypothesis and conclusion.

Answer:

Hypothesis: "The batter gets three strikes."

Conclusion: "The batter is out."

ii) Is the conditional statement true? If it is false, provide a counter-example.

Answer:

Yes, it is true.

iii) Write the converse, inverse, and contrapositive, and determine their truth. Give a counter-example for any statement that is false.

Answer:

Converse: "If the batter is out, the batter got three strikes."

False. The batter may have been tagged out.

Inverse: "If the batter does not get three strikes, then the batter is not out."

False: The batter's pop fly may have been caught and the batter is out.

Contrapositive: "If the batter is not out, then the batter didn't get three strikes."

True.

2. In what case is a conditional statement false?

Answer:

A conditional statement is false if the hypothesis is true but the conclusion is false.

3. You are given the conditional statement, "If there is a severe blizzard, then schools are closed."

a) Consider the four cases (TT, TF, FT, FF) and determine if they are true or false.

Answer:

Case 1: The first case considers that both the hypothesis and conclusion are true. Hypothesis is true—there is a blizzard. The conclusion is true—schools are closed. The conditional statement is true.

Case 2: The hypothesis is true—there is a blizzard, but the conclusion is false—schools are not closed. The conditional statement is then false.

Case 3: The hypothesis is false—there is no blizzard, but the conclusion is true—schools are closed. This is a true statement. Schools are often closed in the summer.

Case 4: The hypothesis is false—there is no blizzard and the conclusion is false—schools are not closed. The original conditional statement is still true since when there is no blizzard, it is true that schools are usually not closed.

Case 4 can be a bit perplexing. How can a false hypothesis and false conclusion make a true statement? In general, if you have a false hypothesis (p is false), you can conclude just about anything you want. So, when p is false, the statement will be true whether the conclusion is true or false. As an example, consider the statement, "If the moon is made of green cheese, then I am the richest person in Manitoba." The hypothesis is false, so the conclusion can be true or false, and the conditional statement still holds true.

b) Complete a truth table to summarize the results.

Answer:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Notice that the statement is false only when the hypothesis is true but the conclusion is false. The results in this truth table can be used for all conditional statements.

4. You are given the conditional statement, "If a number is a multiple of four, then it is a multiple of two."

   a) Construct a truth table for the contrapositive of this statement.

   Answer:

   In a contrapositive statement, the hypothesis is $\sim q$ (not q) and the conclusion is $\sim p$ (not p). Find where $\sim q$ is true but $\sim p$ is false. This statement (Case 2) will be false. All other statements will be true.

p	q	$\sim p$ conclusion	$\sim q$ hypothesis	$\sim q \Rightarrow \sim p$	
T	T	F	F	T	Case 1
T	F	F	T	F	← Case 2
F	T	T	F	T	Case 3
F	F	T	T	T	Case 4

   b) Compare these truth values to the values in the truth table for a conditional statement.

   Answer:

   Find the case in which the hypothesis, p, is true but the conclusion, q, is false. This statement will be false. All others will be true.

   Compared to:

$p \Rightarrow q$
T
F
T
T

   The conditional statement and the contrapositive have the same truth values. As such, the contrapositive is logically equivalent to the conditional truth values. That is why a contrapositive statement is always true whenever the original conditional statement is true.

5. You are given the conditional statement, "If a number has a repeating or terminating decimal, then it can be written as a fraction."

   a) Construct a truth table for the inverse statement.

   Answer:

   In the inverse statement $\sim p \Rightarrow \sim q$, $\sim p$ is the hypothesis and $\sim q$ is the conclusion.

p	q	$\sim p$ hypothesis	$\sim q$ conclusion	$\sim p \Rightarrow \sim q$ inverse
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

   Find the case in which the hypothesis, $\sim p$, is true, but the conclusion, $\sim q$, is false. This statement will be false. All others will be true.

   b) Compare these truth values to the values in the truth table for a converse statement.

   Answer:

   Find the case in which the hypothesis, q, is true, but the conclusion, p, is false. This statement will be false. All others will be true.

   Compared to:

$q \Rightarrow p$
T
T
F
T

   The truth values for converse, $q \Rightarrow p$, and inverse, $\sim p \Rightarrow \sim q$, are equivalent. They are each other's contrapositive so the truth values are the same.

6. Complete the following truth table. Describe any significant patterns or relationships you notice.

Answer:

p	q	$\sim p$	$\sim q$	$p \Rightarrow q$	$q \Rightarrow p$	$\sim p \Rightarrow \sim q$	$\sim q \Rightarrow \sim p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Patterns and relationships that can be noted include:

- The truth values in the negation columns $\sim p$ and $\sim q$ are the exact opposite of the p and q columns.
- The truth values in the columns for the conditional statement $p \Rightarrow q$ and the contrapositive $\sim q \Rightarrow \sim p$ are the same.
- The truth values in the columns for the converse $q \Rightarrow p$ and the inverse $\sim p \Rightarrow \sim q$ are the same.
- If the hypothesis and conclusion are both true, or both false, the statement will be true.
- If the hypothesis is true but the conclusion is false, then the statement is false.
- If the hypothesis is false, the statement will be true.
- The truth of a conditional statement does not imply the truth of its converse or inverse, but if a conditional statement is true, its contrapositive will also be true.

7. A given conditional statement is false.

a) What can you conclude about its hypothesis and conclusion?

Answer:

From the truth table for a conditional statement completed above, row 2 is the only case in which a conditional statement is false. $p \Rightarrow q$ is false only when p is true and q is false. Therefore, the hypothesis, p, is true, and the conclusion, q, is false.

b) What can you conclude about the truth of the contrapositive of the given conditional statement?

Answer:

From row 2 of the truth table, you can see that $\sim q \Rightarrow \sim p$ will also be false because the truth values of a conditional statement and its contrapositive are always the same.

c) What can you conclude about the truth of the converse and inverse of the given conditional statement?

Answer:

From row 2 the truth table, the converse ($q \Rightarrow p$) and inverse ($\sim p \Rightarrow \sim q$) of a false conditional statement will both be true. The converse switches the role of the hypothesis and conclusion. As such, the hypothesis, q, is false and, regardless of the state of the conclusion, p, the converse statement will be true. The converse and inverse statements have the same truth tables, so the inverse statement will also be true.

d) What can you conclude if a conditional statement and its converse are both true?

Answer:

This means that the hypothesis and conclusion will always both be true or both be false. The statement can then be written as a biconditional statement, "p if and only if q" or $p \Leftrightarrow q$.

8. Given the statement, "If you are Canadian, then you are from North America."

a) Write the conditional statement, its inverse, converse, and contrapositive.

Answer:

p: "You are Canadian."

q: "You are from North America."

Conditional ($p \Rightarrow q$):	"If you are Canadian, then you are from North America."
Inverse ($\sim p \Rightarrow \sim q$):	"If you are not Canadian, then you are not from North America."
Converse ($q \Rightarrow p$):	"If you are from North American, then you are Canadian."
Contrapositive ($\sim q \Rightarrow \sim p$):	"If you are not from North America, then you are not Canadian."

b) Determine if the conditional statement, its inverse, converse, and contrapositive are true or false. Give a counter-example if the statement is false.

Answer:

Conditional ($p \Rightarrow q$):	"If you are Canadian, then you are from North America." True.
Inverse ($\sim p \Rightarrow \sim q$):	"If you are not Canadian, then you are not from North America." False. You may be from the United States, which would make you North American, but not Canadian.
Converse ($q \Rightarrow p$):	"If you are from North American, then you are Canadian." False. You may be American.
Contrapositive ($\sim q \Rightarrow \sim p$):	"If you are not from North America, then you are not Canadian." True.

c) Is this a biconditional statement? Explain.

Answer:

For a statement to be biconditional, the conditional statement and its converse must both be true. In this case, the converse is false, so it is not biconditional.

9. Given the statement, "If a triangle is isosceles, then it has two congruent sides."

a) Write the conditional statement, its inverse, converse, and contrapositive.

Answer:

p: "A triangle is isosceles."

q: "It has two congruent sides."

Conditional ($p \Rightarrow q$):	"If a triangle is isosceles, then it has two congruent sides."
Inverse ($\sim p \Rightarrow \sim q$):	"If a triangle is not isosceles, then it does not have two congruent sides."
Converse ($q \Rightarrow p$):	"If a triangle has two congruent sides, then it is isosceles."
Contrapositive ($\sim q \Rightarrow \sim p$):	"If a triangle does not have two congruent sides, then it is not isosceles."

b) Determine if this can be written as a biconditional statement.

Answer:

Conditional ($p \Rightarrow q$): "If a triangle is isosceles, then it has two congruent sides." True.

Converse ($q \Rightarrow p$): "If it has two congruent sides, then the triangle is isosceles." True.

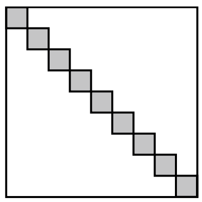
If the conditional statement and its converse both true, it can be written as a biconditional statement: "A triangle is isosceles if and only if it has two congruent sides" or $p \Leftrightarrow q$.

Learning Activity 3.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. What is 11% of $1800?
2. Each shaded square along the diagonal is 1 unit^2. What is the area of the unshaded region?

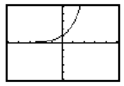


3. At a hockey game, U represents everyone in the arena building. If S represents the people watching the game, what does S' (the complement of set S) represent?
4. Of 100 people surveyed, half of them prefer chewing gum brand A. Half of the rest prefer brand B. How many prefer something else?
5. How many prime numbers are there between 30 and 40?
6. Write in exponential form: $\log 100 = n$
7. What is the y-intercept of $y = e^x$?
8. Solve for x: $25^3 = 5^x$

Answers:

1. $198 (10% of 1800 is 180; 1% of 1800 is 18; 180 + 18 = 198)
2. 72 u^2 (Area of large square: $9 \times 9 = 81$ u^2. Area of shaded regions: $9 \times 1 =$ 9 u^2. Area of unshaded region is $81 - 9 = 72$ u^2.)
3. The people in the building not watching the game. (It could be the people working at the arena or standing in line at the concessions.)
4. 25 ($100 \div 2 = 50$ who prefer brand B; half of the people remaining would be $50 \div 2 = 25$)
5. 2 (Prime numbers other than 2 are odd, so consider 31, 33, 35, 37, and 39. Of these, 33, 35, and 39 are composite, so 31 and 37 are the only prime numbers between 30 and 40.)
6. $10^n = 100$

7. The y-intercept is at 1. (Recall the graph of $y = e^x$ or determine the value of y equal to e^0.)



8. $x = 6$

$$\left(\begin{aligned}\left(5^2\right)^3 &= 5^x\\ 5^6 &= 5^x\\ x &= 6\end{aligned}\right)$$

Part B: Using Logic

Remember, these questions are similar to the ones that will be on your assignments and midterm examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and midterm examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Abe, Bert, Carla, and Diana each choose different natural numbers between 1 and 30. Bert's number is $\frac{1}{7}$th of Abe's number. Abe's number is the product of Bert's and Carla's numbers. Diana's number is the smallest of all the numbers chosen and is the difference between Carla's and Bert's numbers. What number did each person choose?

   Answer:

   Abe's number must be a multiple of 7 (7, 14, 21, or 28).

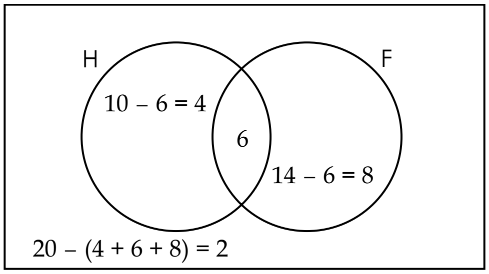
   Bert must have chosen 1, 2, 3, or 4 $\left(\frac{7}{7} = 1, \frac{14}{7} = 2, \frac{21}{7} = 3, \frac{28}{7} = 4\right)$.

   If Abe's number is the product of Bert's and Carla's numbers, Carla must have chosen 7.

   Diana's number is the smallest of the numbers chosen, so it must be less than Bert's number. Diana's number is the difference between Carla's number (7) and Bert's numbers (1, 2, 3, or 4). The possible differences are 6, 5, 4, or 3. But Diana's must be smaller than Bert's number, so Diana chose 3, Bert chose 4, Carla chose 7, and Abe chose 28.

2. In a class of 20 students, 10 play hockey, 14 play football, and 6 play both. How many students play neither hockey nor football? Draw a Venn diagram to illustrate your solution.

   Answer:

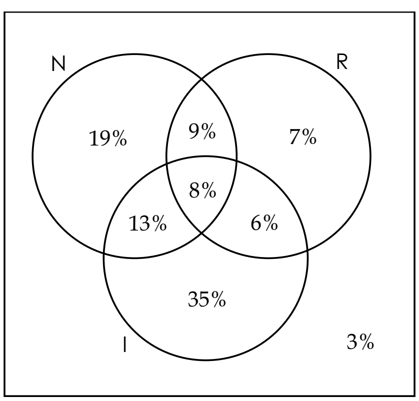
   

   Two students play neither hockey nor football.

3. A survey asked people to identify if they learned about news and information from newspapers (N), the radio (R), or the Internet (I).
   - 49% get their news from the newspaper
   - 62% get their news from the Internet
   - 17% get their news from the newspaper and radio
   - 21% get their news from the newspaper and Internet
   - 8% get their news from the newspaper, radio, and Internet
   - 35% get their news from the Internet but not the newspaper or radio
   - 78% get their news from the Internet or radio

   a) Create a Venn diagram to illustrate the above information.

   Answer:

   b) What percentage of the population surveyed did not get their news and information from any of the three sources listed above?

   Answer:

   $(N \cup R \cup I)' = 100 - 97 = 3\%$

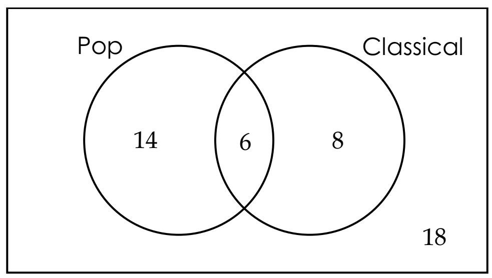
4. In a survey of 46 students, 20 said they listened to pop music and 14 said they listened to classical music. However, the survey allowed the students to select more than one type of music, so some students selected both pop and classical music. Eighteen students said they did not listen to either. How many students like to listen to pop but not classical? How many listen to classical but not pop music? Use a Venn diagram to illustrate your answer.

   Answer:

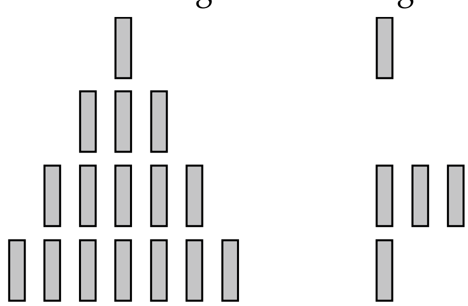
   Eighteen students do not listen to either type of music. So 28 students listen to either pop or classical (46 − 18 = 28). There must be 6 students who like both (20 + 14) − 28 = 6).

   Since 20 − 6 = 14, 14 students listen to pop but not classical.

   Since 14 − 6 = 8, 8 students listen to classical music (but not pop).



5. In the game of Nim, 16 markers are arranged in four rows, as shown on the left in the diagram below. Two players take turns removing any number of markers from any one row. The player who is forced to remove the last marker loses the game. After a number of turns, the remaining tiles are shown in the diagram on the right.



Starting Positions   Your Turn

   If it is your turn and the remaining markers are arranged as shown on the right above, which marker(s) should you remove to win the game?

   Answer:

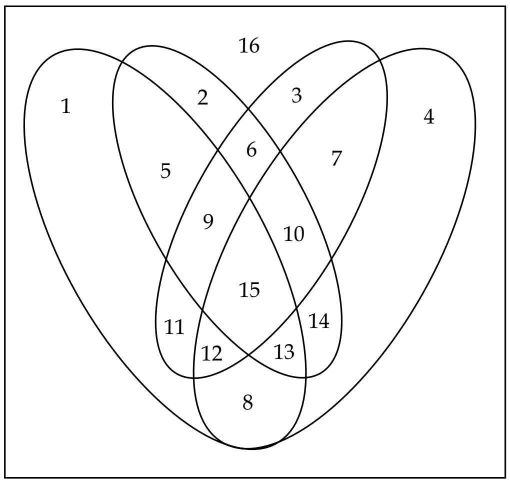
   Take two markers from the second to bottom row. Your partner will then be forced to take the last marker.

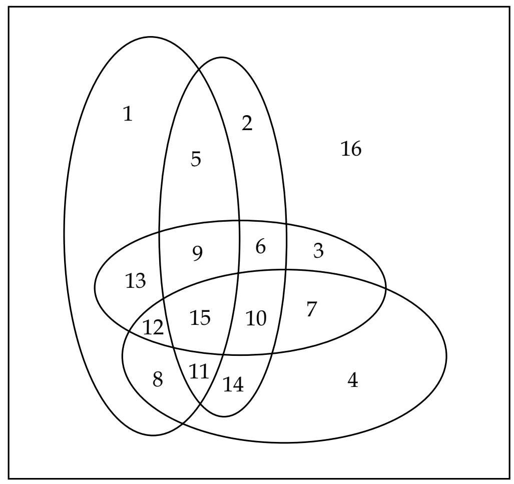
6. What if a Venn diagram had four subsets and you were to colour each possible region a different colour? How would it look and how many colours would be needed if each region was a unique colour? Draw and colour a Venn diagram with four regions.

   Answer:

   Research online or use other mathematics resources if you need help arranging a Venn diagram showing all possible subsets created with 4 overlapping regions. It should have 16 uniquely coloured regions.

   Possible designs may include:





   Do you see the pattern for the number of regions in a Venn diagram? With one circle, there are 2 regions (inside or outside). With two circles there are 4 regions; with three circles there are 8 regions.

Bonus Puzzle (optional):

Two girls (Amanda and Preesha) and three boys (Alvin, Jon, and Cameron) spent a day at the zoo. Each of the five children had a different favourite animal that they wanted to see—chimpanzee, aardvark, baboon, leopard, or gazelle. They each enjoyed a different snack—peanuts, chips, pretzels, nacho chips with cheese, or ice cream. As well, they each purchased a different novelty—a stuffed bunny, a T-shirt, a keychain, a rubber chicken, or a teddy bear.

Use the clues to match each child with his or her favourite animal, snack, and souvenir.

- Preesha didn't care to see the aardvarks. The person who wanted to see them got so excited that they dumped a packet of peanuts.
- One of the girls enjoyed the baboons the most.
- The teddy bear got some nacho cheese on it when its owner got distracted.
- After visiting his favourite animal at the zoo, Jon was wearing a new T-shirt with matching spots.
- Both Amanda and Cameron enjoyed some type of chips for snack.
- The chimpanzee almost got some nachos, since the person who wanted to see that exhibit considered sharing that snack with the animals there.
- The stuffed animals were not purchased by boys.
- The boy who bought the T-shirt got pretzel crumbs all over it.
- The aardvark keychain was purchased by the person who wanted to see that animal.

	aardvark	baboon	chimpanzee	gazelle	leopard	rubber chicken	t-shirt	teddy bear	stuffed bunny	key chain	chips	ice cream	pretzels	nachos	peanuts
Alvin															
Amanda															
Jon															
Cameron															
Preesha															
chips															
ice cream															
pretzels															
nachos															
peanuts															
rubber chicken															
t-shirt															
teddy bear															
stuffed bunny															
key chain															

Fill the chart with your final answer.

Name	Favourite Animal	Snack	Souvenir
Alvin	aardvark	peanuts	keychain
Amanda	chimpanzee	nachos	teddy bear
Jon	leopard	pretzels	T-shirt
Cameron	gazelle	chips	rubber chicken
Preesha	baboon	ice cream	stuffed bunny

Grade 12 Applied Mathematics (40S)

Module 4
Probability

Module 4: Probability

Introduction

Welcome to Module 4 of Grade 12 Applied Mathematics! Probability is a natural extension of the logic module you just completed. You will use the language and notation of logic as you consider the mathematics of predictions. The prediction of an event is not the same thing as simply guessing when something could occur. While a prediction and a guess may both be either correct or incorrect, the difference is that a prediction is based on definitive data, while a guess is based on uncertain information. This module will show you that probability is not the same thing as a promise (so don't blame the weather forecaster when your baseball game gets rained out), and will demonstrate that mathematical predictions are made up of theoretical calculations, include different types of events, can be used in decision making, and can involve subjective judgment.

Some examples in this module will be based on games of chance involving dice or cards. This is not intended as an endorsement for gambling, but rather to provide illustrations of possible outcomes and fair games.

Assignments in Module 4

When you have completed the assignments for Module 4, submit your completed assignments to the Distance Learning Unit either by mail or electronically through the learning management system (LMS). The staff will forward your work to your tutor/marker.

Lesson	Assignment Number	Assignment Title
	Cover Assignment	Strategies
3	Assignment 4.1	Probability and Odds

Resource Sheet

When you write your midterm examination, you are encouraged to take a Midterm Examination Resource Sheet with you into the examination. This sheet will be one letter-sized page, 8½ ″ by 11″, with both sides in your handwriting or typewritten. You will submit it with your examination, but you do not receive any marks for it.

Many students have found that preparing a resource sheet is an excellent way to review. It provides you with a summary of the important facts of each module. You should complete a resource sheet for each module to help with your studying and reviewing. Lesson summaries and module summaries are included for you to use as a guide.

You may use the list of instructions provided below to help you with preparing your resource sheet for the material in Module 4. On this sheet, you should record math terms and definitions, formulas, sample questions, or a list of places where you often make mistakes. You should also identify special areas that require extra attention or review by writing the page numbers.

After you have completed each module's resource sheet, you may summarize the sheets from Modules 1, 2, 3, and 4 to prepare your Midterm Examination Resource Sheet. The midterm examination for this course is based on Modules 1 to 4.

Resource Sheet for Module 4

As you go through the lessons of this module, you may want to consider the following suggestions regarding the creation of a resource sheet.

1. List all the important math terms, and define them if necessary.
2. List all the formulas and perhaps a sample problem that shows how the formula is used.
3. If necessary, write the solutions to some problems, showing in detail how you did the calculations.
4. Copy any questions that represent the key points of the lesson, and perhaps include the solutions as well.
5. Identify the problems you found most difficult, and copy the page numbers onto the resource sheet so that you can review them before writing the examination. You may also copy the problems and the solutions onto your resource sheet, and later write them onto your Midterm Examination Resource Sheet.
6. Write any comments, ideas, shortcuts, or other reminders that may be helpful during an examination.

Writing Your Midterm Examination

You will write the midterm examination when you have completed Module 4 of this course. The midterm examination is based on Modules 1 to 4, and is worth 20 percent of your final mark in the course. To do well on the midterm examination, you should review all the work you complete in Modules 1 to 4, including all the learning activities and assignments. You will write the midterm examination under supervision.

Notes

Module 4 Cover Assignment: Strategies

For this cover assignment you may choose to complete and submit **either** of the following:

- Option 1: Winning Nim

  or

- Option 2: Winning 50ifty

In both options, you are required to consider a game and determine a strategy to win the game. Your assignment must include a description of a winning strategy and a problem-solving strategy.

Notes

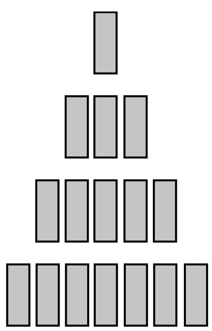
Module 4 Cover Assignment

Strategies

Option 1: Winning Nim

Total: 5 marks

You were introduced to the game of Nim in Module 3. In this game, 16 markers are arranged in four rows, as shown in the diagram below. Two players take turns removing any number of markers from any one row. The player who is forced to remove the last marker loses the game.



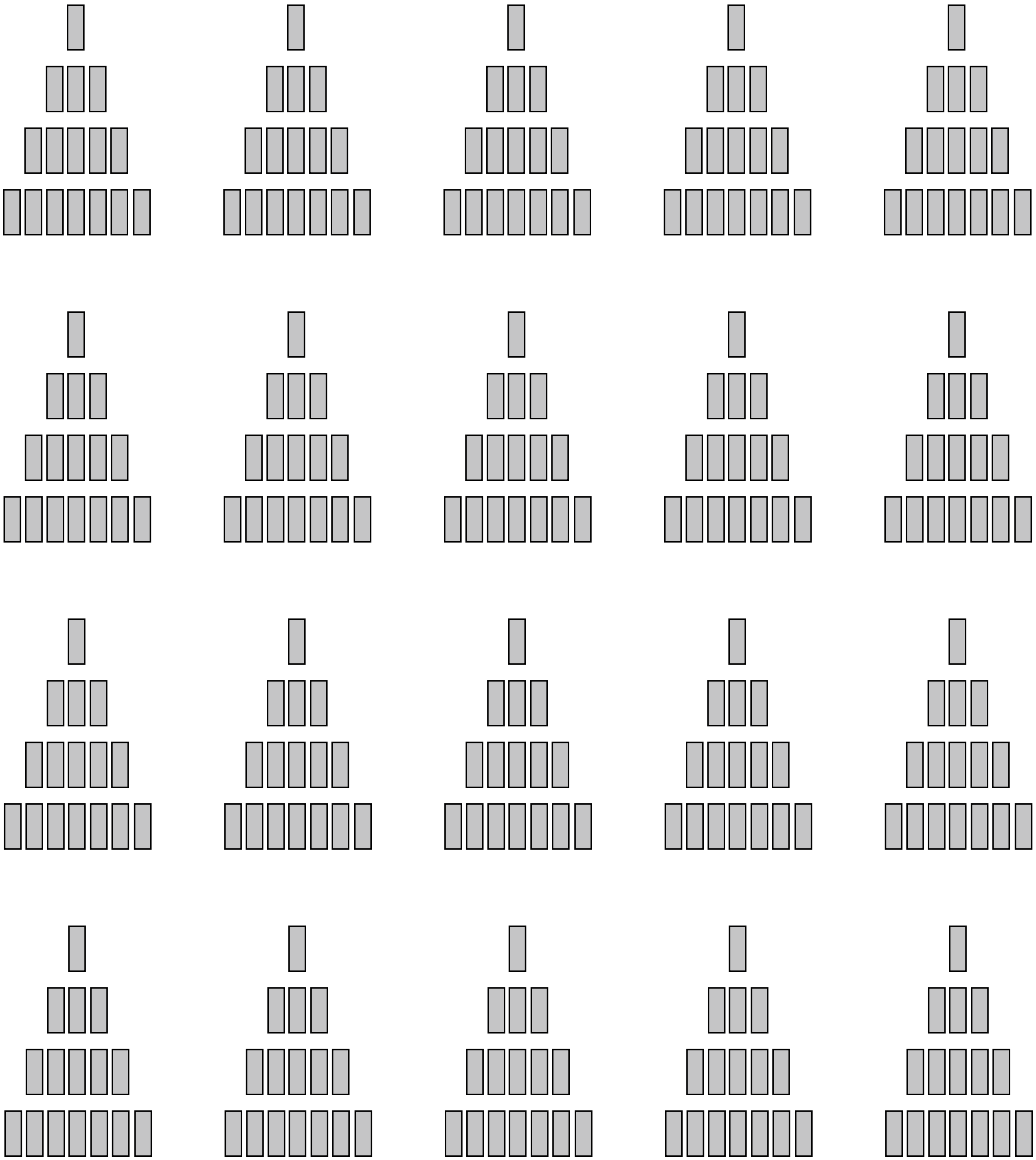
Starting Positions

Find 16 markers, such as pebbles, sticks, chips, beans, or blocks, and arrange them in the starting positions. Alternately, you may play this game using the printed copies on the following page by taking turns crossing off markers. Find someone willing to play this game with you and play it repeatedly. Once you have figured out how to win this game, record your strategies.

1. Determine, explain, and verify your strategy for winning this game.
2. Explain your problem-solving strategy and how that helped you find a winning strategy for the game. Some possible problem-solving strategies include looking for a pattern, making systematic lists, simplifying the original problem, working backwards, and developing an alternate approach.

Note: Remember, submit only one of the two options. You may want to read Option 2 and play both games before deciding which option to submit.

Module 4 Cover Assignment: Strategies (continued)



Module 4 Cover Assignment

Strategies

Option 2: Winning 50ifty

Total: 5 marks

In this game, two players take turns saying a number.

Player A begins by saying either 1, 2, or 3. Player B says the sum of that number plus 1, 2, or 3. Each player in turn increases the previous number by 1, 2, or 3. For example, Player A can choose either 1, 2, or 3 to start. Suppose Player A chooses 2. Player B chooses to add either 1, 2, or 3 and can say either 3, 4, or 5. Player A then adds 1, 2, or 3 to this new number. The player who says "50" wins the game.

Find someone willing to play this game with you and play it repeatedly. Once you have figured out how to win this game, record your strategies.

1. Determine, explain, and verify your strategy for winning this game.
2. Explain your problem-solving strategy and how that helped you find a winning strategy for the game. Some possible problem-solving strategies include looking for a pattern, making systematic lists, simplifying the original problem, working backwards, and developing an alternate approach.

Notes

Lesson 1: Probability of Independent and Dependent Events

Lesson Focus

In this lesson, you will

- ❑ compare, using examples, independent and dependent events
- ❑ determine the probability of an independent or dependent event, given the occurrence of a previous event
- ❑ represent sample space using set notation or graphic organizers
- ❑ solve contextual problems that involve determining the probability of independent or dependent events as well as complementary events

Lesson Introduction

You have experience with probability both in previous courses in school and in your personal lives. Consider these examples:

> “There is no chance of rain today.”
>
> “I will probably get my homework done in an hour.”
>
> “The captain called heads when the referee tossed a coin. If he wins, his team will kick the ball to start the football game.”

Each of these statements is a prediction that can be used in decision making. If the probability of precipitation is 5%, you won’t wear a rain jacket. Based on how long a similar assignment took you to complete, you can estimate when you will finish your homework and make plans for the rest of the day. So that no team has an unfair advantage, a coin is tossed to start the game. A high level of mathematics is required of people who work with pension plans or with home insurance to determine how much money should be set aside to ensure a greater likelihood of being able to pay the costs of a pension or a house fire. In this lesson, you will build on your experience and learn some of the mathematics of probability.

Probability

The study of probabilities involves certainties, impossibilities, and likelihoods. The chance or probability of some event, E, occurring can be denoted as $P(E)$. It is calculated by dividing the number of favourable outcomes or successes by the total number of possible outcomes. It can be written as a fraction, a decimal, or a percent.

$$P(E) = \frac{\text{successful outcomes}}{\text{total possible outcomes}}$$

You may want to add this formula to your resource sheet.

Example 1

When tossing a coin, there are two possible outcomes—the coin can land showing heads or tails.

The probability of it showing a head is $\frac{1}{2}$, since this is one of the two equally likely outcomes.

Solution

$$P(E) = \frac{\text{successful outcomes}}{\text{total possible outcomes}}$$

There is only 1 successful outcome, Heads.

There are 2 possible outcomes, Heads or Tails.

Example 2

When rolling a regular 6-sided die, the six possible outcomes are the numbers from 1 to 6.

a) Find the probability of rolling a 4.

b) Determine the probability of rolling an even number.

c) Calculate the probability of rolling a 7.

Solution

a) There are six possible outcomes that are equally likely: rolling a 1, 2, 3, 4, 5, or 6. The one successful outcome is rolling a 4.

$$P(4) = \frac{\text{successful outcomes}}{\text{total possible outcomes}} = \frac{1}{6}$$

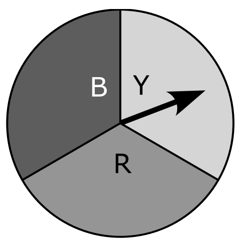
b) There are three even numbers on a regular 6-sided die.

$$P(\text{even number}) = \frac{3}{6} = \frac{1}{2}$$

c) There is no possibility of rolling a 7 on a regular 6-sided die, so $P(7) = \frac{0}{6} = 0.$

Example 3

A spinner has three equal sectors, coloured red, yellow, and blue.



Determine the following:

a) P(yellow)

b) P(red or blue)

c) P(**not** (red or blue))

d) P(red and blue)

e) P(primary colour)

Solution

a) $P(\text{yellow}) = \frac{1}{3}$

All three possible colours are equally likely, but yellow is the only successful outcome of the three possible outcomes.

b) $P(\text{red or blue}) = \frac{2}{3}$

There are two possible successful outcomes out of a total of three possible outcomes.

c) P(**not** (red or blue)) = $\frac{1}{3}$

If the probability of (red or blue) = $\frac{2}{3}$, the probability of (**not** (red or blue)) = $\frac{1}{3}$.

Another way to look at this is to consider that P(**not** (red or blue)) is the same as P(yellow) = $\frac{1}{3}$.

It is important to notice that sum of the probabilities of P(red or blue) and P(**not** (red or blue)) is equal to 1 $\left(\frac{2}{3} + \frac{1}{3} = 1\right)$. If the total probability of something happening plus the probability of it **not** happening is equal to 1, these two probabilities represent a "sure thing."

d) P(red and blue) = 0

It is not possible for the spinner to point to two colours at once, so this outcome is impossible and the probability of it occurring is $\frac{0}{3} = 0$.

e) P(primary colour) = 1

The three colours on the spinner are all primary colours, so the spinner landing on any one of the colours is considered a successful outcome. The number of successful outcomes is 3 and the total number of outcomes possible is 3 so the probability is $\frac{3}{3} = 1$.

Basic Probability Properties

The probability of an event is a value between 0 and 1, where $P(E) = 0$ is an impossibility and $P(E) = 1$ is a sure event. All other probabilities lie between 0 and 1.

Probability can be expressed as a reduced fraction, a decimal, a percent, a ratio, or with words. For example, if the probability of an event,

$P(E) = \frac{\text{successful outcomes}}{\text{total possible outcomes}} = \frac{3}{5}$, this could also be expressed as

$\frac{3}{5} = 0.6 = 60\%$, as a ratio of $3:5$, or by stating "the probability of event E is three out of five."

The probability that an event does **not** occur is the complement of the event, and can be expressed using the same notation you learned in the logic module, $P(E')$. It is calculated as $P(E') = 1 - P(E)$ or $P(E) + P(E') = 1$.

Note that the logic symbols ∪ and ∩, for union (or) and intersection (and), may also be used when expressing probability.

Representing the Sample Space

In the last module, you investigated set theory and learned how sets could be applied to define and solve problems. The set of all elements considered part of a specified situation or population was called the universal set, while a proper subset contained part of the universal set. In probability, the list of "all possible outcomes" represents the universal set while the "successful outcomes" represent a subset.

The universal set of possible outcomes in a given event is also called the **sample space**. It can be expressed as a list of events in braces or by using a chart, a tree diagram, or a Venn diagram.

You may want to add this information to your resource sheet.

Example 1

Using braces, state the sample space of the event: Rolling two regular 6-sided dice.

Solution

The sample space consists of all possible combinations of the number 1 through 6 on the first die with the numbers 1 through 6 on the second die.

S = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

The 36 possible outcomes in this sample space might be more clearly communicated by using a chart or table.

Example 2

a) Using a table, state the sample space of the event: Rolling two dice.

b) State the probability of rolling doubles as a fraction, a decimal (rounded to the nearest thousandth), and a percent (to one decimal place).

Solution

a)

Die #2 \ Die #1	1	2	3	4	5	6
1	1, 1	2, 1	3, 1	4, 1	5, 1	6, 1
2	1, 2	2, 2	3, 2	4, 2	5, 2	6, 2
3	1, 3	2, 3	3, 3	4, 3	5, 3	6, 3
4	1, 4	2, 4	3, 4	4, 4	5, 4	6, 4
5	1, 5	2, 5	3, 5	4, 5	5, 5	6, 5
6	1, 6	2, 6	3, 6	4, 6	5, 6	6, 6

b) The events in the sample space that illustrate rolling doubles are shown in a slightly larger font in the table above. This subset consists of 6 events, so $P(\text{doubles}) = \frac{6}{36}$.

As a reduced fraction, $P(\text{doubles}) = \frac{1}{6}$.

As a decimal, $P(\text{doubles}) = \frac{1}{6} = 0.1666\ldots \approx 0.167$ (the wavy equality symbol means "approximately equal to").

As a percent, $P(\text{doubles}) \approx 0.167 \times 100 = 16.7\%$.

Example 3

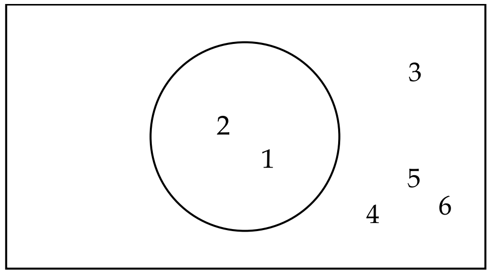
a) Create a Venn diagram to illustrate the sample space of rolling a regular 6-sided die, and the probability of event E (the number rolled will be less than 3).

b) Calculate $P(E)$

c) Calculate $P(E')$

Solution

a)



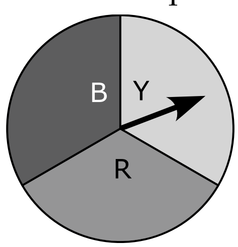
b) $P(E) = \frac{2}{6} = \frac{1}{3}$

c) $P(E') = 1 - P(E) = 1 - \frac{2}{6}$

$$= \frac{6}{6} - \frac{2}{6} \quad = \frac{4}{6} \quad = \frac{2}{3}$$

Example 4

a) Create a tree diagram to illustrate the sample set of selecting red, blue, or yellow on the spinner shown and rolling a regular 6-sided die.

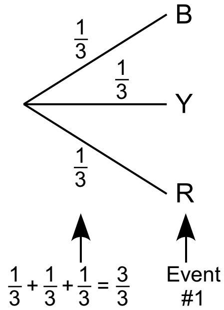


b) What is *P*(red, 3)?

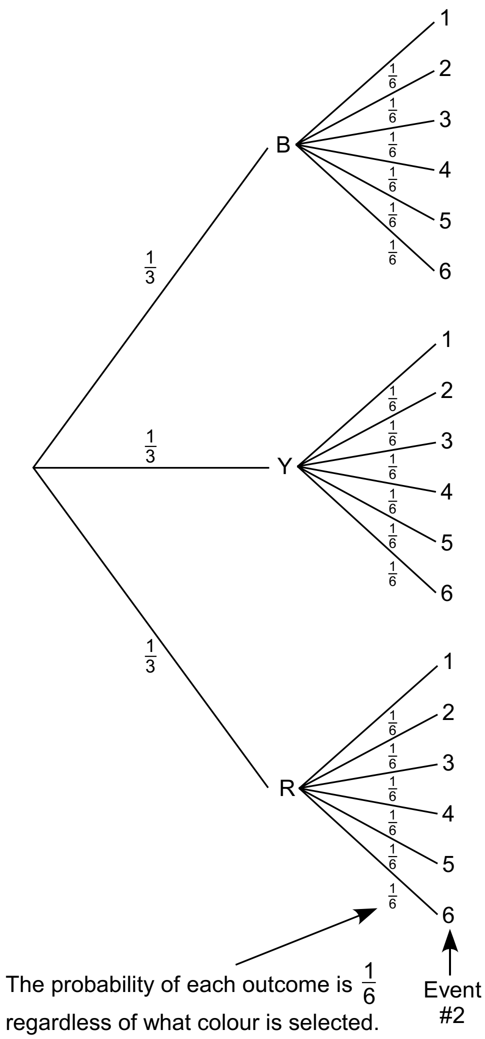
c) What is *P*(yellow or blue and even number)?

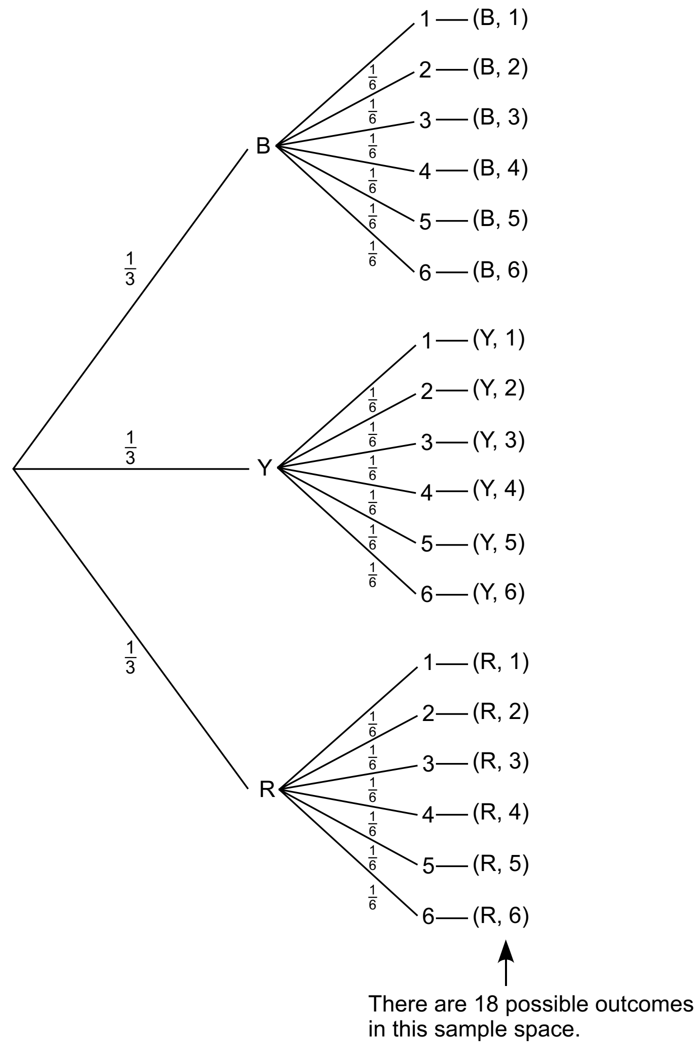
Solution

a) A **tree diagram** is a graphical representation of the sample space, showing each possible event as a "branch." This sample space includes two events—spinning a colour (with three equally likely outcomes, B, Y, or R) and rolling a die (with six equally likely outcomes). The outcomes of each event are drawn vertically. The sum of the probabilities for an event is equal to 1.



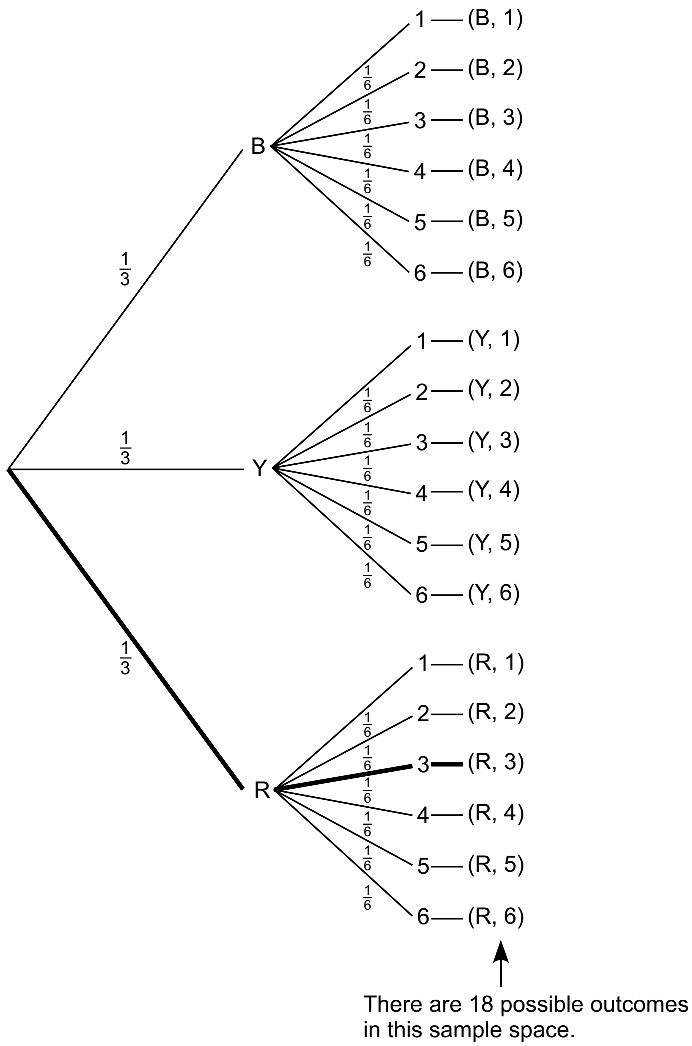
The second event, rolling the die, is independent of the first event, spinning a colour. Regardless of what colour the spinner lands on, the die can land as 1, 2, 3, 4, 5, or 6.





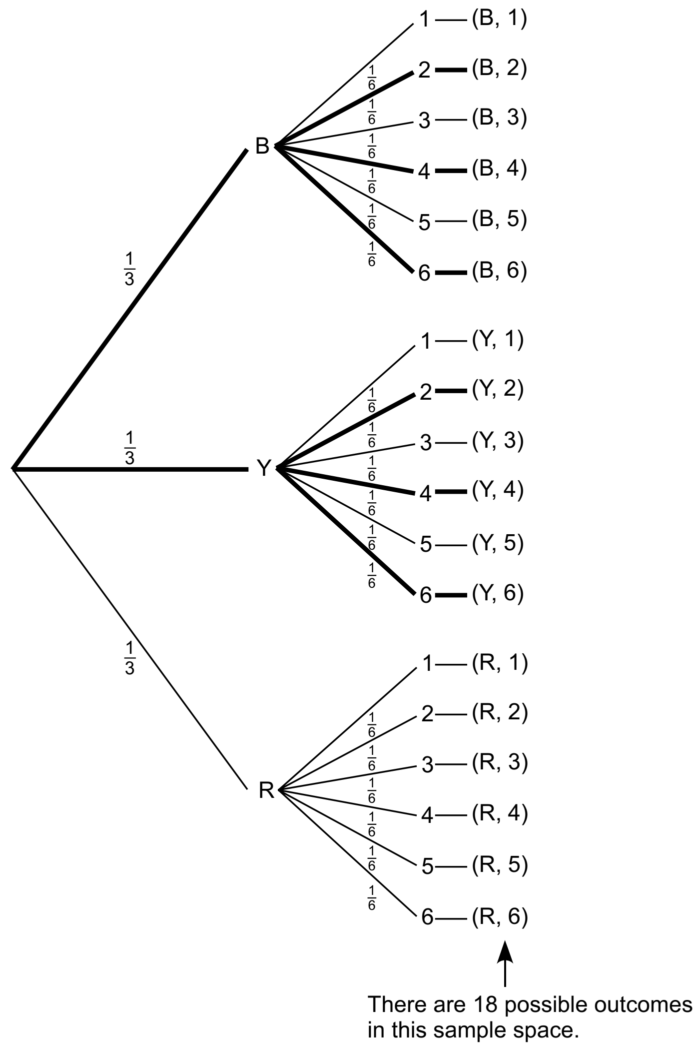
The sample space for this situation can be written by following each possible branch on the tree. Remember that a sample space must always state all possible outcomes.

b) P(red, 3)



Of the 18 possible outcomes in this sample space, only one represents the event (red, 3) so $P(\text{red}, 3) = \dfrac{1}{18}$.

c) P(yellow or blue and even number)



On the tree diagram, 6 of the outcomes (shown with bold lines) in the sample space are yellow or blue and an even number.

$$P((\text{yellow} \cup \text{blue}) \cap \text{even}) = \frac{6}{18} = \frac{1}{3}$$

Calculating the Probability of Independent Events

If two or more events can occur in such a way that the outcomes of any one of them do not affect the possibilities of the outcomes in any of the other events, they are said to be independent. In the example above, the outcomes of rolling the die were not affected by the colour of the spinner. These two events are independent.

The probability of a set of independent events occurring can be calculated by multiplying the probabilities of the separate events. In the previous example, the probability of yellow or blue is $\frac{2}{3}$, the probability of an even dice number is $\frac{3}{6}$. When these fractions are multiplied, you get $\frac{6}{18} = \frac{1}{3}$. This is the same value you obtained by adding up the six events on the tree diagram. Remember, in this context, the symbol, ∩, can be replaced with the word "and."

$$P(A \cap B) = P(A) \times P(B)$$

You may want to add this formula to your resource sheet along with a description of what the symbols mean.

Example 1

A coin is flipped and a die is rolled. What is the probability that the coin will show heads and the die will land on 4?

a) Determine $P(\text{heads} \cap 4)$

b) Verify the result by creating a tree diagram

Solution

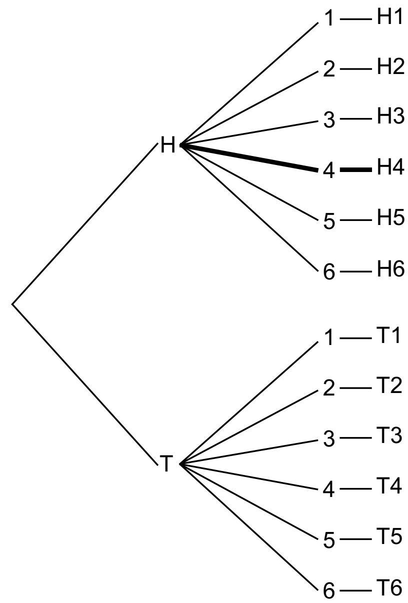
a) The results of the coin toss and die roll do not affect each other, so the events are independent.

The probability of heads on a coin is $\frac{1}{2}$. The probability of rolling a 4 is $\frac{1}{6}$.

$$P(\text{heads} \cap 4) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$

b) The tree diagram illustrating the sample space for tossing a coin and rolling a die shows that there are 12 outcomes and only one of these is heads and 4.

$P(\text{heads} \cap 4) = \dfrac{1}{12}$



Example 2

Marbles are concealed inside two containers. The first container has 2 blue, 2 yellow, and 3 red marbles. The second container has 1 green, 2 yellow, and 2 red marbles. One marble is selected from each container.

a) Determine the probability that the marbles are
   i) both red
   ii) blue and green
b) Create a tree diagram to illustrate the sample space. Indicate the probability along each branch.
c) What is the probability that at least one yellow marble is drawn?
d) What is the probability of not drawing a yellow marble?

Solution

a) i) Probability that both marbles are red.

These events are independent since the first choice in one container does not affect the second choice in the other container.

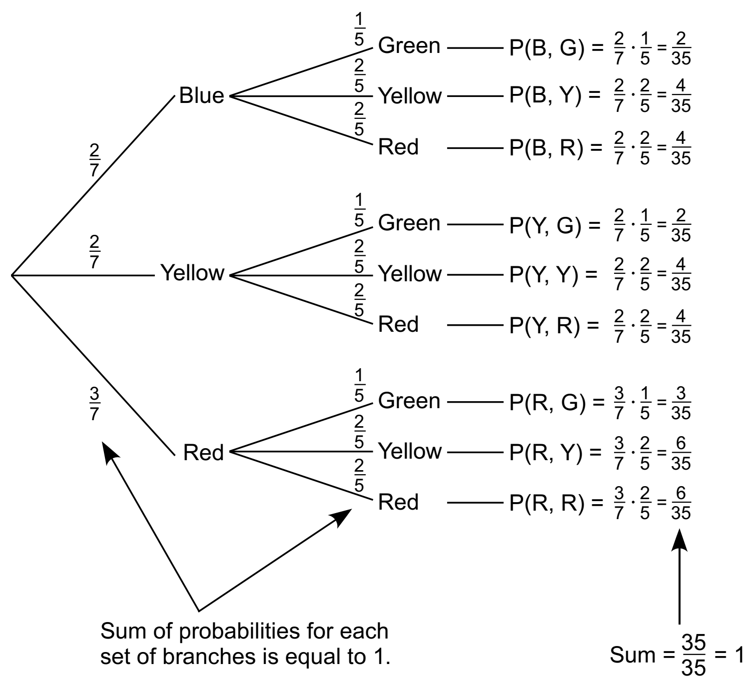
$P(\text{red and red}) = P(\text{red from container 1}) \times P(\text{red from container 2})$

$\text{P(red and red)} = \dfrac{3}{7} \times \dfrac{2}{5} = \dfrac{6}{35}$

ii) Probability that the marbles are blue and green

$$P(\text{blue and green}) = \frac{2}{7} \times \frac{1}{5} = \frac{2}{35}$$

b) A tree diagram that has a branch for each marble would end up having 35 branches. It is possible to draw this, but it is too large and awkward to draw accurately. You may instead draw the diagram using only the colours possible for each event, and include the probability of choosing that colour along each branch. This will make probability calculations more convenient and the diagram easier to read.



You can verify the calculations made in part (a) by following the branches in a horizontal fashion. For the first event, follow the bottom branch for red and continue along the second red branch to verify the product is $\frac{6}{35}$. For the probability of (blue and green), the only outcome in the sample space that matches follows the top branch for blue and continues along the second top branch for green. The product is $\frac{2}{35}$.

c) What is the probability that at least one yellow marble is drawn? From the tree diagram, you can see that it is possible to draw zero, one, or two yellow marbles. In order for there to be at least one yellow marble, you would have to draw either one or two yellow marbles. You may choose yellow on the first pick or the second pick. The probability of choosing at least one yellow is the sum of the all the probabilities of picking one or two yellows.

P(at least one yellow) = P(one yellow or two yellows)
= P(one yellow) + P(two yellows)

$$P(\text{at least one yellow}) = \left(\frac{4}{35} + \frac{2}{35} + \frac{4}{35} + \frac{6}{35}\right) + \left(\frac{4}{35}\right) = \frac{20}{35} = \frac{4}{7}$$

Since the case of picking a yellow first **or** the case of picking a yellow second are both considered successful outcomes, you must add these probabilities together.

> **When using a tree diagram, remember to multiply probabilities** as you go along branches from left to right and **add the probabilities** of different cases from top to bottom.

d) What is the probability of not drawing a yellow marble? The complement of an event is the total number of ways it cannot occur. It is calculated as $P(E') = 1 - P(E)$. Since you know the probability of picking at least one yellow, the probability of not picking yellow is the complement of that event.

P(no yellow) = 1 − P(at least one yellow)

$$P(\text{no yellow}) = 1 - \frac{4}{7} = \frac{7}{7} - \frac{4}{7} = \frac{3}{7}$$

This can be verified by adding all the probabilities along the branches in the tree diagram that do not have yellow.

$$\left(\frac{2}{35} + \frac{4}{35} + \frac{3}{35} + \frac{6}{35}\right) = \frac{15}{35} = \frac{3}{7}$$

Dependent Events

If two events are related so that the occurrence of one event affects the probability of occurrence of the second event, then the second event is said to be **dependent** on the first.

The **probability of dependent events** A and B occurring is found by multiplying the probability of the first event and the probability of the second event, given that the first event has occurred. This is written as $P(A \cap B) = P(A) \times P(B|A)$. The vertical line between the B and A stands for "given that" or "such that." This statement can be read as "The probability of events A and B occurring is the probability of event A multiplied by the probability of event B, given that event A has occurred."

Note that $P(B|A)$ is equal to $P(B)$ if the events A and B are independent. This is true, since the fact that event A happened will have no influence on the probability of event B as the events are independent of each other. With dependent events, the probability of event B is affected by the occurrence of event A, so it can be calculated only if event A has occurred.

You may want to add this information to your resource sheet.

Example 1

A box contains 4 red marbles and 3 green marbles. What is the probability of choosing 2 green marbles if the first one is **not** replaced after it is selected?

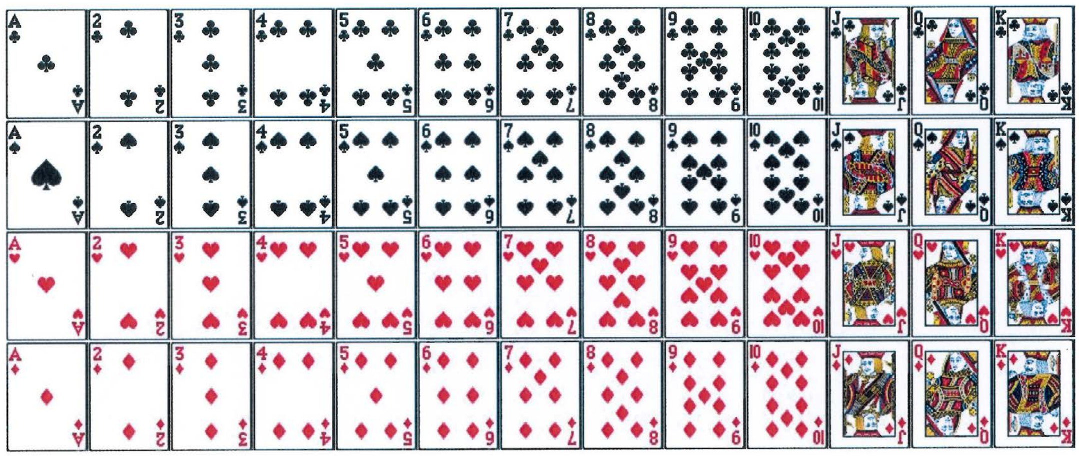
Solution

The probability of selecting a green marble the first time is $\frac{3}{7}$. The probability of selecting a green marble the second time, given that a green was first selected, is $\frac{2}{6}$. This is because you kept the first marble and only two green remain in the box, which now contains only 6 marbles. The probability of the second marble being green depends on what happened with the first marble, so the two events are called dependent events.

The probability of selecting two green marbles if the first one selected is not replaced is $\frac{3}{7} \times \frac{2}{6} = \frac{6}{42} = \frac{1}{7}$.

Example 2

A standard deck of 52 cards has 4 suits, shown in two colours: clubs, ♣, and spades, ♠, are black while hearts, ♥, and diamonds, ♦, are red. Each suit has 13 cards—the Ace has one symbol pictured on it, nine of the cards are numbered indicating 2 to 10, and there are 3 face cards called the Jack, Queen, and King.



Two cards are drawn at random without replacement. Find the probability that

a) both cards are spades.

b) both cards are Kings.

c) both cards are red.

Solution

Drawing a card without replacement and then drawing a second card are dependent events, since the card drawn on the first pick changes the number of cards remaining in the deck.

a) both cards are spades

$$P(\spadesuit, \spadesuit) = P(\text{first } \spadesuit) \times (\text{second } \spadesuit \mid \text{first } \spadesuit)$$

$$P(\spadesuit, \spadesuit) = \frac{13}{52} \times \frac{12}{51}$$

$$P(\spadesuit, \spadesuit) = \frac{156}{2652} = \frac{1}{17}$$

Assuming you have drawn a spade on the first pick, when drawing the second spade, there are only 12 possible spades and only a total of 51 cards remaining.

b) both cards are Kings

$$P(\text{K, K}) = P(\text{first K}) \times (\text{second K} \mid \text{first K})$$

$$P(\text{K, K}) = \frac{4}{52} \times \frac{3}{51}$$

$$P(\text{K, K}) = \frac{12}{2652} = \frac{1}{221}$$

c) both cards are red

$$P(\text{red, red}) = P(\text{first red}) \times (\text{second red} \mid \text{first red})$$

$$P(\text{red, red}) = \frac{26}{52} \times \frac{25}{51}$$

$$P(\text{red, red}) = \frac{650}{2652} = \frac{25}{102}$$

Learning Activity 4.1

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Find a number halfway between $\frac{1}{5}$ and $\frac{1}{6}$.

2. Which of the following is **not** a Pythagorean triple: 3–4–5, 5–12–13, 6–8–10, 7–9–11?

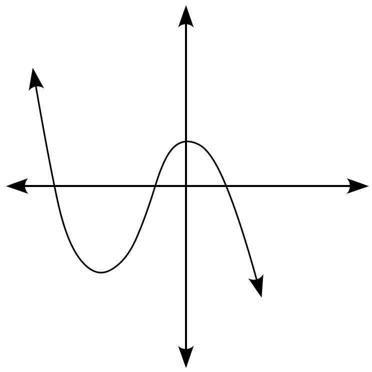
3. The outside surface of a 27 cm^3 cube is painted blue and then cut into 1 cm^3 cubes. How many 1 cm^3 cubes have **no** blue faces?

4. Lori worked 40 hours at $8 an hour and 6 hours at time-and-a-half. What was her gross pay for the week?

5. Sketch the graph of a quadratic function with two roots and a positive leading term coefficient.

continued

6. Describe the end behaviour of the polynomial graph pictured below.



7. Identify the hypothesis and the conclusion, given the conditional statement, "If the three angles are equivalent, then the triangle is equilateral."

8. Write the converse of the statement, "If the three angles are equivalent, then the triangle is equilateral."

Part B: Probability of Independent and Dependent Events

Remember, these questions are similar to the ones that will be on your assignments and midterm examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and midterm examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. A bag contains 2 red marbles and 1 white marble. You reach in and select one marble at random. Determine the following probabilities. Express them in the requested form.
   a) P(red) as a fraction
   b) P(white) as a decimal to the nearest hundredth
   c) P(red or white) as a statement
   d) P(blue) as a percentage

2. A regular 6-sided die is rolled. Determine the probability of the following outcomes.
   a) P(number less than 5 shows)
   b) P(number not greater than 4 shows)

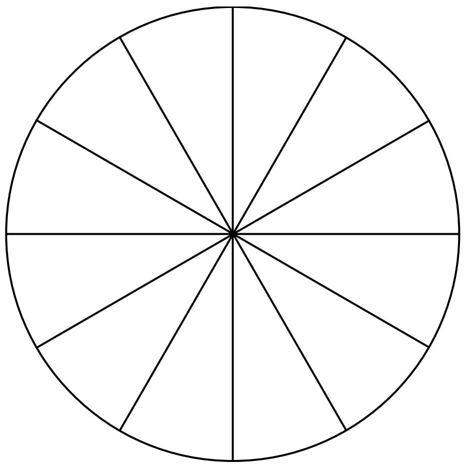
continued

3. A family plans on having three children and either a cat or a dog.
   a) Represent this sample space in two different ways.
   b) Determine the following probabilities.
      i) P(exactly 3 girls)
      ii) P(at least 1 boy)
      iii) P(not a cat)
4. Roll two regular 6-sided dice.
   a) List the sample space consisting of all possible sums of the two numbers rolled.
   b) Is the probability of each outcome in the sample space equal—that is, is each sum equally likely? Explain.
   c) Determine the probability of each of the following events.
      i) P(sum of 6)
      ii) P(even sum)
      iii) P(sum > 9)
5. There is a 25% chance of rain for each of the next two days. What is the probability of no rain on either of the next two days? Draw a tree diagram showing the probabilities along each branch to verify your answer.
6. On a cold winter morning in Flin Flon, the probability that Colette's car will start is 0.45 and the probability that Gabrielle's car will start is 0.62.
   a) Create a sample space for this situation showing all the probabilities. What is the probability that neither car will start?
   b) What is the probability that only one car will start?

continued

Learning Activity 4.1 (continued)

7. Using only the colours black, green, orange, and purple, colour the spinner so that the probability of black is $\frac{1}{3}$, the probability of green is approximately 16.7%, and the probability of orange is equal to the probability of purple.



8. A bag contains 5 white markers, 3 green markers, and 2 red markers. Two markers are drawn successively without replacement. What is the probability (to the nearest thousandth) that
   a) the first marker drawn is white and the second is green?
   b) a green and a white marker are drawn?
   c) both markers are green?

9. A standard deck of 52 cards is well shuffled and 3 cards are dealt without replacement. What is the probability of drawing three spades in a row?

continued

10. Three black, four green, and five white beans are in a cup.
    a) Calculate the probability of the following pairs of events.
        i) Drawing two green beans, if the first bean is replaced after the first pick.
        ii) Drawing two green beans, if the first bean is not replaced after the first pick.
        iii) Drawing a black bean and a white bean, if the first bean is not replaced after the first pick.
        iv) Which of the three pairs of events above are dependent events?
    b) Draw a tree diagram showing the conditional probabilities of choosing two beans if the first is not replaced.
11. Create and solve a problem that involves determining the probability of dependent or independent events.

Lesson Summary

In this lesson, you calculated the probability of an event using the formula $P(E) = \dfrac{\text{successful outcomes}}{\text{total possible outcomes}}$.

When two or more events occur at the same time, they may be dependent or independent. You calculated the probability of independent and dependent events using the **multiplication rules of probability**:

- $P(A \text{ and } B) = P(A) \times P(B)$ when A and B are independent events
- $P(A \text{ and } B) = P(A) \times P(B \mid A)$ when A and B are dependent events

When considering the probability of events where a first case **or** a second case are considered successes, you added the probabilities together.

You solved problems that involved the complement of an event using $P(E') = 1 - P(E)$ or $P(E) + P(E') = 1$.

You represented sample spaces using graphic organizers such as tree diagrams, tables, and Venn diagrams.

Notes

Lesson 2: Odds and Probability

Lesson Focus

In this lesson, you will

- ❑ provide examples of statements of odds and probability
- ❑ explain the relationship between odds and probability, and express relationships both ways
- ❑ explain how subjective judgments and probability or odds may be used in decision making
- ❑ solve contextual problems involving odds or probability

Lesson Introduction

Every day you make choices based on the likelihood of possible outcomes. These choices include decisions about what you wear, how you will get to school, and where you will spend your time. What you do and why you do the things you do are affected by probabilities.

Not all your decisions are the result of objective, mathematically calculated probability. Opinion, feelings, and other influences such as peer pressure and parental expectation all play into your choices. These subjective judgments are an important part of decision making as well. In this lesson, you will learn how probability can be used in decision making

Probability and Odds

The likelihood of an event occurring is not always expressed in terms of probability. It can also be expressed in terms of the odds in favour of it occurring. The probability of an event occurring and the odds in favour of it occurring are not the same. Consider the following example.

Example 1

Three coins are tossed.

a) Determine the probability of all three coins landing heads.

b) Determine the odds in favour of all three coins landing heads.

c) Determine the odds against all three coins landing heads.

Solution

When flipping three coins, there are 8 possible outcomes:

HHH HHT HTH THH TTH THT HTT TTT

a) HHH is one of the 8 possible outcomes, so the probability of all three coins landing heads is $\frac{1}{8}$. This can also be written as the ratio $1:8$.

b) The odds in favour of all three coins landing heads is the ratio of the one outcome, HHH, compared to the remaining 7 outcomes. Therefore, the odds in favour of all three coins landing heads is equal to $1:7$.

c) The odds against all three coins landing heads is the ratio of the seven outcomes (HHT, HTH, THH, TTH, THT, HTT, TTT), compared to the one outcome, HHH. Therefore, the odds against all three coins landing heads is equal to $7:1$.

When you compare the definitions of probability and odds, it becomes clear why they are not the same.

The probability of an event $P(E) = \dfrac{\text{number of ways the event can occur}}{\text{total number of possible outcomes}}$.

The odds in favour of an event $= \dfrac{\text{number of ways the event can occur}}{\text{number of ways the event cannot occur}}$.

You can see in the definitions above that the numerators are the same, but the denominators are different. Therefore, the value you calculate for the probability of an event, compared to the odds of that event, will be different. Also, the probability of an event can be represented by a fraction, decimal, or percent. However, the odds of an event are most commonly represented by a ratio.

The definition of the odds against an event occurring is the reverse of the definition of the odds in favour of it occurring.

The odds against an event $= \dfrac{\text{number of ways the event cannot occur}}{\text{number of ways the event can occur}}$.

You may want to add this information to your resource sheet.

Example 2

Roll a 6-sided number cube and determine the following:

a) The probability of rolling a number less than three

b) The odds of rolling a number less than three

c) The odds against rolling a number less than three

Solution

a) Out of the six possible outcomes, there are two outcomes, 1 and 2, that are less than 3. The probability of rolling a number less than three is $\frac{2}{6}$ or $2:6$, which can be reduced to $1:3$.

b) If two of the six possible outcomes are rolling a number less than three, then there are four outcomes that are not less than three: 3, 4, 5, 6. The odds in favour of rolling a number less than three are $2:4$ or as a reduced ratio, $1:2$.

c) The odds against rolling a number less than three will therefore be $2:1$.

The odds for or against an event are expressed as a part-part relationship, while probability is expressed as a part-whole relationship.

- The probability of an event occurring is always a value between 0 and 1. It may be written as a fraction, decimal, ratio, or percentage.
- The odds of an event occurring can be greater than 1, less than 1, but not less than 0. Odds are usually written as a ratio.
- The sum of the probability of an event occurring and the probability of it not occurring (its complement) is equal to 1.
- The terms of the odds in favour of an event occurring and the odds against an event occurring are reversed.

Decision Making with Probability and Odds

A weather forecast with a probability of precipitation (POP) of 90% may result in a decision to wear a raincoat, unless your raincoat is outdated and ripped, in which case you may ignore your parent's advice and wear a more fashionable sweatshirt to school. A car salesperson may try to convince you to purchase additional warranty on a new car based on the cost of potential repairs. The buyer would need to decide if the cost of the extended warranty was justified by the likelihood and cost of the possible repairs.

Experimental and Theoretical Probability

The probability of an event such as rain or snow (POP) is calculated based on data collected over a period of time. If today's weather conditions are similar to previously observed/recorded conditions that resulted in rain 9 out of 10 times, the experimental probability of precipitation may be reported as 90%. Experimental probability is calculated based on real-world data or trials from an actual experiment. This is different than theoretical probability, which is based on calculations or formulas assuming ideal, consistent conditions without bias.

Example 1

A bag contains 10 marbles of the same size. There are 6 red marbles, 3 black marbles, and 1 white marble. The marbles are mixed thoroughly and one marble is drawn out, its colour recorded, and the marble replaced. This is done a total of 80 times, in four sets of 20 trials, to make it easier to record the results. The results are shown below.

Trial	# of red drawn	# of black drawn	# of white drawn
1	13	5	2
2	11	8	1
3	9	8	3
4	11	6	3
Total	44	27	9

a) Determine the experimental probability of drawing each colour.

b) Determine the theoretical probability of drawing each colour and compare them to the experimental probabilities.

c) How could the differences between theoretical and experimental probabilities affect any decisions you have to make that are based on the results?

Solution

a) Experimental probability of drawing each colour based on the trials and the data in the tables:

$$P(R) = \frac{44}{80} = 0.55$$

$$P(B) = \frac{27}{80} = 0.34$$

$$P(W) = \frac{9}{80} = 0.11$$

b) Theoretical probability of drawing each colour based on the known number and colour of marbles in the bag:

$$P(R) = \frac{6}{10} = 0.6$$

$$P(B) = \frac{3}{10} = 0.3$$

$$P(W) = \frac{1}{10} = 0.1$$

The experimental probabilities are close to the theoretical probabilities, but they are not exactly the same.

c) If you had to make a decision based on the probability of an event and used experimental data, you would want to make sure that the data is based on a sufficient number of trials. If you only picked two marbles and both were black, you may be tempted to say that the probability of picking black is equal to 1, a sure thing, and always predict that black will be drawn. This may not be the best decision.

Over the long run, as more data is collected, the experimental probability will get closer to the theoretical probability.

Basing a decision purely on theoretical probability may not take into account various factors that influence practical situations. When you flip a coin and state that the $P(H) = \frac{1}{2}$, you are assuming that the coin is a perfectly balanced coin, that it is flipped in a random way each time, and that it lands on the same surface with exactly one side up after each flip. The assumptions could go on and on. When such assumptions are made, the resulting probability is said to be theoretical. This is the probability you would expect if it were possible to control all these factors. You may try to test some of these assumptions by comparing observed or experimental data with predicted or theoretical probabilities. For example, when you flip a coin 10 times, the theoretical probability of landing heads is 50%, so you should flip heads five times. However, in your experiments, you may land heads more or less than five times.

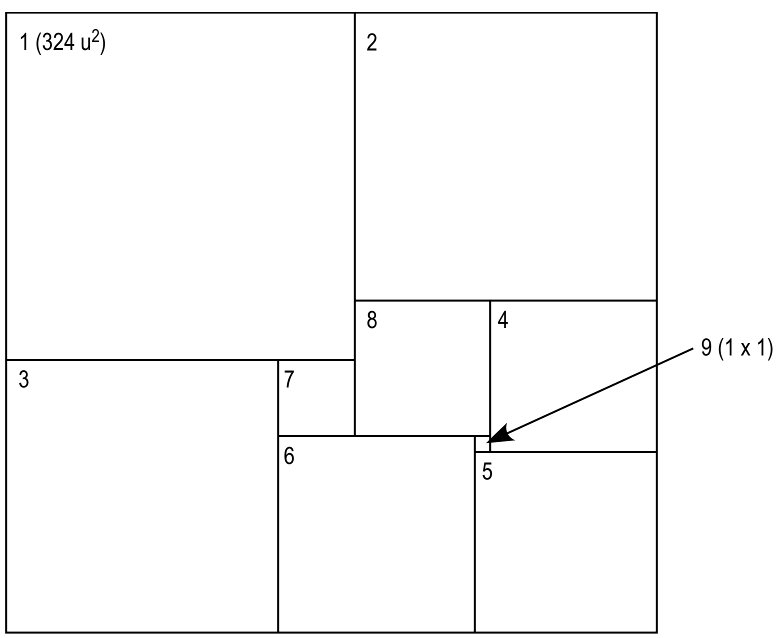
Learning Activity 4.2

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

Zbigniew Moroń, a Polish mathematician, discovered the first squared rectangle in 1925. He found that a rectangle that is 32 × 33 units may be tiled with 9 squares, each with different integer side lengths, the smallest being 1 × 1 unit and the largest having an area of 324 units^2. Find the dimensions of each of the squares numbered 1 to 8 in this rectangle. (Diagram not drawn to scale.)



continued

Part B: Odds and Probability

Remember, these questions are similar to the ones that will be on your assignments and midterm examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and midterm examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Each letter of the word MATHEMATICAL is written on a different card. All the cards are the same shape, size, and colour. The cards are placed face down and shuffled.
   a) Determine the probability of drawing an M. Express it as a fraction and a ratio.
   b) Determine the odds in favour of drawing an M.
   c) Determine the probability of not drawing an M.
   d) Determine the odds against drawing an M.
2. A card is randomly drawn from a standard deck of 52 playing cards.
   a) Determine the probability of drawing a diamond.
   b) Determine the odds in favour of drawing a diamond.
   c) Determine the probability of drawing an Ace.
   d) Determine the odds against drawing an Ace.
3. There are four white, fourteen blue, and five green marbles in a bag. A marble is selected randomly. Find the odds of the following:
   a) The odds against selecting a green marble.
   b) The odds in favour of not selecting a green marble.
   c) The odds in favour of the marble selected being either a white or a blue marble.
   d) What is true about the above odds? Explain.
4. The probability of a soccer game in a particular league going into overtime is 0.125. Find the following:
   a) The odds in favour of a game going into overtime.
   b) The odds in favour of a game not going into overtime.
   c) The approximate number of games in a 100-game season that a team could expect to go into overtime.

continued

5. A royal couple are expecting another baby.
   a) What are the odds in favour of the baby being a girl?
   b) What is the probability of the baby being a boy?
   c) If the baby is a girl, the odds in favour of the baby's name being Alexandra are 2:1. What is the probability the baby will be named Alexandra?

6. A certain type of candy comes in red, blue, and green wrappers. A box of these candies contains four red, five blue, and six green wrapped candies. A candy is drawn from the box and replaced a total of 20 times. The outcomes are shown in the following table.

	Red	Blue	Green
# of times drawn	3	9	8

   a) If another candy is drawn at random from the box, determine the experimental probabilities (calculated to 2 decimal places) for:
      i) $P(\mathrm{R})$ =
      ii) $P(\mathrm{B})$ =
      iii) $P(\mathrm{G})$ =
   b) If another candy is drawn at random from the box, determine the theoretical probabilities (calculated to 2 decimal places) for:
      i) $P(\mathrm{R})$ =
      ii) $P(\mathrm{B})$ =
      iii) $P(\mathrm{G})$ =
   c) Find the sums of the theoretical and experimental probabilities.
   d) If you really like the blue candies but think the red ones taste awful, would you want to randomly choose a candy to eat? Justify your answer.

continued

Learning Activity 4.2 (continued)

7. Alexa purchases a new car and is offered additional warranty coverage for 3 years for $1500. To determine if she needs the extended warranty, Alexa researches and finds the average repair costs and probability of needing a repair during the extended warranty period. Her findings are below.

	Probability of Repair Needed During Extended Warranty Period	**Average Repair Costs**
Mechanical repair	30%	$500
Electrical repair	20%	$1500
Both	2%	$5000

Should she purchase the additional warranty coverage? Justify your decision.

Lesson Summary

In this lesson, you learned the difference between odds and probability and expressed odds for and against an event as a probability, and vice versa. You learned about experimental and theoretical probability. You saw how probability, odds, and subjective judgment can influence decision making and you solved problems involving odds or probability.

Notes

Lesson 3: Mutually Exclusive Events

Lesson Focus

In this lesson, you will

- ❑ classify events as mutually exclusive or non-mutually exclusive and explain the reasoning
- ❑ solve contextual problems that involve the probability of mutually exclusive and non-mutually exclusive events

Lesson Introduction

Finding the probability of a single event can be straightforward. The probability of rolling a 2 on a six-sided number cube (dice) is 1 out of 6. The probability of precipitation may be reported as 30%. In this lesson, you will look at the probability of more than one event happening in the same situation. The probability of the Goldeyes winning their next game may be known. The probability of the Bombers winning their next game may also be known. How are those two probabilities combined if you want to know the probability of either the Bombers or the Goldeyes (or maybe even both!) winning their next games? This is the type of question that is addressed in this lesson.

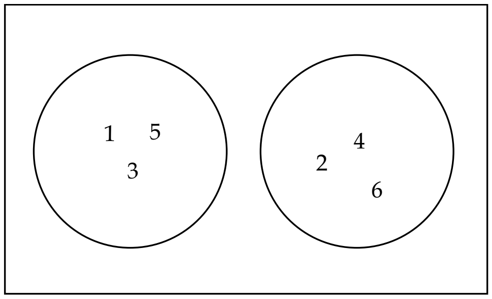
Combining Events

Mutually Exclusive and Overlapping Events

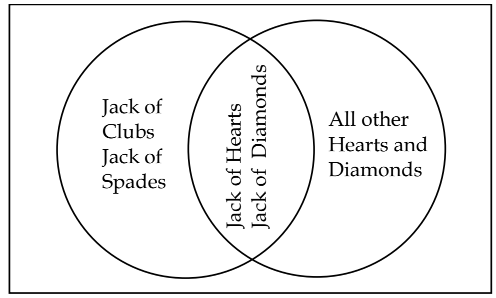
A regular 6-sided number cube is rolled. Event *A* is rolling an odd number. The successful outcomes are to roll a 1, 3, or 5. Event *B* is rolling an even number and consists of the outcomes of rolling a 2, 4, or 6. These events are said to be **mutually exclusive**, since there are no elements in their outcomes that are the same (they are disjoint sets). In other words, these events cannot happen at the same time.

When a card is drawn from a standard deck of 52, event *C* (drawing a red card) is made up of 26 outcomes (13 diamonds plus 13 hearts). Event *D* (drawing a Jack) is made up of four events (Jack of clubs, Jack of diamonds, Jack of hearts, Jack of spades). Events *C* and *D* are said to be overlapping (not mutually exclusive) because the Jack of diamonds and the Jack of hearts appear in the outcome sets of both events. In other words, events *C* and *D* can occur at the same time.

If Venn diagrams are drawn to represent these two situations, they may appear as follows:



Mutually Exclusive



Overlapping
(also called non-mutually exclusive)

Example 1

Classify the following events as either mutually exclusive or non-mutually exclusive (overlapping).

a) Going on vacation and getting a job

b) Tossing a head and tossing tails on a coin

c) Getting the gold medal and the silver medal in an event at the Olympics

d) Going for a run and listening to music

Solution

a) non-mutually exclusive

b) mutually exclusive

c) mutually exclusive

d) non-mutually exclusive

Addition Rules of Probability

To find the probability of event *A* or event *B* occurring, you will typically add the elements in each event; however, you will need to consider whether the events are mutually exclusive or overlapping.

Example 1

In the roll of a 6-sided die (a numbered cube), the probability of rolling a 1, 3, or 5 may be considered the same as the probability of rolling an odd number.

$$P(\text{odd}) = \frac{3}{6} = \frac{1}{2}$$

Alternately, since the events of rolling a 1, rolling a 3, and rolling a 5 are mutually exclusive, you could calculate the probability as:

$$P(\text{odd}) = P(1 \text{ or } 3 \text{ or } 5) = P(1) + P(3) + P(5)$$

$$P(\text{odd}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$$

$$P(\text{odd}) = \frac{3}{6} = \frac{1}{2}$$

If two events are mutually exclusive, then $P(A \text{ or } B) = P(A) + P(B)$.

You have already used this addition rule in an earlier lesson. In Lesson 1 of this module, you considered the probability of picking at least one yellow marble from two separate jars. These events were independent and mutually exclusive and, since there were two possible cases that represented a successful outcome (choosing one yellow marble or choosing two yellow marbles), you added the probability of each case to find the probability of the event. When events are mutually exclusive, you will add their probabilities with no adjustment. Compare this to calculating the probability of non-mutually exclusive (or overlapping) events, where an adjustment will be necessary.

Example 2

A card is drawn from a standard deck of 52 cards. What is the probability of drawing a red card **or** a Jack?

Solution

In this case, the successes are made up of drawing any of the 26 red cards or any of the 4 Jacks. If you simply add these outcomes, you would have 30 successes, but this is not correct.

$$P(R \text{ or } J) \neq P(R) + P(J)$$

$$P(R \text{ or } J) \neq \frac{26}{52} + \frac{4}{52}$$

$$P(R \text{ or } J) \neq \frac{30}{52}$$

Since these events are not mutually exclusive, the two red Jacks have been counted twice (once in the group of red cards and once in the group of Jacks). There are 26 red cards and only 2 other black Jacks for a total of 28 out of 52. Since these events—red and Jack—are overlapping, an adjustment must be made by subtracting the number of outcomes that have been counted twice—the Jack of hearts and the Jack of diamonds.

The correct calculation for these overlapping (non-mutually exclusive) events is:

$$P(R \text{ or } J) = P(R) + P(J) - P(R \text{ and } J)$$

$$P(R \text{ or } J) = \frac{26}{52} + \frac{4}{52} - \frac{2}{52}$$

$$P(R \text{ or } J) = \frac{28}{52} = \frac{7}{13}$$

If two events are overlapping (non-mutually exclusive), then
$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

Note that this formula actually works for both mutually exclusive and overlapping events because $P(A \text{ and } B)$ will be zero for mutually exclusive events.

You may want to add a summary of this information to your resource sheet.

Example 3

Determine if the following events are mutually exclusive or overlapping and calculate the probability of each event.

a) The probability of drawing an Ace or a 10 from a standard deck of 52 cards.

b) The probability of rolling a number less than or equal to 3 or a perfect square number on the roll of a six-sided die (or number cube).

Solution

a) The events of drawing an Ace or a 10 from a standard deck of 52 cards are mutually exclusive. So, you could just count the total number of outcomes in the described event—there are 8 (that is, 4 Aces and 4 tens)—or use the formula:

$$P(\text{Ace or } 10) = P(\text{Ace}) + P(10)$$

$$P(\text{Ace or } 10) = \frac{4}{52} + \frac{4}{52}$$

$$P(\text{Ace or } 10) = \frac{8}{52} = \frac{2}{13}$$

b) The events of rolling a number less than or equal to 3 or a perfect square number on the roll of a 6-sided die are overlapping because both 1 and 4 are perfect square numbers, and 1, 2, and 3 are all less than or equal to 3. So you could just count the total number of outcomes in the described event—there are 4 (that is, 1, 2, 3, 4) or use the formula (subtract so you don't count 1 twice).

$$P(\leq 3 \text{ or perfect square}) = P(\leq 3) + P(\text{perfect square}) - P(\leq 3 \text{ and perfect square})$$

$$P(\leq 3 \text{ or perfect square}) = \frac{3}{6} + \frac{2}{6} - \frac{1}{6}$$

$$P(\leq 3 \text{ or perfect square}) = \frac{4}{6} = \frac{2}{3}$$

Example 4

At a certain high school, 75 male students are randomly selected and surveyed regarding their sports team involvement. Of the 75 students, 15 played competitive baseball, 31 were on hockey teams, and 4 students participated in both of these sports.

a) What is the probability that a randomly selected male student plays baseball or hockey?

b) What is the probability that a randomly selected male student plays neither baseball nor hockey?

c) How many of the students surveyed played only hockey? Create a Venn diagram to support your answer.

Solution

a) Since some students play both sports, this is an overlapping (non-mutually exclusive) event.

$$P(B \text{ or } H) = P(B) + P(H) - P(B \text{ and } H)$$

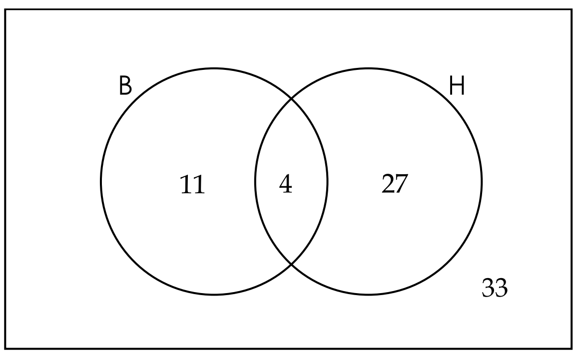
$$P(B \text{ or } H) = \frac{15}{75} + \frac{31}{75} - \frac{4}{75}$$

$$P(B \text{ or } H) = \frac{42}{75} = 0.56$$

There is a 56% chance that a randomly selected male student plays baseball or hockey.

b) The probability of the complement of an event is $1 - P(E)$, so the probability that a randomly selected male student does not play baseball or hockey is $1 - 0.56 = 0.44$ or 44%.

c) Of the students surveyed, 27 played only hockey.



Learning Activity 4.3

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

Answer questions 1 to 7 based on the following conditional statement:

“If the phone works, then the battery is not dead.”

1. Write the converse.
2. Write the inverse.
3. Write the contrapositive.
4. Is the converse true or false?
5. Is the inverse true or false?
6. Is the contrapositive true or false?
7. Is the statement biconditional?
8. Owen’s age is 125% of Avery’s age. If Avery is 16 years old, how old is Owen?

continued

Part B: Mutually Exclusive and Non-Mutually Exclusive Events

Remember, these questions are similar to the ones that will be on your assignments and midterm examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and midterm examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. The probability that Edith has a pet cat is $\frac{5}{14}$. The probability that she has a pet dog is $\frac{3}{7}$. The probability she has both a cat and a dog is $\frac{1}{7}$. What is the probability that Edith has either a cat or a dog for a pet?

2. A card is drawn from a standard deck of 52 playing cards.
   a) What is the probability of drawing the Ace of spades or Jack of clubs?
   b) What is the probability that the card is an Ace or a club?

3. Chad has arranged to meet his girlfriend Stephanie either in the library or in the student lounge. The probability that he meets her in the lounge is $\frac{1}{3}$, and the probability he meets her in the library is $\frac{2}{9}$. What is the probability he meets her in the library or lounge?

4. A factory uses two machines to produce auto parts. Machine A accounts for 40% of the total production while Machine B produces the rest. The production from Machine A is 4% defective while the production from Machine B is 3% defective.
   a) Create a tree diagram to illustrate the sample space in this situation. Include all probabilities.
   b) What is the probability that the factory produces a defective part?

5. From a box containing six blue and four yellow balls, three balls are drawn without replacement.
   a) Find the probability that two balls are blue and one ball is yellow.
   b) Find the probability that the balls are all the same colour.

continued

Learning Activity 4.3 (continued)

6. A customer enters a restaurant. The probability that the customer orders steak or salad is $\frac{8}{11}$. The probability that the customer orders steak is $\frac{2}{11}$, while the probability of ordering salad is $\frac{7}{11}$.
   a) What is the probability that the customer orders both a steak and a salad?
   b) Create a Venn diagram to represent this situation. Include all probabilities.

7. The probability that a randomly selected person wears a hearing aid is $\frac{2}{7}$, and the probability that a person wears a hairpiece is $\frac{1}{6}$. The probability of wearing both is $\frac{2}{42}$. Assuming the two events are independent, find the probability that a person chosen at random is wearing:
   a) a hearing aid and a hairpiece.
   b) a hearing aid or a hairpiece.

8. Create and solve a problem that involves determining the probability of mutually exclusive or non-mutually exclusive events.

Lesson Summary

In this lesson, you classified events as either mutually exclusive or overlapping (non-mutually exclusive), and solved problems involving the probabilities of these events.

Notes

Assignment 4.1

Probability and Odds

Total: 41 marks

This is a hand-in assignment. Clearly show the steps in your solutions on the question sheets below and submit these pages when you send in your assignments for marking. Final answers must include units where appropriate. Answers given without supporting calculations will not be awarded full marks.

1. Kate is taking a true–false test and has no idea of the answers to the last three questions. She decides to guess at the answers.

   a) What is the probability of guessing an answer correctly? *(1 mark)*

   b) Using C for correct and W for wrong, list all the possible outcomes in the sample space for answering the three questions. *(3 marks)*

c) Find the probability that she guesses all three answers correctly. *(1 mark)*

d) Find the probability that at least one of her answers is correct. *(1 mark)*

Assignment 4.1: Probability and Odds (continued)

2. A jar contains 5 blue, 1 purple, and 2 white marbles.

   a) Determine the probability of drawing 3 blue marbles if each marble is selected one at a time and each is returned before the next draw. *(1 mark)*

   b) Determine the probability of drawing 3 blue marbles if each marble is selected one at a time, but they are not replaced before the next draw. *(1 mark)*

   c) Which situation, (a) or (b) above, describes dependent events? Explain. *(1 mark)*

   d) What is the probability of not selecting purple on a single draw? *(1 mark)*

   e) What are the odds in favour of drawing a single blue marble? *(1 mark)*

   f) What are the odds against drawing a white marble? *(1 mark)*

Assignment 4.1: Probability and Odds (continued)

3. Four jets, one from each airline—Air Canada, CalmAir, West Jet, and Bearskin—are circling the Winnipeg airport waiting to land. Assuming they all have equal priority, what is the probability that they land in the following order: West Jet, Bearskin, Air Canada, CalmAir? *(2 marks)*

4. The following table indicates the principal causes of accidental deaths in Canada during one year.

Cause	Number
Motor Vehicle	2500
Fall	850
Burn	200
Drowning	350
Firearms	100
Poisonous Gas	50
Other Poisons	150
Total	**4200**

What is the probability that an accident victim died as a result of a fall? *(1 mark)*

5. On a summer day, the probability that Anne stays outside for more than 3 hours is 80%. If she stays outside for more than 3 hours, the probability that she gets a sunburn is 65%. If she stays outside for less than 3 hours, the probability that she gets a sunburn is 25%.

   a) Create a sample space for this situation showing all the probabilities. *(3 marks)*

   b) Anne thinks that the probability of getting a sunburn on a summer day is 60%. Is she correct? Explain your reasoning using probabilities. *(2 marks)*

6. Bingo chips numbered 1 to 10 are placed in a cup and two are randomly selected without replacement. What is the probability of drawing two odd-numbered chips? *(1 mark)*

7. A bag containing 20 marbles consists of five sets of four marbles and each set is a different colour (white, red, blue, yellow, and green).

   a) Find the probability that three marbles drawn at random, without replacement, are all white. *(2 marks)*

   b) Find the probability that three marbles drawn at random, without replacement, are all the same colour. *(3 marks)*

Assignment 4.1: Probability and Odds (continued)

8. What is the probability in a single toss of two dice that the sum of the numbers will be 6 or 7? *(1 mark)*

9. You draw a single card from a standard deck of 52 cards. Determine the probability it is

   a) a 7 or an Ace. *(1 mark)*

   b) black or a face card (that is, J, Q, or K). *(1 mark)*

Assignment 4.1: Probability and Odds (continued)

10. The probability that Anica will study on a Friday night is 0.3. The probability that she will play piano is 0.7. The probability that she will do at least one of these things on a Friday night is 0.8. Determine the probability that she does both. *(2 marks)*

11. The probability that a randomly selected person has allergies is 15%. Research shows that 8% of people have food allergies while 10% of people have seasonal allergies. What percentage of the population has both food and seasonal allergies?

    Create a Venn diagram to represent this situation. Show all probabilities. *(3 marks)*

12. The odds in favour of the Penguins winning the Stanley Cup are 6 : 5, while the odds in favour of the Blackhawks winning are 4 : 7.

    Determine the probability of each team winning the Stanley Cup, stated in two different ways. *(2 marks)*

13. Erik purchases a new TV for $1200 and a home audio system for $800. The purchase includes a one-year warranty. If he wants to extend the warranty coverage for three more years, it will cost him an additional $200. To determine if he needs the extended warranty, Erik researches and finds the average repair costs and probability of needing a repair during the extended warranty period. His findings are shown below.

	Probability of Repair Needed During Extended Warranty Period	**Average Repair Costs**
TV	17%	$150
Audio System	21%	$200
Both	5%	$350

Should he purchase the additional warranty coverage? Justify your decision, using probability of repairs and costs. *(5 marks)*

Notes

MODULE 4 SUMMARY

Congratulations, you have finished Module 4! In this module, you learned how to calculate the probability of an event, the probability of independent and dependent events, the probability of complementary events, and the probability of mutually exclusive and non-mutually exclusive or overlapping events. You represented all the possible outcomes in a sample space in different ways and solved problems that involved the probability of events. You considered the odds for and against an event and compared that to the probability of the event. You used probability and odds, along with subjective judgement, in decision making.

As you complete Module 4, make sure your module resource sheet is complete and up to date. You will need it as you begin to create your Midterm Examination Resource Sheet.

In the next module, you will learn to apply knowledge you gained regarding exponential growth as you study financial mathematics, including compound interest, loans, and investments.

Submitting Your Assignments

It is now time for you to submit the Module 4 Cover Assignment and Assignment 4.1 to the Distance Learning Unit so that you can receive some feedback on how you are doing in this course. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 4 assignments and organize your material in the following order:

- ❑ Module 4 Cover Sheet (found at the end of the course Introduction)
- ❑ Cover Assignment: Strategies
- ❑ Assignment 4.1: Probability and Odds

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Midterm Examination

Congratulations, you have finished Module 4 in the course. The midterm examination is out of 100 marks and worth 20% of your final mark. In order to do well on this examination, you should review all of your learning activities and assignments from Modules 1 to 4.

You will complete this examination while being supervised by a proctor. You should already have made arrangements to have the examination sent to the proctor from the Distance Learning Unit. If you have not yet made arrangements to write it, then do so now. The instructions for doing so are provided in the Introduction to this module.

You will need to bring the following items to the examination: some pens and/or pencils (2 or 3 of each), metric and imperial rulers, a graphing and/or scientific calculator, and your Midterm Examination Resource Sheet. A maximum of 3 hours is available to complete your midterm examination. When you have completed it, the proctor will then forward it for assessment. Good luck!

Graphing technology (either computer software or a graphing calculator) **is required** to complete the examination. Check with your tutor/marker to be sure your graphing technology is appropriate.

At this point you will also have to combine your resource sheets from Modules 1 to 4 onto one 8½″ × 11″ paper (you may use both sides). Be sure you have all the formulas, definitions, and strategies that you think you will need. This paper can be brought into the examination with you. We suggest that you divide your paper into two quadrants on each side so that each quadrant contains information from one module.

Examination Review

You are now ready to begin preparing for your midterm examination. Please review the content, learning activities, and assignments from Modules 1 to 4.

The midterm practice examination is also an excellent study aid for reviewing Modules 1 to 4.

You will learn what types of questions will appear on the examination and what material will be assessed. Remember, your mark on the midterm examination determines 20% of your final mark in this course and you will have 3 hours to complete the examination.

Midterm Practice Examination and Answer Key

To help you succeed in your examination, a practice examination can be found in the learning management system (LMS). The midterm practice examination is very similar to the actual examination that you will be writing. The answer key is also included so that, when you have finished writing the practice examination, you can check your answers. This will give you the confidence that you need to do well on your examination. If you do not have access to the Internet, contact the Distance Learning Unit at 1-800-465-9915 to get a copy of the practice examination and the answer key.

To get the most out of your midterm practice examination, follow these steps:

1. Study for the midterm practice examination as if it were an actual examination.
2. Review those learning activities and assignments from Modules 1 to 4 that you found the most challenging. Reread those lessons carefully and learn the concepts.
3. Contact your learning partner and your tutor/marker if you need help.
4. Review your lessons from Modules 1 to 4, including all of your notes, learning activities, and assignments.
5. Use your module resource sheets to make a draft of your Midterm Examination Resource Sheet. You can use both sides of an 8½″ by 11″ piece of paper.
6. Bring the following to the midterm practice examination: some pens and/or pencils (2 or 3 of each), metric and imperial rulers, a graphing and/or scientific calculator, and your Midterm Examination Resource Sheet.
7. Write your midterm practice examination as if it were an actual examination. In other words, write the entire examination in one sitting, and don't check your answers until you have completed the entire examination. Remember that the time allowed for writing the midterm examination is 3 hours.
8. Once you have completed the entire practice examination, check your answers against the answer key. Review the questions that you got wrong. For each of those questions, you will need to go back into the course and learn the things that you have missed.
9. Go over your resource sheet. Was anything missing or is there anything that you didn't need to have on it? Make adjustments to your Midterm Examination Resource Sheet. Once you are happy with it, make a photocopy that you can keep.

Notes

Grade 12 Applied Mathematics (40S)

Module 4
Probability

Learning Activity Answer Keys

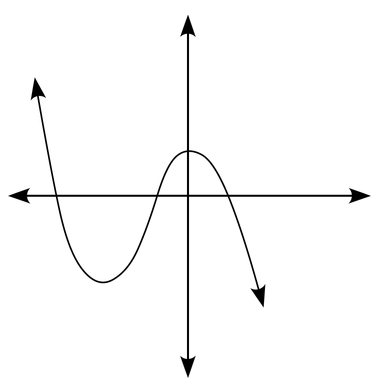
Module 4: Probability

Learning Activity 4.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Find a number halfway between $\frac{1}{5}$ and $\frac{1}{6}$.
2. Which of the following is **not** a Pythagorean triple: 3–4–5, 5–12–13, 6–8–10, 7–9–11?
3. The outside surface of a 27 cm^3 cube is painted blue and then cut into 1 cm^3 cubes. How many 1 cm^3 cubes have **no** blue faces?
4. Lori worked 40 hours at $8 an hour and 6 hours at time-and-a-half. What was her gross pay for the week?
5. Sketch the graph of a quadratic function with two roots and a positive leading term coefficient.
6. Describe the end behaviour of the polynomial graph pictured below.

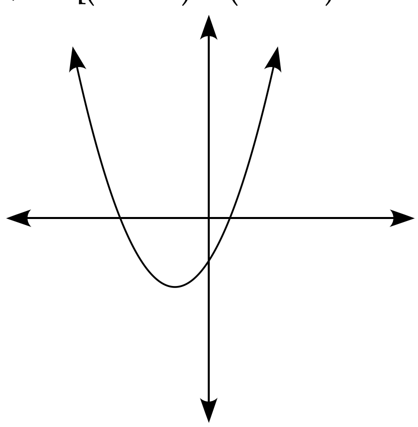


7. Identify the hypothesis and the conclusion, given the conditional statement, "If the three angles are equivalent, then the triangle is equilateral."
8. Write the converse of the statement, "If the three angles are equivalent, then the triangle is equilateral."

Answers:

1. $\frac{11}{60}$ $\begin{pmatrix} \frac{1}{5} = \frac{12}{60} \\ \frac{1}{6} = \frac{10}{60} \end{pmatrix}$

2. 7–9–11
$$\begin{pmatrix} 3^2+4^2 = 5^2 & 5^2 + 12^2 = 13^2 & 6^2 + 8^2 = 10^2 & 7^2 + 9^2 \neq 11^2 \\ 9 + 16 = 25 & 25 + 144 = 169 & 36 + 64 = 100 & 49 + 81 \neq 121 \end{pmatrix}$$

3. 1 (Only the one 1 cm^3 cube at the centre of the original 3 by 3 by 3 cube will have all six sides unpainted.)
4. $392 [(40 × 8) + (6 × 12) = 320 + 72 = 392)]
5. 

   Any parabola that opens upward and crosses the x-axis twice is an acceptable solution.
6. Quadrant II to Quadrant IV (The function goes up to the left in Quadrant II and down to the right in Quadrant IV.)
7. Hypothesis: "The three angles are equivalent."
   Conclusion: "The triangle is equilateral."
8. "If the triangle is equilateral, then the three angles are equivalent." (Switch the order of hypothesis and conclusion.)

Part B: Probability of Independent and Dependent Events

Remember, these questions are similar to the ones that will be on your assignments and midterm examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and midterm examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. A bag contains 2 red marbles and 1 white marble. You reach in and select one marble at random. Determine the following probabilities. Express them in the requested form.
   a) *P*(red) as a fraction
   b) *P*(white) as a decimal to the nearest hundredth
   c) *P*(red or white) as a statement
   d) *P*(blue) as a percentage

   Answers:

   a) $\frac{2}{3}$

   b) $\frac{1}{3} = 0.33$

   c) $\frac{3}{3} = 1$ (The probability of drawing red or white is a certainty.)

   d) 0%

2. A regular 6-sided die is rolled. Determine the probability of the following outcomes.
   a) *P*(number less than 5 shows)

      Answer:

      $$\frac{4}{6} = \frac{2}{3}$$

   b) *P*(number not greater than 4 shows)

      Answer:

      $$1 - P\left(\text{greater than } 4\right)$$

      $$1 - \frac{2}{6}$$

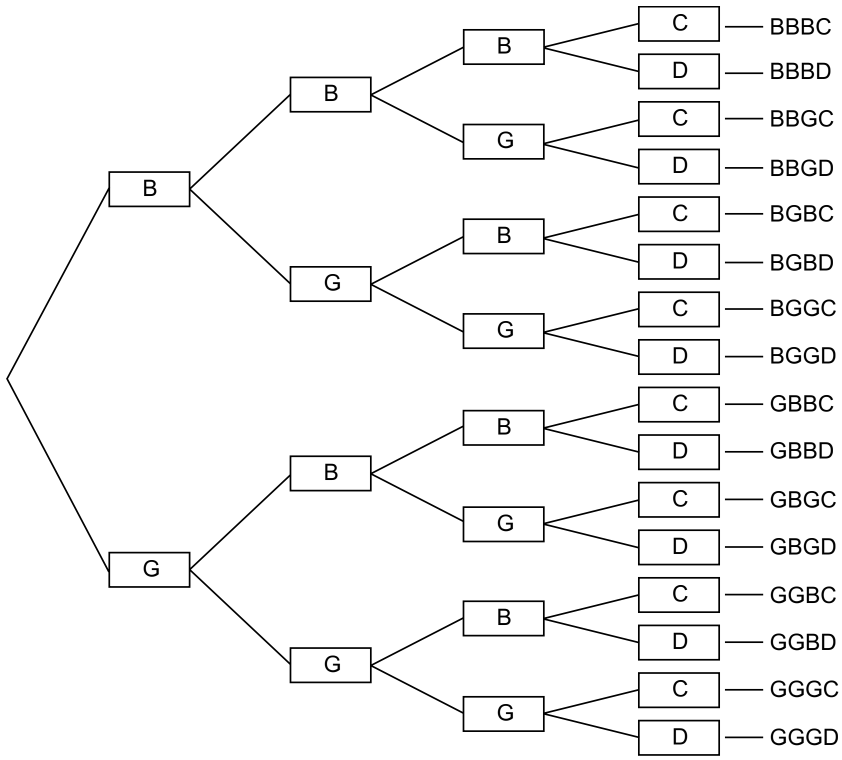
      $$\frac{4}{6} = \frac{2}{3}$$

3. A family plans on having three children and either a cat or a dog.

a) Represent this sample space in two different ways.

Answer:

As a **tree diagram** showing possible outcomes where B represents boy, G represents girl, C represents cat, and D represents dog.



As a **list of possible outcomes** where B represents boy, G represents girl, C represents cat, and D represents dog.

S = {(BBBC), (BBBD), (BBGC), (BBGD), (BGBC), (BGBD), (BGGC), (BGGD), (GBBC), (GBBD), (GBGC), (GBGD), (GGBC), (GGBD), (GGGC), (GGGD)}

b) Determine the following probabilities.

i) *P*(exactly 3 girls)

Answer:

Multiply probabilities of independent events or count 2 events (GGGC and GGGD) out of 16 using the tree diagram or list.

$$P(\text{exactly 3 girls}) = \frac{2}{16} \text{ or}$$

$$P(\text{exactly 3 girls}) = P(\text{G}) \times P(\text{G}) \times P(\text{G}) \times P(\text{C or D})$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{2}{2} = \frac{2}{16} = \frac{1}{8}$$

ii) P(at least 1 boy)

Answer:

$$P(\text{at least 1 boy}) = P(\text{one boy}) + P(\text{two boys}) + P(\text{three boys})$$
$$= \frac{6}{16} + \frac{6}{16} + \frac{2}{16}$$
$$= \frac{14}{16} = \frac{7}{8}$$

or

$$1 - P(3\text{ girls}) = 1 - \frac{2}{6}$$
$$= \frac{14}{16} = \frac{7}{8}$$

iii) P(not a cat)

Answer:

$$P(\text{not a cat}) = \frac{1}{2}$$

4. Roll two regular 6-sided dice.

   a) List the sample space consisting of all possible sums of the two numbers rolled.

      Answer:

      $S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

b) Is the probability of each outcome in the sample space equal—that is, is each sum equally likely? Explain.

Answer:

No, by creating a table showing the possible roll combinations you can see that there are multiple ways to arrive at certain sums. For example, the sum of 2 appears only once, but there are six ways to roll a sum of 7.

Die #1 / Die #2	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

c) Determine the probability of each of the following events.

i) $P(\text{sum of } 6)$

ii) $P(\text{even sum})$

iii) $P(\text{sum} > 9)$

Answers:

Counting the occurrences in the table above:

i) $\frac{5}{36}$

ii) $\frac{18}{36} = \frac{1}{2}$

iii) $\frac{6}{36} = \frac{1}{6}$

5. There is a 25% chance of rain for each of the next two days. What is the probability of no rain on either of the next two days? Draw a tree diagram showing the probabilities along each branch to verify your answer.

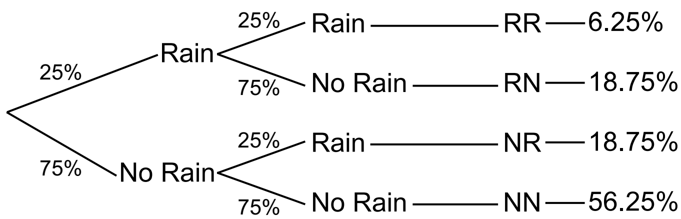
   Answer:

   P(no rain day one and no rain day two)

   = *P*(no rain day 1) × *P*(no rain day two)

   = 0.75 × 0.75

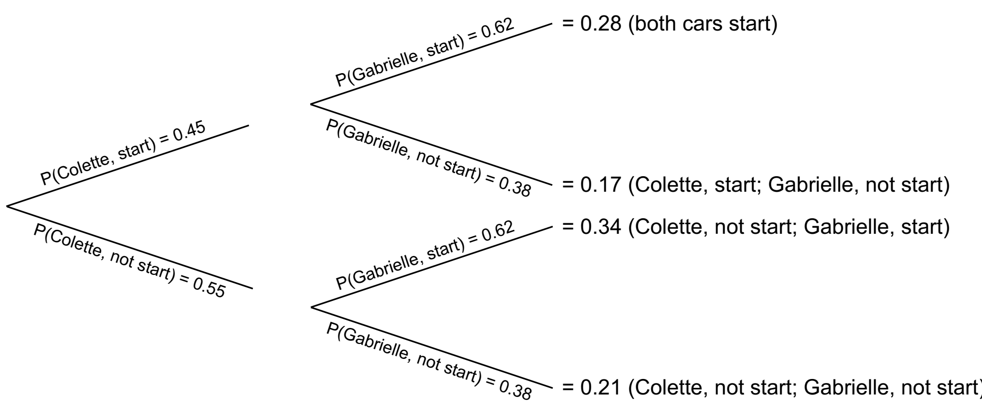
   = 0.5625



6. On a cold winter morning in Flin Flon, the probability that Colette's car will start is 0.45 and the probability that Gabrielle's car will start is 0.62.

   a) Create a sample space for this situation showing all the probabilities. What is the probability that neither car will start?

      Answer:



      P(neither car will start) = 0.21 or 21%

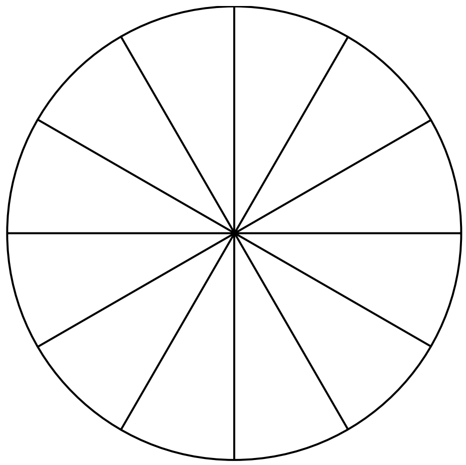
   b) What is the probability that only one car will start?

      Answer:

      P(only one car will start) = 0.17 + 0.34

      P(only one car will start) = 0.51 or 51%

7. Using only the colours black, green, orange, and purple, colour the spinner so that the probability of black is $\frac{1}{3}$, the probability of green is approximately 16.7%, and the probability of orange is equal to the probability of purple.



Answer:

There should be 4 black sectors, 2 green sectors, 3 orange sectors, and 3 purple sectors.

8. A bag contains 5 white markers, 3 green markers, and 2 red markers. Two markers are drawn successively without replacement. What is the probability (to the nearest thousandth) that

   a) the first marker drawn is white and the second is green?

   Answer:

   The probability of the second event depends on the outcome of the first, so the events are dependent.

   $$P(W \text{ and } G) = P(W) \times P(G \mid W)$$

   $$P(W \text{ and } G) = \frac{5}{10} \times \frac{3}{9}$$

   $$P(W \text{ and } G) = \frac{15}{90}$$

   $$P(W \text{ and } G) = 0.167$$

b) a green and a white marker are drawn?

Answer:

The order in which the markers are taken is not mentioned. You must assume that it could be G–W or W–G. Add the two different cases.

$$P(\text{W} - \text{G or G} - \text{W}) = \left(\frac{5}{10} \times \frac{3}{9}\right) + \left(\frac{3}{10} \times \frac{5}{9}\right)$$

$$P(\text{W} - \text{G or G} - \text{W}) = \frac{15}{90} + \frac{15}{90}$$

$$P(\text{W} - \text{G or G} - \text{W}) = \frac{30}{90}$$

$$P(\text{W} - \text{G or G} - \text{W}) = 0.333$$

c) both markers are green?

Answer:

$$P(\text{G1 and G2}) = P(\text{G1}) \times P(\text{G2} \mid \text{G1})$$

$$P(\text{G1 and G2}) = \frac{3}{10} \times \frac{2}{9}$$

$$P(\text{G1 and G2}) = \frac{6}{90}$$

$$P(\text{G1 and G2}) = 0.067$$

9. A standard deck of 52 cards is well shuffled and 3 cards are dealt without replacement. What is the probability of drawing three spades in a row?

Answer:

Representing the first spade as ♠1, the second as ♠2, and the third as ♠3:

$$P(\spadesuit 1,\ \spadesuit 2,\ \spadesuit 3) = P(\spadesuit 1) \times P(\spadesuit 2 \mid \spadesuit 1) \times P(\spadesuit 3 \mid \spadesuit 1 \text{ and } \spadesuit 2)$$

$$P(\spadesuit 1,\ \spadesuit 2,\ \spadesuit 3) = \frac{13}{52} \times \frac{12}{51} \times \frac{11}{50}$$

$$P(\spadesuit 1,\ \spadesuit 2,\ \spadesuit 3) = \frac{1716}{132600}$$

$$P(\spadesuit 1,\ \spadesuit 2,\ \spadesuit 3) = 0.0129$$

10. Three black, four green, and five white beans are in a cup.

a) Calculate the probability of the following pairs of events.

i) Drawing two green beans, if the first bean is replaced after the first pick.

Answer:

$$P(\mathrm{G, G}) = \frac{4}{12} \times \frac{4}{12}$$

$$P(\mathrm{G, G}) = \frac{16}{144} = \frac{1}{9}$$

ii) Drawing two green beans, if the first bean is not replaced after the first pick.

Answer:

$$P(\mathrm{G, G}) = \frac{4}{12} \times \frac{3}{11}$$

$$P(\mathrm{G, G}) = \frac{12}{132} = \frac{1}{11}$$

iii) Drawing a black bean and a white bean, if the first bean is not replaced after the first pick.

Answer:

Since it does not specify in which order the beans must be drawn, you must consider both cases, black then white or white then black.

$$P(\mathrm{B, W \text{ or } W, B}) = \left(\frac{3}{12} \times \frac{5}{11}\right) + \left(\frac{5}{12} \times \frac{3}{11}\right)$$

$$P(\mathrm{B, W \text{ or } W, B}) = \frac{15}{132} + \frac{15}{132}$$

$$P(\mathrm{B, W \text{ or } W, B}) = \frac{30}{132} = \frac{5}{22}$$

To confirm this answer, check the branches on the tree diagram in part (b) of this question.

iv) Which of the three pairs of events above are dependent events?

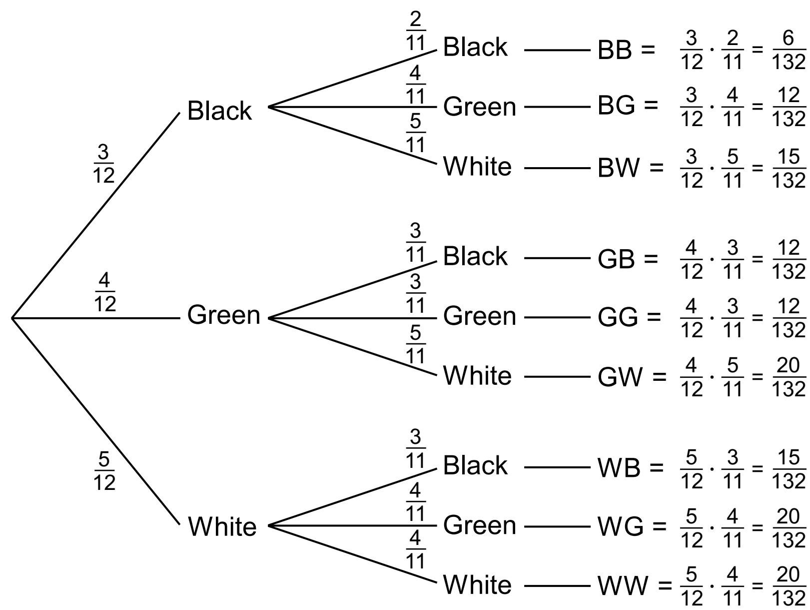
Answer:

The first pair of events, in part (i), is independent because the bean is replaced, so the second event is not affected by the first event.

Parts (ii) and (iii) are dependent, because the outcome of the second is affected by the outcome of the first. That is, you are only choosing from the remaining 11 beans in the cup, not the original 12.

b) Draw a tree diagram showing the conditional probabilities of choosing two beans if the first is not replaced.

Answer:



(conditional probabilities—no replacement)

11. Create and solve a problem that involves determining the probability of dependent or independent events.

Answer:

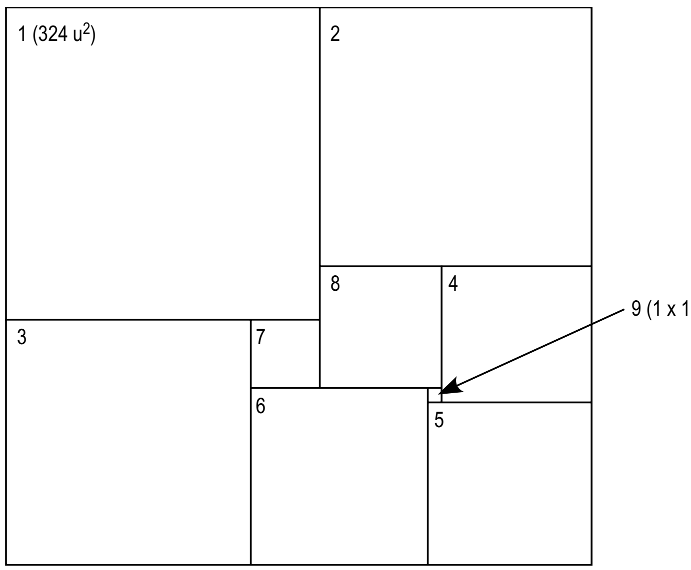
Share the problem and solution you created with your learning partner. Generally, events that involve selecting an object with replacement are independent events and events that involve selecting an object without replacement are dependent events.

Learning Activity 4.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

Zbigniew Moroń, a Polish mathematician, discovered the first squared rectangle in 1925. He found that a rectangle that is 32 × 33 units may be tiled with 9 squares, each with different integer side lengths, the smallest being 1 × 1 unit and the largest having an area of 324 units^2. Find the dimensions of each of the squares numbered 1 to 8 in this rectangle. (Diagram not drawn to scale.)



Answers:

1. $18 \times 18 \left(\sqrt{324} = 18\right)$
2. 15 × 15 (square 2 is slightly larger than square 3, so subtract 18 from longer side length of rectangle, 33 – 18 = 15)
3. 14 × 14 (32 − 18 = 14; subtract 18 from shorter length of rectangle)
4. 8 × 8 (32 − 15 = 17; 17 = 8 + 9; divide remaining shorter side length into two integer values that differ by only one unit)
5. 9 × 9 (sides are one unit longer than sides of square 4)
6. 10 × 10 (33 − (14 + 9) = 10; subtract known side lengths of squares from longer side length of rectangle)
7. 4 × 4 (14 − 10 = 4)
8. 7 × 7 (8 − 1 = 7)

Part B: Odds and Probability

Remember, these questions are similar to the ones that will be on your assignments and midterm examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and midterm examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Each letter of the word MATHEMATICAL is written on a different card. All the cards are the same shape, size, and colour. The cards are placed face down and shuffled.
   a) Determine the probability of drawing an M. Express it as a fraction and a ratio.

      Answer:

      $$P(\text{M}) = \frac{2}{12} = \frac{1}{6} \text{ or } 1:6$$

   b) Determine the odds in favour of drawing an M.

      Answer:

      2:10 or 1:5

   c) Determine the probability of not drawing an M.

      Answer:

      $$P(\text{M}') = \frac{10}{12} = \frac{5}{6} \text{ or } 5:6$$

   d) Determine the odds against drawing an M.

      Answer:

      5:1

2. A card is randomly drawn from a standard deck of 52 playing cards.

   a) Determine the probability of drawing a diamond.

      Answer:

      $P(\text{diamond}) = \frac{13}{52} = \frac{1}{4} \text{ or } 1:4$

   b) Determine the odds in favour of drawing a diamond.

      Answer:

      $13:39$ or $1:3$

   c) Determine the probability of drawing an Ace.

      Answer:

      $P(\text{ace}) = \frac{4}{52} = \frac{1}{13} \text{ or } 1:13$

   d) Determine the odds against drawing an Ace.

      Answer:

      $48:4$ or $12:1$

3. There are four white, fourteen blue, and five green marbles in a bag. A marble is selected randomly. Find the odds of the following:

   a) The odds against selecting a green marble.

      Answer:

      $18:5$

   b) The odds in favour of not selecting a green marble.

      Answer:

      $18:5$

   c) The odds in favour of the marble selected being either a white or a blue marble.

      Answer:

      $18:5$

   d) What is true about the above odds? Explain.

      Answer:

      The odds above are all the same because they represent the same situation, although each is worded differently.

4. The probability of a soccer game in a particular league going into overtime is 0.125. Find the following:

a) The odds in favour of a game going into overtime.

Answer:

The probability of overtime is $0.125 = \frac{1}{8}$, so the probability of no overtime is $\frac{7}{8}$. The odds of it going into overtime are $1:7$.

b) The odds in favour of a game not going into overtime.

Answer:

$7:1$

c) The approximate number of games in a 100-game season that a team could expect to go into overtime.

Answer:

$$\frac{1}{8} = \frac{x}{100}$$

$$8x = 100$$

$$x = 12.5$$

You could expect 12 or 13 games each season to go into overtime.

5. A royal couple are expecting another baby.

a) What are the odds in favour of the baby being a girl?

Answer:

$1:1$

b) What is the probability of the baby being a boy?

Answer:

$1:2$

c) If the baby is a girl, the odds in favour of the baby's name being Alexandra are 2:1. What is the probability the baby will be named Alexandra?

Answer

Let event A be a baby girl, and event B be naming her Alexandra.

$P(A \cap B) = P(A) \times P(B|A)$

The probability the baby is a girl is $\frac{1}{2}$.

The probability the baby will be named Alexandra (given the baby is a girl), is 2:3 or $\frac{2}{3}$.

$$P(A \cap B) = \frac{1}{2} \times \frac{2}{3}$$

$$P(A \cap B) = \frac{1}{3}$$

The probability of the baby being named Alexandra is $\frac{1}{3}$.

6. A certain type of candy comes in red, blue, and green wrappers. A box of these candies contains four red, five blue, and six green wrapped candies. A candy is drawn from the box and replaced a total of 20 times. The outcomes are shown in the following table.

	Red	Blue	Green
# of times drawn	3	9	8

a) If another candy is drawn at random from the box, determine the experimental probabilities (calculated to 2 decimal places) for:

i) $P(R) =$

ii) $P(B) =$

iii) $P(G) =$

Answers:

i) $P(R) = \frac{3}{20} = 0.15$

ii) $P(B) = \frac{9}{20} = 0.45$

iii) $P(G) = \frac{8}{20} = 0.40$

b) If another candy is drawn at random from the box, determine the theoretical probabilities (calculated to 2 decimal places) for:

i) P(R) =

ii) P(B) =

iii) P(G) =

Answers:

i) $P(\text{R}) = \frac{4}{15} = 0.27$

ii) $P(\text{B}) = \frac{5}{15} = 0.33$

iii) $P(\text{G}) = \frac{6}{15} = 0.40$

c) Find the sums of the theoretical and experimental probabilities.

Answer:

The sums of both of these probabilities is equal to 1.

d) If you really like the blue candies but think the red ones taste awful, would you want to randomly choose a candy to eat? Justify your answer.

Answer:

Theoretically, the probability of choosing red or blue is close to the same, but the experimental results show that blue was chosen more often, so I may feel the risk is worth taking. My decision may be influenced by the taste of the green candy (as it is theoretically the most likely one I will pick), the availability of other snacks, and whether or not I have just brushed my teeth.

7. Alexa purchases a new car and is offered additional warranty coverage for 3 years for $1500. To determine if she needs the extended warranty, Alexa researches and finds the average repair costs and probability of needing a repair during the extended warranty period. Her findings are below.

	Probability of Repair Needed During Extended Warranty Period	**Average Repair Costs**
Mechanical repair	30%	$500
Electrical repair	20%	$1500
Both	2%	$5000

Should she purchase the additional warranty coverage? Justify your decision.

Answer:

Determine the probability of needing mechanical or electrical repairs.

P(mech. or elect. repairs) = P(mech.) + P(elect.) − P(mech. and elect.)

P(mech. or elect. repairs) = 0.30 + 0.20 − 0.02

P(mech. or elect. repairs) = 0.48

The probability of needing a repair during the extended warranty time is 48%, which is quite high, so she may want to consider the additional warranty, but should first determine the cost of repairs compared to the cost of the warranty.

To justify this decision, determine the cost or savings possible in this situation.

	Probability of Repair Needed During Extended Warranty Period	**Cost or Savings if Extended Warranty Is Purchased**
Mechanical repair	30%	$500 – $1500 = –$1000 (cost)
Electrical repair	20%	$1500 – $1500 = 0
Both	2%	$5000 – $1500 = $3500 (savings)

Even if Alexa purchases the warranty for $1500, there is a 52% chance she will not need any repairs. It will cost her $1000 more than the mechanical repairs needed, cost the same as electrical repairs, but save her $3500 if both repairs are required. The probability of saving money on both types of repairs is only 2%. The probability of the warranty costing her more or the same as the repairs required is 98%. If she has a friend or family member who can help her do her own oil changes, she may save even more money. She probably does not need to buy the additional warranty.

Learning Activity 4.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

Answer questions 1 to 7 based on the following conditional statement:

“If the phone works, then the battery is not dead.”

1. Write the converse.
2. Write the inverse.
3. Write the contrapositive.
4. Is the converse true or false?
5. Is the inverse true or false?
6. Is the contrapositive true or false?
7. Is the statement biconditional?
8. Owen’s age is 125% of Avery’s age. If Avery is 16 years old, how old is Owen?

Answers:

1. “If the battery is not dead, then the phone works.“ (Switch the order of the hypothesis and conclusion.)
2. “If the phone doesn’t work, then the battery is dead.“ (Negate the hypothesis and the conclusion.)
3. “If the battery is dead, then the phone does not work.“ (Switch the order and negate the hypothesis and conclusion.)
4. False (A counter-example may be found when the battery is charged but the phone does not work because it has been dropped into water.)
5. False (The phone may not work for reasons other than a dead battery.)
6. True
7. No (A statement is biconditional only if the conditional and converse statements are true.)
8. 20 ($1.25 \times 16 = (1 \times 16) + (0.25 \times 16) = 16 + 4 = 20$)

Part B: Mutually Exclusive and Non-Mutually Exclusive Events

Remember, these questions are similar to the ones that will be on your assignments and midterm examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and midterm examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. The probability that Edith has a pet cat is $\frac{5}{14}$. The probability that she has a pet dog is $\frac{3}{7}$. The probability she has both a cat and a dog is $\frac{1}{7}$. What is the probability that Edith has either a cat or a dog for a pet?

   Answer:

   These events are overlapping (non-mutually exclusive).

   $$P(\text{C or D}) = \frac{5}{14} + \frac{3}{7} - \frac{1}{7} = \frac{9}{14}$$

   The probability she has either a cat or a dog is $\frac{9}{14}$ or approximately 64.3%.

2. A card is drawn from a standard deck of 52 playing cards.

   a) What is the probability of drawing the Ace of spades or Jack of clubs?

   Answer:

   These events are mutually exclusive.

   $$P(\text{Ace of spades or Jack of clubs}) = \frac{1}{52} + \frac{1}{52} = \frac{2}{52} = \frac{1}{26}$$

   b) What is the probability that the card is an Ace or a club?

   Answer:

   These events are overlapping (not mutually exclusive).

   $$P(\text{Ace or a club}) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

3. Chad has arranged to meet his girlfriend Stephanie either in the library or in the student lounge. The probability that he meets her in the lounge is $\frac{1}{3}$, and the probability he meets her in the library is $\frac{2}{9}$. What is the probability he meets her in the library or lounge?

   Answer:

   These events are mutually exclusive.

   $P(\text{library or lounge}) = \frac{1}{3} + \frac{2}{9} = \frac{5}{9}$

   The probability of meeting her in the library or lounge is 0.556.

4. A factory uses two machines to produce auto parts. Machine A accounts for 40% of the total production while Machine B produces the rest. The production from Machine A is 4% defective while the production from Machine B is 3% defective.

   a) Create a tree diagram to illustrate the sample space in this situation. Include all probabilities.

   Answer:

   b) What is the probability that the factory produces a defective part?

   Answer:

   The two branches in a tree diagram are always mutually exclusive as you have seen. So, add the two defective branches.

   P(A–D and B–D) = (0.4 × 0.04) + (0.6 × 0.03)

   P(A–D and B–D) = 0.016 + 0.018

   P(A–D and B–D) = 0.034 or 3.4%

5. From a box containing six blue and four yellow balls, three balls are drawn without replacement.

a) Find the probability that two balls are blue and one ball is yellow.

Answer:

BBY, BYB, and YBB are mutually exclusive events that could be represented on a tree diagram.

$$P(\text{2B and 1Y}) = P(\text{BBY}) + P(\text{BYB}) + P(\text{YBB})$$

$$P(\text{2B and 1Y}) = \left(\frac{6}{10} \times \frac{5}{9} \times \frac{4}{8}\right) + \left(\frac{6}{10} \times \frac{4}{9} \times \frac{5}{8}\right) + \left(\frac{4}{10} \times \frac{6}{9} \times \frac{5}{8}\right)$$

$$P(\text{2B and 1Y}) = \frac{120}{720} \times 3$$

$$P(\text{2B and 1Y}) = 0.5$$

b) Find the probability that the balls are all the same colour.

Answer:

BBB and YYY are mutually exclusive events.

$$P(\text{all same colour}) = P(\text{all blue}) + P(\text{all yellow})$$

$$P(\text{all same colour}) = \left(\frac{6}{10} \times \frac{5}{9} \times \frac{4}{8}\right) + \left(\frac{4}{10} \times \frac{3}{9} \times \frac{2}{8}\right)$$

$$P(\text{all same colour}) = \frac{120}{720} + \frac{24}{720}$$

$$P(\text{all same colour}) = 0.2$$

6. A customer enters a restaurant. The probability that the customer orders steak or salad is $\frac{8}{11}$. The probability that the customer orders steak is $\frac{2}{11}$, while the probability of ordering salad is $\frac{7}{11}$.

   a) What is the probability that the customer orders both a steak and a salad?

      Answer:

      These events are not mutually exclusive.

      $$P(\text{steak or salad}) = P(\text{steak}) + P(\text{salad}) - P(\text{steak and salad})$$

      $$\frac{8}{11} = \frac{2}{11} + \frac{7}{11} - P(\text{steak and salad})$$

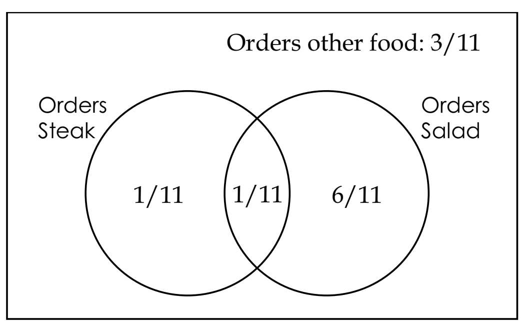
      $$\frac{8}{11} - \frac{2}{11} - \frac{7}{11} = -P(\text{steak and salad})$$

      $$\frac{-1}{11} = -P(\text{steak and salad})$$

      $$\frac{1}{11} = P(\text{steak and salad})$$

   b) Create a Venn diagram to represent this situation. Include all probabilities.

      Answer:



7. The probability that a randomly selected person wears a hearing aid is $\frac{2}{7}$, and the probability that a person wears a hairpiece is $\frac{1}{6}$. The probability of wearing both is $\frac{2}{42}$. Assuming the two events are independent, find the probability that a person chosen at random is wearing:

   a) a hearing aid and a hairpiece.

   Answer:

   Independent events, so multiply probabilities.

   $$P(\text{HA and HP}) = P(\text{HA}) \times P(\text{HP})$$

   $$P(\text{HA and HP}) = \frac{2}{7} \times \frac{1}{6}$$

   $$P(\text{HA and HP}) = \frac{2}{42} = \frac{1}{21} \text{ or approximately } 4.76\%$$

   b) a hearing aid or a hairpiece.

   Answer:

   These events are overlapping, so do not just add probabilities.

   $$P(\text{HA or HP}) = P(\text{HA}) + P(\text{HP}) - P(\text{HA and HP})$$

   $$P(\text{HA or HP}) = \frac{2}{7} + \frac{1}{6} - \frac{2}{42}$$

   $$P(\text{HA or HP}) = \frac{17}{42} \text{ or approximately } 40.5\%$$

8. Create and solve a problem that involves determining the probability of mutually exclusive or non-mutually exclusive events.

   Answer:

   Share your problem and solution with your learning partner. Mutually exclusive events cannot occur at the same time, while non-mutually exclusive, or overlapping events, may occur at the same time.

Grade 12 Applied Mathematics (40S)

Module 5
Financial Mathematics

Module 5: Financial Mathematics

Introduction

Welcome to Module 5. In this module, you will apply your number sense to financial situations and consider interest, loans, credit options, renting versus buying versus leasing, and investment strategies. You will use spreadsheets, graphing technology, and a number of online resources in your investigation, analysis, and evaluation of financial mathematics.

Financial success is important to most people. However, real success is achieved when you are able to manage your money and have it work for you, rather than against you. This module is designed to help you build the skills and understanding needed to make informed financial decisions—a critical life skill.

Assignments in Module 5

When you have completed the assignments for Module 5, submit your completed assignments to the Distance Learning Unit either by mail or electronically through the learning management system (LMS). The staff will forward your work to your tutor/marker.

Lesson	Assignment Number	Assignment Title
	Cover Assignment	Crossing the Canal with Cats
2	Assignment 5.1	Compound Interest
4	Assignment 5.2	Loans and Investments
8	Assignment 5.3	Investment Decisions

Resource Sheet

When you write your final examination, you are encouraged to take a Final Examination Resource Sheet with you into the examination. This sheet will be one letter-sized page, 8½ ″ by 11″, with both sides in your handwriting or typewritten. You will submit it with your examination, but you do not receive any marks for it.

Many students have found that preparing a resource sheet is an excellent way to review. It provides you with a summary of the important facts of each module. You should complete a resource sheet for each module to help with your studying and reviewing. Lesson summaries and module summaries are included for you to use as a guide.

You may use the list of instructions provided below to help you with preparing your resource sheet for the material in Module 5. On this sheet, you should record math terms and definitions, formulas, sample questions, or a list of places where you often make mistakes. You should also identify special areas that require extra attention or review by writing the page numbers.

After you have completed each module's resource sheet, you may summarize the sheets from Modules 5, 6, 7, and 8 to prepare your Final Examination Resource Sheet. The final examination for this course is based on Modules 5 to 8.

Resource Sheet for Module 5

As you go through the lessons of this module, you may want to consider the following suggestions regarding the creation of a resource sheet.

1. List all the important math terms, and define them if necessary.
2. List all the formulas and perhaps a sample problem that shows how the formula is used.
3. If necessary, write the solutions to some problems, showing in detail how you did the calculations.
4. Copy any questions that represent the key points of the lesson, and perhaps include the solutions as well.
5. Identify the problems you found most difficult, and copy the page numbers onto the resource sheet so that you can review them before writing the examination. You may also copy the problems and the solutions onto your resource sheet, and later write them onto your Final Examination Resource Sheet.
6. Write any comments, ideas, shortcuts, or other reminders that may be helpful during an examination.

Module 5 Cover Assignment: Crossing the Canal with Cats

Some problem solutions require working within very specific constraints and require reasoning and logic rather than numerical or algebraic manipulation. You may be familiar with one of the classic problems involving crossing a river in a boat with animals and food (sometimes an animal is also food). The goal is to get everyone or everything across the river safely and in the fewest number of trips. This cover assignment is such a problem. You need to read the problem carefully to be sure you are considering all constraints, and you will need to use logic and reasoning to come up with an efficient answer. Furthermore, similar to other aspects of mathematics, to communicate your reasoning clearly you will need to be thoughtful about the notation you choose.

Notes

Module 5 Cover Assignment

Crossing the Canal with Cats

Total: 5 marks

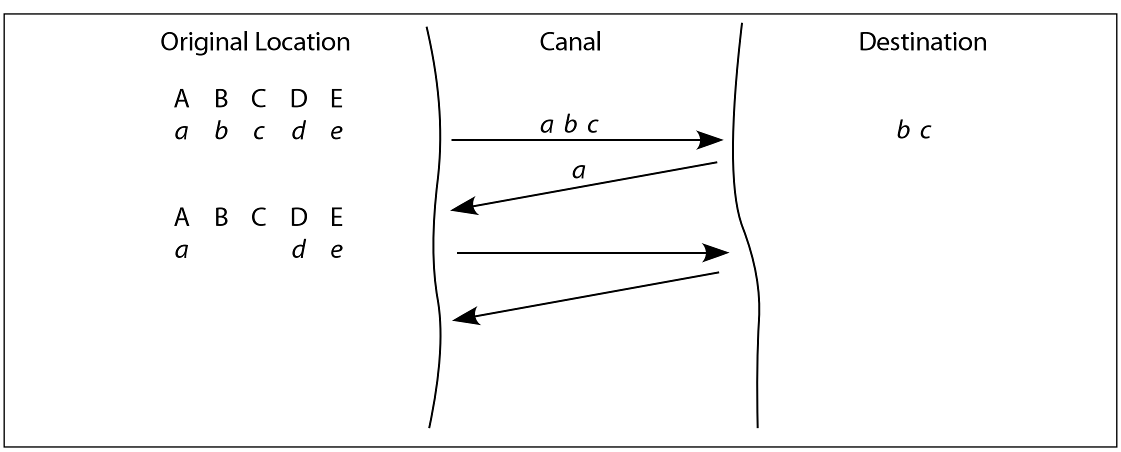
A family of five is on vacation in The Netherlands with their five pet cats (each person owns one cat). To get to the place they want to have lunch, they must cross a canal, but the only way to get to the other side is to use a small boat. Unfortunately, the boat can only hold three living things, regardless of size. The cats are very anxious creatures and cannot tolerate another person, even for a moment, unless its owner is present. Left alone, the cats get along very well with each other. One of the cats is a tiger and has been trained to operate a boat, but the other cats lack this ability. All five family members can operate the boat. How can the five family members and their five cats cross the canal?

Describe your solution clearly and concisely.

For example, you may use letters to denote the family members such as A, B, C, D, E, and use letters such as *a*, *b*, *c*, *d*, e for their respective cats, with the letter *a* being the tiger, capable of navigating the boat.

You can describe who is in the boat for each trip across the canal, who stays on the destination side, and who returns to the original location. For example, begin by saying "Cats *a*, *b*, and *c* go across the canal, cats *b* and *c* remain at the destination and cat *a* returns."

Alternatively, you may represent your solution graphically using a diagram such as the following.



Module 5 Cover Assignment: Crossing the Canal with Cats (continued)

Your task is to determine and describe the minimum number of trips across the canal, including trips both to and from the destination, for the family and the cats to get safely to the other side. Once a family member or cat crosses the canal, he or she does not have to stay on that side until the other cats and owners have arrived. They can cross the canal, return to the original side, and cross again, if necessary, to ensure that each cat is always with its owner when other family members are present.

Marks will be awarded as follows:

- A valid solution with at most 17 crossings clearly described: *5 marks*
- A valid solution with 19+ crossings clearly described: *4 marks*
- Partial solutions: marks awarded at discretion of the tutor/marker

Lesson 1: Simple and Compound Interest

Lesson Focus

In this lesson, you will

- ❑ identify the advantages and disadvantages of simple and compound interest in various situations
- ❑ graph and compare the total interest earned or paid for different compounding periods
- ❑ determine the total interest of a loan, given the principal, interest rate, and number of compounding periods
- ❑ graph and describe the effects of changing the value of one of the variables in a situation involving compound interest
- ❑ solve contextual problems involving compound interest
- ❑ apply the Rule of 72 to solve investment problems and explain the limitations of the rule

Lesson Introduction

You can grow money! Just not on a tree. This lesson is about an important and powerful mathematical concept—compound interest.

When you invest money, compound interest helps your investment to grow and, in the end, you will have more money than what you invested. When you buy on credit or borrow money, compound interest means that your debt can grow quickly. In the end, you will have to pay back more than you originally borrowed.

Knowing how to reap the benefits of compound interest, while avoiding its pitfalls, is an important lesson to learn.

Interest

Interest is a fee paid for borrowing someone else's money.

When you borrow money from a bank for a loan or mortgage, or buy on credit, you are using someone else's money (the bank's or the credit card company's) to pay for something, and you promise to pay them back. These institutions agree to do this because they charge you a fee, called interest.

If you invest in a savings account, you give your money to an investor or bank, who then use it for their own purposes. For the privilege of using your money, the investor or bank pays you. This is also called interest.

There are two ways interest can be calculated—simple interest and compound interest.

Simple Interest

Simple interest is calculated on the initial amount invested or borrowed according to the formula

$I = Prt$

where I = the amount of interest earned or paid

P = the principal or initial amount invested or borrowed

r = is the annual interest rate, written as a decimal

t = length of time, in years

With simple interest, it is assumed that you are not re-investing the interest you earn each year.

It may be helpful for you to include the formulas for simple interest along with the meaning of each variable on your resource sheet.

Example 1

Calculate the amount of interest earned on $5000 at 2.7% per year for 3 years.

Solution

Substitute the given values for the appropriate variables in the formula.

Recall that percentage means "per one hundred," so $2.7\% = \frac{2.7}{100} = 0.027$.

$I = Prt$

$I = (5000)(0.027)(3)$

$I = 405$

The investment earns $405 in simple interest in 3 years.

Example 2

Ruth wants to attend college in 72 months time. If she has $4444 to invest and her bank offers her an annual rate of 3.75% simple interest, what amount will she have for the first year's tuition?

Solution

Convert 72 months into years by dividing by 12.

$$I = Prt$$
$$I = (4444)(0.0375)\left(\frac{72}{12}\right)$$
$$I = 999.9$$

The total amount is equal to the sum of the principal plus the interest, or $A = P + Prt$.

$$A = P + Prt$$
$$A = 4444 + 999.90$$
$$A = 5443.90$$

Ruth will have $5443.90 for tuition.

Example 3

a) How long will it take for $100 to grow into $150 if the simple interest rate is 6% per year?

b) If the investment was held for 30 months, what annual simple interest rate would be needed for $100 to grow to $150?

Solution

a) You need to earn $50 in simple interest.

$$I = Prt$$
$$50 = (100)(0.06)(t)$$
$$\frac{50}{(100)(0.06)} = t$$
$$t = 8.33333$$

It will take approximately 8 years and 4 months to earn $50 simple interest on $100 at 6%.

b) First, convert months to years.

30 months is the same as 30 ÷ 12 = 2.5 years.

$$I = Prt$$
$$50 = (100)(r)(2.5)$$
$$\frac{50}{(100)(2.5)} = r$$
$$r = 0.2 = 20\%$$

You would need a rate of 20% to earn $50 simple interest on $100 in 2.5 years (30 months).

Example 4

Create a spreadsheet showing the value of an investment of $2500 held for 5 years at an annual rate of 4% simple interest. Remember, with simple interest, the interest is only paid on the original $2500 each year.

Solution

$100 interest is added each year for 5 years, so after 5 years the initial investment of $2500 is worth $3000.

Year	Amount Invested	Interest Rate	Interest Earned $I = Prt$	Total Value of Investment $P + I$
1	2500	4%	100	2600
2	2500	4%	100	2700
3	2500	4%	100	2800
4	2500	4%	100	2900
5	2500	4%	100	3000
		Total:	500	

Depending on the spreadsheet program you use, the formulas in the cells may be as follows:

Year	Amount Invested	Interest Rate	Interest Earned $I = Prt$	Total Value of Investment
1	2500	0.04	=B2*C2*1	=B2+D2
2	=B2	0.04	=B3*C3*1	=E2+D3
3	=B2	0.04	=B4*C4*1	=E3+D4
4	=B2	0.04	=B5*C5*1	=E4+D5
5	=B2	0.04	=B6*C6*1	=E5+D6
		Total:	=SUM(D2:D6)	

Compound Interest

There is a small but very significant difference between simple and compound interest. With simple interest, the interest is earned on the initial amount and the interest is paid out rather than being reinvested. As a result, the same amount of interest is added to the total each year. With simple interest, it is assumed that the interest that is paid out each year is kept out of the bank (maybe in your sock drawer).

With compound interest, the interest is earned and reinvested so it is added to the initial amount invested. Each time interest is calculated, the value is added to the investment and future interest calculations are done on the sum of the initial investment ***plus*** the interest. You can imagine that it is much more common for banks to use compound interest when calculating savings and loans because most people will keep their interest in the bank (and not in their sock drawer!).

Example 5

Create a spreadsheet showing the value of a $2500 investment at 4% compounded annually for 5 years.

Solution

Remember to round financial values to two decimal places (the nearest cent).

Year	Amount Invested	Interest Rate	Interest Earned	Total Value of Investment
1	2500.00	0.04	100.00	2600.00
2	2600.00	0.04	104.00	2704.00
3	2704.00	0.04	108.16	2812.16
4	2812.16	0.04	112.49	2924.65
5	2924.65	0.04	116.99	3041.63

The formulas in the cells may be as follows:

Year	Amount Invested	Interest Rate	Interest Earned	Total Value of Investment
1	2500	0.04	=B2*C2*1	=B2+D2
2	=E2	0.04	=B3*C3*1	=E2+D3
3	=E3	0.04	=B4*C4*1	=E3+D4
4	=E4	0.04	=B5*C5*1	=E4+D5
5	=E5	0.04	=B6*C6*1	=E5+D6

The investment is valued at \$3041.63. This is worth \$41.63 more than if the investment was calculated with simple interest (see previous example). Before reading on, spend a few minutes comparing the numbers in the spreadsheet for compound interest with the numbers in the spreadsheet for simple interest.

You may have noticed the main difference in the formulas is in the "amount invested" column. The numbers in that column keep getting bigger each year for compound interest rather than staying at the same number as it did for simple interest.

In this example, the interest on the \$2500 investment is compounded once every year at 4% per year for 5 years. A quick way to do this calculation is to find the product:

$$A = 2500(1.04)(1.04)(1.04)(1.04)(1.04)$$
$$A = 3041.63$$

The factor, 1.04, can be written (1 + 0.04). The 1 is used to add on the whole amount invested and the 0.04 is used to represent the annual percent interest.

When the interest is compounded once every year, the formula for the total amount earned uses an exponent to represent several factors of $(1 + r)$, one factor for each year. The compound interest formula is

$$A = P(1 + r)^t$$

where

A is the total amount of the investment plus interest after t years

P is the principal or initial amount invested

r is the annual interest rate as a decimal

t is the time the money is invested in years

Example 6

Use the formula to determine the value of an initial investment of \$500 after 9 years at a rate of 2.6% compounded annually (once per year). Calculate the amount of interest earned.

Solution

$$\begin{aligned} A &= P(1 + r)^t \\ &= 500(1 + 0.026)^9 \\ &= 629.94 \end{aligned}$$

The investment would be worth approximately \$629.94 after 9 years.

You could verify this amount using the spreadsheet.

The interest earned can be found by subtracting the principal invested, P, from the total amount, A.

$$629.94 - 500 = 129.94$$

Approximately $129.94 is earned in interest.

Note that this formula only works when the interest is compounded once per year. When the compounding period is not one year, the formula is adjusted to calculate the change in the number of times interest is calculated, called the compounding period.

Compounding Periods

A **compounding period** represents how many times per year the interest calculation is done and added to the principal for the next calculation. Typically, interest is compounded annually, semi-annually, quarterly, monthly, weekly, or daily. The number of compounding periods in one year is represented by n in the formula. The value of n in the compound interest formula refers to the number of compounding periods in one year and is shown in the following chart.

Compounding Period	Value of n (number of compounding periods per year)
annual	1
semi-annual	2
monthly	12
semi-monthly	24
bi-weekly	26
weekly	52
daily	365

The effect of the value of n is to increase the number of times interest is compounded over a number of years, t. If an investment is made for 5 years and the compounding period is quarterly, then interest will be compounded nt or $5 \times 4 = 20$ times in the 5 years.

Also, the value of n affects the interest rate for compounding since the interest rate is given as an annual rate, r. If the annual rate of interest is 2% and the compounding period is quarterly, then the interest rate for each quarter will be $\frac{r}{n}$ or $\frac{2\%}{4} = 0.5\%$ for each quarter of a year.

As you can see, these two calculations involving n are a part of the formula for calculating the total amount earned, A, when an initial investment, P, earns compound interest:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Notice that this formula can be used when interest is calculated once per year since n will be 1. Additionally, this same formula can be used to calculate amounts for other compounding periods. It may be helpful for you to include this formula for compound interest along with the meaning of each variable on your resource sheet.

Example 7

Collette invests a graduation gift of $1000 from her grandmother in a term deposit for 10 years at a rate of 2.85% compounded annually. Her twin brother Carl takes his $1000 gift and invests it in a savings account at 2.85% compounded daily. What will be the value of each investment after 10 years? What can you conclude about compounding periods?

Solution

Collette:

Compounded annually, so $n = 1$

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$A = 1000\left(1 + \frac{0.0285}{1}\right)^{(1)(10)}$$

$$A = 1000(1.324472545)$$

$$A = 1324.472545$$

Carl:

Compounded daily, so $n = 365$

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$A = 1000\left(1 + \frac{0.0285}{365}\right)^{(365)(10)}$$

$$A = 1000(1.329747233)$$

$$A = 1329.747233$$

After 10 years, Collette's investment is worth about $1324.47, while Carl's investment is worth about $1329.75 or $5.27 more than Collette's investment. The more often the interest calculation is done, the more often interest is added to the account and subsequent calculations are done based on an increased value. Changing the compounding period to increase the number of times when interest is compounded (annual versus daily) makes a difference in the amount earned, but the difference is small.

Example 8

How much interest is earned on an initial investment of $6750 at 9% for 50 months, compounded semi-monthly?

Solution

Compounded semi-monthly (every half month), so $n = 24$.

Also, 50 months is $\frac{50}{12}$ years.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$A = 6750\left(1 + \frac{0.09}{24}\right)^{(24)\left(\frac{50}{12}\right)}$$

$$A = 6750(1.00375)^{100}$$

$$A = 9814.306153$$

Subtract the initial investment from this amount to determine the amount of interest earned.

$$I = A - P$$

$$I = 9814.306153 - 6750$$

$$I = 3064.306153$$

Approximately $3064.31 is earned in interest.

Example 9

Graph and compare the total interest earned each year on $500 when invested for at least 10 years at 12%, calculated in the following ways: (Total interest earned ($) is the dependent variable and time (years) is the independent variable.)

a) simple interest

b) compounded annually

c) compounded weekly

Solution

a) Graph interest, I, along the vertical axis and time, t, along the horizontal axis. The formula for simple interest is $I = Prt$. To graph this situation, replace the values of P and r.

$$I = (500)(0.12)(t)$$

You can use technology to graph the function $y = (500)(0.12)x$.

b) The formula for compound interest calculates the total amount or value of the investment (interest plus initial investment). To graph the value of the interest only, modify the equation by subtracting the amount of principal to account for this.

$$I = A - P$$

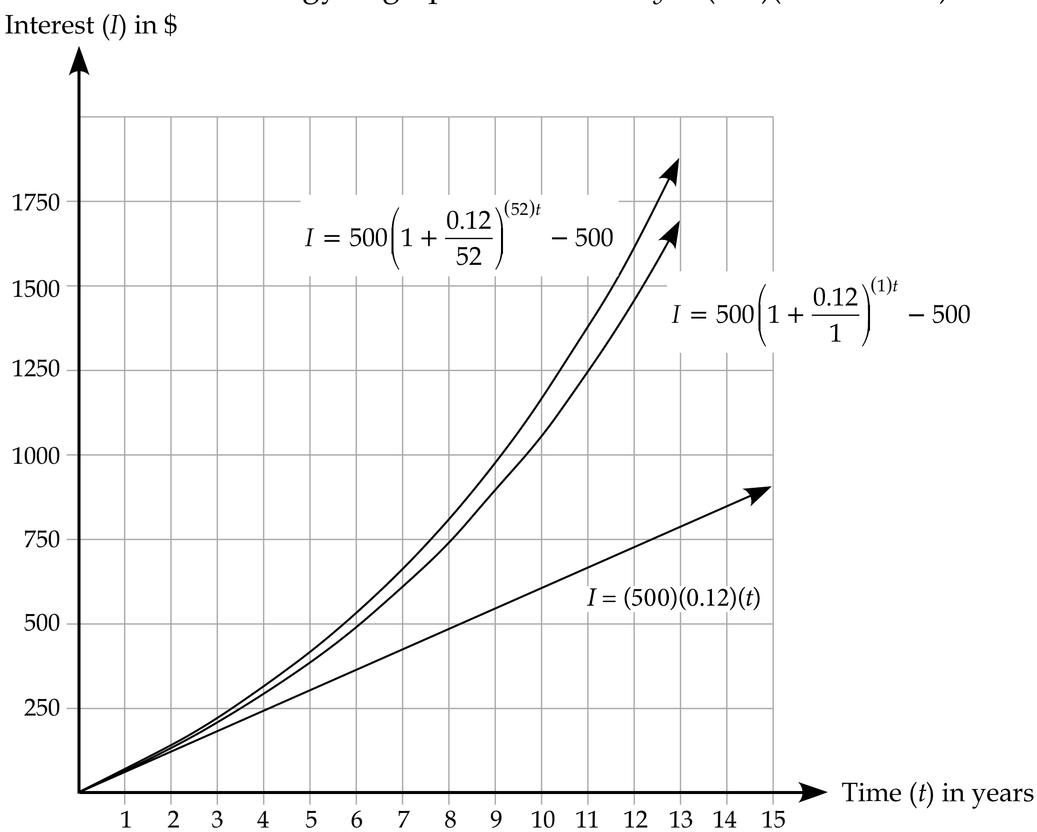
$$I = 500\left(1 + \frac{0.12}{1}\right)^{(1)t} - 500$$

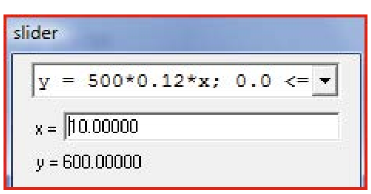
You can use technology to graph the function $y = (500)(1.12)^x - 500$.

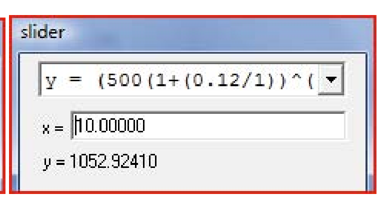
c) The formula for compound interest compounded weekly (52 times per year) becomes:

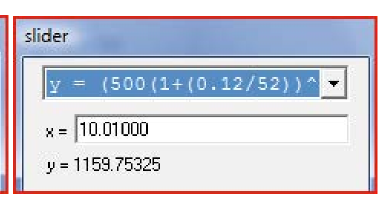
$$I = 500\left(1 + \frac{0.12}{52}\right)^{(52)t} - 500$$

You can use technology to graph the function $y = (500)(1.002307692)^{52x} - 500$.









After 10 years, the investment has earned $600 simple interest, $1052.92 when compounded annually, and $1159.75 when compounded weekly. The more often interest is compounded, the more quickly the investment grows. Notice that there is a big increase in money earned when interest is compounded compared to simple interest. Also notice that there is only a small increase in interest when compounding weekly instead of annually.

Example 10

Benji received $1100 from an income tax return. He wants to invest the amount and double it in the next 5 years. At what rate will he need to invest the money if it is compounded annually?

Solution

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$2200 = 1100\left(1 + \frac{r}{1}\right)^{(1)(5)}$$

$$2200 = 1100(1 + r)^5$$

$$\sqrt[5]{\frac{2200}{1100}} = \sqrt[5]{(1 + r)^5}$$

$$\sqrt[5]{2} = 1 + r$$

$$0.148698355 = r$$

He will need a rate of about 14.9%.

Rule of 72

While you can use the formula for compound interest to determine the rate required to double an investment, a quick way to estimate how long it will take for an investment to double is to use the Rule of 72. All you need to know is the interest rate.

72 = years to double × percent interest rate

where the interest rate is given as a percent, not a decimal.

The nice thing about the number 72 is that it has lots of factors. So, the following estimations are valid:

- An investment at 12% would take about 6 years to double (since $6 \times 12 = 72$)
- An investment at 2% would take about 36 years to double (since $36 \times 2 = 72$)
- Investing for 24 years would require an interest rate of 3% to double ($24 \times 3 = 72$)
- Investing for 9 years would require an interest rate of 8% to double ($9 \times 8 = 72$)
- Investing for 10 years would require an interest rate of 7.2% to double ($10 \times 7.2 = 72$)

It may be helpful for you to include some of this information on your resource sheet.

Example 11

If Benji invests $1100 at 15% for 5 years, compounded annually, will this be enough time for the amount to double?

Solution

$$\frac{72}{15} = 4.8$$

Yes, the amount should double in just under 5 years. For more precise values, it is necessary to use the compounding interest formula.

Note that the Rule of 72 is only approximate and should be used for quick, mental math calculations. It is based on compound interest, not simple interest, and is most accurate for rates greater than 0% and less than 15%. The approximations are okay but not as accurate when higher interest rates are used.

Learning Activity 5.1

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Determine 10% of 835.
2. Determine 10% of 0.24.
3. Determine 5% of 650.
4. Determine 5% of 1.5.
5. How many weeks are in one year?
6. If you make bi-weekly payments on a loan, how many payments will you make in a year?
7. If you make semi-monthly payments on a loan, how many payments will you make in one year?
8. How much simple interest will you earn in one year on $1000 if the rate is 6.5% per year?

Part B: Interest

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Why is the value of n in the compound interest formula different for semi-monthly and bi-weekly compounding periods?
2. Give three examples where compound interest is either charged or earned.

continued

3. Complete the following chart. Use technology and the simple interest or compound interest formula to determine the missing values.

	Amount (A)	Interest (I)	Principal (P)	Rate (r)	Compounding period	Length of Time (t)
a)			\$3000	7.4% (simple)	(not compounded)	5 years
b)			\$8500	4.9%	monthly	6.3 years
c)	\$933			$3\frac{1}{2}\%$	semi-annual	18 months
d)	\$2020		\$1530		weekly	365 days

4. Marcus received an amount of money on his 12th birthday, which he invested at 5% compounded quarterly. On his 20th birthday, he withdrew the total amount of the investment, which was \$1116.10. Determine the initial value of his birthday gift.

5. Kathy has \$2000 to invest. Her choices are to invest in either a GIC account that offers 6% simple interest or a savings account that offers 6% compounded monthly.
   a) Find the amount of money in each of these investment options after 5, 10, and 20 years. Record the values in a spreadsheet.
   b) Graph the amount in each account over time on the same set of axes.
   c) Which investment option should she choose if she is investing for the long term? Explain your reasoning.

6. Graph a representation of the value of an investment of \$5000 held in a savings account at 8.3%, compounded monthly, and compare it to a similar investment at 9.5%, compounded monthly. Determine the value of each investment after 20 years.

7. a) Use graphing technology to compare how long it would take an investment of \$2000 to double in value at 9% simple interest, compared to 9% compounded bi-weekly.
   b) Use the Rule of 72 to determine approximately how long it would take to double the investment if it were compounded annually at 9%.

8. Victoria invests \$5000 in a RESP (registered educational savings plan) for her daughter that has 9.6% interest compounded annually. Eventually she will need \$20,000 to pay for university. For approximately how long will she need to invest the money?

Lesson Summary

In this lesson, you learned about some of the advantages and disadvantages of compound and simple interest. You used the formulas for simple and compound interest to determine the amount of interest earned and the value of an investment. You graphed the amount of interest earned, compared compounding periods, and investigated the effects of changing the value of one of the variables in a situation involving compound interest.

Notes

Lesson 2: Buying on Credit

Lesson Focus

In this lesson, you will

- ❑ compare and explain, using technology, different credit options involving compound interest, including bank or store credit cards or special promotions

Lesson Introduction

In the first lesson, you learned how compound interest can work for you much better than simple interest. If you have the time to leave an investment alone for many years, you can accumulate a lot of interest. Conversely, lending institutions are happy if you borrow money and borrow it for a long time because that means that they are earning compound interest from you. For you, it is best to pay off debt quickly to avoid having a lot of compound interest added to an initial loan. In this lesson, you will look at a variety of credit options, which are different ways to borrow money.

Debt

You have seen how simple and compound interest affect the growth of investments and savings. Credit cards with high interest rates can accumulate debt just as quickly if you carry a balance with the card.

Example 1

Niv and Sam each buy a computer on the first day of college. The computer costs $899. Niv borrows the cost of the computer from her parents who charge 17.5% simple interest annually, while Sam uses a bank credit card that charges 17.5% compounded monthly. Both students pay off their credit card debt when they graduate, three years later. How much do they each end up paying for their computer?

Solution

Niv:

$$I = Prt$$

$$I = (899)(0.175)(3)$$

$$I = 471.975$$

$$A = I + P$$

$$A = 471.975 + 899$$

$$A = 1370.975$$

Niv pays $1370.98 for her computer.

Sam:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$A = 899\left(1 + \frac{0.175}{12}\right)^{(12)(3)}$$

$$A = 1513.971375$$

Sam pays $1513.97 for his computer.

Sam ends up paying about $143 more for his computer than Niv pays for hers, since the interest on his loan is compounded.

Note: Credit cards can be important tools that can help you establish a good line of credit, something that is necessary if you want to buy a car, get a loan, or take out a mortgage on a home in the future. Banks and credit card companies do not actually offer a simple interest option so when you borrow money, you will be paying compound interest. In the above example, the students did not make any payments on their respective loan during the three years. Typically, credit cards and bank loans require regular minimum payments to cover at least the cost of interest and fees. Be very aware of the importance of paying off your total credit card balance each month to avoid the interest charges. Missing or being late for payments on your credit card may jeopardize your chances of getting a loan in the future.

Using a Credit Card

When you make a purchase with a credit card, the credit company gives you a **grace period**. During the grace period, they do not charge interest on your balance. If you pay the entire balance each month before the grace period ends, all interest charges are waived. If you carry an outstanding balance after the grace period, you are charged interest on the entire outstanding balance, calculated per day from the date of purchase. In addition, you lose your grace period for all additional purchases in the next month(s) until your accumulated balance is paid in full. In this case, credit card debt is calculated in the same way as a loan.

The annual interest rate on credit cards is typically between 18–20%, compounded daily. Credit card companies often offer users the option of using a service called "cash advance," which allows you to withdraw cash using the credit card at an ATM or bank. The amount you withdraw becomes a part of your credit card debt. For cash advances, there is no grace period and the interest rate is usually around 25%, compounded daily.

Example 2

Peter purchases new hockey equipment on August 3rd for a total of $1287 and charges the full amount on his store credit card. His credit card statement is printed on the 8th of each month, and he has until the 29th to pay without penalty (his grace period). His minimum payment is given as $28.45, but with his new job Peter feels he can pay back $400 on the 29th of each month (with his first payment in August), until his balance is paid off. The credit card company charges 19.99% annual interest compounded daily. If he makes no additional credit card purchases, calculate the interest charges he must pay with his payments on September 29th, October 29th, and November 29th, and the total interest charged.

Solution

Peter's first payment of $400 on August 29th is within the grace period and so he owes no interest yet.

He now has an outstanding balance of $1287 – $400 = $887. He will be charged interest on this amount for the number of days from August 3rd to his second payment on September 29th, a total of 57 days (August 3rd to August 29th is 26 days, plus August 29th to September 29th is 31 days, for a total of 57 days).

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$A = 887\left(1 + \frac{0.1999}{365}\right)^{(365)\left(\frac{57}{365}\right)}$$

$$A = 915.12$$

On September 29th, he owes $915.12. This amount of $915.12 includes interest of $28.12 ($915.12 − $887.00 = $28.12).

He will reduce the amount he owes ($915.12) by $400.00 with his September payment and he then owes $515.12. He will pay interest on this amount for the number of days from August 3rd to October 29th (87 days) when he makes his next payment.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$A = 515.12\left(1 + \frac{0.1999}{365}\right)^{(365)\left(\frac{87}{365}\right)}$$

$$A = 540.25$$

On October 29th, he owes $540.25. This amount of $540.25 includes interest of $25.13 ($540.25 − $515.12 = $25.13).

After his October 29 payment, his balance is $540.25 − $400.00 = $140.25.

Since Peter waits until November 29th to make his next payment, he will owe interest on a balance of $140.25 for the number of days from August 3rd until November 29th (118 days).

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$A = 140.25\left(1 + \frac{0.1999}{365}\right)^{(365)\left(\frac{118}{365}\right)}$$

$$A = 149.61$$

On November 29th, he owes $149.61. This amount of $149.61 includes interest of $9.36 ($149.61 − $140.25 = $9.36). After making his final payment of $149.61, his balance owing is now zero dollars.

In total, he has made three payments of $400.00 and a final payment of $149.61. His total interest paid has been $28.12 + $25.13 + $9.36 = $62.61.

In the end, he paid $1349.61 to purchase his hockey equipment using a credit card ($1287.00 for the equipment and $62.61 in interest).

Graphing calculators, spreadsheet software, and financial websites can be used to find approximate solutions to credit and interest problems. These applications typically cannot take into account the exact number of days of the grace period or the precise number of days for which interest is charged. However, they can give you an estimated cost of interest and help you to determine how long it may take you to pay off a credit card debt.

Example 3

Jason is buying a new television for $1025, including taxes. Extended warranty costs an additional $55. He can make monthly payments of $80 and has two credit options:

- **Option 1:** A store credit card that offers an introductory interest rate of 14% for 6 months and then 19%, compounded daily.
- **Option 2:** A bank credit card that includes extended warranty on all purchases and an interest rate of 20%, compounded daily.

Use technology to approximate how long it will take Jason to pay off his new television, how much he will pay in interest, and the total cost using both Option 1 and Option 2. Which option would you encourage Jason to use? (Ignore the grace period in your calculations.)

Solution

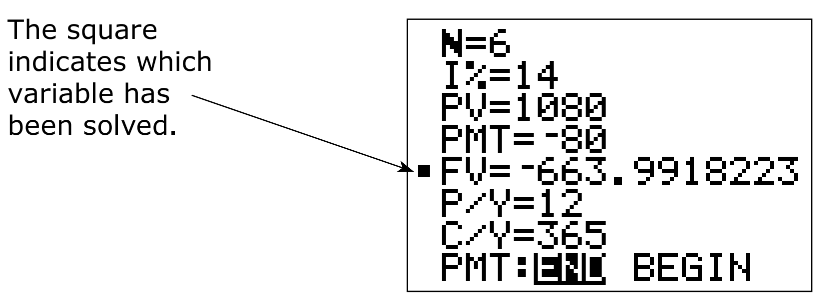
You can use a TVM or Time Value of Money solver (a loan calculator found in the finance menu of a graphing calculator), or find online or mobile apps to help solve credit card problems. Look for an application that allows you to set different numbers of compounding periods and payments per year, as well as the other required variables.

Refer to the Technology Appendix for help in learning how to use the TVM solver of the TI-84 graphing calculator. Other TVM solver apps work in a very similar fashion. You are required to use financial calculation technology for some assignment questions and on your final examination.

Option 1

A store credit card that offers an introductory interest rate of 14% for 6 months and then 19%, compounded daily.

Jason borrows $1080 on the store credit card to cover the cost of the television and warranty ($1025 + $55). An interest rate of 14% is used to determine how much he will still owe after his first 6 payments of $80 each. Input the values into the technology application (TVM solver) of your choice. Using a graphing calculator, it may appear as follows:



The values in the bank (or store) are entered as positive numbers and the values coming out of your pocket are entered (and calculated) as negative numbers. N is the number of payments and I is the interest percent. The abbreviations represent the following:

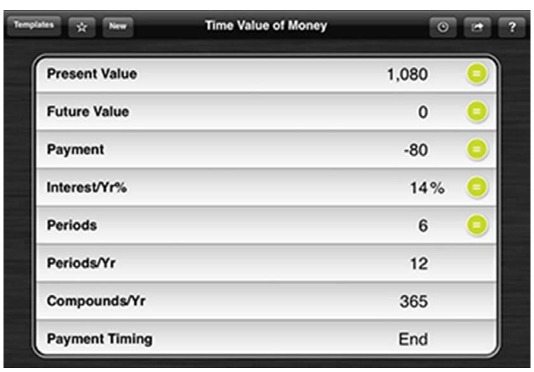
- PV means "present value"
- PMT means "payment amount"
- FV means "future value"
- P/Y means "payment periods per year"
- C/Y means "interest compounding periods per year"

The payment timing is for the end of period.

You should determine what financial calculator software you would like to use and then you may want to include some information about the use of the software on your resource sheet.

Talk to your tutor/marker to be sure your choice of financial technology is suitable for the course and the final examination.

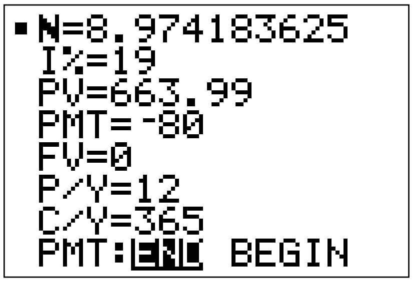
These screen shots are from an iPad app called TVM by powerOne Finance Pro Calculator.



After the six initial payments at 14%, there will be an outstanding balance of $663.99 owed to the store, as listed beside "Future Value" in the screenshot above.

The interest rate now changes to 19%. How many more payments must he make to pay off the balance?

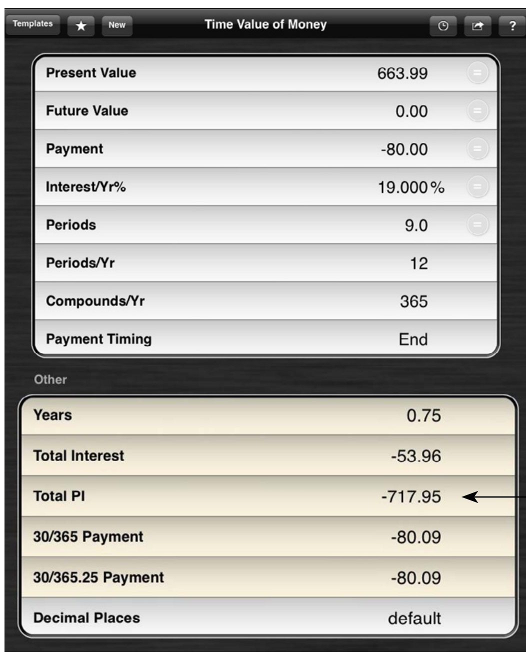
You could use the TVM solver on the graphing calculator (find it in Apps, Finance).



Or, you could use the iPad powerOne TVM solver app.

The TVM solver on the graphing calculator indicates it will require 8.974 periods, which means nine more payments for a total of 15 payments. The last one will be slightly less than $80.

(6 × 80) + (8.974183625 × 80) = 1197.93469



Initial 6 payments of $80 plus $717.95 paid at 19% interest equals total paid for the television and warranty.

6 x 80 + 717.95 = $1197.95

There is a very slight variation in total cost, depending on which technology is used.

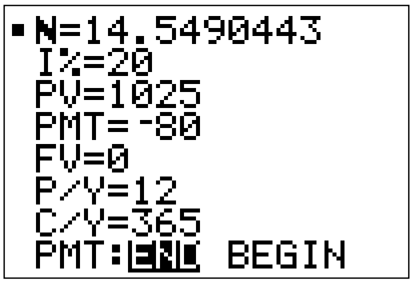
Jason would pay approximately $1197.93 with Option 1.

Option 2

A bank credit card that includes extended warranty on all purchases and an interest rate of 20% compounded daily.

With this option, Jason saves the $55 cost of extended warranty as it is included with the credit card.

Using a graphing calculator to solve for N, the number of payments results in:

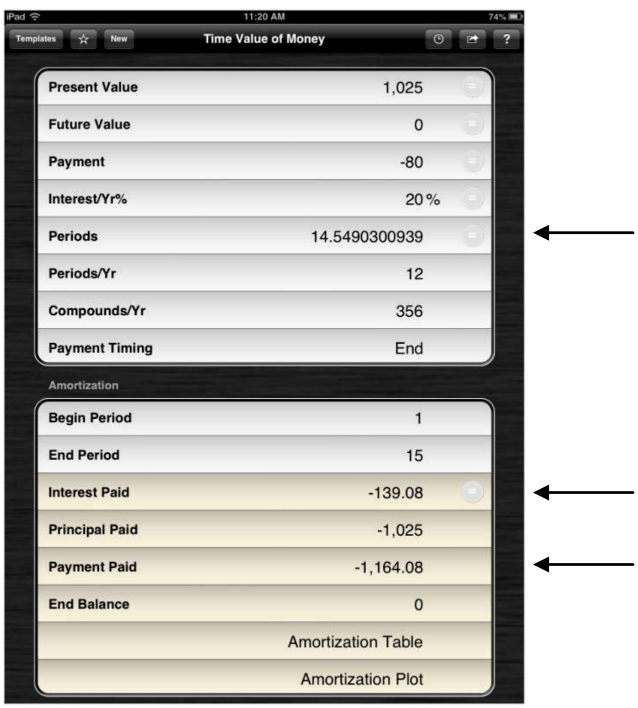


He will make 15 payments, with the last one being slightly less.

$14.5490443 \times 80 = 1163.923544$

Jason will pay approximately $1163.92 for the television, using the credit card ($138.92 is interest).

Using the powerOne TVM solver app, the results vary slightly.



From these calculations, it appears that Jason would save about $34 if he used option 2, the bank credit card, rather than the store credit card. Other factors that may be worthwhile considering are whether the credit cards have annual fees, reward points, or other bonuses.

Example 4

Kate uses a **line of credit** (a pre-approved loan) from her bank to finance the purchase of a used car. She borrows $8300 at 3%, compounded monthly. She wants to use monthly payments to repay her loan in less than five years.

a) Use technology to determine her payment amount for three-, four-, and five-year loan options. Determine the amount of interest she pays in each situation.

b) Comment on the advantages and disadvantages of longer or shorter term loans.

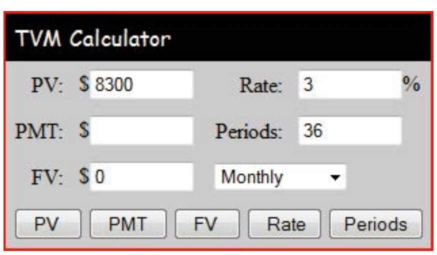
Solution

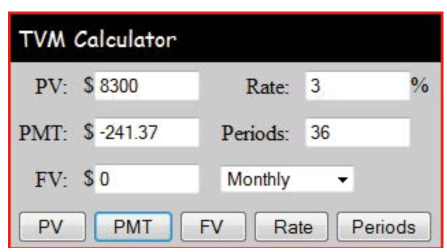
a) Using a TVM solver online, as an app or in a calculator, enter the variables for a three-year loan. In this situation the payments and the compounding period are both monthly, so various other online apps can be used.

You can use a graphing calculator, an app on an iPad, or an app on a computer. The following screenshots show another alternative.

You should check with your tutor/marker to be sure the TVM solver you choose to use is acceptable for the assignments and the final examination.

Source: www.zenwealth.com/BusinessFinanceOnline/TVM/TVMCalcWindow.html

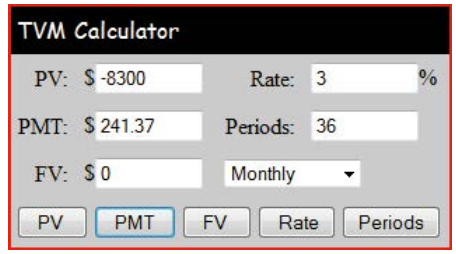




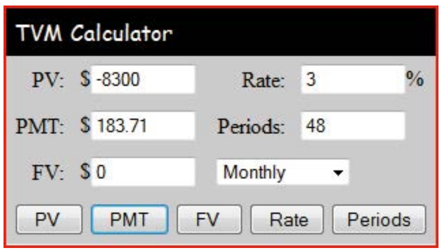
Click the PMT button.

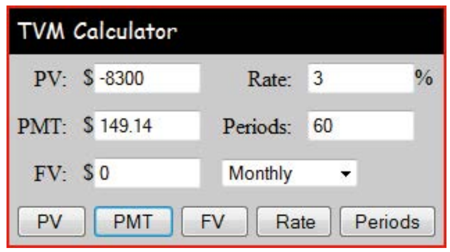
The monthly payment for a 3-year loan (36 payments) is $241.37. This means Kate will pay a total of 36 × $241.37 = $8689.32 for her car. Only $8300 of that amount is principal, so the difference is the amount paid for interest ($8689.32 − $8300 = $389.32).

Note that the amount of payment is given as a negative number, which indicates it reduces the present value of the loan. If the loan amount had initially been given as $-8300 to indicate a balance owing, the payment would have been given as a positive value.



For the four-year and five-year loan options, the payments would be $183.71 and $149.14 respectively.





The amount of interest paid on the four-year loan is (48 × $183.71) – $8300 = $518.08

The amount of interest paid on the five-year loan is (60 × $149.14) – $8300 = $648.40

b) The payment is approximately $100 less per month with a longer term loan ($149.14 for five years compared to $241.37 for three years). In the end, however, a short-term loan means you pay less interest ($648.40 over five years as compared to $389.32 over three years). If Kate can afford a slightly higher payment with a short-term loan, she could save over $250 in interest.

Choosing a three-year loan rather than a five-year term saved $259.08 in interest ($648.40 – $389.32).

Special Promotions

Often stores will try to entice buyers to purchase large items by giving them a "buy now-pay later" deal or by offering incentives to use their own store credit cards.

Example 5

Leroy goes to a furniture store and picks out a leather sectional sofa with a sale price of $1999 for his living room. He is offered his choice of the following promotions:

> **Promo #1: "Do not pay for 12 Months! 0% Interest! No monthly payments!"**
>
> He reads the fine print: *O.A.C. 15% deposit of total sale including taxes, surcharges where applicable, and a processing fee of $69.95 is due at the time of purchase. Balance is due 12 months from the date of purchase.*

> **Promo #2: "12 equal monthly payments"**
>
> Again, he reads the fine print: *O.A.C. 15% deposit of total sale including taxes, surcharges where applicable, and a processing fee of $25.00 is due at the time of purchase. Balance is payable in 12 equal monthly payments (8.334% of purchase price), starting with the statement following the delivery/pick-up.*

a) How much will he pay for the sofa if he accepts either of the promotional deals and pays for it by the due date? The surcharge applies to both promotions and is a merchant fee of $89.95. Which promotion would you suggest? Explain.

b) At the 12-month due date, Leroy finds that he has not saved enough money and cannot pay the full balance owing of $1920.04 with the Promo #1 "Do not pay, No monthly payments" deal. He is charged a deferral fee of $42.50 to convert his debt to a regular credit purchase with 29.9% annual interest, compounded daily. He wants to pay off the credit card in one year. Determine his payment and the total cost to purchase the sofa, considering the entire two-year time frame.

Solution

a) O.A.C. stands for "on approved credit," which means his credit rating must be approved by the company before he qualifies to participate in the promotional offer.

The cost of the sofa, plus 5% GST and 8% PST is \$2258.87.

Promo #1:

To participate in the "Do not pay for 12 months" promo, he must pay \$498.73 at the time of sale, which includes a deposit of 15% of the cost, plus the processing and merchant fees.

$2258.87 \times 0.15 = 338.83$

$338.83 + 69.95 + 89.95 = 498.73$

The amount due at 12 months from the date of purchase is the remaining 85% of the purchase price, \$1920.04.

In total, he must pay \$1920.04 + \$498.73 = \$2418.77.

Promo #2:

To take advantage of the "Equal Payments" promo, he must pay \$453.78, which includes a deposit of 15% of the purchase price, plus fees at the time of purchase.

$2258.87 \times 0.15 = 338.83$

$338.83 + 25 + 89.95 = 453.78$

Each month, he must pay 8.334% of the remainder of the purchase price (less deposit).

$(2258.87 - 338.83) \times 0.08334 = 160.02$

Twelve payments of \$160.02 plus the initial cost of \$453.78 results in a final cost of \$2373.97.

At the end of the 12 months, Promo #2 costs Leroy a little less than Promo #1 but, in both cases, he pays more than the purchase price plus taxes for the sofa. The benefit of Promo #2 is that at the end of the 12 months the sofa is paid for with smaller payments—\$160.02 each. At the end of the 12 months with Promo #1, Leroy must still pay \$1920.04, which might be difficult if he has not been setting some money aside each month in a savings account.

b) If Leroy cannot pay the $1920.04 at the end of the 12-month period and converts it to a credit card purchase, his payment amount can be found using a financial app.

```
 N=12
 I%=29.9
 PV=1920.04
▪PMT=-187.42772…
 FV=0
 P/Y=12
 C/Y=365
 PMT:END BEGIN
```

He will have to make 12 monthly payments of $187.43 to pay back the debt of $1920.04 at 29.9% annual interest, compounded daily. He will also have to pay the deferral fee.

The total cost of the sofa is now $2790.39 (12 × 187.43 + 42.50 + 498.73).

If he had paid for the sofa with cash at the time of sale, he would have been charged $1999 plus taxes for a total of $2258.87. The “Do not pay for 12 Months! 0% Interest! No monthly payments!” promo, plus the credit card payments, cost him an additional $531.52.

Learning Activity 5.2

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Determine 50% 845.6.
2. Determine 50% 0.6.
3. Determine 25% of 845.6.
4. Determine 25% of 0.6.
5. Determine 12.5% of 1600 (a good approximation of the amount of PST and GST charged in Manitoba).

continued

6. Determine 12.5% of 50.
7. A perfect number equals the sum of all its factors, excluding itself. Show that 6 is a perfect number.
8. State the formula for simple interest. Indicate what each variable represents.

Part B: Credit Options

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Darrell purchases two tickets to a football game using his credit card. He is charged $270. If he currently does not carry a balance with his credit card, and makes a payment of $270 on the day he receives his statement, how much interest will he pay on this purchase? His bank charges 18% interest, compounded daily, on all outstanding balances.
2. Ali has a season ticket for her favourite hockey team. It costs $3680, which she adds to her balance on her bank credit card with 17.5% interest, compounded weekly.
   a) How much must her monthly payments be, if she wants to pay off the entire cost of the season ticket in 12 months?
   b) Determine the total cost to purchase the tickets on credit and the amount of interest she pays over the 12 months.
3. Lynn attends the symphony's *12 Masterpiece Concert Series* each year. Her tickets cost $609.70, which she purchases with her line of credit. How long will it take Lynn to pay for her tickets if she makes monthly payments of $85 on the account, which has 19% interest, compounded monthly? Determine the amount of interest she pays.
4. Brooke purchases a graduation ring and arranges to make weekly payments of $20 on a store credit card to cover the cost. She is charged 18% interest, compounded weekly, and makes payments for 25 weeks. How much was the ring? What was the ticket price of the ring, if combined taxes are 13% (PST and GST)?

continued

5. Joshua is considering transferring his existing debt of $3500 to a new credit card or line of credit, based on the introductory offers available. The credit card offers 0% interest for 6 months and then an interest rate of 25%, compounded daily. The line of credit is available at 17%, compounded monthly, with a rebate of $100. Joshua can afford a $400 per month payment. Which option is the better way to repay his debt? Explain your reasoning.

6. Bruce and his wife decide to take a cruise. They find an all-inclusive sell-off vacation deal for $4396, including taxes and fees. They finance the trip with a cash advance on their credit card and plan to make payments of $225 per month. The credit card charges 25%, compounded daily, on cash advances, with no grace period.
   a) How long will it take to pay back the loan?
   b) What will the trip actually cost them?
   c) How much do they pay in interest?

7. Why is it important to read all of the fine print when considering a store promotion?

Lesson Summary

In this lesson, you used technology to compare credit options involving compound interest, including store or bank credit cards and special promotions.

Notes

Assignment 5.1

Compound Interest

Total: 25 marks

This is a hand-in assignment. Please show your work clearly and in an organized manner. Round final answers to 2 decimal places, and include units, if appropriate. If you use technology as a strategy in your solution steps, please indicate which application you are using, the values you input, and a sketch or printout of the results. Answers given without supporting calculations will not be awarded full marks.

1. Sophie invests $3700 in a GIC at 3.9% simple interest. When she withdraws her money, she receives $889.85 interest. For how many months was the GIC investment held? *(2 marks)*

2. Determine the amount of interest earned on $6400 invested at 4.8%, compounded semi-monthly, for 7 years. *(3 marks)*

3. a) Graph and compare the value of an investment of $10,000 at 6%, compounded daily, with an investment of $10,000 at 6%, compounded annually. Print or sketch the graph, showing a minimum of 25 years. Include labels. *(5 marks)*

b) What is the value of each investment after (i) 2 years? (ii) 20 years? *(4 marks)*

Assignment 5.1: Compound Interest (continued)

c) Describe the effect of changing the compounding period. *(2 marks)*

4. Approximately how long would it take $500,000 to double to one million dollars if it were invested at 6%, compounded annually? *(2 marks)*

5. Use technology to determine the monthly payment amount required to repay a credit loan of $12,500, compounded monthly, at 1.6% for 5 years? How much interest is paid? *(3 marks)*

6. Cheryl is travelling and needs a cash advance to purchase a train ticket. She uses her credit card and receives $275 cash. Her credit card company charges 25% interest on cash advances, compounded daily, with no grace period. How much will she have to repay for the cash advance when she gets home in 15 days? *(2 marks)*

7. Tracey went shopping for a new back-to-school wardrobe. She spent $632.77 in her favourite stores and charged it all on a bank credit card. Over the course of the next seven months, she paid off her balance by making payments of $95.15 per month. Use technology to determine the interest rate charged by the bank, compounded weekly. *(2 marks)*

LESSON 3: MORTGAGES AND LOANS

Lesson Focus

In this lesson, you will

- ❑ determine, using technology, the total cost of a loan under a variety of conditions such as different amortization periods, interest rates, and compounding periods or terms

Lesson Introduction

In the previous lesson, you considered credit options, such as bank and store credit cards, special promotions, cash advances, and a line of credit. These credit options are typically used as a short-term loan. You borrow money from the bank, store, or a credit card company in order to purchase something such as furniture, electronics, a trip, education, or even Christmas gifts, and then repay the loan over a short period of time. These types of loans may be possible without a down payment. In this lesson, you will learn about loans that require a longer period of time to repay.

Mortgages and Loans

Some larger purchases, such as a home, a cottage, a vehicle, or an investment loan, may require a down payment and a longer **amortization period** (the number of years required to repay the loan). It is possible to obtain a 5- or 10-year loan, or a mortgage with an amortization period of 5, 10, 15, 20, or even 25 years. Typically, the rate of the mortgage will be negotiated for terms of 3 to 5 years, and then renegotiated, based on the **Bank of Canada prime rate**. The prime rate, or prime lending rate, refers to an index used by banks and lending institutions to calculate the interest rate on loans and investments. Businesses often negotiate a rate at or below the prime rate but individuals usually pay an interest rate that is higher than the prime lending rate. The prime rate fluctuates over time—it has been as high as 22.75% in the 1980s and as low as 2.25% in 2010. At the time of this writing, the prime rate in Canada was 3%.

The calculations for loans and mortgages are similar to credit card loans. Technology may be used to simplify the process.

You may want to include some of these terms on your resource sheet.

Buying a House

Buying a home is often the single most significant purchase a person makes in his or her lifetime. Calculating the mortgage payment is similar to other loan calculations, but the rates may be fixed or variable and the repayment may be affected by a down payment, fees, insurance, closing costs, and additional payment options.

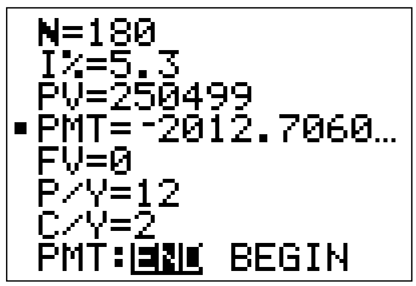
Example 1

Stephanie is interested in purchasing a new home. She has $15,500 saved in a term deposit that she can use as a down payment. The house she is considering is listed at $265,999. The prime rate is 3%. The bank is offering her a 15-year mortgage with a 5-year term at prime plus 2.3%, compounded semi-annually. How much will her monthly payment be in this situation?

Solution

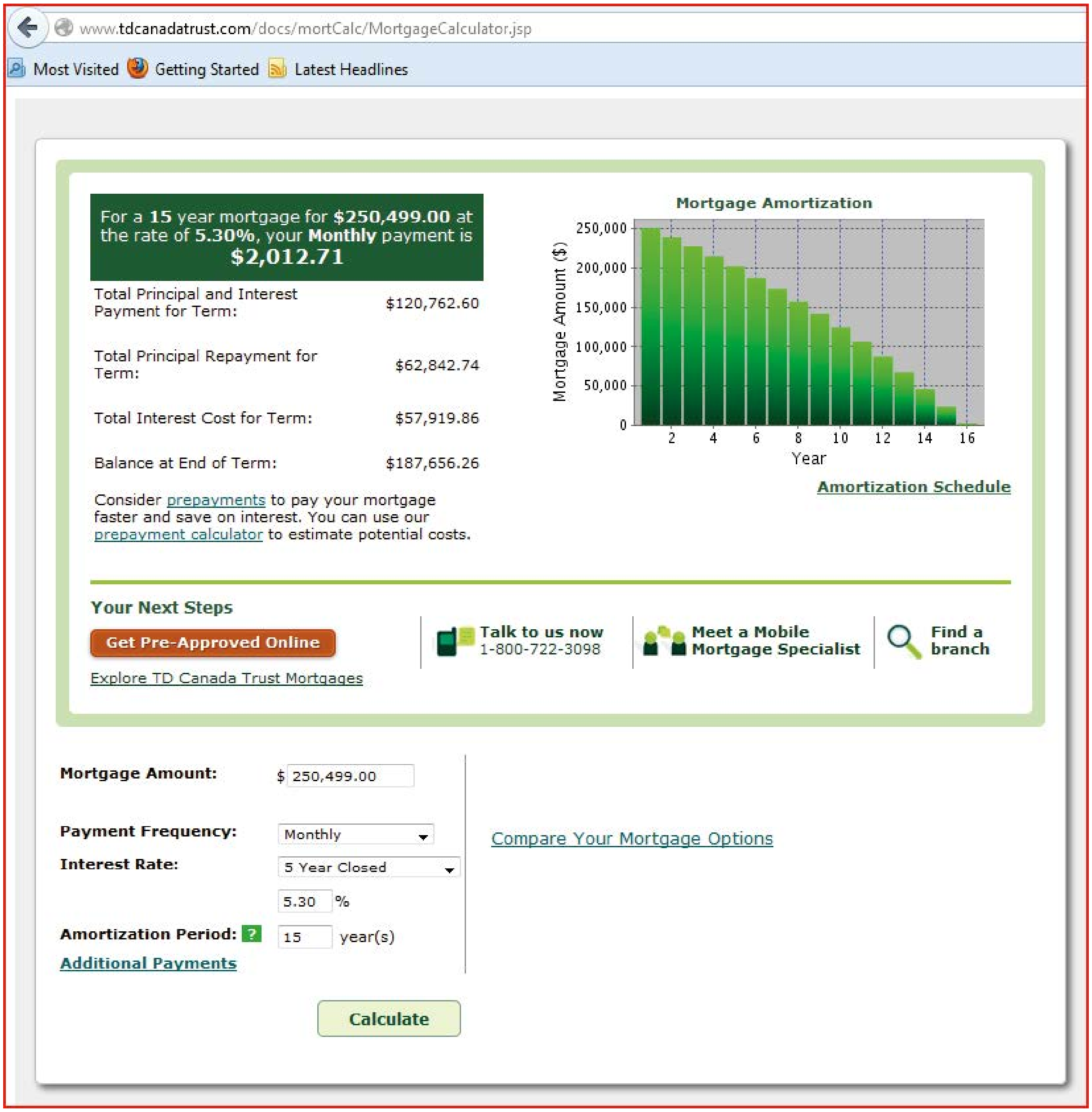
Use technology and enter the information given to solve for the amount of payment.

N=15X12
I%=3+2.3
PV=265999–15500
PMT=
FV=0
P/Y=12
C/Y=2
PMT: END BEGIN



Stephanie's monthly payment will be $2012.71.

This could also have been solved using an online mortgage calculator from a bank or lending institution.



You can see in this sample solution that after the five-year term, Stephanie has an outstanding balance of $187,656.26. This amount would then be renegotiated with her financial institution for another term. She would have the option to make additional payments and secure a new interest rate for the next five years.

For your final examination, you will need to specify one type of technology (e.g., graphing calculator) or one software application on a computer or tablet that you will use for financial calculations. Talk to your tutor/marker to be sure you are using appropriate financial technology.

Example 2

The Wawryk family purchased a cottage, using money received from an inheritance as a down payment. Their mortgage broker was able to secure a 10-year mortgage with a three-year term at prime plus 0.95%, compounded annually. The prime lending rate is 3%. The list price of the cottage was $197,500. They will have to make renovations to the cottage at a cost of $40,000. They paid 8% of the list price in closing costs (lawyer fees, taxes, utilities, and service charges relating to the purchase of the property), and included this in the mortgage. The Wawryks made a down payment of $25,000.

a) Determine the Wawryk’s monthly mortgage payment for the duration of the first term.

b) Calculate their balance at the end of the three-year term.

c) After the three-year term, the mortgage is renegotiated for a five-year term at prime plus 2.3%, compounded annually. They make a lump sum payment of $10,000. How much is their new monthly payment.

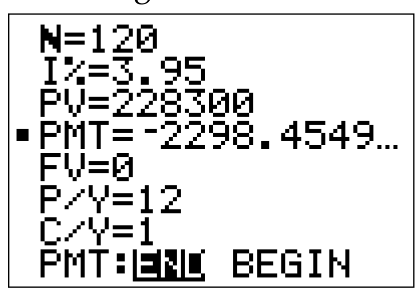
d) The final two-year term is negotiated at prime plus 0.14%, compounded annually. Calculate their new monthly payment?

e) Determine the total cost of purchasing the cottage.

Solution

a) The monthly payment can be determined using a TVM solver.

N=10X12
I%=3+0.95
PV=197500+40000+(0.08*197500)–25000
PMT=
FV=0
P/Y=12
C/Y=1
PMT: END BEGIN



Their monthly payments during the three-year term will be $2298.45.

b) The balance owing at the end of the three-year term (36 months) can be calculated using technology. Refer to the Technology Appendix for help in using various applications.

```
bal(36)
     168833.3499
```

```
N=36
I%=3.95
PV=228300
PMT=−2298.4549
FV=
P/Y=12
C/Y=1
```

Using the graphing calculator, the balance is $168,833.35.

Using the TVM solver, the future value (FV) is $168,833.35.

After the three-year term, the Wawryks have an outstanding balance of $168,833.35.

c) The amount of $168,833.35, less the lump sum payment of $10 000, is renegotiated for a five-year term at 5.3%. There is now seven years remaining in their mortgage. Their new monthly payment is:

```
N=7X12
I%=2.3+3
PV=168833.35–10000
PMT=
FV=0
P/Y=12
C/Y=1
PMT: END BEGIN
```

```
 N=84
 I%=5.3
 PV=158833.3499
▪PMT=-2258.0539…
 FV=0
 P/Y=12
 C/Y=1
 PMT:END BEGIN
```

d) After the five-year term (60 months), the balance owing is $51,377.78405. This is renegotiated for a two-year term at 3.14%, compounded annually. Their payments will be $2210.45.

```
bal(60)
     51377.78405
█
```

```
 N=24
 I%=3.14
 PV=51377.78405
▪PMT=-2210.4548…
 FV=0
 P/Y=12
 C/Y=1
 PMT:END BEGIN
```

e) The total cost of the cottage can be determined by calculating the amount paid each month, plus the down payment and lump sum payment.

$$
\begin{aligned}
& 25,000.00 \\
& 10,000.00 \\
36 \times 2298.45 \;=\; & 82,744.20 \\
60 \times 2258.05 \;=\; & 135,483.00 \\
24 \times 2210.45 \;=\; & 53,050.80 \\
\hline
& \$306,278.00
\end{aligned}
$$

They paid $306,278.00 for the cottage. The initial cost including list price, renovations, and closing costs was $228,300.00, so the difference of $77,978.00 is the interest paid to the bank for the mortgage.

Mortgage Factors

The final cost of a home can be so much higher than the list price! Required taxes, fees, closing costs, optional renovations, and interest all add to the cost of home ownership. Having a mortgage means you will pay interest, but the amount of interest paid on the loan is the one factor over which homeowners can exert influence. Fortunately, with some knowledge about how mortgages work, savvy consumers can make little changes to their payments and affect the total cost in big ways.

Factors that influence the amount of interest paid in a mortgage include the following:

- amount of down payment (more is better)
- frequency of payment (more often is better)
- interest rate (lower is better)
- extra payments (more is better)
- length of amortization (shorter is better)

It may be helpful for you to include some of this information on your resource sheet.

For ease of comparison, consider the following hypothetical situation:

Example 3

Moninder and Gurdeep purchase a home for $250,000. They negotiate a 15-year mortgage with an interest rate of prime plus 2%, compounded semi-annually, where prime is 3%. Calculate their monthly payment, the amount paid for the mortgage, and the amount of interest paid, if they make a down payment of $37,500 compared to a down payment of $62,500. Calculate the final cost of purchase in both cases. What impact does the down payment have on the amount of interest paid?

Solution

```
 N=180
 I%=5
 PV=212500
▪PMT=-1674.7630…
 FV=0
 P/Y=12
 C/Y=2
 PMT:END BEGIN
```

With a down payment of $37,500.00, the monthly payment will be $1674.76.

Amount paid: 180 × $1674.76 = $301,456.80

Amount of interest paid: $301,456.80.00 − $212,500.00 = $88,956.80

Cost to purchase home: $301,456.80 + $37,500 = $338,956.80

```
 N=180
 I%=5
 PV=187500
▪PMT=-1477.7321…
 FV=0
 P/Y=12
 C/Y=2
 PMT:END BEGIN
```

With a down payment of $62,500.00, the monthly payment will be $1477.73.

Amount paid: 180 × $1477.73 = $265,991.40

Amount of interest paid: $265,991.40 − $187,500.00 = $78,491.40

Cost to purchase home: $265,991.40 + $62,500.00 = $328,491.40

With a larger down payment, they save a total of $10,465.40 in interest ($338,956.80 – $328,491.40). The initial cost for the house was the same, $250,000, in both scenarios, but if they have a larger down payment, they borrow less, have smaller monthly payments, and end up paying less interest.

Another thing to consider in this situation is that if the down payment is less than 25% of the purchase price, the bank will automatically charge you a "high ratio mortgage" insurance fee. This could be between 0.5% and 3% of the purchase price. In the case of the smaller down payment in the example above, the bank would add a fee of between $3000 and $4000 to the mortgage to cover the cost of the insurance.

For more information about the additional hidden costs of purchasing a home, search online sites listed under "hidden costs of buying a home."

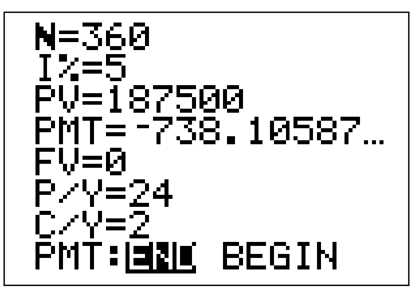
Example 4

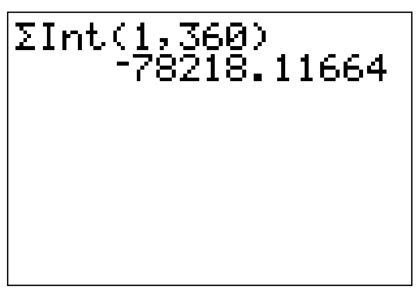
Moninder and Gurdeep are still planning to purchase the home for $250,000, but decide to shop around and find other mortgage options. They negotiate the same 15-year mortgage with an interest rate of prime plus 2%, compounded semi-annually, but consider changing the frequency of their payments from monthly to semi-monthly or bi-weekly. The prime lending rate is 3%. Calculate their payment amount for semi-monthly and bi-weekly schedules. Find the amount of interest paid, if they have a down payment of $62,500. Calculate the final cost of purchase in both cases. What impact does the payment frequency have on the amount of interest paid?

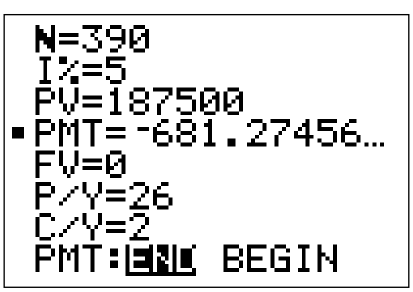
Solution

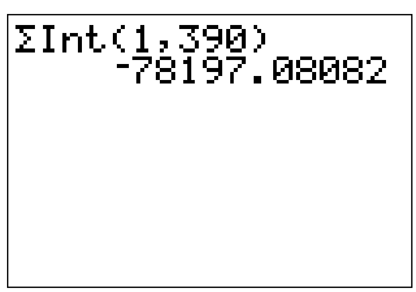
The monthly payment in this situation was determined above to be $1477.73 with $78,491.40 paid in interest (180 × 1477.73 = $265,991.40 paid over 15 years, plus down payment).

If they make two payments per month, they will pay $738.11 twenty-four times per year. Remember to change the value of N to 24 × 15 = 360 (360 × $738.11 = $265,719.60 paid over 15 years, plus down payment).









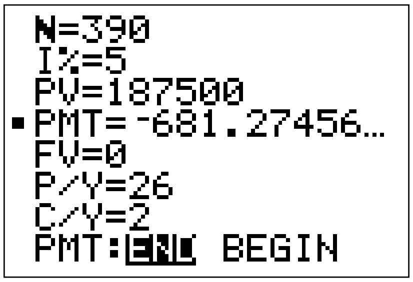
Changing the payment frequency to every two weeks means they will make 52 ÷ 2 = 26 payments per year. This results in 390 payments of $681.27 over the course of the 15 years, for a total of $265,695.30 (plus down payment).

Increasing the frequency of the payments means the payments are smaller, but occur more often. In the end, the biggest difference is that less interest is paid when payments occur more frequently. With monthly payments, $78,491.40 was paid in interest, while $78,197.08 was paid with bi-weekly payments, a difference of $294.32. The interest is less but making bi-weekly rather than monthly payments over 15 years does not make a significant difference. If the amortization period is cut down by several years, you will pay much less interest.

Making payments more often means the home buyer saves money on interest. However, interest is how the bank makes its money. For this reason, banks are not always keen to have mortgages paid off quickly. As a rule, you are locked into making the required payments until the end of the mortgage agreement. In fact, sometimes fees are charged as a penalty if someone pays off their mortgage before the end of the term! It is possible, however, to negotiate a mortgage with the option to make extra annual payments or to increase the amount of the payment on occasion.

Example 5

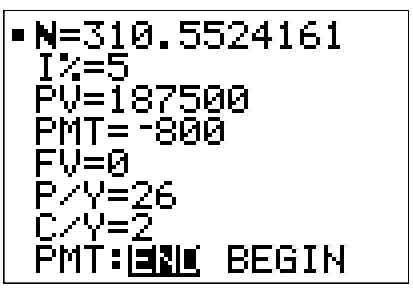
In consultation with a mortgage broker, Moninder and Gurdeep accept a 15-year mortgage, compounded semi-annually, at prime plus 2%, with bi-weekly payments. The prime lending rate is 3%. They make a $62,500 down payment on the $250,000 home. The bank proposes payments of $681.27.



Moninder and Gurdeep decide they can afford to either increase their bi-weekly payments to $800 or make an annual lump sum payment of $3000, in addition to the proposed $681.27 bi-weekly payment. For each of these two scenarios, calculate how long will it take them to pay for their home. How much will it end up costing them?

Solution

If they make bi-weekly payments of $800, they will pay off their mortgage with 311 payments, with the final payment being smaller than the others.



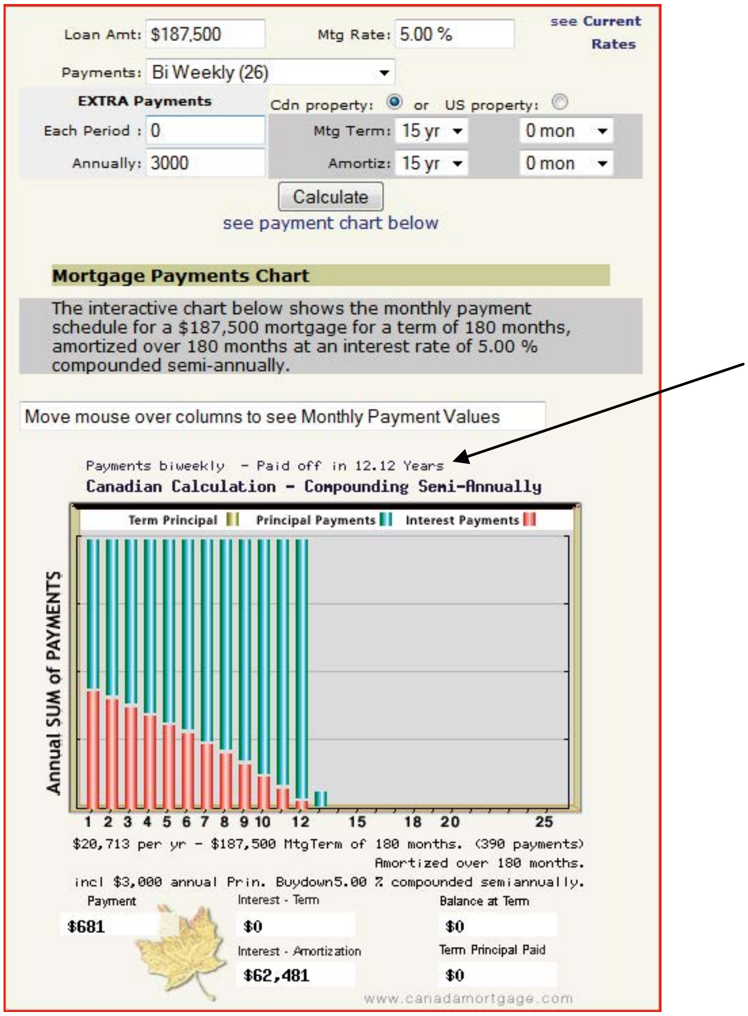
310.5524161 × 800 = $248,441.9329 total paid (plus down payment)

248,441.9329 − 187,500.00 = $60,941.93 interest

An effective way to see the impact of annual lump sum payments is to use an online calculator, such as the one found at this site:

www.canadamortgage.com/calculators/amortization.cgi

With lump sum payments of $3000, the following screenshot shows the loan being repaid in 12.12 years. The total amount paid would be approximately $249,981.00 (plus down payment). The sum of the interest paid is $62,481.00.

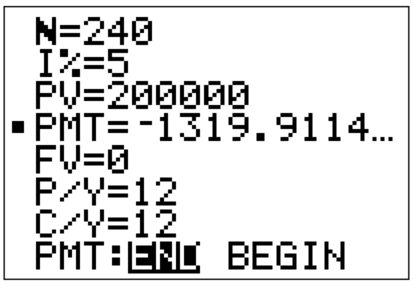

Loan Amt: $187,500
Mtg Rate: 5.00 %
see Current Rates
Payments: Bi Weekly (26)
EXTRA Payments
Cdn property: or US property:
Each Period : 0
Annually: 3000
Mtg Term: 15 yr 0 mon
Amortiz: 15 yr 0 mon
Calculate
see payment chart below
Mortgage Payments Chart
The interactive chart below shows the monthly payment schedule for a $187,500 mortgage for a term of 180 months, amortized over 180 months at an interest rate of 5.00 % compounded semi-annually.
Move mouse over columns to see Monthly Payment Values
Payments biweekly - Paid off in 12.12 Years
Canadian Calculation - Compounding Semi-Annually
Term Principal
Principal Payments
Interest Payments
Annual SUM of PAYMENTS
1 2 3 4 5 6 7 8 9 10 12 15 18 20 25
$20,713 per yr - $187,500 MtgTerm of 180 months. (390 payments)
Amortized over 180 months.
incl $3,000 annual Prin. Buydown5.00 % compounded semiannually.
Payment
$681
Interest - Term
$0
Interest - Amortization
$62,481
Balance at Term
$0
Term Principal Paid
$0
www.canadamortgage.com

Example 6

Craig and Michelle purchase a home for $225,000 with a down payment of $25,000. Create a spreadsheet template to help you analyze the amortization of their 20-year mortgage at 5%, compounded monthly. As the months go by, describe what happens to the amount of each monthly payment that goes towards principal and interest.

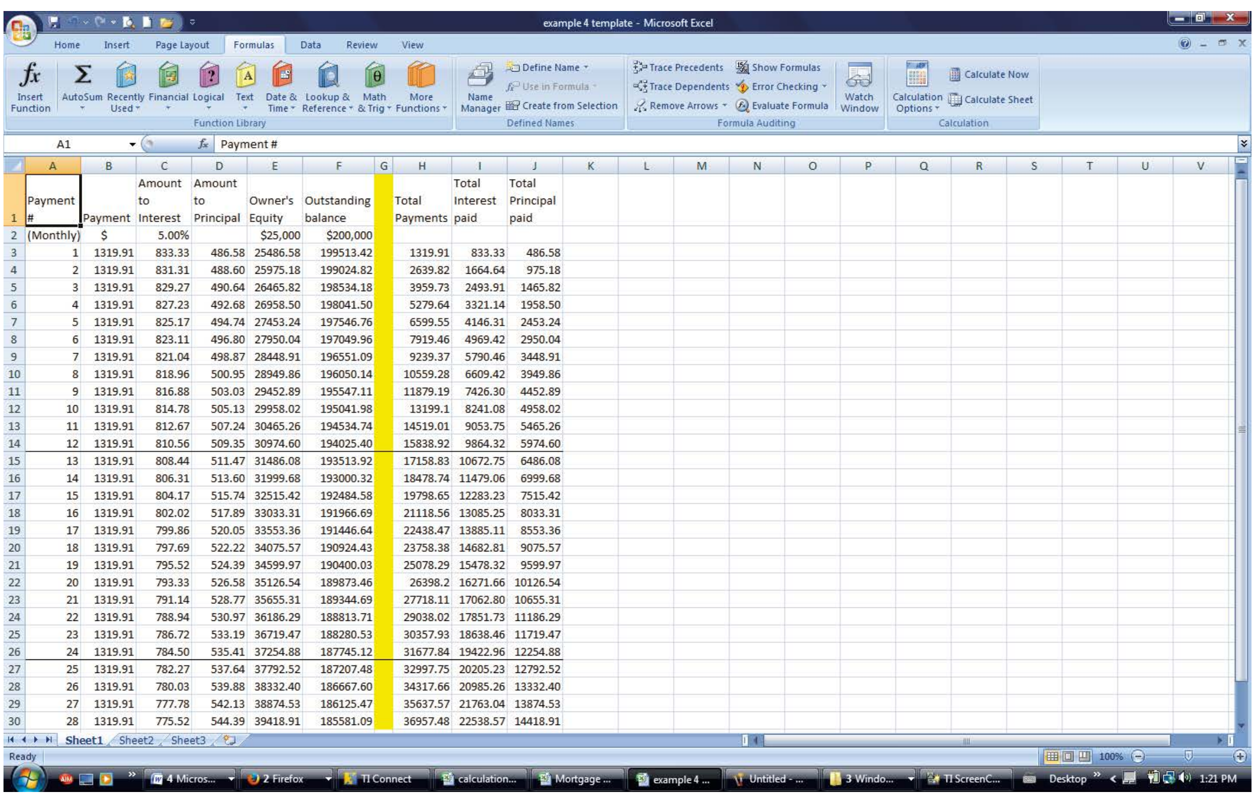
Solution

First, use technology (financial calculator or online app) to determine Craig and Michelle's monthly payment.

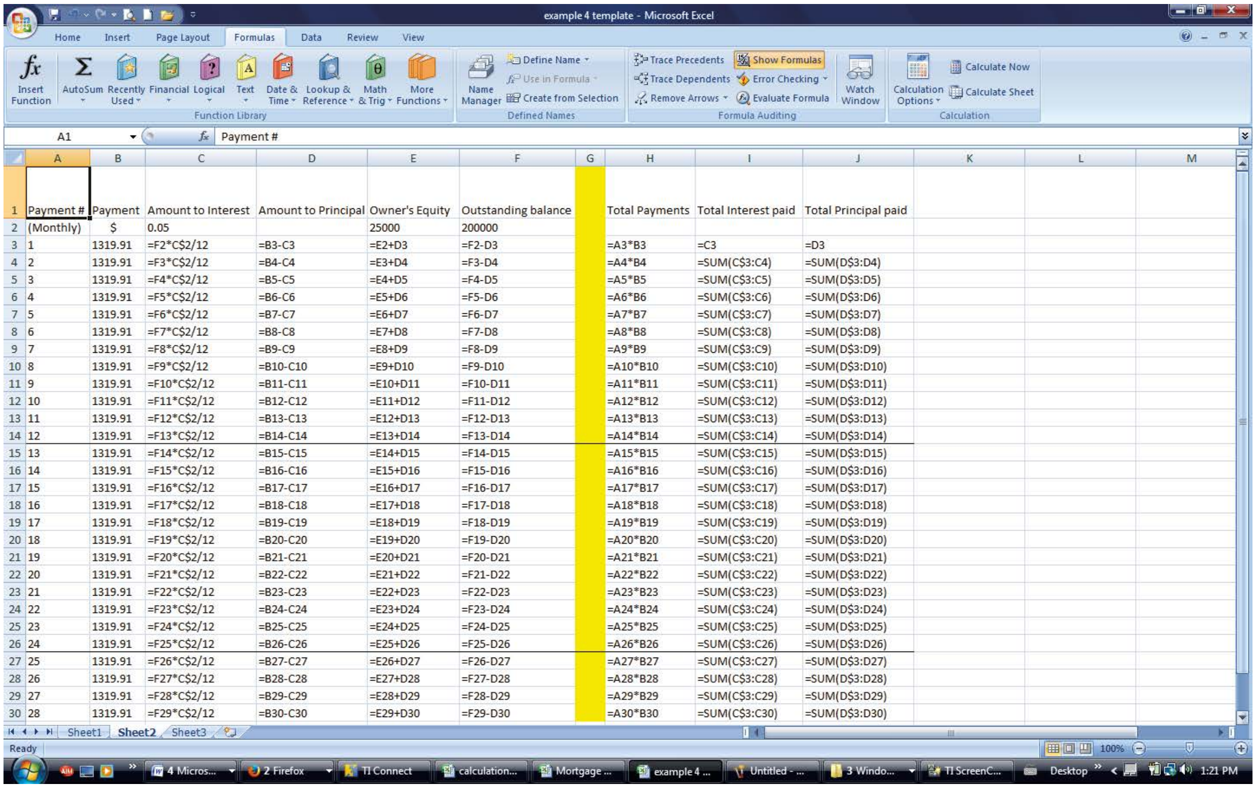


Their monthly payment will be $1319.91.

Open a spreadsheet program and input the required values and formulas. Use the help files available online or in the software program to determine the correct inputs for your choice of program.



Payment #	Payment	Amount to Interest	Amount to Principal	Owner's Equity	Outstanding balance		Total Payments	Total Interest paid	Total Principal paid
(Monthly)	$	5.00%		$25,000	$200,000				
1	1319.91	833.33	486.58	25486.58	199513.42		1319.91	833.33	486.58
2	1319.91	831.31	488.60	25975.18	199024.82		2639.82	1664.64	975.18
3	1319.91	829.27	490.64	26465.82	198534.18		3959.73	2493.91	1465.82
4	1319.91	827.23	492.68	26958.50	198041.50		5279.64	3321.14	1958.50
5	1319.91	825.17	494.74	27453.24	197546.76		6599.55	4146.31	2453.24
6	1319.91	823.11	496.80	27950.04	197049.96		7919.46	4969.42	2950.04
7	1319.91	821.04	498.87	28448.91	196551.09		9239.37	5790.46	3448.91
8	1319.91	818.96	500.95	28949.86	196050.14		10559.28	6609.42	3949.86
9	1319.91	816.88	503.03	29452.89	195547.11		11879.19	7426.30	4452.89
10	1319.91	814.78	505.13	29958.02	195041.98		13199.1	8241.08	4958.02
11	1319.91	812.67	507.24	30465.26	194534.74		14519.01	9053.75	5465.26
12	1319.91	810.56	509.35	30974.60	194025.40		15838.92	9864.32	5974.60
13	1319.91	808.44	511.47	31486.08	193513.92		17158.83	10672.75	6486.08
14	1319.91	806.31	513.60	31999.68	193000.32		18478.74	11479.06	6999.68
15	1319.91	804.17	515.74	32515.42	192484.58		19798.65	12283.23	7515.42
16	1319.91	802.02	517.89	33033.31	191966.69		21118.56	13085.25	8033.31
17	1319.91	799.86	520.05	33553.36	191446.64		22438.47	13885.11	8553.36
18	1319.91	797.69	522.22	34075.57	190924.43		23758.38	14682.81	9075.57
19	1319.91	795.52	524.39	34599.97	190400.03		25078.29	15478.32	9599.97
20	1319.91	793.33	526.58	35126.54	189873.46		26398.2	16271.66	10126.54
21	1319.91	791.14	528.77	35655.31	189344.69		27718.11	17062.80	10655.31
22	1319.91	788.94	530.97	36186.29	188813.71		29038.02	17851.73	11186.29
23	1319.91	786.72	533.19	36719.47	188280.53		30357.93	18638.46	11719.47
24	1319.91	784.50	535.41	37254.88	187745.12		31677.84	19422.96	12254.88
25	1319.91	782.27	537.64	37792.52	187207.48		32997.75	20205.23	12792.52
26	1319.91	780.03	539.88	38332.40	186667.60		34317.66	20985.26	13332.40
27	1319.91	777.78	542.13	38874.53	186125.47		35637.57	21763.04	13874.53
28	1319.91	775.52	544.39	39418.91	185581.09		36957.48	22538.57	14418.91



	A	B	C	D	E	F	G	H	I	J
1	Payment #	Payment	Amount to Interest	Amount to Principal	Owner's Equity	Outstanding balance		Total Payments	Total Interest paid	Total Principal paid
2	(Monthly)	$	0.05		25000	200000				
3	1	1319.91	=F2*C$2/12	=B3-C3	=E2+D3	=F2-D3		=A3*B3	=C3	=D3
4	2	1319.91	=F3*C$2/12	=B4-C4	=E3+D4	=F3-D4		=A4*B4	=SUM(C$3:C4)	=SUM(D$3:D4)
5	3	1319.91	=F4*C$2/12	=B5-C5	=E4+D5	=F4-D5		=A5*B5	=SUM(C$3:C5)	=SUM(D$3:D5)
6	4	1319.91	=F5*C$2/12	=B6-C6	=E5+D6	=F5-D6		=A6*B6	=SUM(C$3:C6)	=SUM(D$3:D6)
7	5	1319.91	=F6*C$2/12	=B7-C7	=E6+D7	=F6-D7		=A7*B7	=SUM(C$3:C7)	=SUM(D$3:D7)
8	6	1319.91	=F7*C$2/12	=B8-C8	=E7+D8	=F7-D8		=A8*B8	=SUM(C$3:C8)	=SUM(D$3:D8)
9	7	1319.91	=F8*C$2/12	=B9-C9	=E8+D9	=F8-D9		=A9*B9	=SUM(C$3:C9)	=SUM(D$3:D9)
10	8	1319.91	=F9*C$2/12	=B10-C10	=E9+D10	=F9-D10		=A10*B10	=SUM(C$3:C10)	=SUM(D$3:D10)
11	9	1319.91	=F10*C$2/12	=B11-C11	=E10+D11	=F10-D11		=A11*B11	=SUM(C$3:C11)	=SUM(D$3:D11)
12	10	1319.91	=F11*C$2/12	=B12-C12	=E11+D12	=F11-D12		=A12*B12	=SUM(C$3:C12)	=SUM(D$3:D12)
13	11	1319.91	=F12*C$2/12	=B13-C13	=E12+D13	=F12-D13		=A13*B13	=SUM(C$3:C13)	=SUM(D$3:D13)
14	12	1319.91	=F13*C$2/12	=B14-C14	=E13+D14	=F13-D14		=A14*B14	=SUM(C$3:C14)	=SUM(D$3:D14)
15	13	1319.91	=F14*C$2/12	=B15-C15	=E14+D15	=F14-D15		=A15*B15	=SUM(C$3:C15)	=SUM(D$3:D15)
16	14	1319.91	=F15*C$2/12	=B16-C16	=E15+D16	=F15-D16		=A16*B16	=SUM(C$3:C16)	=SUM(D$3:D16)
17	15	1319.91	=F16*C$2/12	=B17-C17	=E16+D17	=F16-D17		=A17*B17	=SUM(C$3:C17)	=SUM(D$3:D17)
18	16	1319.91	=F17*C$2/12	=B18-C18	=E17+D18	=F17-D18		=A18*B18	=SUM(C$3:C18)	=SUM(D$3:D18)
19	17	1319.91	=F18*C$2/12	=B19-C19	=E18+D19	=F18-D19		=A19*B19	=SUM(C$3:C19)	=SUM(D$3:D19)
20	18	1319.91	=F19*C$2/12	=B20-C20	=E19+D20	=F19-D20		=A20*B20	=SUM(C$3:C20)	=SUM(D$3:D20)
21	19	1319.91	=F20*C$2/12	=B21-C21	=E20+D21	=F20-D21		=A21*B21	=SUM(C$3:C21)	=SUM(D$3:D21)
22	20	1319.91	=F21*C$2/12	=B22-C22	=E21+D22	=F21-D22		=A22*B22	=SUM(C$3:C22)	=SUM(D$3:D22)
23	21	1319.91	=F22*C$2/12	=B23-C23	=E22+D23	=F22-D23		=A23*B23	=SUM(C$3:C23)	=SUM(D$3:D23)
24	22	1319.91	=F23*C$2/12	=B24-C24	=E23+D24	=F23-D24		=A24*B24	=SUM(C$3:C24)	=SUM(D$3:D24)
25	23	1319.91	=F24*C$2/12	=B25-C25	=E24+D25	=F24-D25		=A25*B25	=SUM(C$3:C25)	=SUM(D$3:D25)
26	24	1319.91	=F25*C$2/12	=B26-C26	=E25+D26	=F25-D26		=A26*B26	=SUM(C$3:C26)	=SUM(D$3:D26)
27	25	1319.91	=F26*C$2/12	=B27-C27	=E26+D27	=F26-D27		=A27*B27	=SUM(C$3:C27)	=SUM(D$3:D27)
28	26	1319.91	=F27*C$2/12	=B28-C28	=E27+D28	=F27-D28		=A28*B28	=SUM(C$3:C28)	=SUM(D$3:D28)
29	27	1319.91	=F28*C$2/12	=B29-C29	=E28+D29	=F28-D29		=A29*B29	=SUM(C$3:C29)	=SUM(D$3:D29)
30	28	1319.91	=F29*C$2/12	=B30-C30	=E29+D30	=F29-D30		=A30*B30	=SUM(C$3:C30)	=SUM(D$3:D30)

As the monthly payments progress, the amount paid remains the same but the amount of each payment that goes towards the principal increases, while the amount that goes towards interest decreases. The reason for this is that the loan is compounded monthly and, after each payment, Marcel owes less to the bank. If the outstanding amount decreases, his interest owed is less as well, and more of each payment can go towards paying back the principal of the loan.

It may be helpful for you to look back at the examples and record important information on your resource sheet.

Learning Activity 5.3

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

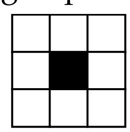
Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. If you earn $150 per week and save 10% of each paycheque, how much would you have saved after one year?

2. Using all four of the digits 2, 3, 4, and 9, some of the four common mathematical operations (+, −, ×, ÷), and brackets, create an expression equal to 42.

3. Given that the area of the shaded square is 16 cm^2, what is the area of the large square?



4. If the odds in favour of an event occurring are 1 : 2, what is the probability of the event occurring?

5. Lu scored $\frac{21}{25}$ on a test, while Min scored $\frac{8}{10}$. Who scored higher?

6. State the formula for annual compound interest. Indicate what each of the variables represents.

7. Illustrate the following on a number line: $\{-3 \le x < 9, x \in \Re\}$

8. Write the inverse, given the conditional statement: “If a batter gets three strikes, then the batter is out.”

continued

Part B: Mortgages and Loans

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Steve and Val take out a $20,000 loan for renovations to their home. They negotiate a 5-year term at prime plus 2.5%, compounded semi-annually. The prime rate is 3%. What will be the amount of their semi-monthly payments? How much interest will they pay on the loan?

2. Frieda and Ernest purchase a retirement condo for $335,999. They must pay additional costs on the new building, which include moving costs, lawyer fees, insurance fees, and other taxes. They add 2% of the purchase price to their mortgage to cover these additional costs. Compare the amount of interest they would pay on this mortgage if they amortized it over 15 years rather than 20 years. Assume they make monthly payments and the mortgage is compounded semi-annually at prime plus 3.1%. The prime rate is 3%.

3. Owen wants to start a restaurant and applies for a bank loan of $65,000. His business plan includes repaying the loan in 10 years with bi-weekly payments. He compares the options at two different banks.
   - Bank A: 8.1%, compounded monthly
   - Bank B: 6.3%, compounded monthly

   How much of a difference will a lower rate make to the bi-weekly payments and to the amount of interest Owen will pay?

4. Garrett and Adria purchase a farmhouse and land for $840,000. They use an inheritance of $200,000 as the down payment and take out a 25-year mortgage at prime plus 4%, compounded semi-annually, for the remainder of the purchase price. The prime rate is 3%. Determine their payment amount and the amount paid in interest if they choose to make bi-weekly payments. Calculate the difference in their payments and the amount of inteest they will pay if they make semi-annual payments.

continued

5. Reta makes $450 payments on her condo mortgage twice a month. Her original mortgage amount was $120,000, and has been compounded monthly at 5.12%.
   a) How long will it take Reta to pay off her mortgage?
   b) Give three suggestions to help Reta reduce the amount of interest she will pay over the life of her mortgage.
6. Marcel purchases a racing snowmobile that costs $7000 on credit. The loan is for four years, compounded monthly, at 18.9%. Use technology to determine:
   a) the monthly payment amount.
   b) the amount he still owes after 2 years.
   c) Use your spreadsheet template to visualize the amortization of Marcel's loan and determine the amount of interest paid on the loan.
7. Brendan takes out a loan to buy a quad ATV and to develop a parcel of land behind his home to create a racetrack. He borrows $8500 from his local credit union at 15% interest, compounded quarterly, for 3 years.
   a) Determine his monthly payments.
   b) Use technology to determine the amount of interest paid in each of the three years. Explain why the amount of interest changed each year.

Lesson Summary

In this lesson, you used technology to determine the total cost of a loan or mortgage under a variety of conditions, such as different amortization periods, interest rates, and compounding periods or terms.

Lesson 4: Investing

Lesson Focus

In this lesson, you will

- ❑ determine, using technology, the total value of an investment when there are regular contributions to the principal
- ❑ graph and compare the total value of an investment with or without regular contributions
- ❑ determine, using technology, possible investment strategies to achieve a financial goal
- ❑ explain the advantages and disadvantages of long-term or short-term investment options
- ❑ explain, using examples, why smaller investments over a longer term may be better than larger investments over a shorter term

Lesson Introduction

Borrowing money from a bank for something you need may be necessary at some point during your life. However, you can sometimes avoid some of the fees associated with borrowing money by careful saving. Saving some money from each paycheque and investing it for a long-term investment is recommended. In addition to spending money and giving money to charities of your choice, you need to consider saving money for your future. If you start early to invest money and have that money earn compound interest, you will be taking advantage of the properties of compound interest. Some people say that you should "pay yourself first," which means you shouldn't just save what is left over at the end of the month; instead, you should put some money in a savings account as soon as you receive your pay, before you spend all of your earnings. This lesson will focus on the details of investing money.

Investing

In Lesson 1, you considered the value of an investment, given an initial amount, over time with simple or compound interest. In Lessons 2 and 3, you investigated how changing the parameters of a loan or mortgage affects the total cost of credit (e.g., the number of compounding periods and payments per year, the interest rate, the amount of the down payment, or the amount of the payment). In this lesson, you will examine how these factors affect investing.

Regular Investing

When you invest a fixed amount on a recurring basis, it is called regular investing. This investment strategy helps you save small amounts of money on a regular basis. This is often easier than coming up with occasional large, lump sum investments.

Example 1

Nick invests $100 per month at 6%, compounded annually. His sister Carly invests the same amount in a savings account with the same interest and compounding period, but makes one deposit of $1200 per year. Use technology to determine the future value of their investments after 20 years.

Solution

Nick:

```
 N=240
 I%=6
 PV=0
 PMT=100
▪FV=-45343.86325
 P/Y=12
 C/Y=1
 PMT:END BEGIN
```

Carly:

```
 N=20
 I%=6
 PV=0
 PMT=1200
▪FV=-44142.70944
 P/Y=1
 C/Y=1
 PMT:END BEGIN
```

Nick makes 12 × 20 contributions of $100 ($24,000). The future value of his investment after 20 years is $45,343.86.

Carly invests the same amount as Nick, $24 000, but makes 20 deposits of $1200 each.

The future value of her investment after 20 years is $44,142.71, which is $1201.15 less than the future value of Nick’s investment.

Investing more frequently allows more time for your investment to grow in value. In addition, it is often easier to part with smaller amounts more often, rather than trying to accumulate a lump sum amount on a yearly basis.

Example 2

Jeremy deposits $250 into an account every three months. Interest is 6.9%, compounded quarterly.

Troy invests $1000 a year as a lump sum deposit. It is invested at 6.9%, compounded annually.

Graph and compare the value of the investments in these two scenarios after 8 and 20 years.

Solution

The formula you used in Lesson 1 to graph the value of investments over time, with compounded interest was $A = P\left(1 + \frac{r}{n}\right)^{nt}$. This formula does not account for regular contributions over time. To calculate or graph the value of an investment with regular contributions, you need to use the formula:

$$A = \frac{R\left[\left(1 + \frac{r}{n}\right)^{nt} - 1\right]}{\frac{r}{n}}$$

where: A is the amount after t years

R is the regular deposit amount

r is the annual interest rate

n is the number of compounding periods per year

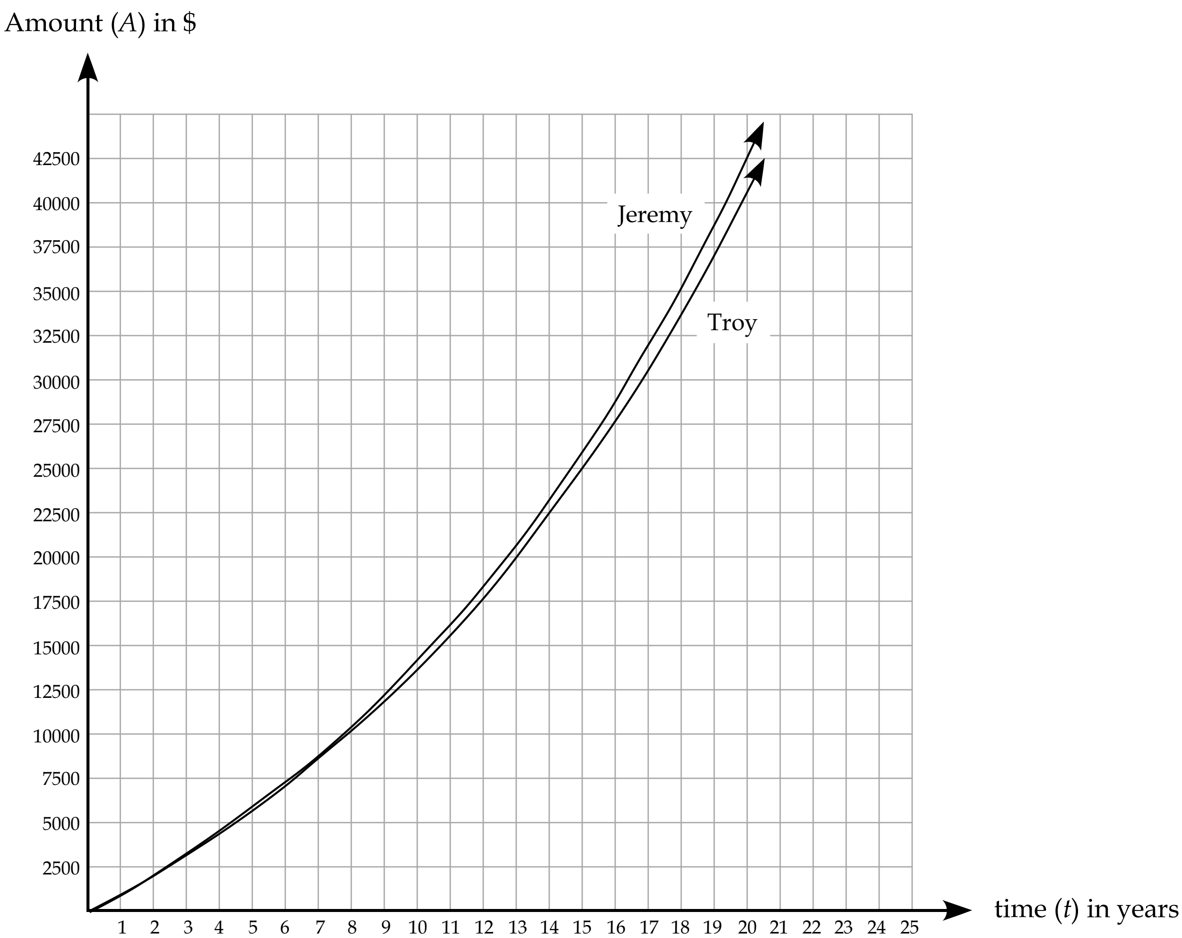
t is the number of years

Note: When using this formula, the number of compounding periods per year and the number of regular payments per year must be the same.

Use the formula $A = \frac{R\left[\left(1 + \frac{r}{n}\right)^{nt} - 1\right]}{\frac{r}{n}}$ to model the amount of each investment over time.

Value of Jeremy's investment over time: $A = \dfrac{250\left[\left(1 + \dfrac{0.069}{4}\right)^{(4)(t)} - 1\right]}{\dfrac{0.069}{4}}$

Value of Troy's investment over time: $A = \dfrac{1000\left[\left(1 + \dfrac{0.069}{1}\right)^{(1)(t)} - 1\right]}{\dfrac{0.069}{1}}$

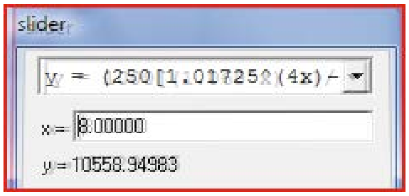


Use the graph or the formula to solve for A (the final amount after t years), when $t = 8$ and $t = 20$.

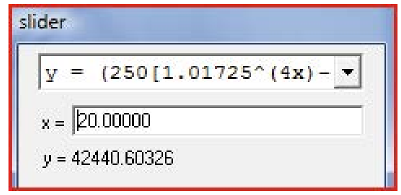
You can enter the formula replacing t with x in a grapher app such as Winplot (as shown) and then trace the value when $x = 8$ and when $x = 20$.

Jeremy's Investment

Value after 8 years: A = $10,558.95



Value after 20 years: A= $42,440.60



Or

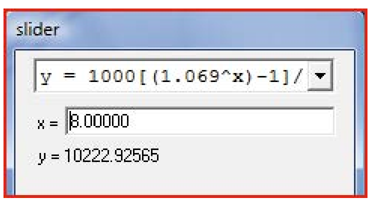
$$A = \frac{250\left[\left(1+\dfrac{0.069}{4}\right)^{(4)(8)} - 1\right]}{\dfrac{0.069}{4}}$$

$$A = \frac{250[1.01725^{32} - 1]}{0.01725}$$

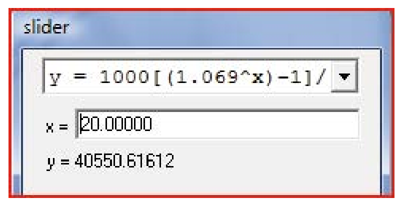
$$A = 10\ 558.95$$

Troy's Investment

Value after 8 years: A = $10,222.93



Value after 20 years: A= $40,550.62



Or

$$A = \frac{1000\left[\left(1+\dfrac{0.069}{1}\right)^{(1)(20)} - 1\right]}{\dfrac{0.069}{1}}$$

$$A = \frac{1000[1.069^{20} - 1]}{0.069}$$

$$A = 40\ 550.62$$

Jeremy and Troy both contributed the same amount of money to their respective savings account, but the value of Jeremy's investment grows a little more quickly since he contributes smaller amounts more often.

Rate of Return

The amount of interest earned on an investment (or charged on a loan) is given as a percentage of the principal. When you invest, you want to make sure that the interest rate is as high as possible in order to get the best return, or profit, on your investment. Conscientious investors will keep track of how well their investments are doing by evaluating the rate of return (ROR) or their return on investment (ROI). These terms refer to a calculation of the ratio of money gained in interest on an investment compared to the amount of capital or principal invested, expressed as a percentage.

The formula for rate of return (ROR) or return on investment (ROI) is:

$$\text{ROR} = \frac{\text{Interest Earned}}{\text{Principal}} = \frac{\left(\begin{matrix}\text{Current value} \\ \text{of portfolio}\end{matrix} - \begin{matrix}\text{Previous value} \\ \text{of portfolio}\end{matrix}\right)}{\text{Previous value of portfolio}}$$

The ROR or ROI is usually expressed as a percentage.

It may be helpful for you to include some of this information and these formulas on your resource sheet.

Example 3

Determine the ROR for the investments held by Nick and Carly in Example 1, and Jeremy and Troy in Example 2.

Solution

First determine the amount of interest gained (profit) and the amount invested (capital cost or principal) for each investor. Use the formula:

$$\frac{\text{Final Value of Investment} - \text{Initial Amount Invested}}{\text{Initial Amount Invested}}$$

Nick and Carly each invested \$24,000 over 20 years. Nick's future value is \$45,343.86, so his gain is \$45,343.86 − \$24,000.00 = \$21,343.86. The value of Carly's investment after 20 years is \$44,142.71. She earned \$20,142.71 in interest.

Nick:

$$\frac{\text{Final Value of Investment} - \text{Initial Amount Invested}}{\text{Initial Amount Invested}}$$

$$= \frac{45\ 343.86 - 24\ 000}{24\ 000}$$

$$= \frac{21\ 343.86}{24\ 000}$$

$= 0.8893275$

Over the 20 years, the rate of return on Nick's investment is 88.9%.

Carly:

$$= \frac{20\ 142.71}{24\ 000}$$

$= 0.83928$

The ROR for Carly's investment is 83.9%.

For Jeremy and Troy, you can calculate the ROR at 8 years and at 20 years.

Jeremy:

8 years:

$$= \frac{10\ 558.95 - 8000}{8000}$$

$= 0.3199$

$= 31.99\%$

20 years:

$$= \frac{42\ 440.60 - 20\ 000}{20\ 000}$$

$= 1.12203$

$= 112.103\%$

Troy:

8 years:

$$= \frac{10\ 222.93 - 8000}{8000}$$

$= 0.2779$

$= 27.79\%$

20 years:

$$= \frac{40\ 550.62 - 20\ 000}{20\ 000}$$

$= 1.0275$

$= 102.75\%$

It is obvious from these examples that the longer an investment is held (8 years versus 20 years), the higher the return. It can also be observed that a higher interest rate (6.9% for Jeremy and Troy compared to 6% for Carly and Nick) resulted in a higher rate of return on the 20-year investments.

Long Term Versus Short Term

You have seen how investing over a longer period of time allows investments to grow. A longer term permits a small amount to become larger, due to the exponential growth of compounding interest.

Example 4

After your class graduated from university, you and a friend started new jobs. Starting with your first paycheque, you decided to take $100 from each of your semi-monthly paycheques and invest it in a RRSP for retirement in 27 years time. The RRSP is compounded annually at 3.1%. Your friend wanted to spend money now and save later, so waited until retirement was only seven years away, panicked, and then invested $1000 per month in an RRSP at 3.1%, compounded annually.

a) Who will have more money to spend during retirement?

b) Determine the rate of return on each investment.

c) What can you conclude about short-term versus long-term investments?

Solution

a) You:

$n = 24 \times 27$

```
 N=648
 I%=3.1
 PV=0
 PMT=100
▪FV=-100581.2312
 P/Y=24
 C/Y=1
 PMT:END BEGIN
```

You will retire with $100,581.23.

Friend:

$n = 12 \times 7$

```
 N=84
 I%=3.1
 PV=0
 PMT=1000
▪FV=-93531.55559
 P/Y=12
 C/Y=1
 PMT:END BEGIN
```

Your friend will have $93,531.56.

b) $\text{ROR} = \dfrac{\text{Final Value of Investment} - \text{Initial Amount Invested}}{\text{Initial Amount Invested}}$

Your rate of return:

$$\text{ROR} = \frac{100\ 581.23 - (100 \times 24 \times 27)}{64\ 800}$$

$$\text{ROR} = 55.22\%$$

Your friends rate of return:

$$\text{ROR} = \frac{93\ 531.56 - (1000 \times 12 \times 7)}{84\ 000}$$

$$\text{ROR} = 11.35\%$$

c) Two significant observations can be made.

- Your ROR is significantly higher than your friend's return and you only invested $64,800, while your friend invested $84,000. You invested almost $20,000 less than your friend, and ended up with almost $7050 more!
- As well, smaller, regular investments of $100 over a longer time frame would likely be easier to save/contribute than $1000 contributions over a shorter period.

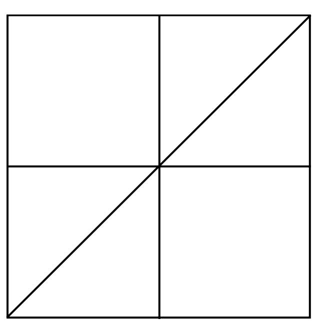
Learning Activity 5.4

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Using each of the digits 2, 5, 6, and 8, some of the four common mathematical operations, and brackets, create an expression equal to 4.
2. The statement "If p then q" may be written in the form "$p \rightarrow q$." Write the contrapositive of "$p \rightarrow q$."
3. A perfect number is a number that equals the sum of all its factors, excluding itself. Show that 28 is a perfect number.
4. Sketch $y = \log(x)$
5. What is the maximum number of Mondays that can occur in the first 45 days of a year?
6. What is the total number of triangles and squares in the following figure?



continued

7. A Pythagorean Triple, such as 3, 4, 5, is a set of three numbers that satisfy the Pythagorean theorem, $a^2 + b^2 = c^2$. What value for k will make this set of numbers a Pythagorean Triple?

   5, k, 13

Part B: Investing

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. a) Jimmy Sky wants to purchase a paddleboard for next summer. He has $500 saved but the board costs $795, plus 13% taxes. He plans to invest his $500 and make monthly deposits in a “Special Project Account” at his bank, offering 0.04% interest, compounded monthly. How much will he have to save each month in order to afford the board nine months from now?

   b) Jimmy decides he can’t afford those monthly contributions, so he will rent a board this summer and invest his $500 until the following summer to purchase his own. His bank will give him 0.09%, compounded monthly, on a 21-month term. How much will he need to set aside each month for 21 months to afford the board the following year?

2. a) Avery has $1200 saved from her summer job and will continue to work part-time during the school year. She plans to save $30 from each of her semi-monthly paycheques in order to buy a used car when she graduates in June, 10 months from now. Her bank offers her a savings account at 1.2%, compounded quarterly. How much will she have available to spend on a car in June?

   b) If she wants to have $2000 by June, what would she have to do differently, in terms of this investment, to reach that goal? Support your answer with TVM solver results (or other software applications).

3. Matthew invests $500 in an escalating GIC with annual compounding that pays 1% for the first year, 1.35% for the second year, 1.75% for the third year, 2.5% for the fourth year, and 4% for the fifth year.

   a) Determine the value of his investment after each year.

   b) Calculate the ROR for this investment after the first and fifth years.

continued

4. a) Michelle would like to buy a condo after graduation from university and begins to save for a down payment. She would like to have $17,000 available. If she can contribute $325 each month during the four years she attends university, what rate will she have to negotiate with her bank to make the down payment possible? Use quarterly compounding.

   b) If Michelle negotiates a slightly higher rate of 4.8%, compounded quarterly, and begins with an initial investment of $1000 while still contributing $325 per month, what will her down payment be?

5. Gordon and Rebecca are saving for retirement. Rebecca contributes $50 from her paycheque bi-weekly into an RRSP at 7.1%, compounded bi-weekly, Gordon contributes $1300 annually into an RRSP at 7.1%, compounded annually.

   a) Graph the value of each of their investments over time and determine the amount saved by Rebecca and Gordon after 25 years.

   b) Describe the advantages and disadvantages of smaller investments over a longer term compared to larger investments over a shorter term.

6. When Cornelius retired, he purchased a sailboat and left on a year-long cruise around the Mediterranean Sea. His adventure cost him $212,000, but he had been saving up for this for his entire career (39 years working at the same job). If he contributed $100 per month to an RRSP that averaged 5.9%, compounded monthly, what was his initial investment?

7. a) Arnold successfully kicks a "pack a week" smoking habit. He takes the $13.75 per week that he would have spent on cigarettes and invests it in a growth fund at 7.4%, compounded weekly. How much will he have in his investment after (i) one year? (ii) ten years?

   b) How much would Arnold have saved if he invested $13.75 per day for 10 years? (Assume 7.4%, compounded daily.)

8. Calculate the rate of return on the investment of a lump sum of $10,000 at 3%, compounded annually, for 15 years.

9. Craig buys a television for $679, using a store credit card. He is given a special promotion of 0% interest for 12 months but is charged a finance fee of $49.99. He makes 12 equal payments of $60.75 before the one year deadline, and so pays no interest. What rate of return does the store get on this purchase by charging the finance fee?

continued

Learning Activity 5.4 (continued)

10. Akeela uses a lay-away plan at a local department store to purchase a necklace for her mother's birthday. The necklace costs $69 plus 13% taxes. Akeela must pay a $5 fee and a down payment of 20% of the purchase price or $20 (whichever is higher), and the store will hold the necklace for her. She must make four equal bi-weekly payments to cover the remaining cost of the necklace.
    a) What is Akeela's final cost for the necklace?
    b) What rate of return does the store receive by offering this service?
11. Daniel purchases a BMX bike for $800, using his credit card. He is charged 19.99% interest, compounded monthly (no grace period), and pays it back with regular payments over the course of 18 months.
    a) Determine his monthly payment.
    b) Determine the rate of return on this credit card purchase.
    c) If Daniel had not made any payments on the bike for 18 months and then paid it off in one lump sum (including interest), what would his rate of return have been?
    d) What can you conclude about making smaller, regular contributions to pay down debt?
12. Ming has a credit card debt of $4100 for furniture purchased for her new condo. The interest rate on her balance is 19.99%, compounded monthly.
    a) If Ming wants to pay off the furniture with regular monthly payments over the next two years, what amount should she pay each month?
    b) Using the regular monthly payment you calculated, how long will it take for her to pay back half of her debt?

Lesson Summary

In this lesson, you used technology to graph and find the total value of an investment when there are regular contributions to the principal. You considered investment strategies to achieve a financial goal and explained the advantages and disadvantages of making smaller investments over a longer term compared to making larger investments over a shorter term.

Assignment 5.2

Loans and Investments

Total: 41 marks

This is a hand-in assignment. Please show your work clearly and in an organized manner. Round final answers to 2 decimal places and include units, if appropriate. If you use technology as a strategy in your solution steps, please indicate what application you are using, the values you input, and a sketch or printout of the results. Answers given without supporting calculations will not be awarded full marks.

1. Derek and Marilyn decide to build a new home. The purchase price is quoted at $325,000. In addition to this, Marilyn and Derek must pay GST of 5% on the purchase price. Inspections, insurance, and legal fees, as well as utility hook-ups and moving costs, add an additional 2.3% of the purchase price. They have saved a down payment of $80,000. They negotiate a mortgage at their bank for 15 years at prime plus 3.15%, compounded monthly. The prime rate is 3%.

   a) Determine:

      i) their monthly payment amount. *(2 marks)*

      ii) the total interest paid on the mortgage. *(1 mark)*

iii) the total cost of purchasing the home. *(2 marks)*

b) If Derek and Marilyn decide to make bi-weekly payments of $1100 instead of monthly payments, how long will it take them to pay off their mortgage? *(2 marks)*

c) How much will they save in interest with the bi-weekly payments, compared to the monthly payments? *(2 marks)*

Assignment 5.2: Loans and Investments (continued)

2. Phil and Kristy would like to purchase a cottage, which costs $192,000. They can either amortize this amount over 15 years or 20 years. The interest rate in both cases is 5.8%, compounded semi-annually, with monthly payments and a down payment of $30,000.

   a) What length of term would you suggest they accept? Support your answer by showing TVM calculations (or other software applications). *(4 marks)*

   b) What other suggestions could you give them to help them reduce the amount of interest they must pay on a mortgage? *(2 marks)*

3. You want to take out a personal loan to finance a trip to Europe. You estimate that you will need $7500 and the bank offers you a 4-year term at 1.98%, compounded quarterly. What will your payments be each month? *(1 mark)*

4. In approximately how many years will an investment that earns 9%, compounded annually, double in value? *(2 marks)*

5. Billy invests $225 per month for 10 years in a mutual fund that averages 4.3%, compounded semi-annually.

   a) What is the total amount of his investment after 10 years? *(2 marks)*

b) Calculate his rate of return. *(2 marks)*

6. a) Mina puts $100 from her monthly paycheque directly into a term deposit at 4.5%, compounded monthly. She intends to retire 25 years from now.

   i) How much will she have invested over the 25 years? *(1 mark)*

   ii) What is the future value of her term deposit? *(2 marks)*

   iii) What is Mina's rate of return? *(2 marks)*

b) Mina's colleague, Francesco, also wants to retire in 25 years but prefers to wait until he is more financially stable before saving money. He plans to wait for 15 years and then begin to put $250 per month into the same sort of account as Mina (4.5%, compounded monthly) for 10 years.

i) How much will he have invested? *(1 mark)*

ii) What is the future value of Francesco's term deposit? *(2 marks)*

iii) What is Francesco's rate of return? *(2 marks)*

c) On the last day of work before they retire, Mina and Francesco compare their investments. What conclusions do you think they could make? *(2 marks)*

7. a) Use technology to graph and compare an investment of $20 per week at 5.7%, compounded weekly, with an investment of $1000 per year, at 5.7%, compounded annually, showing a minimum of 25 years. *(5 marks)*

   b) Determine the value of each investment after 22 years. *(2 marks)*

Notes

Lesson 5: Investment Portfolios

Lesson Focus

In this lesson, you will

- ❑ determine, using technology, possible investment strategies to achieve a financial goal
- ❑ explain the advantages and disadvantages of long-term or short-term investment options
- ❑ determine and compare the strengths and weaknesses of two or more portfolios
- ❑ solve investment problems

Lesson Introduction

One of your first large investments you make might be an investment in a house. In this lesson, you will learn some factors that can be used to help you decide what you can afford. In the last lesson, you saw how making regular contributions to an investment can be a powerful way to grow money over time. This is not the only strategy available to investors. There are many different approaches to investing that allow individual investors to create a strategy and portfolio that best reflect their personal needs, resources, and goals, using a variety of investment products. In this lesson, you will learn about some of the types of investment products that are available to you.

Making Investment Decisions

Investment Strategies

A fundamental principal of budgeting is that you should always begin by "paying yourself first." This means that you should automatically save 10% of your income before allocating money for bills, living expenses, or entertainment. This 10% should be invested as savings for the future (possibly for retirement or as a down payment on a home) or as an emergency fund. You can save this money in a number of ways—invest in a savings account, purchase stocks in a trading account, buy a unit of a mutual fund, purchase a GIC, or buy a bond—or you can investigate other investment options. Making regular investing a priority is as valuable as paying off your credit debt each month.

The questions that remain are “When is the best time to invest?” and “In what should I invest?”

Investing in a House

One of the first major investments that you might consider making is the purchase of property, whether it is a house or a condominium. Part of your decision will be about how much you can afford for mortgage payments each month. One calculation that banks and credit unions use to assess whether you can afford a mortgage payment is the **Gross Debt Service ratio**. It is calculated using the formula:

$$\text{GDS ratio} = \frac{\left(\begin{array}{c}\text{monthly}\\ \text{mortgage}\\ \text{payment}\end{array} + \begin{array}{c}\text{monthly}\\ \text{property}\\ \text{taxes}\end{array} + \begin{array}{c}\text{monthly}\\ \text{heating}\\ \text{costs}\end{array} + \begin{array}{c}\text{one-half}\\ \text{monthly}\\ \text{condo fees}\end{array}\right)}{\text{Gross Monthly Income}}$$

If you are buying a house, then the monthly condo fees do not apply.

The Gross Debt Service (GDS) ratio is often represented as a percent. According to this rule of thumb, your debt level is acceptable if the GDS ratio is at or below 32%.

It may be helpful for you to include some of this information on your resource sheet.

For more details, visit the Canadian Mortgage and Housing Corporation website at https://www.cmhc-schl.gc.ca/en/co/buho/step-by-step/index.cfm.

Example 1

Sandy has a gross salary of $50,000 per year and wants the bank to give her a mortgage with payments of $1000 per month. The annual property taxes are $2400 per year and the estimated heating costs for the year are $800. Should she expect the bank to give her the mortgage?

Solution:

Before calculating the GDS ratio, you need to convert to monthly values. It is important to note that since GDS is a ratio, the formula also works for annual values, but then all the values must be annual values.

Monthly salary = 50000 ÷ 12 = 4167

Monthly property taxes = 2400 ÷ 12 = 200

Monthly heating costs = 800 ÷ 12 = 67

$$\text{GDS ratio} = \frac{1000 + 200 + 67}{4167}$$

$$\text{GDS ratio} = \frac{1267}{4167} = 0.304$$

The Gross Debt Service ratio is 30.4%. Since it is below 32%, the bank is likely to give her the mortgage.

Another factor that banks consider when loaning money to individuals is the **Debt to Equity ratio**. This ratio compares the liabilities of a person or a company with the net worth of the individual or the company. To understand the Debt to Equity ratio, you need to know the meaning of a few terms used by investors.

- **Net worth** is a measure of the finances of an individual or a company. It is the difference between the assets and the liabilities.
- **Assets** are anything of value that is owned by a company or an individual. Examples of assets for individuals include cash, money in a bank, investments (stocks, bonds, mutual funds), vehicles, real estate, and personal items (jewelry, furniture, tools).
- **Liabilities** are any debts an individual or a business owes to someone else. Examples of liabilities include a mortgage, a car loan, money owed to a line of credit, money owed to a credit card company, and a student loan.

To calculate net worth:

net worth = total assets − total liabilities

Equity in a home is a term used to describe the net worth of the home to the owner (the current market value less the remaining mortgage or the portion of the home that has been paid for by the buyer). Since equity and net worth describe the same thing, it is consistent that the Debt to Equity (D/E) ratio for a person is a ratio of liabilities (excluding a mortgage) and net worth. The Debt to Equity ratio is calculated as:

$$\text{D/E ratio} = \frac{(\text{total liabilities} - \text{mortgage})}{\text{net worth}}$$

The D/E ratio is often represented as a percent. When lending institutions such as banks are considering an individual's finances, they consider a healthy Debt to Equity ratio to be at or below 0.5 or 50%.

It may be helpful for you to include some of this information on your resource sheet.

Example 2

Courtney owns a house valued at $275,000 and has a mortgage on the property of $240,000. She has a car worth $25,000 and a car loan of $20,000. She also has $12,000 saved in an RRSP. She would like to borrow $7000 to add a pool to her yard. Will the bank give her the loan based on her Debt to Equity ratio?

Solution:

First, calculate assets, liabilities, and net worth.

Assets = 275000 + 25000 + 12000 = 312000

Liabilities = 240000 + 20000 + 7000 = 267000

Net worth = 312000 − 267000 = 45000

Now you have what you need to calculate the Debt to Equity ratio.

$$\text{D/E ratio} = \frac{(\text{total liabilities} - \text{mortgage})}{\text{net worth}}$$

$$\text{D/E ratio} = \frac{(267000 - 240000)}{45000}$$

$$\text{D/E ratio} = 0.6 = 60\%$$

The bank will not give her the loan because her D/E ratio, including the loan, will be above 50%.

Note that her D/E ratio before borrowing the $7000 is acceptable at 38.5%, as shown by the calculation:

$$\text{D/E ratio} = \frac{(260000 - 240000)}{52000} = 0.385$$

The Debt to Equity ratio is also widely used by investors when choosing companies on a stock market. A healthy Debt to Equity ratio for companies on the stock market may range from 0.25 to 2.0, depending on the type of company. A company that buys and stores manufacturing supplies will have a higher percentage of debt than a company that offers a service and does not need much inventory.

Timing Your Investments

Interest rates, stock prices, and investment products vary based on market performance and economic and political factors. You cannot control these influences, but if you follow a strategy of "regular investing," you can reduce the probability that your purchase will be poorly timed.

Some investors practice the strategy of trying to “buy low, sell high,” but if you are purchasing investment products regularly, the cost of the purchases you make will be averaged between the highs and lows of the market. As a result, you tend to be more in line with the long-term trends of the market.

Another strategy may be to “buy and hold” a certain product for a longer time frame to realize a return, while other investors may “buy and sell often,” purchasing volatile stocks or high-risk investment products and selling as soon as a return has been earned.

The timing of your purchases may be a result of your resources, goals, and tolerance of risk as much as it is a factor of your long- and short-term strategies, but starting sooner, rather than later, is good advice to follow. The next thing you need to determine is whether you want to diversify or to focus on one type of product.

Investment Products

Different financial institutions offer different types of investment products. A strong portfolio is one that includes a variety of investment products. The specific nature of the diversity of a portfolio may change with time, but the types of products available fall under three main categories: safe (cash) investments, fixed (income) products, and equity (growth) investments.

- **Cash investments** include safe options such as term deposits and bank accounts. A **savings account** has a very low rate of interest, often less than 0.5%, while a **chequing account** may have 0% interest and be subject to monthly charges or transaction fees. However, these secure types of accounts are very accessible, your money is not locked away for a fixed period of time, and you can withdraw at any time. Many people require this flexibility for paying regular expenses. Term deposits are short-term investments that are considered safe, but also produce low returns. If you want a higher return on your savings and investing, you may have to balance the safety and accessibility (liquidity) of your funds with the desired return.
- **Fixed income investments** such as bonds and **GICs** (Guaranteed Investment Certificates) offer a fixed rate of return or income for a set time period with limited risk. A GIC investment is guaranteed by the financial institution so you cannot lose your principal and you know exactly how much you will earn during the term. However, the rate is not usually as high as riskier investments. Also, the principal and interest are locked in for a term, decided upon at the outset, and your money cannot be withdrawn or redeemed until that time without penalty.

- **Bonds** are investment products offered by governments and corporations and are considered secure (you lend money to the government or corporation and they pay interest on your investment semi-annually). Bonds provide a steady stream of constant income from an investment in the form of interest over time and are liquid (can be redeemed at any time). However, bonds are generally not used as a way to grow the principal investment.

The investment products listed above guarantee that you will earn some interest on your principal. The investment products described below carry a higher element of risk.

The purchase of **equity investments** such as mutual funds or stocks is another way to invest.

- When you buy **stocks** in a company, you become a part owner of that company and are entitled to a share of the company's profits. Companies sell stocks (or shares) in order to generate funds for the company. As the value of the company grows, the value of the stocks increases and, upon selling the shares, a return on the investment is earned. Some companies also pay dividends—as the stocks increase in value, some of that value is returned to each shareholder in the form of a regular payment. It has been shown that over the long run stocks provide higher returns than other investment products, but, if the company does not do well financially, it is possible for investors to lose part or all of their investment. A tool used by investors to assess the trends in the Canadian stock market is the S&P/TSX Composite Index. This index provides a point value that indicates the overall growth or decrease in price of the 300 largest traded companies on the market. The baseline was set at 1000 points in 1977. In 2014, it averaged around 12 000 points.
- A **mutual fund** is an investment option where you entrust the daily decisions about your investment to a professional fund manager and pay a management fee. A mutual fund is a professional investment program where the resources of many investors are pooled and a variety of securities are collectively purchased. The portfolio may contain stocks, bonds, GICs, savings, growth funds, and other assets. This diversity, combined with professional management, removes the responsibility of the individual for making daily decisions about the investment, but also restricts the freedom to make personal choices. Because a mutual fund investment contains a variety of securities, the risk of losing most or all of your investment is minimized, as it is unlikely that all of the different stocks or bonds will experience great losses simultaneously. Each mutual fund is designed to reflect the stated objectives of the investment, and each investor shares equally in the gains and losses of the fund, based on the number of units of the fund he or she holds.

Equity investments such as stocks and mutual funds are considered growth investments with high potential returns. These investments should be thought of as long-term investments, as it may take several years to realize their potential rate of return (ROR).

It may be helpful for you to include a summary of this information on your resource sheet.

Example 3

A 35-year old plumber has the following investment portfolio. Calculate his total annual return for this year and the average rate of return.

Investment	Amount	ROR	Return
Bank Account	$3788	0.5%	
Savings Bond	$2225	1%	
Mutual Fund	$24,000	4.3%	
RRSP	$9150	7.1%	
Totals			

Solution

Investment	Amount	ROR	Return
Bank Account	$3788	0.5%	18.94
Savings Bond	$2225	1%	22.25
Mutual Fund	$24,000	4.3%	1032.00
RRSP	$9150	7.1%	649.65
Totals	$39,163		1722.84

$$\text{ROR} = \frac{1722.84}{39163}$$

$$\text{ROR} = 4.399\%$$

His average rate of return is 4.4%.

The average rate of return is between the lowest ROR and the highest ROR. However, it is not the mean of 0.5, 1, 4.3, and 7.1, so you cannot just add the numbers and divide by 4.

Example 4

Elena has $1000 invested in a variety of stocks that are listed in the S&P/TSX Composite Index group. On September 3rd, the S&P/TSX Composite Index was 12 654 points, and on October 3rd it was 12 847 points. What was the change in value of Elena's portfolio during that month? How much are her shares worth now?

Solution

$$\text{ROR} = \frac{12847 - 12654}{12654}$$

$$\text{ROR} = 1.525\%$$

The value of her portfolio increased by 1.525% to $1015.25.

Risk Versus Return

When considering the different investment options and strategies available, individual investors need to make a decision regarding the risk they are willing to tolerate, balanced with the rate of return they are hoping to achieve. With a "safe" investment, you have a better chance of protecting your original principal, but the guaranteed rate of return is generally lower than with riskier investment options. Investments with a higher potential rate of return often involve a greater chance of losing all or some of your money. Investors are often encouraged to have diversified portfolios to balance these two aspects of the different products available, or to have one core portfolio as the main investment plan and one experimental portfolio.

Long-Term Versus Short-Term Investments

A young investor may have fewer resources but will have more time to grow the principal value of a deposit, compared to someone who is closer to retirement and has the capital but is looking for a secure income from investments. However, if you have the time to gain back potential losses from higher risk investments, your tolerance for risk might be higher than someone who needs to have a secure income from an investment over the short term. The type of investment products someone chooses depends on current income, debt and expenses, long- and short-term goals, and tolerance for risk.

Example 5

The inheritance from a family member is to be shared among a 15-year-old son, a 45-year-old father, and 75-year-old grandfather. If each receives one-third of the amount, describe how each beneficiary might invest their portion differently in terms of cash, fixed income. and/or equity investment products. Justify your choices.

Solution

Answers will vary, but typical trends of investing at different stages of life may include the following percentages for the allocations of resources:

- The 15-year-old has little need for immediate cash, but will have university and living expenses within the next 3 to 10 years. He may be interested in purchasing a car or a house within the next 10 to 15 years, and he could already be creating a savings account for the future. He may want to keep 10% in cash investments (savings accounts or term deposits) so it is accessible in the near future. He may consider investing 50% of it in equities such as mutual funds or certain RRSPs that minimize risk, as he has the time to realize gains on his principal. The return on these investments could be used as a down payment on a house or a car in 10 to 15 years. With the remaining 40%, he could purchase GICs or bonds, which are secure but will, over the next 50 years, give him a solid savings account for retirement. A young investor would likely want to start with a balanced portfolio, emphasizing growth.

- The 45-year-old father likely has more disposable income already available to him, so should invest his portion for maximum growth in anticipation of retirement. He should diversify his investments, possibly investing 75% in equities—stocks that will give him significant growth opportunities over a shorter time frame, 15% in income investments for secure growth, and 10% in cash investments. As he gets older, however, he may consider moving some of the riskier shares into more secure ventures as he begins to plan for retirement. He will begin by building his investment, then move towards saving it.

- The 75-year-old grandfather needs a portfolio that will provide income so he can enjoy his retirement. He may want to keep 20% in accessible cash investments such as a term deposit or savings account to supplement his pension income. He will likely want to preserve the capital or principal amounts and live off the interest earned on investments. He could consider investing about 30% in equities and 50% in safer GICs or bonds that pay out regular amounts of interest earned.

Other Considerations When Investing

You have seen how compound interest grows the value of an investment over time. Inflation works in the opposite way. A dollar today is worth less than a dollar yesterday, and today's dollar has more purchasing power than tomorrow's dollar. If you plan to save $500,000 for when you retire in about 45 years from now, that money will not be able to purchase the same amount as $500,000 today. If the inflation rate is 2%, a ROR for your investment of 2% will not increase the buying power of your money.

If you engage the services of a mutual fund manager or stock market broker, use the services of a bank or financial institution, or hire a financial advisor, you need to be aware of **management fees, commissions, or service charges** deducted from your account. These costs can cut into the rate of return you receive on an investment.

As an investor, you need to take responsibility for your financial future and shop around, get good information from financial advisors, and make investment decisions that reflect your means, goals, and comfort level. Now is a good a time to begin engaging in some research, either online or in person, to determine what sort of investing strategies other people have used, and the advantages and disadvantages associated with those strategies and products. You can then begin to formulate your own goals.

Learning Activity 5.5

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. State the formula for the area of a triangle.
2. State the formula for the volume of a cylinder.
3. State the formula for the surface area of a cube.
4. List all the factors of 72.

continued

Approximately how long will it take an investment of $9274 to double, if compounded annually at the following rates?

5. 4%

6. 8%

7. 3%

8. 12%

Part B: Investment Portfolios

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Adrian thinks he can afford to pay $1000 per month for a mortgage payment for a property that has property taxes worth $205 per month and heating costs estimated at $65 per month. His gross monthly income is $3750 per month. Should he expect the bank to lend him the money?

2. Brett and Megan have a combined annual salary of $80,000. What is the maximum monthly mortgage payment they could afford if property tax is estimated at $180 per month and heating is estimated at $70 per month?

3. Sam and Caitlyn own a $25,000 car with an outstanding car loan of $15,000. They own a home valued at $225,000 with a mortgage of $210,000. They have student loans totalling $5000. What is their net worth?

4. Judith is a 52-year-old accountant who has invested her money in four different options, each with a different risk factor and rate of return. Determine the average rate of return on her investment portfolio.

Investment	Amount	ROR	Return
Bank Account	$8155	0.3%	
Term Deposit	$11,706	1.1%	
GIC	$64,144	2.9%	
Mutual Fund	$149,360	6.8%	
Totals			

continued

5. For a certain investment portfolio, 60% is held in a variety of S&P/TSX Composite indexed stocks and 40% is in a mutual fund for a period of one year. The portfolio was valued at $26,500 at the start of the year, the mutual fund earned 4.8% compounded semi-annually, and the Index value was 12 566 one year ago and today is 11 977. Determine the current value of the portfolio.

6. Ben's stockbroker charges a flat rate of $45 per transaction. If he purchases and sells the following shares at the listed prices, determine his average rate of return on this portfolio. What advice would you give Ben about his investment?

	Purchase Price of Shares	Number of Shares Purchased	Value of Shares	Broker Fee	Change in Price +/−	Selling Price of Shares	Value of Shares	Gain or Loss	Broker Fee
Daily Java Ltd.	58.22	20			+1.50				
Fashionista Co.	1.57	400			−0.12				
Home Life Concepts	3.65	150			+0.39				
Metals Mining	0.88	1800			+0.09				
		Total							

7. Rachel owns 600 units of a mutual fund. The value when she purchased was $5.99/unit. The first year the rate of return was 3.5%; the second year, the rate was 2.9%; and the third year, the rate was 4.1% (compounded annually). At the end of each year, Rachel must pay a management fee on the fund worth 1.1% of her investment. Calculate the value of her mutual fund and the rate of return if she were to sell the units after the third year.

8. Laura has $5000 saved and can contribute $50 per month from her paycheque to either a GIC, earning 4.5%, compounded monthly, or to her outstanding credit card debt of $5000, which has an interest rate of 19.99%, compounded monthly. What would you suggest she do with the lump sum amount and her monthly contributions over the next 5 years? Support your answer with TVM or other software calculations.

9. Wanda would like to go on a trout fishing trip this fall but is a little short on cash. What investment strategy would you suggest she use with her $500 capital to have enough to fund the trip in three months time? Justify your answer.

continued

10. Compare the following portfolios for strengths and weaknesses. If you were a financial advisor, what advice would you give these investors?

    Marin:

Canada Savings Bond	$29,372
Savings Account	$2,855

    Phillip:

Chequing Account:	$3,800
Term Deposit	$9,100
GIC	$5,827
Stocks	$11,945

11. Suli has the following investment portfolio. Determine the total value of her portfolio and her average ROR after one year.

    - Stocks

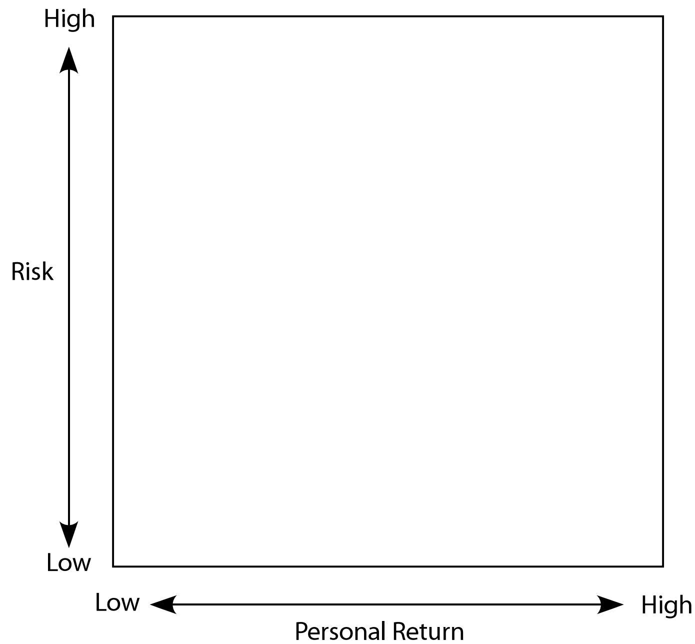
	Stock Purchase Price	Number of Shares Purchased	Value of Shares	Change in Price (+/–) Over One Year	Selling Price of Stock	Value of Shares
Gold E Corporation	1.22	542		+0.79		
Space Science Ltd.	6.46	120		–0.05		
Cell Technology	0.53	847		+1.04		
Best Insurance Co.	27.80	100		+0.63		
		Total				

- Mutual Fund: $13 400 currently invested at 7%, compounded annually.
- Savings account: $3900 at 0.05%, compounded monthly. She saves about $125 per month in this account (after expenses and liabilities are paid).
- Employee RRSP: $20,800 currently invested at 3.5%, compounded semi-annually. She contributes $50 per month.

continued

Learning Activity 5.5 (continued)

12. Place "cash investments," "fixed income investments," and "equity investments" on the chart below, indicating where, in general, they fall in terms of the degree of risk and the potential for growth on return.



Lesson Summary

In this lesson, you solved investment problems. You used technology to determine possible investment strategies to achieve a financial goal. You considered the advantages and disadvantages of long-term and short-term investment options, and compared the strengths and weaknesses of two or more portfolios.

Lesson 6: Buying, Renting, and Leasing

Lesson Focus

In this lesson, you will

- ❑ identify and describe examples of assets that appreciate or depreciate
- ❑ compare, using examples, renting, leasing, and buying
- ❑ justify, for a specific set of circumstances, if renting, buying, or leasing would be advantageous
- ❑ solve, using technology, a contextual problem involving renting, leasing, or buying, and complete a cost-and-benefit analysis

Lesson Introduction

You have looked at factors you need to consider to determine how much money you may be able to borrow from a bank for a mortgage if you decide to purchase a home. Another large purchase that you may consider in the future is buying a vehicle. For both of these items, many people prefer other options instead of purchasing. For example, some people prefer to rent an apartment rather than own a home, and many drivers would rather lease their vehicle than purchase it. In this lesson, you will learn about some of the advantages and disadvantages of renting versus buying a home and of leasing versus buying a vehicle.

Buying or Renting a Home

You have seen how mortgages function and have calculated some of the additional costs of purchasing a home. After the initial set up of a home with furniture and appliances, which may result in credit card debt, homeowners are responsible for many other expenses. In addition to their mortgage payments (consisting of principal plus interest), there are monthly bills to pay, including the telephone, Internet, television, utility bills (electricity to run the lights and electronics and hydro or gas for heat), home insurance, renovations, repairs, and many others as well. In addition, homeowners must pay annual property taxes that cover fees for garbage collection, street upkeep, and other municipal services. Home ownership can seem like a "money pit" where owners keep throwing more and more money "away." So why do they do it?

Appreciating Assets

People need a place to live and the housing market functions as a long-term investment option or strategy. The value of a home typically appreciates or grows, over time. This means you can buy a home, live in it, and then sell it for more than the original purchase price, and in that way see a return on your investment.

However, not everyone wants to, or is able to, purchase a home. Renting is another option for accommodation. Rental options vary and there are financial and other factors to consider when deciding whether to rent or buy.

Regular upkeep of the grounds in an apartment complex is often the responsibility of the landlord, so renters may not have to shovel snow or mow a lawn. The flip side is that renters often do not have private outdoor areas to enjoy and there may be restrictions on having pets. Neighbours in an apartment live in close proximity, so privacy may be an issue for some tenants. Sometimes the monthly rental cost includes utilities, but a parking or recreation fee might be added.

Furnished apartments may be available in some locations, which would cut down on initial costs, but insurance on your personal belongings would be extra. A damage deposit, often equal to one month's rent, is usually required when signing a lease (rental agreement), but this is less than the down payment required to purchase a home. If the apartment is well cared for and clean when you move out, the damage deposit is returned, while the down payment becomes part of the equity in the home.

This brings you to what is likely the biggest financial factor that must be considered when deciding whether to rent or buy a home. Rental payments never create any assets. When you move out of an apartment, you get no return on the rental amounts you have paid. Rental payments are not an investment. There is no appreciation or growth; it is just an expense.

It may be helpful for you to include some of this information on your resource sheet.

Example 1

Indicate which of the following additional costs are incurred with the purchase of a home and/or the rental of an unfurnished property.

Cost	Buy	Rent	Cost	Buy	Rent
Land Transfer Tax			Home insurance		
Mortgage application fee			Tenant insurance		
Appraisal fee			Land survey fee		
Decorating costs			Inspection fee		
Property tax			Adjustments to interest, property tax, utilities		
Mortgage insurance			Monthly utility fees		
Recreation facility fee			Parking fee		
Moving costs			Damage deposit		
Furniture			Appliances		

Solution

Cost	Buy	Rent	Cost	Buy	Rent
Land Transfer Tax	X		Home insurance	X	
Mortgage application fee	X		Tenant insurance		X
Appraisal fee	X		Land survey fee	X	
Decorating costs	X	X	Inspection fee	X	
Property tax	X		Adjustments to interest, property tax, utilities	X	
Mortgage insurance	X		Monthly utility fees	X	X
Recreation facility fee		X	Parking fee		X
Moving costs	X	X	Damage deposit		X
Furniture	X	X	Appliances	X	

Note: Furniture and appliances may or may not be included with a rental property. Depending on the terms of the purchase offer, they may or may not be included with the purchase of a home.

Is it better to rent or buy? Most financial advisors would probably say that it is better to invest in an appreciating asset such as a home rather than rent, but this is not always possible or convenient. As with any investment, you must consider your short- and long-term goals, and balance these against the costs and risks involved when buying or selling.

- A student moving to a new city to go to university for four years, but planning on returning home to work for the summers, may find it more economical to rent an apartment for the eight months of the year when he or she needs a place to live. The student may have to move twice a year, but he or she will not have to pay for a mortgage during the summers.
- A couple employed by the Canadian Armed Forces, stationed in a new city for a four-year placement, might find it more economical to purchase a place in a high-demand neighbourhood. The chances of being able to sell it quickly when transferred to a new location make it a lucrative option.
- A retired couple on a fixed income might be happier with an all-inclusive rental payment compared to making a mortgage payment, plus paying for utilities and repair costs.
- A recent university graduate may not have the liquid assets to come up with a down payment, and so may need to rent for a few years in order to save up to purchase a home.
- An employee required to travel frequently to another city may consider purchasing a small condo in one city and a home in another, in an effort to save hotel costs.

Example 2

You have the opportunity to purchase a home valued at $245,000 with a down payment of $43,000 or rent a similar home for $1550 per month.

a) If the mortgage is amortized over 20 years at 8.5%, compounded monthly, use technology to determine the monthly mortgage payment.

b) If property taxes work out to approximately 1.5% of the market value of the property, how much will you spend on the mortgage and taxes during the first year of the purchase?

c) What is the outstanding balance on the mortgage after one year?

d) If the home appreciates in value by 4% per year, what is the value of the home after one year?

e) What is the owner's equity after one year?

f) If you decide to rent a similar house for the $1550 per month, how much of your income would you spend on housing during the first year?

g) While renting, you take the $43,000 that you would have used as a down payment to purchase the home, and invest it at a rate of 7%, compounded annually. What is the value of the investment after the first year?

h) What is the renter's equity after one year?

i) Create a spreadsheet to calculate, after any number of years of ownership, the following:

- total mortgage payments and amounts paid towards mortgage interest and principal
- appreciated value of the home (and amount of increased equity in home)
- cost of property taxes
- total amount of rent paid in a year
- cost of monthly rent if rent increases at a rate of 5% per year
- value of the investment

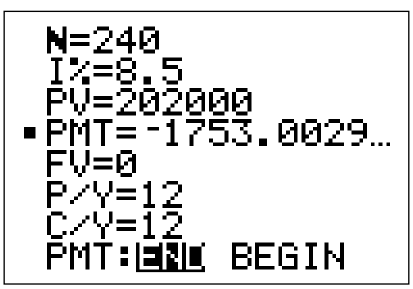
Note: You may want to save this spreadsheet as a template for future investigations.

j) Summarize the information gathered to determine when renting is more advantageous and when purchasing is more advantageous by comparing the buying versus renting costs and equity for the first three years. Be sure to take into consideration the following:

- factors when purchasing: amount of mortgage payment, equity in the property, increase in the value of the home, property tax
- factors when renting: rental increases, investment of the down payment

Solution

a) The monthly mortgage payment will be $1753.



b) $0.015 \times 245000 = 3675$

Property taxes will be $3675 for the first year.

The mortgage costs for the first year would be 12 × 1753 = $21,036.

Mortgage plus taxes for the first year would be $24,711.

c) The balance on the mortgage after the first year is $197,979.74.

```
bal(12)
     197979.737
```

Or, use TVM solver to find future value after 12 payments.

```
  N=12
  I%=8.5
  PV=202000
  PMT=–1753*0029
■ FV=197979.737
  P/Y=12
  C/Y=12
  PMT: END BEGIN
```

d) 1.04 × 245 000 = 254 800

e) The home is valued at $254,800 and a balance of $197,979.74 is still owing. The owner's equity is the difference, $56,820.26.

The home will be valued at $254,800 after the first year.

f) 12 × 1550 = 18 000

You would spend $18,000 on rent.

g)

```
  N=1
  I%=7
  PV=43000
  PMT=0
■ FV=-46010
  P/Y=1
  C/Y=1
  PMT:END BEGIN
```

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$A = 43000\left(1 + \frac{0.07}{1}\right)^{(1)(1)}$$

$$A = 43000 \times 1.07$$

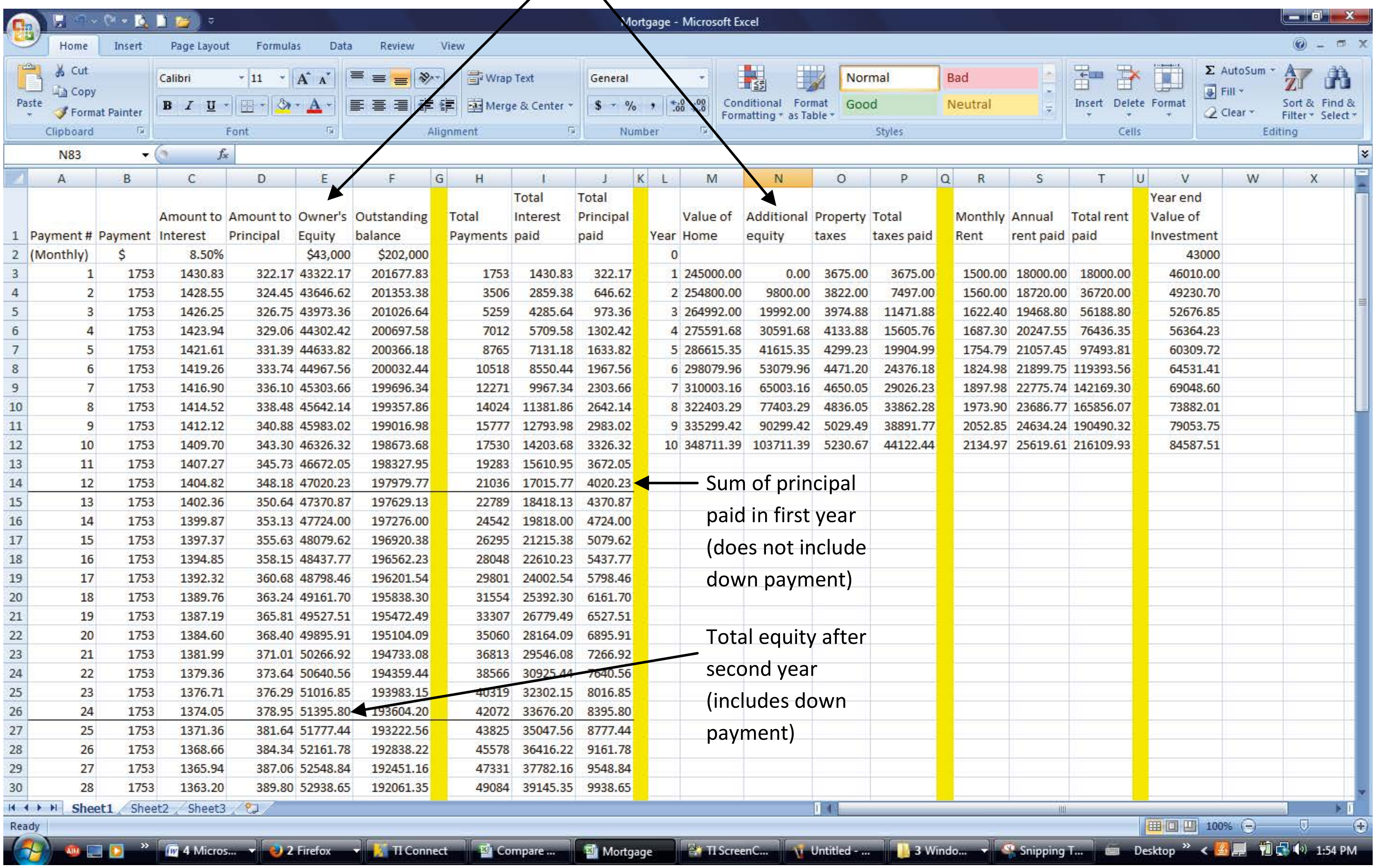
$$A = 46010$$

The investment would be worth $46,010.

h) $0.00

i)

Equity at the end of each year is the total of the value in column E and the value in column N.



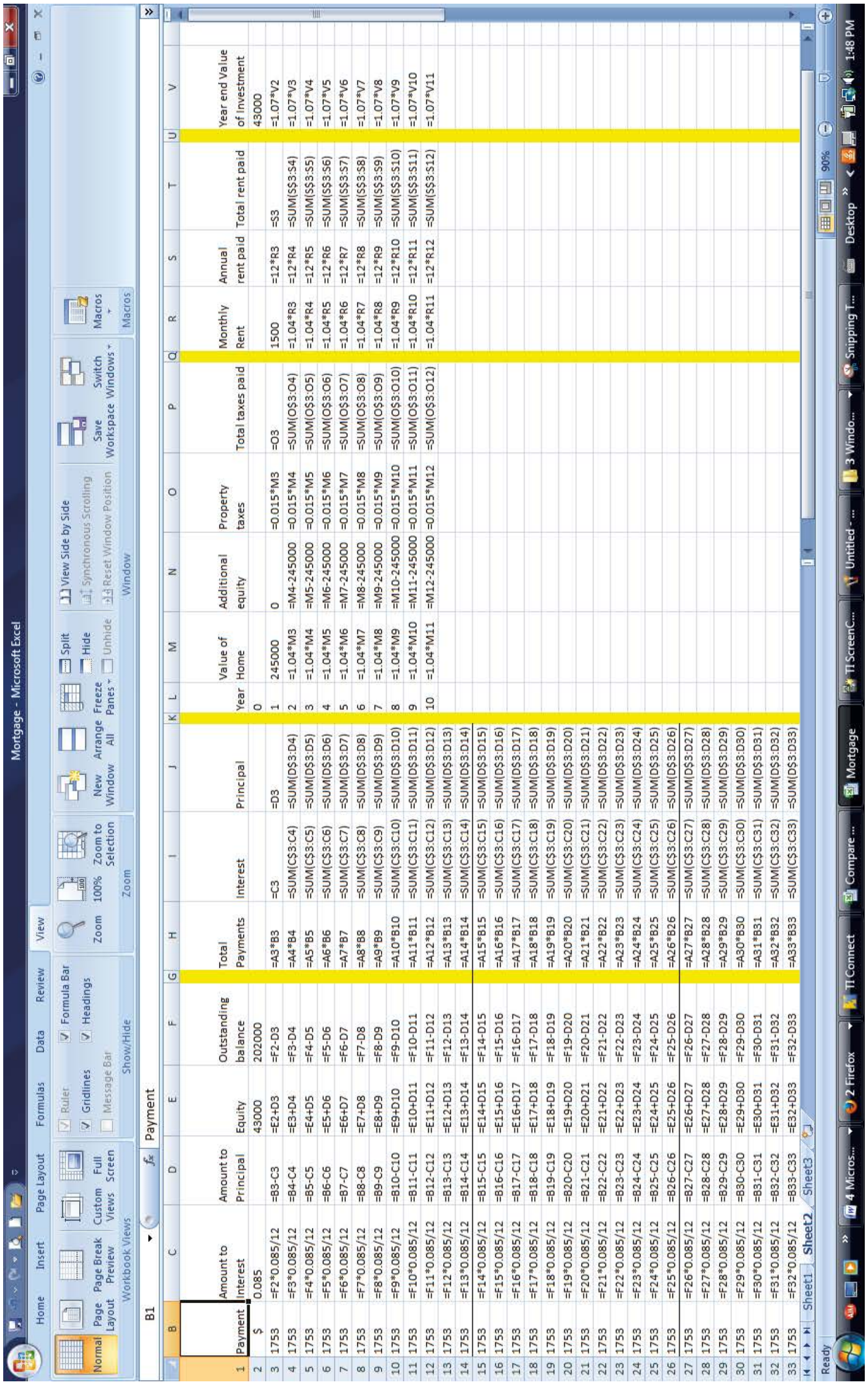
Row	Payment # (Monthly)	Payment $	Amount to Interest	Amount to Principal	Owner's Equity	Outstanding balance	Total Payments	Total Interest paid	Total Principal paid
2			8.50%		$43,000	$202,000			
3	1	1753	1430.83	322.17	43322.17	201677.83	1753	1430.83	322.17
4	2	1753	1428.55	324.45	43646.62	201353.38	3506	2859.38	646.62
5	3	1753	1426.25	326.75	43973.36	201026.64	5259	4285.64	973.36
6	4	1753	1423.94	329.06	44302.42	200697.58	7012	5709.58	1302.42
7	5	1753	1421.61	331.39	44633.82	200366.18	8765	7131.18	1633.82
8	6	1753	1419.26	333.74	44967.56	200032.44	10518	8550.44	1967.56
9	7	1753	1416.90	336.10	45303.66	199696.34	12271	9967.34	2303.66
10	8	1753	1414.52	338.48	45642.14	199357.86	14024	11381.86	2642.14
11	9	1753	1412.12	340.88	45983.02	199016.98	15777	12793.98	2983.02
12	10	1753	1409.70	343.30	46326.32	198673.68	17530	14203.68	3326.32
13	11	1753	1407.27	345.73	46672.05	198327.95	19283	15610.95	3672.05
14	12	1753	1404.82	348.18	47020.23	197979.77	21036	17015.77	4020.23
15	13	1753	1402.36	350.64	47370.87	197629.13	22789	18418.13	4370.87
16	14	1753	1399.87	353.13	47724.00	197276.00	24542	19818.00	4724.00
17	15	1753	1397.37	355.63	48079.62	196920.38	26295	21215.38	5079.62
18	16	1753	1394.85	358.15	48437.77	196562.23	28048	22610.23	5437.77
19	17	1753	1392.32	360.68	48798.46	196201.54	29801	24002.54	5798.46
20	18	1753	1389.76	363.24	49161.70	195838.30	31554	25392.30	6161.70
21	19	1753	1387.19	365.81	49527.51	195472.49	33307	26779.49	6527.51
22	20	1753	1384.60	368.40	49895.91	195104.09	35060	28164.09	6895.91
23	21	1753	1381.99	371.01	50266.92	194733.08	36813	29546.08	7266.92
24	22	1753	1379.36	373.64	50640.56	194359.44	38566	30925.44	7640.56
25	23	1753	1376.71	376.29	51016.85	193983.15	40319	32302.15	8016.85
26	24	1753	1374.05	378.95	51395.80	193604.20	42072	33676.20	8395.80
27	25	1753	1371.36	381.64	51777.44	193222.56	43825	35047.56	8777.44
28	26	1753	1368.66	384.34	52161.78	192838.22	45578	36416.22	9161.78
29	27	1753	1365.94	387.06	52548.84	192451.16	47331	37782.16	9548.84
30	28	1753	1363.20	389.80	52938.65	192061.35	49084	39145.35	9938.65

Row	Year	Value of Home	Additional equity	Property taxes	Total taxes paid	Monthly Rent	Annual rent paid	Total rent paid	Year end Value of Investment
2	0								43000
3	1	245000.00	0.00	3675.00	3675.00	1500.00	18000.00	18000.00	46010.00
4	2	254800.00	9800.00	3822.00	7497.00	1560.00	18720.00	36720.00	49230.70
5	3	264992.00	19992.00	3974.88	11471.88	1622.40	19468.80	56188.80	52676.85
6	4	275591.68	30591.68	4133.88	15605.76	1687.30	20247.55	76436.35	56364.23
7	5	286615.35	41615.35	4299.23	19904.99	1754.79	21057.45	97493.81	60309.72
8	6	298079.96	53079.96	4471.20	24376.18	1824.98	21899.75	119393.56	64531.41
9	7	310003.16	65003.16	4650.05	29026.23	1897.98	22775.74	142169.30	69048.60
10	8	322403.29	77403.29	4836.05	33862.28	1973.90	23686.77	165856.07	73882.01
11	9	335299.42	90299.42	5029.49	38891.77	2052.85	24634.24	190490.32	79053.75
12	10	348711.39	103711.39	5230.67	44122.44	2134.97	25619.61	216109.93	84587.51

Sum of principal paid in first year (does not include down payment)

Total equity after second year (includes down payment)

The formulas for this spreadsheet:



j) The first year of home ownership or rental is costly in both scenarios, but more so in the case of the purchase. The $43,000 amount is either put towards the equity in the home or equity in the investment, but since the annual mortgage payments exceed the cost of renting for the year, and there is no property tax with the rental scenario, the rental option appears better for the first year (see the chart below).

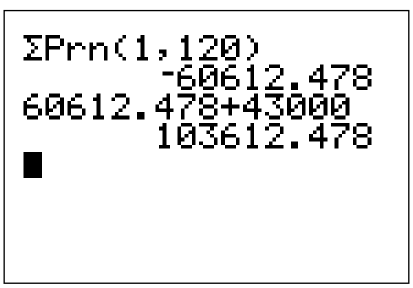
In the second year, the difference is balanced out as the equity in the home increases by an amount that exceeds the return on the investment. In addition, the mortgage payments are split between principal and interest, while the rental payments generate no additional equity for the renter. Even with the cost of property tax, the purchase scenario is more advantageous than renting.

In the third and subsequent years, the renter's investment would continue to grow, but since the increasing rental payments are not contributing to the renter's overall equity, while the portion of the constant mortgage payment amount going towards the principal continues to go up, the homeowner's equity quickly surpasses the renter's equity. If you are planning on staying in the home for several years, purchasing is the better option for the long term. Note that closing costs, furniture, maintenance, and utility costs are not factored into the calculations.

	Purchasing:		Renting:	
Costs in year 1	Down payment	$ 43,000.00	Investment	$ 43,000.00
	Mortgage payments	$ 21,036.00	Rent	$ 18,000.00
	Property tax	$ 3,675.00		
	Total	$ 67,711.00		$ 61,000.00
Credits in year 1	Principal invested	$ 43,000.00	Investment	$ 46,010.00
	Owner's equity	$ 4,020.26		
	Total	$ 47,020.26	Total	$ 46,010.00
Difference	$47,020.26 − $67,711.00 =	−$20,690.74	$46,010 − $61,000.00 =	−$14,990.00
Costs in year 2	Mortgage payments	$ 21,036.00	Rent	$ 18,720.00
	Property tax	$ 3,822.00	Investment	46,010
	Total	$ 24,858.00	Total	$ 64,730.00
Credits in year 2	Owner's equity	$ 51,395.80	Investment	$ 49,230.70
	Increased equity in value of home	$ 9,800.00		
	Total	$ 61,195.80	Total	$ 49,230.70
Difference	$61,195.80 − $24,858.00 =	$ 36.337.80	$49,230.70 − $64,730.00 =	−$15,499.30
Costs in year 3	Mortgage payments	$ 21,036.00	Rent	$ 19,468.80
	Property tax	$ 3,974.88	Investment	$ 49,230.70
	Total	$ 25,010.88	Total	$ 68,699.50
Credits in year 3	Owner's equity	$ 56,158.14	Investment	$ 52,676.85
	Increased equity in value of home	$ 19,992.00		
	Total	$ 76,150.14	Total	$ 52,676.85
Difference	$76,150.14 − $25,010.88 =	$ 51,139.26	$52,676.85 − $68,699.50 =	−$18,022.65

Alternatively, you could summarize by finding the total amount paid over the course of a certain number of years in each scenario and compare the results. Here is a summary of the results after 10 years.

	Buying		**Renting**	
Costs after 10 years	Down payment Mortgage payments (1753 × 12 × 10) Taxes Closing costs Utilities	\$ 43,000.00 \$ 210,360.00 \$ 44,122.44	Investment principal Rent Utilities Parking or fees	\$ 43,000.00 \$ 216,109.93
Total		\$ 297,482.44		\$ 259,109.93
Credits	Owner's equity Appreciation in value of home	\$ 103,612.48 \$ 103,711.39	Value of investment	\$ 84,587.51
Total		\$ 207,323.87		\$ 84,587.51
Difference		\$−90,158.57		\$ −174,522.42



Owner's equity after 10 years from when the home is purchased is a total of the principal paid plus the down payment. You can use the spreadsheet (fill down to month 120 for the 10-year totals), or you can use a financial calculator or app. The answers may vary slightly due to rounding. This value does not include the additional equity gained by the 4% appreciation in the value of the house each year.

You can see that after 10 years there is a significant cost for living in a home or apartment, no matter if you rent or buy. When you compare the total amount paid for rent with the cost of mortgage payments, you can see that rent has exceeded the mortgage payments, but there is no property tax when renting. The amount of equity after 10 years is significantly higher for the homeowner. By putting some of the monthly rental payment towards the equity in a home, purchasing property becomes a good investment. Once the house is paid off, it should keep appreciating in value and the owners may be able to sell it to realize a return. Keep in mind that there are many other factors that play into a decision to buy or rent.

It may be helpful for you to look back at the examples and include a summary of information on your resource sheet.

Buying or Leasing a Vehicle

One of the first major purchases a young person makes may be the acquisition of a vehicle. The decision to purchase a certain make or model, new or used vehicle is often made based on personal preference, the nature of intended use, and financial constraints. A fuel-efficient compact car, a heavy-duty truck, a motorcycle, or a cross-over vehicle with seating for seven are all options on the market today.

Depreciating Assets

It has been said that the moment you drive your new car off the dealer's lot it has lost 20% of its value. **Depreciation** is the loss of value in an asset over time. Wear and tear, age, and unfavorable market conditions all contribute to items becoming less valuable over time. For this reason, purchasing a vehicle is rarely, if ever, considered an investment. It is important then to consider ways to minimize the costs associated with purchasing a depreciating asset.

Having a vehicle is often considered a necessity by many people, as it allows them the freedom to go when and where they would like to go, without depending on transit schedules, paying taxi fares, or fighting the elements while riding a bike or walking. In some professions, it is a requirement of employment as it may be essential to doing the job effectively. For others, it can be considered a business expense alongside other expenses like the leasing of heavy equipment or electronics. For some, however, car ownership might be unaffordable. Fortunately, ownership is only one option in meeting people's transportation needs.

An alternative to purchasing a vehicle is **leasing** a motor vehicle. Leasing a vehicle is like signing a rental agreement for a car or truck. The lessee (client) signs a contract stating that he or she will pay the lessor (owner) a set amount for the use of the asset. At the end of the contractual time period (usually between two and four years), the vehicle is returned to the owner. The payment covers the cost of depreciation, taxes, and interest on the outstanding balance of the full-purchase price. The amount of the payment depends on the length of the lease, the residual (or lease-end) value of the item, and the interest rate offered for the lease. A down payment may or may not be required with a lease. All these factors contribute to the total cost of the lease, which may or may not be less than the cost to purchase.

The advantages of leasing:

- You can drive a new motor vehicle with regular payments that might be lower than loan payments. You may be able to lease a car you could never afford to purchase.
- The maintenance costs are likely low with a lease, as warranty will typically cover any major repairs during the time of the lease.
- At the end of the lease, the vehicle is either purchased outright by the lessee (at a price determined before the lease is signed) or simply returned to the owner. The driver does not have to worry about selling the vehicle at its depreciated value or have concern about its future value. They simply pick a new vehicle to lease and continue the payments.
- Businesses and professionals may be able to claim the lease payments as a business expense.

There is a downside to leasing as well:

- During the term of the lease, the driver is committed to the vehicle and will be charged high termination fees if he or she wants to get out of the lease.
- The payments they have made do not translate into any equity or asset acquisition, and do not help build your credit rating the way a loan payment would.
- Lease agreements typically limit the number of kilometres that you can accumulate on the vehicle and a per-kilometre fee is charged for each additional kilometre. As well, there is a fee for any wear and tear on the vehicle that is seen as being beyond normal wear and tear, such as scratches, dents, or upholstery damage. You cannot customize a leased vehicle with aftermarket parts.
- Other charges may include a lease acquisition fee, a disposition fee, and a refundable damage deposit.

When is it better to lease or buy a vehicle? Again, it depends on personal preference, long- and short-term goals, available finances, and other considerations.

It may be helpful for you to include some of this information on your resource sheet.

Example 3

Carson is considering trading in his old car for a new one. He sees one listed for $19,995. He has $1700 saved for a down payment and the dealer offers him $1200 for the trade-in. He can finance the remainder over four years with bi-weekly payments at 4.3%, compounded monthly.

Find:

a) his bi-weekly payment amount

b) the total paid for the car

c) the interest paid for the car

Solution

GST (5%) and PST (8%) are charged on the price of the vehicle, less the trade-in allowance (tax has already been paid on the used vehicle), and then the down payment is subtracted.

$19,995 − $1200 = $18,795	Taxes are calculated on $18,795, the price after trade-in
$18,795 × 0.13 = $2443.35	13% taxes
$2443.35 + 18,795 = $21,238.35	Total purchase price is $21,238.35
$21,238.35 − $1700 = $19,538.35	Subtract down payment to calculate the total amount financed

a) The bi-weekly (26 payments per year) payment for the four-year term will be $204.63.

```
 N=104
 I%=4.3
 PV=19538.35
▪PMT=-204.626724
 FV=0
 P/Y=26
 C/Y=12
 PMT:END BEGIN
```

b) The total amount paid can be calculated by multiplying the payment amount by the number of payments, plus the trade-in allowance and down payment:

104 × $204.63 = $21,281.52

$21,281.52 + $1200 + $1700 = $24181.52

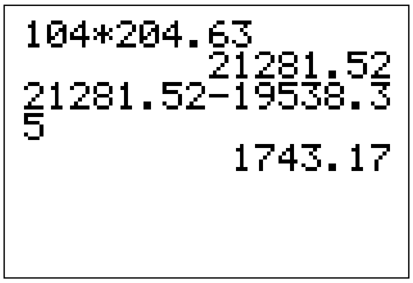
The total amount paid for the car is $24,181.52.

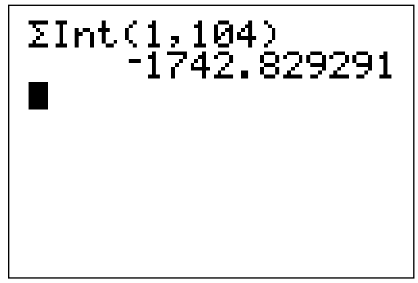
c) The amount of interest is calculated by subtracting the principal amount from the sum of payments:

$21,281.52 − $19,538.35 = $1743.17

Carson pays about $1743.17 in interest.

The TVM solver `Sum of Interest` function may give a slightly different answer based on the number of decimal places used (as shown in the screenshot on the right).





Example 4

If Carson leased the car in the above example for four years, he would make his down payment of $1700 plus trade-in to cover fees and the first month's payment. His subsequent monthly payments would be $295 (includes taxes). The residual value of the car after four years would be 45% of the purchase price. This is the value of the car after the leasing period and the price he could purchase the vehicle for at the end of the lease. Determine the cost of leasing the vehicle for four years and the residual value of the car.

Solution

Carson would make an initial down payment of $1700 plus receive $1200 for the trade-in, and then make 47 monthly payments of $295.

$1700 + $1200 + 47 × $295 = $16,765	Carson would pay $16,765 over the four years.
0.45 × $19,995 = $8,997.75	The residual value of the car.

He would be able to purchase the car for $8,997.75 plus taxes after the lease is up. To own the car at that point, he would have paid $16,765.00 + $8,997.75 = $25,762.75, which is a little more than the total in Example 3.

Learning Activity 5.6

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Using each of the digits 4, 6, 8, and 8, some of the four common mathematical operations, and brackets, create an expression equal to 40.

2. Raj buys 10 bags of oranges, each of which contains 20 oranges. If his family eats 8 oranges a day, how many days will it take him to eat the 10 bags of oranges?

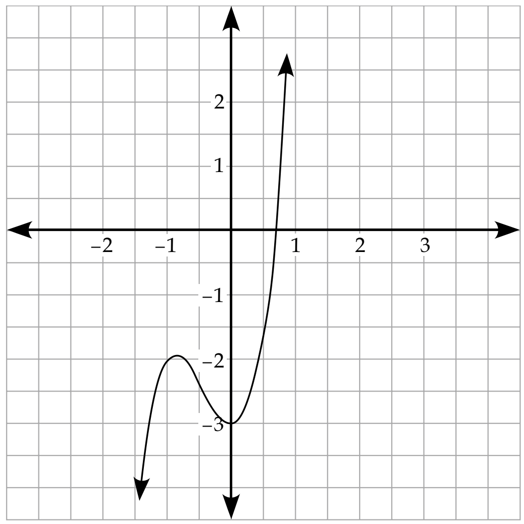
3. A set of integers has a sum of 240 and a mean of 30. How many integers are in the set?

4. Approximate the temperature 82 °F in °C, given the conversion formula, $C = \frac{5}{9}(F - 32)$.

5. Saturn and Morweena are playing six games of tennis against each other. The probability that each of them will win three of the 6 games is $\frac{4}{9}$. What is the probability that one of them will win more games than the other?

continued

6. Describe the end behaviour of the following polynomial graph.



7. Write 1 as a power of 53.

8. Write the converse of $p \rightarrow q$.

Part B: Buying, Renting, and Leasing

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. The residual value of a luxury car is about 40% after a four-year lease. If the purchase price was $84,600, determine its residual value. Calculate the depreciation.

2. Arthur has office equipment valued at $15,000. This year, he can write off depreciation of 7% of the equipment's value as a business expense. Determine the amount that can be claimed as a business expense.

3. The value of a property appreciates at a rate of 3.5% per year. If the property is purchased for $189,000 and then sold 15 years later, what could the home owners propose as a selling price?

4. Rare artwork appreciates in value by about 1.8% per year. Jody's grandmother purchased a painting 70 years ago for $1000. Determine its value now.

continued

5. The value of a mountain bike is calculated to depreciate to $0 over 5 years, using a "straight line method." In this case, the value of the bike decreases by one-fifth of its original value each year. If the bike was purchased for $650, how much does it decrease in value each year?

6. A computer depreciates in value by about 25% per year. Ross spends $800 on a new computer for university. How much is it worth four years later? Calculate the depreciation.

7. The Browns have an opportunity to purchase a retirement home for $50,000, with a down payment of $5000, or rent a similar home for $525 per month.
   a) Use technology to determine their monthly payment for the mortgage if it is negotiated to be 7.5% for 15 years (monthly amortization).
   b) If the property tax works out to approximately 1.5% of the market value of the property, how much will they spend on the mortgage and taxes during the first year of the purchase?
   c) Assuming the value of the home has not changed, how much equity will they have in their home after one year?
   d) If the value of the house appreciates by 2% per year, what is the increase in value of the house after one year? How does this affect their equity?
   e) If instead, they decide to rent a similar house for $525 per month, how much of their income will they have spent on rent during the first year?
   f) If the Browns decide to rent, they would take the $5000 that would have been the down payment on the purchase and invest it in a mutual fund at 7%, compounded annually. How much would the investment be worth after one year?
   g) Create a spreadsheet to determine, after any number of years:
      - the value of the home if it appreciates by 2% per year
      - the taxes due (1.5% of market value of the home)
      - the total amount of rent paid, if the rent increases by 3% per year
      - the value of an investment of $5000 at 7% compounded annually
   h) Summarize the information gathered to determine when renting is more advantageous and when purchasing is more advantageous.

continued

8. Neta gets a new job but must relocate for five years. She can either rent a furnished apartment or purchase a condominium. She has $36,000 equity from the sale of her previous home to be used as a down payment on the purchase of the condo listed at $199,000 or she can invest it in a term deposit at 5% compounded annually. Neta is offered a mortgage at 5.5%, compounded and paid monthly, amortized over 15 years. The condo appreciates in value at 4% per year. Property taxes and condo fees are 1.8% of the property value each year. If she buys the condo, she will need to add about 3% of the purchase price to the mortgage to cover closing and moving costs, and to purchase furniture. Utilities in the condo are about $500 per month and will increase by 2.2% annually. In the apartment, she must pay rent of $975 per month, including utilities and parking. Rent is expected to increase by 2.2% per year.

   Use technology to compare the costs of renting the apartment with the cost of buying the condo, including the impact of using her $36,000.00 equity as either a down payment or an investment. What recommendations would you make to Neta, on the basis of this comparison, for her five-year relocation?

9. Bart has been accepted into a four-year residency program for his medical studies in British Columbia. He can rent an apartment, buy a small fixer-upper house, or stay in a "room and board" dorm. What factors should he consider when making his housing decision?

10. Guillaume is shopping for a new vehicle. He can purchase a new truck for $38,000, including taxes, with a down payment of $4000 and financing at 2.99% for four years for the remainder. Alternatively, he could lease the truck for the down payment of $4000, plus $536 per month for 48 months. The residual value of the truck after four years is 45%. Compare his options by calculating the costs entailed in buying versus leasing this vehicle.

11. Nehal needs to buy a car. She has $3700 saved for a down payment. She is considering either leasing or buying an economical car priced at $12,999, plus 13% tax. With monthly payments, she can finance the car at 1.75% for 36 months or lease it for three years for $175, plus 13% tax per month. The residual value after 36 months is 50%. When the lease is up, Nehal would plan to buy the car with financing at 2.99% over two years. How much would either option cost her? What factors, other than total cost, may influence Nehal's decision?

continued

Learning Activity 5.6 (continued)

12. Derksen Accounting Ltd. upgrades their computer system every two years. If Mr. Derksen orders $5000 worth of new equipment, he could put the total amount on his business credit card and pay it off over two years at 19.99%, compounded monthly. Alternatively, he could lease the system with monthly payments of $155 per month. Taxes of 13% must be added to the purchase price and lease payment amount. Compare the options by calculating the costs associated with each, and make a recommendation to Mr. Derksen about the computer system.

13. Dawson would like to drive a new hybrid car. There is one at a local auto dealer for $25,000. He could trade in his current car for $7000 and he has $2000 saved in a term deposit. The dealership is offering financing at 5% for five years (compounded monthly, monthly payments). Taxes are 13%.

    Find:

    a) his monthly payment amount

    b) the total paid for the car

    c) the interest paid for the car

    d) If Dawson leases the hybrid car for four years, his monthly payment (after the down payment and trade-in) would be $325 plus 13% tax. He would have to pay an acquisition fee of $450 to obtain the lease. The residual value of the car after four years would be 55% of the purchase price. Determine the cost of leasing the vehicle for four years and then financing the residual value of the car over two years at 6%, compounded monthly.

Lesson Summary

In this lesson, you identified and described examples of assets that appreciate or depreciate and compared, using examples, renting, leasing, and buying. You used technology to help you to determine when buying, renting, or leasing might be more advantageous.

Assignment 5.3

Investment Decisions

Total: 53 marks

This is a hand-in assignment. Please show your work clearly and in an organized manner. Round final answers to 2 decimal places, and include units, if appropriate. If you use technology as a strategy in your solution steps, please indicate what application you are using, the values you input, and a sketch or printout of the results. Answers given without supporting calculations will not be awarded full marks.

1. Kenny earns a gross salary of $45,000 per year and wants to buy a house. He thinks he can afford a mortgage payment of $1100 per month. The annual property taxes for a house he is looking at are $2700 per year and the owners say they pay $75 per month for heating the house. Based on his Gross Debt Service ratio, should Kenny expect the bank to grant him the mortgage? Justify your answer. *(3 marks)*

Assignment 5.3: Investment Decisions (continued)

2. Kaydee and Amanda own a house valued at $248,000 and have a mortgage on the property of $178,000. They have paid off their car worth $8000. However, they still owe $3500 on a line of credit at the bank. Based on their Debt to Equity ratio, would the bank give them a loan of $22,000 to buy a new car? *(3 marks)*

3. Kevin receives an income tax return from the government and decides to invest it for the next 12 months. He could either
   - purchase additional shares in a company in which he currently has stock. (The stock has been showing an average growth of 3.6% over the past four years, but is currently selling at −0.12% of the initial purchase price for this year.)
   - purchase a term deposit at 0.75%.

   What do you suggest he do? Explain. *(3 marks)*

Assignment 5.3: Investment Decisions (continued)

4. Explain a connection between long- and short-term investments and return. *(2 marks)*

5. How are risk and return related in investments? *(2 marks)*

Assignment 5.3: Investment Decisions (continued)

6. The return on Melanie's portfolio varies over 3 years, realizing 0.6%, 3.4%, and 8.1% (compounded monthly) per year respectively. If she has an initial investment of $2275 and contributes $40 per month, determine the value of her investment at the end of each year and the rate of return over the three years. *(4 marks)*

Assignment 5.3: Investment Decisions (continued)

7. Bree would like to purchase a snowboard, boots, and bindings for $476.95. She has $300 left in her savings account, and can afford to save $20 per week in a GIC, which earns 4.8% compounded weekly. Her savings account earns 0.6% compounded monthly. What do you suggest she do? How long will it take for her to earn enough to purchase the snowboard? *(2 marks)*

8. Paul and Tabitha are saving to go on a trip to Greece. They would like to go two years from now. They plan to save $300 per month in a savings bond that earns 5.1%, compounded monthly. They have a mutual fund that has been earning 8.5%, compounded semi-annually, to which they will contribute a lump sum, in order to save the remaining amount needed.

   a) Calculate the value of their savings bond after two years. If they estimate that they need $9000 to cover airfare, hotels, food, and transportation costs, how much will they still require to finance their trip? *(2 marks)*

   b) What amount will they need to contribute to their mutual fund so that they can cover the remaining trip in two years? *(1 mark)*

Assignment 5.3: Investment Decisions (continued)

9. For the past 12 years, Betsy-Lou has had $75 deducted each month from her paycheque and deposited into a Canada Savings Bond, earning 4.9%, compounded monthly. For the past 12 years, she has also had a mutual fund that has earned 3.7%, compounded monthly, into which she initially invested $8000.

   a) What is the current value of her portfolio? *(3 marks)*

   b) What is her rate of return over the 12 years? *(2 marks)*

10. When Josephine was born, her parents opened a trust account for her and deposited $100 per month for the first six years of her life. It has earned an average of 5% in interest, compounded monthly. On her sixth birthday, they invested the entire amount in the trust account into a mutual fund, where the interest was compounded annually. Today, Josephine is 18 years old and the investment is worth $18,343.14. What was the interest rate on the mutual fund? *(3 marks)*

Assignment 5.3: Investment Decisions (continued)

11. Compare the investment portfolios of twin brothers Carlos and Juan, 30-year-old IT specialists. Make suggestions of how they may improve their portfolios. *(3 marks)*

    Carlos:

    $16,000 invested in the stocks of five different companies: PetrolCanada, Gold Core, A&F Clothing, Nano Tech, FiveBucks Coffee Co.

    $4000 held in a mutual fund

    $5000 in a savings account

    Juan:

    $10,000 in a term deposit for 5 years

    $3000 in a three-year savings bond issued by Centra Gas

    $12,000 in tech stocks with CellSystems

12. Bethany retired and received a $125,000 payout on vacation time not taken. She wants to invest the amount in an annuity that earns 5.2% interest, compounded monthly, and use the fund as income for the next 15 years. She will withdraw equal monthly payments until the end of the 15 years, at which point the fund will have zero dollars.

    a) How much will her monthly income be? *(1 mark)*

    b) If the annuity earned a rate of 7.9%, compounded monthly, what would her monthly income be? *(1 mark)*

Assignment 5.3: Investment Decisions (continued)

13. Anthony's father purchased 150 shares in a computer technology upstart company in 1980 for $22.00 each. He kept the shares until 2013, when they were worth $489.64 per share.

    a) How much did he earn when he sold them? *(2 marks)*

    b) What was his rate of return? *(1 mark)*

14. You are considering whether to purchase a home or rent an apartment. List two financial reasons why it is preferable to buy a home. List two reasons why it is beneficial to rent an apartment. (*4 marks*)

15. You are comparing offers from a dealership to either buy or lease a car. The price to purchase is $25,950, plus taxes. You have a $3000 down payment for either option.

    The lease is over four years and payments are $385, plus 13% tax per month. The residual value is set at 45%. You would take the option to purchase it after the four years and pay for it outright (include 13% taxes). There is a lease acquisition fee of $539.

    To finance the car with monthly payments over 48 months, the bank offers you 7.5%, compounded monthly.

    a) Find the monthly payment if you finance the purchase of the car. *(2 marks)*

    b) Find the total amount of interest you pay over the loan period. *(2 marks)*

Assignment 5.3: Investment Decisions (continued)

c) Find the total cost to lease the car and buy it out at the end of the term. *(3 marks)*

d) How much do you save by purchasing instead of leasing and then buying it out? *(2 marks)*

e) When might leasing be a better option than buying a depreciating asset such as a car? *(2 marks)*

Notes

Module 5 Summary

Congratulations, you have finished Module 5! In this module, you considered simple and compound interest in financial decision making, the cost of loans, different credit options, renting versus buying versus leasing assets that either appreciate or depreciate, and investment strategies and options. You used spreadsheets, graphing technology, and various online resources in your investigation, analysis, and evaluation of financial mathematics.

In the next module, you will be learning about techniques of counting. You will use the Fundamental Counting Principle, factorial notation, permutations, and combinations to determine how many ways it is possible to arrange a certain number of elements.

Submitting Your Assignments

It is now time for you to submit the Module 5 Cover Assignment and Assignments 5.1 to 5.3 to the Distance Learning Unit so that you can receive some feedback on how you are doing in this course. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 5 assignments and organize your material in the following order:

- ❑ Module 5 Cover Sheet (found at the end of the course Introduction)
- ❑ Cover Assignment: Crossing the Canal with Cats
- ❑ Assignment 5.1: Compound Interest
- ❑ Assignment 5.2: Loans and Investments
- ❑ Assignment 5.3: Investment Decisions

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Notes

Grade 12 Applied Mathematics (40S)

Module 5
Financial Mathematics

Learning Activity Answer Keys

Module 5: Financial Mathematics

Learning Activity 5.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Determine 10% of 835.
2. Determine 10% of 0.24.
3. Determine 5% of 650.
4. Determine 5% of 1.5.
5. How many weeks are in one year?
6. If you make bi-weekly payments on a loan, how many payments will you make in a year?
7. If you make semi-monthly payments on a loan, how many payments will you make in one year?
8. How much simple interest will you earn in one year on $1000 if the rate is 6.5% per year?

Answers:

1. $83.5\left(835\times 0.1 \text{ or } \dfrac{835}{10}\right)$
2. 0.024 (move decimal one place to the left)
3. 32.5 (10% of 650 is 65; 5% is half of 10%, so half of 65 is 32.5)
4. 0.075 (10% of 1.5 is 0.15; half of 0.15 is 0.075)
5. 52
6. 26 (bi-weekly means every 2 weeks; 52 ÷ 2 = 26)
7. 24 (semi-monthly means every half month; 12 months per year, two payments per month; 12 × 2 = 24)
8. $65 (0.065 × 1000 = 65)

Part B: Interest

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Why is the value of n in the compound interest formula different for semi-monthly and bi-weekly compounding periods?

   Answer:

   Semi-monthly means every half month or twice per month (12 months per year × 2 calculations per month = 24 compounding periods). $n = 24$

   Bi-weekly means every two weeks (52 weeks per year ÷ 2 = 26 compounding periods). $n = 26$

2. Give three examples where compound interest is either charged or earned.

   Answer:

   Some examples of where compound interest is earned include some savings accounts, Canada Savings Bonds, and government investment certificates (GICs).

   Examples of where compound interest is charged include on credit card debt, loans (including car, student, home renovation, vacation loans, etc.), and mortgages.

3. Complete the following chart. Use technology and the simple interest or compound interest formula to determine the missing values.

   Answer:

	Amount (A)	Interest (I)	Principal (P)	Rate (r)	Compounding period	Length of Time (t)
a)	$4110	$1110	$3000	7.4% (simple)	(not compounded)	5 years
b)	$11,566.78	$3066.78	$8500	4.9%	monthly	6.3 years
c)	$933	$47.32	$885.68	$3\frac{1}{2}\%$	semi-annual	18 months
d)	$2020	$490	$1530	27.9%	weekly	365 days

a) $I = Prt$

$I = (3000)(0.074)5$

$I = 1110$

$A = I + P$

$A = 1110 + 3000$

$A = 4110$

b) $A = P\left(1 + \frac{r}{n}\right)^{nt}$

$A = 8500\left(1 + \frac{0.049}{12}\right)^{(12)(6.3)}$

$A = 11566.7848$

$I = A - P$

$I = 11566.7848 - 8500$

$I = 3066.7848$

c) $A = P\left(1 + \frac{r}{n}\right)^{nt}$

$933 = P\left(1 + \frac{0.035}{2}\right)^{(2)(1.5)}$

$933 = P(1.053424109)$

$P = \frac{933}{1.053424109}$

$P = 855.6831657$

$I = A - P$

$I = 933 - 855.6831657$

$I = 47.31683432$

d) $A = P\left(1 + \frac{r}{n}\right)^{nt}$

$2020 = 1530\left(1 + \frac{r}{52}\right)^{(52)(1)}$

$\frac{2020}{1530} = \left(1 + \frac{r}{52}\right)^{52}$

$\sqrt[52]{1.320261438} = \sqrt[52]{\left(1 + \frac{r}{52}\right)^{52}}$

$1.005357179 = 1 + \frac{r}{52}$

Subtract 1 from both sides.
Multiply both sides by 52 to isolate r.
$(0.005357179)(52) = r$
$r = 0.2785733052$

Note

You may prefer a graphing solution for part (d). Using graphing technology, graph the function $y = 1530\left(1 + \frac{r}{52}\right)^{52}$ and, to determine the required value of r, also graph $y = 2020$. The solution will be the r-value at the point of intersection.

4. Marcus received an amount of money on his 12th birthday, which he invested at 5% compounded quarterly. On his 20th birthday, he withdrew the total amount of the investment, which was $1116.10. Determine the initial value of his birthday gift.

   Answer:

$$
\begin{aligned}
A &= P\left(1+\frac{r}{n}\right)^{nt} \\
1116.10 &= P\left(1+\frac{0.05}{4}\right)^{(4)(8)} \\
1116.10 &= P(1.0125)^{32} \\
1116.10 &= P(1.488130509) \\
\frac{1116.10}{1.488130509} &= P \\
P &= 750.00
\end{aligned}
$$

   The amount he originally invested was $750.

5. Kathy has $2000 to invest. Her choices are to invest in either a GIC account that offers 6% simple interest or a savings account that offers 6%, compounded monthly.

   a) Find the amount of money in each of these investment options after 5, 10, and 20 years. Record the values in a spreadsheet.

      Answer:

      Use the formula $I = Prt$ for simple interest and $A = P\left(1+\frac{r}{n}\right)^{nt}$ for compound interest.

      Alternatively, you may use a spreadsheet or an online financial app, such as

      - GetSmarterAboutMoney.ca (www.getsmarteraboutmoney.ca/tools-and-calculators/compound-interest-calculator)
      - TheCalculatorSite.com (www.thecalculatorsite.com/finance/calculators/compoundinterestcalculator.php)

Option 1: GIC—6% simple interest

5 years	10 years	20 years
$I = Prt$ $I = (2000)(0.06)(5)$ $I = 600$ Value of the investment after 5 years is principal plus interest: $2000 + 600 = 2600$	$I = Prt$ $I = (2000)(0.06)(10)$ $I = 1200$ Value of the investment after 10 years is: $2000 + 1200 = 3200$	$I = Prt$ $I = (2000)(0.06)(20)$ $I = 2400$ Value of the investment after 20 years is: $2000 + 2400 = 4400$

Option 2: Savings account—6%, compounded monthly

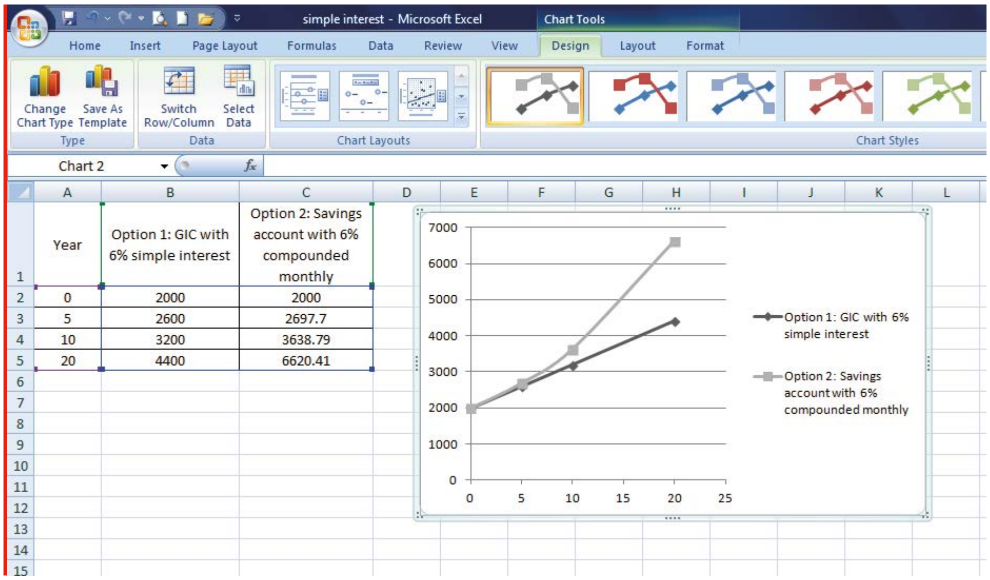
5 years	10 years	20 years
$A = P\left(1 + \frac{r}{n}\right)^{nt}$ $A = 2000\left(1 + \frac{0.06}{12}\right)^{(12)(5)}$ $A = 2000(1.005)^{60}$ $A = 2000(1.348850153)$ $A = 2697.70$	$A = P\left(1 + \frac{r}{n}\right)^{nt}$ $A = 2000\left(1 + \frac{0.06}{12}\right)^{(12)(10)}$ $A = 2000(1.005)^{120}$ $A = 3638.79$	$A = P\left(1 + \frac{r}{n}\right)^{nt}$ $A = 2000\left(1 + \frac{0.06}{12}\right)^{(12)(20)}$ $A = 2000(1.005)^{240}$ $A = 6620.41$

Year	**Option 1: GIC with 6% simple interest**	**Option 2: Savings account with 6% compounded monthly**
0	2000	2000
5	2600	2697.70
10	3200	3638.79
20	4400	6620.41

b) Graph the amount in each account over time on the same set of axes.

Answer:

Use a spreadsheet or graphing technology of your choice to create a graph. An example using Microsoft Excel is shown here.



c) Which investment option should she choose if she is investing for the long term? Explain your reasoning.

Answer:

The GIC with simple interest grows steadily but slowly as the same amount is added to the value each year. The savings account with compound interest grows at a similar rate initially, but more quickly as time goes on. She should choose the option with the compound interest.

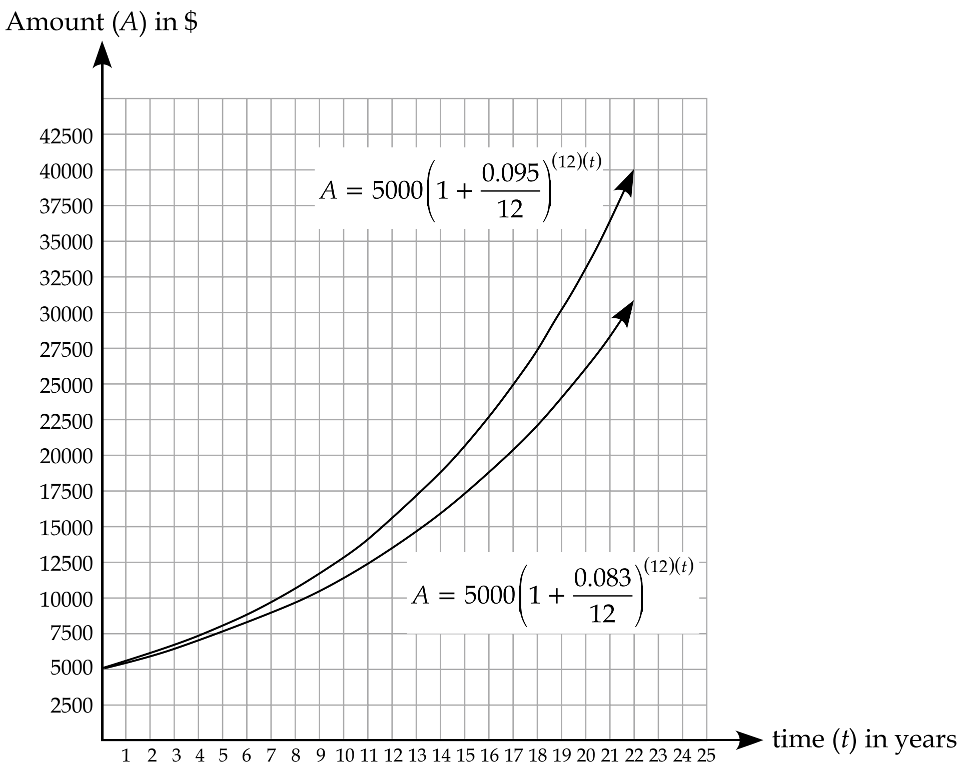
6. Graph a representation of the value of an investment of \$5000 held in a savings account at 8.3%, compounded monthly, and compare it to a similar investment at 9.5%, compounded monthly. Determine the value of each investment after 20 years.

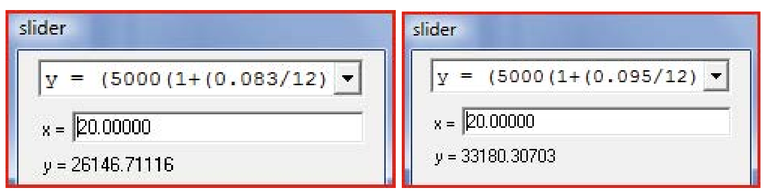
Answer:

Use the formulas to create a function that can be graphed. The total amount, A, is the dependent variable and the time, t, is the independent variable. The functions are:

$$A = 5000\left(1 + \frac{0.095}{12}\right)^{(12)(t)} \quad \text{and} \quad A = 5000\left(1 + \frac{0.083}{12}\right)^{(12)(t)}$$

The graph shown was created using Winplot, but you can use other technology.





After 20 years, the investment at 8.3% is worth $26,146.71, while the investment at 9.5% is worth $33,180.31. A small change in interest rate can have a large impact on the investment.

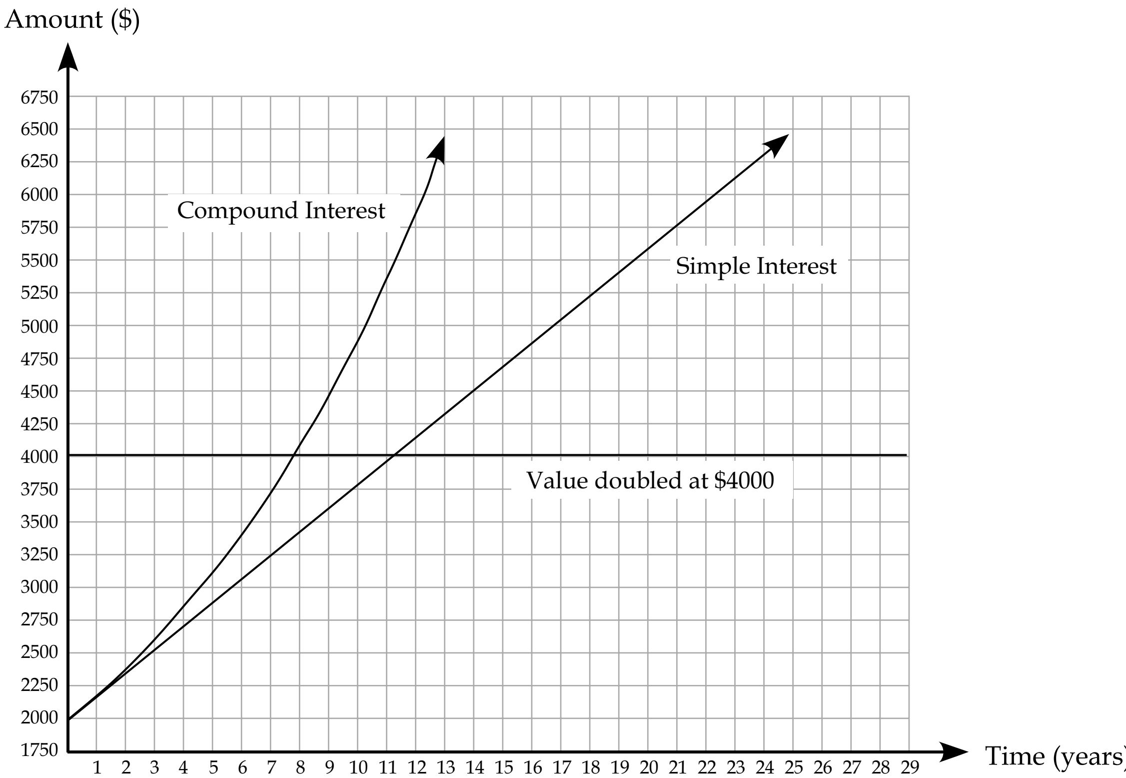
7. a) Use graphing technology to compare how long it would take an investment of $2000 to double in value at 9% simple interest, compared to 9% compounded bi-weekly.

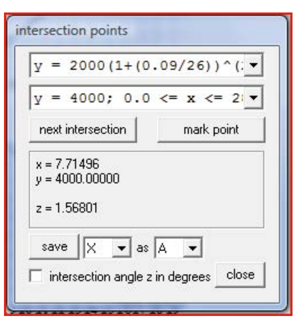
Answer:

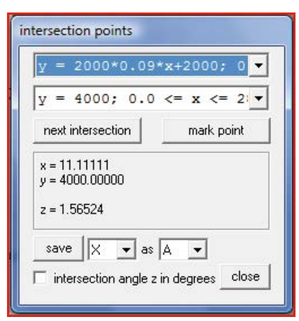
Use the equations $I = Prt$ and $A = P\left(1 + \frac{r}{n}\right)^{nt}$ with the appropriate variables substituted with the given values.

However, you must consider that the simple interest formula calculates the amount of interest only. To determine the value of the investment over time, remember to include the original amount, $A = I + P$.

Simple Interest	Compound Interest
$I = Prt$	$A = P\left(1 + \dfrac{r}{n}\right)^{nt}$
$I + P = Prt + P$	$A_c = 2000\left(1 + \dfrac{0.09}{26}\right)^{26t}$
or	
$\text{Amount} = Prt + P$	
$A_s = (2000)(0.09)(t) + (2000)$	







The lines intersect at approximately (7.7, 4000) and (11.1, 4000). It takes about 7.7 years with compound interest for the investment to double in value, but about 11.1 years with simple interest.

b) Use the Rule of 72 to determine approximately how long it would take to double the investment if it were compounded annually at 9%.

Answer:

$$\frac{72}{9} = 8$$

It would take approximately 8 years to double at 9% compounded annually. Note, this is a good estimate since the actual number of years is 7.7 as calculated in the previous question.

8. Victoria invests $5000 in a RESP (registered educational savings plan) for her daughter that has 9.6% interest compounded annually. Eventually she will need $20,000 to pay for university. For approximately how long will she need to invest the money?

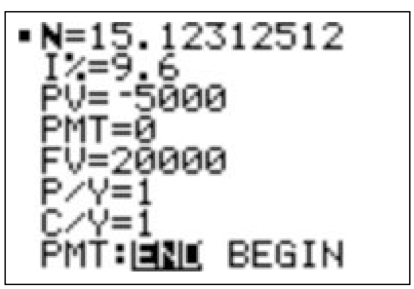
Answer:

Using the Rule of 72, you can approximate how long it will take an investment to double. In this case, however, the investment needs to quadruple, or double twice.

$$\frac{72}{9.6} = 7.5$$

The investment will double to $10,000 in approximately 7.5 years and then double again in another 7.5 years, so Victoria will have $20,000 in the RESP in about 15 years.

You will learn how to use technology and a TVM solver to calculate a more precise answer. The graphing calculator image is shown here.



Learning Activity 5.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Determine 50% 845.6.
2. Determine 50% 0.6.
3. Determine 25% of 845.6.
4. Determine 25% of 0.6.
5. Determine 12.5% of 1600 (a good approximation of the amount of PST and GST charged in Manitoba).
6. Determine 12.5% of 50.
7. A perfect number equals the sum of all its factors, excluding itself. Show that 6 is a perfect number.
8. State the formula for simple interest. Indicate what each variable represents.

Answers:

1. 422.8 (845.6 ÷ 2 = 422.8)
2. 0.3 (0.5 × 0.6 = 0.3)
3. 211.4 (50% of 845.6 is 422.8, half of this amount is 211.4)
4. 0.15 (0.25 × 0.6 = 0.15; or, move the decimal in 0.25 two places to the right, and the decimal in 0.6 one place to the right; multiply 25 × 6 = 150; to compensate, move the decimal in the answer three spaces to the left to get 0.15)
5. 200 (12.5% is half of 25%, which is half of 50%; 1600 ÷ 2 = 800, 800 ÷ 2 = 400, 400 ÷ 2 = 200)
6. 6.25 (12.5% is half of 25%, which is half of 50%; 50 ÷ 2 = 25, 25 ÷ 2 = 12.5, 12.5 ÷ 2 = 6.25)
7. 1 + 2 + 3 = 6 (The factors of 6, excluding itself, are 1, 2, 3; the sum of these numbers is 6)
8. $I = Prt$, where I is the amount of interest earned, P is the principal or initial amount borrowed or invested, r is the annual interest rate as a decimal, and t is the length of time in years.

Part B: Credit Options

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Darrell purchases two tickets to a football game using his credit card. He is charged \$270. If he currently does not carry a balance with his credit card, and makes a payment of \$270 on the day he receives his statement, how much interest will he pay on this purchase? His bank charges 18% interest, compounded daily, on all outstanding balances.

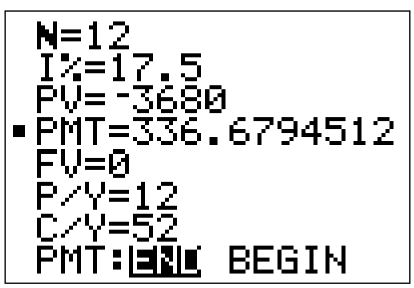
   Answer:

   Interest will be \$0. If Darrell pays the full balance on his credit card before the grace period (usually 20 days from the date on the statement), he is not charged interest.

2. Ali has a season ticket for her favourite hockey team. It costs \$3680, which she adds to her balance on her bank credit card with 17.5% interest, compounded weekly.

   a) How much must her monthly payments be, if she wants to pay off the entire cost of the season ticket in 12 months?

      Answer:

      Notice the payments per year (12) is different than the compounding periods per year (52).

      She must make payments of \$336.68 each month.

   b) Determine the total cost to purchase the tickets on credit and the amount of interest she pays over the 12 months.

      Answer:

      $12 \times 336.68 = 4040.16$ — The total cost is \$4040.16.

      $4040.16 - 3680 = 360.16$ — The amount of interest paid is \$360.16.

3. Lynn attends the symphony's *12 Masterpiece Concert Series* each year. Her tickets cost $609.70, which she purchases with her line of credit. How long will it take Lynn to pay for her tickets if she makes monthly payments of $85 on the account, which has 19% interest, compounded monthly? Determine the amount of interest she pays.

   Answer:

   Use technology to determine the number of payments.

```
▪N=7.674112577
 I%=19
 PV=-609.7
 PMT=85
 FV=0
 P/Y=12
 C/Y=12
 PMT:END BEGIN
```

   She must make 8 payments, with the final payment being smaller than the other 7 payments.

   The total interest she pays is found by calculating:

   $7.674112577 \times 85 = 652.30$

   $652.30 - 609.70 = 42.60$

   She pays $42.60 in interest.

4. Brooke purchases a graduation ring and arranges to make weekly payments of $20 on a store credit card to cover the cost. She is charged 18% interest, compounded weekly, and makes payments for 25 weeks. How much was the ring? What was the ticket price of the ring, if combined taxes are 13% (PST and GST)?

   Answer:

   Solve for the present value using technology.

```
 N=25
 I%=18
▪PV=-478.1843119
 PMT=20
 FV=0
 P/Y=52
 C/Y=52
 PMT:END BEGIN
```

   The original amount charged on the store credit card was $478.18. This includes taxes. To determine the ticket price of the ring, divide by 1.13 (13% taxes).

   $\frac{478.18}{1.13} = 423.17$ The ring was priced at $423.17.

5. Joshua is considering transferring his existing debt of $3500 to a new credit card or line of credit, based on the introductory offers available. The credit card offers 0% interest for 6 months and then an interest rate of 25%, compounded daily. The line of credit is available at 17%, compounded monthly, with a rebate of $100. Joshua can afford a $400 per month payment. Which option is the better way to repay his debt? Explain your reasoning.

   Answer:

   Credit card:

   Joshua can make six monthly payments of $400 with no interest accruing (being added to his balance).

   $6 \times 400 = 2400$

   $3500 - 2400 = 1100$

   He will have a balance of $1100 at 25% interest, compounded daily. Determine how many payments he will have to make to pay this down.

```
▪N=2.862488616
 I%=25
 PV=-1100
 PMT=400
 FV=0
 P/Y=12
 C/Y=365
 PMT:END BEGIN
```

   He will have to make three more payments, although the last one will be slightly smaller.

   $6 \times 400 + 2.862488616 \times 400 = 3544.995446$

   In total, he will have made nine payments totalling $3545.00, of which $45 is interest.

   Line of credit:

   The line of credit offers a rebate of $100. His balance would then be $3400 at 17%, compounded monthly.

```
▪N=9.120965556
 I%=17
 PV=-3400
 PMT=400
 FV=0
 P/Y=12
 C/Y=12
 PMT:END BEGIN
```

   He would have to make 10 payments, with the last one being significantly smaller.

   $9.120965556 \times 400 = 3648.386222$

In total, repaying his debt with the line of credit would cost him $3648.39, of which $148.39 is interest.

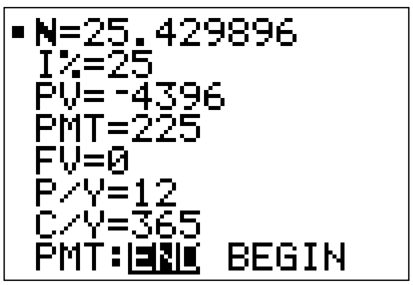
Conclusion:

In this case, the payments are the same in either option, but Joshua is better off taking the credit card over the line of credit, as that way he pays less interest and pays off the debt sooner.

6. Bruce and his wife decide to take a cruise. They find an all-inclusive sell-off vacation deal for $4396, including taxes and fees. They finance the trip with a cash advance on their credit card and plan to make payments of $225 per month. The credit card charges 25%, compounded daily, on cash advances, with no grace period.

   a) How long will it take to pay back the loan?

   Answer:



   It will take them just over two years to pay back the loan. They will make 26 payments.

   b) What will the trip actually cost them?

   Answer:

   25 payments of $225 and a 26th payment of 0.429896 × 225 = 96.73 for a total cost of $5721.73.

   c) How much do they pay in interest?

   Answer:

   $A - P = I$

   The will pay $5721.73 − $4396.00 = $1325.73 in interest.

7. Why is it important to read all of the fine print when considering a store promotion?

   Answer:

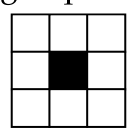
   Stores and credit card companies try to entice consumers to purchase items by offering rewards or incentives, and consumers may not have the current funds to afford these items. The stores and companies charge fees, which may not be clearly explained, and the consumer may end up paying much more than the original ticket price plus taxes. Retailers and credit card companies are providing a service, but are not in the business of helping people out. They want consumers to carry debt so they can charge interest and make money. Be cautious of deals, read the fine print, and, whenever possible, pay the full balance before the grace period expires to save as much of your own money as you can.

Learning Activity 5.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. If you earn \$150 per week and save 10% of each paycheque, how much would you have saved after one year?
2. Using all four of the digits 2, 3, 4, and 9, some of the four common mathematical operations (+, −, ×, ÷), and brackets, create an expression equal to 42.
3. Given that the area of the shaded square is 16 cm^2, what is the area of the large square?



4. If the odds in favour of an event occurring are 1 : 2, what is the probability of the event occurring?
5. Lu scored $\frac{21}{25}$ on a test, while Min scored $\frac{8}{10}$. Who scored higher?
6. State the formula for annual compound interest. Indicate what each of the variables represents.
7. Illustrate the following on a number line: $\{-3 \leq x < 9, x \in \Re\}$
8. Write the inverse, given the conditional statement: "If a batter gets three strikes, then the batter is out."

Answers:

1. \$780 (150 × 10% = 15 per week; 15 × 52 = 15 × 50 + 15 × 2 = 750 + 30 = 780)
2. (9 × 4) + (2 × 3) or ((9 × 2) − 4) × 3 or ((3 × 4) + 9) × 2
   (Other solutions are possible.)
3. 144 cm^2 (The side length of each of the small squares must be $\sqrt{16} = 4$ so the side length of the large square is 3 × 4 = 12; the area of the large square is $12^2 = 144$)
4. $\frac{1}{3}$ $(1 : (1 + 2))$

5. Lu $\left(\frac{21}{25} = \frac{42}{50}; \frac{8}{10} = \frac{40}{50}; \frac{42}{50} > \frac{40}{50}\right)$

6. $A = P\left(1 + \frac{r}{n}\right)^{nt}$

   where A is the amount of the investment plus interest after t years, P is the principal or initial amount invested, r is the annual interest rate as a decimal, n is the number of compounding periods in a year, and t is the number of years.

7. 

8. If a batter does not get three strikes, then the batter is not out (negate the hypothesis and conclusion).

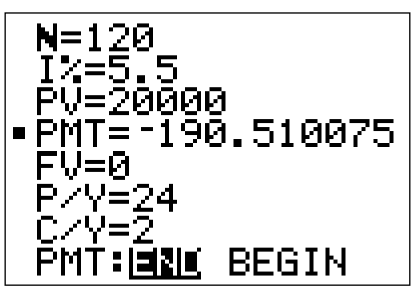
Part B: Mortgages and Loans

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Steve and Val take out a $20,000 loan for renovations to their home. They negotiate a 5-year term at prime plus 2.5%, compounded semi-annually. The prime rate is 3%. What will be the amount of their semi-monthly payments? How much interest will they pay on the loan?

   Answer:

   $n = 5 \times 24$

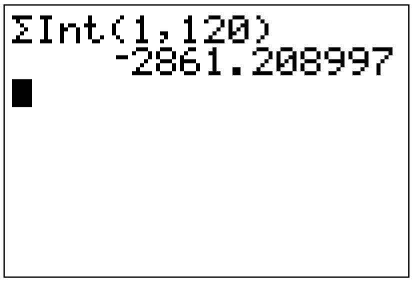
   

   They will make 24 payments per year, for 5 years, at $190.51.

   $190.51 × 120 = $22,861.20 total paid

   $22,861.20 − $20,000.00 = $2861.20 interest paid

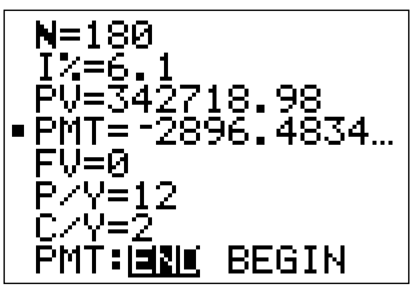
Or, calculate interest on the graphing calculator using the Sum of Interest command as shown.

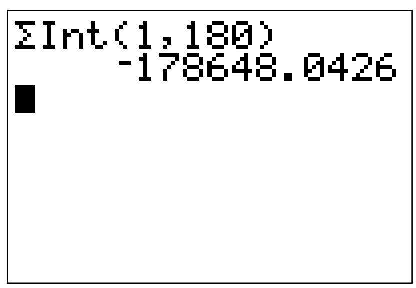


2. Frieda and Ernest purchase a retirement condo for $335,999. They must pay additional costs on the new building, which include moving costs, lawyer fees, insurance fees, and other taxes. They add 2% of the purchase price to their mortgage to cover these additional costs. Compare the amount of interest they would pay on this mortgage if they amortized it over 15 years rather than 20 years. Assume they make monthly payments and the mortgage is compounded semi-annually at prime plus 3.1%. The prime rate is 3%.

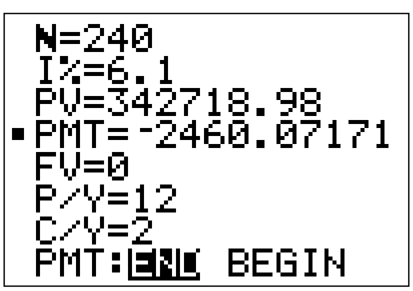
   Answer:

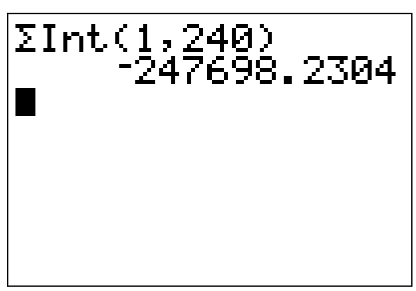
   Use a financial calculator program. The loan is for $335,999 × 1.02 = $342,718.98, over 15 years (or 180 monthly payments). They pay $178,648.04 in interest.





   Over 20 years (or 240 monthly payments), their payments per month are less, but they pay $247,398.23 in interest.





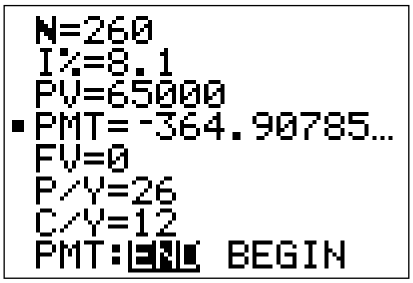
   They pay $69,050.19 more in interest if the amortization period is 20 years rather than 15 years. A shorter amortization period means your payments are slightly larger but you pay the loan back faster and end up paying a lot less interest.

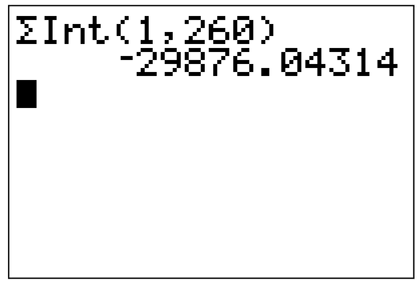
3. Owen wants to start a restaurant and applies for a bank loan of $65,000. His business plan includes repaying the loan in 10 years with bi-weekly payments. He compares the options at two different banks.
   - Bank A: 8.1%, compounded monthly
   - Bank B: 6.3%, compounded monthly

   How much of a difference will a lower rate make to the bi-weekly payments and to the amount of interest Owen will pay?

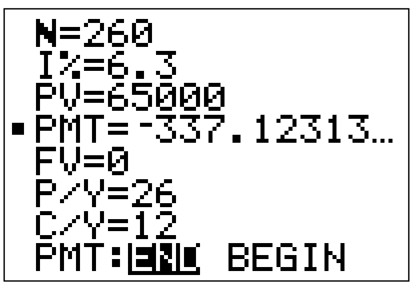
   Answer:

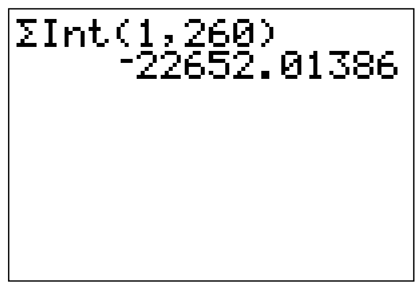
   With Bank A, his payments are bi-weekly (26 times per year) and will be $364.91, and the amount of interest paid will be $29,876.04.





   With Bank B, his payments are bi-weekly (26 times per year) and will be $337.12, and the amount of interest paid will be $22 652.01.



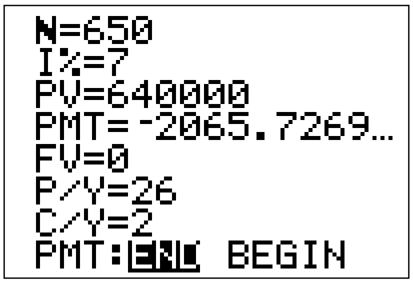


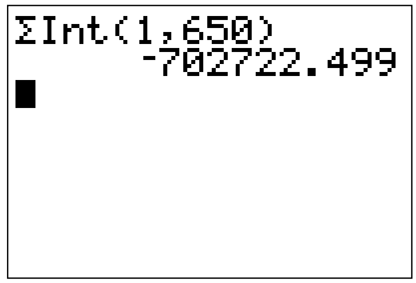
   With a lower interest rate, Owen can save $7224.03.

4. Garrett and Adria purchase a farmhouse and land for $840,000. They use an inheritance of $200,000 as the down payment and take out a 25-year mortgage at prime plus 4%, compounded semi-annually, for the remainder of the purchase price. The prime rate is 3%. Determine their payment amount and the amount paid in interest if they choose to make bi-weekly payments. Calculate the difference in their payments and the amount of interest they will pay if they make semi-annual payments.

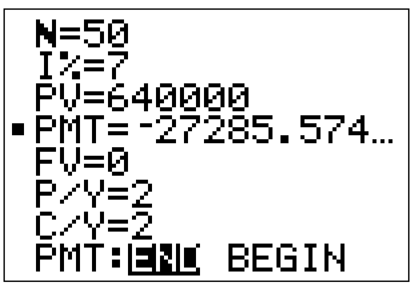
   Answer:

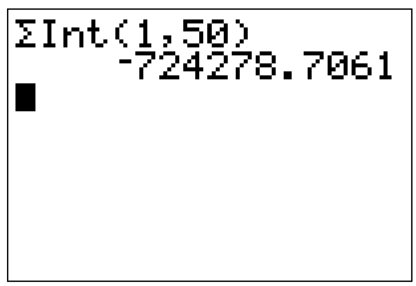
   Bi-weekly (26 per year) payments of $2065.73 means they pay $702,722.50 in interest.





   If they only make two payments per year, the payments will be $27,285.57 each and they end up paying $724,278.71 in interest, $21,556.21 more than if they had made bi-weekly payments.

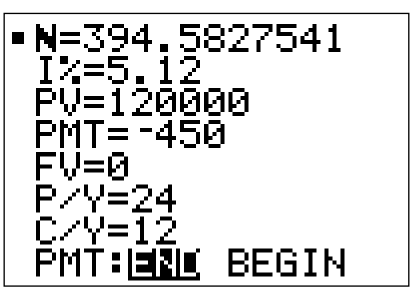




5. Reta makes $450 payments on her condo mortgage twice a month. Her original mortgage amount was $120,000, and has been compounded monthly at 5.12%.

   a) How long will it take Reta to pay off her mortgage?

      Answer:



      She will have to make 395 semi-monthly payments. This will take approximately 16.5 years (395 ÷ 24).

b) Give three suggestions to help Reta reduce the amount of interest she will pay over the life of her mortgage.

Answer:

To reduce the amount of interest she will have to pay, Reta could negotiate with her bank for a better term interest rate, try to pay it off in 10 years rather than 16.5 years, make lump sum payments each year to reduce the principal, double up or increase some or all of her payments, or make bi-weekly payments rather than semi-monthly payments.

6. Marcel purchases a racing snowmobile that costs $7000 on credit. The loan is for four years, compounded monthly, at 18.9%. Use technology to determine:

a) the monthly payment amount.

Answer:

```
 N=48
 I%=18.9
 PV=7000
▪PMT=⁻208.93162…
 FV=0
 P/Y=12
 C/Y=12
 PMT:END BEGIN
```

The monthly payments are $208.93.

b) the amount he still owes after 2 years.

Answer:

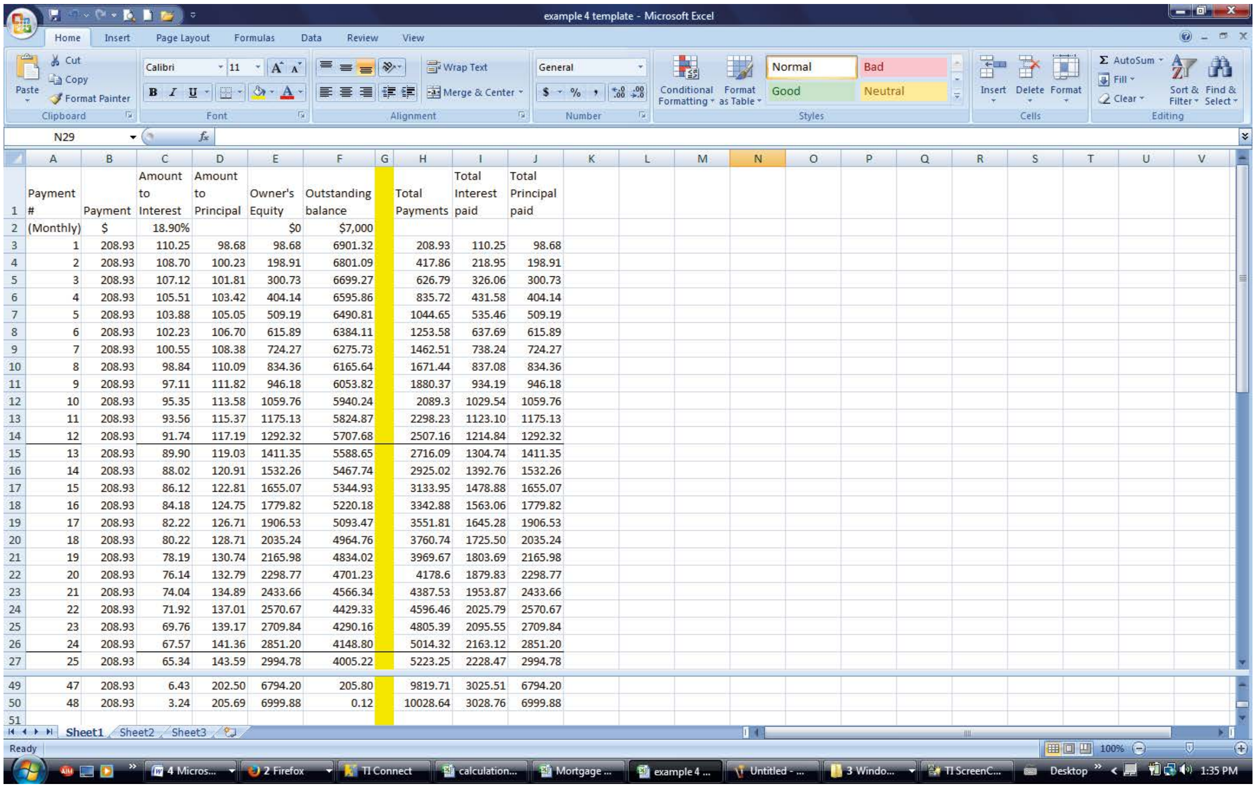
Determine the future value (FV) of the loan after 2 years.

```
 N=24
 I%=18.9
 PV=7000
 PMT=⁻208.93162…
▪FV=⁻4148.757692
 P/Y=12
 C/Y=12
 PMT:END BEGIN
```

After making 24 payments, he still owes $4148.76 on the snowmobile.

c) Use your spreadsheet template to visualize the amortization of Marcel’s loan and determine the amount of interest paid on the loan.

Answer:



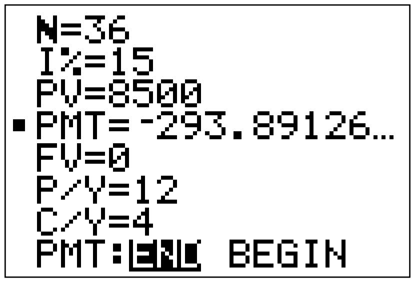
	A	B	C	D	E	F	H	I	J
1	Payment #	Payment	Amount to Interest	Amount to Principal	Owner's Equity	Outstanding balance	Total Payments	Total Interest paid	Total Principal paid
2	(Monthly)	$	18.90%		$0	$7,000			
3	1	208.93	110.25	98.68	98.68	6901.32	208.93	110.25	98.68
4	2	208.93	108.70	100.23	198.91	6801.09	417.86	218.95	198.91
5	3	208.93	107.12	101.81	300.73	6699.27	626.79	326.06	300.73
6	4	208.93	105.51	103.42	404.14	6595.86	835.72	431.58	404.14
7	5	208.93	103.88	105.05	509.19	6490.81	1044.65	535.46	509.19
8	6	208.93	102.23	106.70	615.89	6384.11	1253.58	637.69	615.89
9	7	208.93	100.55	108.38	724.27	6275.73	1462.51	738.24	724.27
10	8	208.93	98.84	110.09	834.36	6165.64	1671.44	837.08	834.36
11	9	208.93	97.11	111.82	946.18	6053.82	1880.37	934.19	946.18
12	10	208.93	95.35	113.58	1059.76	5940.24	2089.3	1029.54	1059.76
13	11	208.93	93.56	115.37	1175.13	5824.87	2298.23	1123.10	1175.13
14	12	208.93	91.74	117.19	1292.32	5707.68	2507.16	1214.84	1292.32
15	13	208.93	89.90	119.03	1411.35	5588.65	2716.09	1304.74	1411.35
16	14	208.93	88.02	120.91	1532.26	5467.74	2925.02	1392.76	1532.26
17	15	208.93	86.12	122.81	1655.07	5344.93	3133.95	1478.88	1655.07
18	16	208.93	84.18	124.75	1779.82	5220.18	3342.88	1563.06	1779.82
19	17	208.93	82.22	126.71	1906.53	5093.47	3551.81	1645.28	1906.53
20	18	208.93	80.22	128.71	2035.24	4964.76	3760.74	1725.50	2035.24
21	19	208.93	78.19	130.74	2165.98	4834.02	3969.67	1803.69	2165.98
22	20	208.93	76.14	132.79	2298.77	4701.23	4178.6	1879.83	2298.77
23	21	208.93	74.04	134.89	2433.66	4566.34	4387.53	1953.87	2433.66
24	22	208.93	71.92	137.01	2570.67	4429.33	4596.46	2025.79	2570.67
25	23	208.93	69.76	139.17	2709.84	4290.16	4805.39	2095.55	2709.84
26	24	208.93	67.57	141.36	2851.20	4148.80	5014.32	2163.12	2851.20
27	25	208.93	65.34	143.59	2994.78	4005.22	5223.25	2228.47	2994.78
49	47	208.93	6.43	202.50	6794.20	205.80	9819.71	3025.51	6794.20
50	48	208.93	3.24	205.69	6999.88	0.12	10028.64	3028.76	6999.88

The values in the spreadsheet are slightly different than the ones calculated using a financial calculator due to the number of decimal places used. The spreadsheet screen is split to show the first two years and the last two months of the fourth year of the loan. Notice there is an outstanding balance of 12 cents that needs to be added to the final payment to clear the debt. The total amount of interest paid on the loan is $3028.76.

7. Brendan takes out a loan to buy a quad ATV and to develop a parcel of land behind his home to create a racetrack. He borrows \$8500 from his local credit union at 15% interest, compounded quarterly, for 3 years.

   a) Determine his monthly payments.

   Answer:



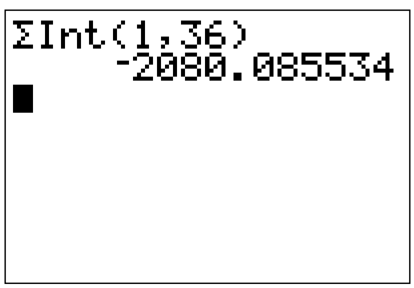
   His monthly payments will be \$293.89.

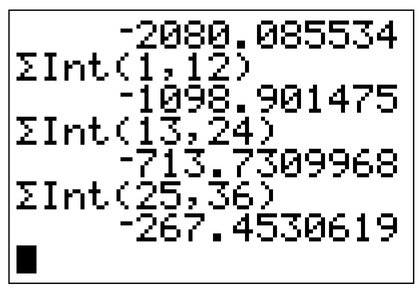
   b) Use technology to determine the amount of interest paid in each of the three years. Explain why the amount of interest changed each year.

   Answer:

   In total, he will pay (\$293.89 × 36) − \$8500 = \$2080.04 in interest.

   Using technology (which calculates the value using more decimal places), the interest amount is determined to be \$2080.09.





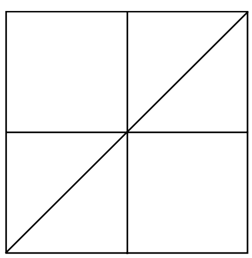
   In the first year, he pays \$1098.90; in the second year, he pays \$713.73; and in the final year, he pays \$267.45. The reason the interest amount decreases each year is that with each payment he makes, a portion of the principal owed is paid off. Each subsequent interest calculation is determined on a decreasing amount of principal, so as time passes, a greater portion of each payment goes towards the principal and less towards interest.

Learning Activity 5.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Using each of the digits 2, 5, 6, and 8, some of the four common mathematical operations, and brackets, create an expression equal to 4.
2. The statement "If p then q" may be written in the form "$p \rightarrow q$." Write the contrapositive of "$p \rightarrow q$."
3. A perfect number is a number that equals the sum of all its factors, excluding itself. Show that 28 is a perfect number.
4. Sketch $y = \log(x)$
5. What is the maximum number of Mondays that can occur in the first 45 days of a year?
6. What is the total number of triangles and squares in the following figure?



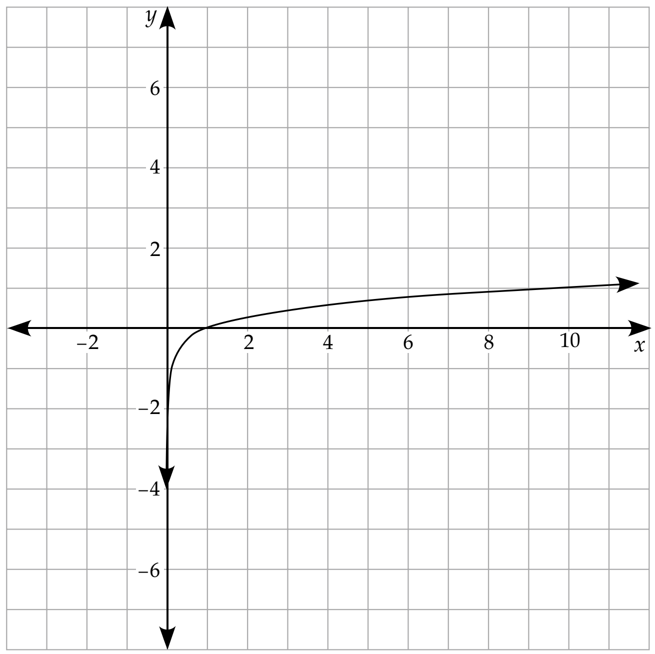
7. A Pythagorean Triple, such as 3, 4, 5, is a set of three numbers that satisfy the Pythagorean theorem, $a^2 + b^2 = c^2$. What value for k will make this set of numbers a Pythagorean Triple?

   5, k, 13

Answers:

1. $6 + 8 - (2 \times 5)$ or $((5 - 2) \times 8) \div 6$ or $(6 - 5) \times (8 \div 2)$ or $((6 \times 2) + 8) \div 5$, or $8 \div (5 - (6 \div 2))$, or $(6 \div (8 - 5)) + 2$
   (Other solutions are possible.)
2. "$\sim q \rightarrow \sim p$" (the symbol ~ implies negation, and the hypothesis and conclusion are switched)
3. 1 + 2 + 4 + 7 + 14 = 28 (The factors of 28, excluding itself, are 1, 2, 4, 7, and 14; the sum of these numbers is 28.)

4. Graph.



5. $7\left(\frac{45}{7} = 6\frac{3}{7};\right.$ if the year begins on a Sunday, there would be 6 full weeks plus Sunday, Monday, and Tuesday of the seventh week)

6. 11 (five squares and six triangles)

7. $$\begin{aligned} 5^2 + k^2 &= 13^2 \\ 25 + k^2 &= 169 \\ k^2 &= 169 - 25 \\ k^2 &= 144 \\ \sqrt{k^2} &= \sqrt{144} \\ k &= 12 \end{aligned}$$

Part B: Investing

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. a) Jimmy Sky wants to purchase a paddleboard for next summer. He has \$500 saved but the board costs \$795, plus 13% taxes. He plans to invest his \$500 and make monthly deposits in a "Special Project Account" at his bank, offering 0.04% interest, compounded monthly. How much will he have to save each month in order to afford the board nine months from now?

    Answer:

    Total is 795 × 1.13 = 898.35.

    ```
     N=9
     I%=.04
     PV=500
    ▪PMT=44.23854329
     FV=-898.35
     P/Y=12
     C/Y=12
     PMT:END BEGIN
    ```

    He will have to save an additional \$44.24 each month.

    Or, since the interest rate is negligible, determine the total needed (898.35 − 500), \$398.35, and the value of 9 equal payments (398.35 ÷ 9), \$44.26.

b) Jimmy decides he can't afford those monthly contributions, so he will rent a board this summer and invest his $500 until the following summer to purchase his own. His bank will give him 0.09%, compounded monthly, on a 21-month term. How much will he need to set aside each month for 21 months to afford the board the following year?

Answer:

```
 N=21
 I%=.09
 PV=500
▪PMT=18.91732475
 FV=-898.35
 P/Y=12
 C/Y=12
 PMT:ENE BEGIN
```

If he waits to buy it the following year and gets a slightly better rate from the bank, he will need to set aside $18.92 each month.

Or, since the interest rate is still very low, determine the total needed, $398.35, and the value of 21 equal payments ($398.35 ÷ 21), $18.97.

2. a) Avery has $1200 saved from her summer job and will continue to work part-time during the school year. She plans to save $30 from each of her semi-monthly paycheques in order to buy a used car when she graduates in June, 10 months from now. Her bank offers her a savings account at 1.2%, compounded quarterly. How much will she have available to spend on a car in June?

Answer:

```
 N=20
 I%=1.2
 PV=1200
 PMT=30
▪FV=-1814.897047
 P/Y=24
 C/Y=4
 PMT:ENE BEGIN
```

She will have $1814.90 saved.

b) If she wants to have $2000 by June, what would she have to do differently, in terms of this investment, to reach that goal? Support your answer with TVM solver results (or other software applications).

Answer:

She would either have to negotiate a better rate of 15.5% (which is very unlikely!):

```
 N=20
▪I%=15.51639352
 PV=1200
 PMT=30
 FV=-2000
 P/Y=24
 C/Y=4
 PMT:END BEGIN
```

Or, make larger contributions to her savings account, $39.21 semi-monthly:

```
 N=20
 I%=1.2
 PV=1200
▪PMT=39.21131728
 FV=-2000
 P/Y=24
 C/Y=4
 PMT:END BEGIN
```

Or make weekly payments (approximately 43 weeks in 10 months):

```
 N=43
 I%=1.2
 PV=1200
 PMT=30
▪FV=-2508.211572
 P/Y=52
 C/Y=4
 PMT:END BEGIN
```

Or begin with a larger initial contribution:

```
 N=20
 I%=1.2
 PV=1500
 PMT=30
▪FV=-2117.907561
 P/Y=24
 C/Y=4
 PMT:END BEGIN
```

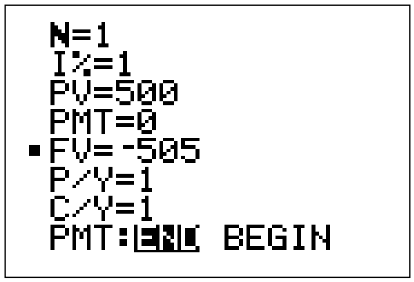
3. Matthew invests $500 in an escalating GIC with annual compounding that pays 1% for the first year, 1.35% for the second year, 1.75% for the third year, 2.5% for the fourth year, and 4% for the fifth year.

   a) Determine the value of his investment after each year.

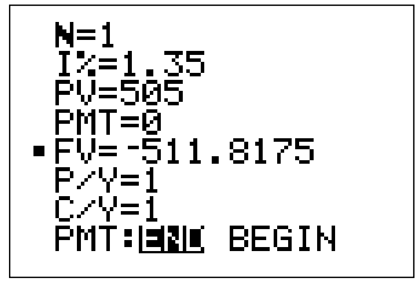
      Answer:

      Find the future value after each year, and that becomes the present value of the following year.

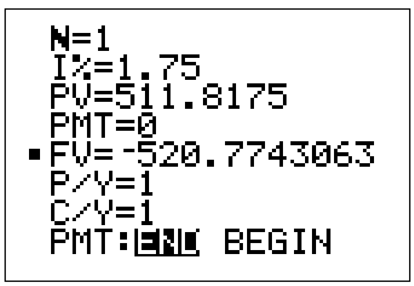
      Year 1



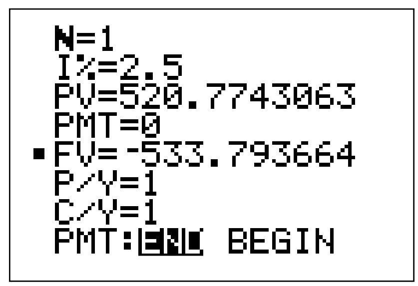
      Year 2



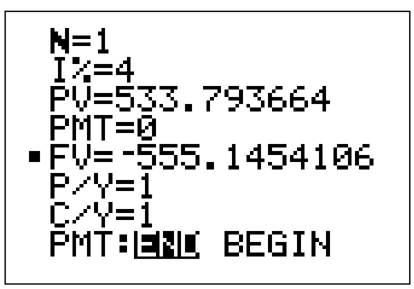
      Year 3



      Year 4



      Year 5



b) Calculate the ROR for this investment after the first and fifth years.

Answer:

After the first year: $\text{ROR} = \dfrac{505 - 500}{500}$

$\text{ROR} = 1\%$

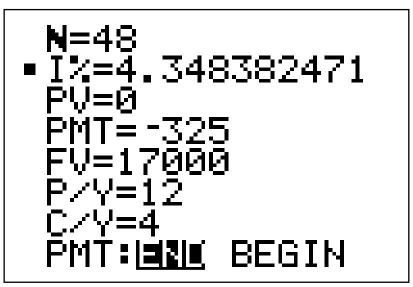
After five years: $\text{ROR} = \dfrac{555.1454106 - 500}{500}$

$\text{ROR} = 11.03\%$

Notice that the average Rate of Return per year over 5 years in the escalating GIC is $11.03 \div 5 \approx 2.2\%$ per year.

4. a) Michelle would like to buy a condo after graduation from university and starts to save for a down payment. She would like to have $17,000 available. If she can contribute $325 each month during the four years she attends university, what rate will she have to negotiate with her bank to make the down payment possible? Use quarterly compounding.

Answer:

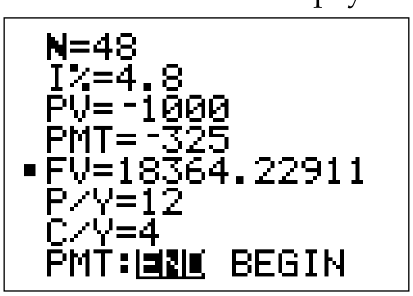


She would need a rate of at least 4.35%.

b) If Michelle negotiates a rate of 4.8%, compounded quarterly, and begins with an initial investment of $1000 while still contributing $325 per month, what will her down payment be?

Answer:

She will have a down payment of $18 364.23 available.



5. Gordon and Rebecca are saving for retirement. Rebecca contributes $50 from her paycheque bi-weekly into an RRSP at 7.1%, compounded bi-weekly. Gordon contributes $1300 annually into an RRSP at 7.1%, compounded annually.

   a) Graph the value of each of their investments over time and determine the amount saved by Rebecca and Gordon after 25 years.

   Answer:

   Use the formula: $A = \dfrac{R\left[\left(1+\dfrac{r}{n}\right)^{nt}-1\right]}{\dfrac{r}{n}}$

   Rebecca:

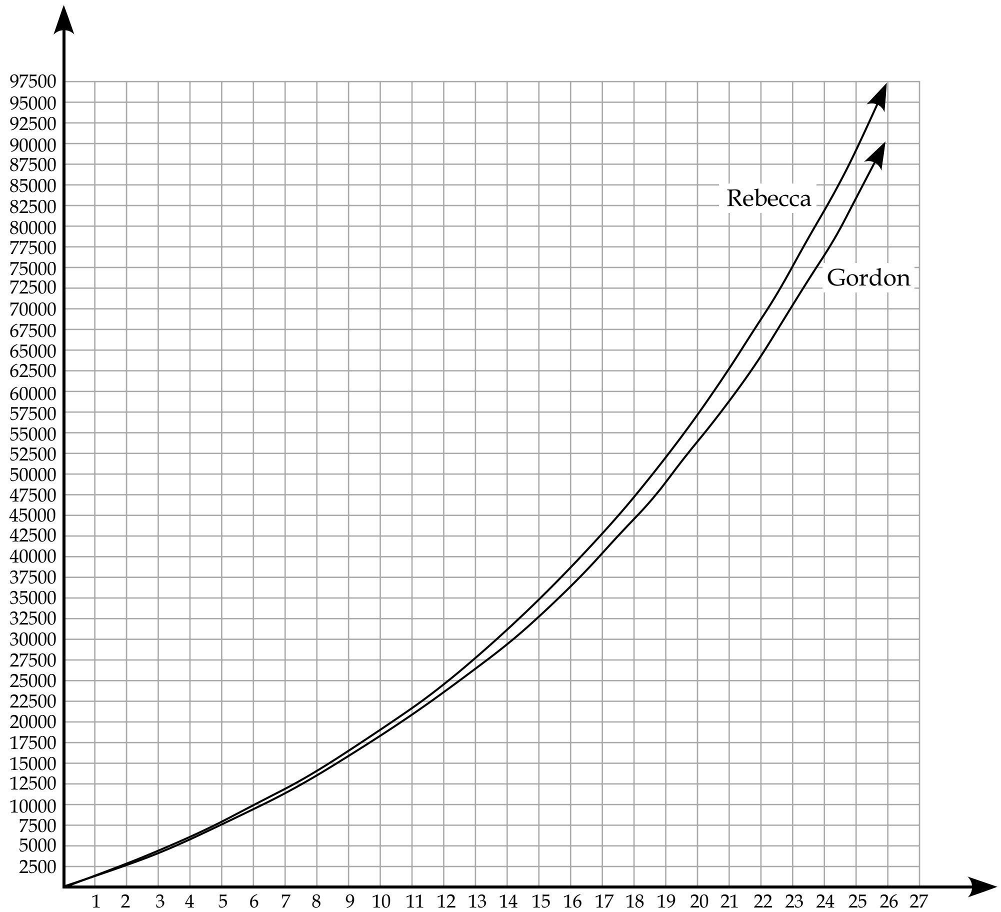
   $$A = \frac{50\left[\left(1+\dfrac{0.071}{26}\right)^{(26)(t)}-1\right]}{\dfrac{0.071}{26}}$$

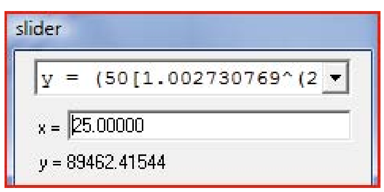
   $$A = \frac{50\left[1.002730769^{26t}-1\right]}{0.002730769}$$

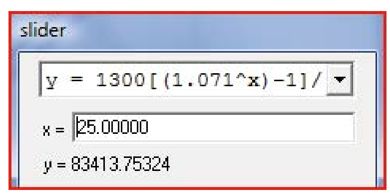
   Gordon:

   $$A = \frac{1300\left[\left(1+\dfrac{0.071}{1}\right)^{(1)(t)}-1\right]}{\dfrac{0.071}{1}}$$

   $$A = \frac{1300\left[1.071^{t}-1\right]}{0.071}$$







After 25 years, Rebecca has $89,462.42 saved while Gordon has $83,413.75 in his RRSP.

b) Describe the advantages and disadvantages of smaller investments over a longer term compared to larger investments over a shorter term.

Answer:

Answers may vary. Smaller amounts may be more convenient to contribute than larger amounts, but you need more time for small investments to increase. If you need to grow money quickly, you may have to invest a larger principal, which could be more difficult to set aside, but it may be for a shorter term, as opposed to locking away smaller contributions of money for a longer term.

6. When Cornelius retired, he purchased a sailboat and left on a year-long cruise around the Mediterranean Sea. His adventure cost him $212,000, but he had been saving up for this for his entire career (39 years working at the same job). If he contributed $100 per month to an RRSP that averaged 5.9%, compounded monthly, what was his initial investment?

   Answer:

   $n = 39 \times 12$

   ```
    N=468
    I%=5.9
   ▪PV=-3063.346972
    PMT=-100
    FV=212000
    P/Y=12
    C/Y=12
    PMT:END BEGIN
   ```

   His initial contribution was $3063.35.

7. a) Arnold successfully kicks a "pack a week" smoking habit. He takes the $13.75 per week that he would have spent on cigarettes and invests it in a growth fund at 7.4%, compounded weekly. How much will he have in his investment after (i) one year? (ii) ten years?

   Answer:

   i) $n = 52 \times 1$ (1 year with 52 weeks per year)

   ```
    N=52
    I%=7.4
    PV=0
    PMT=-13.75
   ▪FV=741.572518
    P/Y=52
    C/Y=52
    PMT:END BEGIN
   ```

   After one year, he will have $741.57 in his investment.

   ii) $n = 52 \times 10$

   ```
    N=520
    I%=7.4
    PV=0
    PMT=-13.75
   ▪FV=10578.4565
    P/Y=52
    C/Y=52
    PMT:END BEGIN
   ```

   After 10 years, he will have saved $10,578.46.

b) How much would Arnold have saved if he invested $13.75 per day for 10 years? (Assume 7.4%, compounded daily.)

Answer:

$n = 365 \times 10$

```
 N=3650
 I%=7.4
 PV=0
 PMT=13.75
▪FV=-74316.72206
 P/Y=365
 C/Y=365
 PMT:END BEGIN
```

He would have accumulated $74,316.72 over 10 years.

8. Calculate the rate of return on the investment of a lump sum of $10,000 at 3%, compounded annually, for 15 years.

Answer:

```
N=15
I%=3
PV=10000
PMT=0
FV=-15579.67417
P/Y=1
C/Y=1
PMT:END BEGIN
```

$$\text{ROR} = \frac{15\,579.67 - 10\,000}{10\,000}$$

$$\text{ROR} = 55.8\%$$

9. Craig buys a television for $679, using a store credit card. He is given a special promotion of 0% interest for 12 months but is charged a finance fee of $49.99. He makes 12 equal payments of $60.75 before the one year deadline, and so pays no interest. What rate of return does the store get on this purchase by charging the finance fee?

Answer:

$$\text{ROR} = \frac{\text{Current Value of Investment} - \text{Previous Value of Investment}}{\text{Previous Value of Investment}}$$

$$\text{ROR} = \frac{(679 + 49.99) - (679)}{679}$$

$$\text{ROR} = \frac{49.99}{679}$$

$$\text{ROR} = 0.0736 \text{ or } 7.4\%$$

10. Akeela uses a lay-away plan at a local department store to purchase a necklace for her mother's birthday. The necklace costs $69 plus 13% taxes. Akeela must pay a $5 fee and a down payment of 20% of the purchase price, and the store will hold the necklace for her. She must make four equal bi-weekly payments to cover the remaining cost of the necklace.

a) What is Akeela's final cost for the necklace?

Answer:

$69 \times 1.13 = \$77.97$

$77.97 + 5 = \$82.97$

In the end, she must pay $82.97 for the necklace.

b) What rate of return does the store receive by offering this service?

Answer:

$$\text{ROR} = \frac{82.97 - 77.97}{77.97}$$

$$\text{ROR} = 6.4\%$$

The store effectively charges 6.4% on this purchase.

11. Daniel purchases a BMX bike for $800, using his credit card. He is charged 19.99% interest, compounded monthly (no grace period), and pays it back with regular payments over the course of 18 months.

a) Determine his monthly payment.

Answer:

```
 N=18
 I%=19.99
 PV=-800
▪PMT=51.8067148
 FV=0
 P/Y=12
 C/Y=12
 PMT:END BEGIN
```

He must pay $51.81 per month for 18 months.

b) Determine the rate of return on this credit card purchase.

Answer:

$$\text{ROR} = \frac{\text{Final Value of Investment} - \text{Initial Amount Invested}}{\text{Initial Amount Invested}}$$

$$\text{ROR} = \frac{(18 \times 51.81) - (800)}{800}$$

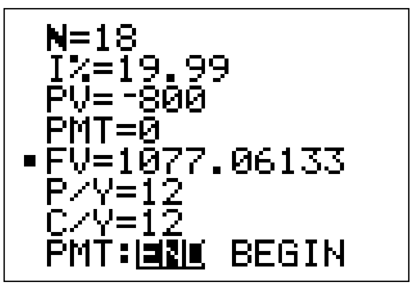
$$\text{ROR} = \frac{932.58 - 800}{800}$$

$$\text{ROR} = \frac{132.58}{800}$$

$$\text{ROR} = 0.1657 \text{ or } 16.6\%$$

c) If Daniel had not made any payments on the bike for 18 months and then paid it off in one lump sum (including interest), what would his rate of return have been?

Answer:



After 18 months at 19.99% interest, he would owe $1077.06.

$$\text{ROR} = \frac{\text{Current Value of Investment} - \text{Previous Value of Investment}}{\text{Previous Value of Investment}}$$

$$\text{ROR} = \frac{(1077.06) - (800)}{800}$$

$$\text{ROR} = \frac{277.06}{800}$$

$$\text{ROR} = 0.3463 \text{ or } 34.6\%$$

d) What can you conclude about making smaller, regular contributions to pay down debt?

Answer:

The rate of return on credit card debt increases quickly without payments, to your disadvantage. By making small regular payments over the term of the loan, you can significantly reduce the amount of interest you pay, resulting in a better rate of return on your loan.

12. Ming has a credit card debt of $4100 for furniture purchased for her new condo. The interest rate on her balance is 19.99%, compounded monthly.

a) If Ming wants to pay off the furniture with regular monthly payments over the next two years, what amount should she pay each month?

Answer:

```
  N=24
  I%=19.99
  PV=4100
 ▪PMT=⁻208.65276…
  FV=0
  P/Y=12
  C/Y=12
  PMT:END  BEGIN
```

Her payments should be $208.65 per month.

b) Using the regular monthly payment you calculated, how long will it take for her to pay back half of her debt?

Answer:

```
 ▪N=13.18180807
  I%=19.99
  PV=4100
  PMT=⁻208.65276…
  FV=⁻2050
  P/Y=12
  C/Y=12
  PMT:END  BEGIN
```

After making her fourteenth payment, she will have eliminated half of the original debt.

Learning Activity 5.5

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. State the formula for the area of a triangle.
2. State the formula for the volume of a cylinder.
3. State the formula for the surface area of a cube.
4. List all the factors of 72.

Approximately how long will it take an investment of $9274 to double, if compounded annually at the following rates?

5. 4%
6. 8%
7. 3%
8. 12%

Answers:

1. $A = \dfrac{bh}{2}$
   (where A = area in units^2, b = length of base, and h = height of triangle)
2. $V = \pi r^2 h$
   (where V = volume in units^3, r = length of radius, and h = height of cylinder)
3. $SA = 6s^2$
   (where SA = surface area in units^2, 6 being the number of square faces, and s = side length of square face)
4. 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72
5. 18 years $\left(\dfrac{72}{4} = 18\right)$
6. 9 years $\left(\dfrac{72}{8} = 9\right)$
7. 24 years $\left(\dfrac{72}{3} = 24\right)$
8. 6 years $\left(\dfrac{72}{12} = 6\right)$

Part B: Investment Portfolios

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Adrian thinks he can afford to pay $1000 per month for a mortgage payment for a property that has property taxes worth $205 per month and heating costs estimated at $65 per month. His gross monthly income is $3750 per month. Should he expect the bank to lend him the money?

   Answer:

   Calculate the Gross Debt Service ratio—it must be less than 32%.

   $$\text{GDS ratio} = \frac{\begin{pmatrix}\text{monthly} & & \text{monthly} & & \text{monthly} & & \text{one-half} \\ \text{mortgage} & + & \text{property} & + & \text{heating} & + & \text{monthly} \\ \text{payment} & & \text{taxes} & & \text{costs} & & \text{condo fees}\end{pmatrix}}{\text{Gross Monthly Income}}$$

   $$\text{GDS ratio} = \frac{1000 + 205 + 65}{3750}$$

   $$\text{GDS ratio} = \frac{1270}{3750} = 0.339$$

   The Gross Debt Service ratio is 33.9%. Since it is above 32%, the bank is not likely to give him the mortgage.

2. Brett and Megan have a combined annual salary of $80,000. What is the maximum monthly mortgage payment they could afford if property tax is estimated at $180 per month and heating is estimated at $70 per month?

   Answer:

   The maximum GDS ratio is 32%. Calculate 32% of gross monthly income.

   Total amount to spend: $0.32 \times (80000 \div 12) = \$2133.33$ monthly

   Mortgage payment: $2133.33 - 180 - 70 = \$1883.33$

   After paying property tax and heating, Brett and Megan can afford $1883.33 for a mortgage each month.

3. Sam and Caitlyn own a $25,000 car with an outstanding car loan of $15,000. They own a home valued at $225,000 with a mortgage of $210,000. They have student loans totalling $5000. What is their net worth?

Answer:

First, find assets and liabilities:

Assets = 25000 + 225000 = 250000

Liabilities = 15000 + 210000 + 5000 = 230000

Net worth = total assets − total liabilities

Net worth = 250000 − 230000

Sam and Caitlyn's net worth is $20000.

4. Judith is a 52-year-old accountant who has invested her money in four different options, each with a different risk factor and rate of return. Determine the average rate of return on her investment portfolio.

Answer:

Investment	Amount	ROR	Return
Bank Account	$8155	0.3%	24.465
Term Deposit	$11,076	1.1%	121.836
GIC	$64,144	2.9%	1860.176
Mutual Fund	$149,360	6.8%	10 156.48
Totals	$232,735		12 162.957

$$\text{ROR} = \frac{12\ 162.957}{232\ 735}$$

$$\text{ROR} = 5.226\%$$

Her average rate of return is 5.2%.

5. For a certain investment portfolio, 60% is held in a variety of S&P/TSX Composite indexed stocks and 40% is in a mutual fund for a period of one year. The portfolio was valued at $26,500 at the start of the year, the mutual fund earned 4.8% compounded semi-annually, and the Index value was 12 566 one year ago and today is 11 977. Determine the current value of the portfolio.

   Answer:

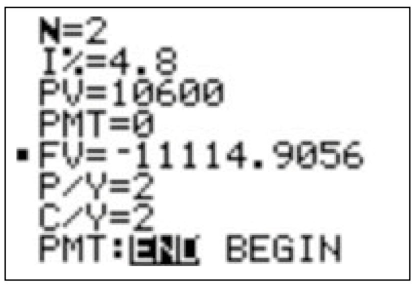
   $$\text{ROR} = \frac{11977 - 12566}{12566} = -0.04687$$

   60% of $26,500 is $15,900. This is invested in stocks that lose about 4.687% of their value.

   0.04687 × 15900 = $745.233 loss.

   Current value of shares: $15,154.73

   40% of $26,500 is $10,600.

   This amount is invested in a mutual fund. Its final value is $11,114.91.

   The portfolio is currently worth $15,154.73 + $11,114.91 = $26,269.64.

6. Ben's stockbroker charges a flat rate of $45 per transaction. If he purchases and sells the following shares at the listed prices, determine his average rate of return on this portfolio. What advice would you give Ben about his investment?

	Purchase Price of Shares	**Number of Shares Purchased**	**Value of Shares**	**Broker Fee**	**Change in Price +/−**	**Selling Price of Shares**	**Value of Shares**	**Gain or Loss**	**Broker Fee**
Daily Java Ltd.	58.22	20	$1164.40	$45	+1.50	59.72	$1194.40	+30.00	$45
Fashionista Co.	1.57	400	$628.00	$45	−0.12	1.45	$580.00	−48.00	$45
Home Life Concepts	3.65	150	$547.50	$45	+0.39	4.04	$606.00	+58.50	$45
Metals Mining	0.88	1800	$1584.00	$45	+0.09	0.97	$1746.00	+162.00	$45
		Total	$3923.90	$180.00			$4126.40	+202.50	$180.00

Including the broker fees to purchase and sell, Ben has an overall loss on this portfolio of

$$\text{ROR} = \frac{4126.40 - (3923.90 + 180 + 180)}{3923.90}$$

$$\text{ROR} = \frac{-157.50}{3923.90}$$

$$\text{ROR} = -4.01\%$$

Ben may want to consider holding these stocks for a longer time frame in order to realize a better return before paying the broker fees.

7. Rachel owns 600 units of a mutual fund. The value when she purchased was $5.99/unit. The first year the rate of return was 3.5%; the second year, the rate was 2.9%; and the third year, the rate was 4.1% (compounded annually). At the end of each year, Rachel must pay a management fee on the fund worth 1.1% of her investment. Calculate the value of her mutual fund and the rate of return if she were to sell the units after the third year.

Answer:

$600 \times 5.99 = 3594.00$

Rachel's initial investment was $3594.00.

```
 N=1
 I%=3.5
 PV=3594
 PMT=0
▪FV=-3719.79
 P/Y=1
 C/Y=1
 PMT:END BEGIN
```

After the first year, the fund is valued at \$3719.79. This amount, less the management fee of (3719.79 × 0.011), \$40.92, is invested for the second year.

```
 N=1
 I%=2.9
 PV=3678.87
 PMT=0
▪FV=-3785.55723
 P/Y=1
 C/Y=1
 PMT:END BEGIN
```

After the second year, it is worth \$3785.56. The management fee will be (3785.56 × 0.011), \$41.64. She invests the difference (3785.56 − 41.64).

```
 N=1
 I%=4.1
 PV=3743.92
 PMT=0
▪FV=-3897.42072
 P/Y=1
 C/Y=1
 PMT:END BEGIN
```

At the end of the third year, she has \$3897.42 less the management fee of \$42.87, or \$3854.55, invested in the fund.

Her earnings over the three years are \$3854.55 − \$3594.00 = \$260.55.

Her three-year rate of return on this mutual fund is 7.25%.

$$\text{ROR} = \frac{3854.55 - 3594.00}{3594.00}$$

$$\text{ROR} = 7.25\%$$

8. Laura has $5000 saved and can contribute $50 per month from her paycheque to either a GIC, earning 4.5%, compounded monthly, or to her outstanding credit card debt of $5000, which has an interest rate of 19.99%, compounded monthly. What would you suggest she do with the lump sum amount and her monthly contributions over the next 5 years? Support your answer with TVM or other software calculations.

   Answer:

   There are many different possible options for her.

   Option 1: She could invest the lump sum and monthly payments in the GIC and carry her credit card debt for 5 years. In this scenario, the GIC would be worth $9616.26 and she would owe $13,473.22 on the credit card.

   GIC:

   ```
    N=60
    I%=4.5
    PV=5000
    PMT=50
   ▪FV=-9616.25671
    P/Y=12
    C/Y=12
    PMT:END BEGIN
   ```

   Credit card:

   ```
    N=5
    I%=19.99
    PV=5000
    PMT=0
   ▪FV=-13473.22286
    P/Y=1
    C/Y=12
    PMT:END BEGIN
   ```

   Option 2: If she invested the $5000 in the GIC and used the $50 as a monthly payment on her credit card, the GIC would be worth $6258.98 and the credit card debt would be $8386.75.

   GIC:

   ```
    N=5
    I%=4.5
    PV=5000
    PMT=0
   ▪FV=-6258.979103
    P/Y=1
    C/Y=12
    PMT:END BEGIN
   ```

   Credit card:

   ```
    N=60
    I%=19.99
    PV=5000
    PMT=-50
   ▪FV=-8386.745907
    P/Y=12
    C/Y=12
    PMT:END BEGIN
   ```

Option 3: If Laura would pay off the entire credit card debt immediately and invest the $50 per month in the GIC, after 5 years she would be debt-free and have $3357.28 invested in the GIC.

GIC:

```
 N=60
 I%=4.5
 PV=0
 PMT=50
▪FV=-3357.277607
 P/Y=12
 C/Y=12
 PMT:END BEGIN
```

Conclusion: Even with the lump sum investment and regular contributions to the GIC, the credit card debt climbs close to $13,500 and quickly exceeds any gains made in the investment. Even with minimum payments to the credit card, the debt surpasses the value of the GIC. I would strongly suggest she pay off her credit card debt immediately and engage in a regular investment strategy as in Option 3.

9. Wanda would like to go on a trout fishing trip this fall but is a little short on cash. What investment strategy would you suggest she use with her $500 capital to have enough to fund the trip in three months time? Justify your answer.

   Answer:

   Considering the very short time frame, Wanda will likely have to invest in equities to see growth. However, since the timeframe is only a few months, she will not have enough time for her investment to grow significantly. Depending on Wanda's preference for high- or low-risk investments, she may purchase stocks or units in a mutual fund, but she will have to research carefully to determine market trends and volatility of the stocks or funds, to match her tolerance. The higher the return she needs on her investment, the higher the risk for the fund she will likely have to invest in.

10. Compare the following portfolios for strengths and weaknesses. If you were a financial advisor, what advice would you give these investors?

Marin:

Canada Savings Bond	$29,372
Savings Account	$2,855

Phillip:

Chequing Account:	$3,800
Term Deposit	$9,100
GIC	$5,827
Stocks	$11,945

Answer:

The diversity in Phillip's account reflects a positive approach to investing. He has some cash, fixed income investment, and equity investment products. These investments give him accessibility (liquidity), growth, and fixed income options. He is prepared to meet small financial emergencies such as a car repair with his chequing account or term deposit, and he will hopefully see some income from the safety of the GIC, if it is held for a length of time. If he watches carefully, a variety of stocks can help him balance his portfolio and help him earn a good return over time, but he will have to monitor the cost of broker fees against his return.

Marin's portfolio is lacking the diversity found in Phillip's portfolio. While a CSB is secure, the principal can only be withdrawn at the anniversary date or she will have to forfeit some interest. If she wants to put a down payment on a house, pay for some university courses, or go on vacation, she may want to transfer some money to a more liquid account. If she can tolerate the risk, she might consider investing a portion of her savings bond in a mutual fund, which would potentially grow the principal more quickly.

11. Suli has the following investment portfolio. Determine the total value of her portfolio and her average ROR after one year.

■ Stocks

	Stock Purchase Price	Number of Shares Purchased	Value of Shares	Change in Price (+/−) Over One Year	Selling Price of Stock	Value of Shares
Gold E Corporation	1.22	542		+0.79		
Space Science Ltd.	6.46	120		−0.05		
Cell Technology	0.53	847		+1.04		
Best Insurance Co.	27.80	100		+0.63		
		Total				

■ Mutual Fund: $13 400 currently invested at 7%, compounded annually.

■ Savings account: $3900 at 0.05%, compounded monthly. She saves about $125 per month in this account (after expenses and liabilities are paid).

■ Employee RRSP: $20,800 currently invested at 3.5%, compounded semi-annually. She contributes $50 per month.

Answer:

	Stock Purchase Price	Number of Shares Purchased	Value of Shares	Change in Price (+/–) Over One Year	Selling Price of Stock	Value of Shares
Gold E Corporation	1.22	542	$661.24	+0.79	2.01	$1089.42
Space Science Ltd.	6.46	120	$775.20	−0.05	6.41	$769.20
Cell Technology	0.53	847	$448.91	+1.04	1.57	$1329.79
Best Insurance Co.	27.80	100	$2780	+0.63	28.43	$2843
		Total	$4665.35			$6031.41

Stocks went from $4665.35 to $6031.41.

```
N=1
I%=7
PV=13400
PMT=0
▪FV=-14338
P/Y=1
C/Y=1
PMT:END BEGIN
```

Mutual fund went from \$13,400 to \$14,338.

```
N=12
I%=.05
PV=3900
PMT=125
▪FV=-5402.294245
P/Y=12
C/Y=12
PMT:END BEGIN
```

Savings account went from \$3900 + 125 × 12 to \$5402.29.

```
N=12
I%=3.5
PV=20800
PMT=50
▪FV=-22144.01839
P/Y=12
C/Y=2
PMT:END BEGIN
```

RRSP went from \$20,800 + 12 × 50 to \$22,144.02.

Value of portfolio:

Amount invested:

\$4665.35 + \$13,400.00 + \$3900.00 + (12 × 125) + \$20,800 + (12 × 50) = \$44,865.35

Value of portfolio at end of year:

\$6031.41 + \$14,338 + \$5402.29 + 22,144.02 = \$47,915.72

Return:

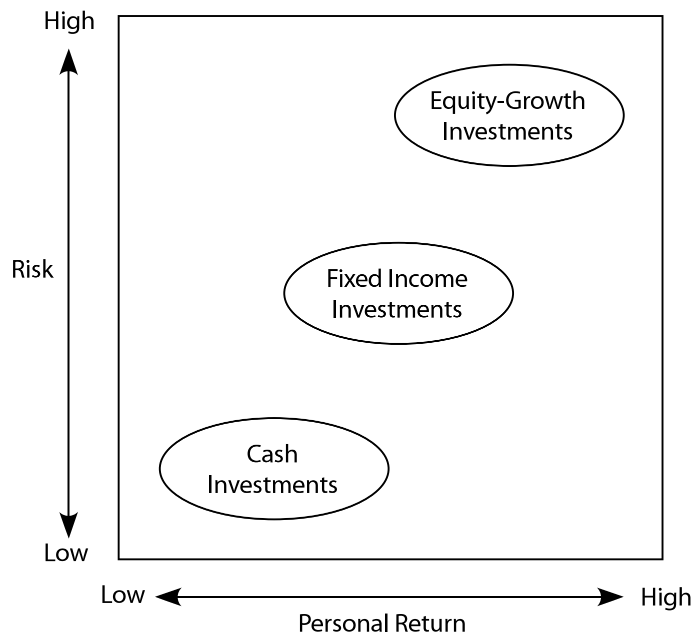
\$47,915.72 − \$44,865.35 = \$3050.37

Rate of return:

$$\frac{3050.37}{44865.35} = 6.8\%$$

12. Place “cash investments,” “fixed income investments,” and “equity investments” on the chart below, indicating where, in general, they fall in terms of the degree of risk and the potential for growth on return.

Answer:



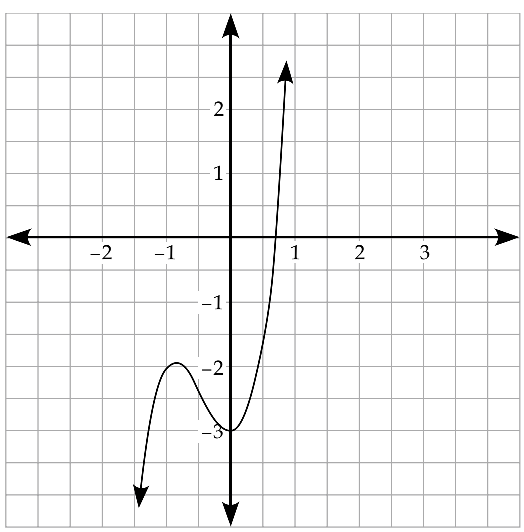
Note that this is a generalization only. There are some equity investments that are lower risk, and the return on other investments may depend on the length of term for which they are held.

Learning Activity 5.6

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Using each of the digits 4, 6, 8, and 8, some of the four common mathematical operations, and brackets, create an expression equal to 40.
2. Raj buys 10 bags of oranges, each of which contains 20 oranges. If his family eats 8 oranges a day, how many days will it take him to eat the 10 bags of oranges?
3. A set of integers has a sum of 240 and a mean of 30. How many integers are in the set?
4. Approximate the temperature 82 °F in °C, given the conversion formula, $C = \frac{5}{9}(F - 32)$.
5. Saturn and Morweena are playing six games of tennis against each other. The probability that each of them will win three of the 6 games is $\frac{4}{9}$. What is the probability that one of them will win more games than the other?
6. Describe the end behaviour of the following polynomial graph.



7. Write 1 as a power of 53.
8. Write the converse of $p \rightarrow q$.

Answers:

1. ((8 − 6) + 8) × 4 or (8 × 8) − (4 × 6) or (8 + 8) + (4 × 6)
   (Other solutions are possible.)
2. 25 (10 × 20 = 200; 200 ÷ 8 = 25)
3. 8 (240 ÷ 30 = 8)
4. Approximately 28 °C (82 − 32 = 50; $\frac{5}{9}$ is just over half, so a bit more than half of 50 is 27 or 28)
5. $\frac{5}{9}$ (There are two possibilities—either each player wins three games or one player wins more than three. Since the probability that one player wins three games is $\frac{4}{9}$, then the probability that one player wins more than three games is $1 - \frac{4}{9} = \frac{5}{9}$.)
6. Quadrant III to Quadrant I
7. $53^0 = 1$ (any base raised to the power of zero is equal to 1)
8. $q \rightarrow p$

Part B: Buying, Renting, and Leasing

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. The residual value of a luxury car is about 40% after a four-year lease. If the purchase price was $84,600, determine its residual value. Calculate the depreciation.

   Answer:

   40% of $84,600 is $33,840 (The residual value is $33,840.)

   $84,600 − $33,840 = $50,760 (It has depreciated by $50,760.)

   Or,

   60% of $84,600 = $50,760 (residual is 40%, so depreciation is 60%)

2. Arthur has office equipment valued at $15,000. This year, he can write off depreciation of 7% of the equipment's value as a business expense. Determine the amount that can be claimed as a business expense.

   Answer:

   $0.07 \times 15000 = 1050$ (He can write off $1050 for depreciation.)

3. The value of a property appreciates at a rate of 3.5% per year. If the property is purchased for $189,000 and then sold 15 years later, what could the home owners propose as a selling price?

   Answer:

   $\$189{,}000 \times 1.035^{15} = \$316{,}640.93$

   They could ask for about $316,641.

4. Rare artwork appreciates in value by about 1.8% per year. Jody's grandmother purchased a painting 70 years ago for $1000. Determine its value now.

   Answer:

   $\$1000 \times 1.018^{70} = \$3486.14$

   The painting is now worth about $3486.

5. The value of a mountain bike is calculated to depreciate to $0 over 5 years, using a "straight line method." In this case, the value of the bike decreases by one-fifth of its original value each year. If the bike was purchased for $650, how much does it decrease in value each year?

   Answer:

   $$\frac{650}{5} = 130$$

   The bike decreases in value by $130 per year.

6. A computer depreciates in value by about 25% per year. Ross spends $800 on a new computer for university. How much is it worth four years later? Calculate the depreciation.

   Answer:

   $\$800 \times 0.75^{4} = \$253.125$ (It retains 75% of its value each year.)

   It is worth about $253.13 after four years.

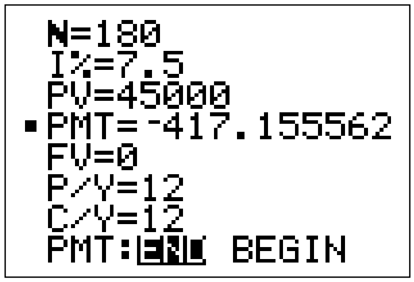
   $\$800 - \$253.125 = \$546.875$

   The depreciation on the computer is $546.88.

7. The Browns have an opportunity to purchase a retirement home for $50,000, with a down payment of $5000, or rent a similar home for $525 per month.

   a) Use technology to determine their monthly payment for the mortgage if it is negotiated to be 7.5% for 15 years (monthly amortization).

      Answer:

      Their payment will be about $417.16. If you used an online mortgage calculator, your answer may be slightly different.

   b) If the property tax works out to approximately 1.5% of the market value of the property, how much will they spend on the mortgage and taxes during the first year of the purchase?

      Answer:

      $0.015 \times \$50,000 = \$750$

      They will pay $750 in property taxes.

      $\$417.16 \times 12 = \$5005.92$

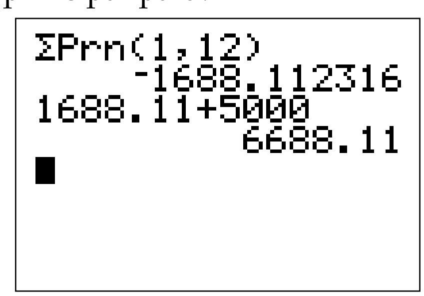
      The mortgage payments for the first year will be $5005.92.

      During the first year, they will spend $5755.92 on the mortgage and taxes.

   c) Assuming the value of the home has not changed, how much equity will they have in their home after one year?

      Answer:

      The amount of equity can be found by adding the down payment amount to the principal amount paid in the first year. You can use the Sum of Principal function of the graphing calculator to find the amount of principal paid.

Or, use the TVM solver to find the future value of the loan after 12 months.

```
  N=12
  I%=7.5
  PV=45000
  PMT=-417.55562
■ FV=
  P/Y=12
  C/Y=12
  PMT: [END] BEGIN
```

The future value (FV) of the mortgage is \$43,311.89, so the equity is \$50,000.00 − \$43,311.89 = \$6,688.11.

They will have equity of about \$6,688.11 after the first year.

d) If the value of the house appreciates by 2% per year, what is the increase in value of the house after one year? How does this affect their equity?

Answer:

$0.02 \times \$50,000 = \$1000$

The house increases in value by \$1000, and is now worth \$51,000. After the first year, their equity would increase by an additional \$1000.

e) If instead, they decide to rent a similar house for \$525 per month, how much of their income will they have spent on rent during the first year?

Answer:

$\$525 \times 12 = \$6300$

They will have spent \$6300 on rent during the first year.

f) If the Browns decide to rent, they would take the \$5000 that would have been the down payment on the purchase and invest it in a mutual fund at 7%, compounded annually. How much would the investment be worth after one year?

Answer:

$\$5000 \times 1.07^1 = \$5350$

The investment would be worth \$5350 after one year.

g) Create a spreadsheet to determine, after any number of years:

- the value of the home if it appreciates by 2% per year
- the taxes due (1.5% of market value of the home)
- the total amount of rent paid, if the rent increases by 3% per year
- the value of an investment of \$5000 at 7% compounded annually

Answer:

	L	M	N	O	P	R	S	T	V
1	Year	Value of Home	Additional equity	Property taxes	Total taxes paid	Monthly Rent	Annual rent paid	Total rent paid	Year end Value of Investment
2	0								5000
3	1	50000.00	0.00	750.00	750.00	525.00	6300.00	6300.00	5350.00
4	2	51000.00	1000.00	765.00	1515.00	540.75	6489.00	12789.00	5724.50
5	3	52020.00	2020.00	780.30	2295.30	556.97	6683.67	19472.67	6125.22
6	4	53060.40	3060.40	795.91	3091.21	573.68	6884.18	26356.85	6553.98
7	5	54121.61	4121.61	811.82	3903.03	590.89	7090.71	33447.56	7012.76
8	6	55204.04	5204.04	828.06	4731.09	608.62	7303.43	40750.98	7503.65
9	7	56308.12	6308.12	844.62	5575.71	626.88	7522.53	48273.51	8028.91
10	8	57434.28	7434.28	861.51	6437.23	645.68	7748.21	56021.72	8590.93
11	9	58582.97	8582.97	878.74	7315.97	665.05	7980.65	64002.37	9192.30
12	10	59754.63	9754.63	896.32	8212.29	685.01	8220.07	72222.44	9835.76
13	11	60949.72	10949.72	914.25	9126.54	705.56	8466.67	80689.11	10524.26
14	12	62168.72	12168.72	932.53	10059.07	726.72	8720.67	89409.79	11260.96
15	13	63412.09	13412.09	951.18	11010.25	748.52	8982.29	98392.08	12049.23
16	14	64680.33	14680.33	970.20	11980.45	770.98	9251.76	107643.84	12892.67
17	15	65973.94	15973.94	989.61	12970.06	794.11	9529.32	117173.16	13795.16
18	16	67293.42	17293.42	1009.40	13979.46	817.93	9815.19	126988.35	14760.82
19	17	68639.29	18639.29	1029.59	15009.05	842.47	10109.65	137098.00	15794.08
20	18	70012.07	20012.07	1050.18	16059.23	867.75	10412.94	147510.94	16899.66
21	19	71412.31	21412.31	1071.18	17130.42	893.78	10725.33	158236.27	18082.64
22	20	72840.56	22840.56	1092.61	18223.03	920.59	11047.09	169283.36	19348.42
23	21	74297.37	24297.37	1114.46	19337.49	948.21	11378.50	180661.86	20702.81
24	22	75783.32	25783.32	1136.75	20474.24	976.65	11719.86	192381.72	22152.01
25	23	77298.98	27298.98	1159.48	21633.72	1005.95	12071.45	204453.17	23702.65
26	24	78844.96	28844.96	1182.67	22816.40	1036.13	12433.60	216886.76	25361.83
27	25	80421.86	30421.86	1206.33	24022.72	1067.22	12806.60	229693.37	27137.16
28	26	82030.30	32030.30	1230.45	25253.18	1099.23	13190.80	242884.17	29036.76

	A	B	C	D	E	G	H	I	K
1	Year	Value of Home	Additional equity	Property taxes	Total taxes paid	Monthly Rent	Annual rent paid	Total rent paid	Year end Value of Investment
2	0								5000
3	1	50000	0	=0.015*B3	=D3	525	=12*G3	=H3	=1.07*K2
4	2	=1.02*B3	=B4-50000	=0.015*B4	=SUM(D$3:D4)	=1.03*G3	=12*G4	=SUM(H$3:H4)	=1.07*K3
5	3	=1.02*B4	=B5-50000	=0.015*B5	=SUM(D$3:D5)	=1.03*G4	=12*G5	=SUM(H$3:H5)	=1.07*K4
6	4	=1.02*B5	=B6-50000	=0.015*B6	=SUM(D$3:D6)	=1.03*G5	=12*G6	=SUM(H$3:H6)	=1.07*K5
7	5	=1.02*B6	=B7-50000	=0.015*B7	=SUM(D$3:D7)	=1.03*G6	=12*G7	=SUM(H$3:H7)	=1.07*K6
8	6	=1.02*B7	=B8-50000	=0.015*B8	=SUM(D$3:D8)	=1.03*G7	=12*G8	=SUM(H$3:H8)	=1.07*K7
9	7	=1.02*B8	=B9-50000	=0.015*B9	=SUM(D$3:D9)	=1.03*G8	=12*G9	=SUM(H$3:H9)	=1.07*K8
10	8	=1.02*B9	=B10-50000	=0.015*B10	=SUM(D$3:D10)	=1.03*G9	=12*G10	=SUM(H$3:H10)	=1.07*K9
11	9	=1.02*B10	=B11-50000	=0.015*B11	=SUM(D$3:D11)	=1.03*G10	=12*G11	=SUM(H$3:H11)	=1.07*K10
12	10	=1.02*B11	=B12-50000	=0.015*B12	=SUM(D$3:D12)	=1.03*G11	=12*G12	=SUM(H$3:H12)	=1.07*K11

h) Summarize the information gathered to determine when renting is more advantageous and when purchasing is more advantageous.

Answer:

During the first year, the amount spent on rent ($6300) would be greater than the amount spent on mortgage payments and taxes ($5755.92), but utilities have not been factored into this calculation. It is possible that the rent payment includes some utilities. As well, there would be some significant additional expenses for the closing costs and general maintenance of the home and yard. If the plan is to stay in the current location for only one year, it is probably better to rent than to buy.

If the Browns rent, their housing costs by the end of the second year would total $12,789, and their investment of $5000 would be worth $5724.50 (earning $724.50 in interest), for a net cost of $12,064.50.

If the Browns purchase the property, by the second year they will have paid $3507.28 towards the principal of the mortgage. This, plus the down payment of $5000, boosts their equity. If the appreciated value in the home is added ($2020), their equity is $10,527.28. Their total costs for the two years include the mortgage payments, taxes, and down payment.

	Buying		**Renting**	
Costs	Down payment	$ 5,000.00	Investment principal	$ 5,000.00
	Mortgage payments (417.16 × 12 × 2)	$ 10,011.84	Rent	$ 12,789.00
	Taxes	$ 1,515.00	Utilities	
	Closing costs		Parking or fees	
	Utilities		COMPARE	
Total Paid		$ 16,526.84		$ 17,789.00
Credits	Owner's equity	$ 8,507.28	Value of investment	$ 5,724.50
	Appreciation in value of home	$ 2,020.00		
Equity		$ 10,527.28		$ 5,724.50
Difference		$ –5,999.56		$ –12,064.50

After two years, their costs are similar, but with the purchase of the home rather than renting their equity is almost twice as high.

8. Neta gets a new job but must relocate for five years. She can either rent a furnished apartment or purchase a condominium. She has $36,000 equity from the sale of her previous home to be used as a down payment on the purchase of the condo listed at $199,000 or she can invest it in a term deposit at 5% compounded annually. Neta is offered a mortgage at 5.5%, compounded and paid monthly, amortized over 15 years. The condo appreciates in value at 4% per year. Property taxes and condo fees are 1.8% of the property value each year. If she buys the condo, she will need to add about 3% of the purchase price to the mortgage to cover closing and moving costs, and to purchase furniture. Utilities in the condo are about $500 per month and will increase by 2.2% annually. In the apartment, she must pay rent of $975 per month, including utilities and parking. Rent is expected to increase by 2.2% per year.

   Use technology to compare the costs of renting the apartment with the cost of buying the condo, including the impact of using her $36,000.00 equity as either a down payment or an investment. What recommendations would you make to Neta, on the basis of this comparison, for her five-year relocation?

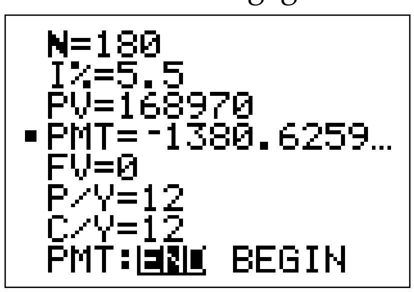
   Answer:

   First, find Neta's mortgage payment:

   3% of the purchase price is added to the mortgage to cover additional costs: 0.03 × $199,000 = $5970.00

   $199,000 + $5970 − $36,000 = $168,970

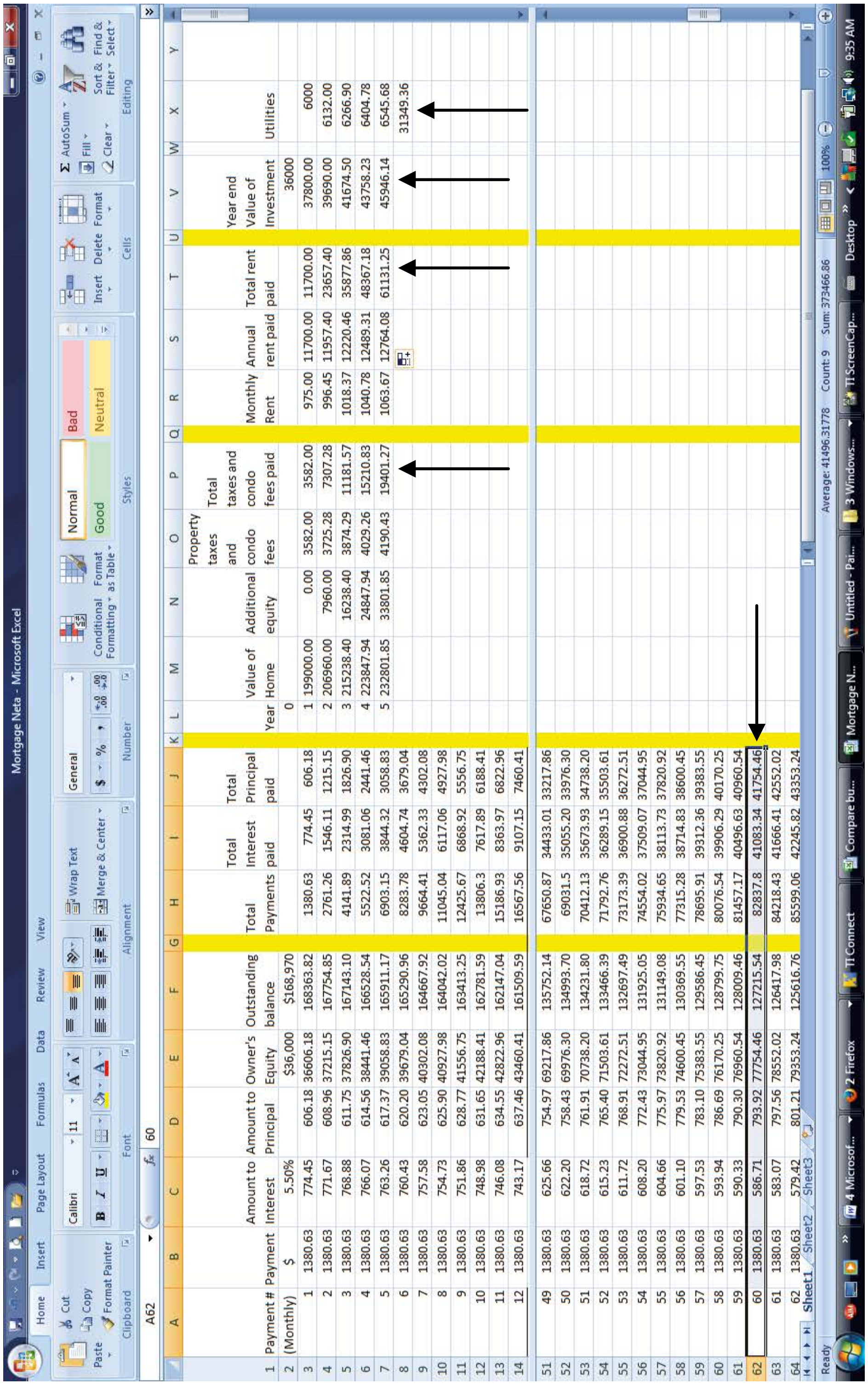
   She will need a mortgage for $168,970.



   Her monthly payments will be $1380.63.

   All the given information is entered into the spreadsheet. The screen is split to show the first and fifth years (the 60th month is the end of the 5th year) of the mortgage, and five years of the other calculations.

   The spreadsheet template used here is the same one used in Example 2 of Lesson 6.



Payment # (Monthly)	Payment $	Amount to Interest 5.50%	Amount to Principal	Owner's Equity $36,000	Outstanding balance $168,970	Total Payments	Total Interest paid	Total Principal paid	Year	Value of Home	Additional equity	Property taxes and condo fees	Total taxes and condo fees paid	Monthly Rent	Annual rent paid	Total rent paid	Year end Value of Investment	Utilities
									0								36000	
1	1380.63	774.45	606.18	36606.18	168363.82	1380.63	774.45	606.18	1	199000.00	0.00	3582.00	3582.00	975.00	11700.00	11700.00	37800.00	6000
2	1380.63	771.67	608.96	37215.15	167754.85	2761.26	1546.11	1215.15	2	206960.00	7960.00	3725.28	7307.28	996.45	11957.40	23657.40	39690.00	6132.00
3	1380.63	768.88	611.75	37826.90	167143.10	4141.89	2314.99	1826.90	3	215238.40	16238.40	3874.29	11181.57	1018.37	12220.46	35877.86	41674.50	6266.90
4	1380.63	766.07	614.56	38441.46	166528.54	5522.52	3081.06	2441.46	4	223847.94	24847.94	4029.26	15210.83	1040.78	12489.31	48367.18	43758.23	6404.78
5	1380.63	763.26	617.37	39058.83	165911.17	6903.15	3844.32	3058.83	5	232801.85	33801.85	4190.43	19401.27	1063.67	12764.08	61131.25	45946.14	6545.68
6	1380.63	760.43	620.20	39679.04	165290.96	8283.78	4604.74	3679.04										31349.36
7	1380.63	757.58	623.05	40302.08	164667.92	9664.41	5362.33	4302.08										
8	1380.63	754.73	625.90	40927.98	164042.02	11045.04	6117.06	4927.98										
9	1380.63	751.86	628.77	41556.75	163413.25	12425.67	6868.92	5556.75										
10	1380.63	748.98	631.65	42188.41	162781.59	13806.3	7617.89	6188.41										
11	1380.63	746.08	634.55	42822.96	162147.04	15186.93	8363.97	6822.96										
12	1380.63	743.17	637.46	43460.41	161509.59	16567.56	9107.15	7460.41										
49	1380.63	625.66	754.97	69217.86	135752.14	67650.87	34433.01	33217.86										
50	1380.63	622.20	758.43	69976.30	134993.70	69031.5	35055.20	33976.30										
51	1380.63	618.72	761.91	70738.20	134231.80	70412.13	35673.93	34738.20										
52	1380.63	615.23	765.40	71503.61	133466.39	71792.76	36289.15	35503.61										
53	1380.63	611.72	768.91	72272.51	132697.49	73173.39	36900.88	36272.51										
54	1380.63	608.20	772.43	73044.95	131925.05	74554.02	37509.07	37044.95										
55	1380.63	604.66	775.97	73820.92	131149.08	75934.65	38113.73	37820.92										
56	1380.63	601.10	779.53	74600.45	130369.55	77315.28	38714.83	38600.45										
57	1380.63	597.53	783.10	75383.55	129586.45	78695.91	39312.36	39383.55										
58	1380.63	593.94	786.69	76170.25	128799.75	80076.54	39906.29	40170.25										
59	1380.63	590.33	790.30	76960.54	128009.46	81457.17	40496.63	40960.54										
60	1380.63	586.71	793.92	77754.46	127215.54	82837.8	41083.34	41754.46										
61	1380.63	583.07	797.56	78552.02	126417.98	84218.43	41666.41	42552.02										
62	1380.63	579.42	801.21	79353.24	125616.76	85599.06	42245.82	43353.24										

After five years, Neta has equity valued at $77,754.46, plus an additional $33,801.85, considering the appreciated value of the home, for a total of $111,556.31.

Her expenses over the five years total $169,588.43, including:

down payment	$36,000.00
mortgage payments	$82,837.80 (1380.63 × 5 × 12)
taxes and condo fees	$19,401.27 (see spreadsheet)
utilities	$31,349.36 (see spreadsheet)

If Neta purchases the condo, the difference between her equity and expenses is:

$111,556.31 – $169,588.43 = –$58,032.12

If Neta rents the apartment for five years, she will have equity in the 5% GIC of $45 946.14 (see spreadsheet).

Her expenses over the five years will be $97,131.25 for the initial investment and rent.

$36,000 + $61,131.25 = $97,131.25 (see spreadsheet)

The difference between Neta's equity and expenses if she rents the furnished apartment is:

$45,946.14 – $97,131.25 = –$51,185.11

	Buying		**Renting**	
Costs	Down payment	$ 36,000.00	Investment principal	$ 36,000.00
	Mortgage payments (1380.63 × 12 × 5)	$ 82,837.80	Rent	$ 61,131.25
	Taxes and condo fees	$ 19,401.27	Utilities	
	Closing costs		Parking or fees	
	Utilities	$ 31,349.43		
Total Paid		$ 169,588.43	← compare →	$ 97,131.25
Credits	Owner's equity	$ 77,754.46	Value of investment	$ 45,946.14
	Appreciation in value of home	$ 33,801.85		
Equity		$ 111,556.31	← compare →	$ 45,946.14
Difference		$ –58,038.06		$ –51,185.11

In either situation, Neta has to pay the cost of living expenses, but they are higher with the condo purchase. The closing costs and additional costs for furniture add a lot of initial expense to condo ownership. She will pay more per month for the mortgage, fees, utilities, and taxes, but in the end, she will have more equity in the condo as compared to renting and investing in the term deposit. The longer she lives in the new location, the smarter it is for her to purchase. If she ends up staying for less than the five years, the start-up costs may prove too heavy and renting the furnished apartment would likely be a smarter financial decision.

9. Bart has been accepted into a four-year residency program for his medical studies in British Columbia. He can rent an apartment, buy a small fixer-upper house, or stay in a "room and board" dorm. What factors should he consider when making his housing decision?

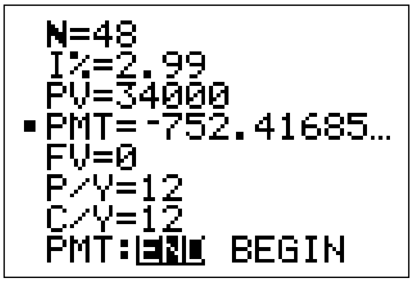
   Answer:

   Bart will have to consider financial factors such as the cost of the monthly mortgage payment, rent, or room and board at the dorm. He will need a down payment for the house and enough money to cover taxes, utilities, repairs, and possibly renovations. The apartment rent may include utilities and the dorm fees will include meals. Another consideration would be the location relative to the university. He may be able to save money on vehicles, parking, and/or transit depending on his proximity to the university. If Bart is studying, he may value a quiet, private house, compared to a busy apartment or a loud dorm room. Depending on the size of the apartment or house, he may be able to split some expenses with roommates. If he does not have the time or the expertise necessary to repair the house, it may be adding more stress on him than he needs during his studies. This is not an exhaustive list of considerations.

10. Guillaume is shopping for a new vehicle. He can purchase a new truck for $38,000, including taxes, with a down payment of $4000 and financing at 2.99% for four years for the remainder. Alternatively, he could lease the truck for the down payment of $4000, plus $536 per month for 48 months. The residual value of the truck after four years is 45%. Compare his options by calculating the costs entailed in buying versus leasing this vehicle.

Answer:

If he purchases the vehicle, Guillaume's monthly payment would be $752.42.



Over the course of the loan, he would pay monthly payments plus a down payment for a total of about $40,116.16.

48 × $752.42 = $36,116.16

$36,116.16 + $4000.00 = $40,116.16

This amounts to about $2116 in interest ($40,116 − $38,000). Once he has paid off the truck, it is worth about $17,100 (residual value is 0.45 × $38,000)—less than half of the amount he paid for it, but it is his.

If he leases the vehicle, the lease would cost him $29,728.00 over the four years.

$536 × 48 = $25,728

$25,728 + $4000 = $29,728

At this point, he has no equity, but he could purchase the truck for $17,100 (either outright or with financing).

The total for him to own the leased vehicle at the end of 4 years is $29,728 + $17,100 = $46,828.

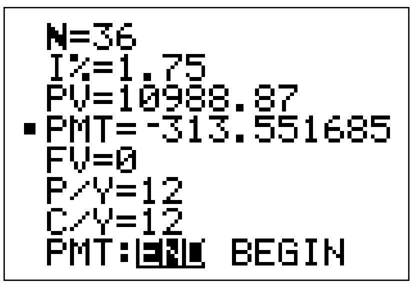
If he wants to own the truck after 4 years, he should purchase rather than lease. If he wants another new vehicle after 4 years, he may prefer to lease the vehicle.

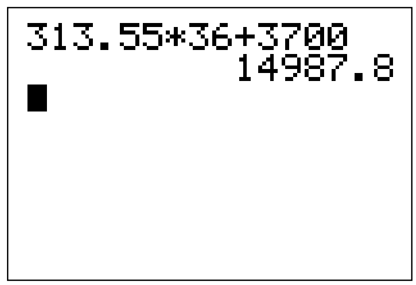
11. Nehal needs to buy a car. She has $3700 saved for a down payment. She is considering either leasing or buying an economical car priced at $12,999, plus 13% tax. With monthly payments, she can finance the car at 1.75% for 36 months or lease it for three years for $175, plus 13% tax per month. The residual value after 36 months is 50%. When the lease is up, Nehal would plan to buy the car with financing at 2.99% over two years. How much would either option cost her? What factors, other than total cost, may influence Nehal's decision?

Answer:

If she chooses to buy, the car costs $12,999 × 1.13 = $14,688.87. Her loan would be for that amount, less the down payment ($14,688.87 − $3,700 = $10,988.87).

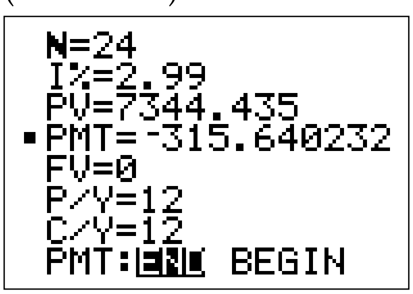
To purchase, Nehal's monthly payments would be $313.55. The total cost would be $14,987.80.





At the end of three years, the car would now be valued at 50% or $6499.50.

If she chooses to lease, the lease would cost $10,819 (36 payments of $175 × 1.13 is $7119, plus a down payment $3700) over the three years. To finance the residual value plus taxes (6499.50 × 1.13 = 7344.435) for 24 months would cost $315.64 per month, or an additional $7575.36 (315.64 × 24).



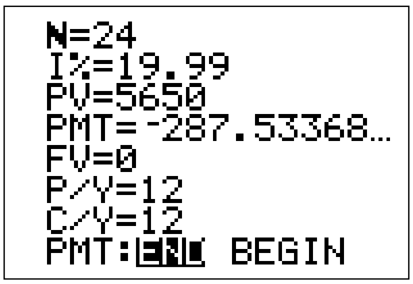
The total cost to lease and then purchase the car would be $18,394.36.

Nehal's lease payments are less than the original loan payments, but in the end it takes two additional years and $3406.56 more to own the car if she leases it first. During the lease period, Nehal may have to restrict the number of kilometres she puts on the car. Financing the car initially would help Nehal to build up her credit rating sooner than waiting three years and then getting financing.

12. Derksen Accounting Ltd. upgrades their computer system every two years. If Mr. Derksen orders $5000 worth of new equipment, he could put the total amount on his business credit card and pay it off over two years at 19.99%, compounded monthly. Alternatively, he could lease the system with monthly payments of $155 per month. Taxes of 13% must be added to the purchase price and lease payment amount. Compare the options by calculating the costs associated with each, and make a recommendation to Mr. Derksen about the computer system.

    Answer:

    To purchase the computers, he would charge $5000 × 1.13 = $5650 on his credit card. His payments would be $287.53, for a total of $6900.72.

    His lease payment would be $155.00 × 1.13 = $175.15.

    Over the two years, he would pay 175.15 × 24 = $4203.60.

    The high interest rate on the credit card, combined with the depreciating value of the computer system, make the purchase option unattractive. The lease payments are less than the loan payments and, at the end of the lease, Mr. Derksen would want to upgrade his system anyway. If he can write off the lease payments as an expense, this may be the better choice for him in this situation. Otherwise, he may want to see about getting a loan from his bank that will offer him a more competitive rate.

13. Dawson would like to drive a new hybrid car. There is one at a local auto dealer for $25,000. He could trade in his current car for $7000 and he has $2000 saved in a term deposit. The dealership is offering financing at 5% for five years (compounded monthly, monthly payments). Taxes are 13%.

    Find:

    a) his monthly payment amount

    Answer:

    GST (5%) and PST (8%) are charged on the price of the vehicle, less the trade-in allowance (tax has already been paid on the used vehicle). Then, the down payment is subtracted.

$25,000 – $7000 = $18 000	Taxes are calculated on $18,000 (the price after trade-in).
$18 000 × 0.13 = $2340	13% taxes.
$2340 + $18,000 = $20,340	Purchase price is $20,340.
$20,340 – $2000 = $18,340	Subtract the down payment to calculate the total amount financed.

    The monthly payment for the five-year term will be $346.10.

    ```
     N=60
     I%=5
     PV=18340
    ▪PMT=-346.098425
     FV=0
     P/Y=12
     C/Y=12
     PMT:END BEGIN
    ```

    b) the total paid for the car

    Answer:

    The total amount paid can be calculated by multiplying the payment amount by the number of payments, plus the trade-in allowance and down payment:

    60 × $346.10 = $20,766 (total payments)

    $20,766 + $7000 + $2000 = $29,766

    The total amount paid for the car is $29,766.

c) the interest paid for the car

Answer:

\$20,766 − \$18,340 = \$2426

Carson pays about \$2426 in interest.

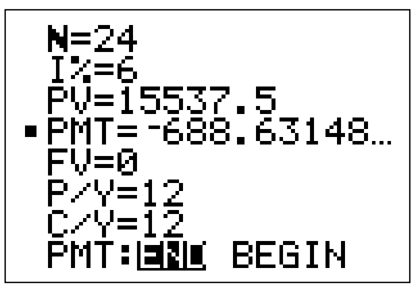
d) If Dawson leases the hybrid car for four years, his monthly payment (after the down payment and trade-in) would be \$325 plus 13% tax. He would have to pay an acquisition fee of \$450 to obtain the lease. The residual value of the car after four years would be 55% of the purchase price. Determine the cost of leasing the vehicle for four years and then financing the residual value of the car over two years at 6%, compounded monthly.

Answer:

Dawson would pay an initial \$7000 + \$2000 + \$450, and 48 monthly payments of \$325, plus 13% tax.

Payments are \$367.25 monthly. He would pay \$27,078 over the four years.

The residual value of the car is 55% or \$13,750. This amount plus 13% taxes is what he would finance for two years at 6%.



The total of his payments to purchase the hybrid would be \$688.63 × 24 = \$16,527.12.

In total, to lease and then buy out the hybrid would cost Dawson \$16,527.12 + \$27,078 = \$43,605.12.

Notes

Grade 12 Applied Mathematics (40S)

Module 6
Techniques of Counting

Module 6: Techniques of Counting

Introduction

Welcome to Module 6. You likely learned to count before you started Kindergarten, so why is a module on counting techniques included in a Grade 12 Applied Mathematics course? Counting techniques, such as the Fundamental Counting Principle, permutations, and combinations, are ways to enumerate or determine the number of elements in a set, or the number of possible outcomes in an experiment, without having to actually count each one. This is a module about counting shortcuts. Who would have thought that everything you need to know about counting, you did not learn in Kindergarten?

This module will continue to develop some of the concepts and skills you used in the probability module.

Assignments in Module 6

When you have completed the assignments for Module 6, submit your completed assignments to the Distance Learning Unit either by mail or electronically through the learning management system (LMS). The staff will forward your work to your tutor/marker.

Lesson	Assignment Number	Assignment Title
	Cover Assignment	Patterns in Numbers
2	Assignment 6.1	Fundamental Counting Principle
5	Assignment 6.2	Applications of Permutations and Combinations

Resource Sheet

When you write your final examination, you are encouraged to take a Final Examination Resource Sheet with you into the examination. This sheet will be one letter-sized page, 8½ ″ by 11″, with both sides in your handwriting or typewritten. You will submit it with your examination, but you do not receive any marks for it.

Many students have found that preparing a resource sheet is an excellent way to review. It provides you with a summary of the important facts of each module. You should complete a resource sheet for each module to help with your studying and reviewing. Lesson summaries and module summaries are included for you to use as a guide.

You may use the list of instructions provided below to help you with preparing your resource sheet for the material in Module 6. On this sheet, you should record math terms and definitions, formulas, sample questions, or a list of places where you often make mistakes. You should also identify special areas that require extra attention or review by writing the page numbers.

After you have completed each module's resource sheet, you may summarize the sheets from Modules 5, 6, 7, and 8 to prepare your Final Examination Resource Sheet. The final examination for this course is based on Modules 5 to 8.

Resource Sheet for Module 6

As you go through the lessons of this module, you may want to consider the following suggestions regarding the creation of a resource sheet.

1. List all the important math terms, and define them if necessary.
2. List all the formulas and perhaps a sample problem that shows how the formula is used.
3. If necessary, write the solutions to some problems, showing in detail how you did the calculations.
4. Copy any questions that represent the key points of the lesson, and perhaps include the solutions as well.
5. Identify the problems you found most difficult, and copy the page numbers onto the resource sheet so that you can review them before writing the examination. You may also copy the problems and the solutions onto your resource sheet, and later write them onto your Final Examination Resource Sheet.
6. Write any comments, ideas, shortcuts, or other reminders that may be helpful during an examination.

Module 6 Cover Assignment: Patterns in Numbers

Numbers can be very intriguing when you stop to consider the infinite patterns that can be found in them. The website "The On-Line Encyclopedia of Integer Sequences® (OEIS®)" is a database of about 200 000 different integer sequences and is found at http://oeis.org/.

If you prefer to see the graphs of 1000 of the sequences, rather than the list of numbers, you can view the YouTube video of The OEIS Movie (8:21) at https://www.youtube.com/watch?v=LCWglXljevY.

Notes

Module 6 Cover Assignment

Patterns in Numbers

Total: 5 marks

Part A: Number Patterns

Consider the following patterns and find the next four numbers in each sequence. Use words to describe how each sequence is generated.

a) 1, 1, 1, 1, ______ , ______ , ______ , ______ . . .

b) 1, 2, 3, 4, ______ , ______ , ______ , ______ . . .

c) 1, 4, 9, 16, ______ , ______ , ______ , ______ . . .

d) 1, 2, 4, 8, 16, ______ , ______ , ______ , ______ . . .

e) 1, 1, 2, 3, 5, 8, ______ , ______ , ______ , ______ . . .

f) 1, 3, 6, 10, 15, ______ , ______ ,______ , ______ . . .

____

____

g) 1, 4, 10, 20, 35, ______ , ______ , ______ , ______ . . .

____

____

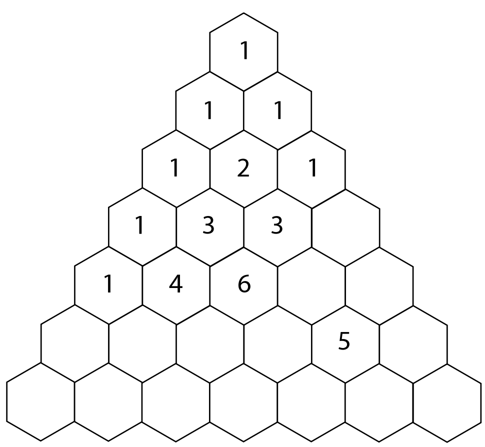
h) 1, 11, 121, 1331, __________ , __________ , __________ , __________ . . .

____

____

Part B: Triangle Number Pattern

Complete and describe the number pattern in the following triangle.



Lesson 1: Fundamental Counting Principle

Lesson Focus

In this lesson, you will

- ❑ represent and solve counting problems, using a graphic organizer
- ❑ generalize, from examples, the Fundamental Counting Principle
- ❑ identify and explain assumptions made in solving a counting problem
- ❑ solve a contextual counting problem, using the Fundamental Counting Principle, and explain the reasoning

Lesson Introduction

You have been counting in mathematics since you first started to work with numbers. When you know the number of items in one set of objects and you add some items from another set of objects, you learned “adding on” as a counting strategy. That strategy is more sophisticated than counting all of the items in the first set again so that you can continue counting all the items in the second set. As well, you learned multiplication as a method of quickly counting several sets of objects with the same number of items in them.

In this lesson, you will continue to learn more sophisticated ways of counting. The strategies that you will learn allow you to count very large sets of numbers, such as the number of ways to create a 6-digit number for a Lotto 649 purchase, or the number of combinations that are possible on a combination lock. These counting strategies are used to make decisions based on probabilities.

Using the Fundamental Counting Principle

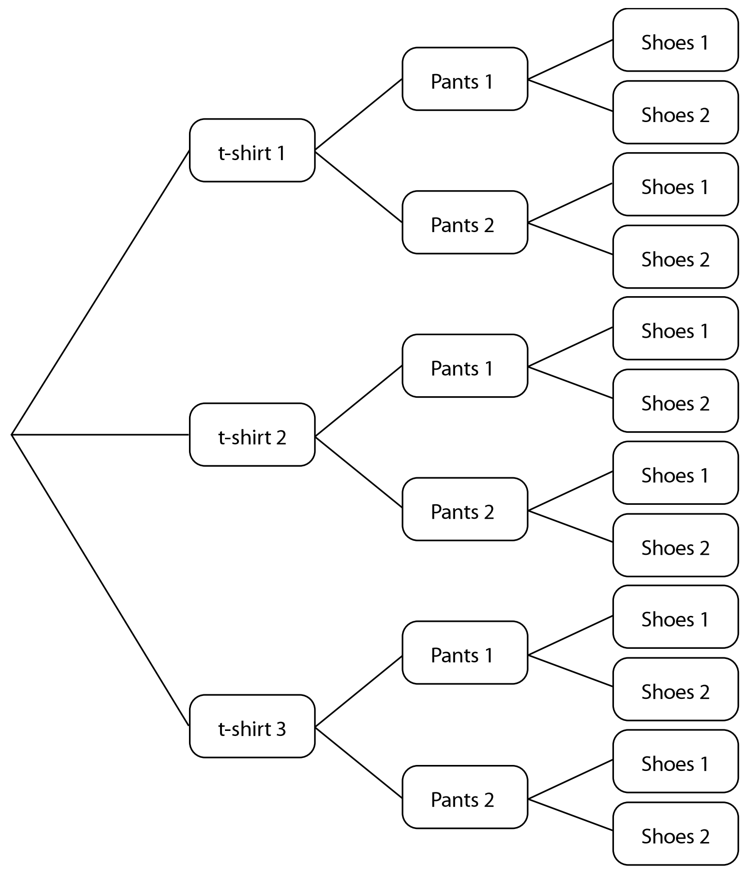
The Fundamental Counting Principle (FCP) allows you to count the number of ways a task can occur, given a series of events.

Example 1

Kelly has three t-shirts, two pairs of pants, and two pairs of shoes. How many different outfits can she wear?

Solution

One way to find all the possible arrangements of tops, pants, and shoes would be to create a tree diagram.



She can put together 12 different possible outfits.

This strategy works for a limited wardrobe, but creating a tree diagram for a situation where there are many more possible arrangements would quickly become inconvenient.

In the probability module, you calculated the likelihood of a specific outcome in the sample space by multiplying the probability of each independent event. A similar calculation can be made to determine the total number of possible outcomes.

The number of ways you can combine three t-shirts with two pairs of pants with two pairs of shoes is given by

# of outfits = (# of t-shirts) × (# of pants) × (# of shoes)

# of outfits = 3 × 2 × 2

# of outfits = 12

In general, the **Fundamental Counting Principle** states that the number of arrangements possible by combining N_1 items with N_2 items with N_3 items, and so on, is given by

$$\text{Arrangements} = N_1 \times N_2 \times N_3 \ldots$$

You may want to add some of this information to your resource sheet.

Example 2

Using the FCP, determine how many possible outcomes there are if you roll a regular 6-sided die and flip a coin.

Solution

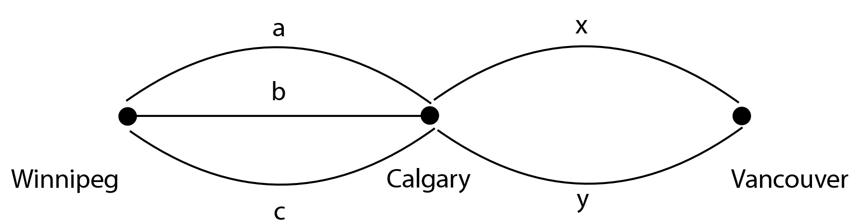
There are two independent events to consider. The die could land in six different ways while the coin has two possible results. Multiply these numbers.

$$\underline{\quad 6 \quad} \times \underline{\quad 2 \quad} = 12$$

The 12 possible arrangements are 1H 2H 3H 4H 5H 6H 1T 2T 3T 4T 5T 6T.

Example 3

Melissa is planning to drive from Winnipeg to Vancouver via Calgary. If there are three roads from Winnipeg to Calgary and two roads from Calgary to Vancouver, how many different routes are possible from Winnipeg to Vancouver, passing through Calgary?



Solution

Melissa needs to make two decisions; which road to take from Winnipeg to Calgary, choosing from road a, b, or c, and which road to take, either x or y, from Calgary to Vancouver.

Step 1: How many decisions must Melissa make? Represent the two decisions with blank lines.

________ ________

Step 2: In how many ways can she make each of these decisions? The first decision has three choices of road. Fill in a 3 on the first blank.

___3___ ________

After making the first decision, she has two choices for her second decision. Fill in the second blank with a 2.

___3___ ___2___

Step 3: Use the FCP to determine the number of ways in which both decisions can be made.

___3___ × ___2___ = 6

There are 6 different routes possible.

The FCP can be extended to include any number of successive decisions.

Example 3: Part 2

If Melissa decided to include the return trip to Winnipeg in her plans, and determined that she did not want to use any road more than once, how many round-trip routes are possible?

Solution

Melissa now has 4 decisions to make: two choices on the way to Vancouver, and two on the return trip. This is represented with four blank lines.

________	________	________	________
W → C	C → V	V → C	C → W

There are three choices of roads from Winnipeg to Calgary and two choices from Calgary to Vancouver. On the return trip, one of the roads between Vancouver and Calgary was used previously so that leaves only one road. Melissa has two remaining roads to choose from between Calgary and Winnipeg.

Using the FCP, you get:

$$\underset{W \rightarrow C}{\underline{\quad 3 \quad}} \times \underset{C \rightarrow V}{\underline{\quad 2 \quad}} \times \underset{V \rightarrow C}{\underline{\quad 1 \quad}} \times \underset{C \rightarrow W}{\underline{\quad 2 \quad}} = 12$$

There are 12 possible round-trip routes if each road can be travelled at most one time.

Counting With Restrictions and Cases

As you saw in the previous example, the number of choices possible for any given decision may be impacted by previous decisions. It is also possible that other factors may restrict specific choices. Since FCP uses multiplication, the commutative and associative properties of mathematics allow you to fill in the blanks representing the decisions in any order.

Example 4

Decorative brass house numbers can be purchased at a local hardware supply store. The only numbers available are 1, 2, 3, 4, and 5.

a) How many different three-digit house numbers can be formed using these numbers, if a number can only be used once?

b) How many of these three-digit house numbers are

  i) even numbers?

  ii) odd?

  iii) greater than 300?

  iv) greater than 300 and even?

Solution

a) Three decisions need to be made, one for each digit. Since each number can only be used once, there are five choices for the first number. After the first choice, there are only four numbers to choose from for the second digit, and then three choices for the last number. Therefore, there are 60 possible house numbers.

$$\underline{\quad 5 \quad} \times \underline{\quad 4 \quad} \times \underline{\quad 3 \quad} = 60$$

b) If there is a restriction on the problem, it is usually best to begin filling the blanks at the restriction.

i) For the number to be even, the last digit must be even. Therefore, you only have two choices for the last digit—it must be either 2 or 4.

________ ________ ___2___

There are four remaining digits to be placed into the next blank (in any order) and three options for the last remaining choice. Therefore, 24 even numbers are possible.

$\underline{\quad 4 \quad} \times \underline{\quad 3 \quad} \times \underline{\quad 2 \quad} = 24$

ii) For the house number to be odd, the last digit must be odd. It must be 1, 3, or 5. Therefore, there are three choices for the final digit. The remaining two blanks may be filled in any order. One of them may be filled with your choice of one of the remaining four digits, and there are three numbers remaining for the last choice.

$\underline{\quad 4 \quad} \times \underline{\quad 3 \quad} \times \underline{\quad 3 \quad} = 36$

There are 36 possible odd numbers. Since odd and even numbers are mutually exclusive, this may also have been calculated as the complement to the answer in part (i). Since you know the total number of possible house numbers and the number of even house numbers, the difference between these two would be the number of odd house numbers.

$60 - 24 = 36$

iii) For a house number to be greater than 300, the first digit must be a 3, 4, or 5. This means there are three choices for the first digit, followed by four choices for the second digit, and three for the last digit.

$\underline{\quad 3 \quad} \times \underline{\quad 4 \quad} \times \underline{\quad 3 \quad} = 36$

iv) For a number to be greater than 300 and even, there are restrictions on both the first and the last digit. Which restriction do you fill in first?

Suppose you begin by filling in the number of choices for the last digit.

________ ________ ___2___

There are two choices for an even last digit.

How many choices remain for the first digit? It depends! If the last digit is the digit 2, then you have three choices remaining for the first digit, namely 3, 4, and 5. However, if the last digit is the digit 4, then there are only two choices for the first digit, namely 3 or 5, as the four has already been used up. The middle digit will use any of the 3 remaining choices.

If you start with filling in the number of choices for the first digit, you encounter the same problem trying to fill in the last digit. To resolve this problem, you have to count the number of choices by considering two cases.

Case 1: If the last digit is the digit 2:

3	×	3	×	1	= 9 possibilities
(three, four, or five)		(three remaining digits)		(this is the digit 2)	

Case 2: If the last digit is the digit 4

2	×	3	×	1	= 6 possibilities
(two choices of 3 or 5)		(three remaining digits)		(one choice of digit 4)	

The total number of house numbers that are greater than 300 and even is the sum of the possibilities in the two cases.

$9 + 6 = 15$

There are 15 possible house numbers.

You may want to add some information you learned from the examples to your resource sheet.

Learning Activity 6.1

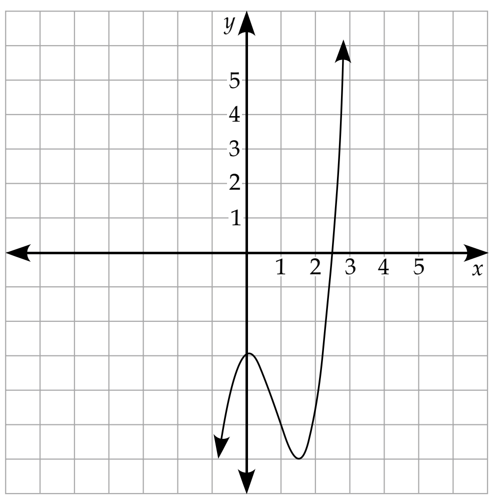
Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. There are five people at a meeting. They all shake hands with one another. How many handshakes take place?
2. Find the number halfway between $\frac{1}{3}$ and $\frac{1}{4}$.

For questions 3 to 8, use the graph of the function $y = 2x^3 - 5x^2 + x - 3$.



3. Estimate the relative maximum of the function.
4. Estimate the relative minimum of the function.
5. Estimate the x–intercept(s).
6. Find the y-intercept.
7. What is the degree of this function?
8. Describe the end behaviour.

continued

Part B: Using the Fundamental Counting Principle

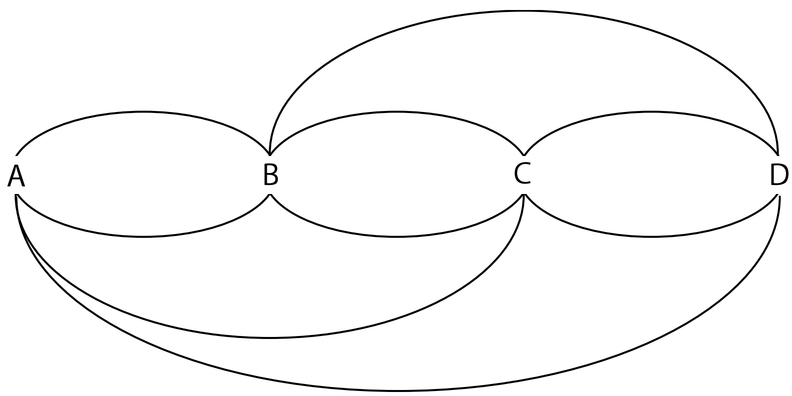
Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Construct a tree diagram showing all of the possible soup and sandwich combinations if the cafeteria offers four types of soup and three types of sandwiches for lunch.
2. How many different double-scoop cones are possible using 12 different flavours of ice cream? State any assumptions you make.
3. How many different ways can you draw five cards from a regular deck of 52 cards, if you keep all five cards together in the order they were drawn?
4. How many different outfits can you make from 3 pairs of shoes, 3 pairs of pants, and 3 shirts? You must choose one of each to make an outfit.
5. A car license plate in Manitoba consists of 3 letters followed by 3 digits.
   a) How many different plates are possible?
   b) What assumptions are you making?
   c) If each of the three letters must be unique and the first digit may not be zero, how many plates are possible?
6. At a dessert buffet in a restaurant, you can make your own ice-cream sundae. You can choose from four flavours of ice cream, six sweet sauces, and eight candy toppings.
   a) If you can choose one item from each category, how many different sundaes are possible?
   b) If you are permitted to choose two different ice cream flavours, two different sauces, and three different toppings, how many unique sundaes are possible?
7. You are shopping for a new car and would consider purchasing either a two-door or a four-door model. You may choose luxury, standard, or turbo options in each vehicle, each of which are available in any of five colours.
   a) From how many vehicles can you choose your new car?
   b) If the luxury model only comes in two colours, how many choices do you have?

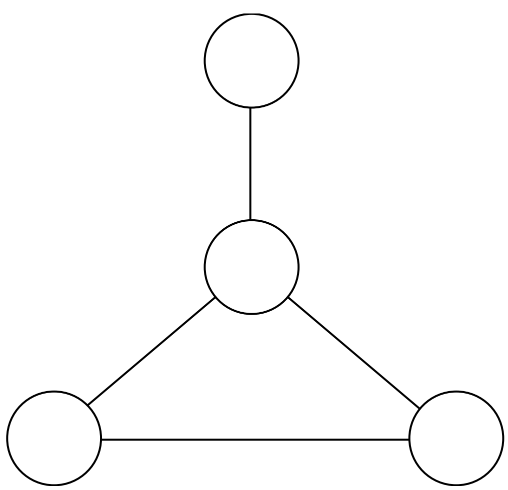
continued

8. Using the letters in the word DRAGONFLY, how many four-letter arrangements can be formed (the arrangements do not need to be real words) if
   a) letters may be used more than once.
   b) letters must not be used more than once.
   c) the arrangement must begin and end with a consonant (Y is considered a vowel in this case) and no repetition of letters is allowed.
   d) the arrangement must alternate vowels and consonants (Y is considered a vowel in this case) and no repetition of letters is allowed.

9. Examine the road map below. In how many ways can you select a route from A to D, if you must always travel in an easterly direction?



10. The circles in the diagram are to be coloured red, blue, or green. In how many ways can this be done, if no two circles joined by a line are coloured the same colour?



Lesson Summary

In this lesson, you represented and solved counting problems, using a graphic organizer. You used the Fundamental Counting Principle to solve problems, including contextual questions with restrictions and different possible cases.

Notes

Lesson 2: Factorial Notation

Lesson Focus

In this lesson, you will

❑ represent the number of arrangements of a number of elements, when taking some or all of them at a time, using factorial notation

❑ determine the value of a factorial

❑ simplify a numeric fraction containing factorials in both the numerator and the denominator

Lesson Introduction

Mathematicians are always looking for shortcuts to reduce the amount of writing required for explanations. As you know, writing a number as a power saves you the time and effort required to write the same number multiplied by itself many times. Similarly, factorial notation was developed to write the product of a series of consecutive natural numbers concisely. As you will see in this lesson, factorial notation works really well to shorten the string of numbers that results from applying the Fundamental Counting Principle.

Factorial Notation

The Fundamental Counting Principle is used to determine the number of ways in which something could occur.

Example 1

In how many ways can you arrange the letters of the alphabet?

Solution

There are 26 letters in the alphabet, so 26 decisions must be made. The number of arrangements could be found by multiplying

$26 \times 25 \times 24 \times 23 \times 22 \times 21 \times 20 \times 19 \times 18 \times 17 \times 16 \times 15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 4.0329 \times 10^{26}$

There are more than 403 septillion arrangements!

In Canada, septillion is 10^{24}. This is not universally accepted. You may want to investigate the names of very large numbers.

While this calculation is quite simple using technology, the possibility of errors when entering the numbers as shown above is quite high. In the previous lesson, you learned about the FCP, a shortcut for counting the number of arrangements. Since mathematicians like shortcuts and try to use little symbols to represent big ideas, the calculation above can be simplified and signified using factorial notation.

Whenever all the whole numbers from a particular number down to 1 are multiplied, it can be indicated using factorial notation, symbolized with an exclamation mark "!." This efficient notation translates a long list of consecutive terms to be multiplied into a concise statement.

$26! = 26 \times 25 \times 24 \times 23 \times 22 \times 21 \times 20 \times 19 \times 18 \times 17 \times 16 \times 15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

26! is read as "twenty-six factorial."

The **factorial notation** of a natural number, n, can be defined as

$$n! = n \times (n - 1) \times (n - 2) \times (n - 3) \times \ldots \times 2 \times 1$$

Also, by definition, $0! = 1$. (This may look weird but you will see that it is consistent with how factorial numbers work together).

Example 2

Use factorial notation to express and determine the number of ways in which the six vowels in the alphabet can be arranged.

Solution

If you consider A, E, I, O, and U to be the vowels, these five letters can be arranged in 5! ways.

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

If the letter Y is included as a vowel, the 6 letters A, E, I, O, U, and Y can be arranged in 6! ways.

$6! = 720$ ways.

Note: Most calculators can compute factorials. Use your manual, check the technology appendix, or access online help for assistance on how to calculate factorials using your choice of technology.

Example 3

King Arthur wants to arrange his 12 knights in single file as they ride out of the castle. In how many ways can he line up his knights?

Solution

12! = 479 001 600

The knights can line up in over 479 million ways.

Example 4

Rashid shuffles a standard deck of playing cards and places them in a stack on the table. In how many ways can the 52 cards be ordered?

Solution

$52! = 8.065817517 \times 10^{67}$

The value of factorials grows quickly, so solutions are often stated in exponential notation.

You may want to add some information about factorial notation to your resource sheet.

Using Factorial Notation in Calculations

You have seen how to calculate the number of ways to arrange a number of items, if all of them are chosen, using factorial notation. This notation may also be used when not all of the items available are chosen.

Example 5

A teacher is making a seating plan for her class. Determine the number of ways

a) ten desks can be filled by ten students.

b) three desks can be filled from amongst ten students.

c) 40 desks can be filled from amongst 60 students.

Solution

a) This is quite straightforward. It can be calculated as 10! = 3 628 800.

b) In this situation, you do not need to multiply all the consecutive natural numbers from ten down to one. You only need to make three decisions, so it can be represented using three blanks instead of factorial notation. How many students could sit in the first desk? the second desk? the third desk?

$$\underline{\quad 10 \quad} \times \underline{\quad 9 \quad} \times \underline{\quad 8 \quad} = 720$$

Compare this answer to the result of $\frac{10!}{7!}$. Can you explain why they are the same?

c) You can see how this question needs to be answered, but the number of blanks required, 40 of them, is very inconvenient to write.

$$\underline{\quad 60 \quad} \times \underline{\quad 59 \quad} \times \underline{\quad 58 \quad} \times \underline{\quad 57 \quad} \times \ldots \times \underline{\quad 22 \quad} \times \underline{\quad 21 \quad}$$

Unfortunately 60! multiplies all the consecutive natural numbers from 60 down to 1. You need to calculate 60! but without including the product of $20 \times 19 \times 18 \times \ldots \times 3 \times 2 \times 1$, or 20!.

Factorial notation is a way to express the product of natural numbers, so it can be manipulated in calculations.

Consider a simpler factorial to illustrate the concept.

Since $5! = 5 \times 4 \times 3 \times 2 \times 1$ and $\frac{5}{5} = 1$, $\frac{5!}{5!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} = 1$:

5! could also be written as $5 \times 4 \times 3!$.

To calculate $\frac{5!}{3!}$, this could be written as $\frac{5 \times 4 \times 3!}{3!} = \frac{5 \times 4 \times \cancel{3!}}{\cancel{3!}} = 20$, since $\frac{3!}{3!} = 1$.

To calculate how many ways 40 desks can be filled from amongst 60 students, you can use 60! written as $60 \times 59 \times 58 \times \ldots \times 22 \times 21 \times 20!$, and then divide this by 20! to eliminate the part that is not needed.

$$\underline{\quad 60 \quad} \times \underline{\quad 59 \quad} \times \underline{\quad 58 \quad} \times \ldots \times \underline{\quad 22 \quad} \times \underline{\quad 21 \quad},$$

rewritten as $\frac{60!}{20!}$, is equal to $\frac{60 \times 59 \times 58 \times \ldots \times 22 \times 21 \times 20!}{20!}$.

This can be calculated as $\frac{60 \times 59 \times 58 \times \ldots \times 22 \times 21 \times \cancel{20!}}{\cancel{20!}}$

or $\frac{60!}{20!} = 3.42 \times 10^{63}$.

There is about 3.42 vigintillion ways to fill 40 desks from the different arrangements of 60 students!

In Canada, vigintillion is 10^{63}. This is not universally accepted. You may want to investigate the names of very large numbers.

Strategies for Solving Counting Problems

So far, you have seen several different types of questions. Different strategies are more efficient with certain types of questions. Use blanks to represent choices when only a few decisions need to be made or when there are restrictions or multiple cases to consider. Use factorial notation when the number of decisions to be made or the number of objects to be arranged is large.

Example 6

Choose the most appropriate strategy to solve the following problems.

a) Seven athletes compete in a race. In how many different ways is it possible to award first, second, and third place?

b) Five friends are going for a ride on a massive roller coaster. The bench in each car seats five riders sitting side by side. In how many ways can the friends be seated in a car if one person insists on sitting at the end of the row?

c) An ice-cream shop offers 12 flavours of ice cream. How many single, double, or triple-scoop cones are possible? State any assumptions you make.

d) In how many ways can the letters in the word FLAMINGO be arranged?

e) How many three letter arrangements can be made using the letters of the word FLAMINGO?

f) Eleven positions are to be filled from among 33 willing volunteers. In how many ways can the jobs be assigned?

Solution

a) Use blank lines to represent the three decisions that need to be made and calculate using the FCP.

$\underline{\quad 7 \quad} \times \underline{\quad 6 \quad} \times \underline{\quad 5 \quad} = 210$

b) Since there is a restriction on one decision, use blank lines to represent the seating choices and calculate the number of arrangements using the FCP.

$\underline{\quad 1 \quad} \times \underline{\quad 4 \quad} \times \underline{\quad 3 \quad} \times \underline{\quad 2 \quad} \times \underline{\quad 1 \quad} = 24$

c) There are three possible cases to consider: single, double, and triple cones.

Single scoop: $\underline{\quad 12 \quad}$

Double scoop: $\underline{\quad 12 \quad} \times \underline{\quad 12 \quad} = 144$

Triple scoop: $\underline{\quad 12 \quad} \times \underline{\quad 12 \quad} \times \underline{\quad 12 \quad} = 1728$

Total number of cones possible: $12 + 144 + 1728 = 1884$

Possible assumptions: In the double and triple cones, you may have more than one scoop of the same flavor. The order in which the flavours are stacked creates a unique cone; that is, a triple-scoop with chocolate, vanilla, and strawberry on top is different than a triple scoop with chocolate, strawberry, and vanilla on top. You might have made different assumptions and arrived at a different answer.

d) $8! = 40\,320$

e) There are eight letters to choose from, to fill three spots.

$\underline{\;\;8\;\;} \times \underline{\;\;7\;\;} \times \underline{\;\;6\;\;} = 336$

Or

$$\frac{8!}{5!} = \frac{8 \times 7 \times 6 \times \cancel{(5 \times 4 \times 3 \times 2 \times 1)}}{\cancel{(5 \times 4 \times 3 \times 2 \times 1)}} = 336$$

f) $\underline{\;\;33\;\;} \times \underline{\;\;32\;\;} \times \underline{\;\;31\;\;} \times \ldots \times \underline{\;\;24\;\;} \times \underline{\;\;23\;\;}$

$$= \frac{33 \times 32 \times 31 \times \ldots \times 24 \times 23 \times 22!}{22!} = \frac{33!}{22!}$$

$$\frac{33!}{22!} = 7.725 \times 10^{15}$$

Arrangements with Identical Items

The examples you have seen so far have involved arrangements of distinct items. This is not always the case.

Example 7

Consider the letters in the word ZOO. In how many ways can the letters be arranged?

Solution

The arrangements, ZOO, OZO, and OOZ, are possible.

Using the FCP, however, the three letters can be placed in three spots, so $\underline{\;\;3\;\;} \times \underline{\;\;2\;\;} \times \underline{\;\;1\;\;} = 3! = 6$ arrangements should be possible.

If you use O_1 and O_2 to designate the two Os, the following arrangements can be created, but there are still only three distinct arrangements:

ZO_1O_2	O_1ZO_2	O_1O_2Z
ZO_2O_1	O_2ZO_1	O_2O_1Z

When counting the number of arrangements possible using items that are not distinct, you have to account for the number of ways the identified items can be arranged and divide the total possible number of arrangements by the number of identified arrangements. This is done because rearranging the identical objects doesn't actually create a different pattern or arrangement.

Since in this example the two similar Os can be arranged in 2! ways, the number of distinct arrangements of the three letters is $\frac{3!}{2!} = \frac{3 \times \cancel{(2 \times 1)}}{\cancel{(2 \times 1)}} = 3$.

The number of permutations of a set of n items containing a identical objects of one kind, b identical items of a second kind, c identical items of a third kind, and so on, is $\frac{n!}{a!b!c!\ldots}$.

You may want to add some of the information from this section to your resource sheet.

Arrangements with Grouped Items

When there are restrictions included in the arrangements of elements, it may be best to represent the choices with lines, fill in the number of ways the decisions can be made, and then use factorial notation to simplify the calculations.

Example 8

Five musical groups (a jazz, a blues, a rock, a country, and a pop band) are competing in a garage band contest. To minimize the time required for equipment setup between acts, the organizers determine that the rock band and the country band must follow one another in the performance lineup. In how many ways can the bands perform?

Solution

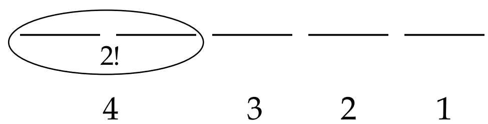
There are five acts to accommodate in the lineup, so begin by making five lines.

____ ____ ____ ____ ____
5 4 3 2 1

Two of the lines are grouped together because two groups must follow each other and cannot be separated, so there are now only four positions to arrange, which can be done in 4! ways.

(____ ____) ____ ____ ____
4 3 2 1

The two grouped bands can be arranged in 2! ways, depending on whether the rock band or the country band plays first.



Therefore, the total number of ways in which the five bands can be lined up, if two must play one after the other, is 4!2! or 48 ways.

Look back at all of the strategies that you were shown in this lesson and add some information about them to your resource sheet.

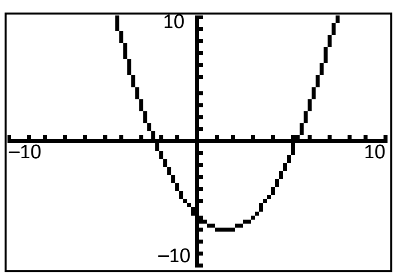
Learning Activity 6.2

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

For questions 1 to 5, use the graph of $y = 0.5x^2 - 1.5x - 6$.



1. Write domain of the quadratic function.
2. Estimate the range of the quadratic function.
3. Describe the end behaviour of the quadratic function.
4. Estimate the zeros.
5. State the y-intercept.

continued

6. Solve for p: $1500 = p(0.05)(10)$
7. If April 30th is a Wednesday, what day of the week is 50 days from April 30th?
8. How many two-digit numbers contain at least one 3?

Part B: Factorial Notation

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Calculate:
   a) 10!
   b) 13!
   c) 20!
2. Simplify the following without the use of a calculator:
   a) $\frac{7!}{5!}$
   b) $\frac{21!}{20!}$
   c) $\frac{3!}{4!}$
   d) $\frac{6!}{3!3!}$
3. There are 33 cars that race in the Indy 500.
   a) In how many ways can the cars cross the finish line?
   b) In how many ways can first, second, and third place finishes be awarded?

continued

Learning Activity 6.2 (continued)

4. There are 50 golfers registered for a charity tournament.
   a) In how many ways can the schedule be organized for the golfers' first tee-off?
   b) The organizer must put the 50 golfers into groups of four. How many ways is it possible to create the first group of four golfers?
5. Three boys and three girls must each take a turn singing a solo. In how many ways can they be arranged, if they must alternate between male and female voices?
6. At the annual summer Corn and Apple Festival, there are three concerts, a children's entertainer, a magician, a pie-eating contest, and a talent-search contest scheduled on the main stage. In how many ways can these events be scheduled if the concerts must follow each other in the line-up?

Lesson Summary

In this lesson, you learned how to simplify and determine the value of factorials. You used factorial notation to represent and determine how many arrangements of a certain number of elements are possible when you take all, or some, of the elements, including elements that may not be distinct.

Assignment 6.1

Fundamental Counting Principle

Total: 21 marks

This is a hand-in assignment. Please show your work clearly and in an organized manner. Round final answers to 2 decimal places, and include units, if appropriate. If you use technology as a strategy in your solution steps, please indicate what application you are using, the values you input, and a sketch or printout of the results. Answers given without supporting calculations will not be awarded full marks.

1. In how many ways can you arrange the letters in the word MONKEY? *(2 marks)*

2. How many arrangements of the letters in the word DOLPHIN are possible, if you must start and end with a consonant? *(2 marks)*

3. In how many unique ways can the letters in the word GIRAFFE be arranged? *(2 marks)*

4. Simplify as a product without factorial notation and evaluate.

   a) $\frac{14!}{9!}$ *(1 mark)*

   b) $\frac{12!}{6!5!4!}$ *(2 marks)*

5. A committee consisting of a treasurer, a secretary, a president, and a vice-president must be elected from a class of 25 students. In how many different ways is this possible? *(2 marks)*

Assignment 6.1: Fundamental Counting Principle (continued)

6. A family consisting of two adults and three children wishes to have a photo taken. In how many ways can they be arranged, if they sit

   a) in a single row? *(1 mark)*

   b) with an adult at either end of the row? *(2 marks)*

   c) alternating child and adult? *(2 marks)*

7. Using the digits 3, 4, 5, 6, 7, and 8, how many even, three-digit numbers, larger than 500, can be made if no repetition of digits is allowed? *(3 marks)*

8. The Grade 12 class has 139 graduating students. In how many ways can a grad committee be chosen if the representatives elected are to fill the positions of chairperson, secretary, treasurer, entertainment advisor, and advertising consultant? State your answer using factorial notation. *(2 marks)*

Lesson 3: Permutations

Lesson Focus

In this lesson, you will

- ❑ determine how many permutations of a number of elements are possible when you take some, or all, of the elements
- ❑ solve contextual problems that involve permutations and probability

Lesson Introduction

Being able to count quickly the number of possible arrangements of elements from a group of items is used in mathematics, statistics, and computer science. In some arrangements, the order in which the elements are arranged affects the number of possible outcomes. When the order of the arrangements matters and affects the counting process, the arrangements are given the name, **permutations**. In this lesson, you will be shown the short-hand notation for permutations that is commonly used in mathematics, and you will learn several applications of this quick counting method.

Permutations

In the previous lessons, you determined the number of ways to arrange a set of items using the Fundamental Counting Principle and factorial notation. In each case, the order or position of the items was important. Consider the phone number 204-189-5367. It has a different arrangement of numbers than 204-189-3576 and 204-198-5367, even though these three phone numbers use the exact same digits. As you can see, the order of the elements is important, so these arrangements are called permutations.

Formula for Permutations

A **permutation** is the arrangement of a number of objects in which *the order of the objects is important*. The number of ways n objects can be placed into r possible positions, written as ${}_nP_r$, is given by the formula ${}_nP_r = \dfrac{n!}{(n-r)!}$ where n is the number of objects you have altogether, and r is the number of items you arrange or the number of blanks or decisions to be made. ${}_nP_r$ is read as "the permutations of n items, taken r at a time." Permutation notation can be written as ${}_nP_r$ or as $P(n, r)$—they mean the same thing.

Example 1

Using the numbers 1 through 39, how many locker combinations are possible if three different numbers are used?

Solution

A locker combination of 15–7–23 is different than the combination of 7–23–15. Since the order of the numbers is important, this is a permutation. If three different numbers from a possible 39 numbers must be chosen, the number of arrangements possible can be found by calculating ${}_{39}P_3$.

${}_{39}P_3$ means 39 × 38 × 37, since there are 39 choices for the first number, 38 for the second number, and 37 for the third number (assuming the same number can only be used once). Using factorial notation and the formula, this calculation is:

$$ {}_{39}P_3 = \frac{39!}{(39-3)!} $$

$$ {}_{39}P_3 = \frac{39!}{(36)!} $$

$$ {}_{39}P_3 = 54\ 834 $$

The number of possible different combinations is 54 834. This assumes that numbers cannot be repeated.

Note: Most calculators can compute permutations. Use the manual, check the technology appendix, or access online help to find out how to calculate permutations using your choice of technology.

You may want to add the permutation formula, an example of how it is used, and any necessary calculator instructions to your resource sheet.

Look back at the examples in the previous lesson where a certain number of items were chosen from a larger number of possible elements. You will see that the calculations you used in those examples fit the formula given above.

Example 2

Determine in how many ways you can make 5 letter arrangements from the letters in the words:

a) THUNDERCLAPS

b) TRAMPOLINE

c) EVENT

Solution

a) THUNDERCLAPS

$$_{n}P_{r} = \frac{n!}{(n-r)!}$$

$$_{12}P_{5} = \frac{12!}{(12-5)!}$$

$$_{12}P_{5} = \frac{12!}{7!}$$

$$_{12}P_{5} = 95040$$

b) TRAMPOLINE

$$_{10}P_{5} = \frac{10!}{(10-5)!}$$

$$_{10}P_{5} = \frac{10!}{5!}$$

$$_{10}P_{5} = 30240$$

c) EVENT

There are two non-distinct letters in this word, so the number of permutations possible must be divided by 2!, the number of ways the two Es can be arranged.

$_{5}P_{5}$ with two identical items = $\dfrac{5!}{(5-5)!(2!)} = 60.$

Example 3

Beth and Shannon invited four friends to go to the movies with them. In how many ways can these six people be seated in a row if

a) there are no restrictions?

b) Beth is seated at the left end of the row and Shannon is seated at the right end of the row?

c) Beth and Shannon want to sit together?

Solution

a) This is an unrestricted permutation of six people taken six at a time. Use the formula for permutations.

$${}_nP_r = \frac{n!}{(n-r)!} \qquad \text{or } \underline{\ 6\ } \times \underline{\ 5\ } \times \underline{\ 4\ } \times \underline{\ 3\ } \times \underline{\ 2\ } \times \underline{\ 1\ } = 720$$

$${}_6P_6 = \frac{6!}{(6-6)!}$$

$${}_6P_6 = \frac{6!}{0!} = 720$$

Recall that 0! = 1. Now you can see why it is consistent that 0! is defined as being equal to one.

b) With restrictions, it is advised that you use blanks to represent the decisions to be made, rather than using the formula. (You may have chosen to represent the first part of this question with blanks as well.)

$$\underset{\text{Beth}}{\underline{\ 1\ }} \times \underline{\ 4\ } \times \underline{\ 3\ } \times \underline{\ 2\ } \times \underline{\ 1\ } \times \underset{\text{Shannon}}{\underline{\ 1\ }} = 24$$

The friends can sit in 24 possible arrangements, if Shannon and Beth sit on their respective ends of the row.

c) If Beth and Shannon want to sit together, you need to consider that the two girls can be arranged in 2! ways. Treat the pair as one object. The five objects can be arranged in 5! ways.

(___ ___) 2! ___ ___ ___ ___

5 4 3 2 1

$${}_5P_5 \times {}_2P_2 = 240$$

Permutations and Probability

If you can determine the number of possible arrangements of a number of elements, you can use that number to calculate the probability of a certain arrangement.

Example 4

Tyler's bike lock combination consists of the four numbers, 3, 4, 8, and 11, in some order. If he cannot remember the exact order of the numbers, what is the probability he gets it right on his first attempt to unlock it?

Solution

Since the order of the numbers for a lock matter, it would be more appropriate, mathematically, to call this a locker "permutation" rather than "combination." Find the number of permutations of the four numbers.

$${}_{n}P_{r} = \frac{n!}{(n-r)!} \qquad \text{or} \qquad \underline{\ 4\ } \times \underline{\ 3\ } \times \underline{\ 2\ } \times \underline{\ 1\ } = 24$$

$${}_{4}P_{4} = \frac{4!}{(4-4)!}$$

$${}_{4}P_{4} = \frac{4!}{0!} = 24$$

To calculate probability, you can find the ratio:

$$P(\text{event happening}) = \frac{\text{number of outcomes in event}}{\text{total number of outcomes possible}}$$

The event is getting the numbers right on his first attempt and there is only one way to do that.

There are 24 possible ways to arrange the numbers.

The probability that Tyler gets the correct order is $P(\text{right order}) = \frac{1}{24}$.

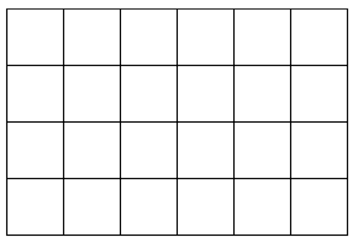
Learning Activity 6.3

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

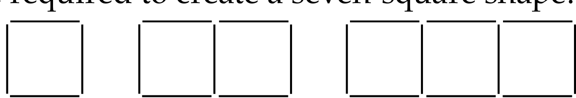
Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. How many squares of all sizes can be found in the following diagram?



2. What is the maximum number of 2″ by 3″ cards that can be cut from a sheet of paper that measures 2′ by 3′?
3. The path around the bases in a baseball field forms a square, with home plate, first base, second base, and third base at the corners. The distance from home plate to first base is 90 feet. Show the calculation required to determine how far the catcher has to throw the ball from home plate to make a play at second base.
4. Consider a pattern of squares made from toothpicks. How many toothpicks are required to create a seven-square shape?



5. How many square shapes could be created with 52 toothpicks?
6. Describe the relationship between the number of toothpicks and the number of square shapes.
7. Evaluate: $\frac{4!}{2!}$
8. Evaluate: $\frac{7!}{4!}$

continued

Part B: Permutations

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. A graduation class has 50 students.
   a) If students are called up randomly to receive their diplomas, in how many ways could they be arranged?
   b) The photographer wants to arrange the students in five rows of ten students each. In how many ways can the front row be arranged?
2. A teacher has 13 assignments to mark. He accidentally drops them down the stairwell and, when he collects them, he stacks them into a single pile. What is the probability that the papers are now arranged in alphabetical order by student name?
3. If you can select 8 courses from the 14 that are offered at your school, how many different class schedules are possible? What assumptions are you making?
4. A briefcase has a combination lock with three numbered disks, and each disk has the digits 1 through 9 along its edge. The disks can be rotated to line up a three-digit code.

   a) How many codes are possible, if digits can be repeated?
   b) How many different codes are possible, if each digit must be unique?
5. a) How many different three-digit numbers can be made using the numbers 0, 3, 6, 7, and 9? State your assumptions.
   b) What is the probability that a randomly selected three-digit number is even, if digits cannot be repeated and zero is not allowed as a first digit?
6. Saylor has eight signal flags that are to be hung in a vertical line on a single pole. If four of the flags are indistinguishable red flags, three are indistinguishable white flags, and one is a blue flag, how many different signals can Saylor create?

continued

7. In how many ways can the letters in the following words be arranged, if all of the letters are used?
   a) HYSTERICAL
   b) POPPY
8. How many five letter arrangements can be made using the letters in the following words?
   a) CAMPGROUND
   b) APRICOT
   c) APPLIED if, at most, one of the Ps is used
9. Erika is decorating for the graduation banquet. She needs to create a fabric banner behind the head table, made with eight different-coloured swatches of fabric. In how many ways can she arrange the swatches, if the black and white coloured swatches may not be placed next to each other?

Lesson Summary

In this lesson, you determined how many permutations of a number of elements are possible when you take some, or all, of the elements, including identical items and restrictions. You solved contextual problems that involve permutations and probability.

Lesson 4: Combinations

Lesson Focus

In this lesson, you will

- ❑ explain, using examples, why order is or is not important when solving problems that involve permutations or combinations
- ❑ generalize strategies and determine the number of combinations of *n* elements taken *r* at a time
- ❑ solve contextual problems that involve probability and combinations

Lesson Introduction

You have seen how the formula for permutations can help you to determine quickly the number of ways in which a number of elements can be arranged, when you take some or all of the elements, and the order of those elements is important.

The order of the elements is not always important. Consider a graduation committee of three members, who will work together to plan a car wash fundraiser. A committee made up of Arnold, Bruce, and Connie is the same as a committee made up of Bruce, Connie, and Arnold. The three committee members do not have special roles or positions and so the order in which they are chosen or arranged is irrelevant. Arrangements of objects where the order **does not** matter are called **combinations** rather than permutations. You will learn about combinations in this lesson.

Combinations

If the letters represent acronyms for company names then the order of the letters matters, since a different order represents a different company. However, if the letters represent the names of people who are to be members of a committee (such as Arnold, Bruce, and Connie), then the order of the names listed on the committee does not matter. ABC is the same as ACB is the same as BCA, BAC, CAB, and CBA. There are six permutations of these three letters, but there is only one combination of the group of three people. There are fewer combinations than permutations of a set of objects.

In the previous lessons, when finding the number of permutations with identical items, you had to divide the total number of permutations by the number of ways those identical items could be arranged. In the case of combinations, when the order of the items is **not** important, you will also have to divide the total number of arrangements by the number of ways the items chosen can be arranged.

$$\frac{\text{number of ways to choose three elements}}{\text{number of ways to arrange the three elements}} = \frac{3!}{3!} = 1$$

Example 1

How many three-person committees are possible if you can select the members from four people? List all the possible combinations of three for A, B, C, and D.

Solution

ABC ABD ACD BCD

There are four ways to choose four people using three at a time.

The number of **permutations** of four items, taken three at a time, is

$${}_{n}P_{r} = \frac{n!}{(n-r)!}$$

$${}_{4}P_{3} = \frac{4!}{(4-3)!} = \frac{4!}{1!}$$

$${}_{4}P_{3} = 4! = 24$$

If you divide this number of arrangements by the number of ways you can arrange the three chosen elements, 3!, you come up with the number of combinations of four items, taken three at a time.

$$\frac{24}{3!} = \frac{24}{6} = 4$$

Formula for Combinations

A **combination** is the arrangement of a number of objects in which the *order of the objects is **not** important*. The number of ways n objects can be placed into r available positions, written as ${}_nC_r$, is given by the formula ${}_nC_r = \frac{n!}{(n-r)!r!}$ where n is the number of objects you have altogether, and r is the number of items you arrange. ${}_nC_r$ is read as "the combinations of n items, taken r at a time" or "the combinations of n choose r." The notations $\binom{n}{r}$ or $C(n, r)$ may also be used.

Note that the formula for combinations is the same as the formula for permutations with the addition of $r!$ to the denominator. Dividing by $r!$ eliminates the number of ways the same chosen items can be arranged in different orders, since they really represent the same group.

You may want to add this information to your resource sheet.

Example 2

A general committee of five people needs to be chosen from amongst 30 employees. In how many ways can this be done?

Solution

Using the Fundamental Counting Principle, $30 \times 29 \times 28 \times 27 \times 26 =$ 17 100 720, so there are over 17 million permutations of 30 people, taken five at a time.

This answer assumes that a committee made up of Arthur, Betty, Chad, Devon, and Eva is a distinctly different committee than the one made up of Eva, Devon, Chad, Betty, and Arthur. If the members of the committee do not have specific roles, such as president, VP, treasurer, secretary, and moderator, the same five members arranged in any way is still the same committee.

Using the formula for combinations, ${}_nC_r = \dfrac{n!}{(n-r)!r!}$, the number of possible arrangements of 30 people, taken five at a time, is

$$ {}_{30}C_5 = \frac{30!}{(30-5)!5!} $$

$$ {}_{30}C_5 = \frac{30 \times 29 \times 28 \times 27 \times 26 \times \cancel{25!}}{\left(\cancel{25!}\right)5!} $$

$$ {}_{30}C_5 = \frac{\cancel{30} \times 29 \times \cancel{28}\,(7) \times 27 \times 26}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times 1} $$

$$ {}_{30}C_5 = 142\ 506 $$

There are 142 506 possible combinations of 30 people taken five at a time.

Note: Most calculators can compute combinations. Use the manual, check the technology appendix, or access online help to find out how to calculate combinations using your choice of technology.

You may want to add the combination formula, an example of how it is used, and any necessary calculator instructions to your resource sheet.

Example 3

A set of coloured markers consists of 12 pens (red, orange, yellow, green, blue, indigo, violet, black, brown, grey, pink, and beige). If Micayla randomly selects three pens at once (no replacement) to colour a picture, what is the probability she chooses red, orange, and yellow?

Solution

Red, orange, and yellow comprise one possible combination of the 12 colours, but red, orange, and yellow is the same as orange, yellow, and red, so the order is not important. Use the formula for combinations and your choice of technology to determine the number of combinations of any three colours of pens from the group of 12 coloured pens (12 choose 3).

$$ {}_nC_r = \frac{n!}{(n-r)!r!} $$

$$ {}_{12}C_3 = \frac{12!}{(12-3)!3!} $$

$$ {}_{12}C_3 = 220 $$

There is only one way to choose the colours red, orange, and yellow, so the probability of randomly selecting those three colours is $\dfrac{1}{220}$.

Example 4

A dodgeball team consists of 20 players—9 female and 11 male. At the start of each round, a team must have six players on the floor, and at least two of them must be female. In how many ways can the players be arranged for a round of dodgeball?

Solution 1

This is a combinations question because the order of players is not important, but the restriction regarding the number of female players must be considered using cases. Use technology to determine the value of the product of each combination.

Case 1: two females and four males

$$_9C_2 \times {}_{11}C_4 = \frac{9!}{(9-2)!2!} \times \frac{11!}{(11-4)!4!} = 11\ 880$$

Case 2: three females and three males

$$_9C_3 \times {}_{11}C_3 = 13\ 860$$

Case 3: four females and two males

$$_9C_4 \times {}_{11}C_2 = 6930$$

Case 4: five females and one male

$$_9C_5 \times {}_{11}C_1 = 1386$$

Case 5: six females and no males

$$_9C_6 \times {}_{11}C_0 = 84$$

The total number of ways to arrange 9 female and 11 male players, chosen 6 at a time, when at least two of them are female, is 11880 + 13860 + 6930 + 1386 + 84 = 34 140.

Solution 2

The order of the players chosen is not important so this is a combination question. The restriction of having at least two female players can be addressed using the complement. The total number of combinations of players, less the number of combinations that have zero or only one female, would result in the number of combinations that fulfill this restriction.

Total possible number of combinations of any 6 players with no conditions (20 choose 6):

$${}_{20}C_6 = 38\ 760$$

Case 1: No females and six males

$${}_{9}C_0 \times {}_{11}C_6 = 462$$

Case 2: One female and five males

$${}_{9}C_1 \times {}_{11}C_5 = 4158$$

The total number of ways to arrange 9 female and 11 male players, chosen 6 at a time, when at least two of them are female, is 38 760 − (462 + 4158) = 34 140.

Learning Activity 6.4

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

Without using a calculator, determine the value of the following factorials.

1. 5!
2. 4!
3. 3!
4. 2!
5. 1!
6. 0!

continued

Simplify the following factorial expressions without using a calculator.

7. $\dfrac{25!}{23!4!}$

8. $\dfrac{14!}{8!9!}$

Part B: Combinations

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. A class of 24 students is sending a delegation of four students to a workshop on environmental issues.
   a) In how many ways can the delegation be selected?
   b) Is the order in which the students are selected important? Explain.
2. Staff members at a school are required to supervise lunchtime activities, sporting events, dances, and band concerts throughout the year. The administration tries to send a different group of four staff members each time they have supervisory duty.
   a) If there are 18 people on staff, in how many ways can a duty group be formed?
   b) The school has a policy that the staff on duty must be equally represented by male and female teachers. If there are 10 female and 8 male teachers, in how many ways can they have distinctly different supervisory teams?
3. How many different hands of five cards are possible from the 52 cards in a standard deck?
4. In a national lottery, players can choose, in any order, six numbers from 1 through 49.
   a) How many combinations of the six numbers are possible?
   b) What is the probability of selecting the winning combination?

continued

5. Eight-ball is a pool game played with 16 distinct billiard balls. If three of the balls are selected at random, how many different combinations can be chosen?

6. Ten people in a tour group want to take a gondola ride along the canals in Venice.
   a) If a gondola can hold up to six people, in how many ways can the tourists be arranged in two identical gondolas?
   b) If there are six seats in each gondola, in how many ways can six people be seated in the first boat?

7. Find the number of ways 16 students can be assigned to two project groups, if each group must have at least 6 students.

8. A committee of five is selected at random from a group of nine people made up of six women and three men.
   a) In how many ways can the committee be selected if there are no restrictions?
   b) What is the probability the committee will include George and Ruth, two of the nine people?
   c) In how many ways can the committee be selected to have exactly two men and three women?
   d) What is the probability the committee will have at least three women?

Lesson Summary

In this lesson, you explained, using examples, why order is or is not important when solving problems that involve permutations or combinations. You used strategies to determine the number of combinations of n elements taken r at a time, and solved contextual problems that involve probability and combinations.

Lesson Focus

In this lesson, you will

- ❑ represent and solve counting problems using a graphic organizer
- ❑ solve contextual problems that involve permutations, combinations, and probability

Lesson Introduction

In this module, you used a variety of strategies to count all the possible outcomes in a given event, taking into account whether order mattered. This lesson will highlight graphic organizers that can help you solve combinations and probability questions, including pathway diagrams and Pascal's Triangle.

Problem Solving with Combinations and Probability

Pathway Problems

Example 1

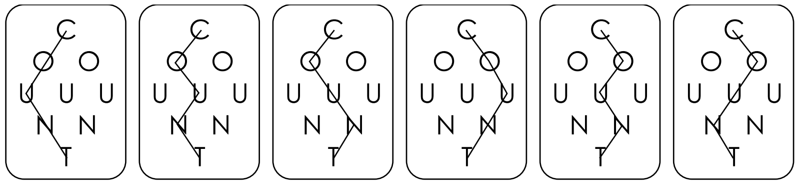
a) How many ways can the word COUNT be found in the array of letters below if you must start at the top, C, and move diagonally down to the bottom, T?

```
    C
  O   O
U   U   U
  N   N
    T
```

b) What is the probability that the middle U is used?

Solution

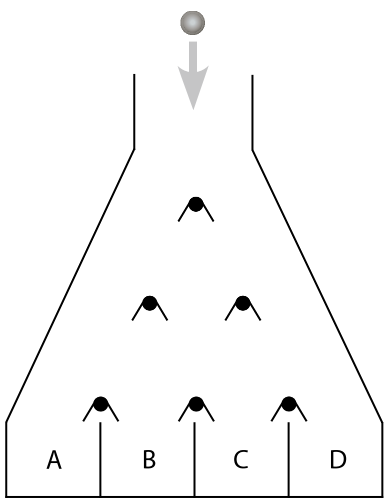
a) There are six possible pathways that spell the word COUNT as shown.



b) Four of the possible pathways use the middle U, so the probability of using it is $\frac{4}{6}$ or $\frac{2}{3}$.

Example 2

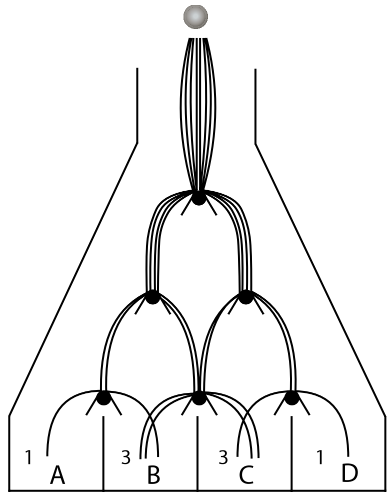
In the game of Plunko, players drop a marble into a slot on top of a triangular grid of pegs. The marble randomly bounces on the pegs, falls either to the left or right, and continues on its downward path until it lands in a bin at the bottom.



a) Trace all of the possible pathways a marble may follow in this Plunko grid. Indicate the number of pathways that lead to each bin along the bottom, labelled A through D, and the total number of pathways possible.

b) On a game show, contestants drop in a marble and win the prize from the bin in which their marble lands. If you were the game show host, in which bin would you place the most expensive prizes? The least expensive prizes? Explain.

Solution

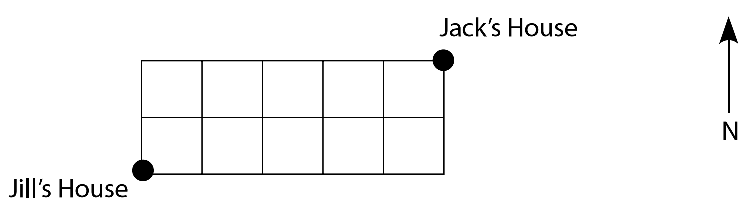


a) There is only one pathway that leads to bin A and one pathway that leads to bin D. There are three pathways to bins B and C. In total, there are eight possible pathways past the three rows of pegs and into a bin ($2^3 = 8$).

b) The marble can take any one of the eight pathways. The probability it lands in A is $\frac{1}{8}$ and the probability it lands in D is $\frac{1}{8}$. The probability of it landing in B is $\frac{3}{8}$ and the probability it lands in C is also $\frac{3}{8}$. Since it is more likely that the marble will land in B or C, as the host, I would put the less expensive prizes in these bins. I would put the more expensive prizes in bins A and D, since it is less likely the marble would land in these bins.

Example 3

Jack must walk seven blocks to get to Jill's house, as shown in the diagram below.



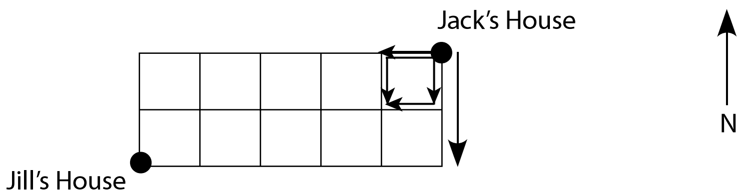
a) If the vertical and horizontal line segments represent streets and avenues, in how many ways can Jack walk to Jill's house without ever backtracking (that is, only walk West and South)?

b) In how many ways can Jack go to Jill's house and return?

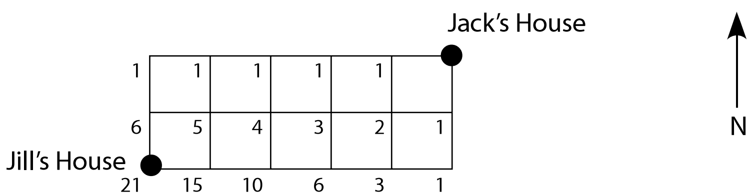
Solution

a) **Method 1:**

You may begin to trace out each possible pathway. A few pathways are started in the diagram below.



An easier method of solving the problem is to find a numerical pattern. Count the number of possible pathways to each intersection, beginning at Jack's house. To get to the next intersection, add the numbers for the two intersections coming to it.



The numbers represent the number of different pathways Jack can walk along to arrive at that corner. There are 21 different pathways that Jack can take to get from his house to Jill's house.

Method 2:

You could solve this problem by labelling the blocks you move west with Ws and the blocks you move south with Ss.

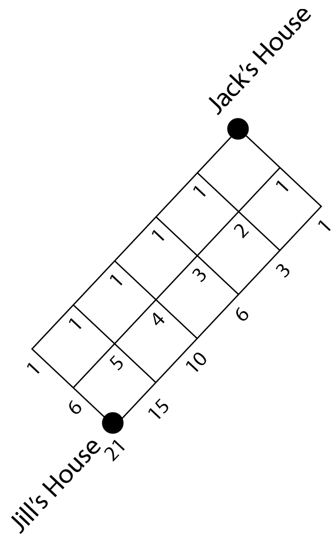
One of the paths could be described as WWWWWSS and another could be described as SWWWSWW. There will always be 5 Ws and 2 Ss. Find the number of arrangements of 5 Ws and 2 Ss to count the pathways.

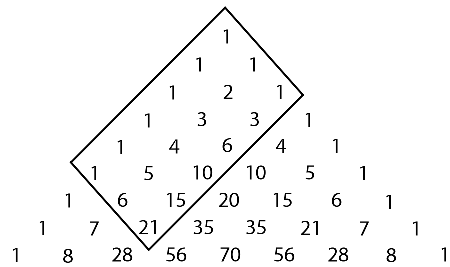
7! ÷ (5! 2!) (divide since these are identical objects)

$$\frac{7 \times 6 \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1} \times 2 \times 1} = \frac{42}{2} = 21 \text{ pathways}$$

b) There are 21 ways to get to Jill's house, and since the two trips are independent events there are also 21 ways to return. So, the number of round-trip pathways is 21 × 21 = 441.

Note: Do you recognize the number pattern when the map diagram is rotated?

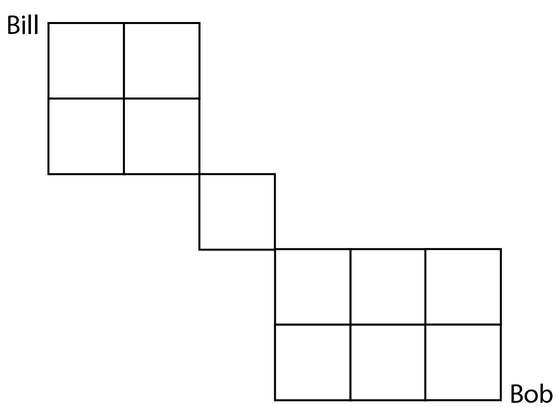




The number at each intersection is the sum of the two numbers above it. This is the same pattern you saw in the cover assignment and it is called Pascal's Triangle.

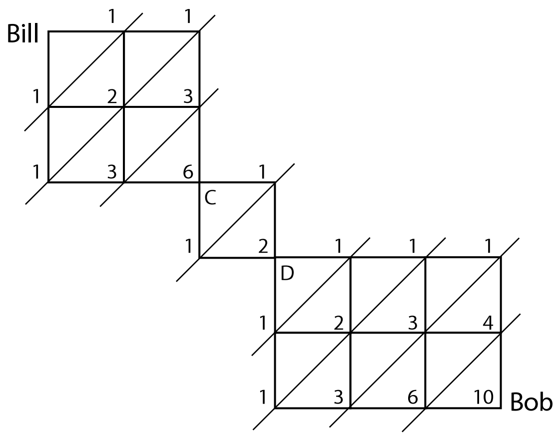
Example 4

Bill and Bob live in the same neighbourhood, as shown in the diagram below. In how many ways can Bill go directly to Bob's house?



There are two intersections where Bill can only use one route.

Label the first restricted intersection that Bill must pass through as C and the second corner he is required to pass through as D, then apply one of the strategies you learned to each section of this diagram. You will find that there are 120 ways to get from Bill's home to Bob's home.



There are six ways for Bill to get to C.

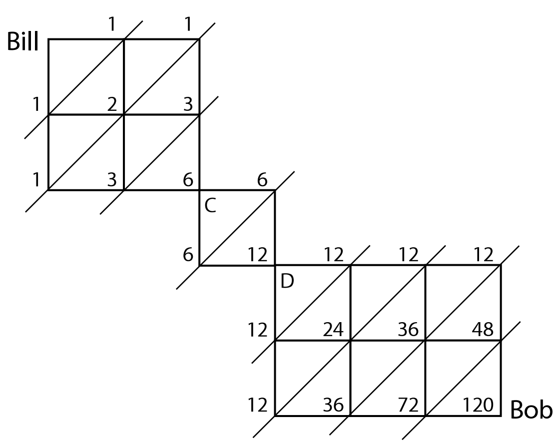
For each of those six ways, there are two ways for Bill to get from C to D.

For each of those two ways, there are 10 possible pathways for Bill to get from D to Bob's house.

If you apply the Fundamental Counting Principal and multiply the possibilities, the total number of possible routes Bill could take is:

$6 \times 2 \times 10 = 120$ ways.

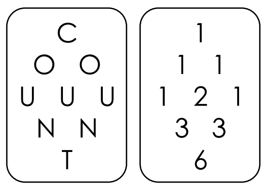
Another way to consider this solution is as follows:

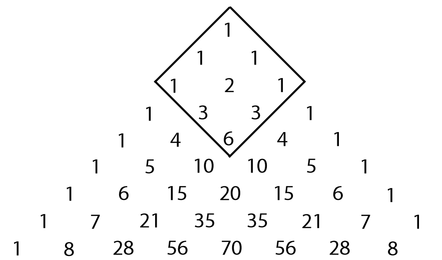


There are 120 possible pathways Bill could take to go directly to Bob's place.

Application of Pascal's Triangle

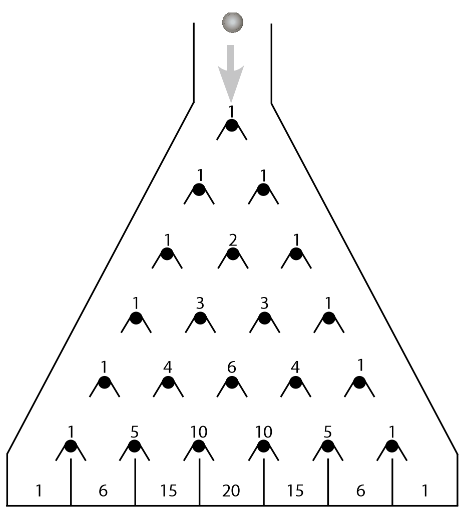
The pattern in the numbers for Pascal's Triangle are evident in each of the examples you have looked at so far in this lesson. Review them and see if you can identify the pattern.

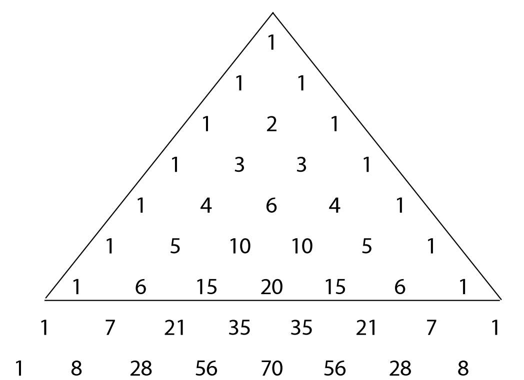




The number of pathways you can take to spell the word COUNT is a model of a portion of Pascal's Triangle.

If the number of pegs and bins in the Plunko game were increased, the number of pathways to each bin would follow the pattern in Pascal's Triangle as well.

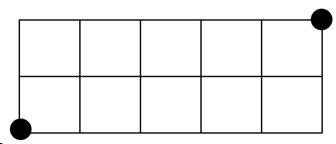




Pascal's Triangle, Combinations, and Probability

What connection does Pascal's Triangle have to combinations and probability?

In solving the pathway problem for Jack and Jill, you had to walk seven blocks, involving two choices to go south and five decisions to go west.



__S__ __S__ __W__ __W__ __W__ __W__ __W__

You could draw lines to represent the seven decisions and arrange two Ss and five Ws. You would need to account for the five identical Ws and two identical Ss. Calculate the permutation as $\frac{7!}{2!5!} = 21$ to find the number of pathways possible. Note that this is the same as the calculation used to determine the combination ${}_7C_2$ or ${}_7C_5$.

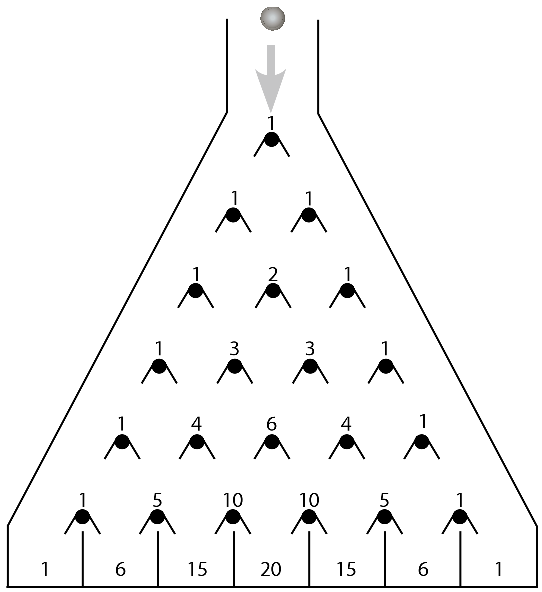
There are seven blocks to travel, choose two of them to the south (in any order), hence ${}_7C_2$.

Alternatively, there are seven blocks to travel, choose 5 of them to the west (in any order), hence ${}_7C_5$.

Starting the count of rows and positions of Pascal's Triangle at 0, you will find the number 21 in row 7 and in position 2 or position 5.

In the Plunko game, at each peg, the marble could fall in one of two directions—either to the right or the left—as it made its way down to the bins.

This type of event is called binary, since there are two equally likely possible choices for each decision.



If there are six lines of pegs the marble must pass, you could draw six lines and choose L or R for each blank.

L or R _______ _______ _______ _______ _______

$2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6$

There are 2^6 possible pathways it could take on the way down.

How many of those pathways have exactly two left bounces? In this case, of the six blanks, two would have Ls and the other four would be Rs (that is, RRRRLL). In how many ways can this be arranged?

Six decisions, two identical Ls, and four identical Rs:

$$\frac{6!}{2!4!} = 15 \text{ or } {}_6C_2 = 15$$

The number of pathways leading to the bins are the values in Row 6 of Pascal's Triangle (recall, the top row with the single 1 is considered Row 0). If you move diagonally to the right four times and then diagonally to the left two times, you reach 15.

Next, you will consider another binary example—tossing coins.

Example 5

Find five different coins, toss them, and record the number of coins with heads showing. In how many different ways can the five coins land if they are tossed all at once?

Solution

Begin with a simpler problem and organize your results in a chart to see if you can find a pattern to help you answer this question.

Each coin can display either "heads" or "tails." If you toss only one coin to start, there would be one blank line with two possible outcomes. The two outcomes are either no heads or one head—that is,

T

H

If you toss two coins, you would have two blank lines or decisions with two possible outcomes for each (H or T), so there would be four results in the sample space. The results would be combinations of zero, one, or two heads showing. The four results can be grouped as:

TT

HT TH

HH

If you toss three coins, there can be zero, one, two, or three heads showing. If you draw three blank lines to represent decisions that can be either H or T, there are 2^3, or eight, ways this can occur. Grouping the eight possible results, you have:

TTT

TTH THT HTT

HHT HTH THH

HHH

If you toss four coins, there can be zero, one, two, three, or four heads showing. The $2^4 = 16$ results can be grouped as:

TTTT

TTTH TTHT THTT HTTT

HHTT HTTH THHT TTHH HTHT THTH

HHHT HHTH HTHH THHH

HHHH

The number of results doubles each time, so tossing five coins should result in $5^2 = 32$ arrangements. Can you figure out how many will have five, four, three, two, one, or no heads showing? Displaying the numbers in a chart helps identify patterns.

Number of Coins	Number of Heads	Grouped Results	Combinations
1	0	T	1
	1	H	1
2	0	TT	1
	1	HT TH	2
	2	HH	1
3	0	TTT	1
	1	TTH THT HTT	3
	2	HHT HTH THH	3
	3	HHH	1
4	0	TTTT	1
	1	TTTH TTHT THTT HTTT	4
	2	HHTT HTTH THHT TTHH HTHT THTH	6
	3	HHHT HHTH HTHH THHH	4
	4	HHHH	1
5	0		1
	1		5
	2		10
	3		10
	4		5
	5		1

The number of possible arrangements of a certain number of heads when tossing coins follows the pattern of numbers in the rows of Pascal's Triangle!

If you toss five coins, there can be five, four, three, two, one, or no heads showing. There will be the following:

- one arrangement of five coins showing zero heads $_5C_0 = 1$
- five arrangements of five coins showing one head $_5C_1 = 5$
- ten arrangements of five coins showing two heads $_5C_2 = 10$
- ten arrangements of five coins showing three heads $_5C_3 = 10$
- five arrangements of five coins showing four heads $_5C_4 = 5$
- one arrangement of five coins showing five heads $_5C_5 = 1$

The numbers in Pascal’s Triangle can be written as combinations of the row number, choosing the position in the row.

${}_0C_0$	Row 0	1
${}_1C_0$ ${}_1C_1$	Row 1	1 1
${}_2C_0$ ${}_2C_1$ ${}_2C_2$	Row 2	1 2 1
${}_3C_0$ ${}_3C_1$ ${}_3C_2$ ${}_3C_3$	Row 3	1 3 3 1
${}_4C_0$ ${}_4C_2$ ${}_4C_2$ ${}_4C_3$ ${}_4C_4$	Row 4	1 4 6 4 1
${}_5C_0$ ${}_5C_1$ ${}_5C_2$ ${}_5C_3$ ${}_5C_4$ ${}_5C_5$	Row 5	1 5 10 10 5 1
${}_6C_0$ ${}_6C_1$ ${}_6C_2$ ${}_6C_3$ ${}_6C_4$ ${}_6C_5$ ${}_6C_6$	Row 6	1 6 14 20 15 6 1
${}_7C_0$ ${}_7C_1$ ${}_7C_2$ ${}_7C_3$ ${}_7C_4$ ${}_7C_5$ ${}_7C_6$ ${}_7C_7$	Row 7	1 7 21 35 35 21 7 1

Use technology to confirm for yourself that the combinations and the corresponding values in Pascal’s Triangle are identical.

Example 6

Use Pascal’s Triangle to determine the probability of getting exactly five heads with seven coin tosses.

Solution

Consider Row 7 of Pascal’s Triangle.

$$1 + 7 + 21 + 35 + 35 + 21 + 7 + 1 = 128 \text{ or } 2^7 = 128$$

There are 128 ways that seven coins can land, grouped in eight different arrangements.

Starting the count at zero, the fifth value in that row represents the combination of seven elements, so choose five.

$${}_7C_5 = 21$$

There are 21 possible ways that seven coins can land to show five heads.

The probability of tossing five heads with seven coins is $\frac{21}{128}$.

Example 7

Souvenirs at the football stadium kiosk include banners, clappers, horns, hats, T-shirts, and posters. The game-day sale offers any three items for $25. Determine how many different combinations of souvenirs are possible for this sale.

Solution

You could use your calculator to find that ${}_6C_3 = 20$. Alternatively, from Pascal's Triangle, it can be determined that ${}_6C_3 = 20$. There are 20 different arrangements of six items when choosing three of them. Find row 6, and then find the third value in that row. Remember that the outside values are considered row 0.

Learning Activity 6.5

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

The triangle pattern of numbers you have considered in this module is called Pascal's Triangle, after the French mathematician and philosopher Blaise Pascal. It is an interesting arrangement of numbers and within it, many other number patterns may be found.

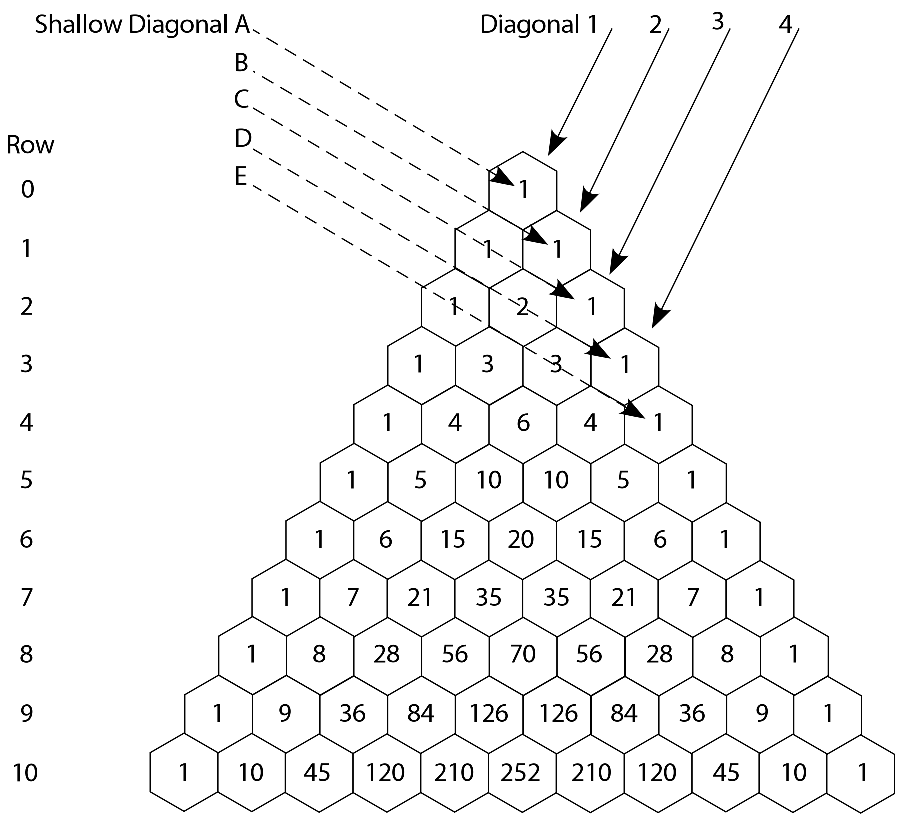
A copy of Pascal's Triangle, labeled with Rows 0 to 10, Diagonals 1 to 4, and Shallow Diagonals A to E is provided below. Note that diagonals and shallow diagonals are not the same. Diagonals pass directly through each number and proceed either left or right to the end of the diagonal row. Shallow diagonals pass through each number and also between each number, and then continue left or right to the end of the diagonal row. The shallow diagonals are, therefore, all the diagonals plus all the diagonal lines between them, as shown in the diagram.

continued

Learning Activity 6.5 (continued)

Locate and highlight each of the following number patterns in the rows and diagonals, or find combinations of groups of cells whose sums represent the number patterns on the graphic below:

1. Ones: 1, 1, 1, 1, . . .
2. Natural numbers: 1, 2, 3, 4, . . .
3. Square numbers: 1, 4, 9, 16, . . .
4. Powers of 2: 2^0, 2^1, 2^2, 2^3, . . . (Hint: Find the value of these powers.)
5. Fibonacci Sequence: 1, 1, 2, 3, 5, 8, . . .
6. Triangle numbers: 1, 3, 6, 10, 15, . . .
7. Tetrahedral numbers: 1, 4, 10, 20, 35, . . .
8. Powers of 11: 11^0, 11^1, 11^2, 11^3, . . . (Hint: Find the values of these powers.)



Note: The triangle is symmetrical, so identical diagonals are found in two orientations. Other patterns are possible.

continued

Learning Activity 6.5 (continued)

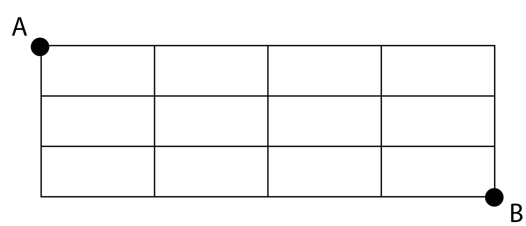
Part B: Applications

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. How many ways can the word MATHEMATICS appear in the following array, if you must spell the word by starting with the top, M, and moving downward to the final, S?

```
          M
        A   A
      T   T   T
    H   H   H   H
  E   E   E   E   E
M   M   M   M   M   M
  A   A   A   A   A
    T   T   T   T
      I   I   I
        C   C
          S
```

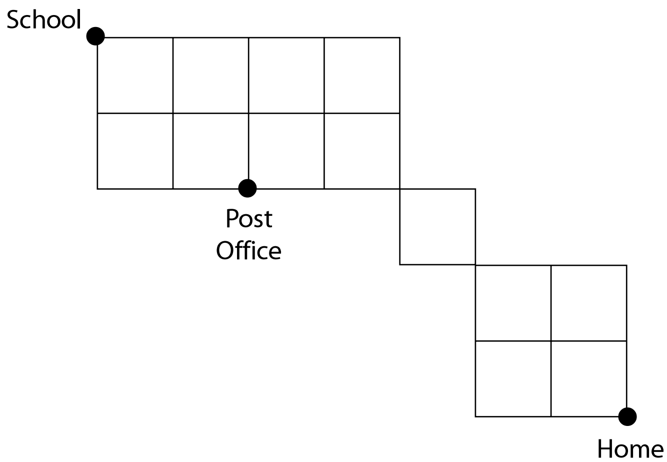
2. How many ways are there to get from A to B if you may only go right or down?



continued

Learning Activity 6.5 (continued)

3. Use the diagram below to answer the questions that follow.



   a) How many ways are there for you to go from school to home through the following grid of roadways, if you go directly home (right or down)?
   b) What is the probability that you will go by way of the Post Office?

4. A family is planning to have five children.
   a) How many different arrangements in the order of boys and girls are possible in a family with five children?
   b) How many different types of groupings of boys and girls are possible?
   c) In how many different ways can a family of five children have exactly three boys?
   d) What is the probability that the five children will be all boys?

Lesson Summary

In this lesson, you considered how combinations, probability, and Pascal's Triangle are connected. You solved binary problems using Pascal's Triangle and used problem-solving strategies and graphic organizers to find solutions to pathway problems.

Assignment 6.2

Applications of Permutations and Combinations

Total: 30 marks

This is a hand-in assignment. Please show your work clearly and in an organized manner. Round final answers to 2 decimal places, and include units, if appropriate. If you use technology as a strategy in your solution steps, please indicate what application you are using, the values you input, and a sketch or printout of the results. Answers given without supporting calculations will not be awarded full marks.

1. In how many ways can five books be arranged on a bookshelf? *(1 mark)*

2. In how many ways can two letters be placed in five mailboxes if
   a) the two letters must be placed in different mailboxes? *(1 mark)*

   b) the two letters can be placed in any mailbox? *(1 mark)*

Assignment 6.2: Applications of Permutations and Combinations (continued)

3. In how many ways can the letters in the following words be arranged?
   a) BOYFRIENDS *(1 mark)*

   b) HEXAHYDROXYCYCLOHEXANE *(2 marks)*

4. In how many ways can the letters in the word TRAPEZOIDS be arranged
   a) if you must use six different letters? *(1 mark)*

   b) if you use any six letters and repetitions are allowed? *(1 mark)*

   c) if the six-letter arrangement must start and end with a consonant, and repetitions are not allowed? *(1 mark)*

Assignment 6.2: Applications of Permutations and Combinations (continued)

5. Seven marbles, one of each colour in the rainbow, are concealed in a box. One marble at a time is drawn out, without replacement, until all the marbles have been revealed. What is the probability that the marbles will be withdrawn in the same order as the colours in a rainbow that you see in the sky? (Recall the mnemonic ROYGBIV for the order of the colours in a rainbow—Red, Orange, Yellow, Green, Blue, Indigo, Violet.) *(2 marks)*

6. Explain the significance of the number represented by ${}_7P_{13}$. *(2 marks)*

7. When programming the screen lock on his mobile device, Peter decided to create a six-digit code rather than a four-digit code because he thought it would be harder for someone else to figure out.

   a) If he can use the digits 1 through 9, determine how many codes are possible with a code of four digits and with a code of six digits. State your assumptions. *(4 marks)*

   b) What are the odds that a person could randomly try a four-digit code and get it right on the first attempt? *(1 mark)*

Assignment 6.2: Applications of Permutations and Combinations (continued)

8. In how many ways can six police officers be selected for special duty from a group of 15 police officers? *(1 mark)*

9. An examination consists of ten questions, of which the student must answer eight.
   a) If a student may choose any eight questions, in how many ways can this be done? *(1 mark)*

   b) If the student must answer at least four of the first five questions, in how many ways can the student answer the required eight questions? *(2 marks)*

Assignment 6.2: Applications of Permutations and Combinations (continued)

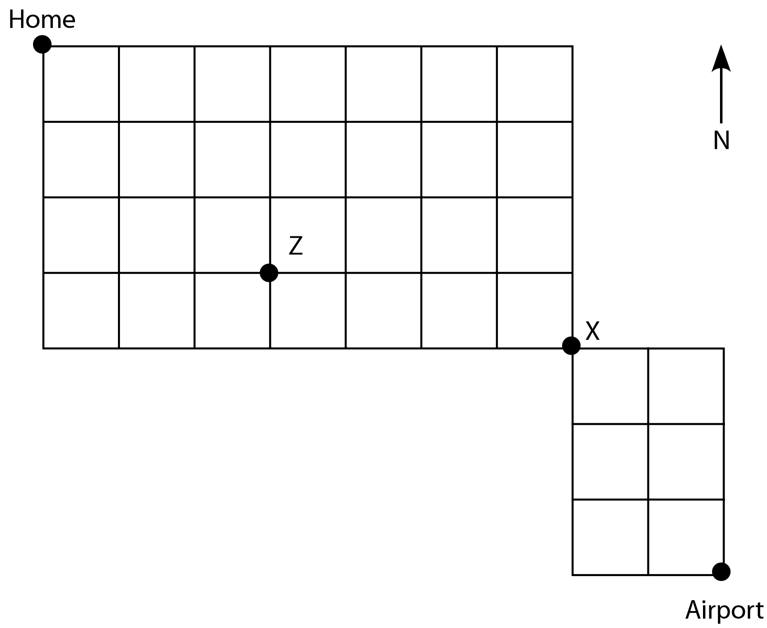
10. Calvin has 13 socks in his drawer: seven black and six white. He selects five socks at random.

    a) In how many ways can he select any five socks? *(1 mark)*

    b) What is the probability that he selects two black and three white socks? *(2 marks)*

11. Tom Hirton's Donut Shop makes 17 different varieties of doughnuts. They sell sample boxes with six different doughnuts in each box. Is it practical for the shop to stock one of each possible sample box? Support your decision. *(1 mark)*

12. The following diagram represents a map of city streets and avenues, which are all one-way streets heading either south or east. Ryan and Malika need to take a taxi from home to the airport.



a) Calculate how many different routes the taxi could follow to take them from their home to the airport. Assume the taxi driver can only drive south or east, and must pass through X. *(2 marks)*

b) What is the probability that the taxi passes through intersection Z on the way? *(2 marks)*

Notes

Module 6 Summary

Congratulations, you have finished Module 6. In this module, you learned about counting techniques such as the Fundamental Counting Principle, factorial notation, permutations, and combinations. You used graphic organizers and problem-solving strategies to find solutions to pathway problems and found the probability of certain events happening.

In the next module, you will be learning about sinusoidal functions.

Submitting Your Assignments

It is now time for you to submit the Module 6 Cover Assignment and Assignments 6.1 and 6.2 to the Distance Learning Unit so that you can receive some feedback on how you are doing in this course. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 6 assignments and organize your material in the following order:

- ❑ Module 6 Cover Sheet (found at the end of the course Introduction)
- ❑ Module 6 Cover Assignment: Patterns in Numbers
- ❑ Assignment 6.1: Fundamental Counting Principle
- ❑ Assignment 6.2: Applications of Permutations and Combinations

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Notes

Grade 12 Applied Mathematics (40S)

Module 6
Techniques of Counting

Learning Activity Answer Keys

MODULE 6: TECHNIQUES OF COUNTING

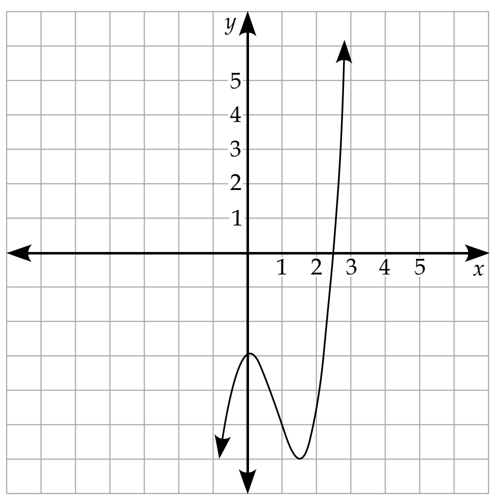
Learning Activity 6.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. There are five people at a meeting. They all shake hands with one another. How many handshakes take place?
2. Find the number halfway between $\frac{1}{3}$ and $\frac{1}{4}$.

For questions 3 to 8, use the graph of the function $y = 2x^3 - 5x^2 + x - 3$.



3. Estimate the relative maximum of the function.
4. Estimate the relative minimum of the function.
5. Estimate the x–intercept(s).
6. Find the y-intercept.
7. What is the degree of this function?
8. Describe the end behaviour.

Answers:

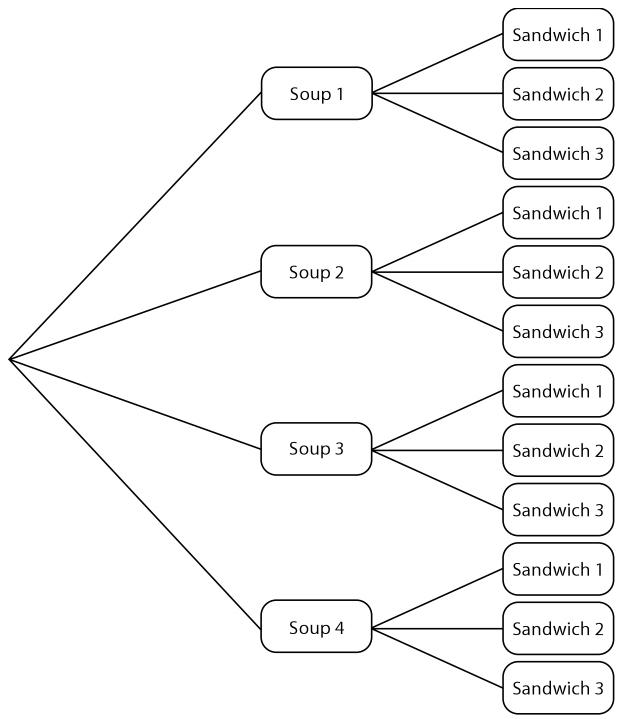
1. 10 handshakes (Person A shakes hands with four other people; Person B, already having shaken A's hand, would shake three other people's hands; Person C would have two additional handshakes; Person D would have one more handshake; and Person E has already shaken everyone else's hand; $4 + 3 + 2 + 1 + 0 = 10$)
2. $\frac{7}{24}$ $\left(\frac{1}{3} = \frac{8}{24}, \frac{1}{4} = \frac{6}{24}, \text{ halfway between these two numbers is } \frac{7}{24}\right)$
3. −3 (at the point (0.107, −2.95))
4. −6 (at the point (1.56, −6.015))
5. 2.5 (at the point (2.5, 0))
6. −3 (at the point (0, −3))
7. Three (it is a cubic function)
8. Quadrant III to Quadrant I

Part B: Using the Fundamental Counting Principle

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Construct a tree diagram showing all of the possible soup and sandwich combinations if the cafeteria offers four types of soup and three types of sandwiches for lunch.

   Answer:

   There are 12 different lunch combinations possible, since $4 \times 3 = 12$.

2. How many different double-scoop cones are possible using 12 different flavours of ice cream? State any assumptions you make.

   Answer:

   __12__ × __12__ = 144

   With this answer, it was assumed that it is possible to get two scoops of the same flavour of ice cream on one cone. Also, it was assumed that a cone made with chocolate on the bottom and vanilla on the top is different than a cone with vanilla on the bottom and chocolate on the top.

3. How many different ways can you draw five cards from a regular deck of 52 cards, if you keep all five cards together in the order they were drawn?

   Answer:

   $\underline{\quad 52 \quad} \times \underline{\quad 51 \quad} \times \underline{\quad 50 \quad} \times \underline{\quad 49 \quad} \times \underline{\quad 48 \quad} = 311\ 875\ 200$

   You make five choices, beginning with 52 options for the first choice, 51 for the second, and so on.

4. How many different outfits can you make from 3 pairs of shoes, 3 pairs of pants, and 3 shirts? You must choose one of each to make an outfit.

   Answer:

   $\underline{\quad 3 \quad} \times \underline{\quad 3 \quad} \times \underline{\quad 3 \quad} = 27$

   There are 3 possibilities for each of shoes, pants, and shirts.

5. A car license plate in Manitoba consists of 3 letters followed by 3 digits.

   a) How many different plates are possible?

      Answer:

      $\underline{\quad 26 \quad} \times \underline{\quad 26 \quad} \times \underline{\quad 26 \quad} \times \underline{\quad 10 \quad} \times \underline{\quad 10 \quad} \times \underline{\quad 10 \quad} = 17\ 576\ 000$

   b) What assumptions are you making?

      Answer:

      Some possible assumptions are:

      - repeating letters and numbers are permitted
      - zero and the letter O may both be used
      - all of the letters and digits are possible options in each of the 3 spots
      - three-letter combinations that make real words are also used
      - other assumptions are possible

   c) If each of the three letters must be unique and the first digit may not be zero, how many plates are possible?

      Answer:

      $\underline{\quad 26 \quad} \times \underline{\quad 25 \quad} \times \underline{\quad 24 \quad} \times \underline{\quad 9 \quad} \times \underline{\quad 10 \quad} \times \underline{\quad 10 \quad} = 14\ 040\ 000$

6. At a dessert buffet in a restaurant, you can make your own ice-cream sundae. You can choose from four flavours of ice cream, six sweet sauces, and eight candy toppings.

   a) If you can choose one item from each category, how many different sundaes are possible?

      Answer:

      __4__ × __6__ × __8__ = 192

      It is possible to make 192 different ice-cream sundaes, if they are made with one flavour of ice cream, one sauce, and one candy topping.

   b) If you are permitted to choose two different ice cream flavours, two different sauces, and three different toppings, how many unique sundaes are possible?

      Answer:

      You now have seven choices to make.

      __4__ × __3__ × __6__ × __5__ × __8__ × __7__ × __6__ = 120 960

      You could create one of 120 960 different sundaes.

7. You are shopping for a new car and would consider purchasing either a two-door or a four-door model. You may choose luxury, standard, or turbo options in each vehicle, each of which are available in any of five colours.

   a) From how many vehicles can you choose your new car?

      Answer:

      __2__ × __3__ × __5__ = 30

   b) If the luxury model only comes in two colours, how many choices do you have?

      Answer:

      Case 1: Luxury

2	×	1	×	2	= 4
two or four door		luxury		colour	

      Case 2: Other

2	×	2	×	5	= 20
two or four door		standard or turbo		colour	

      Total available: 20 + 4 = 24

8. Using the letters in the word DRAGONFLY, how many four-letter arrangements can be formed (the arrangements do not need to be real words) if

   a) letters may be used more than once.

      Answer:

      $\underline{\ \ 9\ \ } \times \underline{\ \ 9\ \ } \times \underline{\ \ 9\ \ } \times \underline{\ \ 9\ \ } = 6561$

   b) letters must not be used more than once.

      Answer:

      $\underline{\ \ 9\ \ } \times \underline{\ \ 8\ \ } \times \underline{\ \ 7\ \ } \times \underline{\ \ 6\ \ } = 3024$

   c) the arrangement must begin and end with a consonant (Y is considered a vowel in this case) and no repetition of letters is allowed.

      Answer:

      $\underset{\text{consonant}}{\underline{\ \ 6\ \ }} \times \underline{\ \ 7\ \ } \times \underline{\ \ 6\ \ } \times \underset{\text{consonant}}{\underline{\ \ 5\ \ }} = 1260$

   d) the arrangement must alternate vowels and consonants (Y is considered a vowel in this case) and no repetition of letters is allowed.

      Answer:

      The arrangement may begin with a vowel or a consonant. Two cases must be considered.

      Case 1: Vowel first

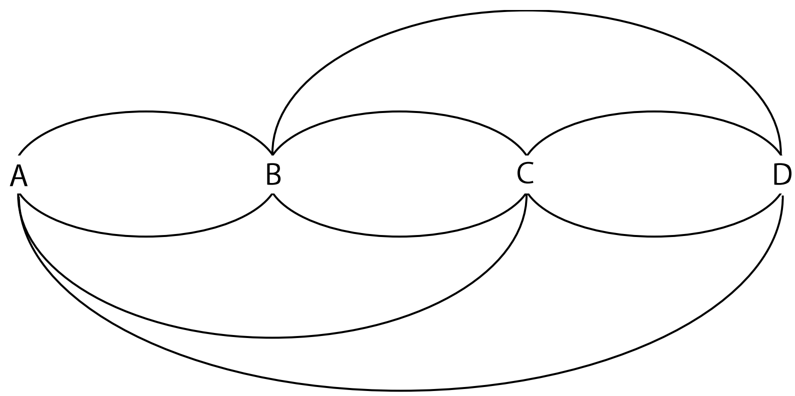
      $\underset{\text{V}}{\underline{\ \ 3\ \ }} \times \underset{\text{C}}{\underline{\ \ 6\ \ }} \times \underset{\text{V}}{\underline{\ \ 2\ \ }} \times \underset{\text{C}}{\underline{\ \ 4\ \ }} = 180$

      Case 2: Consonant first

      $\underset{\text{C}}{\underline{\ \ 6\ \ }} \times \underset{\text{V}}{\underline{\ \ 3\ \ }} \times \underset{\text{C}}{\underline{\ \ 5\ \ }} \times \underset{\text{V}}{\underline{\ \ 2\ \ }} = 180$

   Possible arrangements: 180 + 180 = 360

9. Examine the road map below. In how many ways can you select a route from A to D, if you must always travel in an easterly direction?



Answer:

There are three cases to consider.

Case 1: Direct routes

$\underset{\text{A to D}}{\underline{\quad 1 \quad}}$

Case 2: One stopover

$\underset{\text{A to B}}{\underline{\quad 2 \quad}} \times \underset{\text{B to D}}{\underline{\quad 1 \quad}} = 2$

$\underset{\text{A to C}}{\underline{\quad 1 \quad}} \times \underset{\text{C to D}}{\underline{\quad 2 \quad}} = 2$

Case 3: Two stopovers

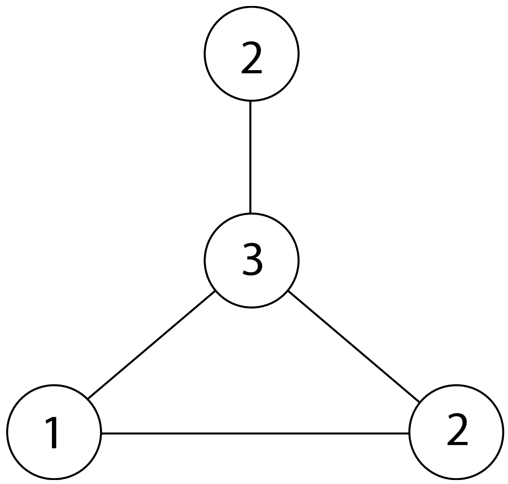
$\underset{\text{A to B}}{\underline{\quad 2 \quad}} \times \underset{\text{B to C}}{\underline{\quad 2 \quad}} \times \underset{\text{C to D}}{\underline{\quad 2 \quad}} = 8$

Total number of routes: 1 + 2 + 2 + 8 = 13

There are 13 possible routes between A and D.

10. The circles in the diagram are to be coloured red, blue, or green. In how many ways can this be done, if no two circles joined by a line are coloured the same colour?

Answer:



Instead of lines to represent decisions, this diagram has circles. Begin with the circle with the most restrictions, the centre circle. You have a choice of three colours for this circle. That leaves two options for the circle at the top and two options for one of the circles along the bottom. The last circle along the bottom must be coloured the one remaining colour.

$2 \times 3 \times 2 \times 1 = 12$

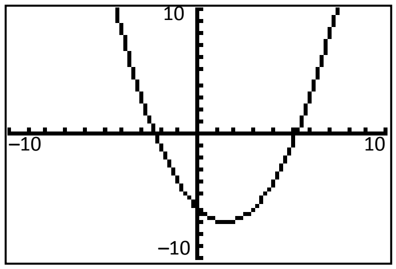
There are 12 different ways this diagram may be coloured.

Learning Activity 6.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

For questions 1 to 5, use the graph of $y = 0.5x^2 - 1.5x - 6$.



1. Write domain of the quadratic function.
2. Estimate the range of the quadratic function.
3. Describe the end behaviour of the quadratic function.
4. Estimate the zeros.
5. State the y-intercept.
6. Solve for p: $1500 = p(0.05)(10)$
7. If April 30th is a Wednesday, what day of the week is 50 days from April 30th?
8. How many two-digit numbers contain at least one 3?

Answers:

1. $\{x \in \Re\}$
2. $\{y \mid y \geq -7.125, y \in \Re\}$ (you may have estimated $y \geq -7$)
3. Quadrant II to Quadrant I
4. −2.275, 5.275 (you may have estimated −2.5 and 5)
5. −6
6. $p = 3000 \left(p = \dfrac{1500}{(0.05)(10)} \text{ or } \dfrac{1500}{0.5}\right)$

7. Thursday ($7 \times 7 = 49$, so 50 days is 7 weeks plus one day)

8. 18 (13, 23, 30, 31, 32, 33, . . ., 34, 35, 36, 37, 38, 39, 43, 53, 63, 73, 83, 93; there are 10 with a 3 in the tens digit and 8 others with a 3 in the ones digit)

Part B: Factorial Notation

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Calculate:
   a) 10!
   b) 13!
   c) 20!

   Answers:
   a) 10! = 3 628 800
   b) 13! = 6 227 020 800
   c) $20! = 2.43 \times 10^{18}$

2. Simplify the following without the use of a calculator:

   a) $\frac{7!}{5!}$

   b) $\frac{21!}{20!}$

   c) $\frac{3!}{4!}$

   d) $\frac{6!}{3!3!}$

   Answers:

   a) $\frac{7!}{5!} = \frac{7!}{5!} = \frac{7 \times 6 \times \cancel{5!}}{\cancel{5!}} = 42$

   b) $\frac{21!}{20!} = \frac{21!}{20!} = \frac{21 \times \cancel{20!}}{\cancel{20!}} = 21$

   c) $\frac{3!}{4!} = \frac{3!}{4!} = \frac{\cancel{3!}}{4 \times \cancel{3!}} = \frac{1}{4}$

   d) $\frac{6!}{3!3!} = \frac{6 \times 5 \times 4 \times \cancel{3!}}{(3 \times 2 \times 1) \times \cancel{3!}}$

   $= \frac{\cancel{6} \times 5 \times 4}{\cancel{3} \times \cancel{2} \times 1} = 20$

3. There are 33 cars that race in the Indy 500.
   a) In how many ways can the cars cross the finish line?

      Answer:

      $33! = 8.6833 \times 10^{36}$

   b) In how many ways can first, second, and third place finishes be awarded?

      Answer:

      $$\frac{33!}{30!} = 32\ 736 \text{ or } 33 \times 32 \times 31 = 32\ 736$$

      There are 32 736 possible ways to arrange first, second, and third place among 33 racers.

4. There are 50 golfers registered for a charity tournament.
   a) In how many ways can the schedule be organized for the golfers' first tee-off?

      Answer:

      $50! = 3.0414 \times 10^{64}$

   b) The organizer must put the 50 golfers into groups of four. How many ways is it possible to create the first group of four golfers?

      Answer:

      $$\frac{50!}{46!} = 5\ 527\ 200$$

5. Three boys and three girls must each take a turn singing a solo. In how many ways can they be arranged, if they must alternate between male and female voices?

   Answer:

   Since there is an equal number of boys and girls, they may alternate starting with either a boy or a girl.

   $(3 \times 3 \times 2 \times 2 \times 1 \times 1) + (3 \times 3 \times 2 \times 2 \times 1 \times 1) = 36 + 36 = 72$

   B G B G B G or G B G B G B

6. At the annual summer Corn and Apple Festival, there are three concerts, a children's entertainer, a magician, a pie-eating contest, and a talent-search contest scheduled on the main stage. In how many ways can these events be scheduled if the concerts must follow each other in the line-up?

Answer:

There are seven events but three are grouped together, so that leaves five positions to fill. The three concerts can be arranged in 3! ways.

(____ ____ ____) ____ ____ ____ ____

3!

5 4 3 2 1

$5!3! = 720$

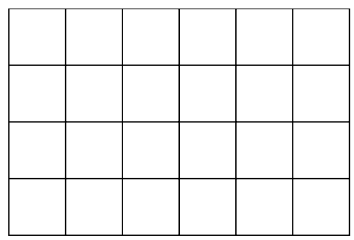
The acts can be arranged in 720 ways, if the three bands must play one after the other.

Learning Activity 6.3

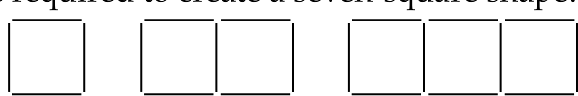
Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. How many squares of all sizes can be found in the following diagram?



2. What is the maximum number of 2″ by 3″ cards that can be cut from a sheet of paper that measures 2′ by 3′?
3. The path around the bases in a baseball field forms a square, with home plate, first base, second base, and third base at the corners. The distance from home plate to first base is 90 feet. Show the calculation required to determine how far the catcher has to throw the ball from home plate to make a play at second base.
4. Consider a pattern of squares made from toothpicks. How many toothpicks are required to create a seven-square shape?

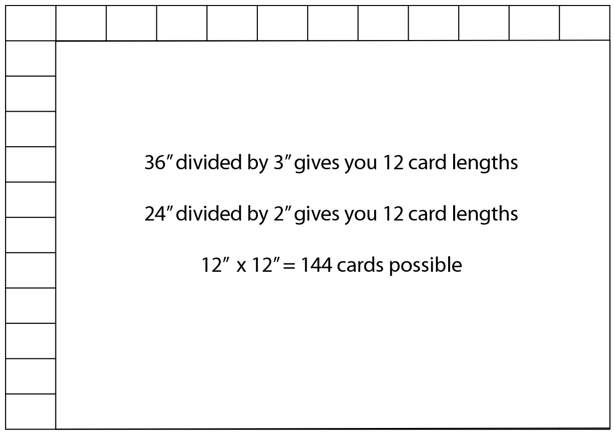


5. How many square shapes could be created with 52 toothpicks?
6. Describe the relationship between the number of toothpicks and the number of square shapes.
7. Estimate: $\frac{4!}{2!}$
8. Estimate: $\frac{7!}{4!}$

Answers:

1. 50 squares (1 × 1 squares: 6 × 4 = 24; 2 × 2 squares: 5 × 3 = 15; 3 × 3 squares: 4 × 2 = 8; 4 × 4 squares: 3 × 1 = 3)

2. 144 (2′ = 24″, 3″ = 36″, 24″ ÷ 2″ = 12, 36″ ÷ 3″ = 12, 12 × 12 = 144)



3. $\sqrt{90^2 + 90^2} = 127.28$ feet

4. 22 (there are 10 toothpicks in the three-square shape; adding one square requires three more toothpicks be added; to create a seven-square shape, you need to add four more squares or 12 more toothpicks; 10 + 12 = 22)

5. 17 (each square is made from three toothpicks except the first square, which requires four toothpicks; 52 divided by 3 is 17 with one left over, so you could made 17 squares with 52 toothpicks)

6. # of toothpicks = 3 × (# of squares) + 1

   or

   # of squares = (# of toothpicks − 1) ÷ 3

7. 12 $\left(\dfrac{4 \times 3 \times \cancel{2!}}{\cancel{2!}} = 12\right)$

8. 210 $\left(\dfrac{7 \times 6 \times 5 \times \cancel{4!}}{\cancel{4!}} = \dfrac{7 \times 30}{1} = 210\right)$

Part B: Permutations

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. A graduation class has 50 students.
   a) If students are called up randomly to receive their diplomas, in how many ways could they be arranged?

      Answer:

      $${}_{50}P_{50} = \frac{50!}{(50-50)!} = 50!$$

      $$50! = 3.04140932 \times 10^{64}$$

   b) The photographer wants to arrange the students in five rows of ten students each. In how many ways can the front row be arranged?

      Answer:

      $${}_{n}P_{r} = \frac{n!}{(n-r)!}$$

      $${}_{50}P_{10} = \frac{50!}{(50-10)!} = \frac{50!}{40!}$$

      $${}_{50}P_{10} = 3.727604302 \times 10^{16}$$

2. A teacher has 13 assignments to mark. He accidentally drops them down the stairwell and, when he collects them, he stacks them into a single pile. What is the probability that the papers are now arranged in alphabetical order by student name?

   Answer:

   The 13 papers can be arranged in 13! ways, only one of which is in correct alphabetical order.

   $${}_{13}P_{13} = 13!$$

   $$13! = 6\ 227\ 020\ 800$$

   The probability of the papers being in correct alphabetical order is $\frac{1}{6\ 227\ 020\ 800}$.

3. If you can select 8 courses from the 14 that are offered at your school, how many different class schedules are possible? What assumptions are you making?

   Answer:

   $$_{14}P_8 = \frac{14!}{(14-8)!}$$

   $$_{14}P_8 = \frac{14!}{6!} = 121\ 080\ 960$$

   This assumes that all 14 classes are offered in each of the 8 slots.

4. A briefcase has a combination lock with three numbered disks, and each disk has the digits 1 through 9 along its edge. The disks can be rotated to line up a three-digit code.

   a) How many codes are possible, if digits can be repeated?

      Answer:

      Use the Fundamental Counting Principle.

      $$\underline{\quad 9 \quad} \times \underline{\quad 9 \quad} \times \underline{\quad 9 \quad} = 729$$

   b) How many different codes are possible, if each digit must be unique?

      Answer:

      $$\underline{\quad 9 \quad} \times \underline{\quad 8 \quad} \times \underline{\quad 7 \quad} = 504$$

      Or

      $$_{9}P_3 = \frac{9!}{(9-3)!} = \frac{9!}{6!}$$

      $$_{9}P_3 = 504$$

5. a) How many different three-digit numbers can be made using the numbers 0, 3, 6, 7, and 9? State your assumptions.

Answer:

A three-digit number does not start with zero. Otherwise, it would be a two-digit number.

If digits can be repeated, but zero is not acceptable as the first digit, then 100 arrangements are possible.

$\underline{\quad 4 \quad} \times \underline{\quad 5 \quad} \times \underline{\quad 5 \quad} = 100$

If digits cannot be repeated but zero is not acceptable as the first digit, then 48 arrangements are possible.

Case 1 (last digit is zero): $\underset{\text{any of 4}}{\underline{\quad 4 \quad}} \times \underset{\text{any of 3}}{\underline{\quad 3 \quad}} \times \underset{\text{zero}}{\underline{\quad 1 \quad}} = 12$

Case 2 (last digit is not zero): $\underset{\text{not zero, not last digit}}{\underline{\quad 3 \quad}} \times \underset{\text{any of 3}}{\underline{\quad 3 \quad}} \times \underset{\text{not zero}}{\underline{\quad 4 \quad}} = 36$

There are 12 + 36 = 48 possible arrangements.

b) What is the probability that a randomly selected three-digit number is even, if digits cannot be repeated and zero is not allowed as a first digit?

Answer:

As found in part (a), there are 48 total arrangements possible. Even numbers will end in zero or six. Consider the two cases:

Case 1: Numbers that end with 0:

$\underline{\quad 4 \quad} \times \underline{\quad 3 \quad} \times \underset{\text{zero}}{\underline{\quad 1 \quad}} = 12$

Case 2: Numbers that end with 6:

$\underset{\text{not zero, not six}}{\underline{\quad 3 \quad}} \times \underline{\quad 3 \quad} \times \underset{\text{six}}{\underline{\quad 1 \quad}} = 9$

There are 12 + 9 = 21 even numbers. The probability of selecting an even number is therefore $\frac{21}{48} = \frac{7}{16}$.

6. Saylor has eight signal flags that are to be hung in a vertical line on a single pole. If four of the flags are indistinguishable red flags, three are indistinguishable white flags, and one is a blue flag, how many different signals can Saylor create?

   Answer:

   Saylor needs to arrange eight flags, taken eight at a time.

   $$_{n}P_{r} = \frac{n!}{(n-r)!}$$

   $$_{8}P_{8} = \frac{8!}{(8-8)!} = \frac{8!}{0!} = 8!$$

   To account for the repetitions of the red and white flags, divide by the number of arrangements of the indistinguishable items.

   $$\frac{8!}{4!3!} = 280$$

   Saylor can create 280 signals with her flags.

7. In how many ways can the letters in the following words be arranged, if all of the letters are used?

   a) HYSTERICAL

      Answer:

      10! = 3 628 800

   b) POPPY

      Answer:

      There are three identical letters.

      $$\frac{5!}{3!} = \frac{5 \times 4 \times 3!}{3!} = 20$$

8. How many five letter arrangements can be made using the letters in the following words?

a) CAMPGROUND

Answer:

$$_{n}P_{r} = \frac{n!}{(n-r)!}$$

$$_{10}P_{5} = \frac{10!}{(10-5)!} = \frac{10!}{5!}$$

$$_{10}P_{5} = 30\ 240$$

b) APRICOT

Answer:

$$\frac{7!}{(7-5)!} = \frac{5040}{2}$$

$$= 2520$$

c) APPLIED if, at most, one of the Ps is used

Answer:

$$\frac{6!}{(6-5)!} = \frac{720}{1}$$

$$= 720$$

This problem would be more challenging if both Ps could be used, since there would be two cases that depend on how the letters are chosen. The first case is calculated as shown above. The second case is when the letters chosen include both Ps and 3 other letters. In that case, you need to take the permutations of the identical Ps into account (that is, after choosing which letters to use in addition to two Ps, divide by 2! for repeating Ps).

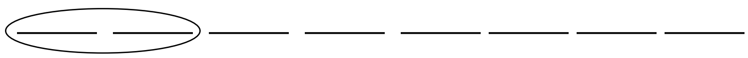
9. Erika is decorating for the graduation banquet. She needs to create a fabric banner behind the head table, made with eight different-coloured swatches of fabric. In how many ways can she arrange the swatches, if the black- and white-coloured swatches may not be placed next to each other?

Answer:

The number of possible arrangements is restricted by the grouping of two of the elements. The solution to this problem can be found by determining the complement—that is, the number of arrangements where the black and white ARE positioned next to each other—and subtracting that from the total number of permutations.

The total number of arrangements is ${}_8P_8 = 40\ 320$.

If lines are drawn to represent the eight decisions and two are grouped,



the number of arrangements of the pair of objects is 2! and, counting the group as one object, there are 7 objects to arrange. So, the number of arrangements when the black and white are positioned next to each other is

$$7!2!$$

or

$${}_7P_7 \times {}_2P_2 = 10\ 080$$

Therefore, the number of possible arrangements when black and white are not grouped next to each other is the complement:

$$40\ 320 - 10\ 080 = 30\ 240$$

Learning Activity 6.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

Without using a calculator, determine the value of the following factorials.

1. 5!
2. 4!
3. 3!
4. 2!
5. 1!
6. 0!

Simplify the following factorial expressions without using a calculator.

7. $\dfrac{25!}{23!4!}$
8. $\dfrac{14!}{8!9!}$

Answers:

1. 120 ($5 \times 4 \times 3 \times 2 \times 1 = 20 \times 6 = 120$)
2. 24 ($4 \times 3 \times 2 \times 1 = 12 \times 2 = 24$)
3. 6
4. 2
5. 1
6. 1 (It is recommended that you memorize the values for the factorials from 0 to 5.)
7. 25 $\left(\dfrac{25!}{23!4!} = \dfrac{25 \times \cancel{24} \times \cancel{23!}}{\cancel{23!} \times \cancel{4!}} = 25\right)$
8. $\dfrac{143}{24}$ $\left(\dfrac{14!}{8!9!} = \dfrac{\cancel{14}\left(\cancel{2}\right) \times 13 \times \cancel{12} \times 11 \times \cancel{10} \times \cancel{9!}}{8 \times \cancel{7} \times \cancel{6}(3) \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times 1 \times \cancel{9!}} = \dfrac{13 \times 11}{8 \times 3}\right)$

Part B: Combinations

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. A class of 24 students is sending a delegation of four students to a workshop on environmental issues.

   a) In how many ways can the delegation be selected?

   Answer:

   ${}_{24}C_4 = 10626$

   b) Is the order in which the students are selected important? Explain.

   Answer:

   No, the students do not have specific roles or positions on the delegation so the order is not important. The number of arrangements can be determined using the formula for combinations.

2. Staff members at a school are required to supervise lunchtime activities, sporting events, dances, and band concerts throughout the year. The administration tries to send a different group of four staff members each time they have supervisory duty.

   a) If there are 18 people on staff, in how many ways can a duty group be formed?

   Answer:

   ${}_{18}C_4 = 3060$

   b) The school has a policy that the staff on duty must be equally represented by male and female teachers. If there are 10 female and 8 male teachers, in how many ways can they have distinctly different supervisory teams?

   Answer:

   $\underbrace{{}_{10}C_2}_{\text{female}} \times \underbrace{{}_{8}C_2}_{\text{male}} = 1260$

3. How many different hands of five cards are possible from the 52 cards in a standard deck?

   Answer:

   ${}_{52}C_5 = 2\,598\,960$

4. In a national lottery, players can choose, in any order, six numbers from 1 through 49.

   a) How many combinations of the six numbers are possible?

      Answer:

      The order is not important so this is a combinations question.

      $$_{49}C_6 = \frac{49!}{(49-6)!6!} = 13\ 983\ 816$$

   b) What is the probability of selecting the winning combination?

      Answer:

      The probability of choosing all six numbers correctly is about one in 14 million.

5. Eight-ball is a pool game played with 16 distinct billiard balls. If three of the balls are selected at random, how many different combinations can be chosen?

   Answer:

   $_{16}C_3 = 560$

6. Ten people in a tour group want to take a gondola ride along the canals in Venice.

   a) If a gondola can hold up to six people, in how many ways can the tourists be arranged in two identical gondolas?

      Answer:

      The first gondola can have either five or six people, and the second will have either five or four, respectively. Once the first gondola is filled, all the remaining tourists must get into the second gondola.

      Case 1: $_{10}C_5 \times {}_5C_5 = 252 \times 1 = 252$

      Case 2: $_{10}C_6 \times {}_4C_4 = 210 \times 1 = 210$

      $252 + 210 = 462$

      Or

      $_{10}C_5 + {}_{10}C_6 = 462$

      There are 462 possible ways to arrange the ten tourists in the two gondolas.

b) If there are six seats in each gondola, in how many ways can six people be seated in the first boat?

Answer:

Now, there are six positions so it becomes a permutation question. There are 720 permutations of six items, taken six at a time.

$$_6P_6 = \frac{6!}{(6-6)!}$$

$$6! = 720$$

7. Find the number of ways 16 students can be assigned to two project groups, if each group must have at least 6 students.

Answer:

The first group could have 6, 7, 8, 9, or 10 students with all the remaining students in the second group. These are all different cases since they don't happen at the same time. The different cases need to be added together.

$$_{16}C_6 + {}_{16}C_7 + {}_{16}C_8 + {}_{16}C_9 + {}_{16}C_{10} = 51\ 766$$

8. A committee of five is selected at random from a group of nine people made up of six women and three men.

a) In how many ways can the committee be selected if there are no restrictions?

Answer:

$_9C_5 = 126$

b) What is the probability the committee will include George and Ruth, two of the nine people?

Answer:

There is one possible way to select George and Ruth ($_2C_2 = 1$). The three other people on the committee must be selected from the remaining seven.

$$_2C_2 \times {}_7C_3 = 35$$

To find the probability, divide the number of outcomes in the event by the total number of outcomes with no restrictions.

$$P(\text{G \& R on committee}) = \frac{35}{126} = \frac{5}{18}$$

The probability that George and Ruth are on the committee is about 28%.

c) In how many ways can the committee be selected to have exactly two men and three women?

Answer:

Find the product of the number of possibilities of choosing two men and three women.

$${}_3C_2 \times {}_6C_3 = 60$$

men women

d) What is the probability the committee will have at least three women?

Answer:

The cases to be considered are committees of three women and two men, four women and one man, and five women.

Case 1: 3 women and 2 men: ${}_6C_3 \times {}_3C_2 = 60$

Case 2: 4 women and 1 man: ${}_6C_4 \times {}_3C_1 = 45$

Case 3: 5 women and 0 men: ${}_6C_5 \times {}_3C_0 = 6$

Add the cases: 60 + 45 + 6 = 111

$$P(\text{at least three women}) = \frac{111}{126}$$

Learning Activity 6.5

Part A: BrainPower

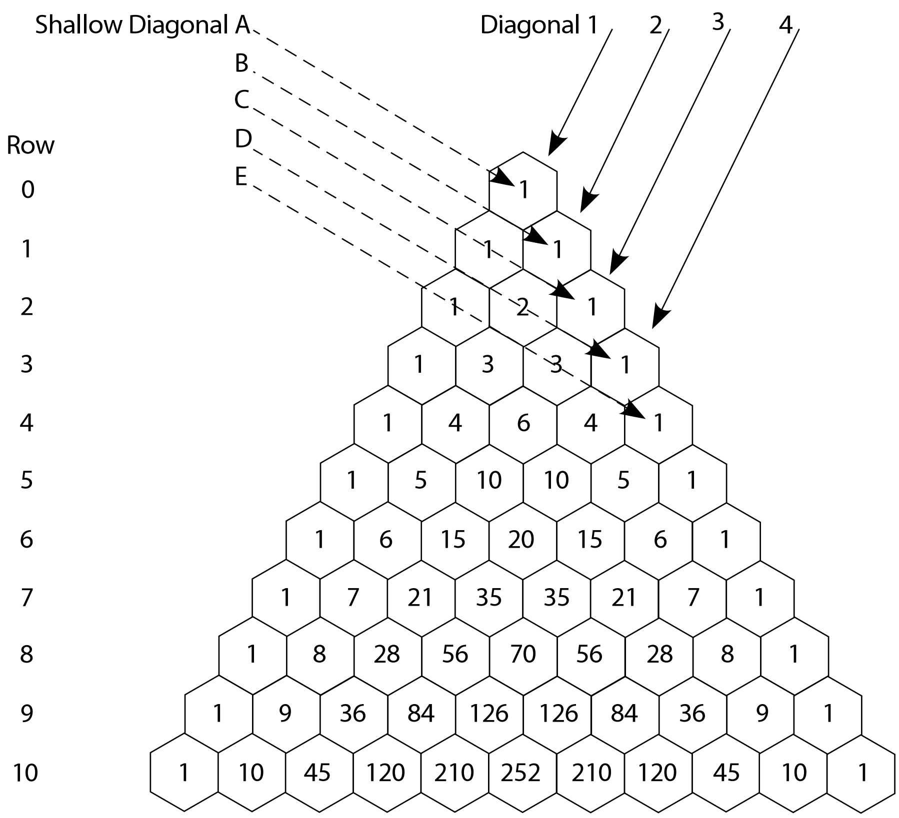
The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

The triangle pattern of numbers you have considered in this module is called Pascal's Triangle, after the French mathematician and philosopher Blaise Pascal. It is an interesting arrangement of numbers and within it, many other number patterns may be found.

A copy of Pascal's Triangle, labeled with Rows 0 to 10, Diagonals 1 to 4, and Shallow Diagonals A to E is provided below. Note that diagonals and shallow diagonals are not the same. Diagonals pass directly through each number and proceed either left or right to the end of the diagonal row. Shallow diagonals pass through each number and also between each number, and then continue left or right to the end of the diagonal row. The shallow diagonals are, therefore, all the diagonals plus all the diagonal lines between them, as shown in the diagram.

Locate and highlight each of the following number patterns in the rows and diagonals, and/or sums of numbers on the graphic below:

1. Ones: 1, 1, 1, 1, . . .
2. Natural numbers: 1, 2, 3, 4, . . .
3. Square numbers: 1, 4, 9, 16, . . .
4. Powers of 2: 2^0, 2^1, 2^2, 2^3, . . . (Hint: Find the value of these powers.)
5. Fibonacci Sequence: 1, 1, 2, 3, 5, 8, . . .
6. Triangle numbers: 1, 3, 6, 10, 15, . . .
7. Tetrahedral numbers: 1, 4, 10, 20, 35, . . .
8. Powers of 11: 11^0, 11^1, 11^2, 11^3, . . . (Hint: Find the values of these powers.)

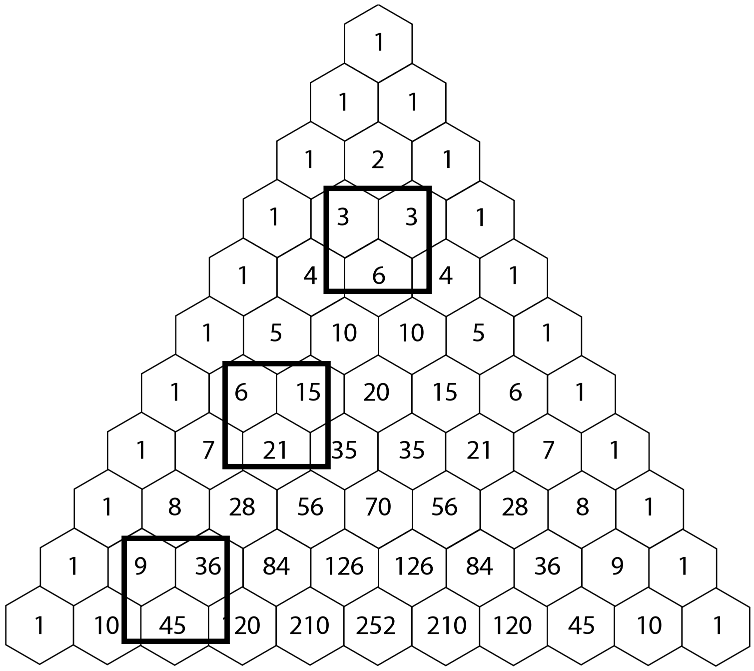


Note: The triangle is symmetrical, so identical diagonals are found in two orientations. Other patterns are possible.

Answers:

1. Along left and right edges or outer diagonals, diagonal #1 (begins at Row 0)
2. In diagonal 2 (begins at Row 1)
3. Found in the sum of two successive numbers in diagonal 3
   (1 + 3 = 4, 3 + 6 = 9, 6 + 10 = 16 . . .)

   Also, see squares on diagram below. Take a natural number in diagonal 2. The sum of the two numbers that are to the right and just below that natural number in diagonal 3 is the square of that natural number (e.g., $6^2 = 15 + 21$).



Square numbers:
$3^2 = 3 + 6$
$6^2 = 15 + 21$
$9^2 = 36 + 45$

4. Horizontal rows add to powers of two.

Sum of row 0 is 1	$2^0 = 1$
Sum of row 1 is 1 + 1 = 2	$2^1 = 2$
Sum of row 2 is 1 + 2 + 1 = 4	$2^2 = 4$
Sum of row 3 is 1 + 3 + 3 + 1 = 8	$2^3 = 8$

Sum of each row is 2 raised to power of the row number.

5. Sum of numbers in the "shallow" diagonals A to E . . . Remember that shallow diagonals pass through numbers and between numbers.
6. Found in diagonal 3 (begins at Row 2)
7. Found in diagonal 4 (begins at Row 3)

8. If each row is considered a single number, using each element as a digit in that number (and carrying over when the element itself has more than one digit), the number is equal to 11 raised to the power of the row number. Remember to start from the right, or "ones," and move left to carry over when there is more than one digit.

Row 0: 1	$11^0 = 1$
Row 1: 1 1	$11^1 = 11$
Row 2: 1 2 1	$11^2 = 121$
Row 3: 1 3 3 1	$11^3 = 1331$
Row 4: 1 4 6 4 1	$11^4 = 14641$
Row 5: 1 5 10 10 5 1 (1, 5 + 1, 0 + 1, 0, 5, 1)	$11^5 = 161051$
Row 6: 1 6 15 20 15 6 1 (1, 6 + 1, 5 + 2, 0 + 1, 5, 6, 1)	$11^6 = 1771561$

Part B: Applications

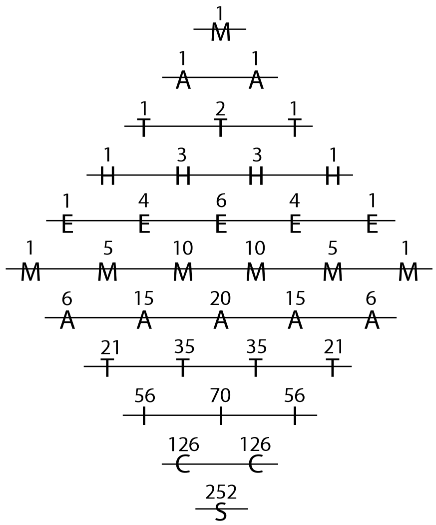
Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. How many ways can the word MATHEMATICS appear in the following array, if you must spell the word by starting with the top, M, and moving downward to the final, S?

```
          M
        A   A
      T   T   T
    H   H   H   H
  E   E   E   E   E
M   M   M   M   M   M
  A   A   A   A   A
    T   T   T   T
      I   I   I
        C   C
          S
```

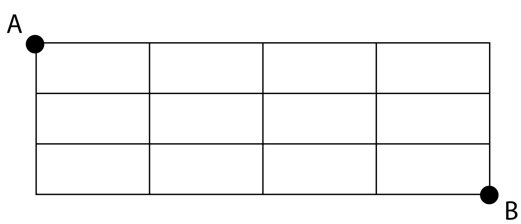
Answer:

There are 252 ways.



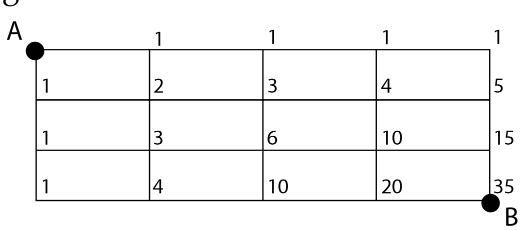
Or, consider this to be a part of Pascal's Triangle in row 10 (start counting at zero) in the middle of the row. So, calculate ${}_{10}C_5 = 252$.

2. How many ways are there to get from A to B if you may only go right or down?



Answer:

Method 1: There are 35 ways to go from A to B as indicated by in the diagram below.



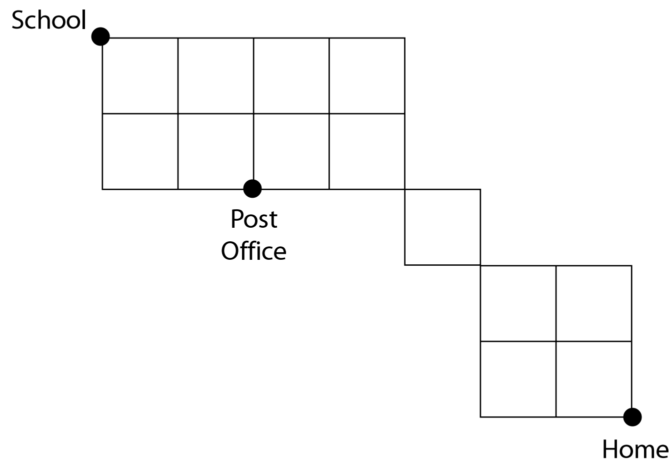
Method 2: Find the number of permutations of the directions 4 units Right and 3 units Down as represented by RRRRDDD.

$$\frac{7!}{4!3!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(4 \times 3 \times 2 \times 1)(3 \times 2 \times 1)} = 35$$

Method 3: Find the combination of the 7 blocks and choose any 4 of them to be to the right.

$_7C_4 = 35$

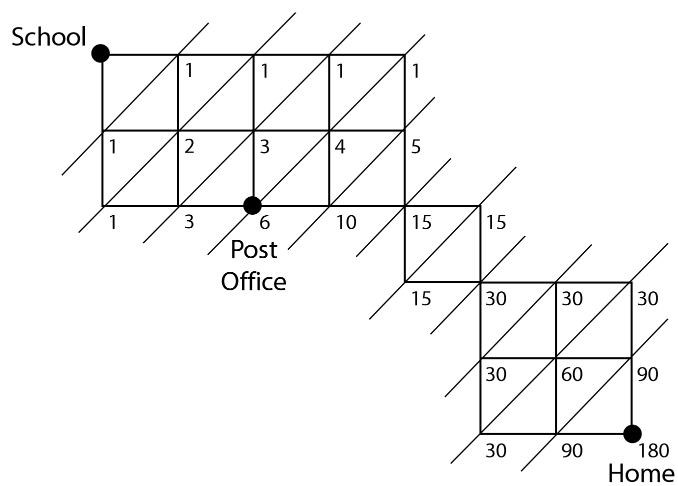
3. Use the diagram below to answer the questions that follow.



a) How many ways are there for you to go from school to home through the following grid of roadways, if you go directly home (right or down)?

Answer:

Method 1: There are a total of 180 ways, as indicated by the diagram below.



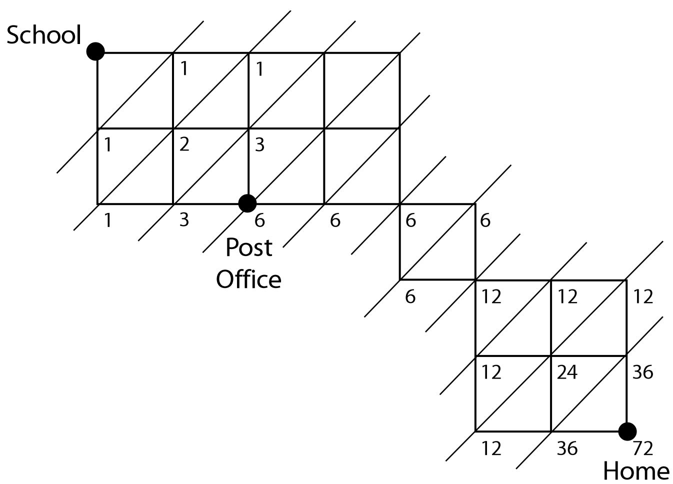
Method 2: Going Right 4 and Down 2 gets you to the bottom right of the first rectangle grid. Calculate the number of ways as $\frac{6!}{4!2!} = 15$.

Going down 1 and right 1 gets you to the bottom right of the middle square in 2 ways. Going down 2 and right 2 gets you to the bottom of the final square grid. Calculate the number of ways as $\frac{4!}{2!2!} = 6$. Using the Fundamental Counting Principal, the total number of ways is $15 \times 2 \times 6 = 180$ ways.

b) What is the probability that you will go by way of the Post Office?

Answer:

The probability that you will go by way of the Post Office is found by determining the number of ways of going home by way of the Post Office and dividing that by the total number of possible routes home. The number of routes that pass by the Post Office is 72, as indicated by the diagram below.



Or, use the Fundamental Counting Principle and multiply $6 \times 2 \times 6 = 72$.

$P(\text{Going by Post Office}) = \frac{72}{180} = \frac{2}{5}$

4. A family is planning to have five children.

a) How many different arrangements in the order of boys and girls are possible in a family with five children?

Answer:

Each of the five children could be either a boy or a girl.

$\underline{\quad 2 \quad} \times \underline{\quad 2 \quad} \times \underline{\quad 2 \quad} \times \underline{\quad 2 \quad} \times \underline{\quad 2 \quad} = 2^5 = 32$

Or, from Row 5 of Pascal's Triangle: $1 + 5 + 10 + 10 + 5 + 1 = 32$

There are 32 different ways the order of boys and girls could occur in a family of five children.

b) How many different types of groupings of boys and girls are possible?

Answer:

There are six different types of groupings possible:

- five girls and zero boys
- four girls and one boy
- three girls and two boys
- two girls and three boys
- one girl and four boys
- zero girls and five boys

c) In how many different ways can a family of five children have exactly three boys?

Answer:

${}_5C_3 = 10$

Here is a list of the possibilities:

- BBBGG BBGGB BGGBB GGBBB BGBGB
- GBBBG GBBGB BBGBG BGBBG GBGBB

d) What is the probability that the five children will be all boys?

Answer:

$2^5 = 32$

${}_5C_1 = 1$

$P(\text{five boys}) = \frac{1}{32}$

Notes

Grade 12 Applied Mathematics (40S)

Module 7
Sinusoidal Functions

Module 7: Sinusoidal Functions

Introduction

Welcome to Module 7. By now, you have used many different mathematical models to represent real-world situations. When events happen in a repetitive or periodic manner, such as the height of tides, the time of sunrise, or the average monthly temperatures in Manitoba, a sinusoidal function may be used to model it. In this module, you will represent data using sinusoidal functions and use the functions to solve problems. You will describe the characteristics of the graphs and equations of sinusoidal functions and interpret the graphs that model situations.

Assignments in Module 7

When you have completed the assignments for Module 7, submit your completed assignments to the Distance Learning Unit either by mail or electronically through the learning management system (LMS). The staff will forward your work to your tutor/marker.

Lesson	Assignment Number	Assignment Title
	Cover Assignment	Playing Fair
2	Assignment 7.1	Sinusoidal Function Models

Resource Sheet

When you write your final examination, you are encouraged to take a Final Examination Resource Sheet with you into the examination. This sheet will be one letter-sized page, 8½ ″ by 11″, with both sides in your handwriting or typewritten. You will submit it with your examination, but you do not receive any marks for it.

Many students have found that preparing a resource sheet is an excellent way to review. It provides you with a summary of the important facts of each module. You should complete a resource sheet for each module to help with your studying and reviewing. Lesson summaries and module summaries are included for you to use as a guide.

You may use the list of instructions provided below to help you with preparing your resource sheet for the material in Module 7. On this sheet, you should record math terms and definitions, formulas, sample questions, or a list of places where you often make mistakes. You should also identify special areas that require extra attention or review by writing the page numbers.

After you have completed each module's resource sheet, you may summarize the sheets from Modules 5, 6, 7, and 8 to prepare your Final Examination Resource Sheet. The final examination for this course is based on Modules 5 to 8.

Resource Sheet for Module 7

As you go through the lessons of this module, you may want to consider the following suggestions regarding the creation of a resource sheet.

1. List all the important math terms, and define them if necessary.
2. List all the formulas and perhaps a sample problem that shows how the formula is used.
3. If necessary, write the solutions to some problems, showing in detail how you did the calculations.
4. Copy any questions that represent the key points of the lesson, and perhaps include the solutions as well.
5. Identify the problems you found most difficult, and copy the page numbers onto the resource sheet so that you can review them before writing the examination. You may also copy the problems and the solutions onto your resource sheet, and later write them onto your Final Examination Resource Sheet.
6. Write any comments, ideas, shortcuts, or other reminders that may be helpful during an examination.

Module 7 Cover Assignment: Playing Fair

A game is considered fair if every player has an equal chance of winning. Some games may seem fair but in reality are not. This cover assignment involves a game played by two players.

Play the following game with a partner and record your results in the table. You can be Player A and your partner can be Player B.

Notes

Module 7 Cover Assignment

Playing Fair

Total: 5 marks

The game involves choosing marbles from a box. Place two white marbles and two black marbles in a box. (Alternatively, you may use two blue bingo chips and two red bingo chips.) Without looking in the box, Player A randomly chooses two of the marbles.

If the two marbles chosen are of the same colour, Player A wins 1 point for that round. If the two marbles chosen are each a different colour, Player B wins 1 point for that round.

Player A returns the marbles to the box and for the next round, Player B randomly selects two marbles. If the two marbles are of the same colour, Player A again wins 1 point. If the two chosen marbles are each a different colour, Player B again wins 1 point.

Player A and Player B take turns choosing marbles.

Play 30 rounds of this game and total the number of points each player wins. Does this game seem fair? Explain.

Give your analysis and opinion in a well written paragraph. Justify your answer based on the calculation of the theoretical and experimental probabilities of either player winning this game. Please show your work clearly and in a well-organized manner. Include complete explanations.

Module 7 Cover Assignment: Playing Fair (continued)

Round #	1	2	3	4	5	6	7	8	9	10
Player A's points (same colour)										
Player B's points (different colours)										

Round #	11	12	13	14	15	16	17	18	19	20
Player A's points (same colour)										
Player B's points (different colours)										

Round #	21	22	23	24	25	26	27	28	29	30	Total Points
Player A's points (same colour)											
Player B's points (different colours)											

Lesson 1: Periodic Functions

Lesson Focus

In this lesson, you will

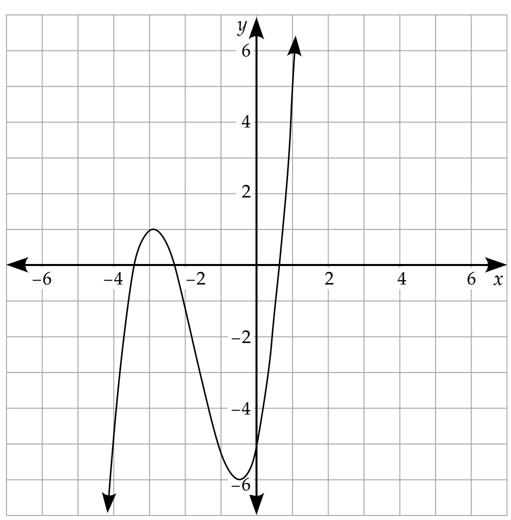
- ❑ represent data using sinusoidal functions
- ❑ describe the characteristics of sinusoidal functions by analyzing their graphs
- ❑ graph data and determine the sinusoidal function that best approximates the data
- ❑ interpret the graph of a sinusoidal function that models a situation, and explain the reasoning
- ❑ solve, using technology, a contextual problem that involves data that is best represented by graphs of sinusoidal functions, and explain the reasoning

Lesson Introduction

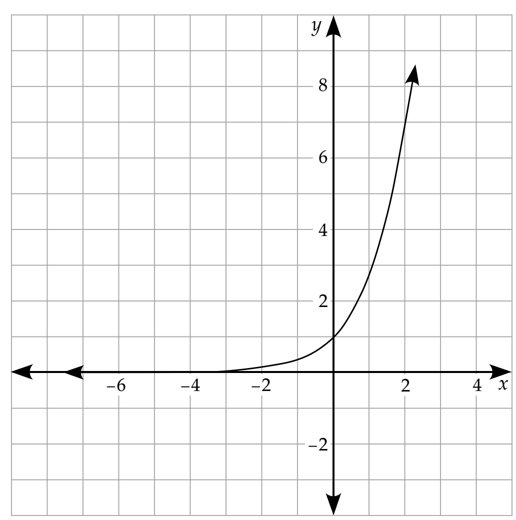
You began this course by considering the graphs and equations of polynomial, exponential, and logarithmic functions. When you plotted data points, you used technology to find the regression equation that best represented them, and used that to solve problems related to that function.

Recall the following four types of graphs:

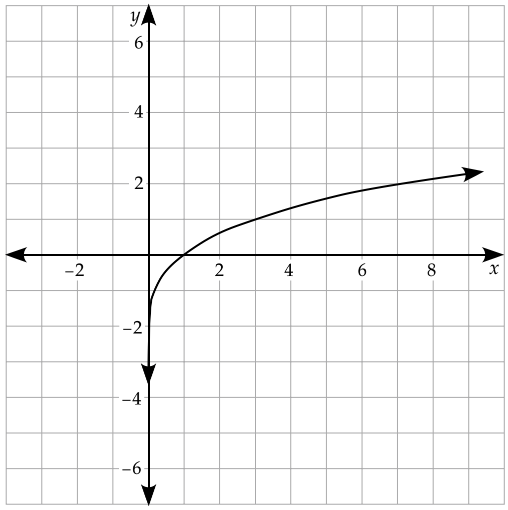
Cubic



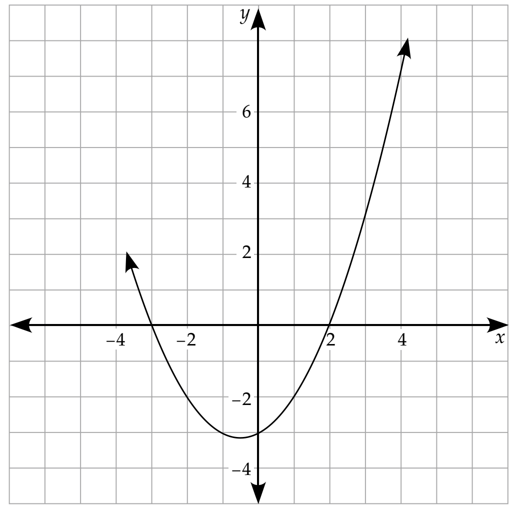
Exponential



Logarithmic



Quadratic



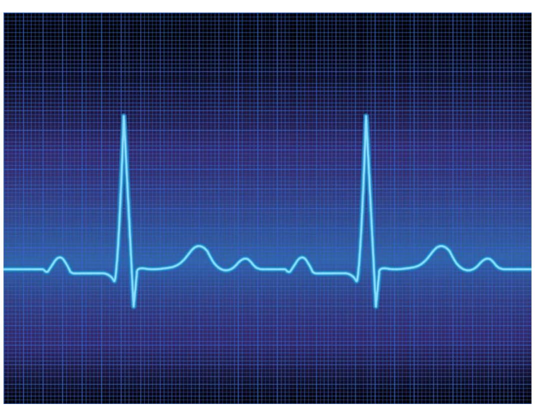
While each of these graphs has specific characteristics that describe its shape and behaviour, none of them illustrates a repeating pattern. In this lesson, you will learn about a function type called a sinusoidal function that is useful for modelling repeating, periodic phenomena.

Modelling Periodic Data with Sinusoidal Functions

Periodic Data

Data that contain cycles that repeat at regular intervals is called **periodic data**.

If a doctor wanted to check the electrical activity in your heart, he or she may request a test called an electrocardiogram, which translates your heart's electrical activity into a line tracing on paper. The repeating heart beat creates a wave-like pattern.



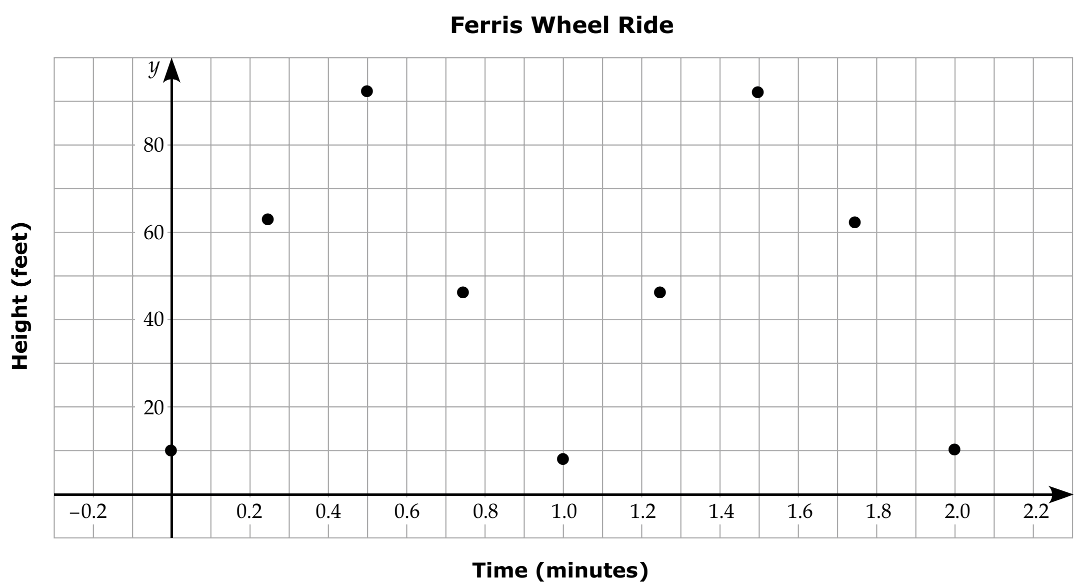
In this lesson, you will consider data that has a repeating cyclic or periodic pattern, and the equations that can be used to best model these sinusoidal functions.

Example 1

Tessa is riding on a Ferris wheel. Her height above ground as she circles around the ride over time is recorded in the chart below. Use technology to graph the data points.

Time (minutes)	Height (feet)
0	10.24
0.25	62.5
0.5	92.15
0.75	45.56
1	7
1.25	46.21
1.5	92.27
1.75	61.87
2	10

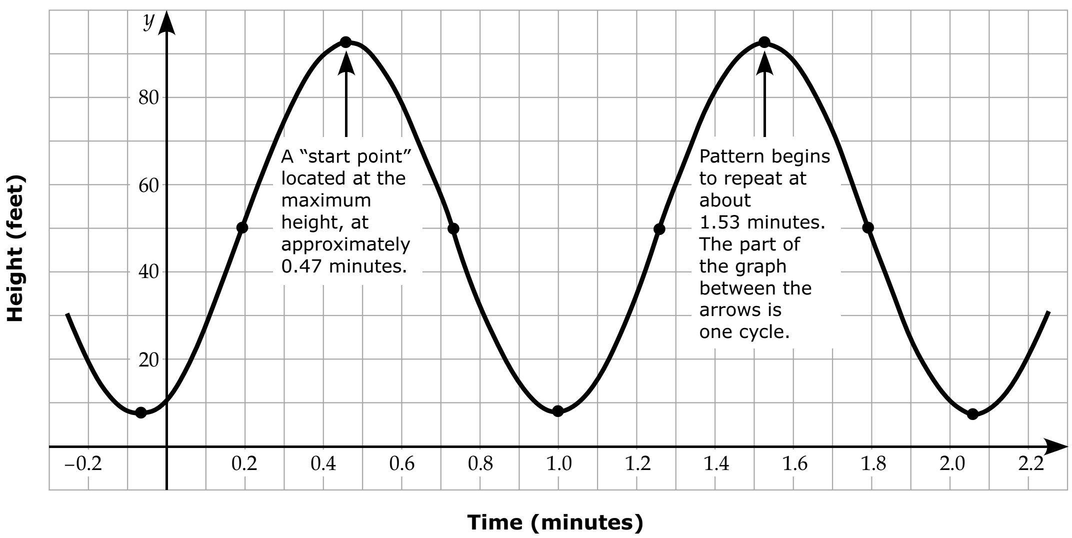
Solution



Tessa's height above ground as she rides the Ferris wheel is a repeating, wave-like pattern. During one revolution on the ride, she would be as high as about 92 feet and as low as about 7 feet. This type of cyclic, repetitive data is called periodic data.

The dots on the above graph could be connected by drawing a smooth curve because this height data is continuous at every moment of the ride.

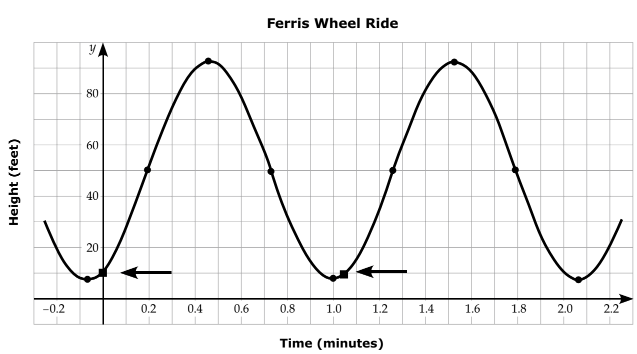
Ferris Wheel Ride



One revolution would take her just over one minute. This can be deduced by choosing any "start point" and then locating when that point repeats itself after one full cycle. The period is said to be the length of the shortest repeating interval of the function. For example, you may choose the highest point on the graph, and then find when the line returns to the highest point. The period of the above graph is the difference between the x-values of these points.

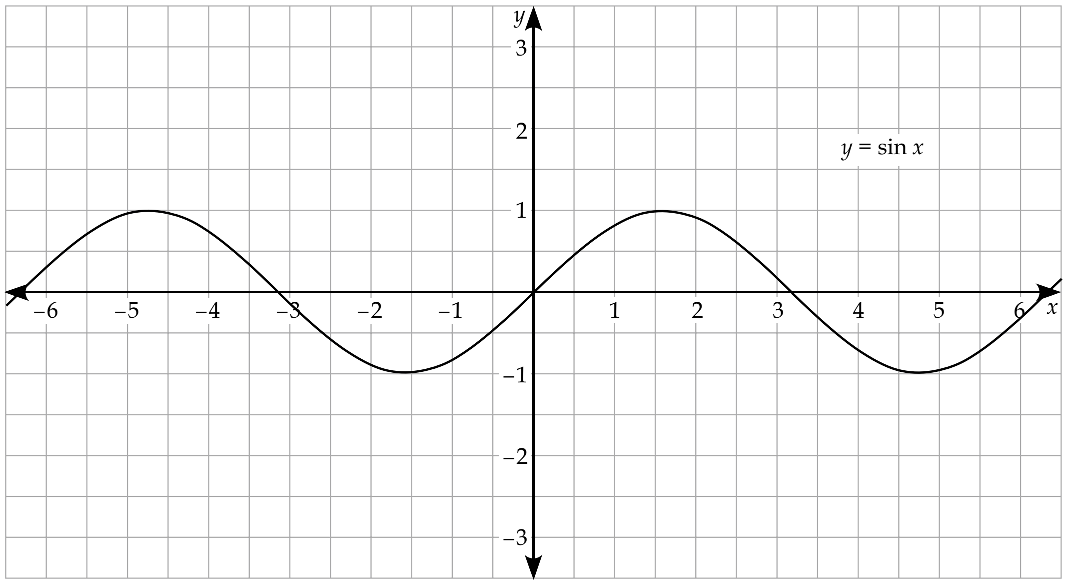
$$1.53 - 0.47 = 1.06 \text{ minutes}$$

The **cycle** is the repeated pattern. Another cycle could be defined by the points where time = 0 seconds, when she is at 10.24 feet (see data table given above) and the ride is moving upwards. The next time she is at that same height and moving in an upwards direction is when time is at 1.06 seconds, corresponding to the time length of the period.



Sinusoidal Functions

To represent periodic data with an equation of best fit, the graph of the sine function, $y = \sin x$, can be used. The sine curve, shown below, models cyclic, periodic, or sinusoidal data in radian measure.



6.28 radians is one revolution of a circle and of a sinusoidal curve, and it is the period of the function $y = \sin x$

Note: Radians are units for measuring angles, similar to degrees.

In a circle, there are 360 degrees, or 2π radians.

- $360° = 2\pi \cong 6.28$ radians
- $180° = \pi \cong 3.14$ radians

When using a graphing calculator such as the TI-83 or TI-83 Plus or other graphing technology to graph periodic data, it is common to use radians rather than degrees. Change the mode on your graphing technology to radian mode (sometimes stated as RAD), rather than degree mode (DEG).

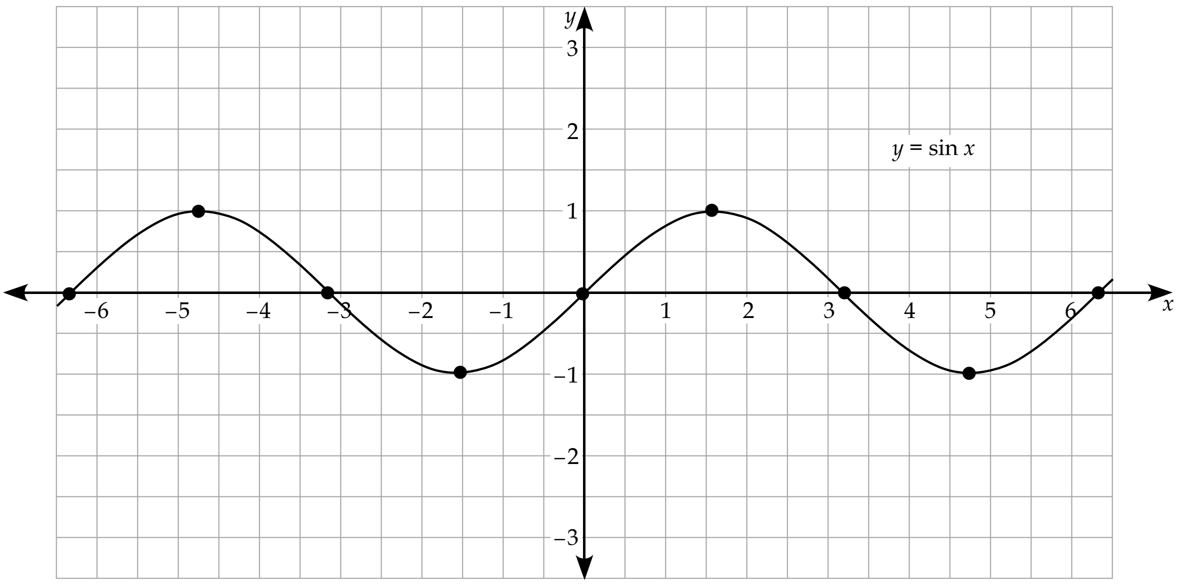
The characteristics of the sine graph are described in terms of its maximum and minimum values, the equation of the median, its amplitude, the range, and its period.

Example 2

Determine the maximum, minimum, range, median, amplitude, and period of the graph of the function $y = \sin x$, where x is in radians.

Solution

Use technology to graph $y = \sin x$ and determine the coordinates of the maximum and minimum points.



Maximum points at (−4.71, 1) and (1.57, 1)

Minimum points at (−1.57, −1) and (4.71, −1)

The period of this graph will be the shortest, repeating interval, found by calculating the difference between the x-values of two corresponding points on the graph: choose any one point on the graph and find the next point where the graph begins to repeat the same pattern. For example, you may take consecutive maximum (or minimum) points.

$$1.57 - (-4.71) = 6.28$$

The period of the sine graph is 6.28 radians. This means one cycle repeats in 2π radians. (Recall that 2π radians is equivalent to 360°, so this represents one full rotation of the circle.)

The **range** is from the minimum to the maximum y-values. In set notation, this can be denoted as $\{y \mid -1 \le y \le 1, y \in \Re\}$.

The **median** is the line midway between the minimum and maximum. It can be seen on this graph as coinciding with the x-axis and it can be calculated as the average of the maximum and minimum as follows:

$$\text{median} = \frac{\text{maximum } y\text{-value} + \text{minimum } y\text{-value}}{2}$$

$$\text{median} = \frac{1 + (-1)}{2}$$

$$\text{median} = \frac{0}{2} = 0$$

The equation of the median in the graph of the sine function is the horizontal line, $y = 0$.

The **amplitude** is the vertical distance between the median line and the maximum value. It is 1 unit and can be read from the graph or calculated:

Amplitude = maximum y-value − median

Amplitude = 1 − 0

Amplitude = 1 unit

Using Sinusoidal Regression Equations to Model Periodic Data

Using the example above, the graph of the data relating Tessa's height over time as she rides the Ferris wheel can be modeled by the equation $y = 43 \sin (5.9x - 1.18) + 50$. This equation is determined by using technology to find the regression equation that best fits the data points given. Refer to the technology help files associated with the graphing application of your choice, or the Technology Appendix, to review how to find a regression equation.

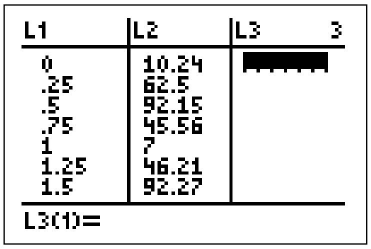
Example 3

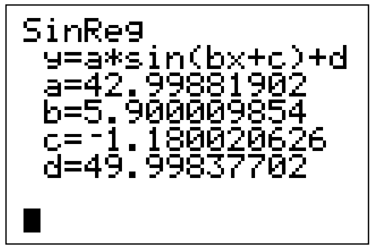
Use technology to confirm that the equation above models the data from the Ferris wheel ride.

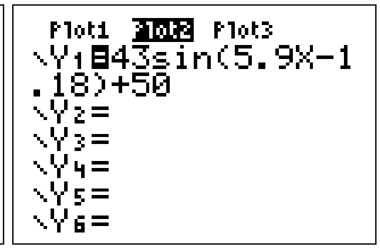
Use the graph of the sinusoidal function to determine the coordinates of a maximum and a minimum point along the path that Tessa travels on the Ferris wheel. Use these values to determine the maximum, minimum, range, median, amplitude, and period of the sinusoidal function that describes this situation.

For sinusoidal graphs use Radian MODE.

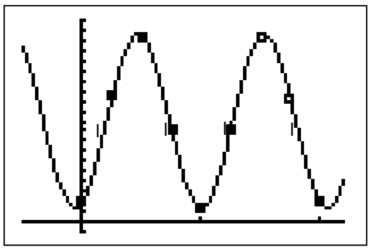
Plot the points. Find the Sine Regression equation.

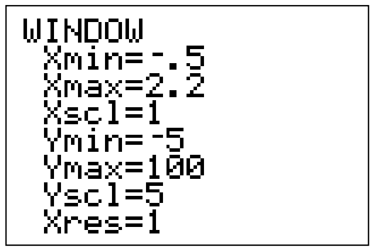


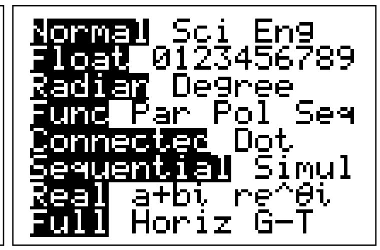




Adjust the window settings to view the points and graph.



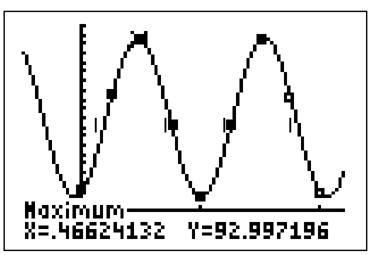


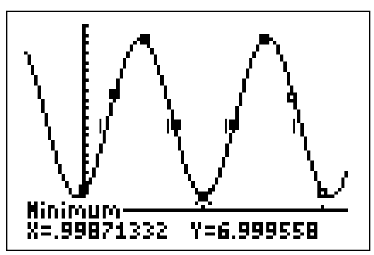


The equation $y = 43 \sin (5.9x - 1.18) + 50$ represents the data very well. The curve passes through all of the given points.

Note: The sine regression equation used to create the graph has used values rounded to two decimal places. The use of more decimal places will make slight differences in the calculations of coordinates.

Use technology to determine the maximum and minimum y-values along the curve.





There is a maximum at (0.466, 93) and a minimum at (1, 7). This means that at 0.466 minutes into the ride, Tessa is at the highest point of the ride, and at the one-minute mark she is at the lowest point of the ride.

The maximum height is 93 feet and the minimum height is 7 feet.

The range is $\{y \mid 7 \leq y \leq 93, y \in \Re\}$.

The median is halfway between the maximum and minimum, so it is the average and is calculated as follows:

$$\text{median} = \frac{\text{maximum } y\text{-value} + \text{minimum } y\text{-value}}{2}$$

$$\text{median} = \frac{93 + 7}{2}$$

$$\text{median} = \frac{100}{2}$$

The median is 50 feet.

Amplitude = maximum y-value − median

Amplitude = 93 − 50

Amplitude = 43 feet

The distance from the median up to the maximum is 43 feet and down to the minimum is 43 feet.

The distance from a maximum point to the next minimum point represents half of a period.

Half period = 1 − 0.466

Half period = 0.534

If half of a period is 0.534 minutes, then one full cycle occurs in a period of 1.068 minutes.

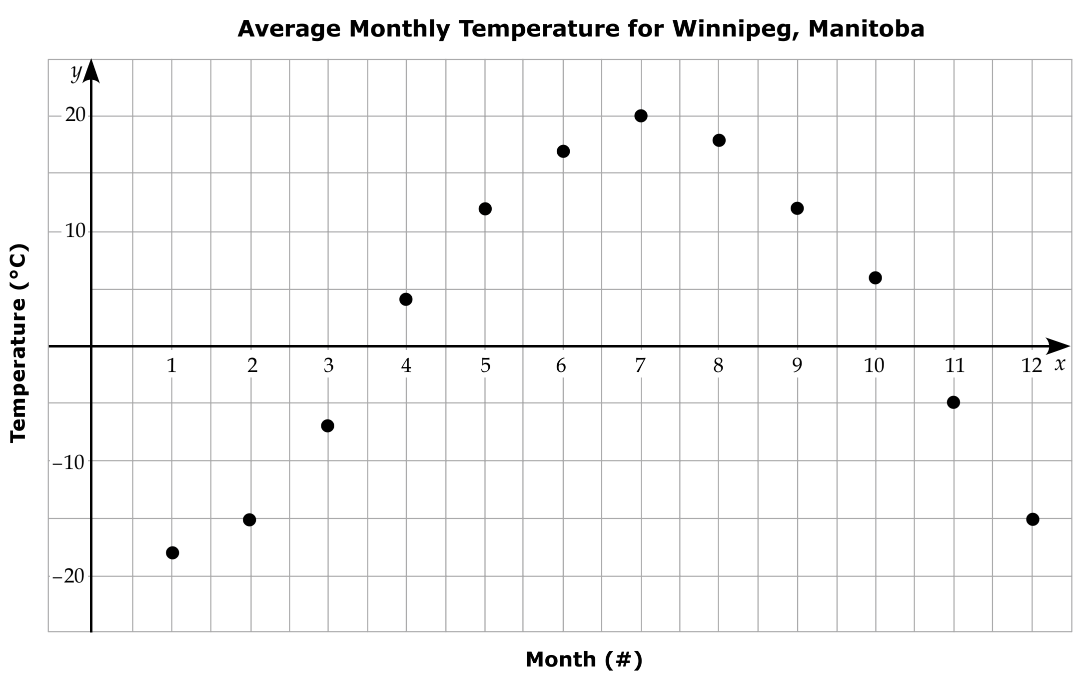
Example 4

The chart below states the average monthly temperature in Winnipeg, Manitoba, as given on the website www.winnipeg.climatemps.com/.

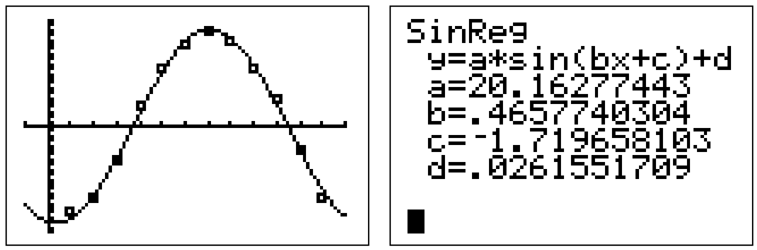
Month (#)	Average Temperature (°C)
January (1)	–18
February (2)	–15
March (3)	–7
April (4)	4
May (5)	12
June (6)	17
July (7)	20
August (8)	18
September (9)	12
October (10)	6
November (11)	–5
December (12)	–15

Use technology to graph the data. Determine if it is periodic. Find the sinusoidal regression equation that best fits the data. Determine the period, maximum average monthly temperature, minimum average monthly temperature, the range of temperatures, and the average annual temperature.

Solution

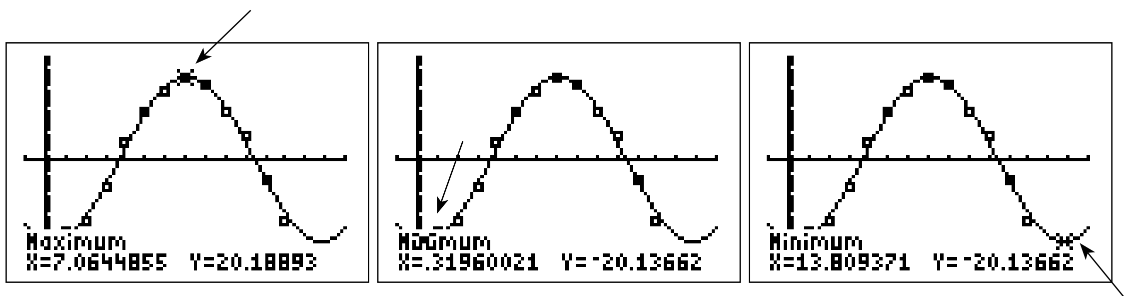


The data points would repeat this pattern each year so the data is periodic. Use technology to determine the sinusoidal regression equation in the form $y = a \sin(bx + c) + d$, that best models this data.



$y = 20.2 \sin(0.466x - 1.72) + 0.026$ is the sinusoidal regression equation that best models this data.

The maximum temperature, according to the curve of best fit, is at (7, 20.19). This implies July is the hottest month of the year with an average temperature of 20.19 °C.



The minimum temperature is found at (0.32, −20.14). Since January is plotted as month 1, this would relate approximately to sometime between December and January. Since you are graphing one average temperature per month, there are some limits to the application of the data.

The range of temperatures for Winnipeg would be from −20.1 to +20.2 degrees. In set notation, this can be denoted as $\{y \mid -20.1 \le y \le 20.2, y \in \Re\}$.

The average annual temperature is the median, found by calculating the sum of the maximum and minimum y-values, divided by two.

$$\frac{20.2 + (-20.1)}{2} = 0.05$$

The average annual temperature in Winnipeg is 0.05 °C.

The amplitude is the vertical distance between the maximum and the median.

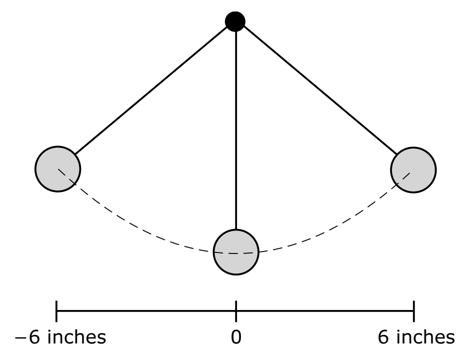
$$20.2 - 0.05 = 20.15$$

The amplitude is 20.15 degrees.

To find the period, highlight one cycle on the graph. You could use the curve from one minimum to the next. You know that the annual period is 12 months but, because the curve is based on data from only one year, it appears that in order to account for the data as given, the period is closer to 13.5 months ($13.8 - 0.3 = 13.5$). This could be due to a month in the year having had an extreme average temperature that affected the fit of the curve. To have a more accurate regression equation, the data should include average monthly temperatures from several years. Ideally, the graph should go through one complete cycle every 12 months. Winnipeg weather can be a bit unpredictable at the best of times, so this graph is only a good approximation.

Example 5

The pendulum on a grandfather clock swings from far left to far right in 1 second. It swings a horizontal distance of 6 inches in either direction from the centre.



a) Complete the following chart with data based on this information, with the pendulum starting at the centre and moving first to the right.

Time (seconds)	0	0.5	1	1.5	2	2.5	3
Distance from centre (inches)							

b) Use technology to plot the points and find the regression equation that best models this situation.

c) Determine the maximum, minimum, range, the equation of the median line, the amplitude, and period of the sinusoidal function that models this situation.

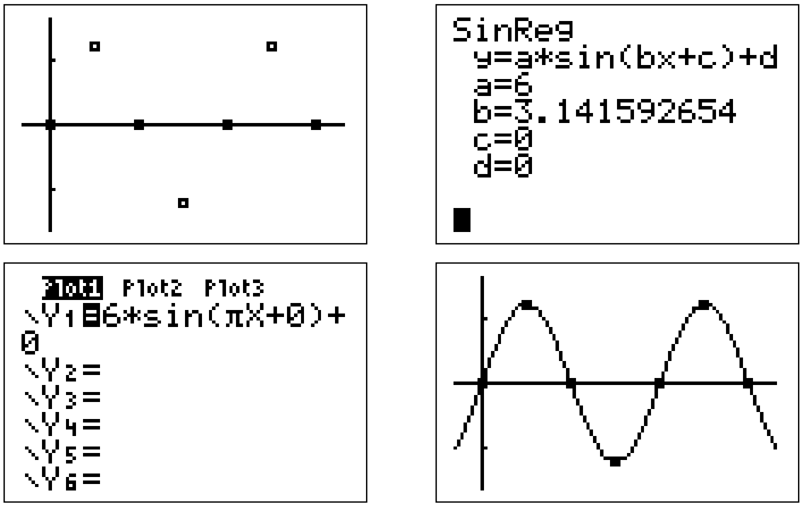
d) What is the pendulum's horizontal distance from the centre at 0.3 seconds? 3.8 seconds?

Solution

a) Assuming the pendulum crosses the centre when time is 0 seconds and is moving to the right in the positive direction, the distance of the pendulum from the centre at half-second intervals is as follows:

Time (seconds)	0	0.5	1	1.5	2	2.5	3
Distance from centre (inches)	0	6	0	−6	0	6	0

b) The wave-like pattern evident in these points would best be modelled by the sinusoidal regression equation $y = 6 \sin (3.14x)$ or $y = 6 \sin (\pi x)$.



c) The maximum y-value is 6 and the minimum y-value is −6.

The range can be denoted as $\{y \mid -6 \leq y \leq 6, y \in \Re\}$.

The median would be at the midpoint between the maximum and minimum. The equation of the median line is $y = 0$.

The amplitude is 6 inches.

The period is 2 seconds. The graph of the line is at the median and increasing at 0 seconds and returns to this point at the 2 second mark. This represents the pendulum swinging through one complete cycle; for example, from centre to the right, back across to the left, and back to the centre, or from left to right and back to left.

d) The pendulum's horizontal distance from the centre at a given time can be found using technology. Solve for $x = 0.3$ seconds and $x = 3.8$ seconds.

You can interpolate using the regression equation. At 0.3 seconds, the pendulum is a horizontal distance of 4.85 inches from the centre.

To extrapolate a value (beyond the given data points), you may need to adjust the window settings to view more of the curve of best fit. The pendulum will be a horizontal distance of −3.53 inches from the centre after 3.8 seconds. The positive and negative value denotes a direction to the right or left, respectively, of centre.

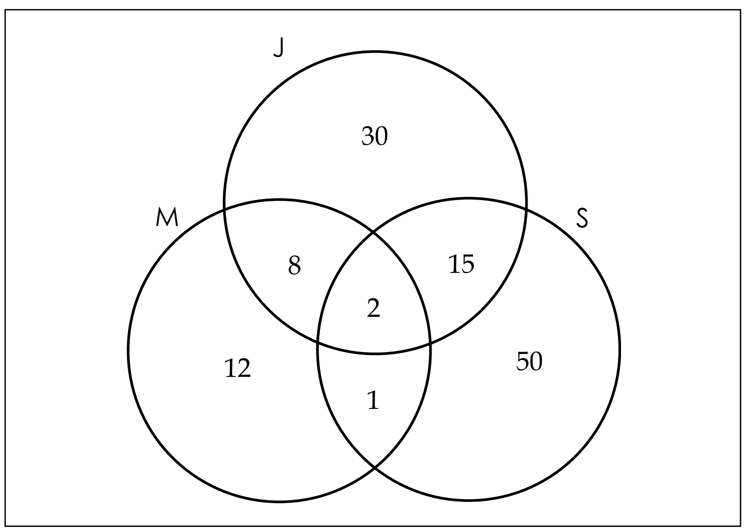
Learning Activity 7.1

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

Answer the following questions based on the information in the given Venn diagram. J is the set of students with an after-school job, M is the set of students who take music lessons, and S is the set of students who play on a sports team.



1. How many students take music lessons?
2. How many students play on a sports team but do not have a job?
3. How many students take music lessons and have a job?
4. How many students do not have a job?
5. How many students take music lessons or play on a sports team?
6. How many students work, take music lessons, and have a job?
7. How many students have a job but do not play sports?
8. What is the total number of students represented in this diagram?

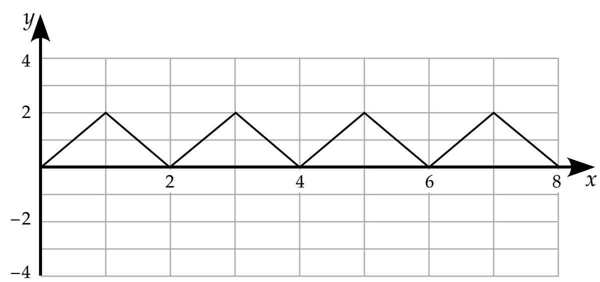
continued

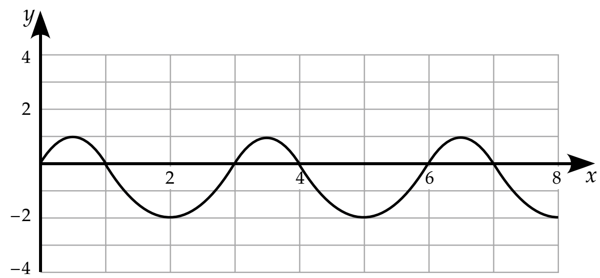
Learning Activity 7.1 (continued)

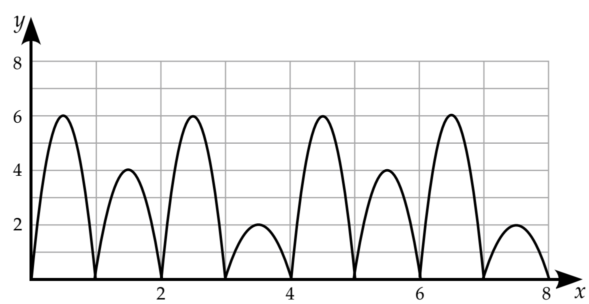
Part B: Periodic Functions

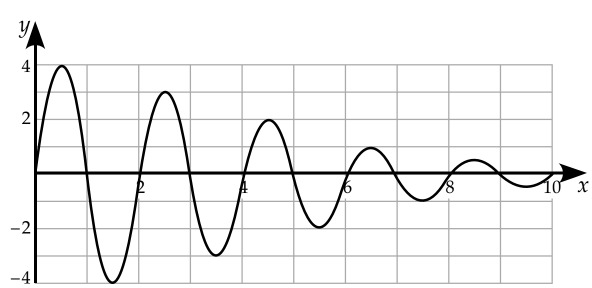
Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Determine if the following graphs represent periodic data. If the data is periodic, approximate the period. Highlight one cycle on each graph.

   a) 

   b) 

   c) 

   d) 

continued

2. A toddler is jumping on a trampoline. The surface of the trampoline is 75 cm above the ground. The little boy jumps up 52 cm above the mat surface, and stretches the mat down the same distance. He completes one bounce every 2.5 seconds. Start the graph at the median height at time = 0 seconds.

   a) Sketch a labelled graph of a sinusoidal function to represent this situation.

   b) State the equation of the median, and identify the maximum and minimum y-values, the amplitude, and period.

3. The temperature of water in the Bay of Fundy changes over the course of one year. It is about 10° C in summer and about 2° C in winter.

   a) Sketch a graph to approximate this situation using a sinusoidal function. Use January as month #1.

   b) State the maximum, minimum, range, median, amplitude, and period of this function.

4. During the year 2013, the time of sunrise on the first day of each month in the town of Banff, AB was recorded.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Time (hhmm)	0843	0816	0724	0615	0513	0431	0430	0506	0554	0641	0733	0822
Time (decimal)	8.72											

   a) Complete the chart by converting the time in minutes to decimal equivalents (i.e., 8:43 am is 8.71667 hours since $\frac{43}{60} = 0.71667$).

   b) Use technology to plot the data, using January as month 1 and December as month 12.

   c) Use technology to find a regression equation that best fits the data points. State the equation and graph it using technology.

   d) Use the regression equation to determine the maximum and minimum y-values, the equation of the median, the amplitude, and the period of this sinusoidal function. Explain the significance of the values you find.

continued

5. The predicted height of the tide in Whale Cove, Nunavut, on October 29, 2013, is given in the chart below. (Hour 0 = midnight, hour 12 = noon)

Hour	0	1	2	3	4	5	6	7	8	9	10	11
Depth (m)	2.9	3.0	2.8	2.6	2.2	1.8	1.5	1.3	1.4	1.6	2.0	2.4

Hour	12	13	14	15	16	17	18	19	20	21	22	23
Depth (m)	2.8	3.0	3.0	2.9	2.5	2.1	1.7	1.5	1.4	1.6	1.8	2.2

Source: http://www.tides.gc.ca/eng/station?sid=5055

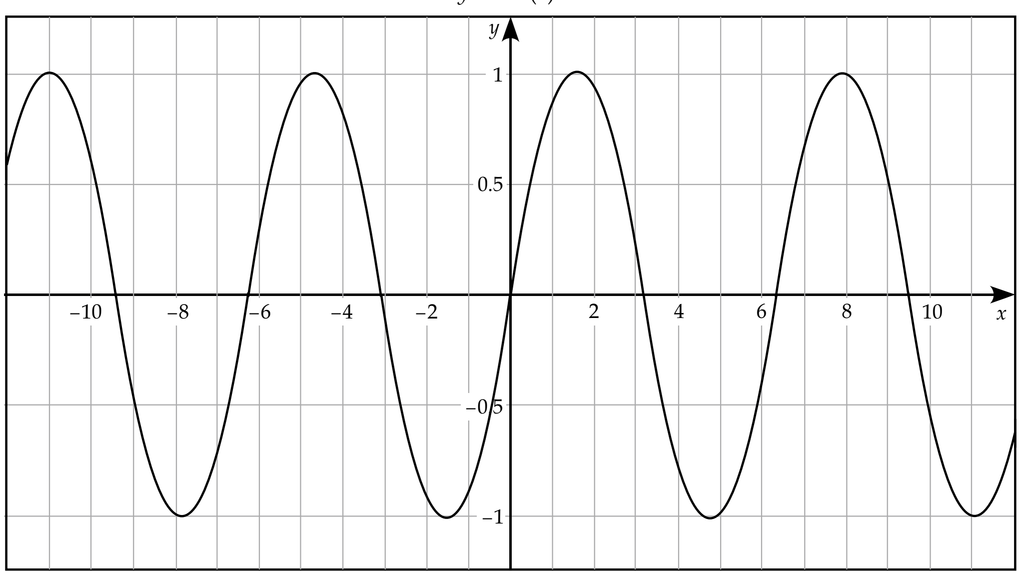
a) Use technology to plot the points and determine the regression equation that best models the tide depth over time.

b) Use the sinusoidal equation to determine the maximum and minimum depths, median depth, amplitude, and period of the tide.

c) A kayaker likes to have at least 2 m of water in the cove when she goes out paddling. Determine the time frame within which she can happily go kayaking.

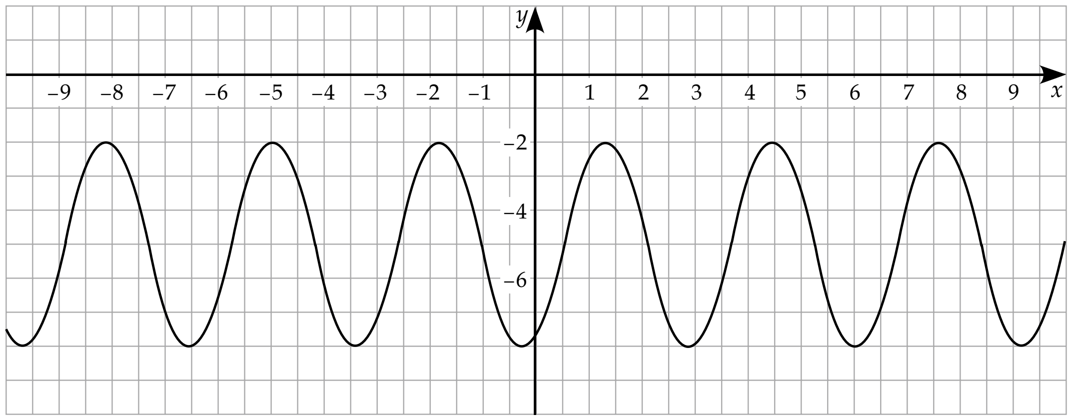
6. Illustrate the properties or characteristics of the graph of the sinusoidal function $y = \sin x$, shown below, by labelling a maximum and a minimum point, using arrows to indicate the range, amplitude, and period, drawing in the midline, and highlighting one cycle.

$y = \sin(x)$



continued

7. Given the graph of the sinusoidal function, $y = 3 \sin (2x - 1) - 5$, determine its maximum and minimum values, the equation of the median, the amplitude, and the period.



Lesson Summary

In this lesson, you represented data using sinusoidal functions and described the characteristics of sinusoidal functions by analyzing their graphs. You used technology to graph data and determine the sinusoidal function that best approximates the data, and interpreted the graph of a sinusoidal function that models a situation to solve contextual problems.

Lesson 2: Sinusoidal Functions

Lesson Focus

In this lesson, you will

- ❑ represent data using sinusoidal functions
- ❑ describe the characteristics of sinusoidal functions by analyzing their graphs and equations
- ❑ match equations in a set to their corresponding graphs
- ❑ graph data and determine the sinusoidal function that best approximates the data
- ❑ interpret the graph of a sinusoidal function that models a situation and explain the reasoning
- ❑ solve, using technology, a contextual problem that involves data that is best represented by graphs of sinusoidal functions, and explain the reasoning

Lesson Introduction

Sinusoidal waves can be used to model tidal waves in an ocean, sound in the atmosphere, electricity in a wire, and electromagnetic waves all around us. Electromagnetic waves include radio waves, microwaves, light rays, and X-rays.

The appearance of a model's graph changes depending on the application being studied. For example, light and sound can be modelled by sinusoidal waves but the frequency of light is many times higher than the frequency of sound. That means that the time for one period of a wave of sound is many times longer than one period of a wave of light. The amplitude of a sound wave affects the loudness with which it is heard. The length of the period of a light wave affects the colour of light that is seen by your eyes. For example, the period of a red light wave is longer than the period of a blue light wave.

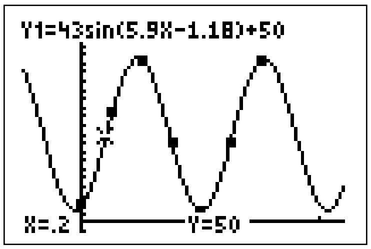
In the last lesson, you described the graph of a sinusoidal function in terms of its period, maximum and minimum y-values, its range, median, and amplitude, and you used technology to determine a sinusoidal regression equation in the form $y = a \sin (bx + c) + d$. In this lesson, you will look more closely at the effect each of the variables in the general sinusoidal function (a, b, c, and d) has on the appearance of the function curve.

Characteristics of the Equation of a Sinusoidal Function

How do the variables *a*, *b*, *c*, and *d* relate to the amplitude, maximum and minimum, period, and median? Look back over the examples in the previous lesson and see if you can make some connections.

Amplitude, Maximum, Minimum, and Median

In the graph of Tessa's ride on the Ferris wheel, her height over time was modelled by the sinusoidal function $y = 43 \sin (5.9x - 1.18) + 50$.



You determined her maximum height to be 93 feet and the minimum height to be 7 feet. The median is 50 feet, the amplitude is 43 feet, and the period of the function is 1.06 minutes.

Compare the function $y = 43 \sin (5.9x - 1.18) + 50$ with the general form of the sinusoidal function $y = a \sin (bx + c) + d$. The value of a is the amplitude, and the horizontal line, $y = d$, is the median. The maximum is $d + a$ and the minimum is $d - a$.

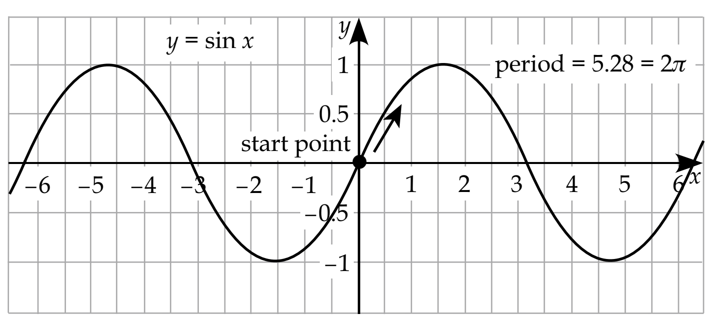
Period and Phase Shift

In order to find the value of the period, you must remember that the period of the basic sine graph is 2π.

To find the period of the function using the variables in the equation $y = 43 \sin (5.9x - 1.18) + 50$, use the formula, period $= \dfrac{2\pi}{b}$.

$$\text{period} = \frac{2\pi}{5.9}$$

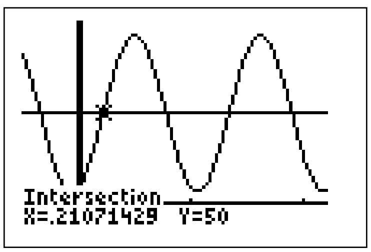
$$\text{period} = 1.06$$



In this way, the variable, b, in the sinusoidal equation can be used to determine the period of the function. Alternatively, if you know the period of the function, you can determine the variable, b, by using the related formula, $b = \dfrac{2\pi}{\text{period}}$.

In the basic sine graph $y = \sin x$, when $x = 0$, the curve of the line is at the median ($y = 0$) and increasing. Not all of the graphs of sinusoidal regression equations you have found so far have been at the median and increasing at (0, 0). This start point on the basic sine graph can be used as a reference point when evaluating the **phase shift** or horizontal shift of the graphs of other sinusoidal functions.

In the graph of Tessa's Ferris wheel ride, the closest reference point where the curve of the line is at the median and increasing is at the point (0.2. 50). This can be found using technology and the intersection of the sinusoidal curve of the regression equation with the median line.



The phase shift can be determined from the sinusoidal equation $y = 43 \sin (5.9x - 1.18) + 50$ using the formula, phase shift $= -\dfrac{c}{b}$.

$$\text{phase shift} = -\frac{c}{b}$$

$$\text{phase shift} = -\left(\frac{-1.18}{5.9}\right)$$

$$\text{phase shift} = +0.2$$

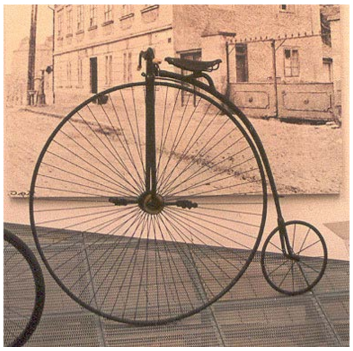
The curve of the basic sine graph has been shifted 0.2 units **to the right**.

If the calculation for $-\dfrac{c}{b}$ results in a negative value, the graph has moved horizontally towards the left.

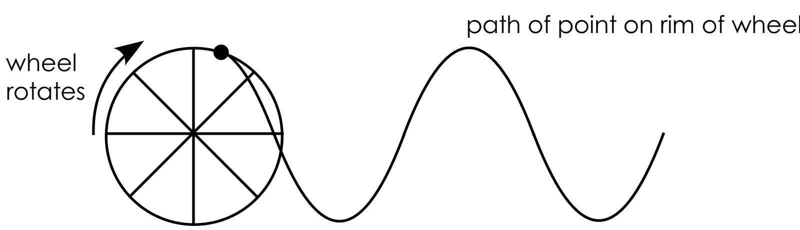
You may want to make a note on your resource sheet of the formulas relating the general sinusoidal function, $y = a \sin(bx + c) + d$, with the amplitude, the median line, the maximum and minimum values, the period, and the phase shift.

Example 1

In the late 1800s, the first machine to be called a bicycle was invented. It had a very large front wheel, with a seat above it, and a small back wheel. It became known as the Penny-Farthing, inspired by the names of two British coins of different sizes. The large front wheel helped cyclists achieve higher speeds, but going uphill, hitting a bump or trying to stop quickly was difficult, and often dangerous, resulting in what became known as "a header."



A popular model of the Penny-Farthing bicycle had a front wheel with a diameter of 1.35 m and a 0.45 m rear wheel. If the front wheel makes one complete rotation every second, the path of a specific point on the rim of the wheel can be modelled using a sinusoidal function.



a) State the maximum and minimum height of the point on the rim, the equation of the median line, amplitude, and period for this situation.

b) Sketch a graph showing the height of a specific point on the rim of the front tire over time.

c) Use technology to find the sinusoidal regression equation that best models this situation.

Solution

From the given information, you can determine the following:

a) maximum: $y = 1.35$

minimum: $y = 0$

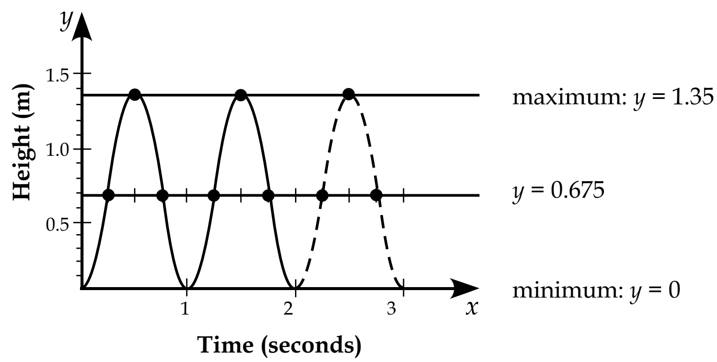
median: $y = 0.675$

amplitude: 0.675

period: one second

b) One possible graph of this situation may look like this:

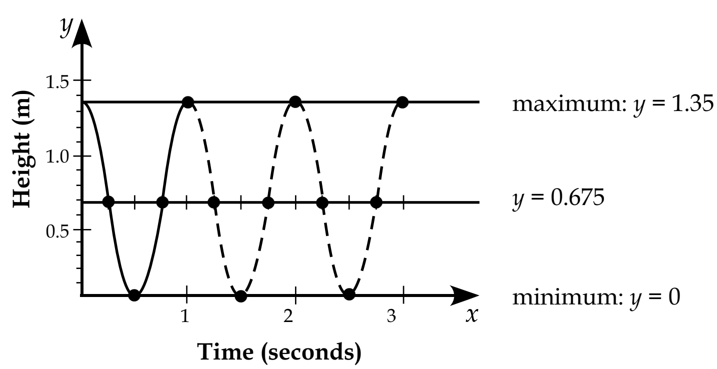
Graph 1:



This graph assumes the point is at ground level when time = 0 seconds.

Another acceptable graph may assume the point begins at the maximum height at time = 0 seconds.

Graph 2:

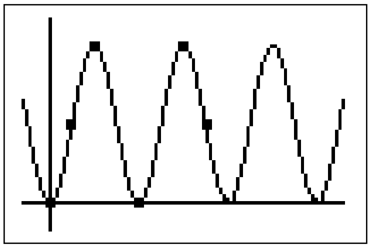


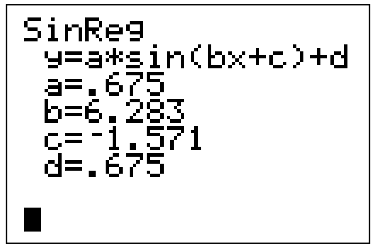
There are, in fact, an infinite number of possible graphs, depending on where the point on the wheel is to begin, when time = 0 seconds. The maximum, minimum, median, amplitude, and period all remain the same; it is just as if the graph of the curve has shifted horizontally, and so starts at a different point. This phase shift is evident in the regression equations that model these two different possible start points.

c) To determine the equation, input some of the critical points and find the sinusoidal regression function using graphing technology.

Graph 1:

Time (seconds)	0	0.25	0.5	1	1.5	1.75
Height (m)	0	0.675	1.35	0	1.35	0.675





Recall:

$\pi \cong 3.14$

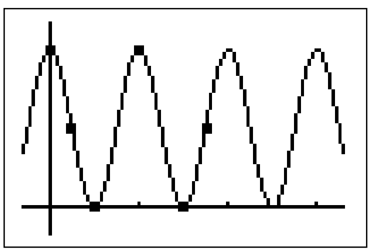
$2\pi \cong 6.283$

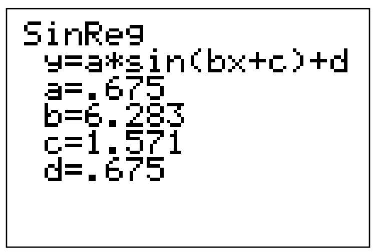
$\frac{\pi}{2} \cong 1.571$

Equation 1: $y = 0.675 \sin\left(2\pi x - \frac{\pi}{2}\right) + 0.675$

Graph 2:

Time (seconds)	0	0.25	0.5	1	1.5	1.75
Height (m)	1.35	0.675	0	1.35	0	0.675





Equation 2: $y = 0.675 \sin\left(2\pi x + \frac{\pi}{2}\right) + 0.675$

The difference between Equation 1 and 2 is the value for *c*. This accounts for the horizontal shift of the curve. The other variables in the two equations have the same values.

amplitude: $a = 0.675$ m

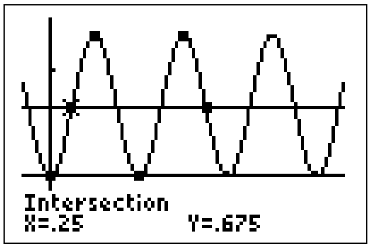
median: $d = 0.675$ m

maximum: $d + a = 1.35$ m

minimum: $d - a = 0$ m

period: $\frac{2\pi}{b} = \frac{2\pi}{2\pi} = 1$

Phase shift is the distance from $x = 0$ to the nearest point on the curve where it is at the median ($y = 0.675$) and increasing. For graph 1, this is at $x = 0.25$.



$$y = 0.675 \sin\left(2\pi x - \frac{\pi}{2}\right) + 0.675$$

Using the formula, phase shift = $-\frac{c}{b}$, this can be calculated:

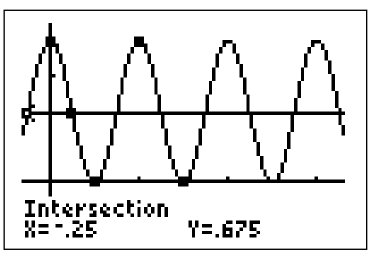
$$\text{phase shift} = -\frac{c}{b}$$

$$\text{phase shift} = -\frac{\left(-\frac{\pi}{2}\right)}{2\pi}$$

$$\text{phase shift} = +0.25$$

The basic sine graph is shifted 0.25 units to the **right**.

For graph and equation 2



$$y = 0.675 \sin\left(2\pi x + \frac{\pi}{2}\right) + 0.675$$

$$\text{phase shift} = -\frac{c}{b}$$

$$\text{phase shift} = -\frac{\left(\frac{\pi}{2}\right)}{2\pi}$$

$$\text{phase shift} = -0.25$$

The basic sine graph is shifted 0.25 units to the **left**.

Example 2

The average hours of daylight in the city of Churchill, Manitoba, over the course of one year are given in the chart below.

Month	Jan	Feb	Mar	Apr	May	June	Jul	Aug	Sep	Oct	Nov	Dec
Hours: Minutes	07:08	09:15	11:48	14:28	16:54	18:17	17:39	15:28	12:50	10:09	07:44	06:25

a) Convert the hours and minutes to the decimal equivalent of an hour and plot the data points using technology.

b) Determine the equation of a sinusoidal function that best represents this situation.

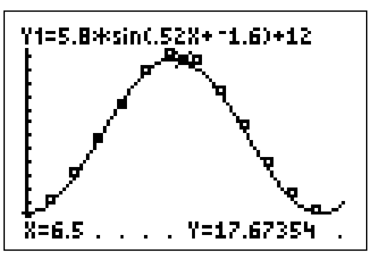
c) Determine the amplitude, equation of the median, maximum, minimum, period, and phase shift for this sinusoidal function, using the values for *a*, *b*, *c*, and *d* in the regression equation.

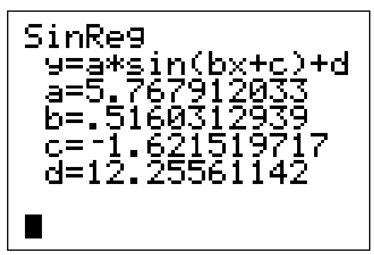
d) Find the approximate date of the longest and shortest days in Churchill, Manitoba.

Solution

a)

Month (#)	Jan (1)	Feb (2)	Mar (3)	Apr (4)	May (5)	June (6)	Jul (7)	Aug (8)	Sep (9)	Oct (10)	Nov (11)	Dec (12)
Hours: Minutes	07:08	09:15	11:48	14:28	16:54	18:17	17:39	15:28	12:50	10:09	07:44	06:25
Decimal Equivalent	7.13	9.25	11.8	14.47	16.9	18.28	17.65	15.47	12.83	10.15	7.73	6.42





b) This situation can be approximated using the sinusoidal regression equation $y = 5.8 \sin (0.5x - 1.6) + 12$.

c) For more accurate answers, use as many decimal places as possible in your calculations.

amplitude: $a = 5.767912033$, or 5 hours, 46 minutes

median: $d = 12.25561142$, or 12 hours, 15 minutes

maximum: $d + a = 18.02352345$, or 18 hours, 1 minute

minimum: $d - a = 6.487699387$, or 6 hours, 29 minutes

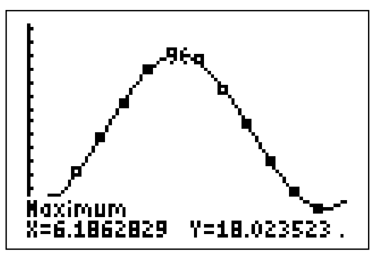
period: $\frac{2\pi}{b} = \frac{2\pi}{0.5160312939} = 12.17597727$, approximately 12 months

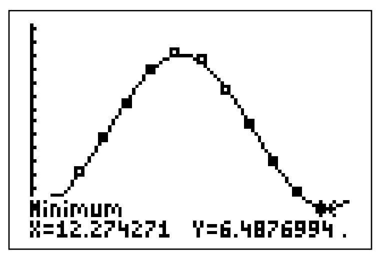
phase shift: $-\frac{c}{b} = -\frac{-1.621519717}{0.5160312939}$

$= +3.142289501 \cong +\pi$

The basic sine graph is shifted 3.14 units to the **right**.

d) Keeping in mind that the original data graphed was for monthly averages, not individual dates, the day with the most hours of daylight is represented with the point (6.19, 18.02) on the curve of the regression equation. This corresponds to approximately June 6th. The day with the fewest daylight hours occurs on approximately December 9. These dates should ideally fall on the summer and winter solstices, June 21 and December 21 respectively.

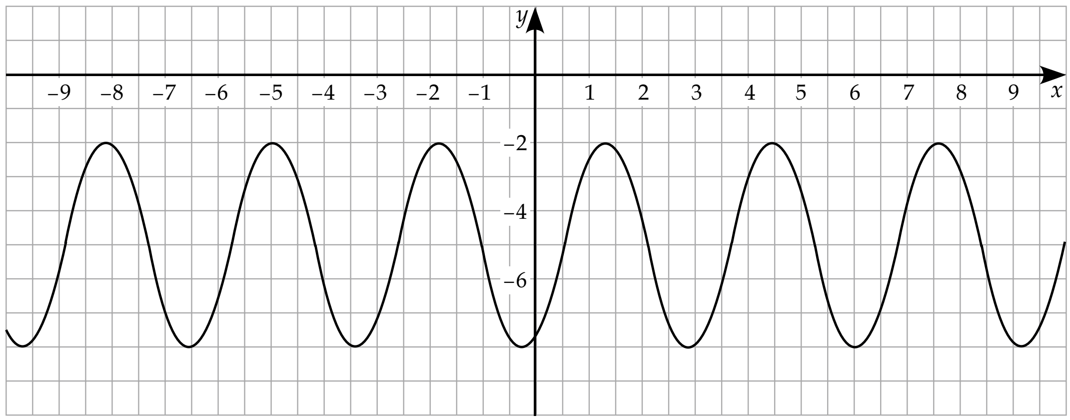




Example 3

Given the sinusoidal function, $y = 3 \sin (2x - 1) - 5$, determine its maximum and minimum values, the equation of the median, the amplitude, and its period using the values for a, b, c, and d.

Solution



$a = 3 \quad b = 2 \quad c = -1 \; d = -5$

amplitude: $a = 3$ units

equation of the median is at $y = \text{d}$: $y = -5$

maximum: $d + \text{a} = -2$

minimum: $d - a = -8$

period: $\dfrac{2\pi}{b} = \dfrac{2\pi}{2} = \pi \cong 3.14$

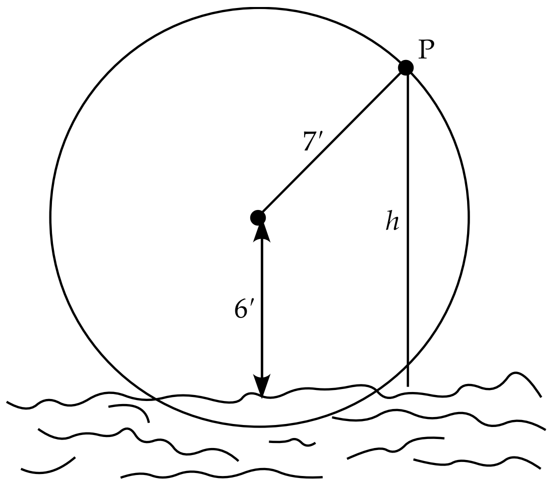
phase shift: $-\dfrac{c}{b} = -\left(\dfrac{-1}{3}\right)$

$\cong +0.333$

This tells you that in order to create the graph of $y = 3 \sin (2x - 1) - 5$, the basic sine graph, $y = \sin x$, is shifted vertically 5 units down so the median is at $y = -5$ instead of at $y = 0$. The amplitude is increased from 1 unit to 3 units, the period is compressed (one cycle in now completed in π or 3.14 radians rather than 2π or 6.28 radians), and the graph is shifted horizontally 0.333 units to the right.

Example 4

A water wheel with radius 7.0 feet has 1.0 feet submerged and rotates at 6.0 revolutions per minute. You start your stopwatch. Five seconds later, a point, P, on the rim of the wheel is at its greatest height above the surface of the water.



Find a sinusoidal function to model the height, h, of the point, P, above the surface of the water in terms of time, t, in seconds the stopwatch reads.

Solution

From the given information, you can determine that the amplitude of the function is the same as the radius of the wheel, so $a = 7$. Assume that $h = 0$ at the surface of the water, and the minimum height of the point as it rotates below the surface of the water will be at $h = -1$. This means the median height, d, will be at $h = 6$ and the maximum height of the point, P, will be at $h = 13$.

The period of the function must be 10 seconds if the wheel makes six revolutions per minute (60 seconds ÷ 6 = 10).

$$\text{period} = \frac{2\pi}{b}$$

$$b = \frac{2\pi}{\text{period}}$$

$$b = \frac{2\pi}{10} = \frac{\pi}{5}$$

The value of c in the sinusoidal function indicates the phase shift of the graph. You need to determine at what time, t, the point, P, is at the median and increasing. You know that it is at the maximum height when the stopwatch reads 5 seconds. If the period is 10 seconds, it would have been at the minimum at $t = 0$, and at the median and increasing when $t = 2.5$ seconds.

$$\text{phase shift} = \frac{-c}{b}$$

$$-c = (\text{phase shift}) \times b$$

$$c = -(\text{phase shift}) \times b$$

$$c = -2.5 \times \frac{\pi}{5}$$

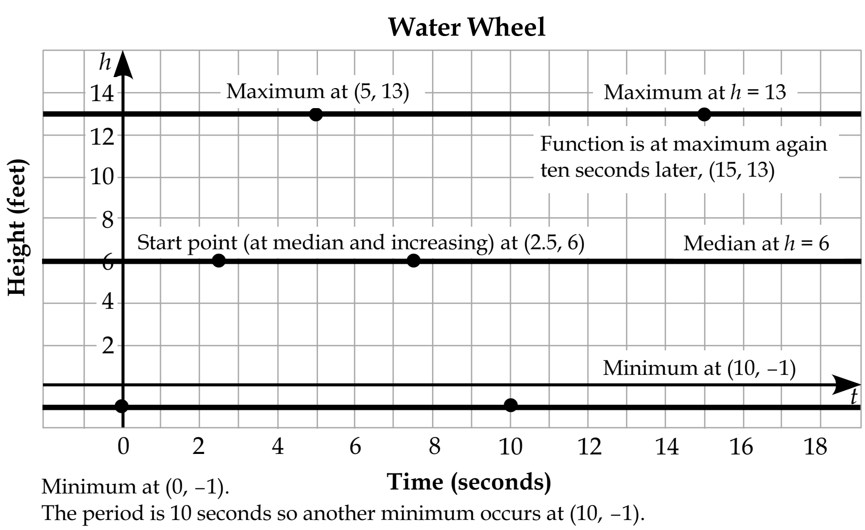
$$c = -\frac{\pi}{2}$$

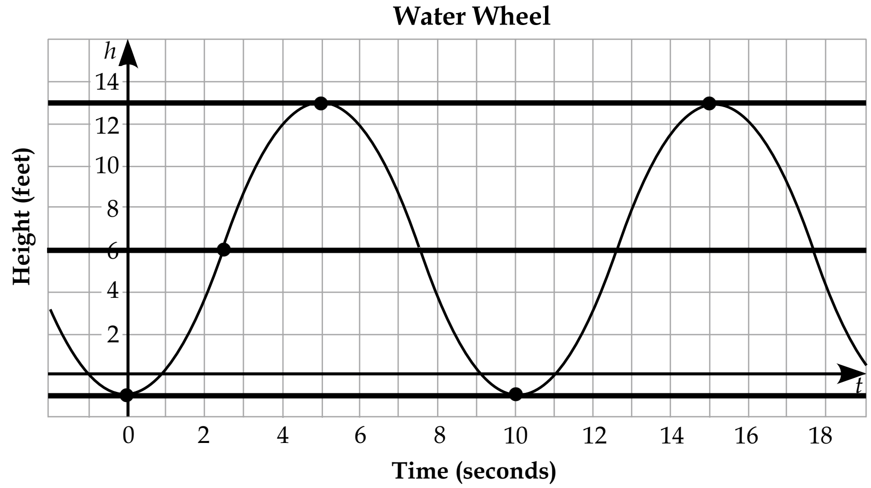
The sinusoidal equation that models this situation is
$h(t) = 7\sin\left(\frac{\pi}{5}t - \frac{\pi}{2}\right) + 6.$

Alternatively, you could find the sinusoidal function by determining the times and heights for several points on the curve and then finding the regression equation.

$y = 7 \sin (0.628x - 1.571) + 6$

Plot the points determined above and sketch the curve of the graph.





You may want to record some information on your resource sheet about determining a sinusoidal function from a scenario or from a graph.

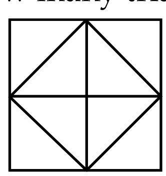
Learning Activity 7.2

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. How many triangles are in the figure below?



2. Find the cost of 60 trinkets if one dozen cost $44.40.
3. How many radians are in one complete rotation of a circle?
4. How many degrees are in a straight line?
5. How many degrees are in an acute angle?
6. What is the sum of the angles in a parallelogram?
7. If two angles are complementary, the sum of their angles is ________°.
8. If two angles are supplementary, the sum of their angles is ________°.

continued

Part B: The Equation of Sinusoidal Function

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Match the equations in the following set to their corresponding graphs. Explain the reasoning used to make your decisions.

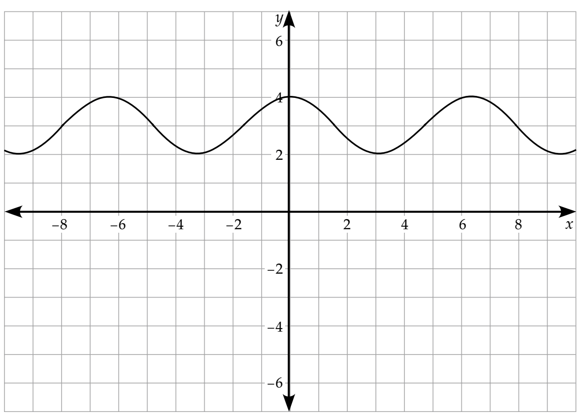
   a) $y = 2\sin x$

   b) $y = \sin(4x)$

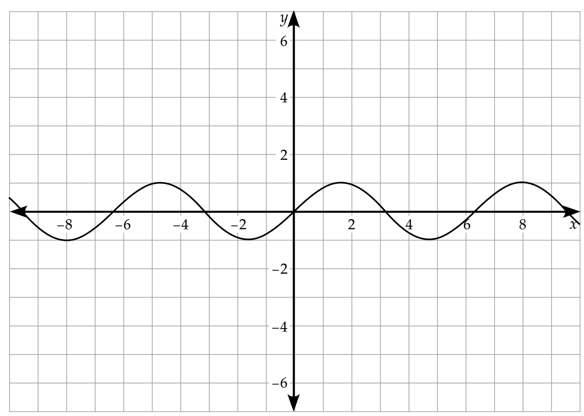
   c) $y = \sin\left(x + \frac{\pi}{2}\right) + 3$

   d) $y = \sin x$

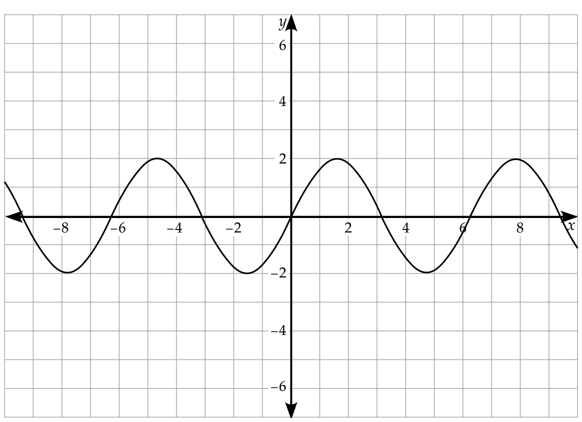
i)



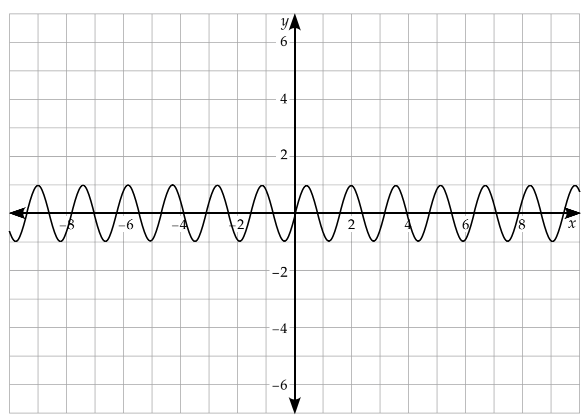
ii)



iii)



iv)



continued

2. The table shows the time of sunset at New York, New York, on the first day of each month.

New York, New York												
Date	Jan 1	Feb 1	Mar 1	Apr 1	May 1	Jun 1	Jul 1	Aug 1	Sep 1	Oct 1	Nov 1	Dec 1
Time of Sunset (p.m.)	4:39	5:13	5:47	7:20	7:52	8:21	8:31	8:17	7:28	6:38	5:52	4:29
Day of Year (#)	1	32										
Time of Sunset	16.65											

Source: http://www.tropicalweather.net/sunrisesunset_Northeast.htm

a) Complete the table above by converting the hours and minutes to decimal fractions of one hour on the 24-hour clock, and by calculating which day of the year the first day of each month represents.

b) Use technology to graph the data and determine the sinusoidal regression equation that best fits the time of sunset.

c) What is the median time of sunset?

d) How much does the maximum time of sunset vary from the median time of sunset?

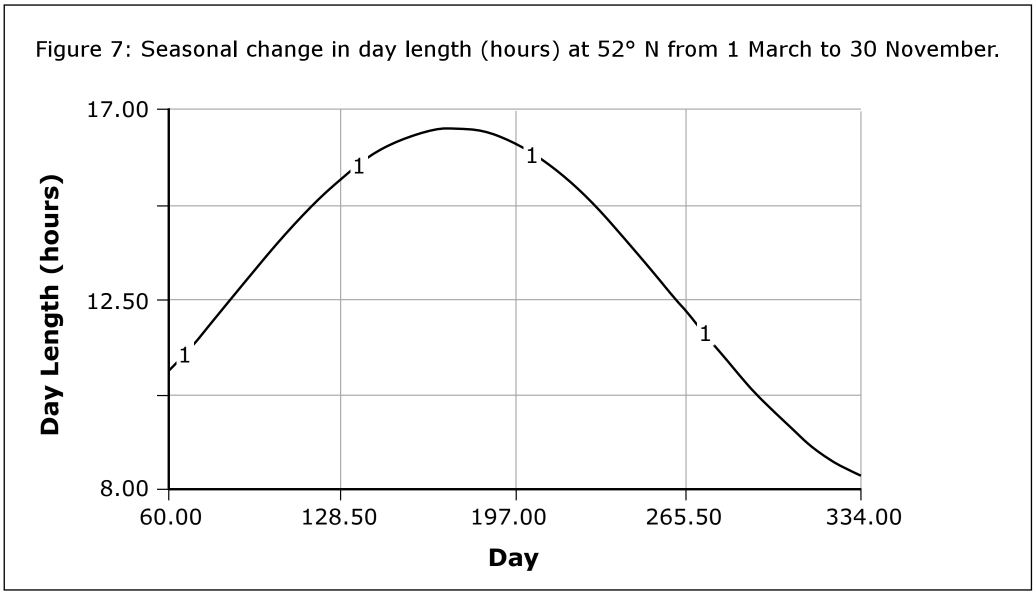
e) What is the latest time of sunset during the year?

f) Use technology to predict the time of sunset on April 15th.

g) Use technology to determine on which day of the year the latest sunset occurs.

continued

3. The following graph was published by the Department of Fisheries and Oceans as part of a study of climate and other factors on zooplankton populations in the Hecate Straight, off the coast of British Columbia.



Source: http://www.dfo-mpo-gc.ca/Library/324983.pdf

a) Interpolate the coordinates of five points from the graph and list them in the table below. You may want to draw additional grid lines at regular intervals on the graph to help you estimate values.

Day (#)	Day Length (hours)

b) Plot the points using technology and determine the sinusoidal regression equation that models this situation.

c) What is the average day length in the Hecate Straight?

d) The summer solstice occurred on June 20th in the year that this data was recorded. What was the length of day on June 20?

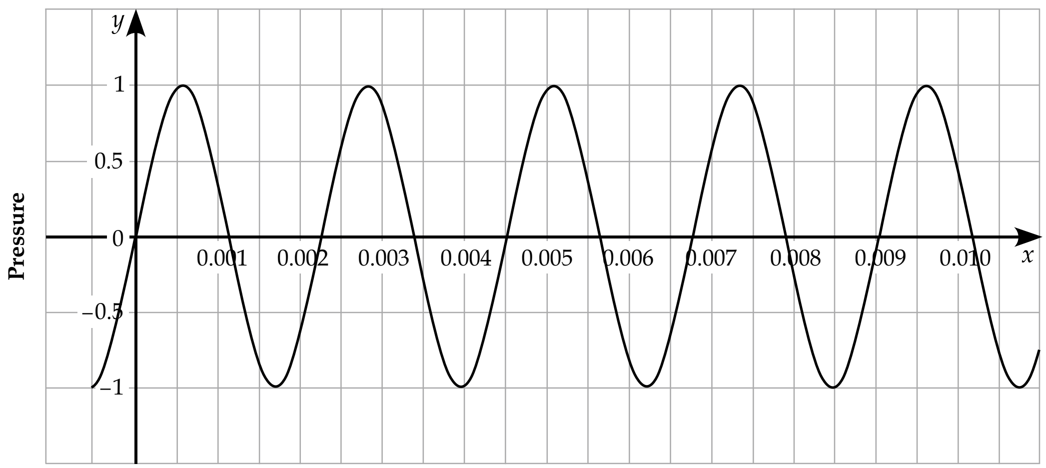
e) Calculate the approximate period for this data.

continued

Learning Activity 7.2 (continued)

4. When you listen to music, you are listening to sound waves. On a piano, the A note above middle C produces a wave according to the equation, $y = \sin(880\pi x)$, illustrated in the graph below. What is the amplitude and period of the sound wave of the note A? How many times per second does the sound wave created by the note A vibrate?

The Note A



5. A point on the rim of the large front wheel of a Penny-Farthing bicycle travels according to the equation $y = 0.675 \sin\left(2\pi x - \frac{\pi}{2}\right) + 0.675$. A point on the rim of the smaller rear wheel travels according to the equation $y = 0.225 \sin\left(6\pi x - \frac{\pi}{2}\right) + 0.225$.

   a) Graph both equations on the same grid.
   b) Compare the amplitude, maximum and minimum, median, and period of each function.
   c) Determine the circumference of the large wheel. How far does the bike travel in one second? How long will it take the biker to go 1 km? Approximately how fast is the biker travelling?

continued

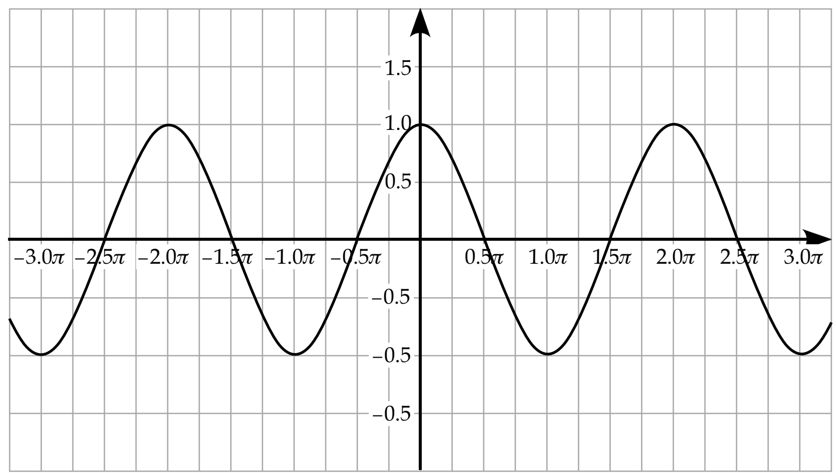
6. The depth, d, of water in a channel varies with time, t, according to the equation $d(t) = 2.5 \sin(0.523t) + 2.9$, where d is the depth of water in metres, and t is the number of hours since midnight.
   a) Create a graph showing the depth of water over 24 hours. Label the graph appropriately.
   b) What is the period of this function? What does it represent in terms of the varying depth of water?
   c) Use the equation to determine the average depth of water in the channel, the maximum depth, and the minimum depth.
   d) If a fishing boat requires a depth of at least 1.75 m to navigate across the channel, determine when the boat may safely cross.
7. A mass is attached to a spring that hangs from a hook above a tabletop, where the mass is allowed to move freely. The up and down motion of the mass is described by a sinusoidal function with an upper height of 65 cm and a lower height of 15 cm above the table. Once in motion, the mass completes one oscillation per second. (Ignore the friction on the spring and assume the mass stays in motion over time.)
   a) Sketch a graph of the sinusoidal function describing the height of the mass as it oscillates, starting from the lowest point.
   b) Write a sinusoidal equation to describe the height, h, of the mass above the table with respect to time, t.

continued

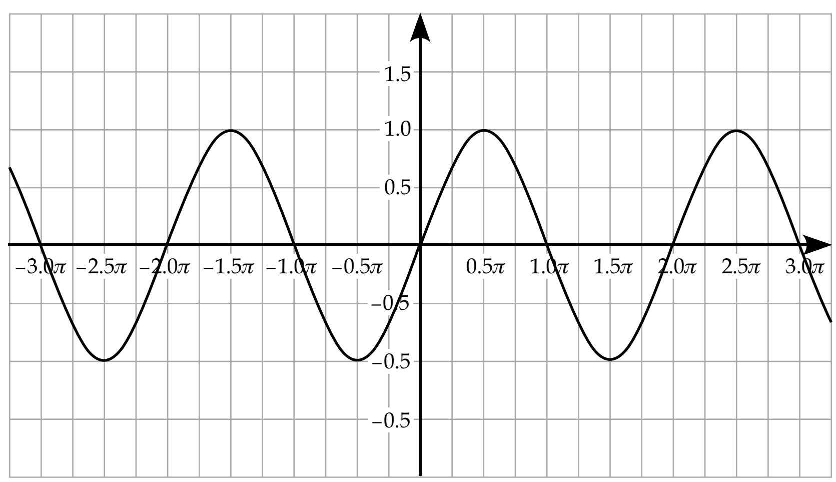
Learning Activity 7.2 (continued)

8. Sinusoidal functions can be written in terms of both sine and cosine functions. Compare the amplitude, median, maximum, minimum, period, and phase shift of the cosine graph compared to the sine graph, shown below.

y = cos x



y = sin x



Lesson Summary

In this lesson, you represented data using sinusoidal functions and described their characteristics by analyzing their graphs and equations. You matched equations in a set to their corresponding graphs and solved, using technology, contextual problems that involved data that is best represented by graphs of sinusoidal functions.

Assignment 7.1

Sinusoidal Function Models

Total: 33 marks

This is a hand-in assignment. Please clearly show your work and include units with final answers. If you are using technology to graph data and determine the regression equations, please include a printout of the graphs and equations or sketch and label the images neatly in your answers. Answers given without supporting calculations will not be awarded full marks.

1. The height above ground level, *h*, in metres, of a passenger on a Ferris wheel ride over time, *t*, in seconds, is modelled by the sinusoidal function $h(t) = 9.5\sin\left(\frac{\pi}{75}(t) - \frac{\pi}{2}\right) + 13.5$.

   a) State the values for a, *b*, *c*, and *d* of the general form of the sinusoidal function. *(2 marks)*

   b) Determine the maximum and minimum height above ground attained by the rider, the rider's variation from the average ride height during the ride, and the time it takes to complete one rotation on the Ferris wheel. *(5 marks)*

Assignment 7.1: Sinusoidal Function Models (continued)

2. Match the equations in the following set to their corresponding graphs. Explain the reasoning used to make your decisions. *(10 marks)*

a) $y = 3\sin(4x)$

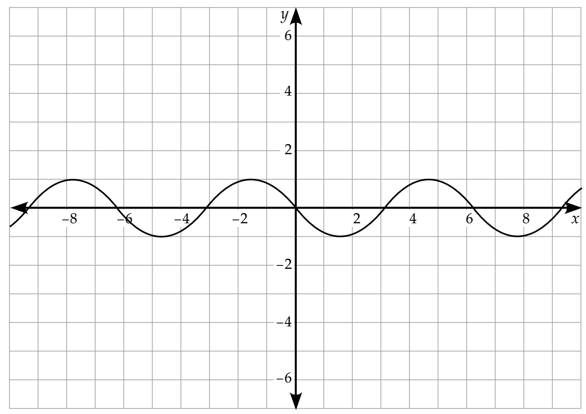
b) $y = 4\sin(x) + 3$

c) $y = \sin(\pi x) - 4$

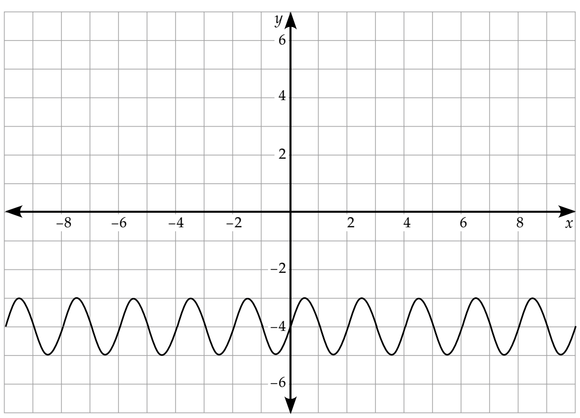
d) $y = -\sin(x)$

e) $y = 3\sin\left(\pi(x) - \left(\frac{\pi}{4}\right)\right)$

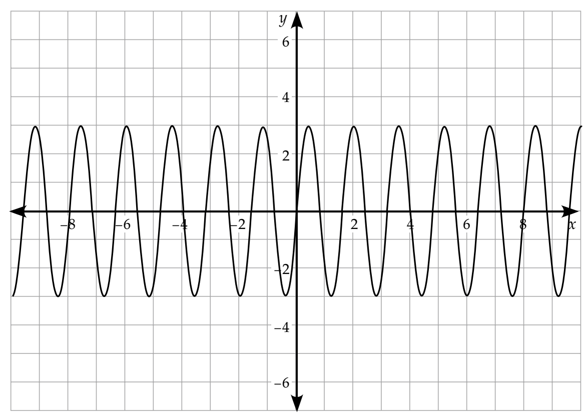
A



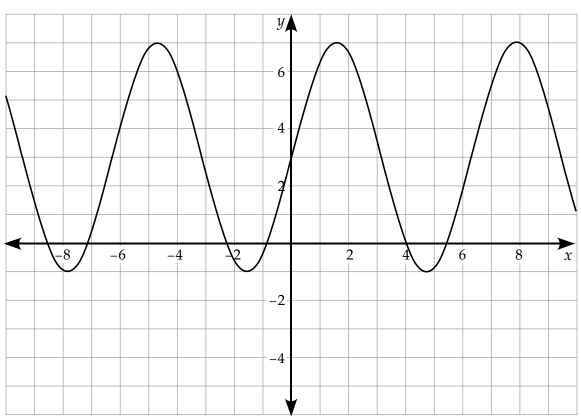
B



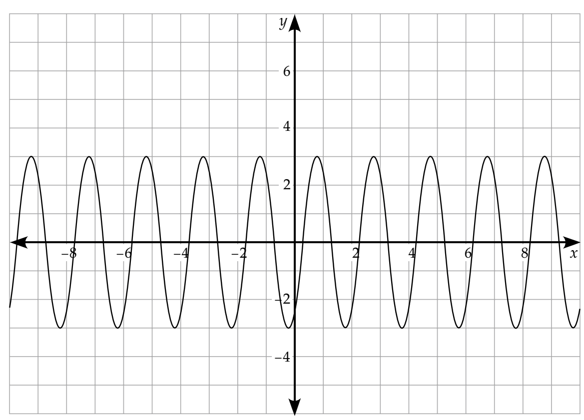
C



D



E



Assignment 7.1: Sinusoidal Function Models (continued)

a) $y = 3 \sin (4x)$ Graph ____________
Explanation:

b) $y = 4 \sin (x) + 3$ Graph ____________
Explanation:

c) $y = \sin (\pi x) - 4$ Graph ____________
Explanation:

d) $y = -\sin (x)$ Graph ____________
Explanation:

e) $y = 3 \sin \left(\pi (x) - \left(\frac{\pi}{4}\right)\right)$ Graph ____________

Explanation:

Assignment 7.1: Sinusoidal Function Models (continued)

3. Norman and Lydia retired and moved to the Caribbean, where they enjoyed the beautiful sunrises each morning. The time sunrise occurred on six days during the year is recorded in the table.

Aruba						
1st Day of Month	Jan. 1	Mar. 1	May 1	July 1	Oct. 1	Dec. 1
Time of Sunrise (a.m.)	7:02	6:45	6:21	6:14	6:44	7:03
Day of Year (#)	1	60				
Time of Sunrise	7.033	6.75				

a) Complete the table above by converting the date to the number of the day and the time to the decimal equivalent on the 24-hour clock. *(2 marks)*

b) Use technology to plot the points and determine the sinusoidal regression equation that best models the time of sunrise each day over the year. Sketch the graph below or include a printout. Include labels and units on the graph. *(4 marks)*

c) What is the average time of sunrise during the year? *(1 mark)*

d) Find the time of the earliest and latest sunrise during the year, and the dates on which these occur. How much do these times vary from the median time of sunrise during the year? *(5 marks)*

e) What period is represented by this function? *(2 marks)*

Assignment 7.1: Sinusoidal Function Models (continued)

f) Use the regression equation to approximate the time of sunrise on your birthday. Show your work. *(2 marks)*

Module 7 Summary

Congratulations, you have finished the seventh module in the course! In this module, you used technology to represent periodic data and used sinusoidal functions to model the situations and solve problems. You analyzed the graphs and equations of sinusoidal functions and described their characteristics. You matched equations in a set to their corresponding graphs and solved, using technology, contextual problems that involved data that is best represented by graphs of sinusoidal functions.

Submitting Your Assignments

It is now time for you to submit the Module 7 Cover Assignment and Assignment 7.1 to the Distance Learning Unit so that you can receive some feedback on how you are doing in this course. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 7 assignments and organize your material in the following order:

- ❑ Module 7 Cover Sheet (found at the end of the course Introduction)
- ❑ Module 7 Cover Assignment: Playing Fair
- ❑ Assignment 7.1: Applications of Sinusoidal Function Models

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Notes

Grade 12 Applied Mathematics (40S)

Module 7
Sinusoidal Functions

Learning Activity Answer Keys

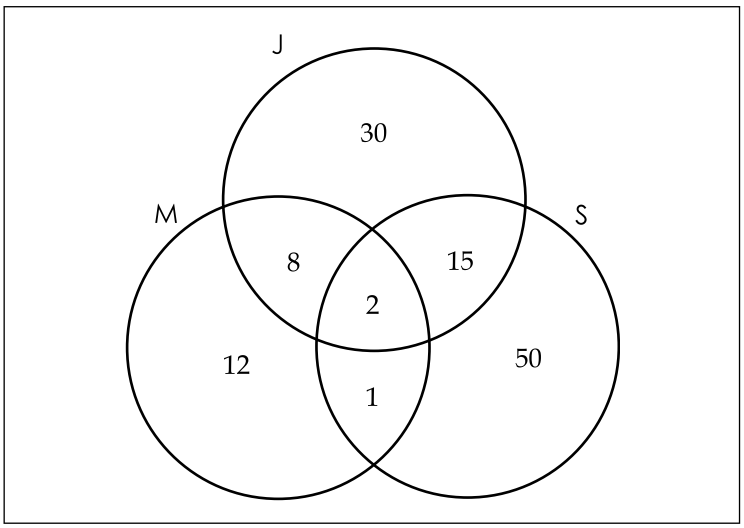
MODULE 7:
SINUSOIDAL FUNCTIONS

Learning Activity 7.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

Answer the following questions based on the information in the given Venn diagram. J is the set of students with an after-school job, M is the set of students who take music lessons, and S is the set of students who play on a sports team.



1. How many students take music lessons?
2. How many students play on a sports team but do not have a job?
3. How many students take music lessons and have a job?
4. How many students do not have a job?
5. How many students take music lessons or play on a sports team?
6. How many students work, take music lessons, and have a job?
7. How many students have a job but do not play sports?
8. What is the total number of students represented in this diagram?

Answers:

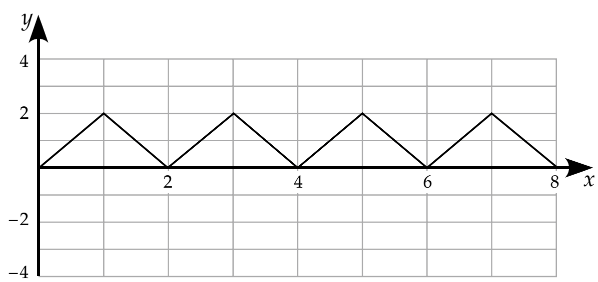
1. 23
2. 51
3. 10
4. 63
5. 88
6. 2
7. 38
8. 118

Part B: Periodic Functions

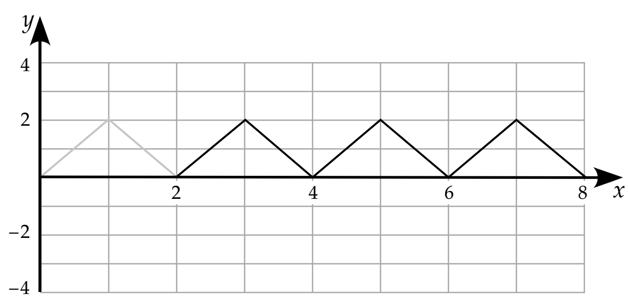
Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Determine if the following graphs represent periodic data. If the data is periodic, approximate the period. Highlight one cycle on each graph.

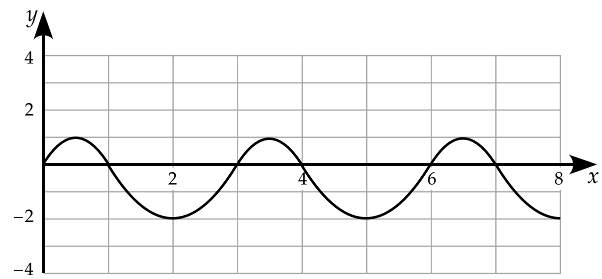
   a)



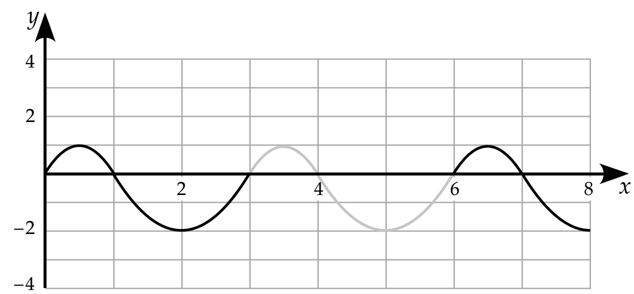
Answer:



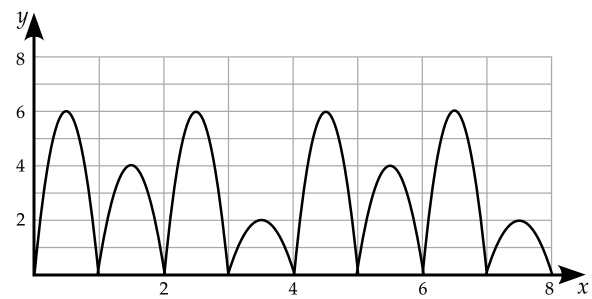
The graph is periodic. The grey part of the graph represents one period. The period is 2 units.

b) 

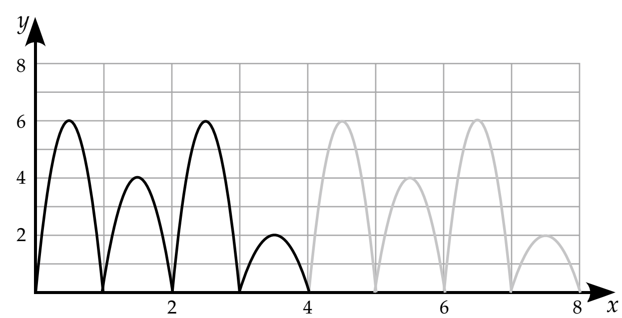
Answer:



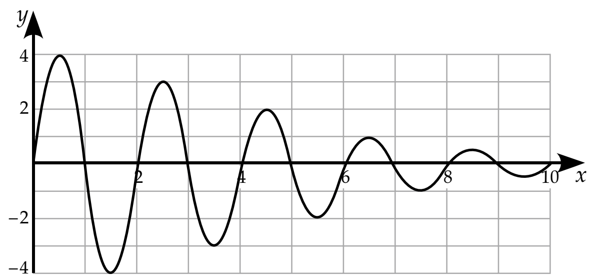
The graph is periodic. The grey part of the graph represents one period (notice that you do not have to start at the beginning). The period is 3 units (goes from an "x-value" of 3 to a value of 6).

c) 

Answer:



The graph is periodic. The grey part of the graph represents one period. The period is 4 units.

d) 

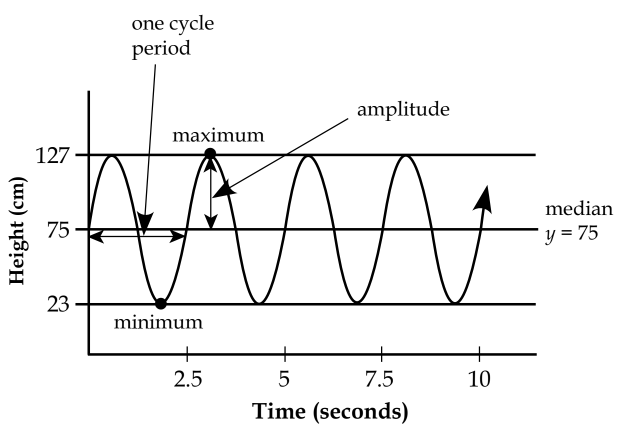
Answer:

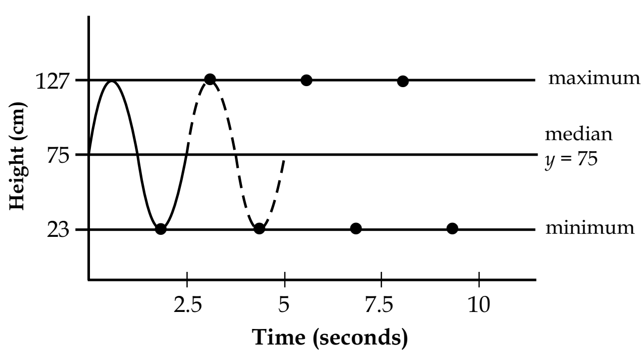
The graph is not periodic since the amplitude decreases over time.

2. A toddler is jumping on a trampoline. The surface of the trampoline is 75 cm above the ground. The little boy jumps up 52 cm above the mat surface, and stretches the mat down the same distance. He completes one bounce every 2.5 seconds. Start the graph at the median height at time = 0 seconds.

   a) Sketch a labelled graph of a sinusoidal function to represent this situation.

   Answer:





Note that the line should look like a smooth curve, not a zigzag line. You may want to sketch a line at the maximum and minimum, as well as the median, and mark points to indicate each period, half period, and the maximum and minimum points of each cycle to help you.

b) State the equation of the median, and identify the maximum and minimum y-values, the amplitude, and period.

Answer:

The equation of the median line is $y = 75$. The amplitude is 52.

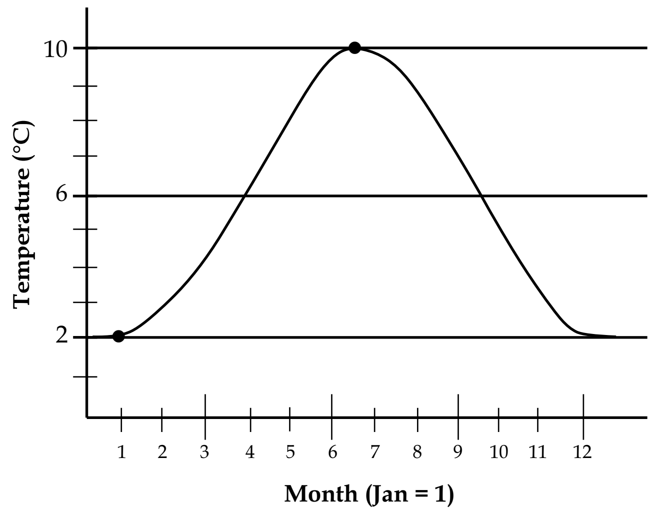
The maximum y-value is 127 and the minimum is $y = 23$.

The period of this function is 2.5 seconds.

3. The temperature of water in the Bay of Fundy changes over the course of one year. It is about 10° C in summer and about 2° C in winter.

a) Sketch a graph to approximate this situation using a sinusoidal function. Use January as month #1.

Answer:



b) State the maximum, minimum, range, median, amplitude, and period of this function.

Answer:

maximum: $y = 10$

minimum: $y = 2$

range: $\{y \mid 2 \leq y \leq 10\}$

median: $y = 6$

amplitude: 4

period: 12 months or one year

4. During the year 2013, the time of sunrise on the first day of each month in the town of Banff, AB was recorded.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Time (hhmm)	0843	0816	0724	0615	0513	0431	0430	0506	0554	0641	0733	0822
Time (decimal)	8.72											

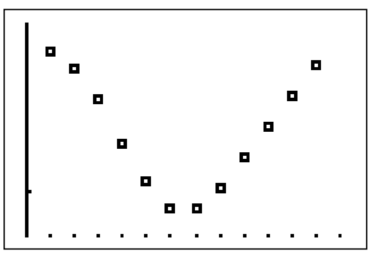
a) Complete the chart by converting the time in minutes to decimal equivalents (i.e., 8:43 am is 8.71667 hours since $\frac{43}{60} = 0.71667$).

Answer:

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Time (hhmm)	0843	0816	0724	0615	0513	0431	0430	0506	0554	0641	0733	0822
Time (decimal)	8.72	8.27	7.4	6.25	5.22	4.52	4.5	5.1	5.9	6.68	7.55	8.37

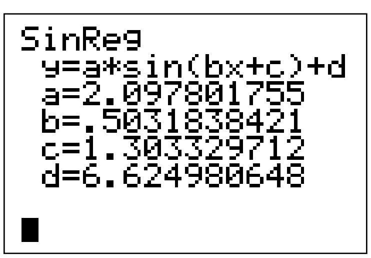
b) Use technology to plot the data, using January as month 1 and December as month 12.

Answer:

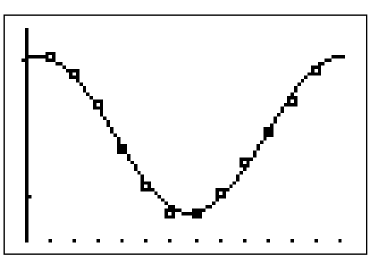


c) Use technology to find a regression equation that best fits the data points. State the equation and graph it using technology.

Answer:

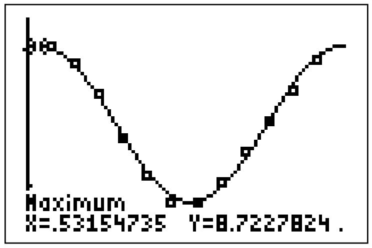


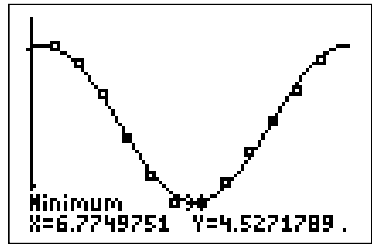
$y = 2.1 \sin (0.5x + 1.3) + 6.6$



d) Use the regression equation to determine the maximum and minimum *y*-values, the equation of the median, the amplitude, and the period of this sinusoidal function. Explain the significance of the values you find.

Answer:





The maximum is 8.72 and the minimum is 4.53. These times correspond to the hours and minutes of 8:43 a.m. and 4:32 a.m.

Median: $\frac{8.72 + 4.53}{2} = 6.625$

The median time is 6.625. The equation of the horizontal median line is $y = 6.625$. This means the average annual time of sunrise is 6:38 a.m.

Amplitude: $8.72 - 6.625 = 2.095$. There is a variation from the median time of sunrise of up to 2 hours and 5.7 minutes, during the course of the year.

One half of a period would be from a maximum to the next minimum. In this case, the difference in *x*-values would be $6.77 - 0.53 = 6.24$.

One cycle would then occur in a period of $6.24 \times 2 = 12.48$ months. Keep in mind that this data is graphed based on the first day of each month. A month may have between 28 and 31 days, and a leap year will have an extra day, so while the period of this data should be one year, the equation is slightly skewed in response to this feature of our calendar.

5. The predicted height of the tide in Whale Cove, Nunavut, on October 29, 2013, is given in the chart below. (Hour 0 = midnight, hour 12 = noon)

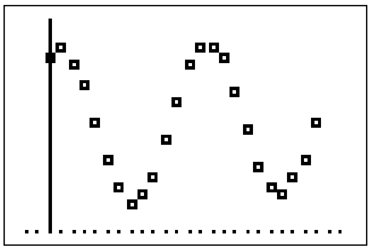
Hour	0	1	2	3	4	5	6	7	8	9	10	11
Depth (m)	2.9	3.0	2.8	2.6	2.2	1.8	1.5	1.3	1.4	1.6	2.0	2.4

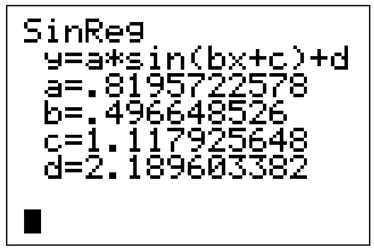
Hour	12	13	14	15	16	17	18	19	20	21	22	23
Depth (m)	2.8	3.0	3.0	2.9	2.5	2.1	1.7	1.5	1.4	1.6	1.8	2.2

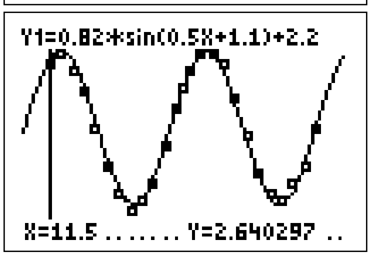
Source: http://www.tides.gc.ca/eng/station?sid=5055

a) Use technology to plot the points and determine the regression equation that best models the tide depth over time.

Answer:



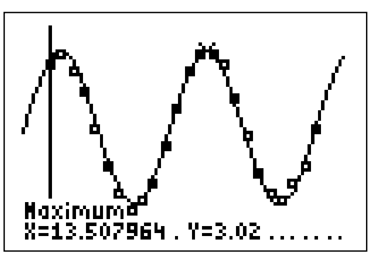


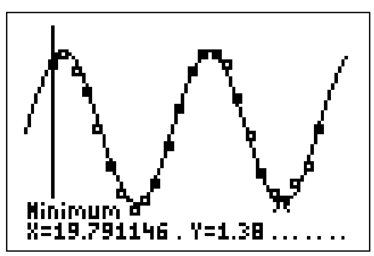


The sinusoidal regression equation of $y = 0.82 \sin (0.5x + 1.1) + 2.2$ models the depth of water in Whale Cove on October 29, 2013.

b) Use the sinusoidal equation to determine the maximum and minimum depths, median depth, amplitude, and period of the tide.

Answer:





According to the sinusoidal function, the maximum y-value is 3.02 m. The minimum y-value is 1.38 m. These values represent the depth of the water at high and low tide in Whale Cove.

Median: $\dfrac{3.02 + 1.38}{2} = 2.2$

The average depth of water in the cove is 2.2 m.

The equation of the median line would be $y = 2.2$

Amplitude: $3.02 - 2.2 = 0.82$

The amplitude of this function is 0.82 m.

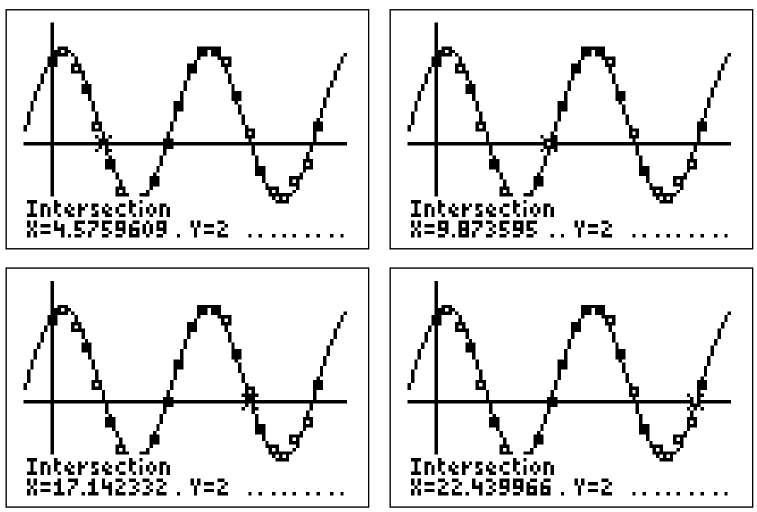
One-half of a period would be from the x-value of a maximum to the x-value at next minimum. In this case, the difference in x-values would be $19.97 - 13.51 = 6.46$.

One cycle would then occur in a period of $6.46 \times 2 = 12.92$ hours, or 12 hours and 55 minutes.

This value represents the time it takes for the water level in the cove to go through one cycle of high and low tides.

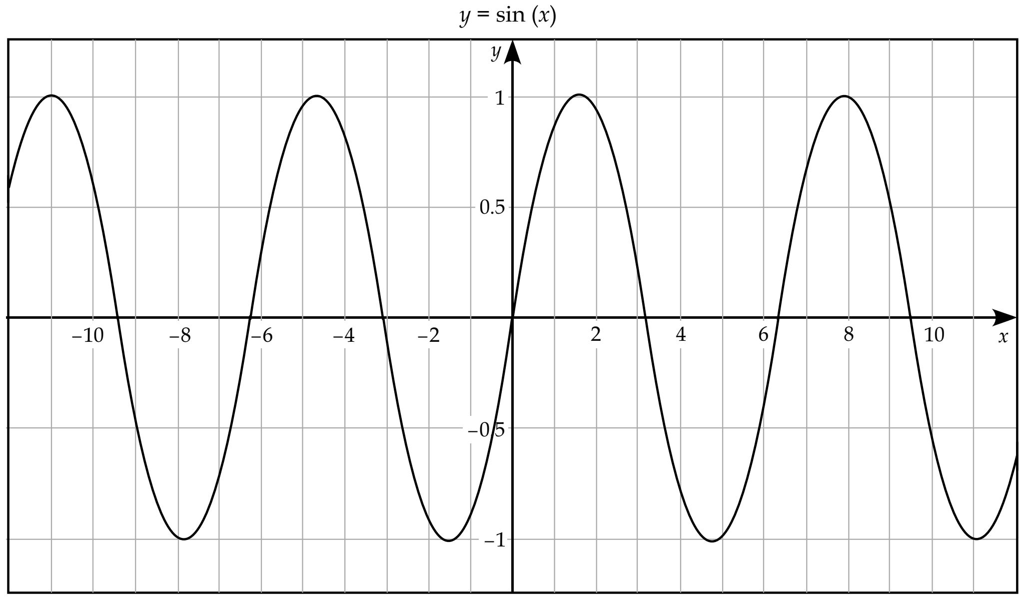
c) A kayaker likes to have at least 2 m of water in the cove when she goes out paddling. Determine the time frame within which she can happily go kayaking.

Answer:

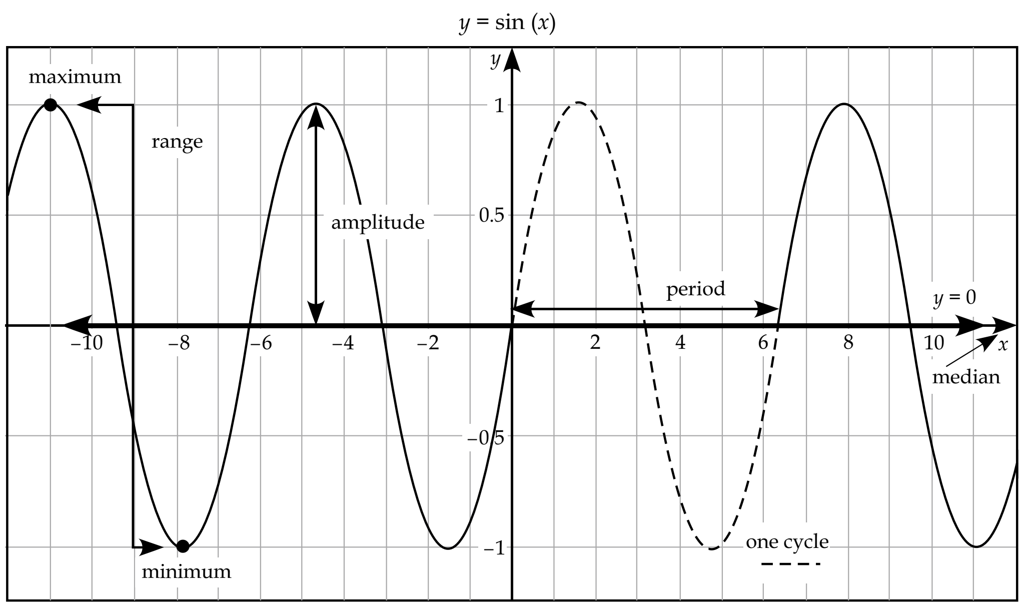


Plot a line at $y = 2$. This represents a depth of 2 m. The paddler can go out when the graph of the sinusoidal function is above this line. On October 29, 2013, this occurs in the early morning until 4.576 or 4:35 a.m., between 9.87 and 17.14 or 9:52 a.m. and 5:08 p.m. or after 22.44 or 10:26 p.m.

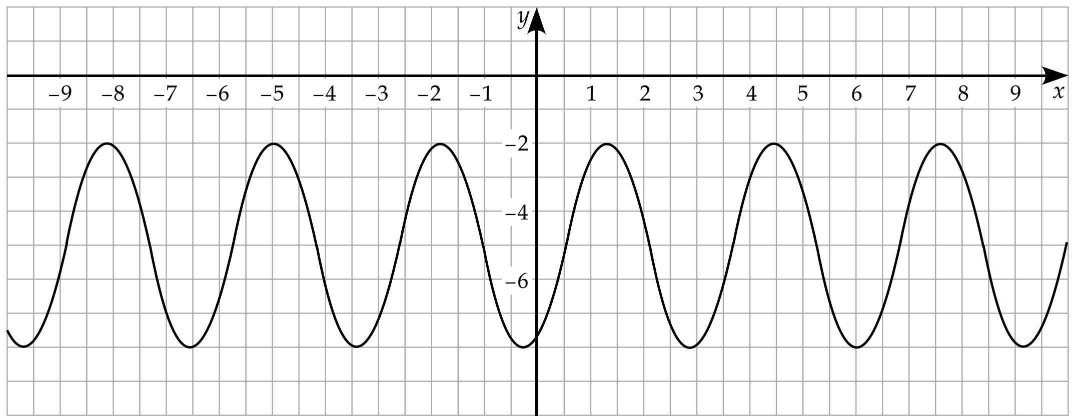
6. Illustrate the properties or characteristics of the graph of the sinusoidal function $y = \sin x$, shown below, by labelling a maximum and a minimum point, using arrows to indicate the range, amplitude, and period, drawing in the median line, and highlighting one cycle.



Answer:



7. Given the graph of the sinusoidal function, $y = 3 \sin(2x - 1) - 5$, determine its maximum and minimum values, the equation of the median, the amplitude, and the period.



Answer:

Maximum y-value = −2

Minimum y-value = −8

Equation of the median: $y = \dfrac{-2 + (-8)}{2}$

$y = -5$

Amplitude: $-2 - (-5) = 3$

Period: Choose a start point and the next consecutive point where the cycle begins to repeat. Calculate the difference between the x-values.

Sample solutions:

(−5, −2) and (−1.9, −2)

$-1.9 - (-5) = 3.1$

or

(0.7, −4) and (3.8, −4)

$3.8 - 0.7 = 3.1$

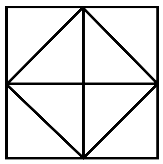
The period is approximately 3.1 units. This corresponds to a period of π radians.

Learning Activity 7.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. How many trianges are in the figure below?



2. Find the cost of 60 trinkets if one dozen cost $44.40.
3. How many radians are in one complete rotation of a circle?
4. How many degrees are in a straight line?
5. How many degrees are in an acute angle?
6. What is the sum of the angles in a parallelogram?
7. If two angles are complementary, the sum of their angles is ______°.
8. If two angles are supplementary, the sum of their angles is ______°.

Answers:

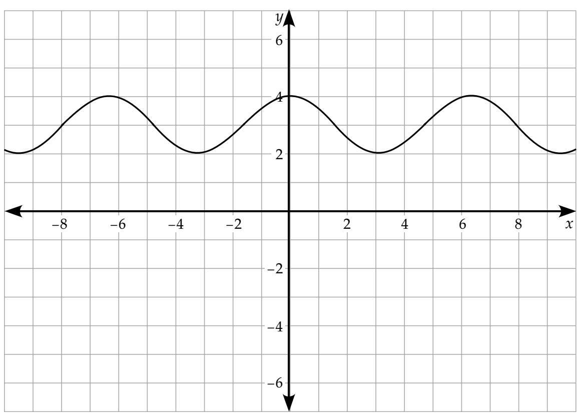
1. 12
2. $222.00 (six of the trinkets would cost $22.20; ten times $22.20 is $222)
3. 2π or 6.28 radians
4. 180°
5. An acute angle is any angle less than 90°.
6. 360°
7. 90°
8. 180°

Part B: The Equation of Sinusoidal Function

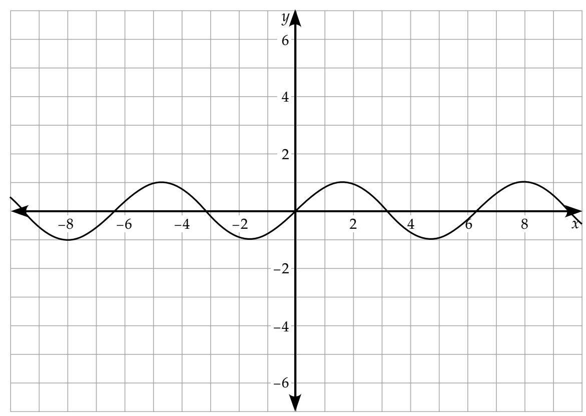
Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Match the equations in the following set to their corresponding graphs. Explain the reasoning used to make your decisions.
   a) $y = 2\sin x$
   b) $y = \sin(4x)$
   c) $y = \sin\left(x + \frac{\pi}{2}\right) + 3$
   d) $y = \sin x$

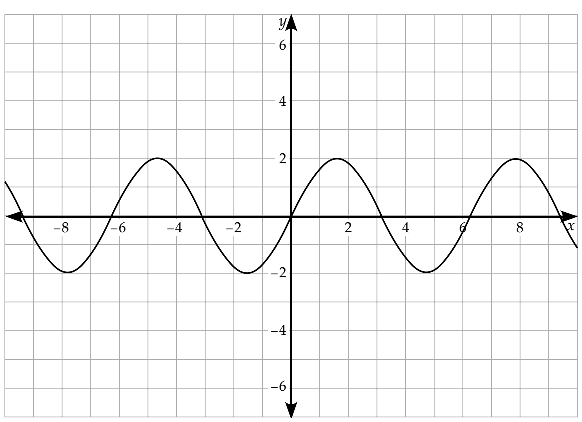
i)



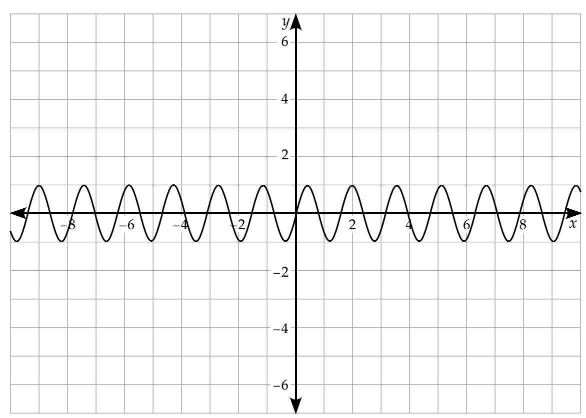
ii)



iii)



iv)



Answers:

a) $y = 2 \sin x$

Matches with graph (iii).

The graph has the same period and median as the basic sine curve, without any phase shift. The amplitude, *a*, is 2 units.

b) $y = \sin (4x)$

Matches with graph (iv).

The amplitude and median of this graph are the same as the basic sine curve. The period is compressed so that four cycles occur in 2π or one period is about 1.57 or $\frac{\pi}{2}$.

$$\text{period} = \frac{2\pi}{b}$$

$$\frac{\pi}{2} = \frac{2\pi}{b}$$

$$b = \frac{4\pi}{\pi}$$

$$b = 4$$

c) $y = \sin\left(x + \frac{\pi}{2}\right) + 3$

Matches with graph (i).

The median is at $y = 3$, so $d = 3$. The reference point where the curve of the graph is at the median and increasing is shifted to the left about 1.57 or $\frac{\pi}{2}$ units.

d) $y = \sin x$

Matches with graph (ii).

This graph illustrates the basic sine curve. The equation of the median is at $y = 0$, so $d = 0$. The amplitude, *a*, is 1. There is no vertical or horizontal shift. At (0, 0), the curve of the graph is at the median and increasing. The period of the graph is 2π.

2. The table shows the time of sunset at New York, New York, on the first day of each month.

New York, New York												
Date	Jan 1	Feb 1	Mar 1	Apr 1	May 1	Jun 1	Jul 1	Aug 1	Sep 1	Oct 1	Nov 1	Dec 1
Time of Sunset (p.m.)	4:39	5:13	5:47	7:20	7:52	8:21	8:31	8:17	7:28	6:38	5:52	4:29
Day of Year (#)	1	32										
Time of Sunset	16.65											

Source: http://www.tropicalweather.net/sunrisesunset_Northeast.htm

a) Complete the table above by converting the hours and minutes to decimal fractions of one hour on the 24-hour clock, and by calculating which day of the year the first day of each month represents.

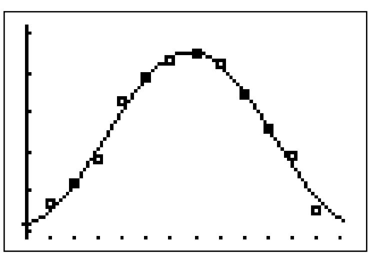
Answer:

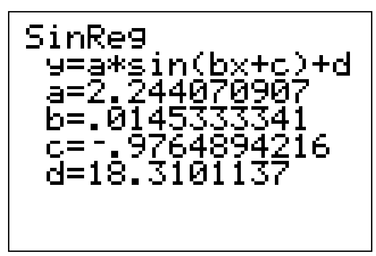
New York, New York												
Date	Jan 1	Feb 1	Mar 1	Apr 1	May 1	Jun 1	Jul 1	Aug 1	Sep 1	Oct 1	Nov 1	Dec 1
Time of Sunset (p.m.)	4:39	5:13	5:47	7:20	7:52	8:21	8:31	8:17	7:28	6:38	5:52	4:29
Day of Year (#)	1	32	60	91	121	152	182	213	244	274	305	335
Time of Sunset	16.65	17.22	17.78	19.33	19.87	20.35	20.52	20.28	19.47	18.63	17.87	16.48

Source: http://www.tropicalweather.net/sunrisesunset_Northeast.htm

b) Use technology to graph the data and determine the sinusoidal regression equation that best fits the time of sunset.

Answer:





The sinusoidal regression equation that best models this data is $y = 2.204 (\sin (0.0145x - 0.976) + 18.310$.

c) What is the median time of sunset?

Answer:

The median is the same as the value for *d*. 18.310 represents a time of 18:19 or 6:19 p.m.

d) How much does the maximum time of sunset vary from the median time of sunset?

Answer:

This refers to the amplitude of this sinusoidal function. From the variable a in the equation, you can determine that the time of sunset varies about 2.204 hours, or by a time of 2 hours, 12 minutes.

e) What is the latest time of sunset during the year?

Answer:

The maximum time of sunset can be found by adding the values for $a + d$.

2.204 + 18.310 = 20.514 or about 8:31 p.m.

f) Use technology to predict the time of sunset on April 15th.

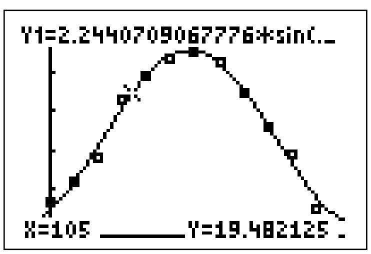
Answer:

April 15 would be day 105. Solve the equation for $x = 105$ or use technology to determine the y-value of the coordinate point where $x = 105$.

$$y = 2.204\ \sin(0.0145x - 0.976) + 18.310$$

$$y = 2.204\ \sin(0.0145(105) - 0.976) + 18.310$$

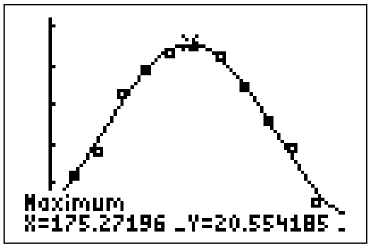
$$y = 19.455$$



The time of sunset would be at approximately 7:27 p.m. or 7:29 p.m. The difference in the calculations is due to the number of decimal places used. Greater accuracy is achieved when values used in calculations have as many decimal places as possible. It is best to round only final answers, not the values used in further calculations.

g) Use technology to determine on which day of the year the latest sunset occurs.

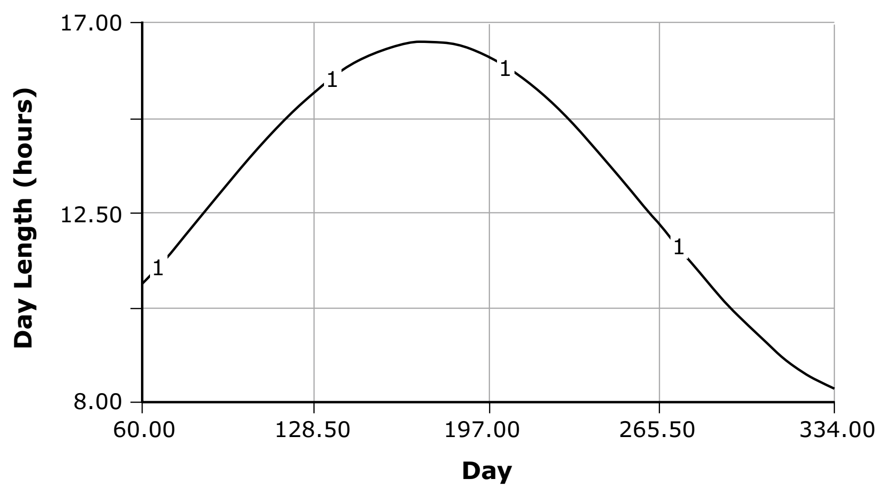
Answer:



The latest sunset of the year occurs on approximately day 175. This relates to June 24, close to the summer solstice.

3. The following graph was published by the Department of Fisheries and Oceans as part of a study of climate and other factors on zooplankton populations in the Hecate Straight, off the coast of British Columbia.

Figure 7: Seasonal change in day length (hours) at 52° N from 1 March to 30 November.



Source: http://www.dfo-mpo-gc.ca/Library/324983.pdf

a) Interpolate the coordinates of five points from the graph and list them in the table below. You may want to draw additional grid lines at regular intervals on the graph to help you estimate values.

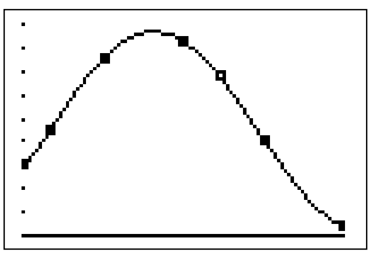
Answer:

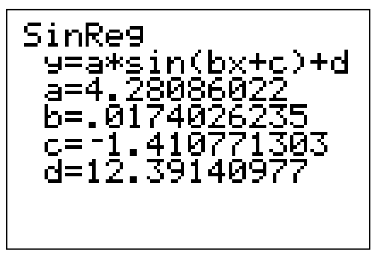
Answers will vary. Approximate values of possible points include the following:

Day (#)	Day Length (hours)
60	11
84	12.5
128.5	15.5
197	16.25
230	14.75
265.5	12
334	8.3

b) Plot the points using technology and determine the sinusoidal regression equation that models this situation.

Answer:





$y = 4.281 \sin (0.0174x - 1.411) + 12.391$

c) What is the average day length in the Hecate Straight?

Answer:

The median of the function that represents the average day length is 12.391 or about 12 hours, 24 minutes.

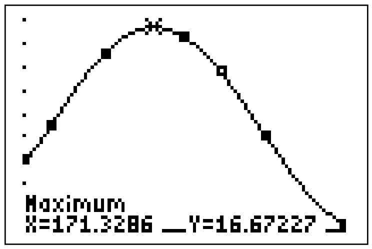
d) The summer solstice occurred on June 20th in the year that this data was recorded. What was the length of day on June 20?

Answer:

The longest day will have the maximum day length. This is found by adding the values of a and d.

4.281 + 12.391 = 16.672 or about 16 hours and 40 minutes.

This can be confirmed by using technology to find the coordinates of the maximum.



e) Calculate the approximate period for this data.

Answer:

$$\text{period} = \frac{2\pi}{b}$$

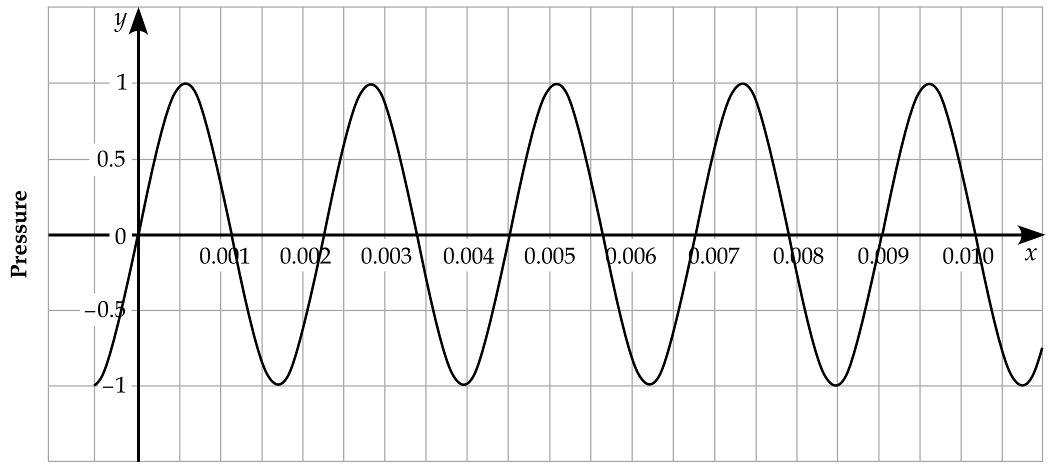
$$\text{period} = \frac{2\pi}{0.0174}$$

$$\text{period} = 361$$

The period is approximately one year.

4. When you listen to music, you are listening to sound waves. On a piano, the A note above middle C produces a wave according to the equation, $y = \sin(880\pi x)$, illustrated in the graph below. What is the amplitude and period of the sound wave of the note A? How many times per second does the sound wave created by the note A vibrate?

The Note A



Time (seconds)

Answer:

The amplitude is 1, and the period is $\frac{2\pi}{b}$.

$$\text{period} = \frac{2\pi}{b}$$

$$\text{period} = \frac{2\pi}{880\pi}$$

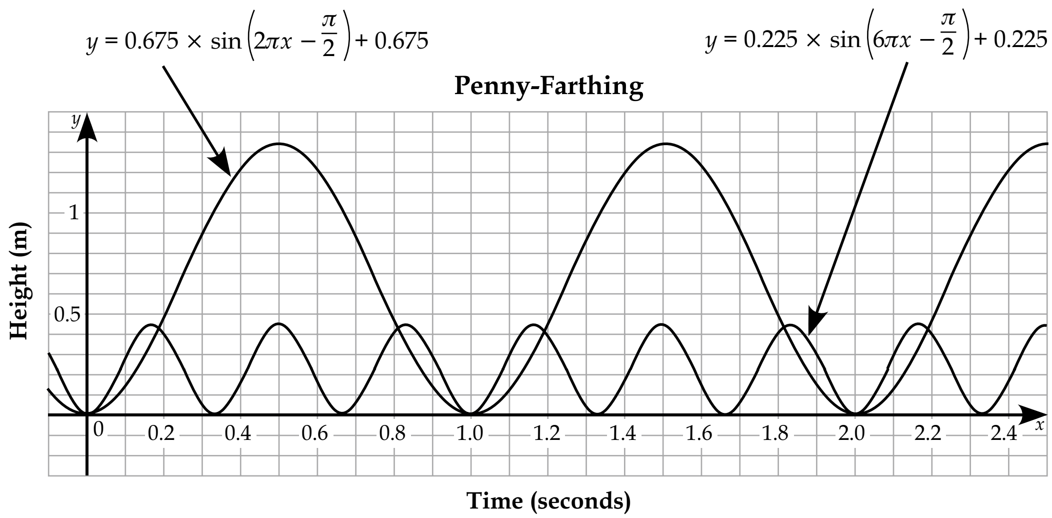
$$\text{period} = \frac{1}{440} \text{ seconds}$$

The sound wave for the note of A vibrates 440 times per second.

5. A point on the rim of the large front wheel of a Penny-Farthing bicycle travels according to the equation $y = 0.675 \sin\left(2\pi x - \frac{\pi}{2}\right) + 0.675$. A point on the rim of the smaller rear wheel travels according to the equation $y = 0.225 \sin\left(6\pi x - \frac{\pi}{2}\right) + 0.225$.

a) Graph both equations on the same grid.

Answer:



b) Compare the amplitude, maximum and minimum, median, and period of each function.

Answer:

	Large Front Wheel	**Small Rear Wheel**	
Amplitude	0.675 m	0.225 m	the radius of the wheel
Maximum	1.35 m	0.45 m	the diameter of the wheel
Minimum	0 m	0 m	wheels moving along the ground
Median	0.675	0.225	centre of the wheel
Period	1 second	0.33 seconds	time for the wheel to rotate once (The smaller wheel rotates three times for every one rotation of the larger wheel.)

c) Determine the circumference of the large wheel. How far does the bike travel in one second? How long will it take the biker to go 1 km? Approximately how fast is the biker travelling?

Answer:

The circumference of the large wheel is found by using the formula $C = \pi d$.

$$C = \pi d$$
$$C = \pi(1.35)$$
$$C = 4.24 \text{ m}$$

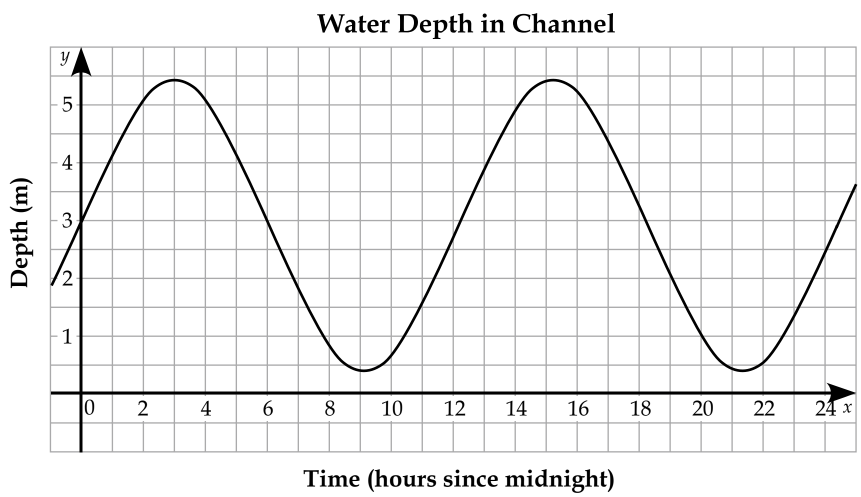
The bike travels 4.24 m per second. There are 1000 m in one kilometre, so it would take the biker $\frac{1000}{4.24} = 235.85$ seconds or $\frac{235.85}{60} = 3.93$ minutes to go 1 km.

If the biker is covering 1 km in approximately 4 minutes, that is approximately $\frac{60}{4} = 15$ km/hr.

6. The depth, d, of water in a channel varies with time, t, according to the equation $d(t) = 2.5 \sin(0.523t) + 2.9$, where d is the depth of water in metres, and t is the number of hours since midnight.

   a) Create a graph showing the depth of water over 24 hours. Label the graph appropriately.

   Answer:



b) What is the period of this function? What does it represent in terms of the varying depth of water?

Answer:

$$\text{period} = \frac{2\pi}{b}$$

$$\text{period} = \frac{2\pi}{0.523}$$

$$\text{period} = 12.01$$

The period is about 12 hours. This means the depth of the water in the channel goes through one cycle of high and low depths every 12 hours, or two cycles of high and low depths each day.

c) Use the equation to determine the average depth of water in the channel, the maximum depth, and the minimum depth.

Answer:

The average depth is the same value as the equation of the median, or d: 2.9 m.

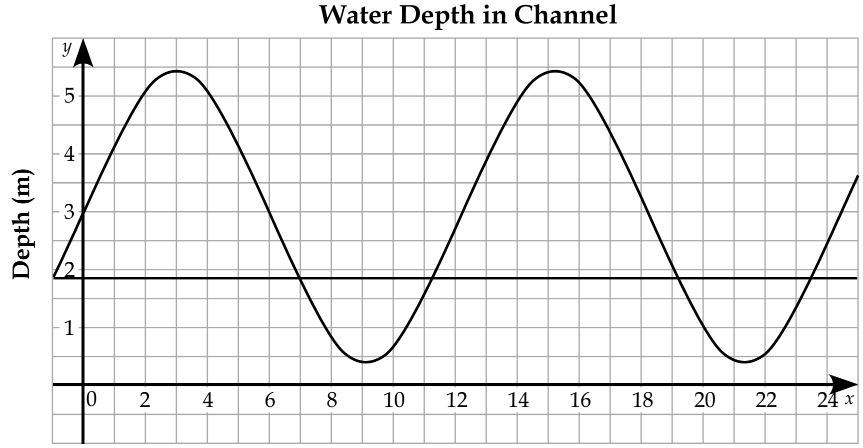
The maximum is the median plus the amplitude, d + a: 2.9 + 2.5 = 5.4 m.

The minimum is the median less the amplitude: $d - s$: 2.9 − 2.5= 0.4 m.

d) If a fishing boat requires a depth of at least 1.75 m to navigate across the channel, determine when the boat may safely cross.

Answer:

Water Depth in Channel



Time (hours since midnight)

Find the points of intersection between the line y = 1.75 and the curve of the sine function. These occur at (6.92, 1.75), (11.10, 1.75), (18.93, 1.75), (23.11, 1.75). The boat may cross the channel before 6:55 a.m., between 11:06 a.m. and 6:56 p.m. or after 11:07 p.m.

7. A mass is attached to a spring that hangs from a hook above a tabletop, where the mass is allowed to move freely. The up and down motion of the mass is described by a sinusoidal function with an upper height of 65 cm and a lower height of 15 cm above the table. Once in motion, the mass completes one oscillation per second. (Ignore the friction on the spring and assume the mass stays in motion over time.)

   a) Sketch a graph of the sinusoidal function describing the height of the mass as it oscillates, starting from the lowest point.

      Answer:

      Draw in lines to represent the maximum at $h = 65$ cm, the minimum at $h = 15$ cm, and the median.

      $$\text{median} = \frac{\text{maximum} + \text{minimum}}{2}$$

      $$\text{median} = \frac{65 + 15}{2}$$

      $$\text{median} = 40 = d$$

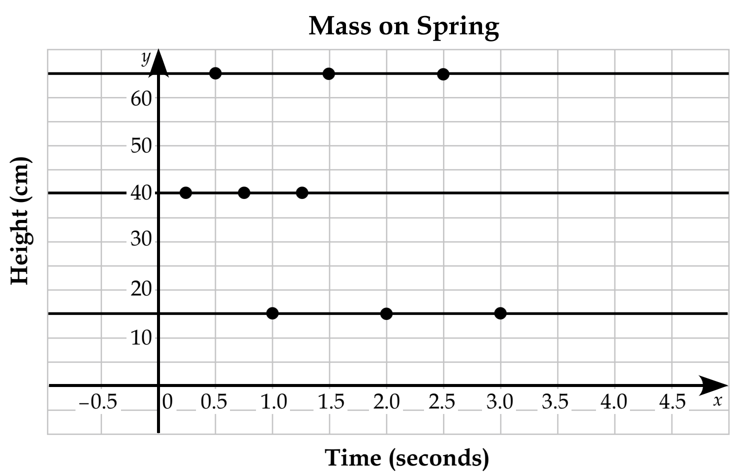
      The period is 1 second.

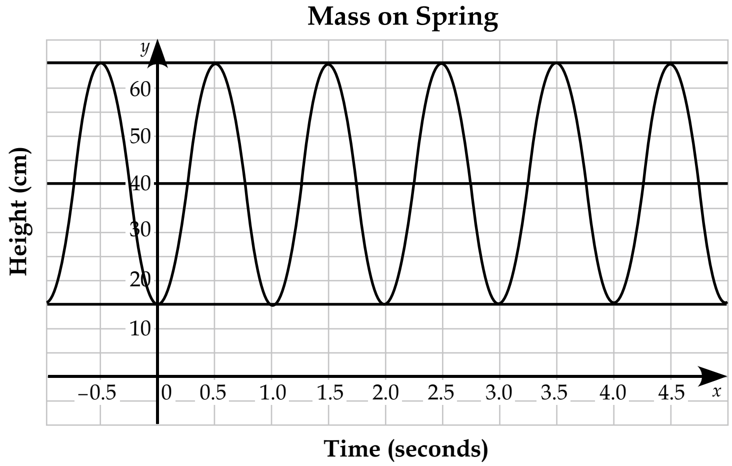
      Plot points at the minimum at intervals of 1 second, beginning at $t = 1$ sec.

      Plot points at the maximum at intervals of 1 second, beginning at $t = 0.5$ sec.

      Plot point at the median at intervals of 0.5 seconds, beginning at $t = 0.25$ sec.

      Join the dots with a smooth curve as this data is continuous.





b) Write a sinusoidal equation to describe the height, *h*, of the mass above the table with respect to time, *t*.

Answer:

maximum = 65

minimum = 15

median = d = 40

$$\text{amplitude} = \frac{\text{maximum} - \text{minimum}}{2}$$

$$\text{amplitude} = \frac{65 - 15}{2}$$

$$\text{amplitude} = 25 = a$$

$$\text{phase shift} = \frac{-c}{b}$$

$$c = -(\text{phase shift} \times b)$$

$$c = -(0.25 \times 2\pi)$$

$$c = \frac{-\pi}{2}$$

The sinusoidal function that models the height of the mass over time is

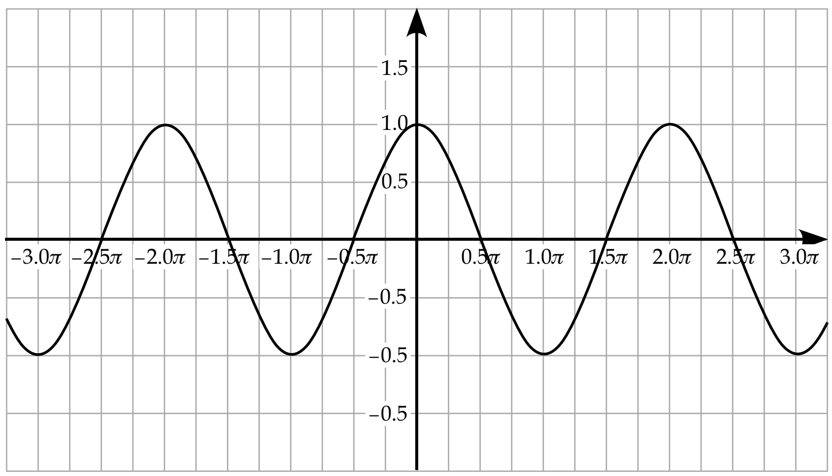
$$h(t) = 25 \sin\left(2\pi(t) - \frac{\pi}{2}\right) + 40.$$

Alternatively, this function could be found by entering the coordinates of several points as found on the graph in part (a) into graphing technology. Then find the sinusoidal regression function.

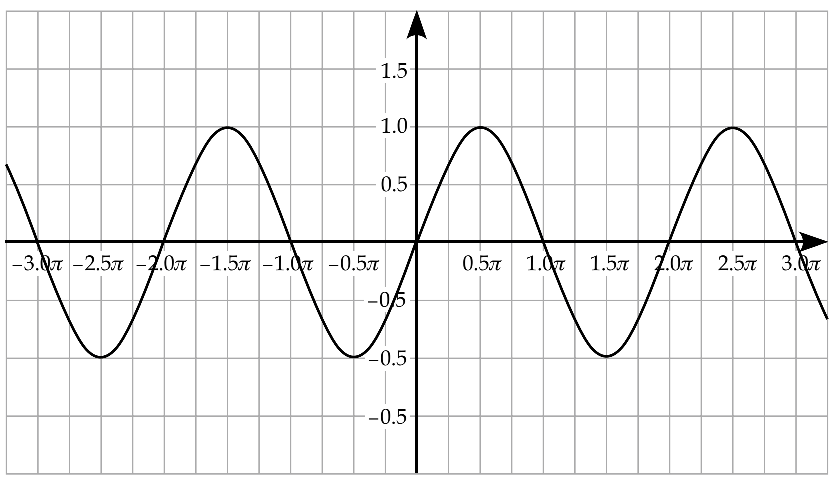
$$y = 25 \sin(6.28t - 1.57) + 40$$

8. Sinusoidal functions can be written in terms of both sine and cosine functions. Compare the amplitude, median, maximum, minimum, period, and phase shift of the cosine graph compared to the sine graph, shown below.

$y = \cos x$



$y = \sin x$



Answer:

In the graphs of $y = \sin x$ and $y = \cos x$, the amplitude = 1, median = 0, maximum = 1, minimum = −1, and period = 2π. These characteristics are identical in both curves. The difference between the two graphs is in the phase shift. When $x = 0$, the graph of $y = \cos x$ is at the maximum, while the graph of $y = \sin x$ is at the median and increasing. The closest point on the graph of $y = \cos x$, where the curve is at the median and increasing is found at $\left(-\frac{\pi}{2}, 0\right)$. This phase shift is the only difference in the graphs of the basic sine curve and cosine curve.

Grade 12 Applied Mathematics (40S)

Module 8
Design and Measurement

Module 8: Design and Measurement

Introduction

Welcome to Module 8. In the senior years Applied Mathematics program you have been given opportunities to solve problems involving surface area and volume of 3-dimensional objects, using both metric and imperial units. You have considered scale diagrams and used proportional reasoning to understand the relationships between scale factors, areas, and volumes in 2-D and 3-D objects. In addition, you have used trigonometry to solve triangle problems. In this module, you will put all of these skills, and others, to use to analyze objects, shapes, and processes to solve cost and design problems. This will also involve using the dimensions of objects and the costs of items and processes to solve design problems within a given budget.

Assignments in Module 8

When you have completed the assignments for Module 8, submit your completed assignments to the Distance Learning Unit either by mail or electronically through the learning management system (LMS). The staff will forward your work to your tutor/marker.

Lesson	Assignment Number	Assignment Title
	Cover Assignment	Container Conundrum
2	Assignment 8.1	Design and Cost Decisions
3	Assignment 8.2	Working within a Project Budget

Resource Sheet

When you write your final examination, you are encouraged to take a Final Examination Resource Sheet with you into the examination. This sheet will be one letter-sized page, 8½ ″ by 11″, with both sides in your handwriting or typewritten. You will submit it with your examination, but you do not receive any marks for it.

Many students have found that preparing a resource sheet is an excellent way to review. It provides you with a summary of the important facts of each module. You should complete a resource sheet for each module to help with your studying and reviewing. Lesson summaries and module summaries are included for you to use as a guide.

You may use the list of instructions provided below to help you with preparing your resource sheet for the material in Module 8. On this sheet, you should record math terms and definitions, formulas, sample questions, or a list of places where you often make mistakes. You should also identify special areas that require extra attention or review by writing the page numbers.

After you have completed each module's resource sheet, you may summarize the sheets from Modules 5, 6, 7, and 8 to prepare your Final Examination Resource Sheet. The final examination for this course is based on Modules 5 to 8.

Resource Sheet for Module 8

As you go through the lessons of this module, you may want to consider the following suggestions regarding the creation of a resource sheet.

1. List all the important math terms, and define them if necessary.
2. List all the formulas and perhaps a sample problem that shows how the formula is used.
3. If necessary, write the solutions to some problems, showing in detail how you did the calculations.
4. Copy any questions that represent the key points of the lesson, and perhaps include the solutions as well.
5. Identify the problems you found most difficult, and copy the page numbers onto the resource sheet so that you can review them before writing the examination. You may also copy the problems and the solutions onto your resource sheet, and later write them onto your Final Examination Resource Sheet.
6. Write any comments, ideas, shortcuts, or other reminders that may be helpful during an examination.

Writing Your Final Examination

You will write the final examination when you have completed Module 8 of this course. The final examination is based on Modules 5 to 8, and is worth 25 percent of your final mark in the course. To do well on the final examination, you should review all the work you complete in Modules 5 to 8, including all the learning activities and assignments. You will write the final examination under supervision.

Notes

Module 8 Cover Assignment: Container Conundrum

When we make a purchase that comes in a box, we are usually most concerned with what is inside, but the container is also a product of careful design and construction. The size, shape, and durability of the contents, as well as the importance of controlling costs by preventing unnecessary waste of materials, are all factors that must be considered in creating containers for shipment and sale of products.

Use your creativity and knowledge of volume and area to create containers with different dimensions.

This is a hand-in assignment. Please show your work clearly and in an organized manner. Round final answers to 2 decimal places, and include units, if appropriate. Explain you answers fully.

Notes

Module 8 Cover Assignment

Container Conundrum

Total: 5 marks

You have been given 100 in.^{2} of a new type of packaging material with which you must construct a container with a large volume suitable for shipping goods by mail.

a) Determine the shape and size of **two** possible containers that both make good use of 100 in.^{2} of material. Sketch a diagram of each with the dimensions labelled. Show your calculations for the volume and surface area of each container. *(4 marks)*

b) Write a brief paragraph outlining the process you used to find the container with the larger volume. Indicate any assumptions you made in the construction of your container. *(0.5 mark)*

c) If you were to ship items in one of your containers by mail, which of your two designs would you choose? Explain your answer. *(0.5 mark)*

Lesson Focus

In this lesson, you will

- ❑ solve a problem involving perimeter, area, and volume using dimensions
- ❑ identify and correct errors in a solution to a problem
- ❑ estimate the solutions to complex measurement problems using simplified models

Lesson Introduction

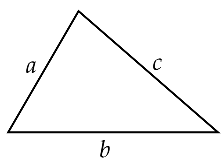
To calculate the perimeter, area, surface area, and volume of basic geometric shapes, composite shapes, or complex shapes, formulas are used. As you go through this lesson, you may want to include these formulas on your resource sheet. Many of these formulas will be familiar to you from your previous experiences with mathematics.

Perimeter, Area, and Volume

Two-dimensional (2-D) Shapes

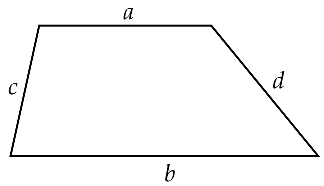
Perimeter is the distance around the outside of a 2-D figure. The formulas for perimeter will involve adding the lengths of all sides. The perimeter of a circle is also called its circumference. The units for perimeter are length units, such as centimetres, inches, feet, metres, and kilometres.

Triangle



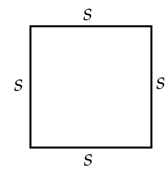
$P = a + b + c$

Trapezoid



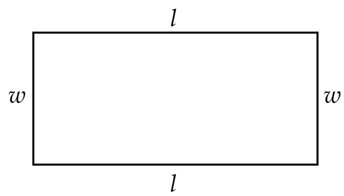
$P = a + b + c + d$

Square



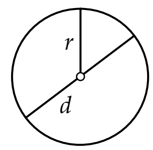
$P = 4s$

Rectangle



$P = 2l + 2w$

Circle

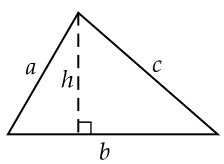


$C = 2\pi r$ or $C = \pi d$

Area is the calculation of the number of squares that are required to cover the surface of a 2-dimensional figure. The units for area are square units such as square centimetres, square inches, square feet, square metres, or square kilometres.

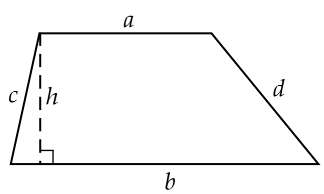
Area formulas for basic geometric figures are as follows:

Triangle



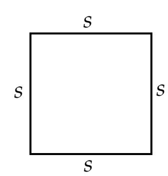
$A = \frac{1}{2}bh$

Trapezoid



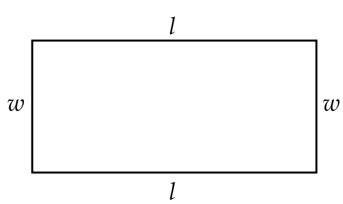
$A = \frac{1}{2}(a + b)h$

Square



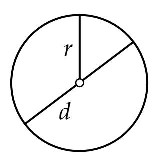
$A = s^2$

Rectangle



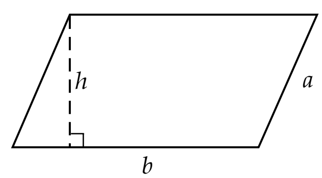
$A = lw$

Circle



$A = \pi r^2$

Parallelogram



$A = bh$

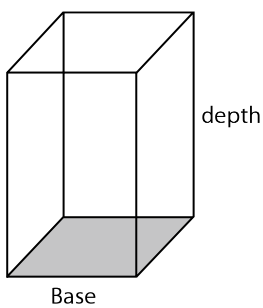
Area units are square units, such as, cm^2, $in.^2$, $ft.^2$, m^2, or km^2.

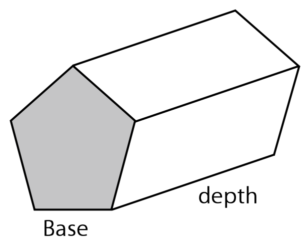
Regular 3-dimensional (3-D) Shapes

Three-dimensional objects have **surface area** and **volume**. Many 3-D objects can be classified as either prisms or pyramids.

A **prism** is a shape where the base of the figure is a polygon and the shape of the base is maintained consistently for each cross-section throughout the height (or depth) of the shape. Just like thinly slicing a block of cheese vertically, the shape and size of the original end of the block of cheese is the same as the new cross-section revealed after slicing. The ends (or bases) of a prism can be any shape and the sides are usually rectangles. The sides can be parallelograms, but we are not considering those types of prisms in this course.

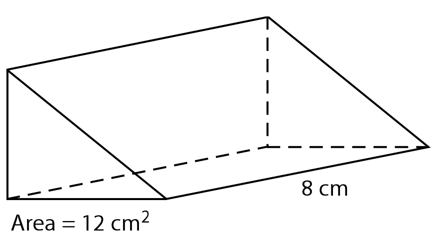
Prism:





$V = Bd$ where B is the area of the base and d is the depth (or height).

In the first diagram, the base at the bottom is the shape of a rectangle. Horizontal slices of the prism will have cross-sections that are the same size and shape as the rectangle base. In the second diagram, the base on the end is the shape of a pentagon. Vertical slices of the prism will have cross-sections that are the same size and shape as the pentagon base. You can think of having thin slices of the base stacked together to form the prism.



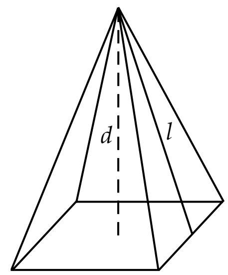
Every vertical slice of the triangular prism shown in this example will have a base that is the same shape as the triangle end.

Given that the depth of the prism is 8 cm and the area of the triangle end is 12 cm^2, the volume can be found.

The volume of this triangular prism is $12 \times 8 = 96$ cm^3.

A **pyramid** is a 3-D figure in which the polygon base of the figure reduces linearly through its height (or depth) to a single point. The sides of a pyramid are triangles.

Pyramid:



$V = \frac{1}{3}Bd$ or $\frac{Bd}{3}$ where B is the area of the base and d is the depth (or height). Note that l is the slant height of the isosceles triangle that makes up the lateral face of the pyramid.

The volume of an object is a measure of the number of cubes that are required to fill the object. The units for volume are various sizes of cubes such as cubic inches, cubic centimetres, cubic feet, cubic metres, or cubic yards. To calculate the volume of regular 3-D objects, classify the object as either a prism or a pyramid.

The volume of a prism is calculated by multiplying the area of the base (in square units) by the height (or depth) of the figure ($V = Bd$ or $V = Bh$).

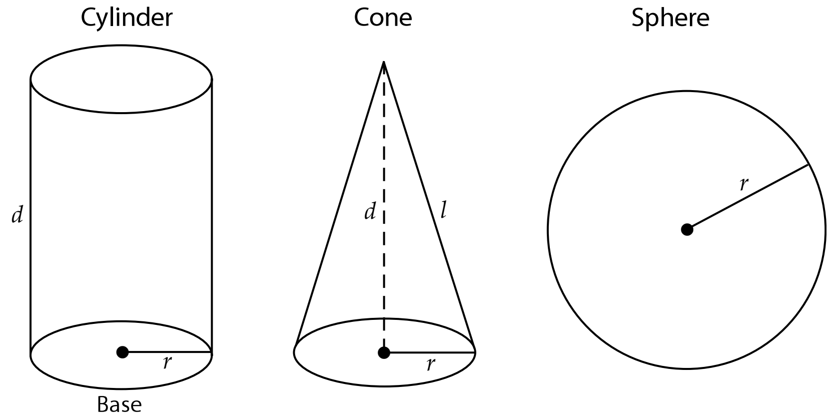
The volume of a pyramid is related to the volume of a prism with the same base and depth—the volume of a pyramid is one-third the volume of the related prism. The volume of a pyramid can be calculated by multiplying one-third of the area of its base (in square units) by the height (or depth) of the figure $\left(V = \frac{1}{3}Bd \text{ or } V = \frac{1}{3}Bh\right)$.

The units of volume are cubic units or units^3.

The **total surface area** of a 3-D figure can be calculated by adding the 2-D area of each surface. The **lateral surface area** is the sum of the areas of each face, excluding the base. Use the appropriate formula(s) for calculating the area of each face and then add them together. The same as other area measures, the units of surface area are square units or units^2.

Circular 3-D Shapes

If the base of a 3-D figure is circular rather than a regular polygon, the 3-D figure may be a sphere, a prism-like cylinder, or a pyramid-like cone. Since squares don't fit perfectly into circles and cubes don't fit perfectly into spheres, their formulas for area and volume require the use of the irrational number, π. Remember that π is the ratio of the circumference and the diameter of a circle. To maximize the accuracy of your answers, use as many decimal places as possible in your calculations and round only final answers.



Cylinder	Cone	Sphere
P = perimeter of the base = $2\pi r$ B = area of the base = πr^2 d = height (or depth) r = radius	$P = 2\pi r$ $B = \pi r^2$ d = height r = radius l = slant height $d^2 + r^2 = l^2$	r = radius
$V = Bd$	$V = \frac{1}{3}Bd$	$V = \frac{4}{3}\pi r^3$
Total S.A. Cylinder $= Pd + 2B$	Total S.A. Cone $= \frac{1}{2}Pl + B$	Total S.A. Sphere $= 4\pi r^2$

Converting Units

When calculating the perimeter, area, surface area, or volume of an object, all dimensions must be stated in the same units. You may be required to convert units before performing calculations. Some common conversions that you may want to include on your resource sheet include the following:

Metric System	Imperial System	Conversions between Systems	
10 mm = 1 cm 100 cm = 1 m 1000 m = 1 km	12″ (inches) = 1′ (foot) 36″ = 1 yard 3′ = 1 yard 5280 ′ = 1 mile	1 cm = 0.3937″ 1 m = 1.0936 yd. 1 km = 0.6214 mi.	1″ = 2.54 cm 1′ = 0.3048 m 1 yd. = 0.9144 m 1 mi. = 1.6093 km

You can use the same conversions (above) to help you convert square units and cubic units. Here are some examples:

10 mm = 1 cm implies that:

10^2 mm^2 = 1^2 cm^2 when simplified means 100 mm^2 = 1 cm^2

10 mm = 1 cm implies that:

10^3 mm^3 = 1^3 cm^3 when simplified means 1000 mm^3 = 1 cm^3

3 feet = 1 yard implies that;

3^2 feet^2 = 1^2 yard^2 when simplified means 9 feet^2 = 1 yard^2

3 feet = 1 yard implies that:

3^3 feet^3 = 1^3 yard^3 when simplified means 27 feet^3 = 1 yard^3

Here are some examples where you need to solve problems involving perimeter, area, and volume using a variety of dimensions.

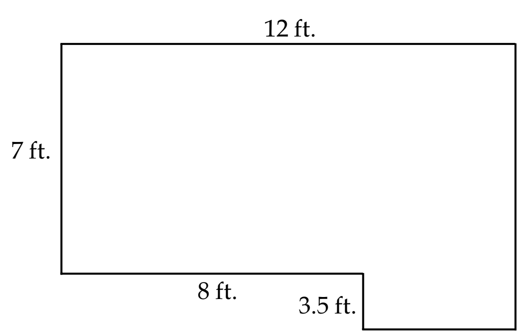
Example 1

Sara would like new flooring and baseboards in her living room.

a) Sara would like to install the new baseboards around the perimeter of the room. What length will she need to purchase?

b) If baseboards cost $4 per linear foot, how much will the baseboards cost her?

c) How many square feet of flooring will she need? How many square metres will she need?



Solution

a) The baseboards go around the entire perimeter of the room. The missing side lengths are 10.5 ft. (7 + 3.5) and 4 ft. (12 − 8). Add together each length to find the total.

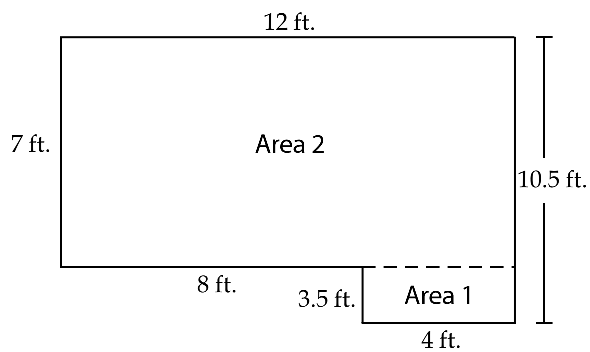
$$7 + 12 + 10.5 + 4 + 3.5 + 8 = 45$$

She will need to install 45 feet of baseboards.

b) $\$4 \times 45$ ft. = $180.00

The baseboards will cost her $180, plus taxes.

c) The amount of flooring needed to cover the living room is a question of area. This 2-D floor plan is not a regular shape. When working with composite figures, you can divide the area into regular shapes and then add the quantities together to find the total area. This can be done in a variety of ways. One possible solution is to calculate the area of the small, lower right-hand rectangle and the area of the large upper portion, and add these values together. Use the given values to determine the missing values.



$A_1 = l \times w$	$A_2 = l \times w$
$A_1 = 3.5 \times 4$	$A_2 = 7 \times 12$
$A_1 = 14$	$A_2 = 84$
$14 + 84 = 98$	

The total area requiring flooring is 98 square feet.

To convert 98 $ft.^2$ to m^2, first convert to square yards. There are 9 $ft.^2$ in 1 $yd.^2$.

$$98 \div 9 = 10.89 \text{ yd.}^2$$

Since 1 yd. = 0.9144 m, 1 $yd.^2$ = 0.8361 m^2.

Therefore, 10.89 $yd.^2$ = 10.89×0.8361 = 9.105 m^2.

She will require 9.105 m^2 of flooring.

Example 2

One of the original highways in the United States was Route 66, which ran from Chicago, Illinois to Santa Monica, California. It covered 2448 miles. How many kilometres is this?

Solution

Use proportional reasoning to solve for the unknown quantity. Set up the conversion ratio, with the unit you want your final answer to be stated in as the value in the numerator (top). The second ratio, with the values from the question, is set up so that the units match (i.e., km in numerator, mi. in denominator). Solve for the unknown quantity.

$$\frac{1\text{ km}}{0.6214\text{ mi.}} = \frac{x\text{ km}}{2448\text{ mi.}}$$

$$(2448)\frac{1}{0.6214} = \frac{x}{\cancel{2448}}\left(\cancel{2448}\right)$$

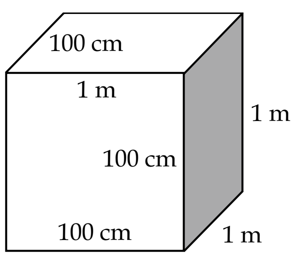
$$x = 3939.49$$

Route 66 was originally about 3939 km.

Example 3

Use a diagram to show that 1 m^3 = 1 000 000 cm^3.

Solution



The volume of this cube is calculated as:

$V = l \times w \times h$

$V = 1\text{ m} \times 1\text{ m} \times 1\text{ m}$

$V = 1\text{ m}^3$

or

$V = l \times w \times h$

$V = 100\text{ cm} \times 100\text{ cm} \times 100\text{ cm}$

$V = 1\ 000\ 000\text{ cm}^3$

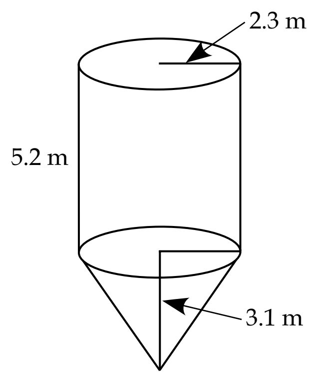
So 1 m^3 = 1 000 000 cm^3

Example 4

Given the diagram of the following tank (not drawn to scale):

a) Find the surface area of the tank.

b) Find the volume of the tank.



Solution

This object is composed of a cylinder on top of a cone.

a) The surface area includes the circular top, flat side of the cylinder, and slanted side of the cone.

$A_{\text{circle}} = \pi r^2$

$A_{\text{circle}} = \pi(2.3)^2$

$A_{\text{circle}} = 16.61902514 \text{ m}^2$

The flat side of the cylinder is a rectangle 5.2 m high and as wide as the perimeter (circumference) of the circle top.

Find the circumference of the circle.

$C = 2\pi r$

$C = 2\pi(2.3)$

$C = 14.45132621 \text{ m}$

Find the area of the flat side of the cylinder.

$A = Ph$

$A = 14.45132621 \times 5.2$

$A = 75.14689627 \text{ m}^2$

The formula for the total surface area of a cone is given previously as:

$$\text{Total SA}_{\text{cone}} = \frac{1}{2}Pl + B,$$

where P is the perimeter, l is the slant height, and B is the area of the base.

In this case, you only want to include the side area, not the area of the base.

$$SA_{cone} = \frac{1}{2}Pl$$

To calculate the slant height, l, of the cone, use the Pythagorean theorem, $h^2 + r^2 = l^2$, where h is the height of the cone, and r is the radius.

$$h^2 + r^2 = l^2$$

$$(3.1)^2 + (2.3)^2 = l^2$$

$$l^2 = 14.9$$

$$l = 3.860051813$$

The perimeter (or circumference) of the circle, calculated above, is C = 14.45132621 m.

$$SA_{cone} = \frac{1}{2}Pl$$

$$SA_{cone} = \frac{1}{2}(14.45132621)(3.860051813)$$

$$SA_{cone} = 27.89143397$$

The total surface area of the tank is the sum of the areas of each face.

$$\text{Total } SA_{tank} = 16.61902514 + 75.14689627 + 27.89143397$$

$$\text{Total } SA_{tank} = 119.6573554$$

The total surface area of the tank is approximately 119.7 m^2.

b) The volume of this tank is comprised of the volumes of the cylindrical and conical parts.

$$V_{cylinder} = Bd,$$

where B is the area of the base and d is the depth (or height) of the cylinder.

$$V_{cylinder} = (16.61902514)(5.2)$$

$$V_{cylinder} = 86.41893073$$

$$V_{cone} = \frac{1}{3}Bd,$$

where B is the area of the base and d is the depth (or height) of the cone.

$$V_{cone} = \frac{1}{3}(16.61902514)(3.1)$$

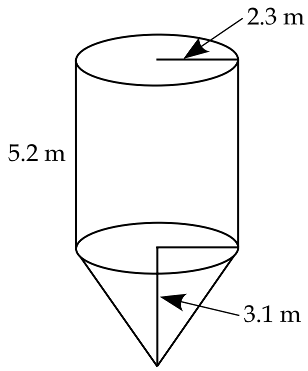
$$V_{cone} = 17.17299264$$

$$\text{Total } V_{tank} = 86.41893073 + 17.17299264 = 103.5919234$$

The volume of this tank is about 103.6 m^3.

Example 5

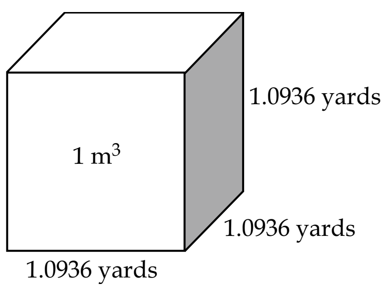
If a farmer wanted to fill up the tank from Example 4 with fertilizer, which is sold in cubic yards, how much fertilizer would he need to order?



Solution

You need to convert the answer from Example 4, 103.6 m^3, to cubic yards. First, convert 1 m^3 into a number of $yd.^3$. It may help to visualize this.

1 m = 1.0936 yd.



$$1\ m^3 = 1.0936 \text{ yards} \times 1.0936 \text{ yards} \times 1.0936 \text{ yards}$$

$$1\ m^3 = (1.0936)^3 \text{ yards}^3$$

$$1\ m^3 = 1.307902906 \text{ yards}^3$$

You can use a proportion as follows:

$$\frac{1.307902906 \text{ yd.}^3}{1 \text{ m}^3} = \frac{x \text{ yd.}^3}{103.6 \text{ m}^3}$$

$$(103.6)\frac{1.307902906}{1} = \frac{x}{\cancel{103.6}}\left(\cancel{103.6}\right)$$

$$x = 135.4987$$

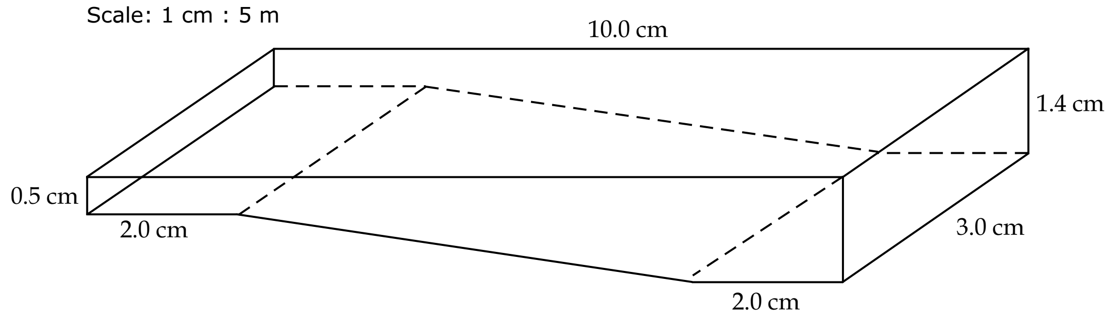
The farmer would need to order 135.5 cubic yards.

Example 6

The scale diagram shown below represents a pool drawn according to the given scale, 1 cm : 5 m. The diagram measurements are given.

a) Find the volume of water in the pool in m^3 if it is filled to within 0.3 m from the top.

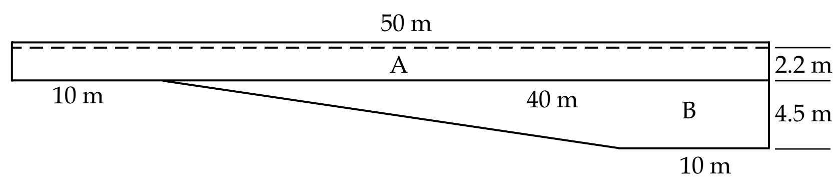
b) Find the surface area of the inside of the pool.



Solution

Convert the diagram measurements to actual measurements using the scale factor of 1 cm = 5 m. These measurements are recorded on the cross-section diagram shown below.

Cross-Section of Pool



a) The shape of the cross-section is consistent throughout the width of the pool, so this can be considered the base of a prism shape. To calculate the volume of a prism, the formula given is $V = Bd$ where B is the area of the base and d is the depth of the prism (which is in this case the width of the pool, 3.0 cm = 15 m).

The shape of the base is composed of a rectangle, A, and a trapezoid, B. The height of rectangle A is 2.2 m since the water is 0.3 m from the top.

$V_A = Bd$

$V_A = (50)(2.2)(15)$

$V_A = 1650$

Recall: $A_{\text{trapezoid}} = \dfrac{(a+b)h}{2}$

where *a* and *b* are the lengths of the parallel sides and *h* is the height.

$$V_B = Bd$$

$$V_B = \frac{(a+b)h}{2}d$$

$$V_B = \frac{(40+10)(4.5)}{2}15$$

$$V_B = 1687.5$$

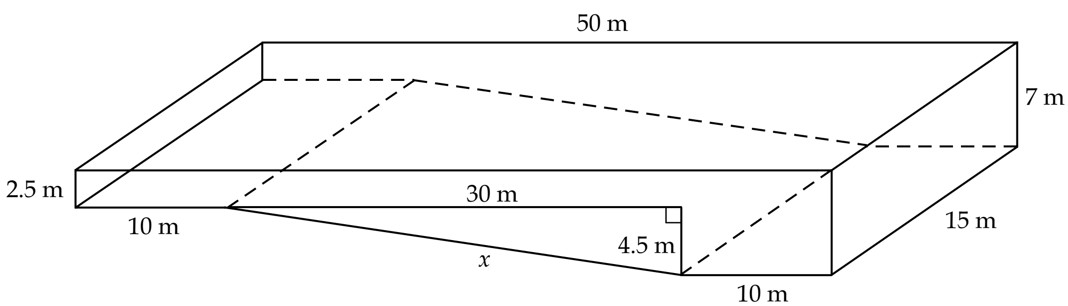
$$1650 + 1687.5 = 3337.5$$

The volume of water required to fill the pool to within 0.3 m from the top is 3337.5 m^3.

b) The surface area of the inside of the pool is the sum of the areas of each face.

The floor is a rectangle with width of 15 m and a length of 10 + 10 + sloped edge.

In order to determine the length of the sloped edge in the cross-section of the pool, use the Pythagorean theorem.



$30^2 + 4.5^2 = x^2$

$920.25 = x^2$

$x = 30.33562262$

Floor area: $(10 + 30.33562262 + 10) \times 15 = 755.0343394$

Deep end wall area: $7 \times 15 = 105$

Shallow end wall area : $2.5 \times 15 = 37.5$

Side areas: $(10 \times 7) + (2.5 \times 40) + \dfrac{(4.5 \times 30)}{2} = 237.5$

Since there are two walls, add this value twice.

Total surface area: $755.0343394 + 105 + 37.5 + 2(237.5) = 1372.534339$

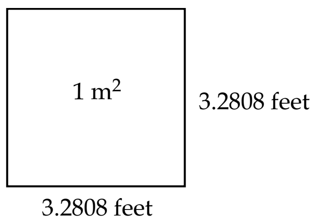
The surface area inside the pool is approximately 1372.5 m^2.

Example 7

Commercial painters purchase paint in gallon pails. One gallon of paint covers approximately 200 square feet of pool surface. How many gallons should the painters purchase in order to have enough for two coats of paint on the pool from Example 6?

Solution

The painters need to paint two coats on an area of 1372.5 m^2. This means they need to purchase enough paint for 2745 m^2. To calculate the number of gallons of paint needed, convert this area measurement to square feet.



Note: 1 m = 1.0936 yd.

1 yd. = 3 ft., so 1.0936 yd. = 3.2808 ft.

Therefore, 1 m = 3.2808 ft. and 1 m^2 = 10.7636 $ft.^2$

$$\frac{10.7636 \text{ ft.}^2}{1 \text{ m}^2} = \frac{x \text{ ft.}^2}{2745 \text{ m}^2}$$

$$x = 29546.21552 \text{ ft.}^2$$

$$\frac{29546.21552 \text{ ft.}^2}{200 \text{ ft.}^2/\text{gallon}} = 147.73 \text{ gallons}$$

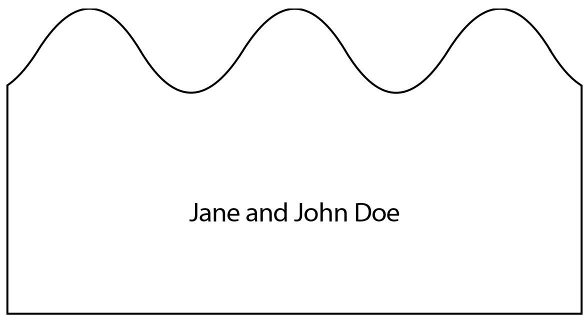
The painters will need to purchase at least 148 gallons of paint to give this pool two coats of paint.

Measurements of Complex Objects

You have completed length, area, surface area, and volume calculations on both regular and composite 2-D and 3-D shapes. Objects are not always constructed using regular shapes such as triangles, rectangles, trapezoids, parallelograms, circles, hexagons, octagons, or others. You can, however, use the perimeter, area, surface area, and volume of regular-shaped objects to estimate the approximate corresponding values in complex objects.

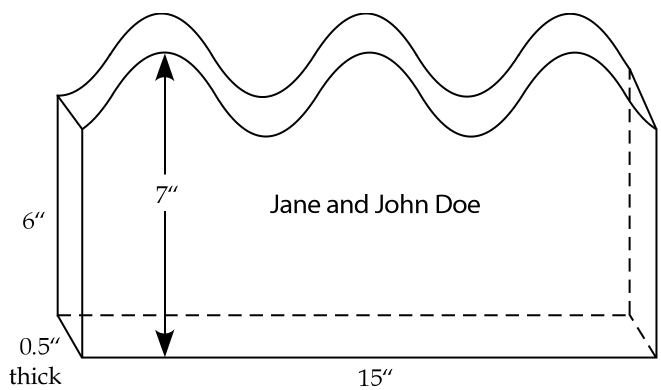
Example 8

A signmaker makes signs for families in the shape shown below.



This sign could be thought of as being rectangular in shape. However, the waved top side makes it irregular and not a true rectangle. If you need to know the perimeter, area, surface area, or volume of the sign, an approximation you could use is that of a rectangle. Methods of compensating for error will be discussed in later examples.

Measurements were given as follows:



You want to know the surface area of the sign in order to calculate the amount of paint needed to paint the complete sign.

Solution

You can treat the sign as a rectangle where l = 15″ and w = 6.5″ (average of the 6″ and 7″ measurements). The surface area of the front and back of the sign would then be approximately $2 \times 15 \times 6.5 = 195$ in.^{2}.

The area of the sides could be estimated using the approximate height of 6.5″ multiplied by the depth of 0.5″. Add this value twice for left and right sides. The top and bottom areas could be first estimated as 15″ × 0.5″ and then doubled.

Total surface area = $195 + 2(6.5 \times 0.5) + 2(15 \times 0.5)$

Total surface area = 216.5 in.^{2}

Since you know that the top edge is longer than the bottom edge, due to its waved shape, you may round the total up to bring it to 220 in.^{2} to give a reasonable approximation of the total surface area of this sign.

Learning Activity 8.1

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Simplify: $\frac{8!}{5!3!}$
2. You have a circle-shaped pie and need to cut it into exactly seven pieces. How can this be accomplished using exactly three straight slices with a knife? (The pieces may be unequal in size.)
3. You have a round cake and need to cut it into exactly eight pieces. How can this be accomplished using exactly three straight slices with a knife?
4. Write using factorial notation: $7 \times 6 \times 5 \times 4$
5. If 4 identical items cost $2.32, how much will 5 of the items cost?
6. A drink and candy cost $1.10. The drink costs one dollar more than the candy. How much does the candy cost?
7. Determine the next three letters in this sequence: O T T F F ___ ___ ___
8. Terry is a firefighter and works 12-hour shifts for four days and then has four days off. He starts work today after having yesterday off. How many days will he work over the next 30 days?

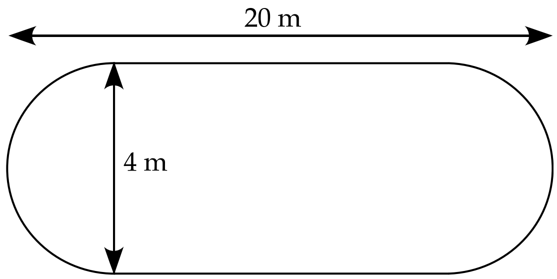
continued

Part B: Perimeter, Area, and Volume

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

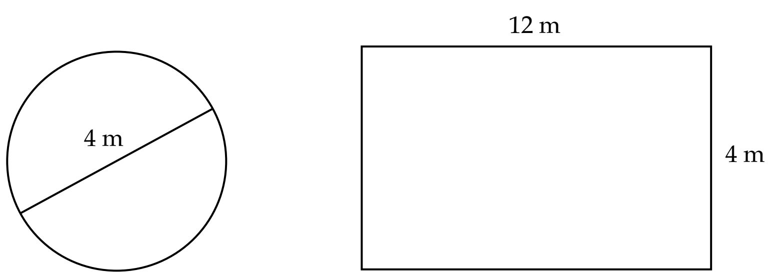
1. Compare the volume of a prism with the volume of a pyramid, which has the same base and height as the prism. Give an example to support your answer and show your calculations.

2. An ice rink has the following dimensions:



Given the task of finding the area covered by ice in the skating rink, pictured above, Aisha wrote the following solution. What went wrong? Find the error(s) Aisha made and correct them. Find the actual area of the ice.

There are two semi-circles (which make a circle) and a rectangle.



$A_{\text{circle}} = \pi r^2$
$A_{\text{circle}} = \pi(2)^2$
$A_{\text{circle}} = 4\pi$
$A_{\text{circle}} = 12.57 \text{ m}^2$

$A_{\text{rectangle}} = l \times w$
$A_{\text{rectangle}} = 4 \times 12$
$A_{\text{rectangle}} = 48 \text{ m}^2$

The total area of the ice surface is $12.57 + 48 = 60.57 \text{ m}^2$.

continued

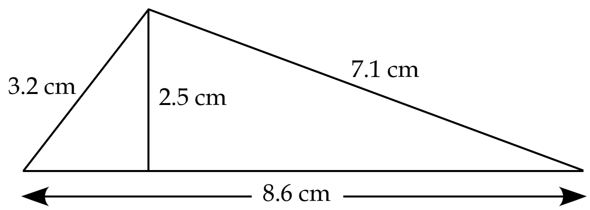
Learning Activity 8.1 (continued)

3. Using the formulas for perimeter/circumference, area, surface area, and volume of 2-D and 3-D figures, calculate the indicated values. State appropriate units.

   a) Find:

      i) perimeter

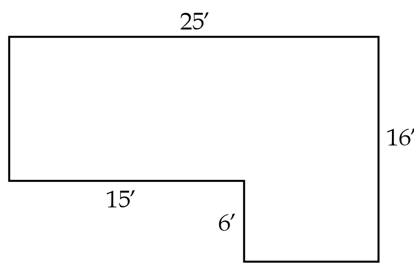
      ii) area



   b) Find:

      i) perimeter

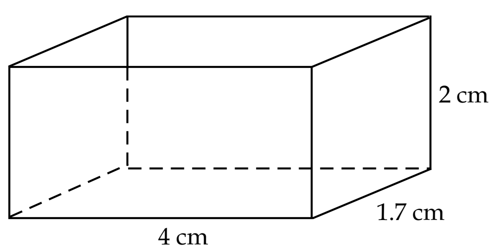
      ii) area



   c) Find:

      i) surface area of the open box in $ft.^2$

      ii) volume of the box in $ft.^3$



(Diagram measurements are shown.)

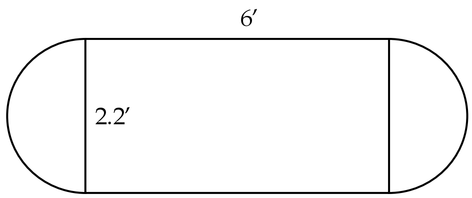
Scale: 1 cm = 2 ft.

continued

d) Find:

i) surface area of the tank

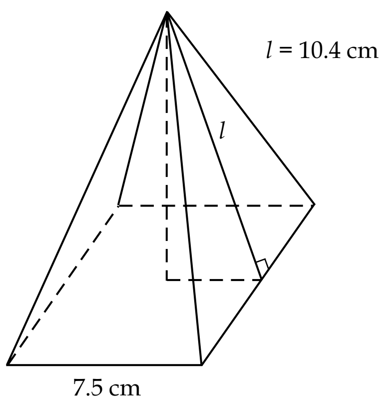
ii) volume of the tank



e) Find:

i) surface area of the square pyramid

ii) volume of the square pyramid



4. The volume of a cone is 375.6 cm^3. The circumference of the base is 33.9 cm. Find:

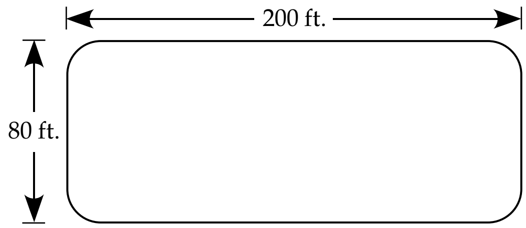
a) the height of the cone

b) the total surface area of the cone

continued

Learning Activity 8.1 (continued)

5. A hockey rink is shaped as follows:



   a) Estimate the area of the ice surface.
   b) Estimate the volume of ice in the rink if the ice is two inches thick. Calculate in cubic yards.

6. A water tank is a sphere with a diameter of 3.6 m. Estimate the volume of water in the tank if the depth of the water is 24 cm.

Lesson Summary

In this lesson, you solved problems involving perimeter, area, and volume of 2-D and 3-D figures. You corrected errors in a solution to a problem that involved the dimensions of objects. Using simplified models, you estimated the solutions to complex measurement problems.

Notes

Lesson 2: Costing a Project

Lesson Focus

In this lesson, you will

- ❑ solve a problem involving perimeter, area, and volume using dimensions and unit prices
- ❑ solve a problem involving estimation and costing for objects, shapes, or processes when a design is given
- ❑ identify and correct errors in a solution to a problem that involves costing for objects, shapes or processes

Lesson Introduction

Cost analysis is a factor in the completion of a construction project. This can take the form of simply determining the cost of the construction, as you will do in this lesson. Alternatively, it may involve determining the scope and design of a project when it is constrained by a given budget, as you will consider in the next lesson. In any case, staying within the budget on a given project is an important aspect of any construction.

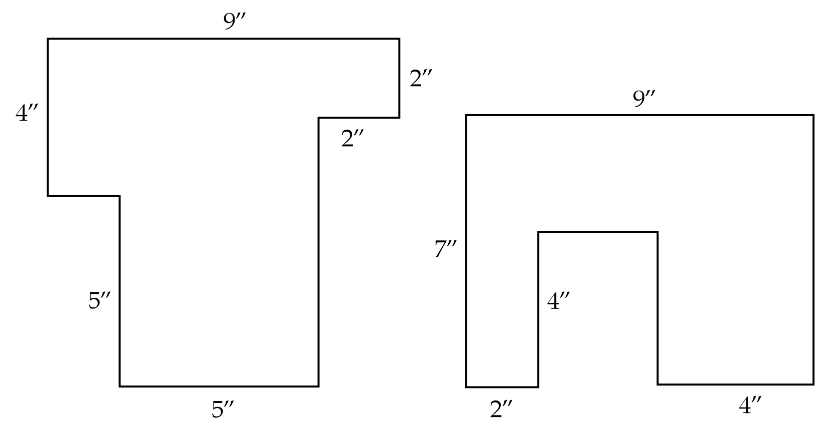
Costing a Project

Cost Analysis

When solving cost-analysis problems, it is important to show all the steps in your solution process and organize your work carefully. Make sure units are consistent when considering dimensions and measurements, and carefully consider length, area, and volume conversions within and between measurement systems. Remember that to purchase construction materials, you often must buy whole units, rather than partial amounts (i.e., if you need 1.3 pails of paint, you must buy two pails), and taxes will vary depending on whether you purchase goods or services. There are many ways to solve design problems—be creative. If your answers vary from what is shown, analyze both solutions and determine the advantages and disadvantages for each and assumptions on which each solution is based.

Example 1

A tinsmith is constructing parts for the heating ducts in houses. A pattern for the two pieces that he fits together to create these parts is shown in the following diagram. Each part consists of both pieces.

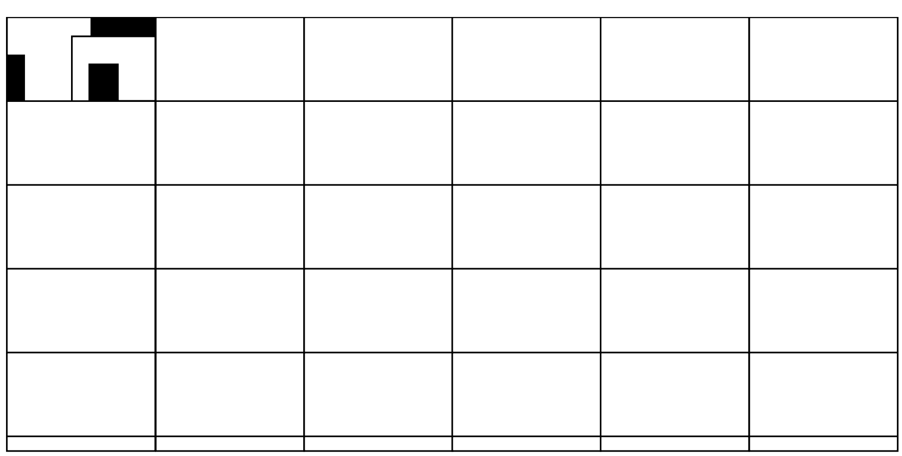


a) If he buys tin in 4′ × 8′ sheets, how many of the parts can he make from one sheet of tin?

b) If the 4′ × 8′ sheet of tin costs $120 and he pays himself $15 labour for each part that he makes, how much should he charge for each part in order to cover the cost of material and labour?

Solution

a) If the two pieces are put together so that the shorter one fits under the taller one, the overall rectangular dimensions will be 16″ long by 9″ wide. If these pieces are placed on the sheet of tin so that the 16″ side runs along the 8′ or 96″ length of the sheet, the tinsmith can cut exactly six sets of pieces in one row. The 9″ side will then run along the 4′ or 48″ side of the sheet, and it will be possible to cut five parts from one column. There will be a total of 6 × 5 = 30 sets of pieces or 30 parts cut from one sheet (see diagram).



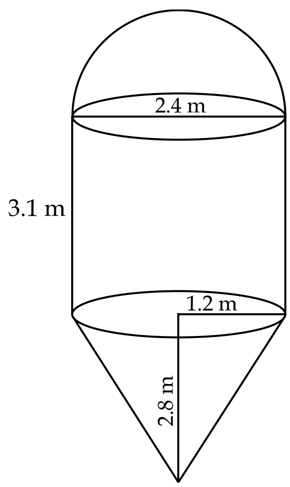
b) Cost of materials for each part is $\frac{\$120}{30} = \$4.00$.

He should charge \$4.00 + \$15.00 = \$19.00 for each part.

He would charge more than this to make a profit.

Example 2

A grain bin is made of aluminum in the following shape:



Aluminum costs \$6.00 a square metre for the half spherical top and cylindrical wall. The aluminum for the conical bottom costs \$9.50 per square metre. Paint for the bin costs \$8.75 a litre and a litre covers 12 m^2. How much would it cost to build the bin if labour costs are ignored and it is painted with two coats of paint? Include 5% GST and 8% PST.

Solution

Slant height (l) of the cone is calculated using the Pythagorean theorem.

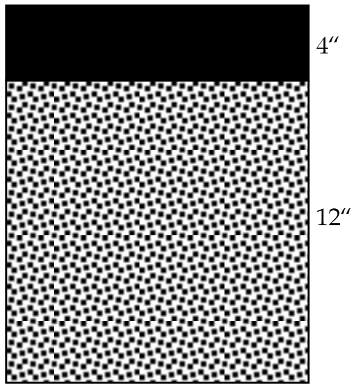
$l^2 = 2.8^2 + 1.2^2$

$l = 3.05$

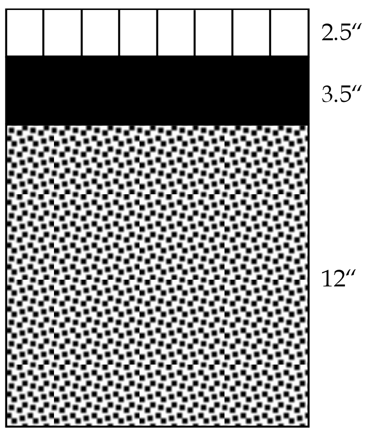
Surface area:	Top	9.05 m^2	$(4\pi(1.2)(1.2) \div 2)$
	Cylinder	23.37 m^2	$(2\pi(1.2)(3.1))$
	Cone	11.498 m^2	$\left(\frac{1}{2}(2\pi)(1.2)(3.05)\right)$
	Total	43.9 m^2	
Cost of metal:	Top	$9.05 \times \$6.00 = 54.30$	
	Cylinder	$23.37 \times \$6.00 = \$140.22$	
	Cone	$11.498 \times \$9.50 = \$109.23$	
	Total	\$303.75	
Cost of paint:	# of litres	$= \frac{(43.9 \times 2)}{12} = 7.32$ or 8 litres	
	Cost	$8 \times \$8.75 = \$70.00$	
Total cost:	$\$(303.75 + 70) \times 1.13 = \$422.28$		

Example 3

A driveway is to be hard-surfaced in one of two ways. The first method is to use concrete 4″ thick over a layer of gravel 1 ft. deep.



The second method is to use paving stones, which must be placed over 1 ft. of crushed limestone covered by 3.5″ of sand. The paving stones are 2.5″ thick.



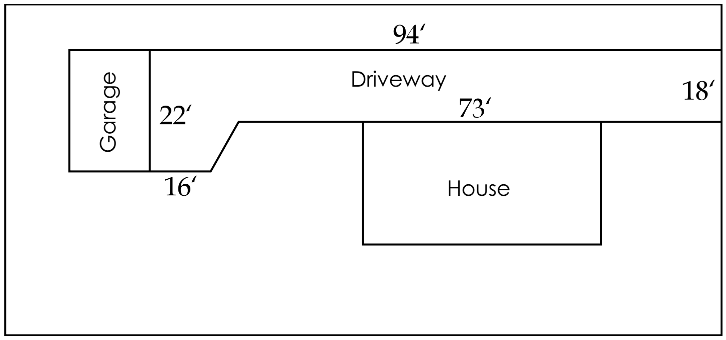
For each method of hard surfacing, the driveway will have to be excavated to the necessary depth.

Excavation costs:	$60.00/yd.^{3}

The costs of materials are as follows:

Gravel:	12.00/yd.^{3}
Concrete:	$130.00/yd.^{3}
Crushed limestone:	$15.00/yd.^{3}
Sand:	$4.00/yd.^{3}
Paving stones:	$2.25/ft.^{3}

If labour costs and taxes are ignored in the pricing, and you should allow 10% extra for paving stones to allow for breakage and cutting, compare costs for the two methods to complete the driveway shown in the following diagram.



Solution

The total area of the driveway can be found by adding the area of the rectangle that has a length of 94′ and a width of 18′ to the area of the trapezoid with $a = 16'$, $b = 94 - 73 = 21'$, and $h = 22 - 18 = 4'$. The result is 1692 + 74 = 1766 ft.^{2}. The necessary volumes can be found by multiplying this area by the required depth.

Case 1: Concrete and Gravel

Recall: 1 yd.^{3} = 27 ft.^{3}

Excavation costs (16″ deep): $1766 \times \frac{16}{12} = 2354.7 \text{ ft.}^3$

$\frac{2354.7}{27} = 87.2 \text{ yd.}^3$

$87.2 \times \$60.00 = \$5232.59$

Gravel costs (12″ deep): $1766 \times \frac{12}{12} = 1766 \text{ ft.}^3$

$\frac{1766}{27} = 65.4 \text{ yd.}^3$

$65.4 \times \$12.00 = \$784.89$

Concrete costs (4″ deep): $1766 \times \frac{4}{12} = 588.67 \text{ ft.}^3$

$\frac{588.67}{27} = 21.8 \text{ yd.}^3$

$21.8 \times \$130.00 = \$2834.32$

Total costs: $\$5232.59 + \$784.89 + \$2834.32$

$= \$8851.80$

In many cases, you may be required to buy whole numbers of cubic yards of gravel and concrete. It is appropriate to think of your answer as an estimate of the cost. You may even write "Estimated cost is $8860.00."

Case 2: Limestone, Sand, and Paving Stones

Excavation costs (18″ deep):

$$1766 \times \frac{18}{12} = 2649 \text{ ft.}^3$$

$$\frac{2649}{27} = 98.1 \text{ yd.}^3$$

$$98.1 \times \$60.00 = \$5886.67$$

Limestone costs (12″ deep):

$$1766 \times \frac{12}{12} = 1766 \text{ ft.}^3$$

$$\frac{1766}{27} = 65.4 \text{ yd.}^3$$

$$65.4 \times \$15.00 = \$981.00$$

Sand costs (2.5″ deep):

$$1766 \times \frac{3.5}{12} = 515.08 \text{ ft.}^3$$

$$\frac{515.08}{27} = 19.1 \text{ yd.}^3$$

$$19.1 \times \$4.00 = \$76.31$$

Paving stone costs:

$$1766 \times 1.10 = 1942.6 \text{ ft.}^2$$

$$1942.6 \times \$2.25 = \$4370.85$$

Total costs:

$$\$5886.67 + \$981.00 + \$76.31 + \$4370.85$$

$$= \$11\,314.83$$

Estimated cost is $11,350.00.

Learning Activity 8.2

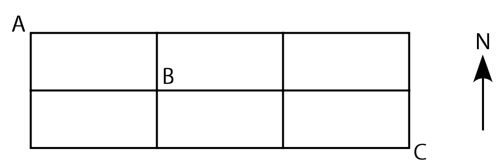
Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. All of Mr. Popper's pets are penguins, except one. All his pets are parrots, except one. How many pets does he have?
2. Knowing there are 24 hours in a day, calculate the number of hours in a week.
3. Estimate the area of a semi-circle with a 20 cm diameter.
4. How many cm^2 are equal to one m^2?
5. Ten striped socks and ten checkered socks are all mixed up in a drawer. If you close your eyes and randomly pick out one sock at a time, what is the smallest number of socks you will have to pull out of the drawer to guarantee you have a matching pair?

Use the following diagram to answer the Questions 6 to 8. The diagram shows a partial map of the pathways in a city park.



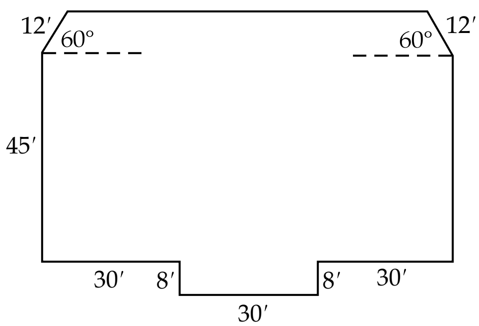
6. If a jogger begins at point A and runs only south and east, in how many different ways can she arrive at point B?
7. If a jogger begins at point A and runs only south and east, in how many different ways can she arrive at point C?
8. What is the probability that the jogger passes by point B on her way to point C?

continued

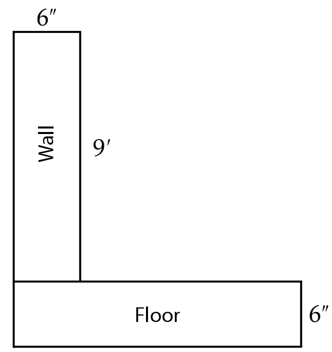
Part B: The Costs of Construction

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. An organization is pouring a concrete basement for a new office building. The dimensions are as follows:



Top View of the Floor

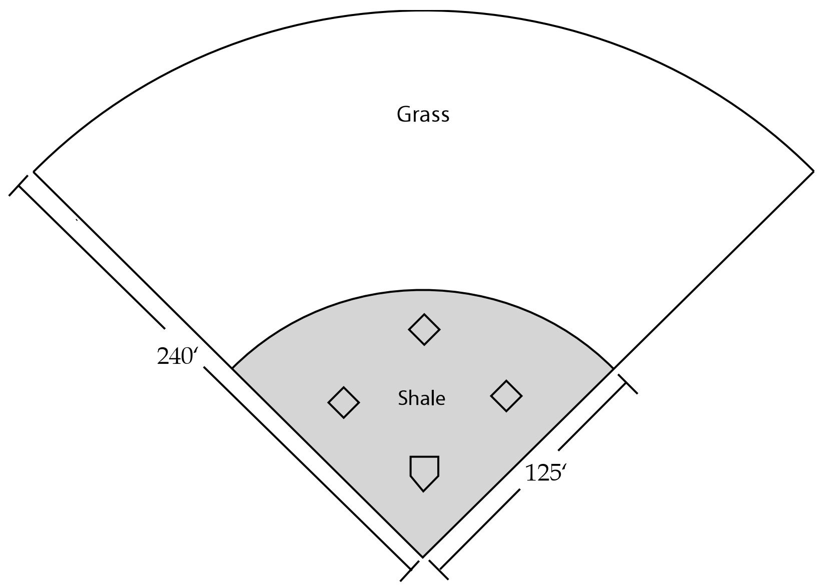


Side View of Basement Wall

The walls and floor are made of concrete and are six inches thick. The basement walls sit on top of the floor, as shown in the side-view diagram. It takes two workers and one foreman one hour to pour 200 cubic feet of concrete. Concrete costs \$120 per cubic yard. The workers make \$19.00 per hour and the foreman makes \$24.50 per hour. GST of 5% and PST of 8% apply to the concrete, and only GST applies to the wages paid. How much will it cost to build the basement?

continued

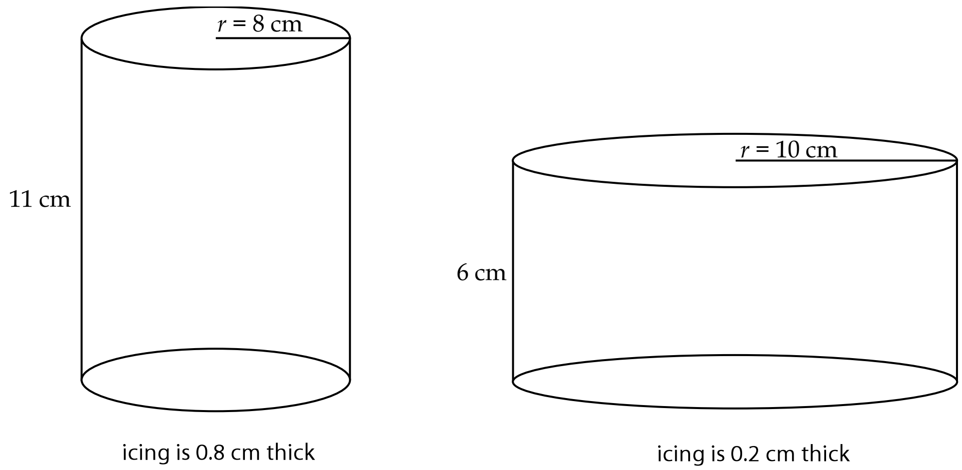
2. A ballpark is to be built according to the diagram. A fence is to be put up around the entire park. Fencing costs $1.35 per running foot. The infield area, which is to have shale on it, needs to be dug down 18″ so that one foot of gravel can be put down as a base and then 6″ of shale put on top. The cost of excavation is $5.00 per cubic yard. Gravel costs $7.00 per cubic yard and shale costs $9.50 per cubic yard. The remaining area of the park needs to have 2″ of topsoil on it and sod put on top of the soil. Topsoil costs $20 per cubic yard and sod costs $1.25 per square foot. The grass needs to have fertilizer as soon as it is laid. The fertilizer costs $15.60 per bag and a bag will cover 75 square yards. If PST at 8% and GST at 5% apply to all materials used and only GST applies to the excavation costs, find the total cost, excluding labour, for this project.



continued

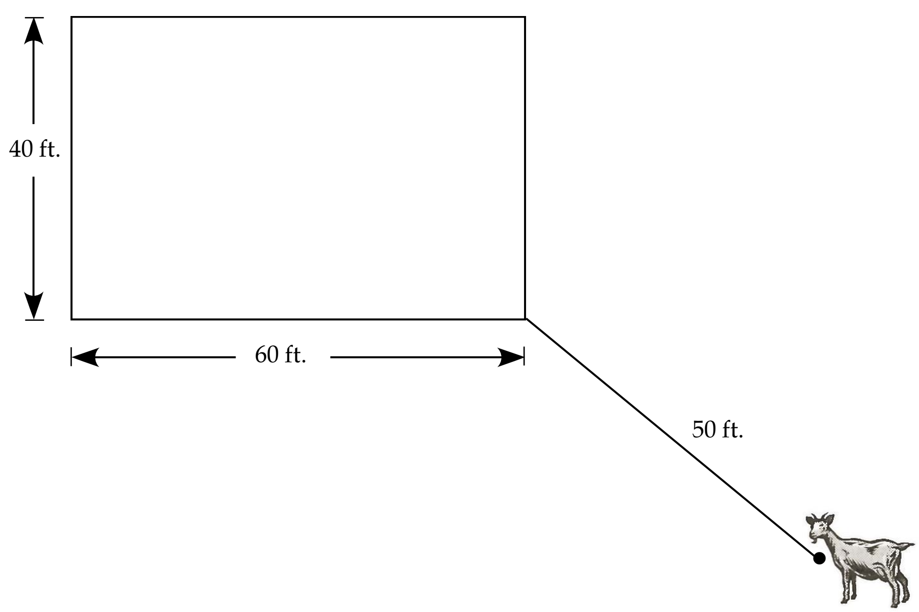
Learning Activity 8.2 (continued)

3. A bake shop has two differently sized cakes, as shown.



   The higher cake has a thick icing; the shorter cake has a thin icing. There is no icing on the bottom of the cakes. The icing costs are $2.00 for each 100 cm^3. Five mL of cake batter makes 16 cm^3 of cake and the batter costs $1.25 per 100 mL. The cakes are placed in square-based boxes with lids that allow a 2 cm space on each side and the top of each cake. If the boxes are made from cardboard that costs $3.00 per square metre, what is the total cost of each cake, including its box? Which is the better buy based on price per unit volume?

4. A goat is tied to the corner of a barn with a 50-foot rope. The barn measures 60 feet by 40 feet. Calculate the total area outside of the barn that can be reached by the goat. Show your work.



continued

Learning Activity 8.2 (continued)

5. You have been hired to finish someone's living room. The room is 23′ by 14′, and has two doorways and four windows.
   a) Calculate the cost of paint for the walls of the living room, including taxes. Use one coat of paint primer and two coats of finishing paint. (GST = 5%, PST = 8%)
      - The walls are 8 feet high.
      - The doors measure 3 ft. × 6 ft. 8 in.
      - The windows measure 4 ft. × 2 ft.
      - One gallon of paint primer costs $29.95 and covers 300 $ft.^2$
      - One gallon of finishing paint costs $44.95 and covers 400 $ft.^2$

      Paint and primer must be purchased in whole units.
   b) Calculate the number of boxes of wood flooring needed to cover the living room floor.
      - One box of flooring covers 2.4 square yards.
      - You will need an extra 5% of flooring to account for waste.

Lesson Summary

In this lesson, you solved problems involving perimeter, area, and volume using dimensions and unit prices. You solved problems involving estimation and costing for objects, shapes, or processes when a design was given.

Notes

Assignment 8.1

Design and Cost Decisions

Total: 22 marks

This is a hand-in assignment. Please clearly show your work and include units with final answers. Answers given without supporting calculations will not be awarded full marks.

1. You have been asked to install floor tiles and paint your aunt's bathroom based on the following information:
   - The floor measures 5 ft. × 7 ft.
   - The walls are 8 ft. high.
   - The door measures 80 in. × 30 in.
   - The window measures 24 in. × 30 in.

   a) You must cover the entire bathroom floor with tiles. Each tile measures 1 ft. × 1 ft. You will need an extra 5% of tiles to account for waste. How many tiles will you need to purchase for the project? *(2 marks)*

Assignment 8.1: Design and Cost Decisions (continued)

b) You must apply two coats of paint to the walls of the bathroom. The door and the window will not be painted. Determine the total area to be painted. How many cans of paint will you need to purchase if one can covers 100 $ft.^2$? Show your work. *(4 marks)*

Assignment 8.1: Design and Cost Decisions (continued)

2. A cake mix will produce 230 cubic inches of batter. You are using cylinder-shaped baking cups that have a diameter of 3 inches and a depth of 2 inches for the batter. How many cupcakes will you be able to make? Show your work. *(3 marks)*

3. To install an in-ground hot tub in his backyard, Neil must hire a crew of excavators with machinery to dig a 6 ft. by 8 ft. area to a depth of 4 ft. The excavators charge $\$120/\text{m}^3$ to do the work. Neil determines that it will cost $56.54 to have the hole dug based on the following calculations. The crew gives him a quote of $700 to do the work. What errors did Neil make in his calculations? Correct the errors and determine if the quote from the crew is accurate. *(4 marks)*

   Volume of dirt to be removed: $6 \times 8 \times 4 = 192 \text{ ft.}^3$

   $1 \text{ ft.} = 0.3048 \text{ m}$ so $1 \text{ ft.}^3 = 0.0283 \text{ m}^3$

   $$\frac{1 \text{ ft.}^3}{0.0283 \text{ m}^3} = \frac{x}{192 \text{ ft.}^3}$$

   $$x = \frac{1}{0.0283}(192)$$

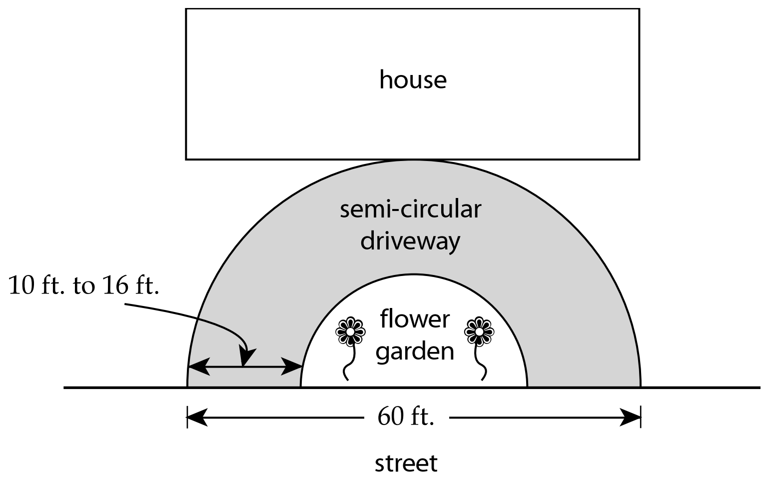
   $$x = 6784.45 \text{ m}^3$$

   Calculate cost as $6784.45 \div 120$.

   It should cost about $56.54 to have the hole dug.

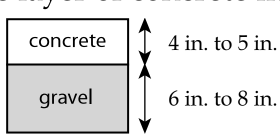
Assignment 8.1: Design and Cost Decisions (continued)

4. The Beliveau family has asked you to construct a semi-circular driveway in front of their house. They have provided you with the following diagram. (Diagram is not drawn to scale.)



Here is the information that you will need:

- a hole must be dug to accommodate the depth of gravel and concrete
- the driveway must be between 10 ft. and 16 ft. wide
- the layer of gravel must be between 6 in. and 8 in. thick
- the layer of concrete must be between 4 in. and 5 in. thick



- excavation costs $5.00 per cubic yard of soil removed
- gravel costs $18.00 per cubic yard
- concrete costs $170.00 per cubic yard

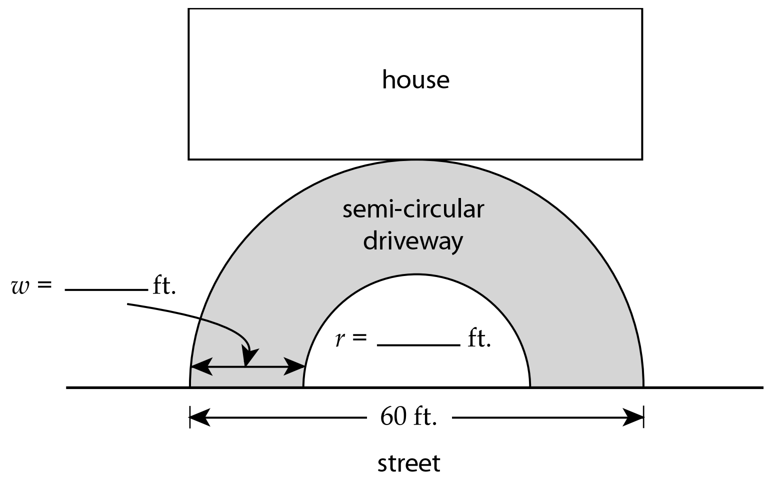
Note: 1 cubic yard = 27 cubic feet

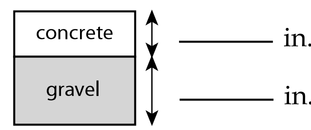
All materials and labour for excavation must be purchased in whole units.

Only GST is charged on excavation, while both GST and PST are charged on the gravel and the concrete. (GST = 5%, PST = 8%)

Assignment 8.1: Design and Cost Decisions (continued)

a) Indicate the width and interior radius of the driveway as well as the depths of the gravel and concrete on the diagrams below. (Diagrams are not drawn to scale.) *(1 mark)*





b) Calculate the total cost to construct the semi-circular driveway. *(5 marks)*

Assignment 8.1: Design and Cost Decisions (continued)

c) The Beliveau family asks you to add a flower garden with cedar wood chips, flowers, shrubs, and lighting between the driveway and the street. You are given the following instructions:

- The flower garden must be covered with cedar wood chips that cost $7.49 per bag. Each bag covers 12 $ft.^2$ for the required depth.
- The cost for the flowers and shrubs is $150.00.
- Excavation is not required.
- Lighting must be placed along the entire edge of the flower garden, including along the street. The lights are connected by an electrical cable that is 12 ft. long. Each cable with its attached lights costs: $32.99.

All materials must be purchased in whole units.

Both GST and PST are charged on all the supplies for the flower garden. (GST = 5%, PST = 8%)

Calculate the total cost to add the flower garden, including taxes. *(3 marks)*

Notes

Lesson 3: Designing within a Budget

Lesson Focus

In this lesson, you will

- ❑ design an object, shape, layout, or process within a specified budget

Lesson Introduction

In construction, you must consider the costs as you create your design. Major construction designs may undergo a number of adjustments before construction begins. These changes may be due to budgeting constraints. If there is a set amount of funding available, the project may need to be modified to fit within the restrictions. How can you design a project and stay within the budget? One strategy may be to find the cost per square unit or cubic unit of the item under construction and work backwards from the budget amount to set the maximum or minimum values. Keep in mind that when working with area and volume, changing one dimension slightly may have a far-reaching impact on further calculations.

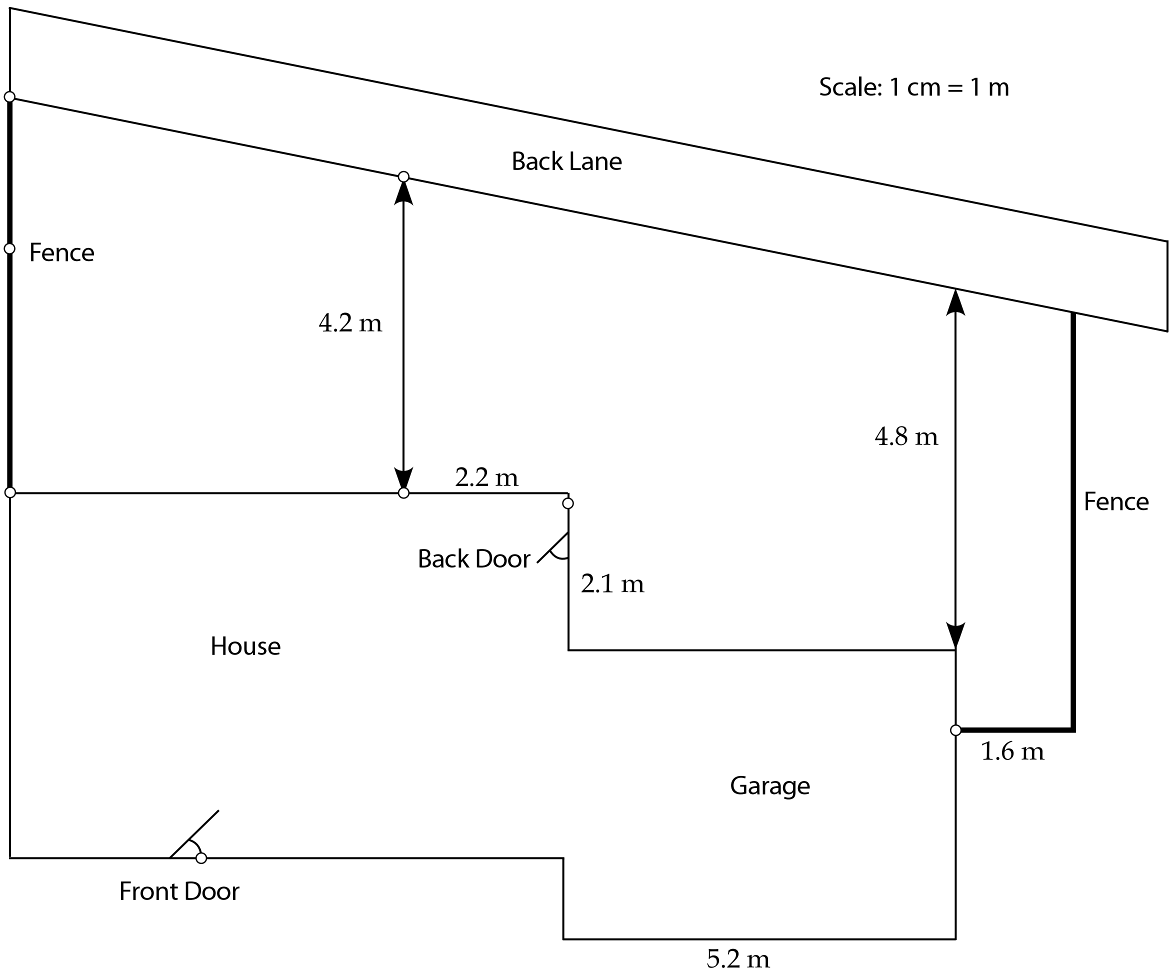
Designing within a Budget

Consider a Project Budget

In design questions involving a budget, there is room for your creativity and imagination. Be sure to state your assumptions and show all your work in an organized manner as you work through the examples, and compare your process to the sample solutions provided. Self evaluation will be an important aspect of completing the assignments in this lesson.

Example 1

You plan to build a patio in the backyard, shown below, and have budgeted $1000.00 to complete this construction project. The patio is to be made of paving stones in such a way so that it is as large as possible.



Paving stones cost $50.00/m^2 and must be put on top of a 10 cm–deep foundation of sand. Sand costs $25.50/m^3. You must purchase an additional 10% of the paving stones to account for breakage and cutting of the stones. Include $75.00 for incidental costs. GST is 5% and PST is 8%. Since you are doing the work yourself, there will be no labour costs.

Design the largest functional patio that remains within budget. Provide a cost analysis of your patio. Identify and explain any assumptions you make. Make a scale diagram of your patio on the diagram of the yard above, using the scale indicated.

Solution

In order to find the possible size of the patio, you need to find the cost per square metre for the patio. Once you have this figure, you can then divide it into the available money to find the maximum possible area for the patio. The only task remaining will be to draw into the diagram any functional patio that fits the area. Functional issues could be: access to the back door, not too narrow or too wide, not on the back lane, etc.

Cost per metre calculation:

Sand costs: $\$25.50/m^3 \times 0.10m = \$2.55/m^2$.

Stones cost $\$50.00 \times 1.10 = \$55.00/m^2$ with the waste included.

Total cost per $m^2 = (\$55.00 + \$2.55) \times 1.13 = \$65.03$ with taxes.

Available money is $1000.00.

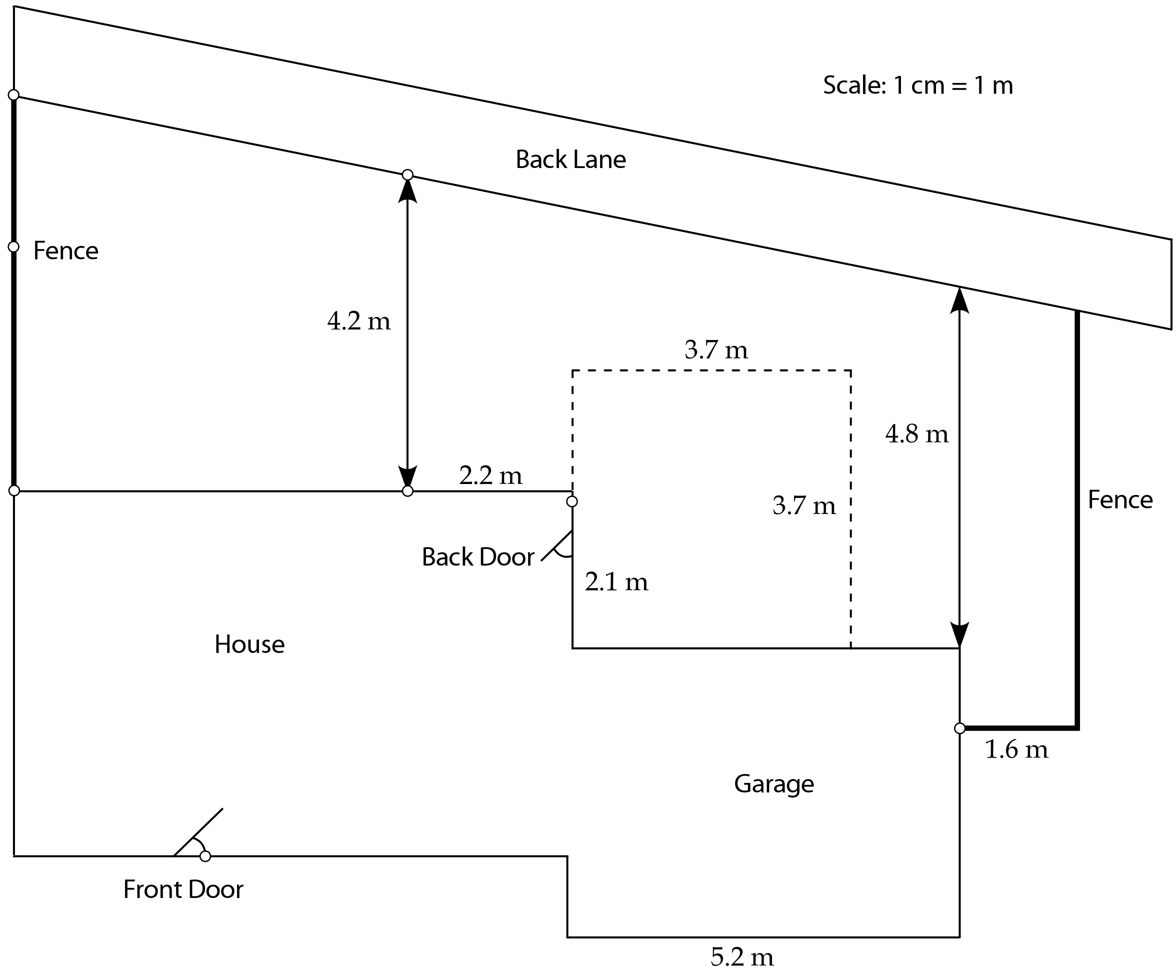
Subtract the incidentals with taxes included.

$\$75.00 \times 1.13 = \$84.75$.

Available: $\$1000.00 - \$84.75 = \$915.25$.

Maximum area of your patio is: $\dfrac{\$915.25}{\$65.03} = 14.07\ m^2$

This would indicate that your patio cannot be larger than $14.07\ m^2$. The shape is your own design. The design in the diagram is the square that has side length of $\sqrt{14.07} \approx 3.7$ m. Note that it has access to the back door and fits in the backyard. The patio is indicated in dotted line segments.



Other designs are possible.

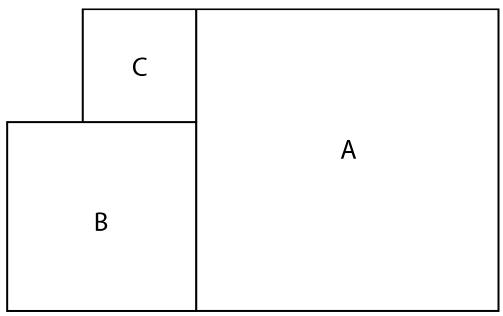
Learning Activity 8.3

Complete the following, and check your answers in the learning activity answer keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Figures A, B, and C are squares. If the perimeter of A is 36 units and the perimeter of C is 16 units, what is the perimeter of B?



2. State the formula for the area of a circle.
3. State the formula for the volume of a cylinder.
4. In how many ways can three boys wear five different baseball caps? (Assume they only wear one at a time.)

In a magic square, the sums of the entries in each row, column, and diagonal are all equal. In the magic square below, find the value of:

B	17	7
D	**A**	**C**
11	1	15

5. A
6. B
7. C
8. D

continued

Part B: Designing a Project within a Budget

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Your neighbours want to build a rectangular rink in their backyard. In order to do so, they must
   a) hire a person with levelling equipment to level the ground. The cost of this is $2.50 per square metre.
   b) flood it with water so that it will create ice to a depth of 12 cm. Ice expands to approximately 1.1 times the volume of water when it freezes. Water costs $2.00 per cubic metre.
   c) paint the ice white with ice paint that costs $1.05 per square metre.
   d) allow $25.00 for miscellaneous materials.
   e) allow for 5% GST and 8% PST for the water and other materials. Only GST applies to the levelling.

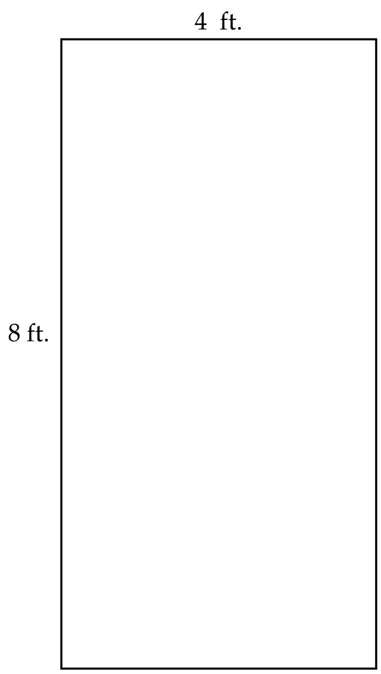
   Design and draw to scale a rink that would cost no more than $1000.00. Provide the scale on your diagram.

continued

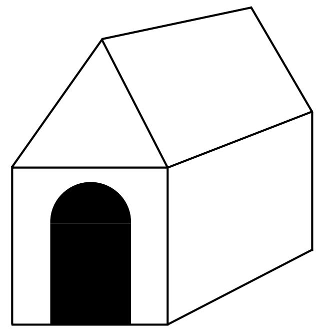
Learning Activity 8.3 (continued)

2. Design and find the dimensions of the largest functional doghouse of the type indicated that can be made from one 4-ft. × 8-ft. sheet of plywood. Show how the plywood would be cut to allow for your design. Indicate the cuts to scale in the scale diagram of the sheet of plywood that follows:

   Scale: 1 cm = 1 ft.



type of doghouse



3. Your track and field coach wants a long-jump pit with a concrete runway for the school. He has volunteers who will build it, but he needs your help to plan the design and calculate the cost of the materials.

   The long-jump pit must meet the following specifications:

   - The perimeter of the pit must be lined with boards to keep the sand separated from the grass.
   - The pit must be between 6 and 10 metres long inclusively.
   - The pit must be twice as wide as the runway.
   - The depth of the sand in the pit must be between 40 and 60 cm inclusively.
   - The runway must be at least 20 metres long and at least 1 metre wide.
   - The concrete in the runway must be between 10 and 15 cm thick inclusively.

continued

Learning Activity 8.3 (continued)

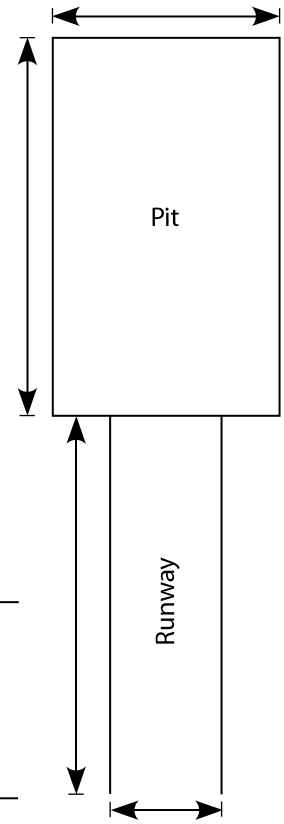
Indicate the dimensions of the pit and runway on the following diagram.

width of pit = ________

length of pit = ________

length of runway = ________

width of runway = ________



a) You will need to give the coach an estimate for the total cost of the materials using the following information:
   - A hole will need to be dug for the pit.
   - The depth of the hole should be equal to the depth of sand you plan on using.
   - Sand costs $28.50/m^3$.
   - 4.9 metre-long boards cost $32.50 each.
   - Concrete costs $130.00/m^3$.
   - The total cost for the long-jump pit must be between $1200.00 and $1500.00, including taxes. (GST = 5%, PST = 8%)

All materials must be purchased in whole units.

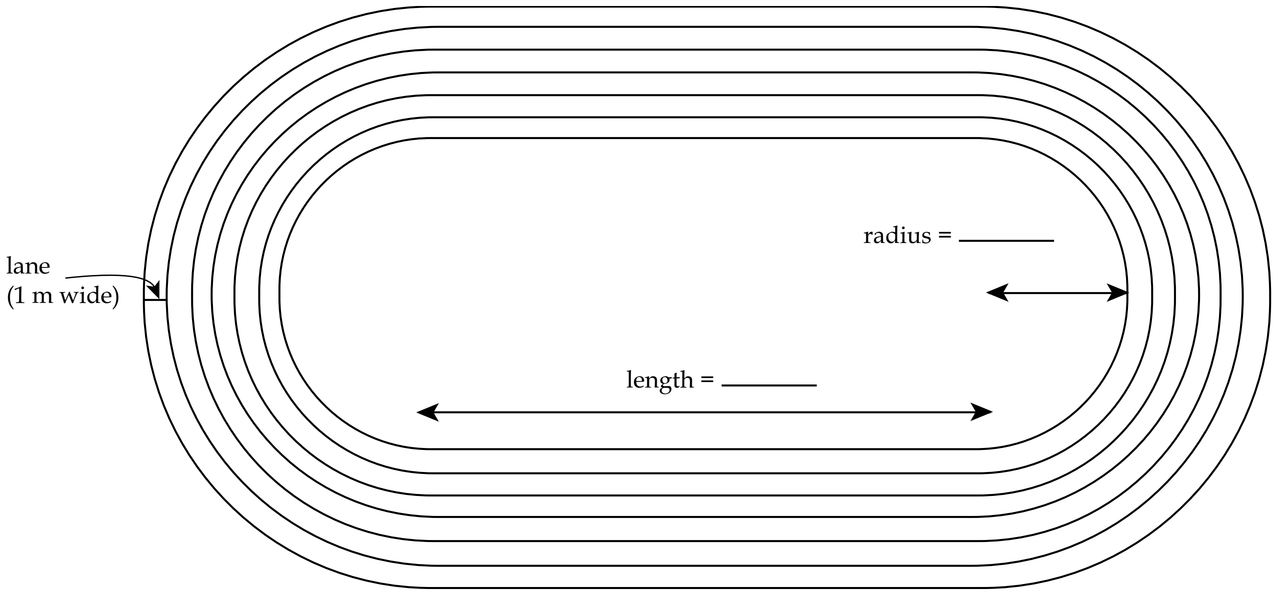
Keeping in mind the budget for this project, determine the total cost of the long-jump pit and runway, including taxes. Show your work.

continued

b) The coach also asks you to plan the design for a 6-lane track.

- The perimeter of the inside edge of the track must be 300 metres.
- Each lane must be 1 metre wide.

Determine the length of the straight portion and the radius of the curve for the track. Calculate the minimum dimensions of the rectangular field required to build the track. Show your work.



Lesson Summary

In this lesson, you designed objects, shapes, layouts, or processes within a specified budget.

Notes

Assignment 8.2

Working within a Project Budget

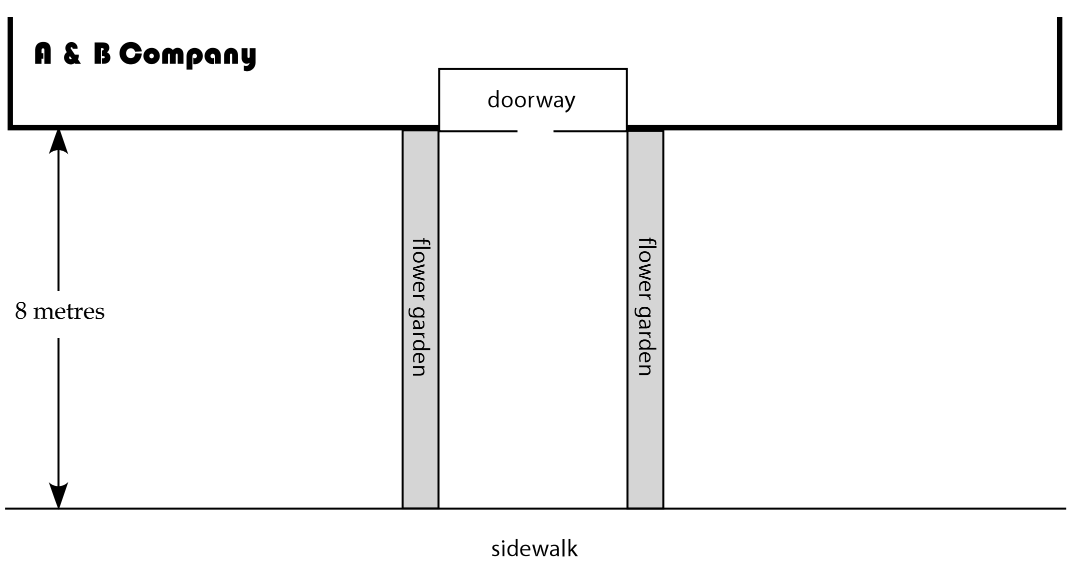
Total: 10 marks

This is a hand-in assignment. You can choose to submit **either** Question 1 **or** Question 2. Please organize your work neatly and show all calculations. State any assumptions you make. Include the appropriate units with your final answers. Answers given without supporting calculations will not be awarded full marks.

1. The A & B Company has hired you to do some landscaping in front of their building. The owners want you to design a walkway according to the following conditions. The walkway must
   - be 8 metres long and extend from the doorway to the sidewalk
   - be built using paving stones
   - have an area between 8 m^2 and 24 m^2 inclusively
   - have flower gardens on both sides

   All items must be purchased in whole units. The cost of the items and a sketch of the building are shown below.

Item	Cost
paving stones (25 cm × 25 cm)	\$8.50 each
soil	\$40.00 per m^3
cement	\$107.00 per m^3
water pump	\$325.00



Assignment 8.2: Working within a Project Budget (continued)

a) State the dimensions of your walkway design. *(1 mark)*

b) Determine the number of paving stones for your walkway. Show your work. *(2 marks)*

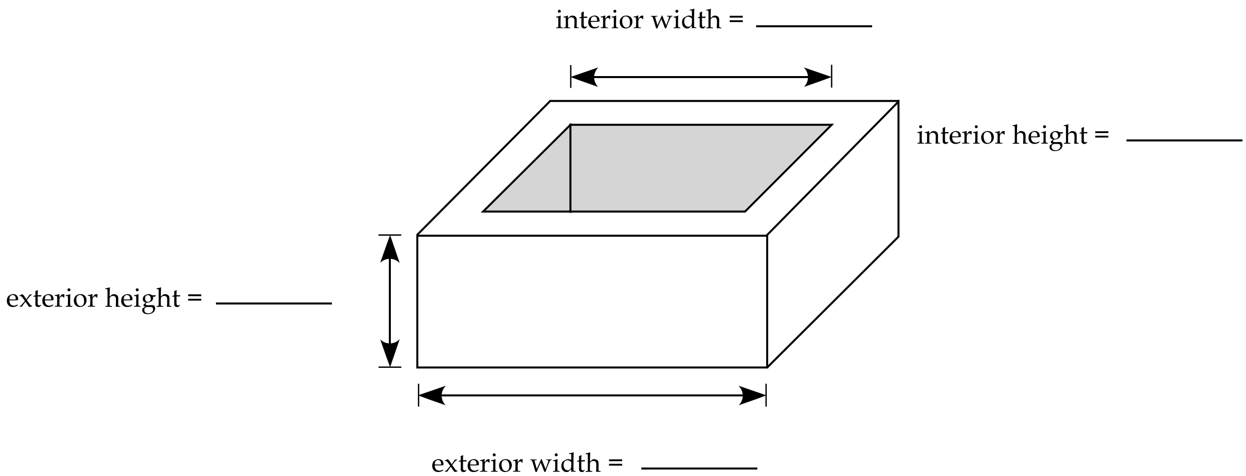
c) Determine the amount of soil you need for the flower gardens if each garden is 50 cm wide and requires a thickness of 30 cm of soil. Show your work. *(2 marks)*

Assignment 8.2: Working within a Project Budget (continued)

d) After completing the walkway and flower gardens, the owners want you to add a water fountain with a square base on the lawn. It will be made of cement. Here is the information that you will need:

- You can spend between $400.00 and $600.00.
- The interior of the fountain must be 50 cm deep.
- The walls of the fountain must be at least 25 cm thick.
- The square base of the fountain must be at least 25 cm thick.

Indicate the dimensions of your fountain, including units, on the diagram below. Determine the total cost of the cement and the water pump for your fountain, not including taxes. Show your work. *(5 marks)*



Assignment 8.2: Working within a Project Budget (continued)

Use this page for Question 1, part (d), if required.

Assignment 8.2: Working within a Project Budget (continued)

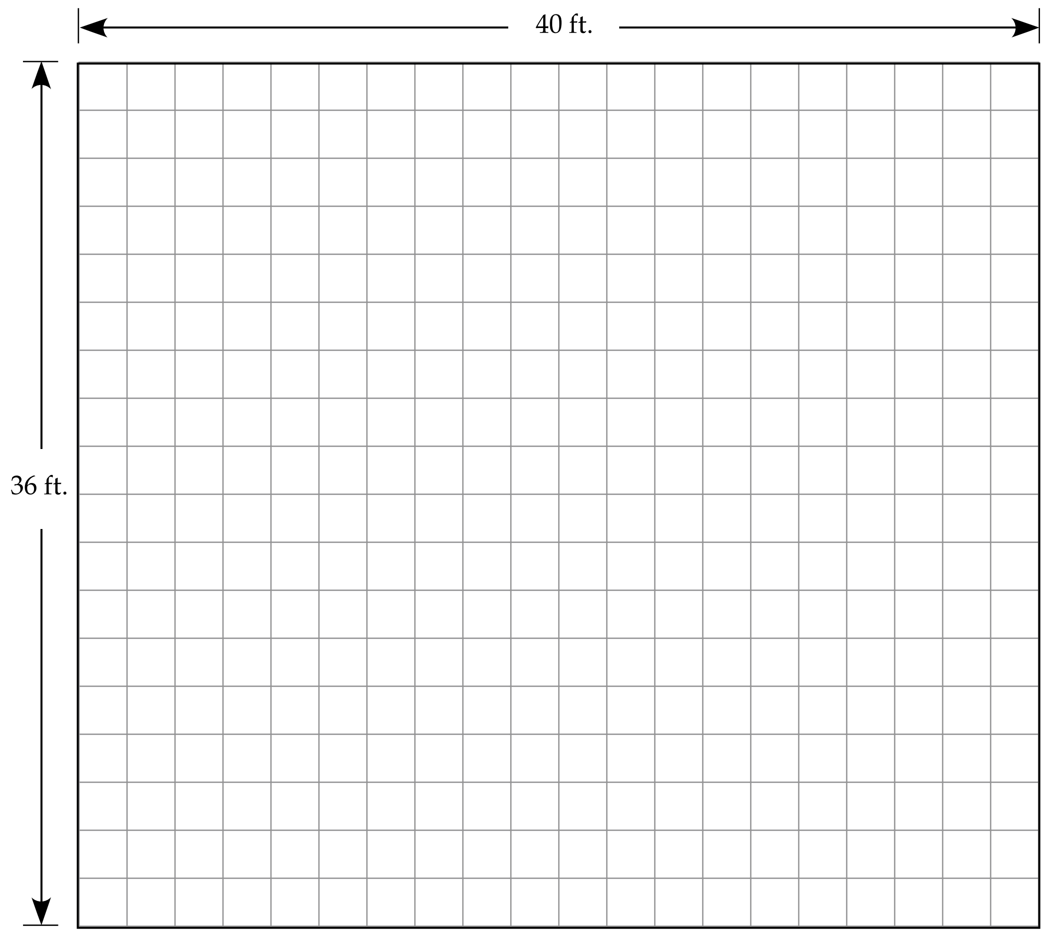
Remember: Complete **only one** of Question 1 **or** Question 2 for hand-in.

2. You have been asked to design the layout for a new community fitness centre. The centre measures 36 feet by 40 feet and must meet the following specifications:
   - The women's washroom, with change room, must have an area between 150 $ft.^2$ and 200 $ft.^2$.
   - The men's washroom, with change room, must have an area between 120 $ft.^2$ and 170 $ft.^2$.
   - The hot tub room must measure 8 ft. by 12 ft.
   - The mechanical room must have an area between 50 $ft.^2$ and 100 $ft.^2$.
   - The rest of the centre must be an open space reserved for equipment and for clients to move around.
   - Each room must have a door that is 36 inches wide and 80 inches high.
   - The entrance to the fitness centre must have a double door that is 60 inches wide and 80 inches high.

Assignment 8.2: Working within a Project Budget (continued)

a) Complete a scale diagram of the centre. Draw all the walls with a solid line and indicate the location of all the doors. When you are considering distances and areas, ignore the thickness of the walls. Label each room and indicate the dimensions. *(3 marks)*

Scale: 1 square = 2 feet



Assignment 8.2: Working within a Project Budget (continued)

b) You have also been asked to install equipment in the open space, allowing a certain amount of space for each piece of equipment to be used safely. You can spend between $12,000.00 and $16,000.00, not including taxes, and must have at least one of each of the pieces indicated in the chart below.

Complete the last column of the chart. Indicate where each piece of equipment will be located on your diagram in (a). Use the labels provided and draw lines around each piece to indicate the area of safe use. Determine the total cost of the equipment, not including taxes. *(2 marks)*

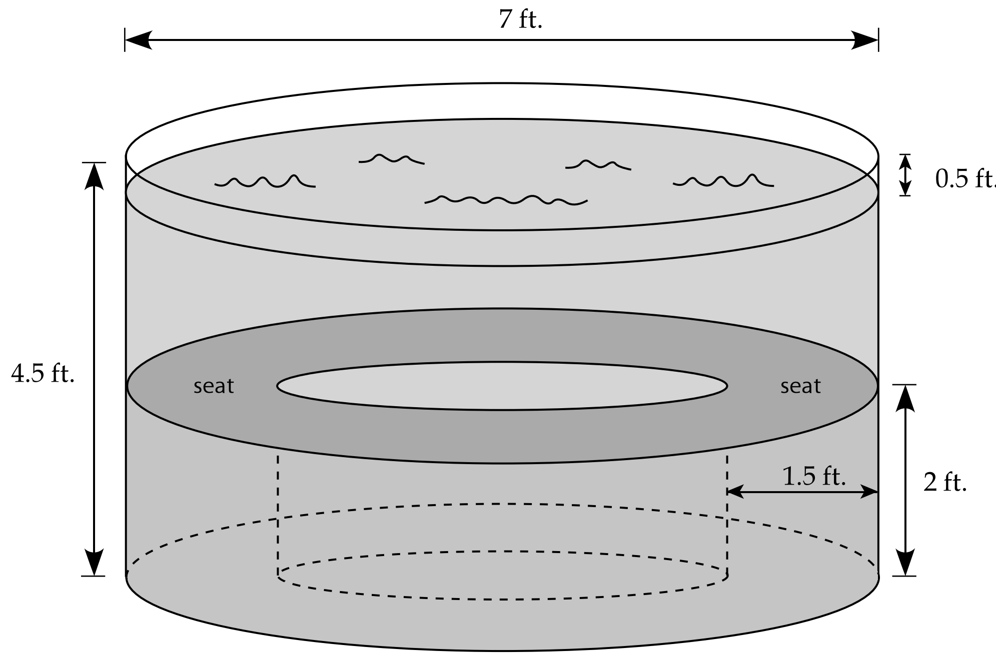
Equipment	Label	Cost (not including taxes)	Area of Safe Use	Quantity
Bench press	BP	$850.00	8 ft. × 8 ft.	
Leg press	LP	$650.00	8 ft. × 8 ft.	
Squat rack	SR	$550.00	8 ft. × 8 ft.	
Treadmill	T	$1550.00	8 ft. × 6 ft.	
Exercise bench	EB	$550.00	6 ft. × 4 ft.	
Elliptical trainer	ET	$1450.00	6 ft. × 4 ft.	
Stationary bicycle	SB	$1575.00	6 ft. × 4 ft.	

Assignment 8.2: Working within a Project Budget (continued)

c) The walls surrounding the open space reserved for the equipment are 10 feet high and must be painted. The doors do not need to be painted. You will need one coat of primer and two coats of paint. How many cans of each will be needed if one can of primer covers 300 ft.^{2} and one can of paint covers 400 ft.^{2}? *(3 marks)*

Assignment 8.2: Working within a Project Budget (continued)

d) The dimensions for the hot tub are indicated on the diagram below. Determine the volume of water necessary to fill the hot tub to 0.5 feet from the top. Show your work. (Diagram is not drawn to scale.) *(2 marks)*



Notes

Module 8 Summary

Congratulations on finishing the last module in this course!

In this module, you learned to analyze objects, shapes, and processes to solve cost and design problems. You solved problems involving perimeter, area, and volume, using dimensions and unit prices. You solved problems involving estimation and costing for objects, shapes, and processes when a design was given. You used simplified models to estimate solutions for complex measurement problems and designed objects, shapes, layouts, or processes within a specified budget.

Submitting Your Assignments

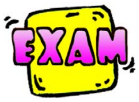
It is now time for you to submit the Module 8 Cover Assignment and Assignments 8.1 and 8.2 to the Distance Learning Unit so that you can receive some feedback on how you are doing in this course. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 8 assignments and organize your material in the following order:

- ❑ Module 8 Cover Sheet (found at the end of the course Introduction)
- ❑ Module 8 Cover Assignment: Container Conundrum
- ❑ Assignment 8.1: Design and Cost Decisions
- ❑ Assignment 8.2: Working within a Project Budget

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Final Examination



Congratulations, you have finished Module 8 in the course. The final examination is out of 100 marks and worth 25% of your final mark. In order to do well on this examination, you should review all of your learning activities and assignments from Modules 5 to 8.

You will complete this examination while being supervised by a proctor. You should already have made arrangements to have the examination sent to the proctor from the Distance Learning Unit. If you have not yet made arrangements to write it, then do so now. The instructions for doing so are provided in the Introduction to this module.

You will need to bring the following items to the examination: some pens and/or pencils (2 or 3 of each), metric and imperial rulers, a graphing and/or scientific calculator, financial application technology, and your Final Examination Resource Sheet. A maximum of 3 hours is available to complete your final examination. When you have completed it, the proctor will then forward it for assessment. Good luck!

Graphing and financial applications technology (either computer software or a graphing calculator) **are required** to complete the examination. Check with your tutor/marker to be sure your graphing technology and financial applications technology are appropriate.

At this point you will also have to combine your resource sheets from Modules 5 to 8 onto one 8½″ × 11″ paper (you may use both sides). Be sure you have all the formulas, definitions, and strategies that you think you will need. This paper can be brought into the examination with you. We suggest that you divide your paper into two quadrants on each side so that each quadrant contains information from one module..

Examination Review

You are now ready to begin preparing for your final examination. Please review the content, learning activities, and assignments from Modules 5 to 8.

The final practice examination is also an excellent study aid for reviewing Modules 5 to 8.

You will learn what types of questions will appear on the examination and what material will be assessed. Remember, your mark on the final examination determines 25% of your final mark in this course and you will have 3 hours to complete the examination.

Final Practice Examination and Answer Key

To help you succeed in your examination, a practice examination can be found in the learning management system (LMS). The final practice examination is very similar to the actual examination that you will be writing. The answer key is also included so that, when you have finished writing the practice examination, you can check your answers. This will give you the confidence that you need to do well on your examination. If you do not have access to the Internet, contact the Distance Learning Unit at 1-800-465-9915 to get a copy of the practice examination and the answer key.

To get the most out of your final practice examination, follow these steps:

1. Study for the final practice examination as if it were an actual examination.
2. Review those learning activities and assignments from Modules 5 to 8 that you found the most challenging. Reread those lessons carefully and learn the concepts.
3. Contact your learning partner and your tutor/marker if you need help.
4. Review your lessons from Modules 5 to 8, including all of your notes, learning activities, and assignments.
5. Use your module resource sheets to make a draft of your Final Examination Resource Sheet. You can use both sides of an 8½″ by 11″ piece of paper.
6. Bring the following to the final practice examination: some pens and/or pencils (2 or 3 of each), metric and imperial rulers, a graphing and/or scientific calculator, financial application technology, and your Final Examination Resource Sheet.
7. Write your final practice examination as if it were an actual examination. In other words, write the entire examination in one sitting, and don't check your answers until you have completed the entire examination. Remember that the time allowed for writing the final examination is 3 hours.
8. Once you have completed the entire practice examination, check your answers against the answer key. Review the questions that you got wrong. For each of those questions, you will need to go back into the course and learn the things that you have missed.
9. Go over your resource sheet. Was anything missing or is there anything that you didn't need to have on it? Make adjustments to your Final Examination Resource Sheet. Once you are happy with it, make a photocopy that you can keep.

Notes

Grade 12 Applied Mathematics (40S)

Module 8
Design and Measurement

Learning Activity Answer Keys

Module 8: Design and Measurement

Learning Activity 8.1

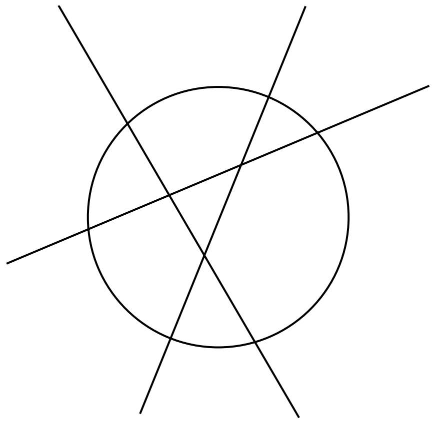
Part A: BrainPower

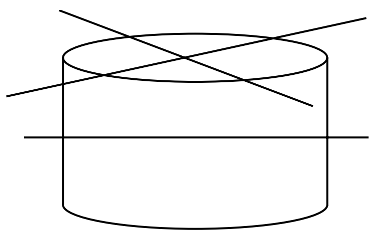
The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Simplify: $\frac{8!}{5!3!}$
2. You have a circle-shaped pie and need to cut it into exactly seven pieces. How can this be accomplished using exactly three straight slices with a knife? (The pieces may be unequal in size.)
3. You have a round cake and need to cut it into exactly eight pieces. How can this be accomplished using exactly three straight slices with a knife?
4. Write using factorial notation: $7 \times 6 \times 5 \times 4$
5. If 4 identical items cost $2.32, how much will 5 of the items cost?
6. A drink and candy cost $1.10. The drink costs one dollar more than the candy. How much does the candy cost?
7. Determine the next three letters in this sequence: O T T F F ___ ___ ___
8. Terry is a firefighter and works 12-hour shifts for four days and then has four days off. He starts work today after having yesterday off. How many days will he work over the next 30 days?

Answers:

1. 56 $\left(\dfrac{8 \times 7 \times \cancel{6} \times \cancel{5!}}{\cancel{5!} \times \cancel{3} \times \cancel{2} \times 1} = 56\right)$

2. 

3. 

4. $\dfrac{7!}{3!}$

5. \$2.90 (Two of the items would cost \$1.16, so one of the items would cost \$0.58; \$2.32 + 0.58 = 2.90.)

6. \$0.05 (\$0.05 + \$1.05 = \$1.10)

7. S S E (The sequence is the first letter of the names of the counting numbers: One, Two, Three, Four, Five, Six, Seven, Eight.)

8. 16 (Terry works in 8-day cycles, with the first four days of each cycle at work. Since 8 × 4 = 32, the last two days of the fourth cycle (when he is off) will be after the 30 days. He will work four days in each of the four cycles, for a total of 16 days. Another way to consider this is to think that $\dfrac{30}{8} = 3\dfrac{6}{8}$, so Terry works three full cycles (12 shifts) plus the first six days of the next cycle. Of the first six days of a cycle, he works on four of them, so 12 + 4 = 16.)

Part B: Perimeter, Area, and Volume

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Compare the volume of a prism with the volume of a pyramid, which has the same base and height as the prism. Give an example to support your answer and show your calculations.

   Answer:

   The volume of the prism will be three times the volume of a pyramid with the same base and height. The formulas for the volume of the prism and pyramid are the same, except the pyramid is multiplied by $\frac{1}{3}$, making it one-third the volume of a prism with the same base.

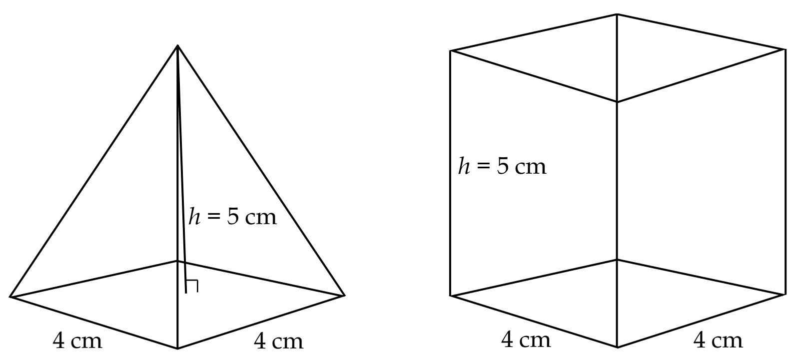
   Examples will vary. A sample solution:

   $$V_{\text{pyramid}} = \frac{Bd}{3} \qquad V_{\text{prism}} = Bd$$

   where B is the area of the base and d is the depth (or height)

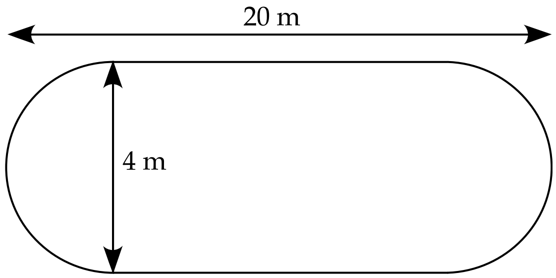
   $$V_{\text{pyramid}} = \frac{(4 \times 4)(5)}{3} \qquad V_{\text{prism}} = (4)(4)(5)$$

   $$V_{\text{pyramid}} = \frac{80}{3} \qquad V_{\text{prism}} = 80$$

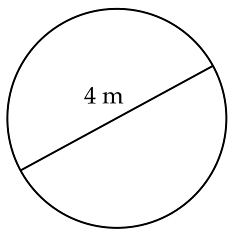
   The volume of the pyramid is $\frac{80}{3}$ cm^3, while the volume of the prism is 80 cm^3.

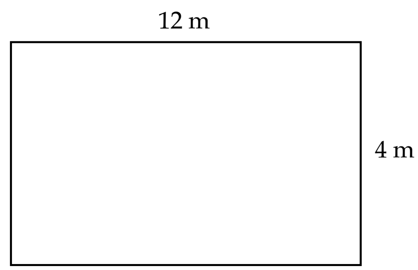
2. An ice rink has the following dimensions:



Given the task of finding the area covered by ice in the skating rink, pictured above, Aisha wrote the following solution. What went wrong? Find the error(s) Aisha made and correct them. Find the actual area of the ice.

There are two semi-circles (which make a circle) and a rectangle.





$A_{circle} = \pi r^2$

$A_{circle} = \pi(2)^2$

$A_{circle} = 4\pi$

$A_{circle} = 12.57 \text{ m}^2$

$A_{rectangle} = l \times w$

$A_{rectangle} = 4 \times 12$

$A_{rectangle} = 48 \text{ m}^2$

The total area of the ice surface is $12.57 + 48 = 60.57 \text{ m}^2$.

Answer:

The problem with Aisha's solution is the calculation of the length of the rectangle. The rink is 20 m from end to end. To find the length of the rectangular part, subtract the radius of 2 m twice.

$20 - 2(2) = 16$

The area of the ice rink is therefore correctly calculated as:

$$A_{\text{circle}} = \pi r^2 \qquad A_{\text{rectangle}} = l \times w$$

$$A_{\text{circle}} = \pi (2)^2 \qquad A_{\text{rectangle}} = 4 \times 16$$

$$A_{\text{circle}} = 4\pi \qquad A_{\text{rectangle}} = 64 \text{ m}^2$$

$$A_{\text{circle}} \cong 12.57 \text{ m}^2$$

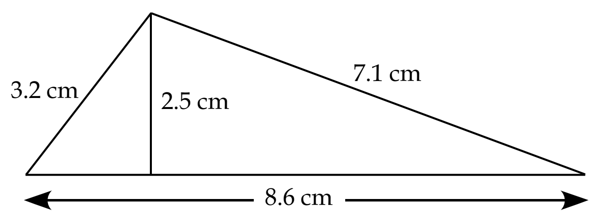
The total area of the ice surface is 12.57 + 64 = 76.57 m^2.

3. Using the formulas for perimeter/circumference, area, surface area, and volume of 2-D and 3-D figures, calculate the indicated values. State appropriate units.

   a) Find:

      i) perimeter

      ii) area



Answer:

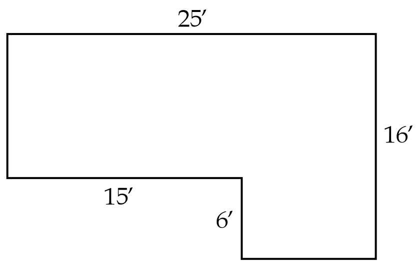
i) perimeter = 18.9 cm (8.6 + 7.1 + 3.2)

ii) area = 10.75 cm^2 (8.6 × 2.5 ÷ 2)

b) Find:

i) perimeter

ii) area



Answer:

i) Missing sides are 10′ (16 − 6) and 10′ (25 − 15).

Add sides: 10 + 25 + 16 + 10 + 6 + 15.

Perimeter = 82′

ii) Area of rectangle on left is 150 ft.^{2} (15 × 10).

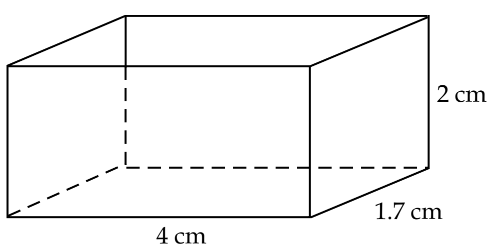
Area of rectangle on right is 160 ft.^{2} (10 × 16).

Area = 310 ft.^{2}

c) Find:

i) surface area of the open box in ft.^{2}

ii) volume of the box in ft.^{3}



(Diagram measurements are shown.)

Scale: 1 cm = 2 ft.

Answer:

i) Actual measurement using the scale are:

8 ft. by 3.4 ft. by 4 ft.

Front: $8 \times 4 = 32$

Side: $3.4 \times 4 = 13.6$

Bottom: $8 \times 3.4 = 27.2$

Add the area of 5 sides (no top):

32 + 32 + 13.6 + 13.6 + 27.2

Surface area = 118.4 ft.^{2}

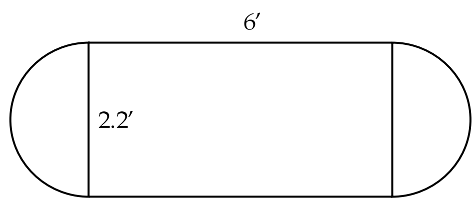
ii) Volume = $8 \times 3.4 \times 4$

Volume = 108.8 ft.^{3}

d) Find:

i) surface area of the tank

ii) volume of the tank



Answer:

i) Sphere area = $4\pi(1.1)^2 = 15.205$

Cylinder area = $2\pi(1.1)(6) = 41.469$

Surface area = 15.205 + 41.469 = 56.674 ft.^{2}

ii) Sphere volume = $\frac{4}{3}\pi\left(1.1\right)^3 = 5.575$

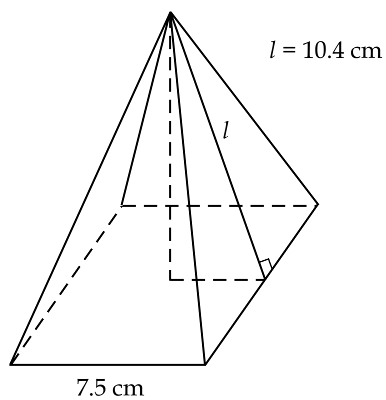
Cylinder volume = $\pi(1.1)^2(6) = 22.808$

Volume = 5.575 + 22.808 = 28.383 ft.^{3}

e) Find:

i) surface area of the square pyramid

ii) volume of the square pyramid



Answer:

The height of the pyramid is found using the Pythagorean theorem.

The base of the right triangle is 7.5 ÷ 2 = 375.

$h^2 + 3.75^2 = 10.4^2$

$h = 9.7$

i) Area of one triangle side = $\frac{(7.5)(10.4)}{2} = 39$

Area of base = 7.5 × 7.5 = 56.25

Add areas: 39 × 4 + 56.25

Total surface area = 212.25 cm^2

ii) Volume = 181.875 cm^3

4. The volume of a cone is 375.6 cm^3. The circumference of the base is 33.9 cm. Find:

a) the height of the cone

Answer:

If the circumference of the base is 33.9 cm, the radius can be found using the formula $C = 2\pi r$.

$33.9 = 2\pi r$

$r = \frac{33.9}{2\pi}$

$r = 5.395352571$

The radius is about 5.4 cm, but you should always use as many decimal places as possible when using this value in further calculations.

The volume of a cone is found using the formula $V_{\text{cone}} = \frac{1}{3}Bd$, where B is the area of the base and d is the depth (or height) of the cone. The base of a cone is a circle.

The area of a circle is found using $A = \pi r^2$, so the formula for the volume of this cone would be:

$$375.6 = \frac{1}{3}\left(\pi \times 5.395352571^2\right)(d)$$

Solve for the height using this formula.

$375.6 = 30.48374203d$

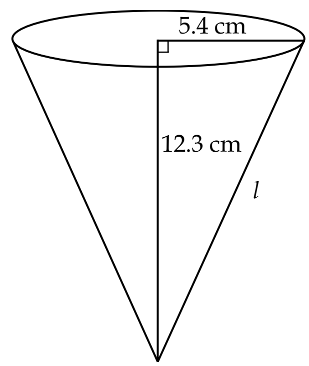
$d = 12.32132196$

The height of this cone is about 12.3 cm.

b) the total surface area of the cone

Answer:

To calculate the total surface area, you need to know the slant length of the cone and substitute it into the formula, Total $SA_{\text{cone}} = \frac{1}{2}Pl + B$, where P is the perimeter (or circumference) of the base, l is the slant length, and B is the area of the base. The perimeter is given as 33.9 cm, and the area of the base is $\pi \times 5.395352571^2 = 91.45122608$ cm^2.



$r^2 + d^2 = l^2$

$5.3953525712 + 12.321321962 = l^2$

$l = 13.45082913$

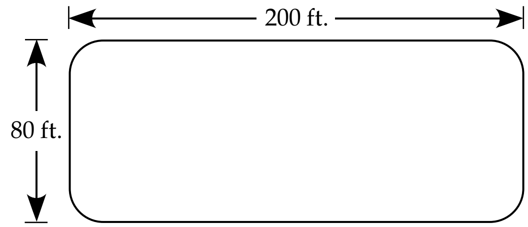
$$\text{Total } SA_{\text{cone}} = \frac{1}{2}Pl + B$$

$$\text{Total } SA_{\text{cone}} = \frac{1}{2}(33.9)(13.45082913) + (91.45122608)$$

$$\text{Total } SA_{\text{cone}} = 319.4427798$$

The total surface area of this cone is about 319.4 cm^2.

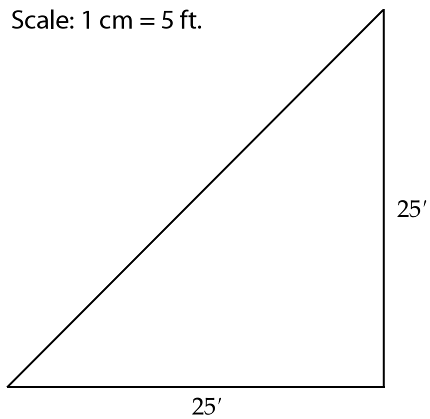
5. A hockey rink is shaped as follows:



a) Estimate the area of the ice surface.

Answer:

If the rink is approximated to be a rectangle with length of 200 ft. and width of 80 ft., the area would be $200 \times 80 = 16\,000$ ft.^{2}. The rounded corners can be approximated to be triangles with a base of 25 ft. and a height of 25 ft., as shown in the following diagram:



The area of one of these triangles would be $0.5 \times 25 \times 25 = 312.5$ ft.^{2}.

Since there are four of these triangles, the total approximate area that needs to be subtracted is $4 \times 312.5 = 1250$ ft.^{2}.

This leaves an approximate area of $16\,000 - 1250 = 14\,750$ ft.^{2}.

b) Estimate the volume of ice in the rink if the ice is two inches thick. Calculate in cubic yards.

Answer:

The volume of ice is found by multiplying the area of the ice by the depth of ice. The depth is 2 inches or $\frac{2}{12} = \frac{1}{6}$ feet.

$$V = 14\,750 \times \frac{1}{6} = 2458.3 \text{ ft.}^3$$

Conversion: 3 ft. = 1 yd. and $(3)^3$ ft.^{3} = 1^3 yd.^{3}

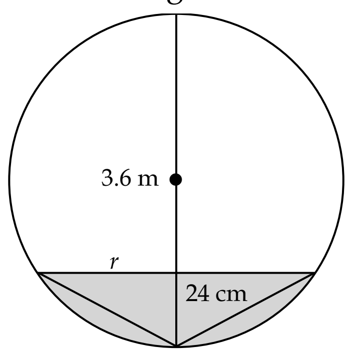
Volume in cubic yards will be $\frac{2458.3}{27} = 91.0 \text{ yd.}^3$.

This is an estimate of the volume due to the approximation of the area, but should be close to the actual volume.

6. A water tank is a sphere with a diameter of 3.6 m. Estimate the volume of water in the tank if the depth of the water is 24 cm.

   Answer:

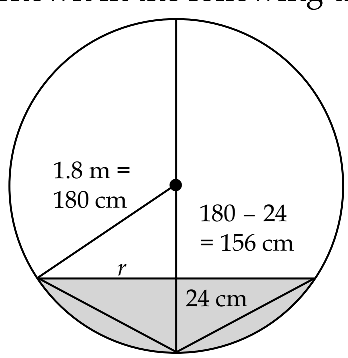
   The water in the bottom of the tank can be seen as a conical shape, as shown in the following:



The volume of a cone is found by $V = \frac{1}{3}\pi \cdot r^2 \cdot h$.

The height of the water is 24 cm.

The radius of the conical-shaped water is found by creating a right triangle, as shown in the following diagram and using the Pythagorean theorem.



By the Pythagorean theorem:

$$180^2 = 156^2 + r^2$$

$$r^2 = 180^2 - 156^2$$

$$r = \sqrt{8064} = 89.8$$

Radius = 89.8 cm

Therefore, $V = \frac{1}{3}\pi \cdot 89.8^2 \cdot 24 = 202\ 671.4\ \text{cm}^3$.

An estimate of 10% could be added for adjustment for the two small segments not included in the volume of the cone, giving an approximate volume of $202\ 671.4 \times 1.10 = 223\ 000\ \text{cm}^3$.

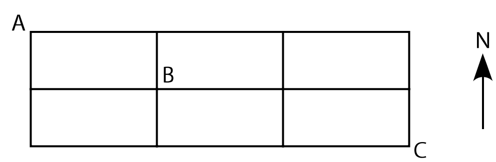
Learning Activity 8.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. All of Mr. Popper's pets are penguins, except one. All his pets are parrots, except one. How many pets does he have?
2. Knowing there are 24 hours in a day, calculate the number of hours in a week.
3. Estimate the area of a semi-circle with a 20 cm diameter.
4. How many cm^2 are equal to one m^2?
5. Ten striped socks and ten checkered socks are all mixed up in a drawer. If you close your eyes and randomly pick out one sock at a time, what is the smallest number of socks you will have to pull out of the drawer to guarantee you have a matching pair?

Use the following diagram to answer the Questions 6 to 8. The diagram shows a partial map of the pathways in a city park.

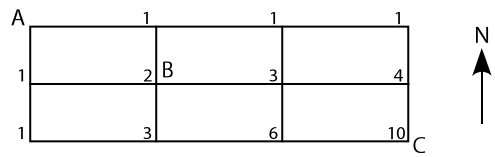


6. If a jogger begins at point A and runs only south and east, in how many different ways can she arrive at point B?
7. If a jogger begins at point A and runs only south and east, in how many different ways can she arrive at point C?
8. What is the probability that the jogger passes by point B on her way to point C?

Answers:

1. Two (one penguin and one parrot)
2. 168 ($20 \times 7 + 4 \times 7 = 140 + 28$)
3. 150 cm^2 ($\pi r^2 \div 2 \approx 3 \times 10^2 \div 2 = 300 \div 2$)
4. 10 000 (100 cm = 1 m, then $(100)^2$ cm^2 = 1 m^2)
5. Three
6. 2

7. 10



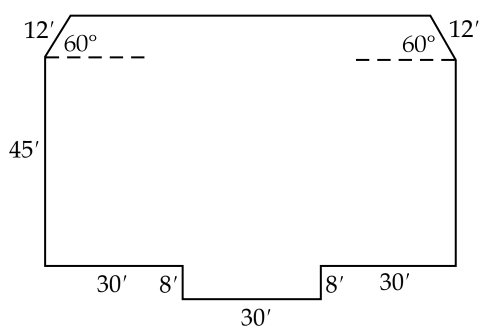
or $\quad {}_5C_3 = \dfrac{5 \times 4}{2 \times 1} = 10$

8. $\dfrac{2}{10} = 0.2 = 20\%$

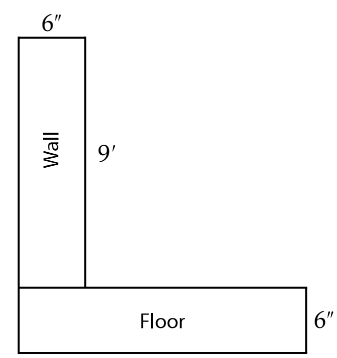
Part B: The Costs of Construction

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. An organization is pouring a concrete basement for a new office building. The dimensions are as follows:



Top View of the Floor



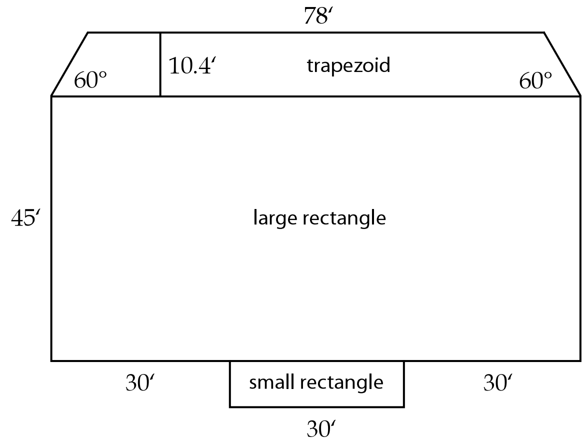
Side View of Basement Wall

The walls and floor are made of concrete and are six inches thick. The basement walls sit on top of the floor, as shown in the side-view diagram. It takes two workers and one foreman one hour to pour 200 cubic feet of concrete. Concrete costs $120 per cubic yard. The workers make $19.00 per hour and the foreman makes $24.50 per hour. GST of 5% and PST of 8% apply to the concrete, and only GST applies to the wages paid. How much will it cost to build the basement?

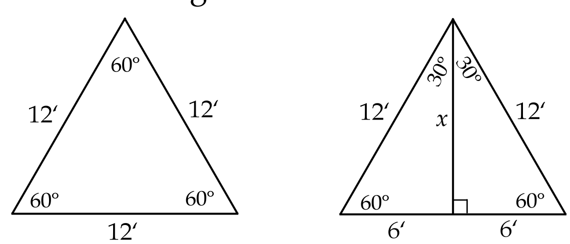
Answer:

$V_{\text{concrete floor}} = (A_{\text{floor}}) \times (\text{Thickness of concrete})$

$A_{\text{floor}} = A_{\text{trapezoid}} + A_{\text{large rectangle}} + A_{\text{small rectangle}}$



The height and top-side length of the trapezoid are found by using the equilateral triangles in each corner as follows:



$a^2 + b^2 = c^2$

or

$x^2 = 12^2 - 6^2$

$x^2 = 108$

$x \cong 10.39230485$

$90' - 2(6) = 78'$

$A_{\text{floor}} = A_{\text{trapezoid}} + A_{\text{large rectangle}} + A_{\text{small rectangle}}$

$A_{\text{floor}} = 0.5(78 + 90)(10.39230485) + (90 \times 45) + (30 \times 8)$

$A_{\text{floor}} = 872.953607 + 4050 + 240$

$A_{\text{floor}} = 5162.953607$

The floor is 6″ thick, which is 0.5 feet.

$V_{\text{floor}} = 5162.953607 \times 0.5$

$V_{\text{floor}} = 2581.476804$

The volume of concrete needed for the floor is about 2581.5 cubic feet.

$$V_{walls} = \text{perimeter of walls} \times \text{thickness of concrete} \times \text{height of wall}$$

$$V_{walls} = (78 + 12 + 45 + 30 + 8 + 30 + 8 + 30 + 45 + 12) \times 0.5 \times 9$$

$$V_{walls} = 1341$$

The volume of concrete needed for the walls is about 1341 cubic feet.

The total amount of concrete required is 2581.5 + 1341 or about 3922.5 $ft.^3$. Concrete costs $120 per cubic yard. Remember, 1 $yd.^3$ = 27 $ft.^3$.

$$\frac{3922 \text{ ft.}^3}{27} = 145.278 \text{ yd.}^3$$

The cost of concrete, including taxes, is:

$$145.278 \times 120 \times 1.13 = 19699.697$$

The total cost of concrete will be about $19,700.00.

Labour costs:

Number of hours required to pour the concrete:

$$\frac{3922.5 \text{ ft.}^3}{200} = 19.6 \text{ hrs.}$$

They will likely have to pay for 20 hours of labour for each of the two workers and for the foreman. Include GST.

$$(2(20 \times 19) + (20 \times 24.50) \times 1.05 = 1312.50$$

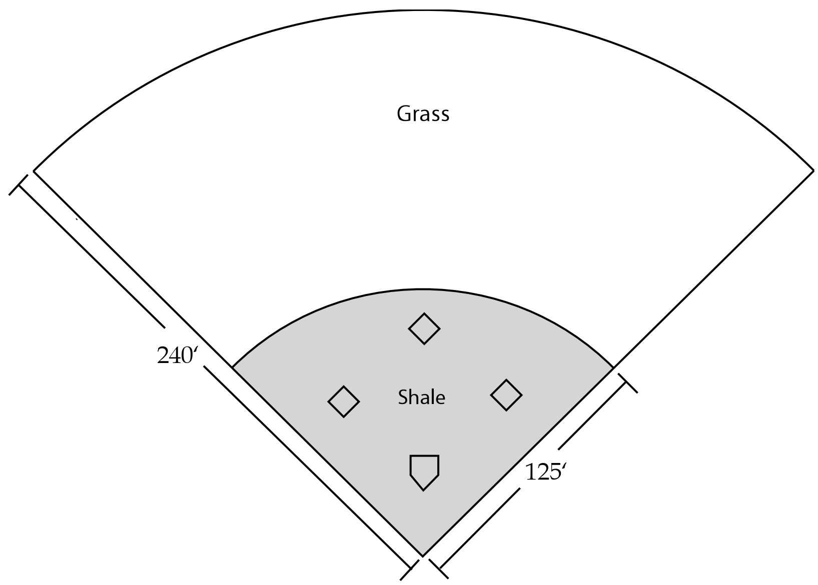
Labour costs are about $1312.50.

Total cost for the basement:

$19,700.00 + $1312.50 = $21,012.50

Note: Approximations have been made at various stages of the calculations, in measurements and through rounding. When purchasing supplies and paying for labour, it is common to round up to the next whole number, rather than paying for partial amounts. It would be appropriate to give an estimate of about $21,500 for this project.

2. A ballpark is to be built according to the diagram. A fence is to be put up around the entire park. Fencing costs $1.35 per running foot. The infield area, which is to have shale on it, needs to be dug down 18″ so that one foot of gravel can be put down as a base and then 6″ of shale put on top. The cost of excavation is $5.00 per cubic yard. Gravel costs $7.00 per cubic yard and shale costs $9.50 per cubic yard. The remaining area of the park needs to have 2″ of topsoil on it and sod put on top of the soil. Topsoil costs $20 per cubic yard and sod costs $1.25 per square foot. The grass needs to have fertilizer as soon as it is laid. The fertilizer costs $15.60 per bag and a bag will cover 75 square yards. If PST at 8% and GST at 5% apply to all materials used and only GST applies to the excavation costs, find the total cost, excluding labour, for this project.



Answer:

Total costs are:

Fence Costs:

$$\text{Perimeter} = 240 + 240 + \frac{1}{4} \cdot 2\pi \cdot 240 = 857 \text{ ft.}$$

Cost = 857 × $1.35 × 1.13 = $1307.35

Excavation Costs:

The depth is 18″ or 1.5 feet.

$$\text{Volume of earth to be removed} = \frac{1}{4} \cdot \pi \cdot (125)^2 \cdot 1.5 = 18\ 407.8 \text{ ft.}^3$$

$$\text{Volume in yd.}^3 = \frac{18\ 407.8}{27} = 681.8 \text{ yd.}^3$$

Cost = 681.8 × \$5.00 × 1.05 = \$3579.45

Gravel Costs:

$$\text{Volume of gravel} = \frac{1}{4} \cdot \pi \cdot (125)^2 \cdot 1 = 12\ 271 \text{ ft.}^3$$

$$\text{Volume in yd.}^3 = \frac{12\ 271.8}{27} = 454.5 \text{ yd.}^3$$

Cost = 454.5 × \$7.00 × 1.13 = \$3595.10

Shale Costs:

$$\text{Volume of shale} = \frac{1}{4} \cdot \pi \cdot (125)^2 \cdot 0.5 = 6135.9 \text{ ft.}^3$$

$$\text{Volume in yd.}^3 = \frac{6135.9}{27} = 227.3 \text{ yd.}^3$$

Cost = 227.3 × \$9.50 × 1.13 = \$2440.07

Topsoil Costs:

The topsoil is 2″ deep or $\frac{2}{12} = \frac{1}{6}$ ft.

$$\text{Volume of topsoil} = \left[\frac{1}{4} \cdot \pi \cdot (240)^2 - \frac{1}{4} \cdot \pi \cdot (125)^2\right] \cdot \frac{1}{6} = 5494.5 \text{ ft.}^3$$

$$\text{Volume in yd.}^3 = \frac{5494.5}{27} = 203.5 \text{ yd.}^3$$

Cost = 203.5 × \$20.00 × 1.13 = \$4599.10

Sod Costs:

$$\text{Area of sod} = \left[\frac{1}{4} \cdot \pi \cdot (240)^2 - \frac{1}{4} \cdot \pi \cdot (125)^2\right] = 32\ 967.1 \text{ ft.}^2$$

Cost = 32 967.1 × \$1.25 × 1.13 = \$46,566.03

Fertilizer Costs:

Area of fertilizer $= \left[\frac{1}{4} \cdot \pi \cdot (240)^2 - \frac{1}{4} \cdot \pi \cdot (125)^2\right] = 32\ 967.1$ ft.^{2}

There are 3^3 or 9 ft.^{2} in one yd.^{2}.

Area in yd.^{2} $= \frac{32\ 967.1}{9} = 3663.0$ yd.^{2}

Bags of fertilizer $= \frac{3663}{75} = 48.8$ or 49 bags

Cost = 49 × \$15.60 × 1.13 = \$863.77

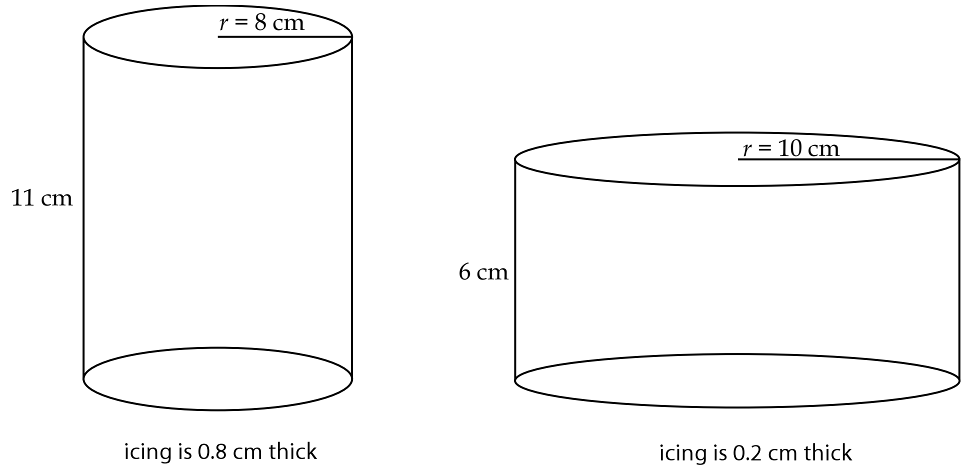
Total Ball Park Costs:

= 1307.35 + \$3579.45 + \$3595.10 + 2440.07 + \$4599.10 + \$46,566.03 + \$863.77

= \$62,950.87

Note: Approximations and rounding have been made at various stages of the calculation. It would be appropriate to give an estimate of \$63,500.00.

3. A bake shop has two differently sized cakes, as shown.



The higher cake has a thick icing; the shorter cake has a thin icing. There is no icing on the bottom of the cakes. The icing costs are \$2.00 for each 100 cm^3. Five mL of cake batter makes 16 cm^3 of cake and the batter costs \$1.25 per 100 mL. The cakes are placed in square-based boxes with lids that allow a 2 cm space on each side and the top of each cake. If the boxes are made from cardboard that costs \$3.00 per square metre, what is the total cost of each cake, including its box? Which is the better buy based on price per unit volume?

Answer:

Tall Cake:

Volume = $\pi \times (8)^2(11) = 2211.7$ cm^3

Batter costs:

Volume of batter = $2211.7 \div 16 \times 5 = 691.2$ mL

Cost of batter = $691.2 \div 100 \times \$1.25 = \$8.64$

Icing costs:

= S.A. × Icing thickness ÷ 100 × \$2.00

$= [(2\pi \times 8)(11) + (\pi \times 8^2)] \times 0.8 \div 100 \times \$2.00 = \$12.06$

Box costs:

10 000 cm^2 = 1 m^2

= S.A. of the box ÷ 10 000 × \$3.00

$= [(20 \times 20 \times 2) + (13 \times 20 \times 4)] \div 10\,000 \times \$3.00$

$= 1840 \div 10\,000 \times \$3.00 = \$0.55$

Note: Dimensions may vary depending on if icing depth was included.

Total Cost of Tall Cake:

$8.64 + $12.06 + $0.55 = $21.25

Short Cake:

Volume = $\pi \times (10)^2(6) = 1885.0\ \text{cm}^3$

Batter costs:

Volume of batter = $1885 \div 16 \times 5 = 589.1$ mL

Cost of batter = $589.1 \div 100 \times \$1.25 = \$7.36$

Icing costs:

= S.A. × Icing thickness ÷ 100 × $2.00

$= [(2\pi \times 10(6) + (\pi \times 10^2] \times 0.2 \div 100 \times \$2.00 = \$2.76$

Box costs:

= S.A. of the box ÷ 10 000 × $3.00

$= [(24 \times 24 \times 2) + (8 \times 24 \times 4)] \div 10\,000 \times \$3.00$

$= 1920 \div 10\,000 \times \$3.00 = \$0.58$

Total Cost of Short Cake:

7.36 + 2.76 + 0.58 = $10.70

Cost per Unit Volume:

Cost ÷ Total Volume of Cake and Icing

Tall Cake:

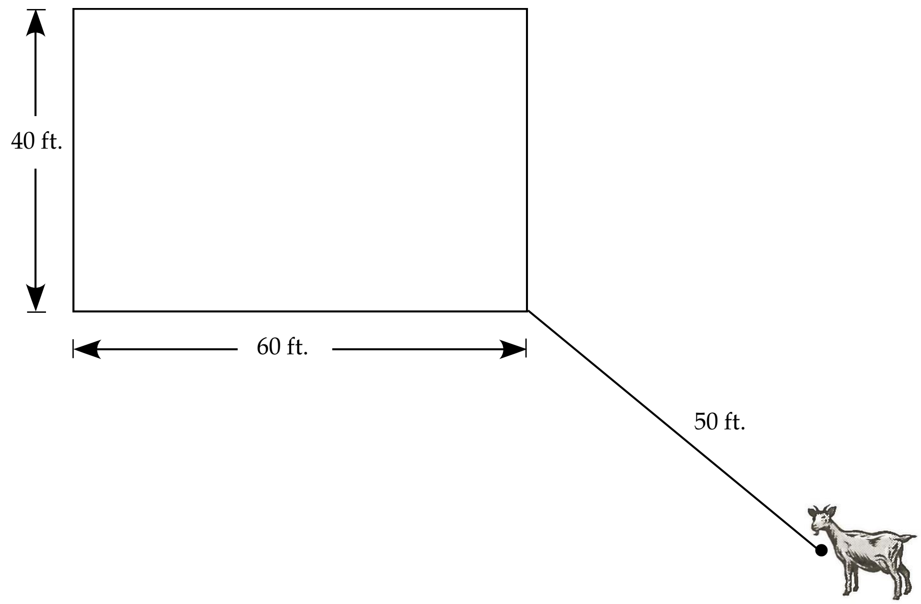
Cost per unit volume = $\$21.25 \div (2211.7 + 603.2) = \$0.00755$

Short Cake:

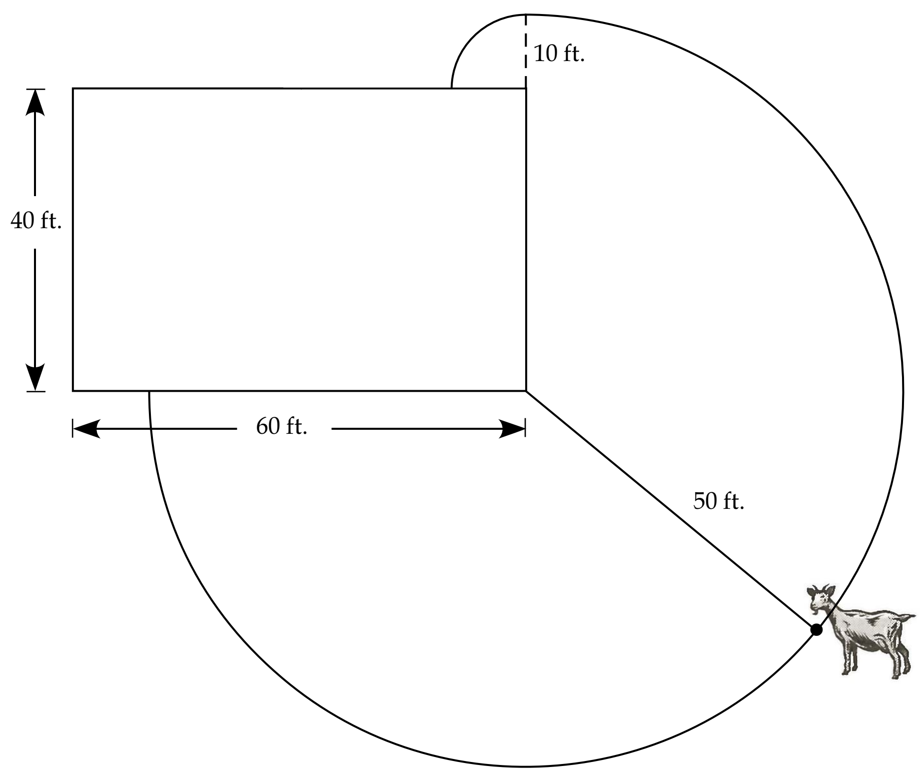
Cost per unit volume = $\$10.70 \div (1885.0 + 138.2) = \$0.00529$

The short cake costs less per unit volume. This is the better buy when only volume is taken into account.

4. A goat is tied to the corner of a barn with a 50-foot rope. The barn measures 60 feet by 40 feet. Calculate the total area outside of the barn that can be reached by the goat. Show your work.



Answer:



The goat can reach an area that is $\frac{3}{4}$ of a circle with a 50 ft. radius, and $\frac{1}{4}$ of a circle with a 10 ft. radius.

$$\text{Total area} = \frac{3\pi\left(50\text{ ft.}\right)^2}{4} + \frac{\pi\left(10\text{ ft.}\right)^2}{4}$$

$$= 5890.49 + 78.54 = 5969.03\text{ ft.}^2$$

5. You have been hired to finish someone's living room. The room is 23′ by 14′, and has two doorways and four windows.

   a) Calculate the cost of paint for the walls of the living room, including taxes. Use one coat of paint primer and two coats of finishing paint. (GST = 5%, PST = 8%)

      - The walls are 8 feet high.
      - The doors measure 3 ft. × 6 ft. 8 in.
      - The windows measure 4 ft. × 2 ft.
      - One gallon of paint primer costs $29.95 and covers 300 $ft.^2$
      - One gallon of finishing paint costs $44.95 and covers 400 $ft.^2$

      Paint and primer must be purchased in whole units.

      Answer:

      Surface area = four walls − windows − doors

      $SA = 2(23 \times 8) + 2(14 \times 8) - 4(4 \times 2) - 2(3 \times 6.667)$

      $SA = 592 - 32 - 40$

      SA = 520 sq. ft.

      To cover the walls with one coat of primer, you will need 2 gallons.

      $$\frac{520}{300} = 1.73 \text{ or } 2$$

      To cover the walls with two coats of paint, you will need 3 gallons.

      $$520 \times 2 = 1040$$

      $$\frac{1040}{400} = 2.6$$

      Cost = $(2 \times 29.95 + 3 \times 44.95) \times 1.13$

      Cost = 220.07

      The cost of paint for the walls will be $220.07.

b) Calculate the number of boxes of wood flooring needed to cover the living room floor.

- One box of flooring covers 2.4 square yards.
- You will need an extra 5% of flooring to account for waste.

Answer:

Area of floor = L × W

Area = 23 × 14

Area = 322 ft.^{2}

Convert sq. ft. to sq. yd. using 1 yd.^{2} = 9 ft.^{2}

$$\frac{322}{9} = 35.778 \text{ yd.}^2$$

To account for waste, add 5% to this quantity.

35.778 × 1.05 = 37.567 yd.^{2}

$$\frac{37.567}{2.4} = 15.653 \text{ boxes}$$

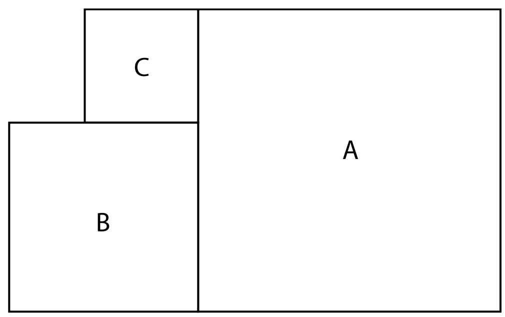
You will need to purchase 16 boxes of flooring.

Learning Activity 8.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Figures A, B, and C are squares. If the perimeter of A is 36 units and the perimeter of C is 16 units, what is the perimeter of B?



2. State the formula for the area of a circle.
3. State the formula for the volume of a cylinder.
4. In how many ways can three boys wear five different baseball caps? (Assume they only wear one at a time.)

In a magic square, the sums of the entries in each row, column, and diagonal are all equal. In the magic square below, find the value of:

B	17	7
D	**A**	**C**
11	1	15

5. A
6. B
7. C
8. D

Answers:

1. 20 units (the side length of A must be $\frac{36}{4} = 9$ units, side length C must be 4 units, so side length B must be $9 - 4 = 5$ units; the perimeter would then be $5 \times 4 = 20$ units)
2. $A = \pi r^2$
3. $V = \pi r^2 h$
4. 60 ($5 \times 4 \times 3 = 60$)
5. 9 (the sum of the bottom row is 27, so the diagonal containing 11 and 7 must include 9)
6. 3 ($27 - 15 - 9$)
7. 5 ($27 - 15 - 7$)
8. 13 (using first column, $27 - 11 - 3$)

Part B: Designing a Project within a Budget

Remember, these questions are similar to the ones that will be on your assignments and final examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and final examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Your neighbours want to build a rectangular rink in their backyard. In order to do so, they must
   a) hire a person with levelling equipment to level the ground. The cost of this is $2.50 per square metre.
   b) flood it with water so that it will create ice to a depth of 12 cm. Ice expands to approximately 1.1 times the volume of water when it freezes. Water costs $2.00 per cubic metre.
   c) paint the ice white with ice paint that costs $1.05 per square metre.
   d) allow $25.00 for miscellaneous materials.
   e) allow for 5% GST and 8% PST for the water and other materials. Only GST applies to the levelling.

   Design and draw to scale a rink that would cost no more than $1000.00. Provide the scale on your diagram.

Answer:

Budget: \$1000.00

Miscellaneous costs: \$25.00 × 1.13 = \$28.25

Amount left for rink: \$1000.00 − \$28.25 = \$971.75

Cost per square metre:

Levelling costs: \$2.50 × 1.13 = \$2.825

Ice costs: $\frac{12}{100} \times \frac{\$2.00}{1.1} \times 1.13 = \$0.2712$

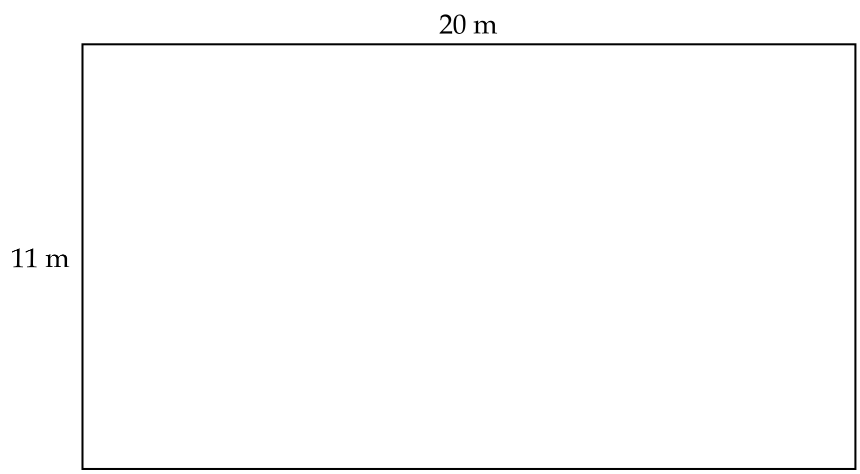
Paint costs: \$1.05 × 1.13 = \$1.1865

Total costs per metre: \$2.825 + \$0.2712 + \$1.865 = \$4.28

Maximum area of rink: 971.75 ÷ 4.28 = 227.04 m^2

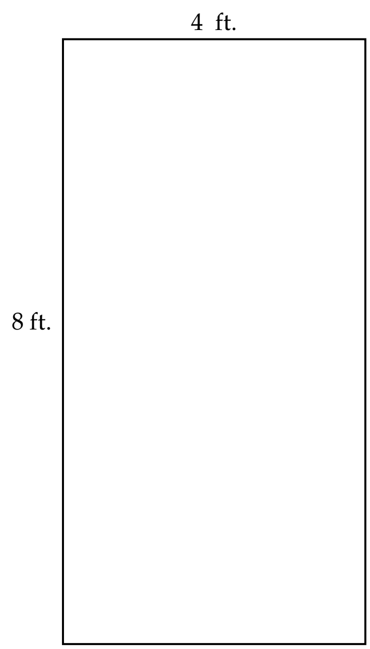
Many sizes of rectangular rinks are possible with a total area of less than 227.04 m^2. One such possibility could be a rink that measures 20 m long and 11 m wide. This would produce an area of 220 m^2, which falls within the budgeted amount.

Scale: 1 cm = 2 m

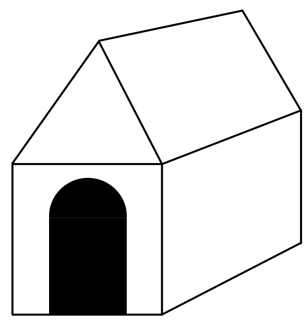


2. Design and find the dimensions of the largest functional doghouse of the type indicated that can be made from one 4-ft. × 8-ft. sheet of plywood. Show how the plywood would be cut to allow for your design. Indicate the cuts to scale in the scale diagram of the sheet of plywood that follows:

Scale: 1 cm = 1 ft.



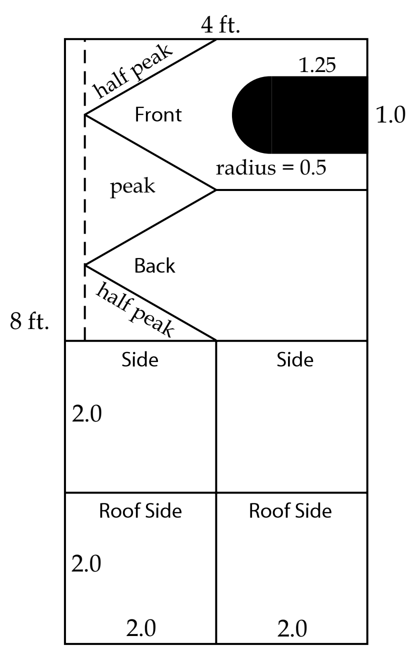
type of doghouse

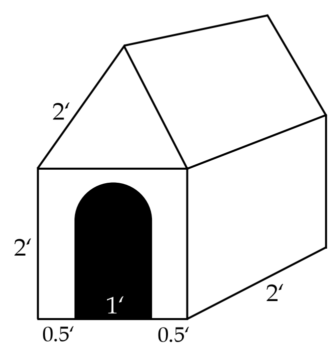


Answer:

Other designs are possible.

Scale: 1 cm = 1 ft.





This doghouse has no floor.

The slant of the front triangle is 2 ft. and so the peak is about 1.7 ft. above the sides.

Dimensions of this doghouse are:

Width: 2 ft.

Length: 2 ft.

Height of walls: 2 ft.

Total height of doghouse = 3.7 ft.

Entrance is 1 ft. wide and is a semicircle on top of a rectangle.

Total height of entrance = 1.75 ft.

3. Your track and field coach wants a long-jump pit with a concrete runway for the school. He has volunteers who will build it, but he needs your help to plan the design and calculate the cost of the materials.

   The long-jump pit must meet the following specifications:

   - The perimeter of the pit must be lined with boards to keep the sand separated from the grass.
   - The pit must be between 6 and 10 metres long (inclusive).
   - The pit must be twice as wide as the runway.
   - The depth of the sand in the pit must be between 40 and 60 cm (inclusive).
   - The runway must be at least 20 metres long and at least 1 metre wide.
   - The concrete in the runway must be between 10 and 15 cm thick inclusively.

Indicate the dimensions of the pit and runway on the following diagram.

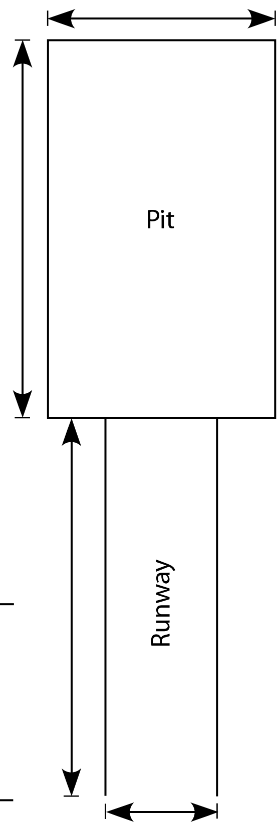
Answer:

width of pit = 3 m

length of pit = 8 m

length of runway = 30 m

width of runway = 1.5 m



This is a sample solution. Please check your calculations carefully and compare your process, steps, and calculations with the sample solution provided. Your answers must be within the limits, as outlined in the information above.

a) You will need to give the coach an estimate for the total cost of the materials using the following information:

- A hole will need to be dug for the pit.
- The depth of the hole should be equal to the depth of sand you plan on using.
- Sand costs $28.50/m^3$.
- 4.9 metre-long boards cost $32.50 each.
- Concrete costs $130.00/m^3$.
- The total cost for the long-jump pit must be between $1200.00 and $1500.00, including taxes. (GST = 5%, PST = 8%)

All materials must be purchased in whole units.

Keeping in mind the budget for this project, determine the total cost of the long-jump pit and runway, including taxes. Show your work.

Answer:

This is a sample solution. Your answers may vary based on the dimensions stated above. Please check your calculations carefully and compare your process, steps, and calculations with the sample solution provided.

Sand:

Depth of hole:

50 cm

Volume of sand:

$V = (8)(3)(0.5)$

$V = 12 \text{ m}^3$

Cost of sand:

$12 \text{ m}^3 \times \$28.50 = \$342.00$

Pit:

Perimeter:

$P = 2(8) + 2(3)$

$P = 22 \text{ m}$

Number of boards::

$22 \div 4.9 = 4.49$

Cost of boards:

$5 \times \$32.50 = \$162.50$

Runway:

Thickness: 12 cm

Length: 30 m

Width: 1.5 m

Volume of concrete:

$V = (0.12)(30)(1.5)$

$V = 5.4 \text{ m}^3$

Cost of concrete:

$6 \text{ m}^3 \times \$130.00 = \$780.00$

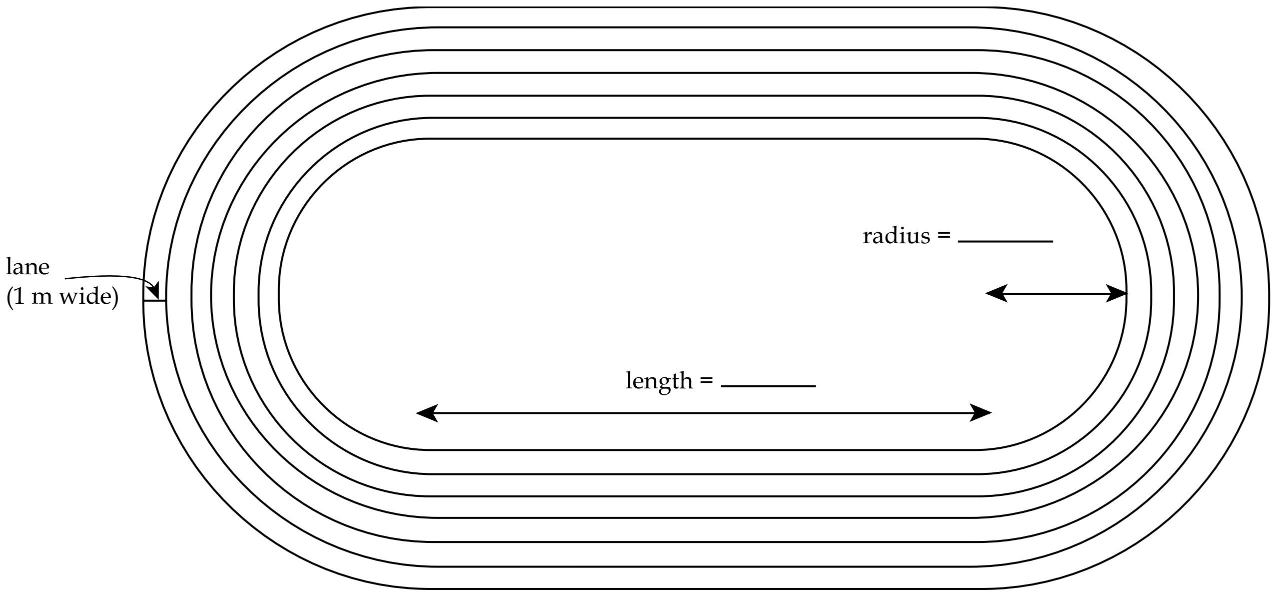
Total Cost:

\$342.00 + \$162.50 + \$780.00 =	\$1284.50
GST:	+ \$64.23
PST:	+ \$102.76
	\$1451.49

b) The coach also asks you to plan the design for a 6-lane track.

- The perimeter of the inside edge of the track must be 300 metres.
- Each lane must be 1 metre wide.

Determine the length of the straight portion and the radius of the curve for the track. Calculate the minimum dimensions of the rectangular field required to build the track. Show your work.



Answer:

Two possible solutions are given below. Other answers are possible. Please check your calculations carefully and compare your process, steps, and calculations with the sample solution provided.

Option 1:

Choose length of straight portion and calculate the radius needed to meet distance requirement.

Length: 100 m

Perimeter (inside edge) = $2\pi r + 2l$

300 m = $(2)(\pi)(r) + (2)(100)$

$r = 15.92$ m

Field Dimensions:

Minimum length $= l + (2)(6) + 2r$
$= 100 + 12 + (2)(15.92)$
$= 143.84$ m

Minimum width $= 2r + (2)(6)$
$= 2(15.92) + 12$
$= 43.84$ m

Option 2:

Choose the radius and calculate the length of the straight portion to meet the distance requirement.

Radius = 25 m

Perimeter (inside edge) $= 2\pi r + 2l$

300 m $= (2)(\pi)(25) + (2)(l)$

$l = 71.46$ m

Field Dimensions:

Minimum length $= l + (2)(6) + 2r$
$= 71.46 + 12 + (2)(25)$
$= 133.46$ m

Minimum width $= 2r + (2)(6)$
$= 2(25) + 12$
$= 62$ m

Notes

Grade 12 Applied Mathematics (40S)

Technology Appendix

TECHNOLOGY APPENDIX

The Technology Appendix provides basic information to help you learn how to use certain technology software and applications to answer the questions in this course.

You **are not expected** to use these applications, nor are you limited to the ones indicated below. There are many technology options available for use. These are simply examples and hints on how to use a variety of applications.

You **are expected** to use technology in this course. You may choose whatever calculator, software, apps, programs, or online applications that are available to you, and which meet the expected outcomes for this course. You are expected to read the handbook or access the online help provided with your choice of technology in order to learn how to use it effectively to fulfill the requirements of this course.

Technology is constantly evolving, and it is understood that after this course is printed, changes will be made to the technology applications explained below. It is your responsibility to learn how to use the upgraded versions or to find alternate applications if the ones outlined below cease to be available.

This appendix only provides basic keystrokes and examples of how to graph equations, find the coordinates of the vertex, find the x-intercepts, y-intercept, and solve for points on the line, as well as intersection points, given more than one graph. It shows how to use the TVM (Time Value of Money) solver on the TI-83 plus graphing calculator. Applications beyond what are outlined here are possible, and you are encouraged to explore and use the technology of your choice to its fullest potential.

On the midterm and final examinations, you will be expected to have access to technology. You must indicate which application or software you are using and show your work by including a printout of the screen, sketching the screen, or indicating the input and output values.

Check with your tutor/marker to be sure the graphing technology and the financial technology you are using will be appropriate for assignments and the final examination.

There are many different websites and apps available for use. Some are free and others must be purchased. Some places to explore include:

- Geogebra www.geogebra.org/cms/
- WinPlot http://faculty.madisoncollege.edu/alehnen/winptut/Install_Winplot.html
- Desmos https://www.desmos.com/calculator
- Meta Calculator www.meta-calculator.com/online/
- khan academy www.khanacademy.org/
- Graphmatica http://download.cnet.com/Graphmatica/3000-2053_4-10031384.html
- purple math www.purplemath.com/
- Texas Instruments TI-83 Plus graphing calculator online emulator—you can search for "TI-83 Flash Debugger" to find an online emulator, license requirements, and instructions for downloading.
- Texas Instruments TI-83/TI-84 Plus Graphing Calculator

Graphing Technology

There are many computer apps that can be used for graphing, such as Winplot, Geogebra, and Desmos. The following instructions show you how to graph on a TI-83 or TI-84 graphing calculator.

Graph an Equation

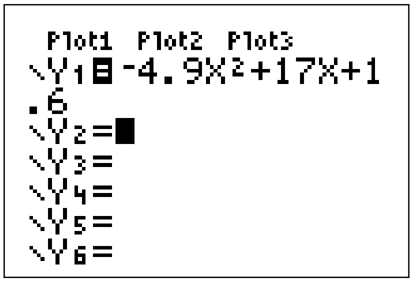
Graph the polynomial function $h = -4.9r^2 + 17t + 1.6$.

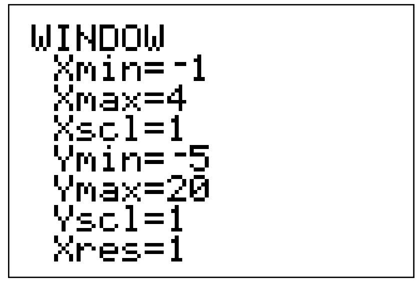
To access the equation editor, press [Y=].

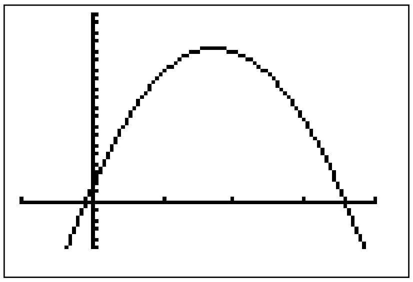
Enter the equation using [X, T, θ, *n*] for the variable and [x^2] for the power of 2.

Press [WINDOW] to set the maximum and minimum values of x and y, and the increments for the scale along the axes.

To view the graph, press GRAPH .

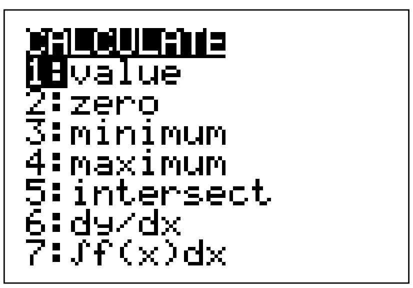


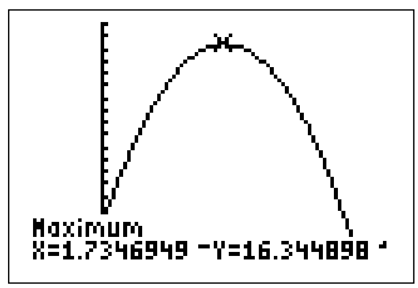




Determine the coordinates of the vertex.

In this case, the vertex is at a maximum *y*-value.





2nd CALC Maximum (above the TRACE key)

Using the blue arrow keys, move the cursor just to the left of the vertex and press ENTER .

Move the cursor just to the right of the vertex and press ENTER .

Move the cursor as close to the maximum point as possible and press ENTER .

Find the *x*-intercepts.

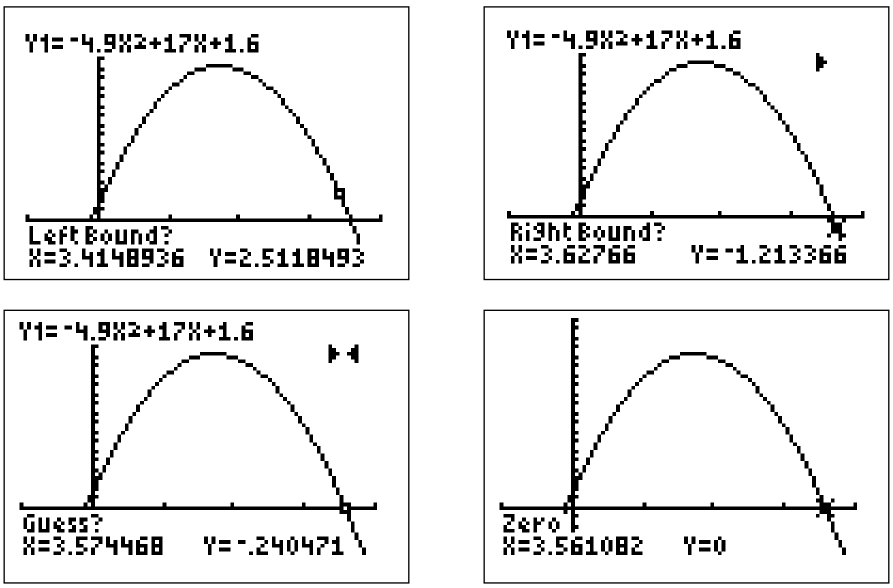
On the TI-83, the keystrokes are:

  2nd CALC Zero (above the TRACE key)

The line is read from left to right so to determine the value of the *x*-intercept near the right side of the screen, use the blue arrow keys to move the cursor just above the intercept on the right and press ENTER.

Move the cursor just below the intercept and press ENTER.

Move the cursor reasonably close to the intercept and press ENTER.



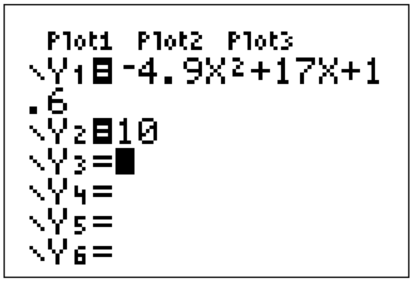
The line of a graph is read from left to right, so the left and right bounds may be above or below the *x*-axis, depending on the end behaviour of the graph.

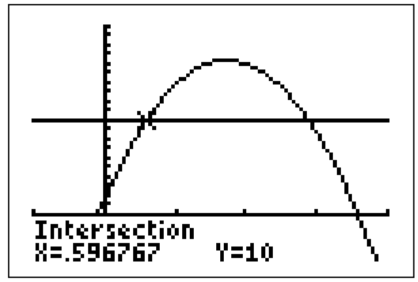
Find the point of intersection between two graphs.

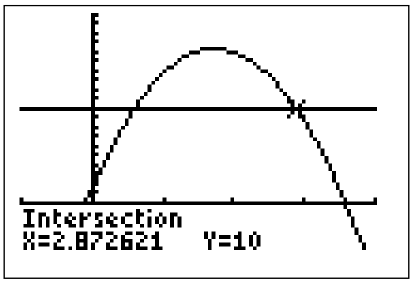
Enter the two equations into the equation editor and view the graph.

On the graphing calculator press 2nd CALC Intersect (above the TRACE key).

Move the cursor along each line as close to the point of intersection as possible. Press ENTER each time. Repeat for the other point of intersection.



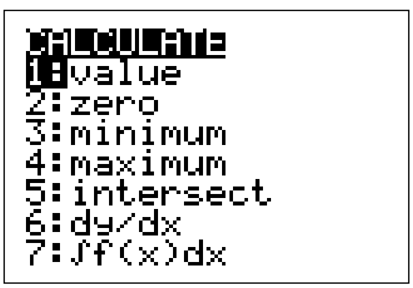


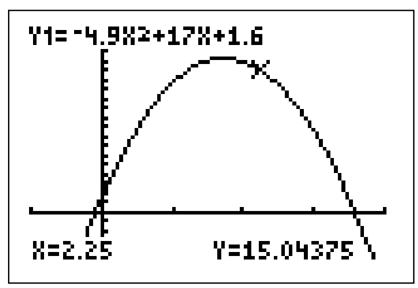


Finding a y-value for a given x-value:

2nd CALC Value (above the TRACE key)

To find the y-value of a given x-coordinate (e.g., $x = 2.25$), type the number 2.25 and then ENTER.

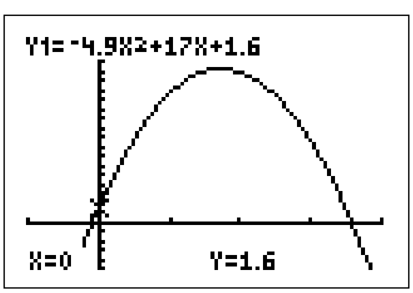




Find the y-intercept.

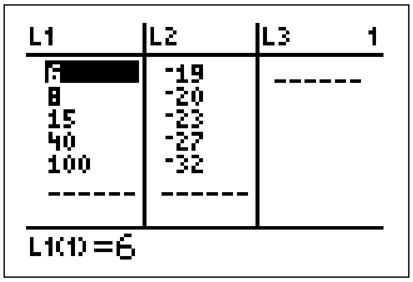
The y-intercept is found where $x = 0$.

On the graphing calculator, press 2nd CALC Value and type the value of 0. Press ENTER.



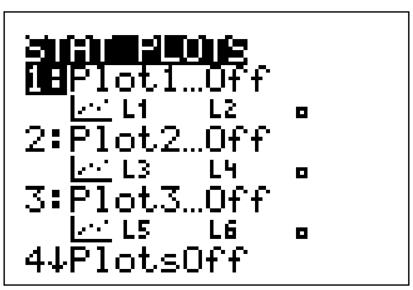
Plotting Points in a Scatterplot

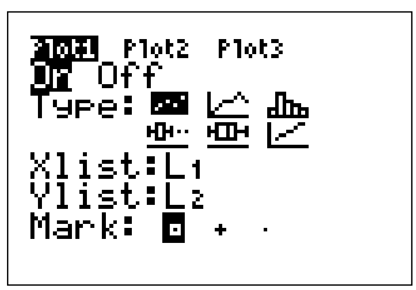
Press STAT EDIT to access the Lists. To clear an existing list of values, move the cursor to the heading L_1, press CLEAR and the blue down arrow, ▼. Do not press DELETE. Under L_1 for list 1, enter the values for the independent variable. Enter the dependent data in L_2. Make sure that you have exactly the same number of values in each list.

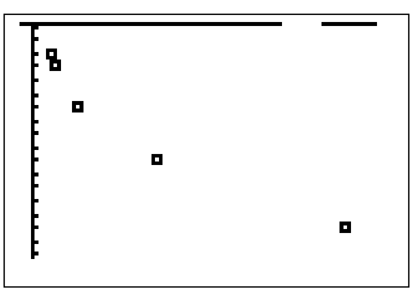


To activate the scatterplot, press 2nd STAT PLOT (above the Y= key). You may see that the plots are off.

To turn one of the plots on, press 1 and toggle the plot On by pressing ENTER. Make sure the first graph type is highlighted and the Xlist is L_1 and the Ylist is L_2. These can be changed by pressing 2nd 1 or 2nd 2. Select the large square point marker. To fit the window to the data, press ZOOM 9. Make sure no previously graphed equations in the Y= register or STAT PLOTs are toggled on. You may change the WINDOW and press GRAPH to display the data if you prefer a different window setting.







Determine a Regression Equation

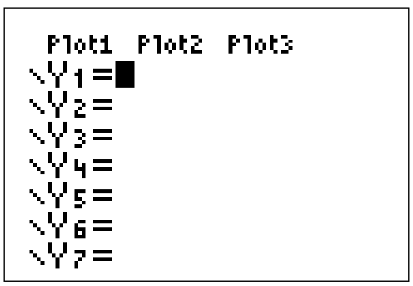
The equation of a line or curve of best fit is also called a regression equation. It models or represents the relationship between the variables and can be used to estimate values. You can use the TI-83 graphing calculator to find different types of regression equations. You can also use a computer app, such as Geogebra.

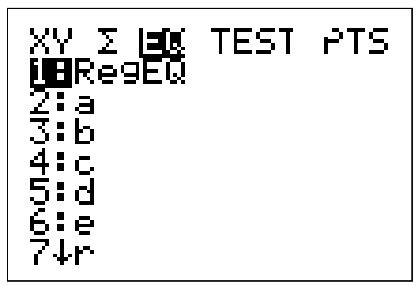
STAT Use the blue arrow to the right ▶ to select CALC and notice the list of possible regression equations. To determine a quadratic regression equation, select 5, the QuadReg.

QuadReg shows up on your home screen. Identify which lists of values you want the regression equation calculated on by pressing 2nd 1 , 2nd 2 ENTER for L_1 and L_2. The comma is above the 7 key.

You can manually enter the regression equation into the Y= register or copy and paste it using the variable memory.

Pressing Y= clears all existing equations. Position cursor next to \Y1=.

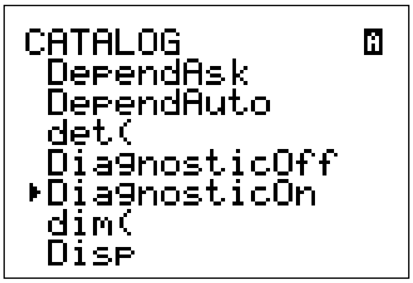


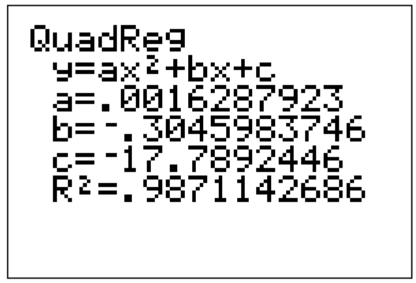


VARS 5 ▶ ▶ to highlight EQ ENTER and GRAPH.

The Coefficient of Determination, R^2 Value

To turn on the diagnostic function, press [2nd] [0] to access the catalogue. Notice the A in the top right corner indicating the ALPHA register is activated. Press [x^{-1}] (it has a green D above it) to get to the D section of the alphabetical listing and use the blue arrow keys to scroll down until you see the triangle indicator pointing to DiagnosticOn. Press [ENTER] and then [ENTER] again.

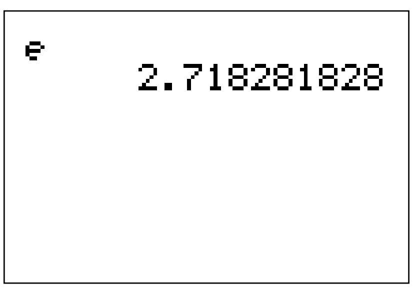




Not all technology applications display this value. It is not required for this course, but it is a useful tool for helping judge how well the function and the data correlate.

Value of *e*

[2nd] [e] [ENTER] (above the ÷ key)

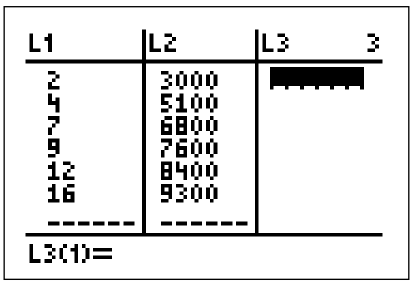


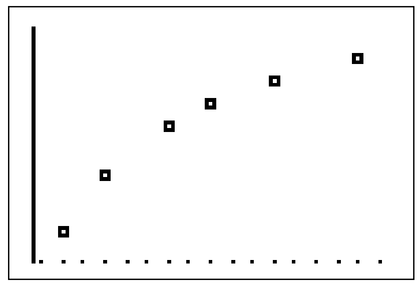
Graphing LOG Functions

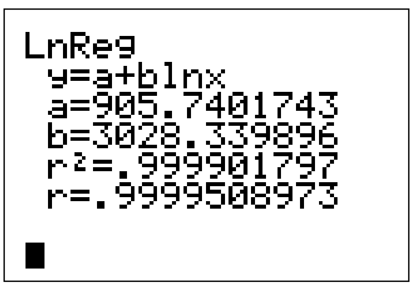
Since the graphing calculator uses base 10 as the default, the keystrokes to input a base 10 log function are [Y=] [LOG] [X] [)] [GRAPH]. Set an appropriate window to view the graph.

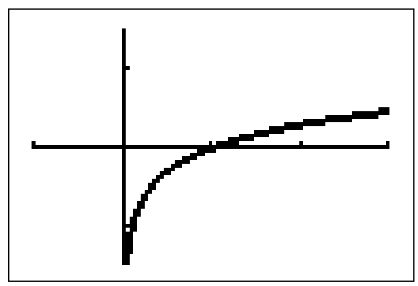
Log or Natural Log Regression Equation

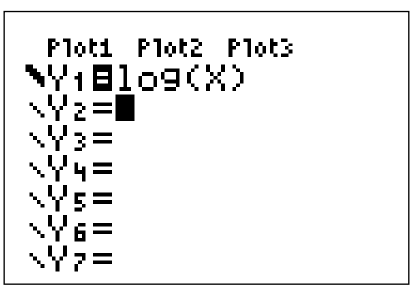
Input the data using the [STAT] editor, graph it using the [2nd] function [STAT PLOT] register, and display it by selecting [ZOOM] [9]. To find the regression equation, use [STAT] [▶] CALC and [9] :LnReg, indicating into which lists the data is entered.

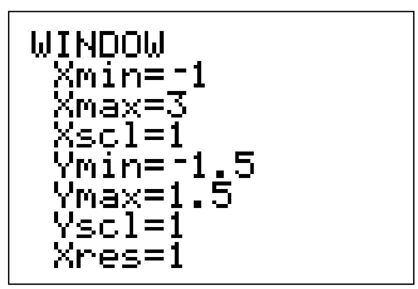








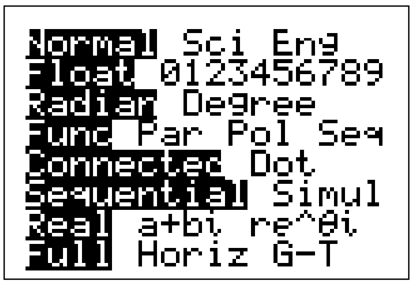


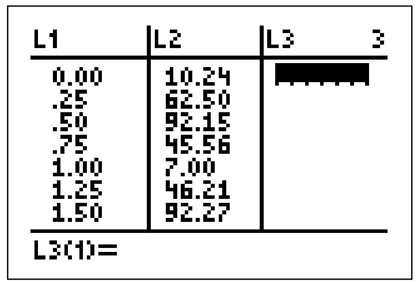


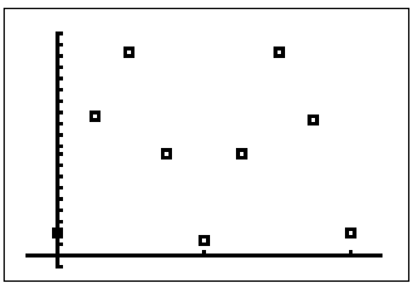
Find a Sinusoidal Regression Equation

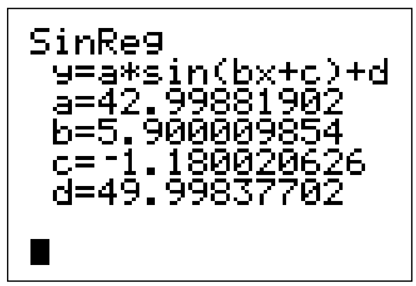
Input the data using the STAT editor, graph it using the 2nd function STAT PLOT register, and display it by selecting ZOOM 9. To find the regression equation, use STAT ▶ CALC and ALPHA C (on the PRGM key) or arrow down ▼ until you see C:SinReg ENTER. Copy or paste the equation into the Y= register, adjust the window settings, and GRAPH.

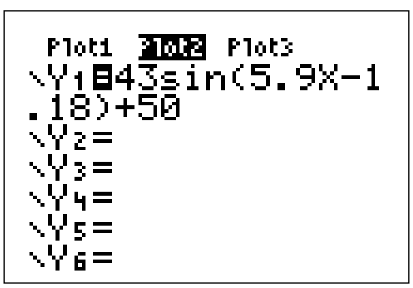
Make sure your calculator is in the appropriate MODE, either radians or degrees, depending on the units with which you are graphing.

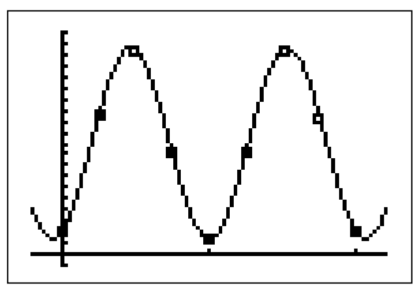












Factorial Notation

Enter the number for which you want to find the factorial. To find 6!, the key strokes are

[6] [MATH] [◀] to access the PRB (probability) menu [4] [ENTER].

```
6!
                720
```

Permutations

The ${}_nP_r$ button on the TI-83 plus graphing calculator is found in the probability register. To calculate ${}_{39}P_3$, input

[39] [MATH] [◀] to access the PRB (probability) menu [2] [3] [ENTER]

```
39 nPr 3
              54834
█
```

Combinations

The ${}_nC_r$ button on the TI-83 plus graphing calculator is found in the probability register. To calculate ${}_{30}C_5$, input

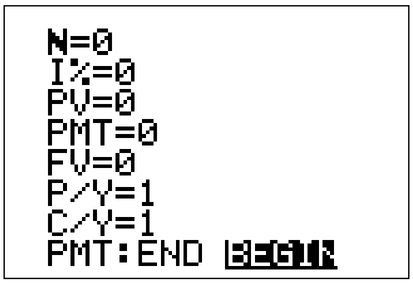
[30] [MATH] [◀] to access the PRB (probability) menu [3] [5] [ENTER]

```
30 nCr 5
             142506
```

TVM Solver

There are Time Value of Money (TVM) solvers available as computer software or as an app. These instructions show the use of the TI-83 and TI-84 graphing calculator. Other TVM solvers are similar.

The finance register is found on the TI-83 Plus calculator in the [APPS] menu. Select [1:Finance] and [1:TVM Solver]. It displays the Time Value of Money (TVM) variables.



N = number of payment periods

I% = interest rate per year, written as a percent

PV = present value

PMT = payment amount

FV = future value

P/Y = payment periods per year

C/Y = compounding periods per year

PMT: END BEGIN = toggle to indicate if the payment is made at the beginning or end of the compounding period

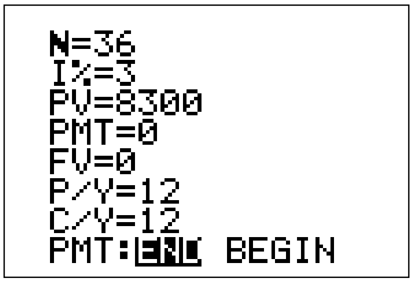
Note: Cash flow in and out (i.e., PMT payment to reduce the PV present value of a loan, or PMT contributions to grow the FV of an investment) will have opposite signs (one positive and one negative).

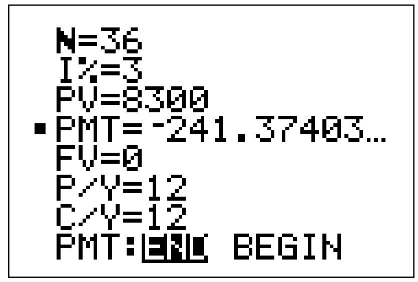
Given four of the values, the calculator solves for the fifth value. Move the cursor to the value you want to solve for by using the blue arrow keys, and press [ALPHA] [ENTER] or [SOLVE].

Example:

Kate takes out a bank loan to finance the purchase of a used car. She borrows $8300 at 3%, compounded monthly. She wants to repay her loan in three years. Determine the amount of her monthly payment.

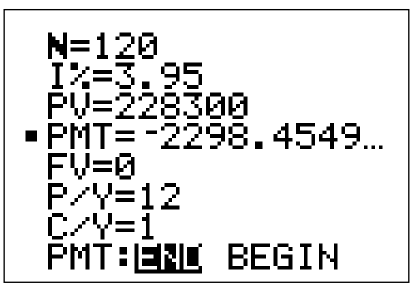
[ALPHA] [SOLVE] with cursor next to PMT.



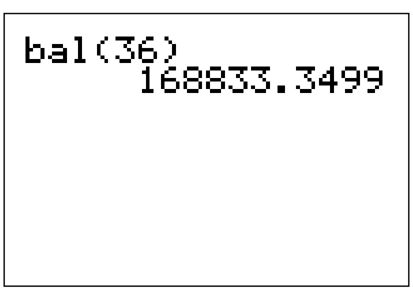


The graphing calculator can be used to determine the balance owing, the sum of principal paid, or sum of interest paid after a set number of payments for a given amortization schedule. It uses the values stored for N, I%, PV, and PMT in the TVM solver.

For example, if the information stored in the TVM solver is



and you want to determine the balance after three years worth of monthly payments (3 × 12 = 36 payments) [APP] [ENTER] [9], enter the number of periods [)] [ENTER].

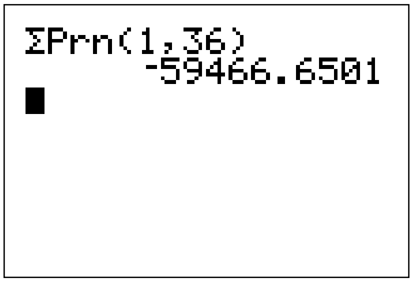


To determine the sum of principal paid in a certain interval of payments

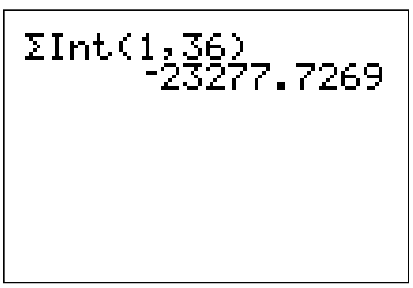
[APP] [ENTER] [0], enter the range of payments you want to calculate.

For example, if you want to know the amount of principal paid from the first to 36th payment, enter number [1] [,] [36] [)] [ENTER].

∑ is called sigma, and it is a symbol for summation. The sum of principal paid in the 1st to 36th payments is $59,466.65.



To calculate the sum of the interest paid in a certain interval of payments, press [APP] [ENTER] [ALPHA] [A] (or use the blue arrows to scroll down the list and press enter at A:∑Int(, to calculates the sum of interest). Enter the range of payments you want to calculate. For example, if you want to know the amount of interest paid from the first to 36th payment, enter number [1] [,] [36] [)] [ENTER].



The amount of interest paid from the 1st to 36th payment is $23,277.73.

The total amount paid in the first 36 payments can be determined by multiplying 36 × payment amount.

$36 \times 2298.4549 = 82744.3764$

Verify that ∑Prn + ∑Int = total paid.

Financial Calculations

Some online options for financial calculations include the following. Please note that different programs and apps may use different calculations and answers may vary slightly. The number of times the interest is compounded may be preset in some apps.

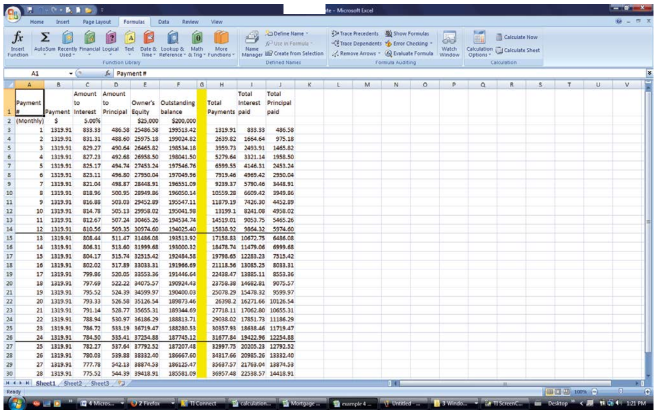
- www.fcac-acfc.gc.ca/eng/resources/toolCalculator/index-eng.asp (Financial Consumer Agency of Canada)
- www.getsmarteraboutmoney.ca/en/tools_and_calculators/calculators/Pages/AllCalculators.aspx#.Ue2GKawdT1E
- http://web.tmxmoney.com/calculators.php
- www.thecalculatorsite.com/finance/calculators/credit-card-payment-calculators.php
- www.zenwealth.com/BusinessFinanceOnline/TVM/TVMCalcWindow.html
- www.calculatorsoup.com/
- www.financialcalculator.org/credit-cards/credit-card-comparison-calculator

Financial institutions such as the TD Bank, the Royal Bank, BMO, CIBC, credit unions, as well as others, may have financial calculator apps on their websites. Use a search engine to find appropriate apps.

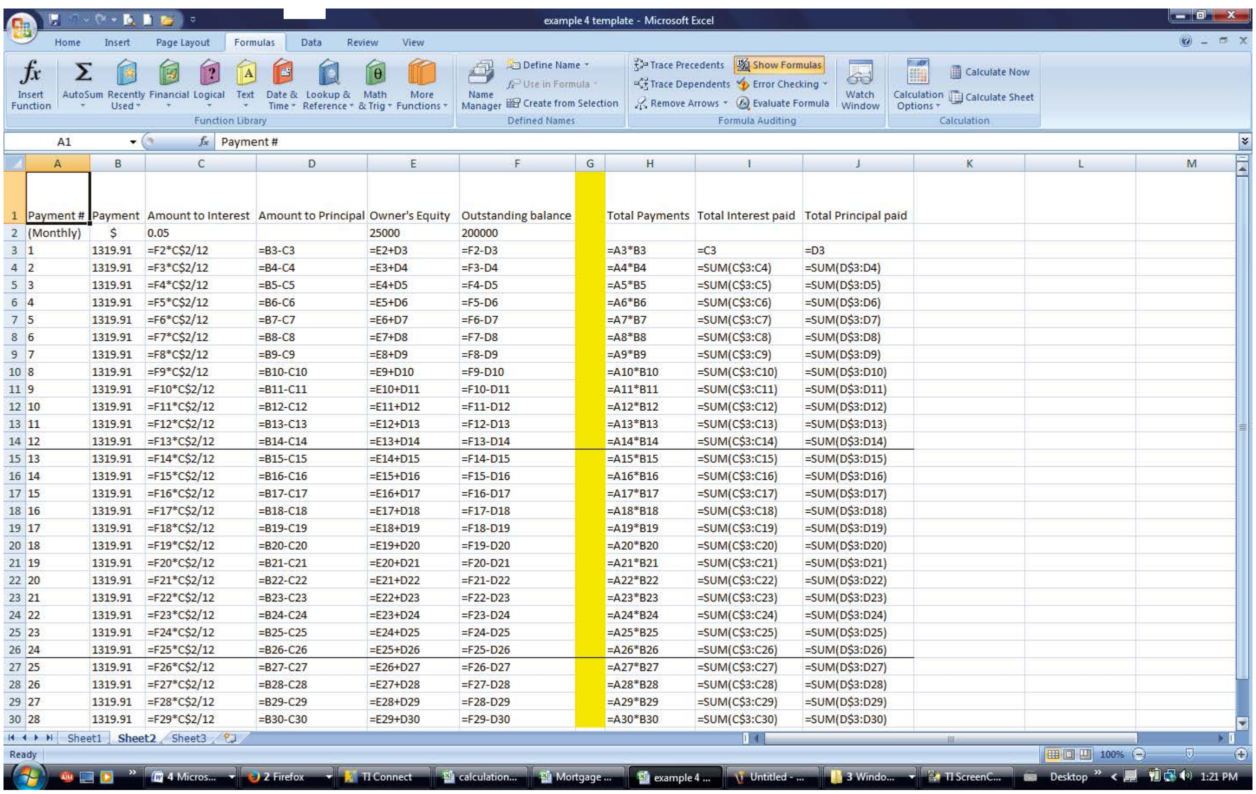
Investing information can be found at sites such as:

- www.getsmarteraboutmoney.ca/en/managing-your-money/planning/investing-basics/Pages/what-are-some-common-investment-strategies-i-may-read-about.aspx#.UkM8Y38dT1E
- www.investopedia.com/
- www.tmx.com/en/index.html (Toronto Stock Exchange)

Spreadsheet programs can be used to help in financial calculations. You may set up a template for mortgage calculations or to assist you in comparing the costs of renting, buying, and/or leasing.



	A	B	C	D	E	F	G	H	I	J
1	Payment #	Payment	Amount to Interest	Amount to Principal	Owner's Equity	Outstanding balance		Total Payments	Total Interest paid	Total Principal paid
2	(Monthly)	$	5.00%		$25,000	$200,000				
3	1	1319.91	833.33	486.58	25486.58	199513.42		1319.91	833.33	486.58
4	2	1319.91	831.31	488.60	25975.18	199024.82		2639.82	1664.64	975.18
5	3	1319.91	829.27	490.64	26465.82	198534.18		3959.73	2493.91	1465.82
6	4	1319.91	827.23	492.68	26958.50	198041.50		5279.64	3321.14	1958.50
7	5	1319.91	825.17	494.74	27453.24	197546.76		6599.55	4146.31	2453.24
8	6	1319.91	823.11	496.80	27950.04	197049.96		7919.46	4969.42	2950.04
9	7	1319.91	821.04	498.87	28448.91	196551.09		9239.37	5790.46	3448.91
10	8	1319.91	818.96	500.95	28949.86	196050.14		10559.28	6609.42	3949.86
11	9	1319.91	816.88	503.03	29452.89	195547.11		11879.19	7426.30	4452.89
12	10	1319.91	814.78	505.13	29958.02	195041.98		13199.1	8241.08	4958.02
13	11	1319.91	812.67	507.24	30465.26	194534.74		14519.01	9053.75	5465.26
14	12	1319.91	810.56	509.35	30974.60	194025.40		15838.92	9864.32	5974.60
15	13	1319.91	808.44	511.47	31486.08	193513.92		17158.83	10672.75	6486.08
16	14	1319.91	806.31	513.60	31999.68	193000.32		18478.74	11479.06	6999.68
17	15	1319.91	804.17	515.74	32515.42	192484.58		19798.65	12283.23	7515.42
18	16	1319.91	802.02	517.89	33033.31	191966.69		21118.56	13085.25	8033.31
19	17	1319.91	799.86	520.05	33553.36	191446.64		22438.47	13885.11	8553.36
20	18	1319.91	797.69	522.22	34075.57	190924.43		23758.38	14682.81	9075.57
21	19	1319.91	795.52	524.39	34599.97	190400.03		25078.29	15478.32	9599.97
22	20	1319.91	793.33	526.58	35126.54	189873.46		26398.2	16271.66	10126.54
23	21	1319.91	791.14	528.77	35655.31	189344.69		27718.11	17062.80	10655.31
24	22	1319.91	788.94	530.97	36186.29	188813.71		29038.02	17851.73	11186.29
25	23	1319.91	786.72	533.19	36719.47	188280.53		30357.93	18638.46	11719.47
26	24	1319.91	784.50	535.41	37254.88	187745.12		31677.84	19422.96	12254.88
27	25	1319.91	782.27	537.64	37792.52	187207.48		32997.75	20205.23	12792.52
28	26	1319.91	780.03	539.88	38332.40	186667.60		34317.66	20985.26	13332.40
29	27	1319.91	777.78	542.13	38874.53	186125.47		35637.57	21763.04	13874.53
30	28	1319.91	775.52	544.39	39418.91	185581.09		36957.48	22538.57	14418.91



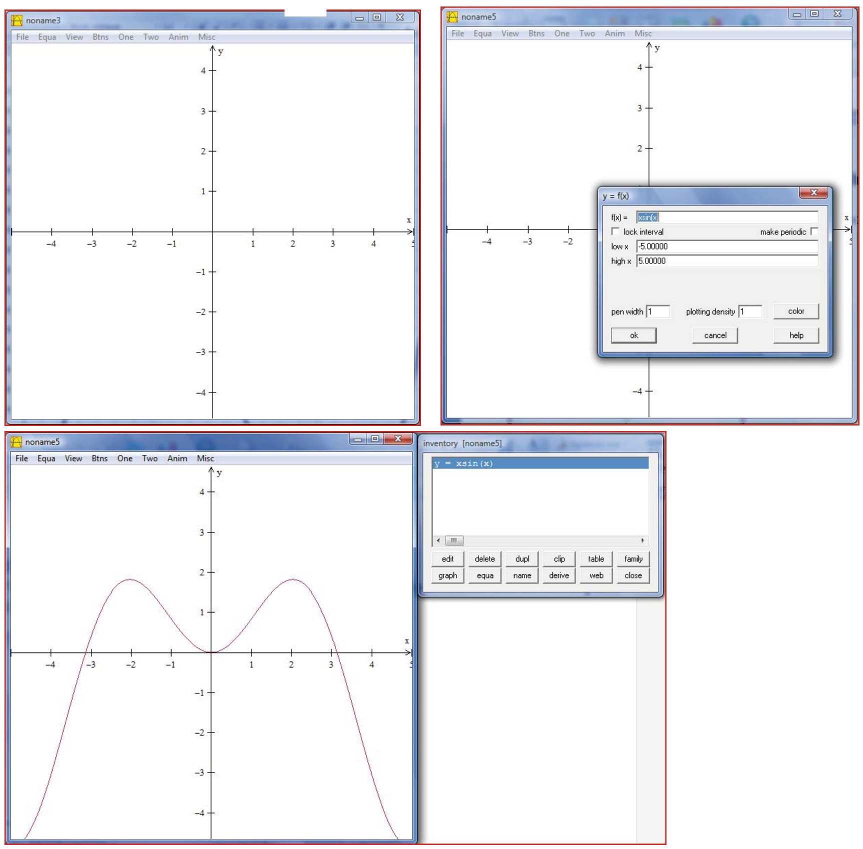
	A	B	C	D	E	F	G	H	I	J
1	Payment #	Payment	Amount to Interest	Amount to Principal	Owner's Equity	Outstanding balance		Total Payments	Total Interest paid	Total Principal paid
2	(Monthly)	$	0.05		25000	200000				
3	1	1319.91	=F2*C$2/12	=B3-C3	=E2+D3	=F2-D3		=A3*B3	=C3	=D3
4	2	1319.91	=F3*C$2/12	=B4-C4	=E3+D4	=F3-D4		=A4*B4	=SUM(C$3:C4)	=SUM(D$3:D4)
5	3	1319.91	=F4*C$2/12	=B5-C5	=E4+D5	=F4-D5		=A5*B5	=SUM(C$3:C5)	=SUM(D$3:D5)
6	4	1319.91	=F5*C$2/12	=B6-C6	=E5+D6	=F5-D6		=A6*B6	=SUM(C$3:C6)	=SUM(D$3:D6)
7	5	1319.91	=F6*C$2/12	=B7-C7	=E6+D7	=F6-D7		=A7*B7	=SUM(C$3:C7)	=SUM(D$3:D7)
8	6	1319.91	=F7*C$2/12	=B8-C8	=E7+D8	=F7-D8		=A8*B8	=SUM(C$3:C8)	=SUM(D$3:D8)
9	7	1319.91	=F8*C$2/12	=B9-C9	=E8+D9	=F8-D9		=A9*B9	=SUM(C$3:C9)	=SUM(D$3:D9)
10	8	1319.91	=F9*C$2/12	=B10-C10	=E9+D10	=F9-D10		=A10*B10	=SUM(C$3:C10)	=SUM(D$3:D10)
11	9	1319.91	=F10*C$2/12	=B11-C11	=E10+D11	=F10-D11		=A11*B11	=SUM(C$3:C11)	=SUM(D$3:D11)
12	10	1319.91	=F11*C$2/12	=B12-C12	=E11+D12	=F11-D12		=A12*B12	=SUM(C$3:C12)	=SUM(D$3:D12)
13	11	1319.91	=F12*C$2/12	=B13-C13	=E12+D13	=F12-D13		=A13*B13	=SUM(C$3:C13)	=SUM(D$3:D13)
14	12	1319.91	=F13*C$2/12	=B14-C14	=E13+D14	=F13-D14		=A14*B14	=SUM(C$3:C14)	=SUM(D$3:D14)
15	13	1319.91	=F14*C$2/12	=B15-C15	=E14+D15	=F14-D15		=A15*B15	=SUM(C$3:C15)	=SUM(D$3:D15)
16	14	1319.91	=F15*C$2/12	=B16-C16	=E15+D16	=F15-D16		=A16*B16	=SUM(C$3:C16)	=SUM(D$3:D16)
17	15	1319.91	=F16*C$2/12	=B17-C17	=E16+D17	=F16-D17		=A17*B17	=SUM(C$3:C17)	=SUM(D$3:D17)
18	16	1319.91	=F17*C$2/12	=B18-C18	=E17+D18	=F17-D18		=A18*B18	=SUM(C$3:C18)	=SUM(D$3:D18)
19	17	1319.91	=F18*C$2/12	=B19-C19	=E18+D19	=F18-D19		=A19*B19	=SUM(C$3:C19)	=SUM(D$3:D19)
20	18	1319.91	=F19*C$2/12	=B20-C20	=E19+D20	=F19-D20		=A20*B20	=SUM(C$3:C20)	=SUM(D$3:D20)
21	19	1319.91	=F20*C$2/12	=B21-C21	=E20+D21	=F20-D21		=A21*B21	=SUM(C$3:C21)	=SUM(D$3:D21)
22	20	1319.91	=F21*C$2/12	=B22-C22	=E21+D22	=F21-D22		=A22*B22	=SUM(C$3:C22)	=SUM(D$3:D22)
23	21	1319.91	=F22*C$2/12	=B23-C23	=E22+D23	=F22-D23		=A23*B23	=SUM(C$3:C23)	=SUM(D$3:D23)
24	22	1319.91	=F23*C$2/12	=B24-C24	=E23+D24	=F23-D24		=A24*B24	=SUM(C$3:C24)	=SUM(D$3:D24)
25	23	1319.91	=F24*C$2/12	=B25-C25	=E24+D25	=F24-D25		=A25*B25	=SUM(C$3:C25)	=SUM(D$3:D25)
26	24	1319.91	=F25*C$2/12	=B26-C26	=E25+D26	=F25-D26		=A26*B26	=SUM(C$3:C26)	=SUM(D$3:D26)
27	25	1319.91	=F26*C$2/12	=B27-C27	=E26+D27	=F26-D27		=A27*B27	=SUM(C$3:C27)	=SUM(D$3:D27)
28	26	1319.91	=F27*C$2/12	=B28-C28	=E27+D28	=F27-D28		=A28*B28	=SUM(C$3:C28)	=SUM(D$3:D28)
29	27	1319.91	=F28*C$2/12	=B29-C29	=E28+D29	=F28-D29		=A29*B29	=SUM(C$3:C29)	=SUM(D$3:D29)
30	28	1319.91	=F29*C$2/12	=B30-C30	=E29+D30	=F29-D30		=A30*B30	=SUM(C$3:C30)	=SUM(D$3:D30)

WinPlot Graphing Utility

This program can be downloaded for free and is useful for graphing and analyzing functions.

Graph an Equation

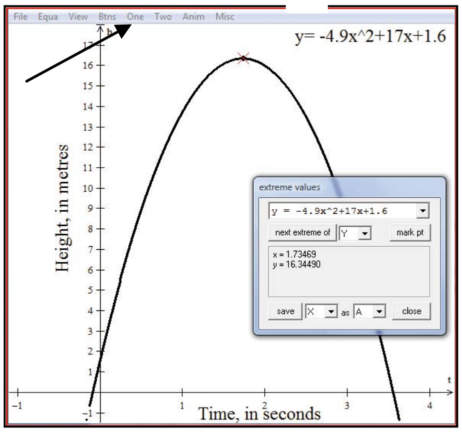
Click WINDOW and 2-dim with Use defaults selected. Select the Equa menu tab and choose Explicit. Enter the equation, specify the max and min *x*-values, and click OK. The Inventory window allows you to edit the equation.

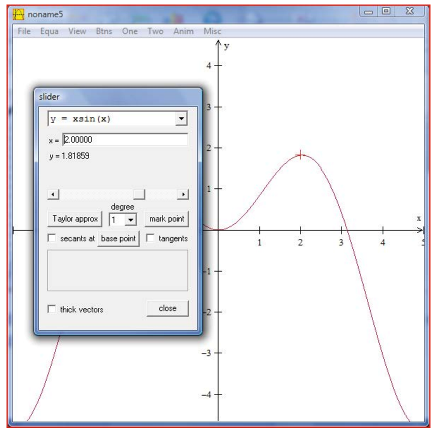


Use the View menu tab to adjust the window, show the grid, and add labels.

Find Values (and Vertex)

To find values of coordinate points along the graph, choose the One tab on the top menu. Select Extremes ... to locate the minimum and maximum *y*-values, such as the vertex.

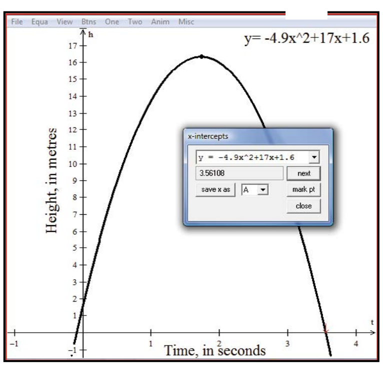




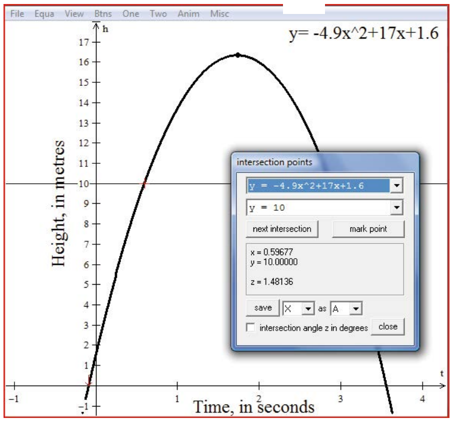
To find the *y*-value of a coordinate when the *x*-value is known, select the One tab and choose Slider. Enter the known value for *x* and press return.

The *x*-intercepts can be found by selecting the One tab and the Zeros ... function.

The first intercept in this example is stated is at −.09169. Select Next and the value of the second intercept is given as 3.56108.



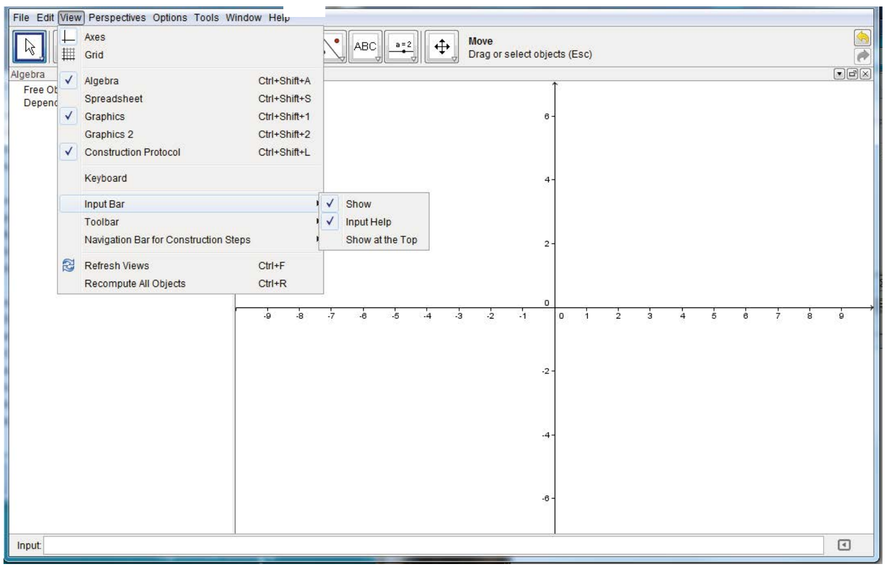
To find a point of intersection between two graphs, use the Two tab on the menu bar and select Intersections ... The first intersection of the lines is at 0.59677 and the second is at 2.87262.



Geogebra Graphing Utility

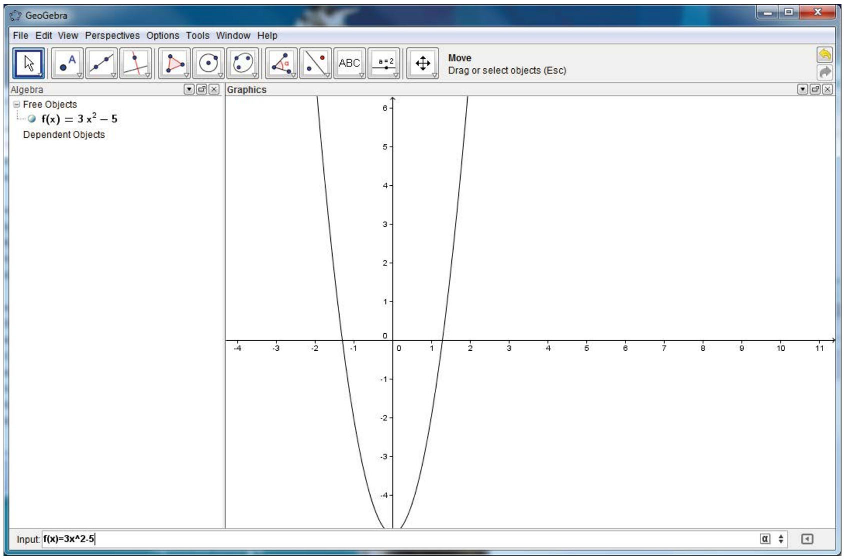
This program can be downloaded at www.geogebra.org. It can be used for graphing and analyzing functions and finding regression equations given a set of points. You can find detailed instructions for Geogebra on their website (www.geogebra.org).

The basic structure of the Geogebra window is shown in the image below. Under the Menu bar at the top, you can see (in the image below) the Algebra window on the left, the Graphics window on the right, and the Input Bar at the bottom. If you don't see one of the windows or the input bar, you can change that by using the View drop-down menu.



Graphing a Function

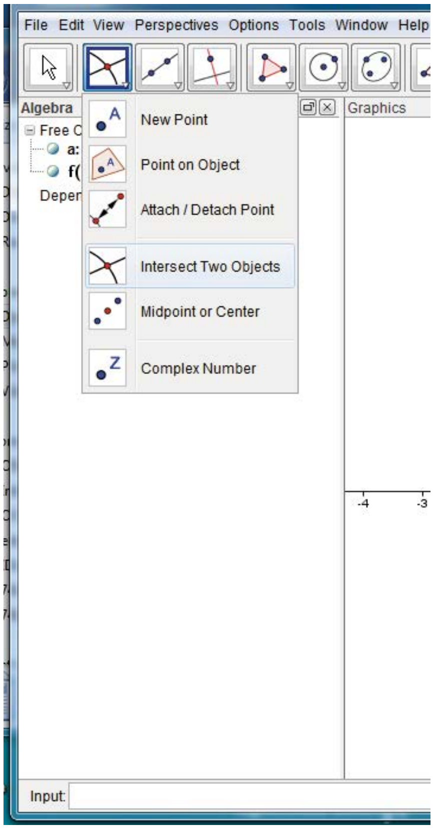
You can enter a function to graph using the Input Bar by typing a function equation such as $f(x) = x * \sin(x)$ or $y = 3x \wedge 2 - 5$, as shown in the image below. Note the "^" symbol is used to indicate an exponent is to follow.

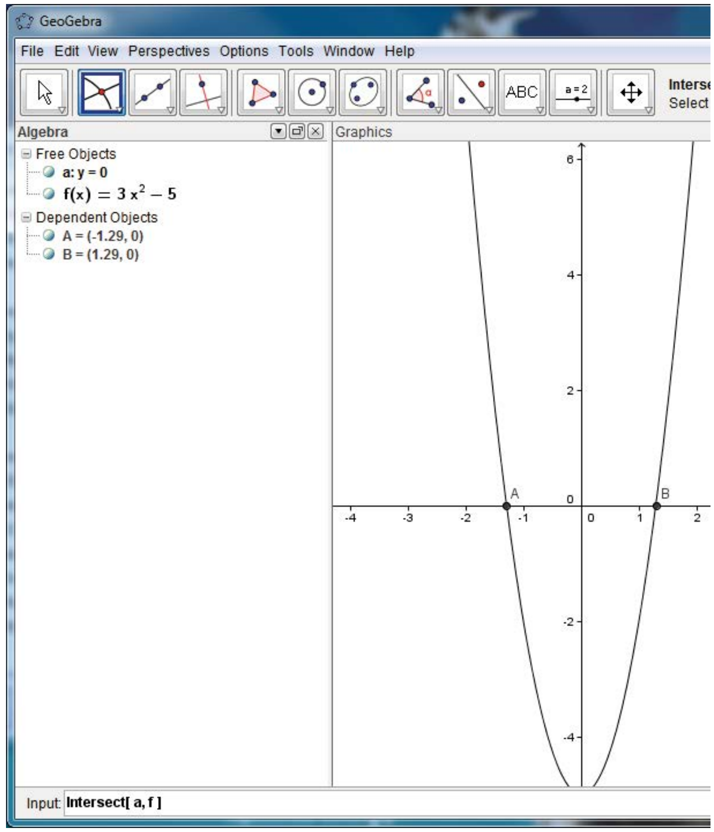


Finding *x*-intercepts (and Other Function Intersections)

After the function is graphed, you can find the *x*-intercepts by graphing the function $y = 0$, which is given the function name, "*a*" by the Geogebra program. Then choose the option Intersect Two Objects from the menu bar. Select $y = 0$ and $f(x) = 3x^2 - 5$ as your two objects to intersect. See the windows shown below. As an alternative to using the menu bar, you can enter the command directly into the Input Bar. As you begin to type "intersect," you will notice after entering the first few letters, "inters," that a list of options becomes available. One of the options is Intersect[<Object>, <Object>]. Press Enter on that option to have it displayed in the Input Bar and then replace one <Object> with the name *a* and the other <Object> with the name *f* (different letters may be assigned in your window). In the Input Bar, you will see Intersect[a,f]. Press enter and you will see the window as shown on the right.

The *x*-intercepts are plotted in the graphics window and their values are listed in the algebra window. In this case, one *x*-intercept is −1.29 and the other *x*-intercept is 1.29.



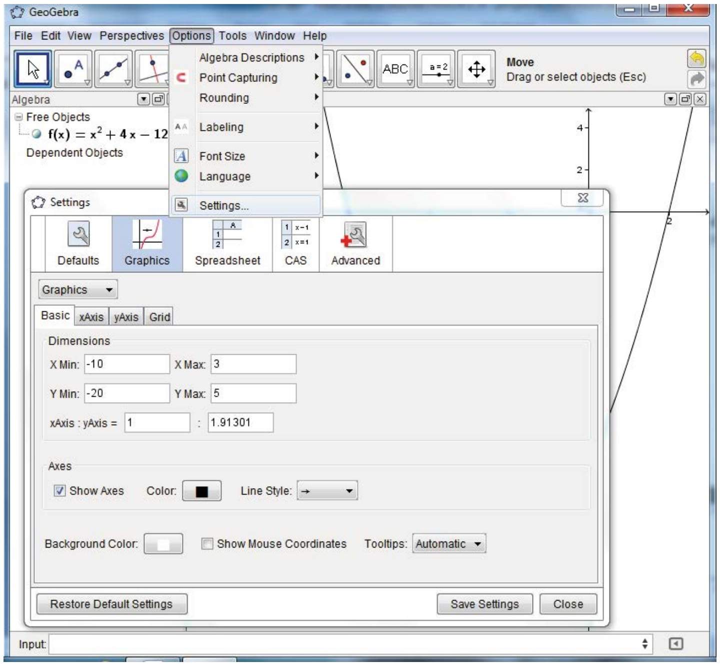


Finding the y-intercept

The process is the same as when finding *x*-intercepts. After the function is graphed, you can find the *y*-intercept by graphing the relation $x = 0$, which will be given a function name (maybe "*b*") by the Geogebra program. Find the *y*-intercept by using Intersect Two Objects from the menu and selecting the function under investigation and the vertical line, $x = 0$.

Changing the Graphics Window Settings

Under the Options menu, select Settings to open a window. Choose the Graphics settings. Enter the desired values for X Min and X Max and for Y Min and Y Max (e.g., 5). The values chosen for the window shown are XMin = –10, XMax = 3, YMin = –20 and YMax = 5.



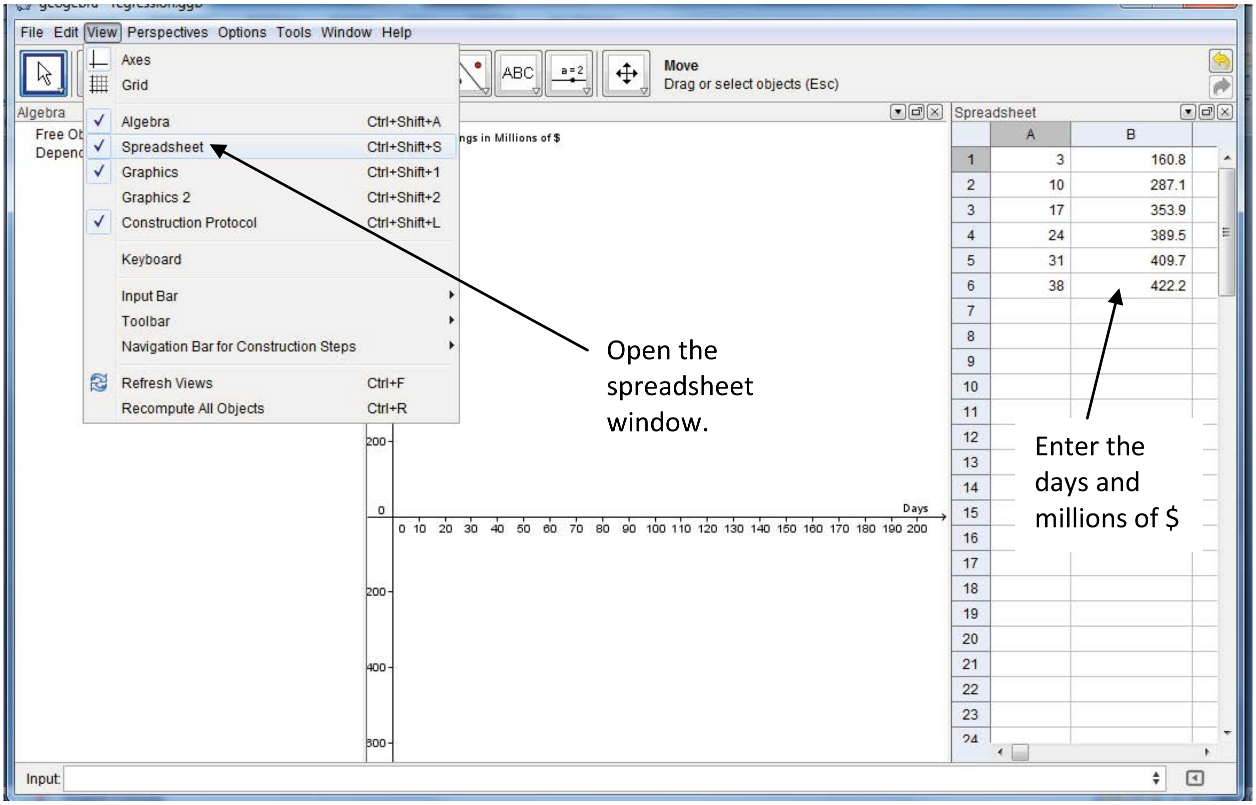
Note: You can also change the min and max axis values directly on the graphics window by selecting the Move Graphics View arrows and then clicking on one axis at a time and dragging it toward or away from the origin (0, 0).

Finding a Regression Equation using Geogebra

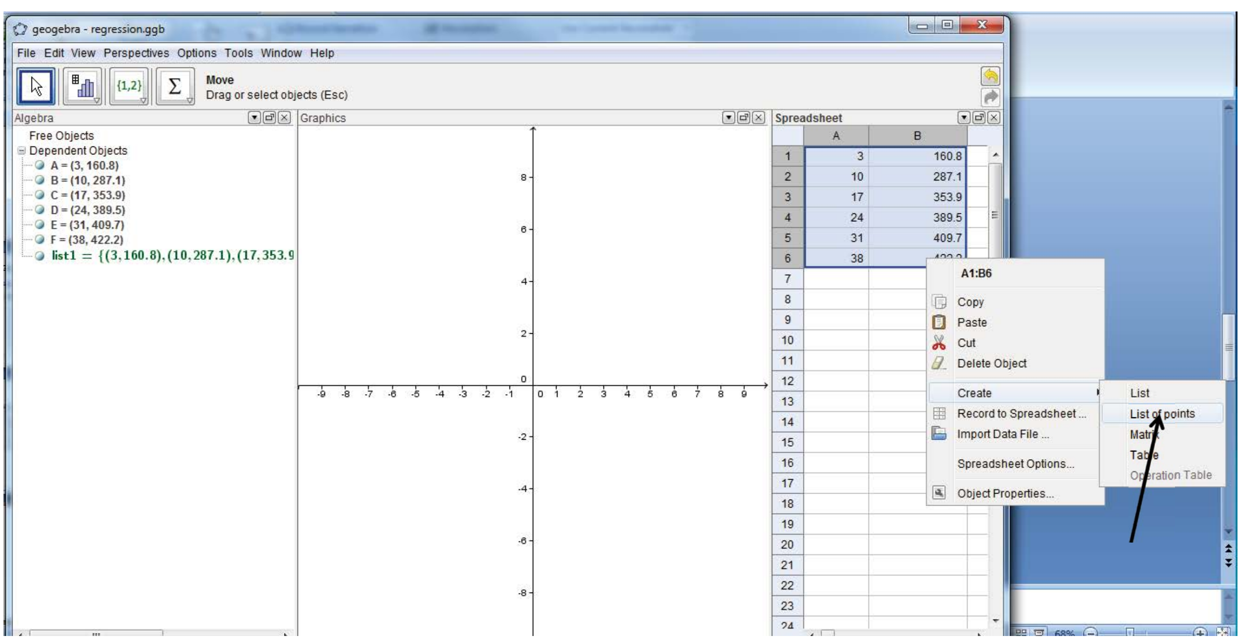
You can use Geogebra to enter data points and determine a line of best fit through those data points. The details for using the program can be found at www.geogebra.org. As well, the example below shows the required steps. The example uses the data in the table below, which shows gross earnings at the box office for a movie.

Days	3	10	17	24	31	38
Gross (million $)	160.8	287.1	353.9	389.5	409.7	422.2

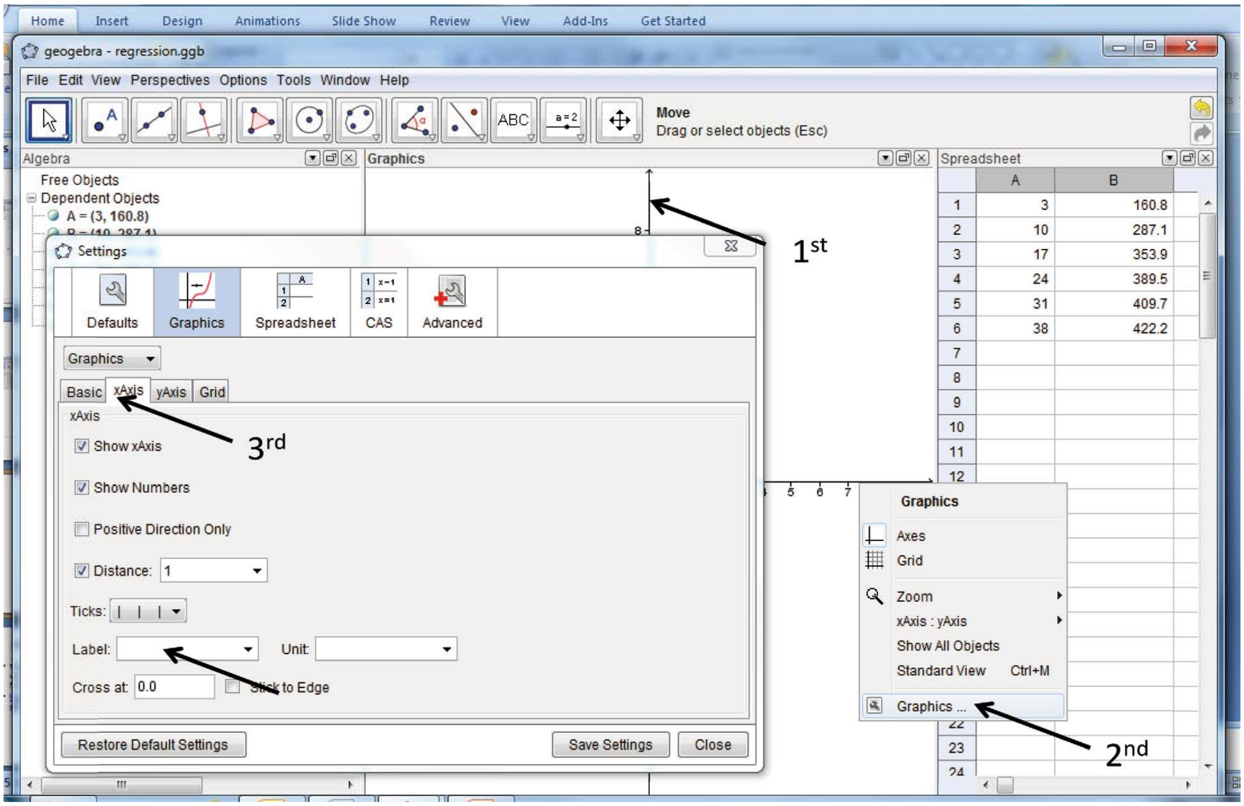
Open Geogebra and go to the View menu. Select the Algebra window, the Graphics window, and the Spreadsheet window, and enter the data in two columns of the Spreadsheet window as shown below.



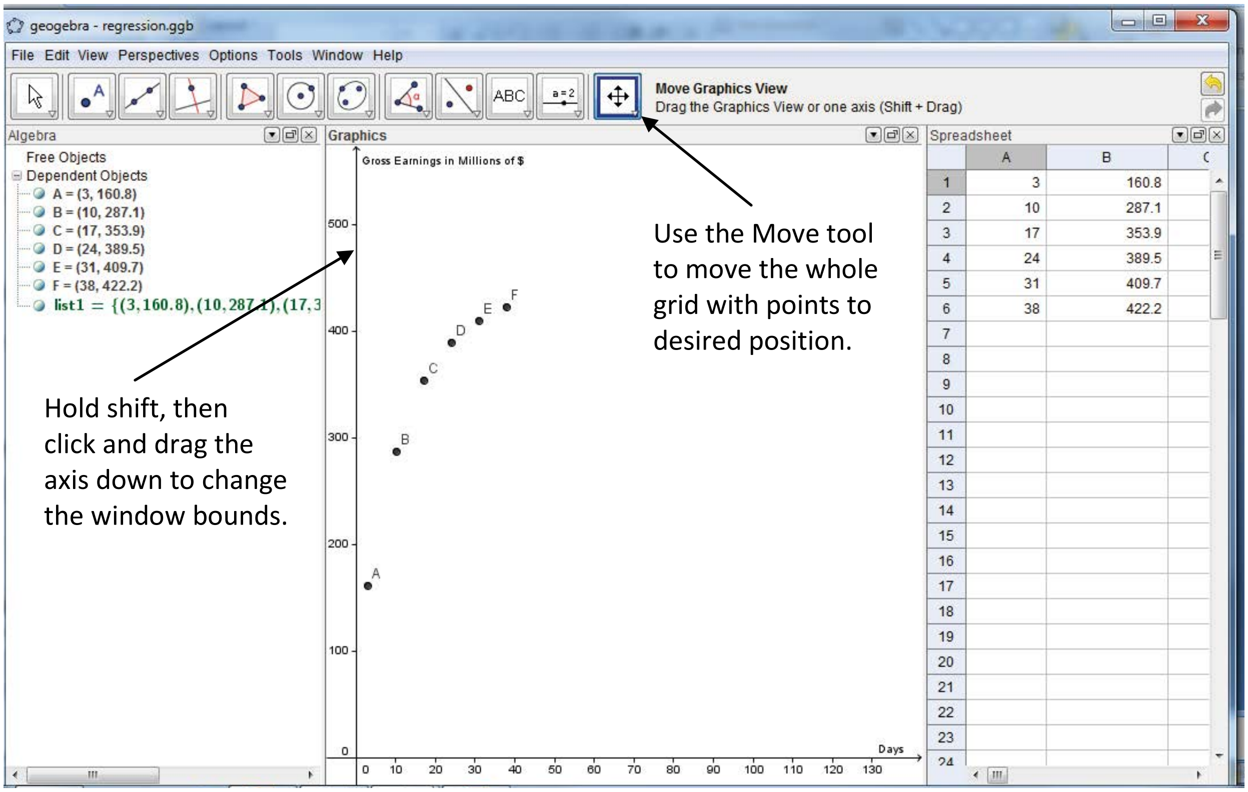
Create a list of coordinates by clicking and dragging to select the desired spreadsheet cells, and then right click to create the List of points. All of the points and the list appear in the Algebra window as shown below.



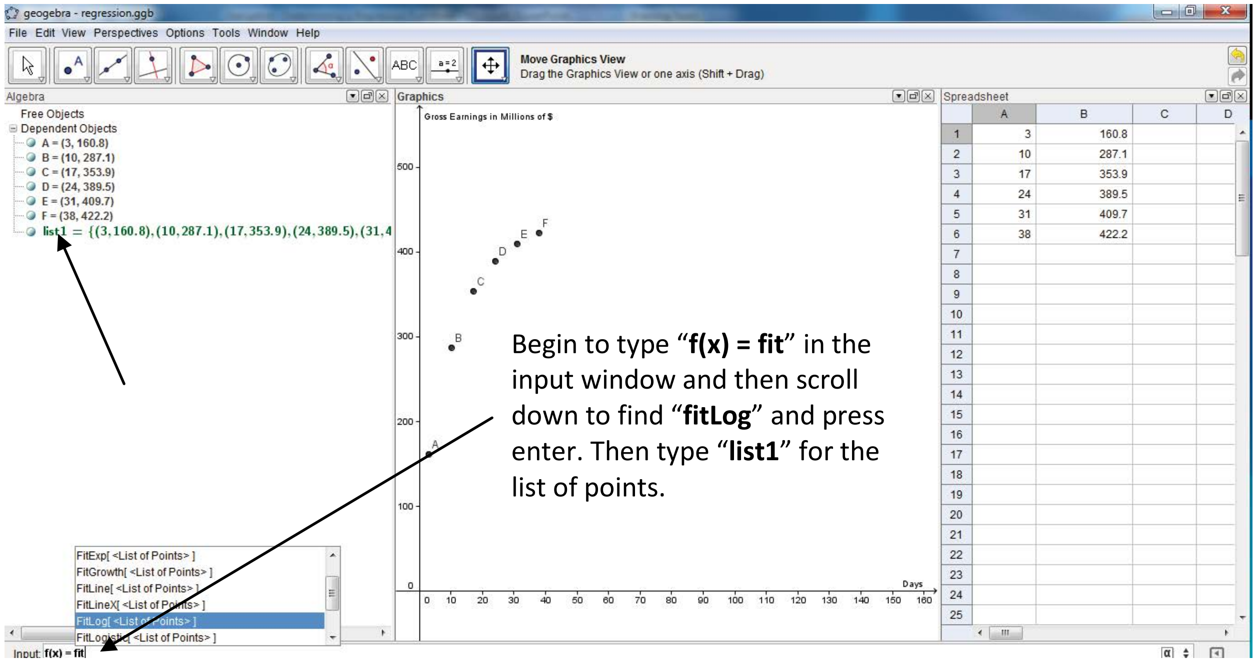
You have the flexibility to change the Graphics window details. Right click on one of the axes (1st) to open the Graphics window (2nd). Set the basic *x*-axis and *y*-axis parameters that you want (3rd).



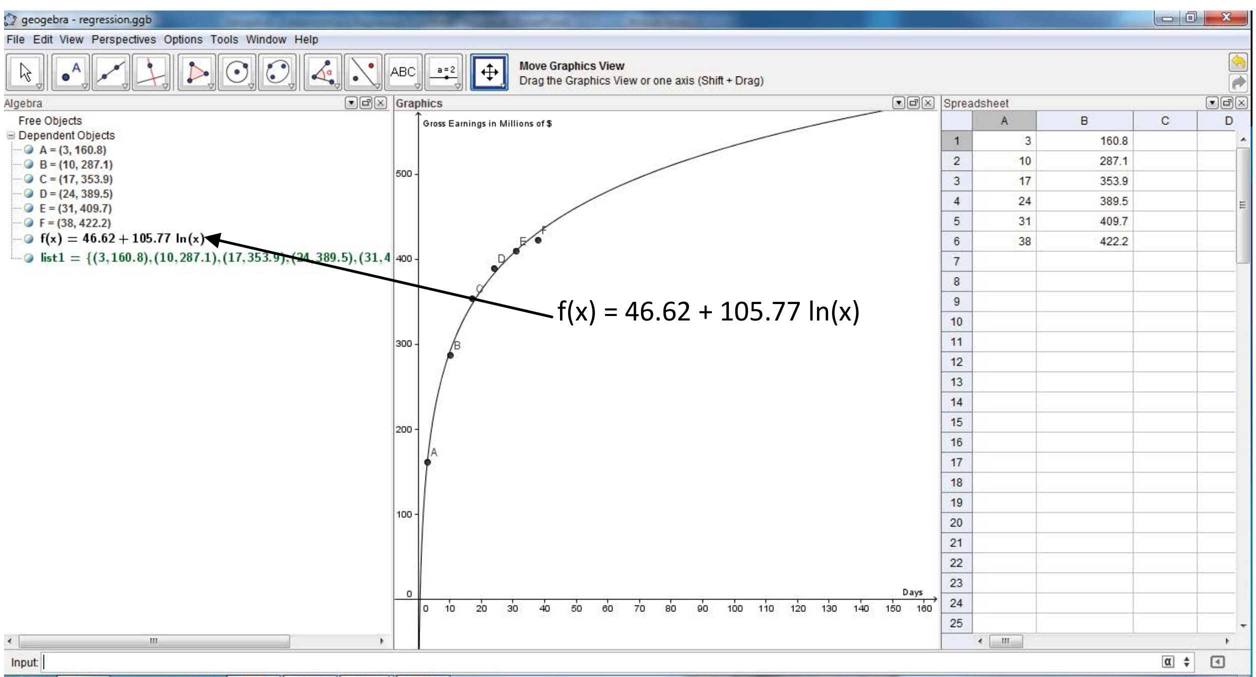
You can adjust the grid in the window by selecting the **Move** tool as shown. Then adjust the viewing window by holding shift, then click and drag the y-axis to change the window bounds. Do the same with the x-axis.



You can try a variety of function types. This example shows the use of regression to find the best fit of a Log function to the list of data points.



The regression function is graphed in the Graphics window and the equation is displayed in the Algebra window.



By specifying a list of points, you can find the best fit of other function types using the input line (for example, fitExp, fitLine, fitLog, fitPoly, or fitSin). When you use fitPoly, you must also specify the degree of the polygon (2 for quadratic, 3 for cubic, etc), along with the list of points.

Grade 12 Applied Mathematics (40S)

Midterm Practice Examination

Grade 12 Applied Mathematics

Midterm Practice Examination

Name: ______________________________________

Student Number: ______________________________

Attending ❑ Non-Attending ❑

Phone Number: ______________________________

Address: ____________________________________

____

____

For Marker's Use Only

Date: ____________________________________

Midterm Mark: ______ /100 = _________ %

Comments:

Instructions

The midterm examination is based on Modules 1 to 4 of the Grade 12 Applied Mathematics course. It is worth 20% of your final mark in this course.

Time

You will have a maximum of **3.0 hours** to complete the midterm examination.

Format

The format of the examination will be as follows:

Part A: Functions	44 marks
Part B: Mathematics Research Project	5 marks
Part C: Logical Reasoning	22 marks
Part D: Probability	29 marks
Total	100 marks

(see over)

Notes:

You are allowed to bring the following to the examination: pens/pencils (2 or 3 of each), metric and imperial rulers, a graphing and/or scientific calculator, and your Midterm Exam Resource Sheet. Your Midterm Exam Resource Sheet must be handed in with the examination. Graphing technology (either computer software or a graphing calculator) **is required** to complete this examination.

Show all calculations and formulas used. Use all decimal places in your calculations and round the final answers to the correct number of decimal places. Include units where appropriate. Clearly state your final answer. Final answers without supporting calculations or explanations will **not** be awarded full marks. Indicate equations and/or keystrokes used in calculations.

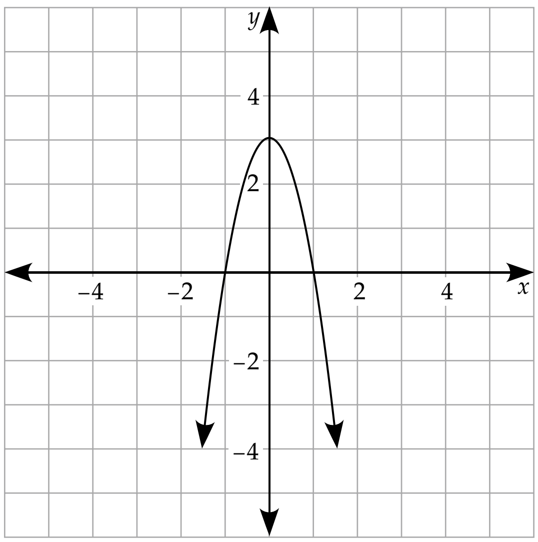
When using graphing technology, include a screenshot or printout of graphs **or** sketch the image and indicate the window settings (maximum and minimum *x*- and *y*-values), increments, and axis labels, including units.

Name: ____________________________________

Answer all questions to the best of your ability. Show all your work.

Part A: Functions *(44 marks)*

1. Given the function graphed below, complete the table with the required information. *(8 marks)*



Type of Function	
Degree	
Coordinates of x-intercept(s)	
Coordinates of y-intercept(s)	
End behaviour	
Absolute or relative maximum and/or minimum—state y-value	
Domain	
Range	

2. The loudness of a sound, L, measured in decibels (dB), is defined by the formula $L = 10 \log\left(\frac{I}{I_o}\right)$, where $\left(\frac{I}{I_o}\right)$ is the intensity of a sound, I, in W/cm^2, compared to the minimum sound intensity, I_o, your ear can detect. *(5 marks)*

   a) The volume at a concert is measured to be 115 dB. Calculate the ratio, $\left(\frac{I}{I_o}\right)$, for this sound in W/cm^2. *(3 marks)*

Name: ____

b) The value of $\left(\frac{I}{I_o}\right)$ for a jet engine is approximately $3.16227766 \times 10^{12}$ W/cm^2.

How many decibels is the loudness of a jet engine? *(2 marks)*

3. When an object is projected vertically upward, its height, h, in metres, after t seconds, is given by the equation $h = -4.9t^2 + vt + s$, where v is the initial velocity of the object and s is the initial height (if any) above the ground. *(9 marks)*

   a) From your location at the top of a cliff, 55 m above a lake, you throw a rock up into the air with a velocity of 15 m/sec. It lands in the water directly below you. Write an equation that models this situation. *(1 mark)*

b) Sketch a graph of this equation, showing the height of the rock as a function of time. Label the axes of the graph. *(3 marks)*

Name: ______________________________

c) Determine the maximum height the rock attains and the number of seconds it takes to attain the maximum height. Show your work or explain your strategy. Round your final answers to the nearest tenth. *(3 marks)*

d) How long is the rock in the air? Show your work or explain your strategy. Round your final answer to the nearest tenth. *(2 marks)*

4. The village of Winkler was first established in 1906 and was recognized as a city in 2002. The city population recorded during various years between 1911 and 2011 is reported below. *(11 marks)*

Year	Year #	Population
1911	5	458
1921	15	812
1931	25	1,005
1941	35	957
1951	45	1,331
1961	55	2,529
1991	85	6,400
2001	95	7,999
2011	105	10,670

Source: http://en.wikipedia.org/wiki/Winkler,_Manitoba

a) Plot the population data on a graph using the Year # as the independent variable. Sketch and label your graph below. *(4 marks)*

b) Using technology, find the exponential regression equation that best fits this data. *(2 marks)*

Name: ____

c) Winkler was officially incorporated as a town when the population was approximately 1835 inhabitants. According to this function model, during what year did this happen? *(3 marks)*

d) Using the exponential regression equation or technology, determine the approximate population of the city of Winkler during its centennial anniversary in 2006. *(2 marks)*

5. Match each the following equations with its corresponding graph. Write the letter of the equation below the correct graph. *(6 marks)*

a) $y = \log(x) + 5$

b) $y = 5^x$

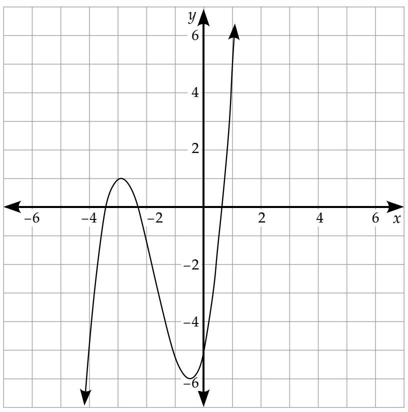
c) $y = \ln x$

d) $y = e^x$

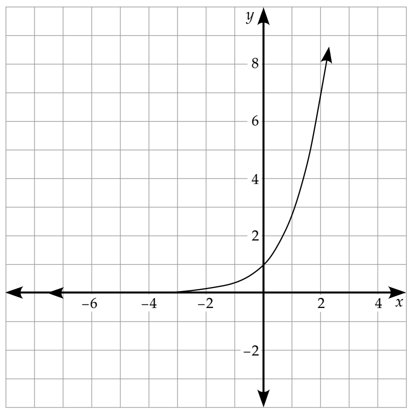
e) $y = 0.5(x^2 + x - 6)$

f) $y = x^3 + 5x^2 + 4x - 5.1$

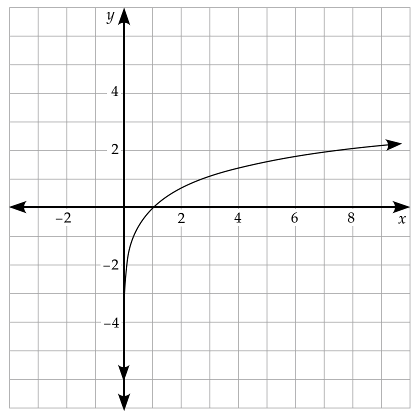
A



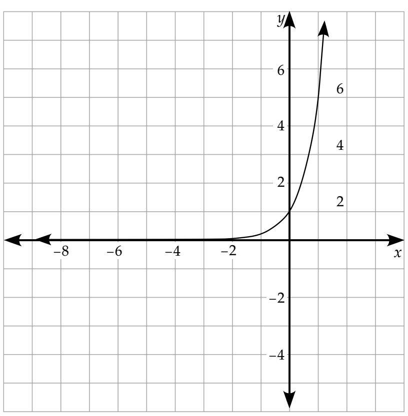
B



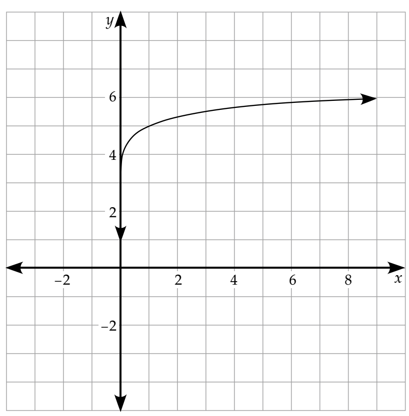
C



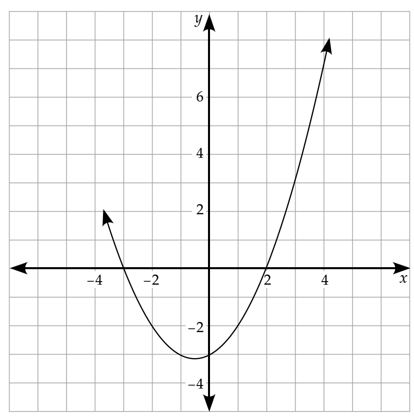
D



E



F



Name: ______________________________

6. Use graphing technology to sketch the graph of $y = \log(x)$ and complete the table of information. *(5 marks)*

Domain	
Range	
x-intercept	
Equation of asymptote	

Part B: Mathematics Research Project *(5 marks)*

7. List **five** things to consider when assessing the accuracy, reliability, and relevance of data and information. *(5 marks)*

Name: ____

Part C: Logical Reasoning *(22 marks)*

8. Create a Venn diagram to represent the following information and answer the question below. *(5 marks)*

   Students at a private dance studio may take ballet, hip hop, or tap dance classes.

   3 students at the studio take all three.

   7 students take tap and hip hop.

   5 are in ballet and hip hop classes.

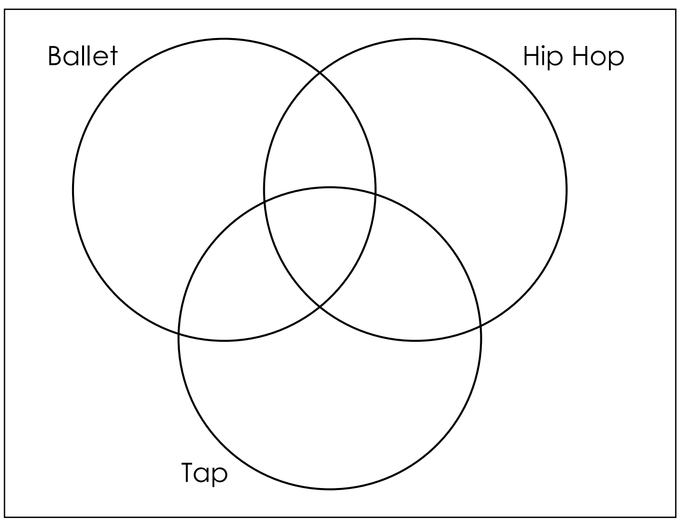
   All students who take tap also take either ballet or hip hop.

   21 of the students take hip hop or tap.

   14 are in ballet classes.

   12 students take tap.

   How many students attend the studio? ____________________



9. In a class of 25 students, 8 are on student council, 5 are on the volleyball team, and 2 students are on both the council and the team. How many students are neither on the council nor on the team? Justify your answer. *(3 marks)*

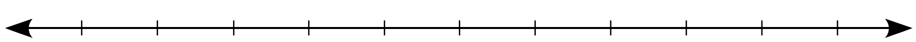
10. Sets A, B, and C are defined as:

    $A = \{x \mid x < 5, x \in \Re\}$

    $B = \{x \mid x \geq -2, x \in \Re\}$

    $C = \{x \mid -3 < x \leq 2, x \in \Re\}$

    a) Graph sets A, B, and C using the number line below. *(3 marks)*



    b) Using set notation, define the following: *(3 marks)*

    i) $A \cap B =$

    ii) $B \cup C =$

    iii) $A \cap B' =$

Name: ____

11. Answer the following questions based on the conditional statement, "If a polygon is an octagon, then it has 8 sides." *(8 marks)*

   a) State the hypothesis and the conclusion of this statement. *(1 mark)*

   b) Complete the following truth table for the conditional statement. Justify your results by describing the possible states of the hypothesis and conclusion for each of the four possible cases. *(5 marks)*

   Answer:

Hypothesis	Conclusion	Conditional Statement

   c) Write the converse of the given statement. *(1 mark)*

   d) Is the given statement biconditional? Explain. *(1 mark)*

Part D: Probability *(29 marks)*

12. An event consists of tossing three coins and noting if they land Heads, H, or Tails, T. Write the sample space, S, of this event. *(1 mark)*

13. Each of the letters in the words PRACTICE EXAM are written on individual cards and placed in a hat. *(5 marks)*

    a) One card is randomly drawn from the hat. Determine the probability of drawing a C. Write the final answer as a reduced fraction. *(1 mark)*

    b) One card is randomly drawn from the hat. Determine the probability of drawing a vowel. Write the final answer as a percentage. *(1 mark)*

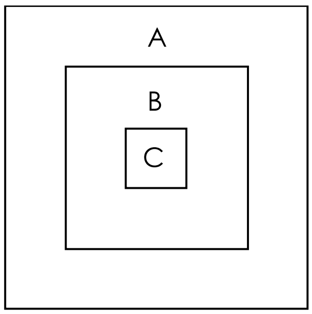
    c) One card is drawn, the letter is noted, and the card is replaced. A second card is drawn. Determine the probability of drawing the P and then the M. Write the final answer as a fraction. *(1 mark)*

    d) One card is drawn, the letter is noted, and then a second card is drawn without replacing the first card. What is the probability of drawing two consonants? Write the final answer as a decimal to the nearest thousandth. *(1 mark)*

    e) One card is drawn at random from the hat. Determine the probability that it is not the X. *(1 mark)*

Name: ___________________________________

14. A dart is thrown randomly and lands on a square dartboard with three areas, as pictured below. Note that the sides of square A are 5 times as long as the sides of square C, and the sides of square B are 3 times as long as the sides of square C.



Calculate: *(4 marks)*

a) Probability that a dart lands in area A

b) Probability that a dart lands in area B

c) Odds in favour of a dart landing in area B

d) Odds against a dart landing in area C

15. An experiment consists of randomly drawing a card from a deck of 50 cards numbered from 1 to 50. What is the probability of drawing a card that is an even number or a multiple of 5? *(4 marks)*

16. Three bags contain marbles:

Bag 1	Bag 2	Bag 3
3 green	2 green	4 green
2 red	1 red	3 white
1 white	5 black	2 black

Answer the following questions regarding one or more of the bags. *(11 marks)*

a) Using Bag 1, determine the probability of drawing a green marble. *(1 mark)*

b) Using Bag 2, determine the odds in favour of drawing a green marble. *(1 mark)*

c) Using Bag 3, determine the odds against drawing a black marble. *(1 mark)*

Name: ______________________________

d) Create a tree diagram to represent the sample space for the event of drawing one marble from Bag 1 and then one marble from Bag 2. Include the probability of drawing each colour along each branch. *(5 marks)*

e) When choosing one marble from each of Bag 1 and Bag 2, what is the probability of choosing two marbles of the same colour? *(2 marks)*

f) When choosing one marble from each of Bag 1 and Bag 2, what is the probability of choosing two marbles of different colours? *(1 mark)*

17. A customer enters a restaurant. The probability that the customer orders one of either a steak or a salad is $\frac{8}{11}$. The probability that the customer orders steak is $\frac{2}{11}$, while the probability of ordering salad is $\frac{7}{11}$. What is the probability that the customer orders both a steak and a salad? *(4 marks)*

Grade 12 Applied Mathematics (40S)

Midterm Practice Examination
Answer Key

Midterm Practice Examination
Answer Key

Name: ______________________________

Student Number: ______________________________

Attending ❑ Non-Attending ❑

Phone Number: ______________________________

Address: ______________________________

____

____

For Marker's Use Only

Date: ______________________________

Midterm Mark: ______ /100 = ______ %

Comments:

Instructions

The midterm examination is based on Modules 1 to 4 of the Grade 12 Applied Mathematics course. It is worth 20% of your final mark in this course.

Time

You will have a maximum of **3.0 hours** to complete the midterm examination.

Format

The format of the examination will be as follows:

Part A: Functions	44 marks
Part B: Mathematics Research Project	5 marks
Part C: Logical Reasoning	22 marks
Part D: Probability	29 marks
Total	100 marks

(see over)

Notes:

You are allowed to bring the following to the examination: pens/pencils (2 or 3 of each), metric and imperial rulers, a graphing and/or scientific calculator, and your Midterm Exam Resource Sheet. Your Midterm Exam Resource Sheet must be handed in with the examination. Graphing technology (either computer software or a graphing calculator) **is required** to complete this examination.

Show all calculations and formulas used. Use all decimal places in your calculations and round the final answers to the correct number of decimal places. Include units where appropriate. Clearly state your final answer. Final answers without supporting calculations or explanations will **not** be awarded full marks. Indicate equations and/or keystrokes used in calculations.

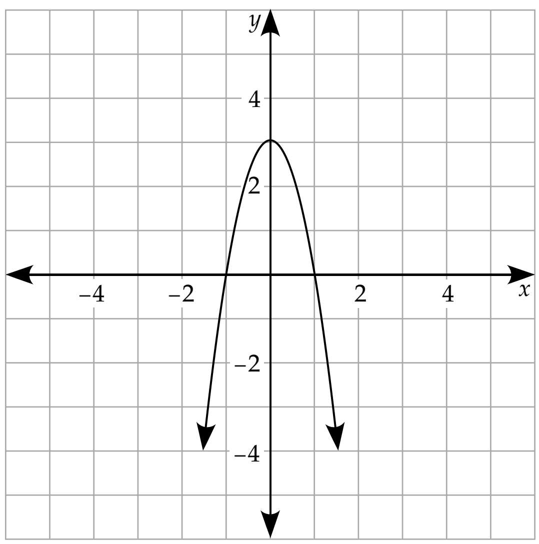
When using graphing technology, include a screenshot or printout of graphs **or** sketch the image and indicate the window settings (maximum and minimum *x*- and *y*-values), increments, and axis labels, including units.

Name: ____

Answer all questions to the best of your ability. Show all your work.

Part A: Functions *(44 marks)*

1. Given the function graphed below, complete the table with the required information. *(8 marks)*



Answer: (Module 1, Lesson 1)

Type of Function	Quadratic
Degree	2
Coordinates of x-intercept(s)	$(-1, 0)$, $(1, 0)$
Coordinates of y-intercept(s)	$(0, 3)$
End behaviour	Quadrant III to Quadrant IV
Absolute or relative maximum and/or minimum—state y-value	Absolute maximum at $y = 3$
Domain	$\{x \mid x \in \Re\}$
Range	$\{y \mid y \le 3, y \in \Re\}$

2. The loudness of a sound, L, measured in decibels (dB), is defined by the formula $L = 10 \log \left(\frac{I}{I_o}\right)$, where $\left(\frac{I}{I_o}\right)$ is the intensity of a sound, I, in W/cm^2, compared to the minimum sound intensity, I_o, your ear can detect. *(5 marks)*

   a) The volume at a concert is measured to be 115 dB. Calculate the ratio, $\left(\frac{I}{I_o}\right)$, for this sound in W/cm^2. *(3 marks)*

   Answer: (Module 1, Lesson 6)

$$L = 10 \log \left(\frac{I}{I_o}\right)$$

$$115 = 10 \log \left(\frac{I}{I_o}\right)$$

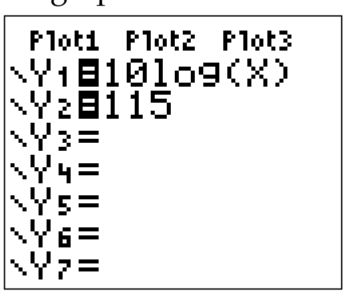
$$\frac{115}{10} = \frac{10}{10} \log \left(\frac{I}{I_o}\right)$$

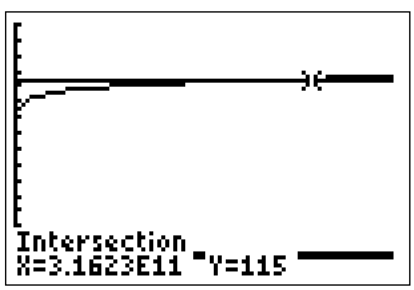
$$11.5 = \log \left(\frac{I}{I_o}\right)$$

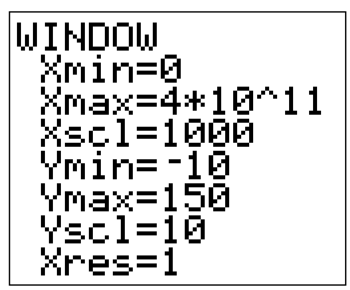
$$10^{11.5} = \left(\frac{I}{I_o}\right)$$

$$\left(\frac{I}{I_o}\right) = 3.16227766 \times 10^{11}$$

Or graphical solution:







The value of the ratio, $\left(\frac{I}{I_o}\right)$, of this sound is approximately 3.16×10^{11} W/cm^2.

b) The value of $\left(\frac{I}{I_o}\right)$ for a jet engine is approximately $3.16227766 \times 10^{12}$ W/cm^2.

How many decibels is the loudness of a jet engine? *(2 marks)*

Answer: (Module 1, Lesson 6)

$$L = 10 \log\left(\frac{I}{I_o}\right)$$

$$L = 10 \log\left(3.16227766 \times 10^{12}\right)$$

$$L = 125$$

The loudness of a jet engine is 125 dB.

3. When an object is projected vertically upward, its height, *h*, in metres, after *t* seconds, is given by the equation $h = -4.9t^2 + vt + s$, where *v* is the initial velocity of the object and *s* is the initial height (if any) above the ground. *(9 marks)*

a) From your location at the top of a cliff, 55 m above a lake, you throw a rock up into the air with a velocity of 15 m/sec. It lands in the water directly below you. Write an equation that models this situation. *(1 mark)*

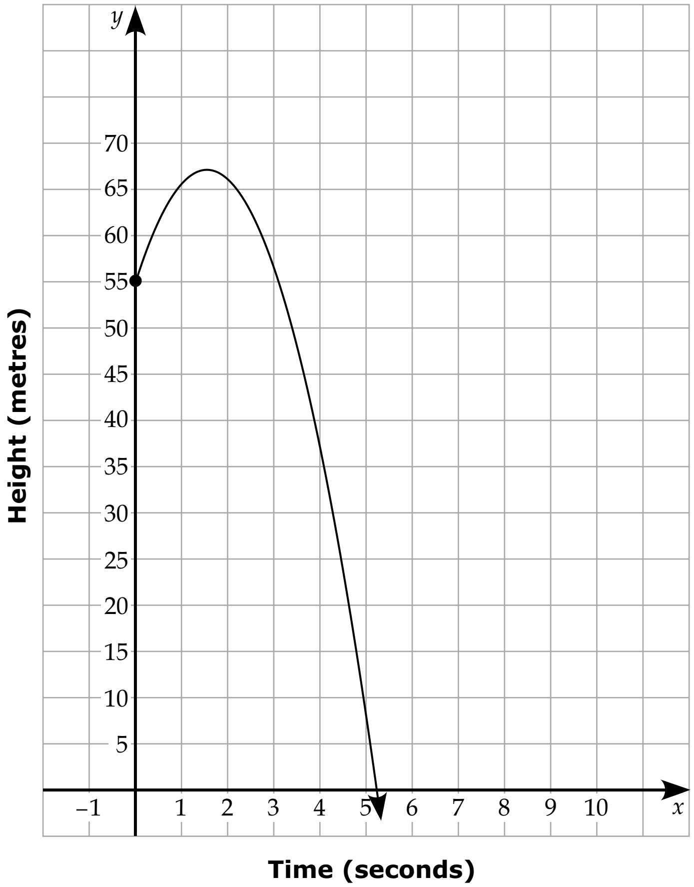
Answer: (Module 1, Lesson 2)

$h = -4.9t^2 + vt + s$

$h = -4.9t^2 + (15)t + (55)$

b) Sketch a graph of this equation, showing the height of the rock as a function of time. Label the axes of the graph. *(3 marks)*

Answer: (Module 1, Lesson 2)

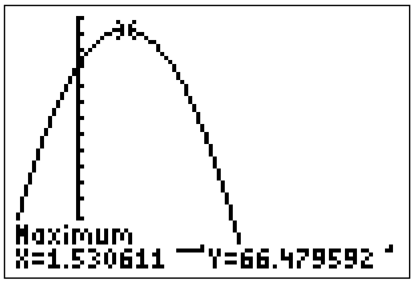


Note: Students may use technology and either sketch or print the screenshot. The graph must have scales and labels on axes, have a parabola shape, and have the correct y-intercept.

c) Determine the maximum height the rock attains and the number of seconds it takes to attain the maximum height. Show your work or explain your strategy. Round your final answers to the nearest tenth. *(3 marks)*

Answer: (Module 1, Lesson 2)

This solution shows a TI-84 graphing calculator screen finding the maximum point using 2nd– Calc, Maximum. The coordinates of the vertex are (1.53, 66.48).

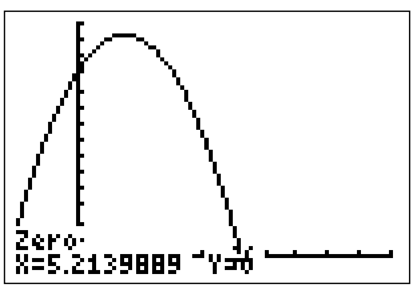


The rock reaches a height of 66.5 m, 1.5 seconds after it is released.

d) How long is the rock in the air? Show your work or explain your strategy. Round your final answer to the nearest tenth. *(2 marks)*

Answer: (Module 1, Lesson 2)

This solution shows a TI-84 graphing calculator screen finding the *x*-intercept using 2nd– Calc, Zero.



The rock hits the water after 5.2 seconds.

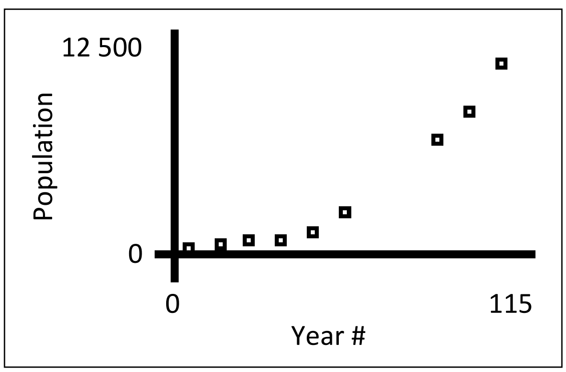
4. The village of Winkler was first established in 1906 and was recognized as a city in 2002. The city population recorded during various years between 1911 and 2011 is reported below. *(11 marks)*

Year	Year #	Population
1911	5	458
1921	15	812
1931	25	1,005
1941	35	957
1951	45	1,331
1961	55	2,529
1991	85	6,400
2001	95	7,999
2011	105	10,670

Source: http://en.wikipedia.org/wiki/Winkler,_Manitoba

a) Plot the population data on a graph using the Year # as the independent variable. Sketch and label your graph below. *(4 marks)*

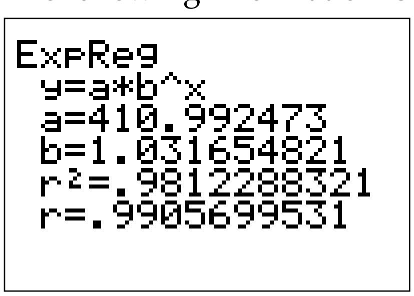
Answer: (Module 1, Lesson 4)



b) Using technology, find the exponential regression equation that best fits this data. *(2 marks)*

Answer: (Module 1, Lesson 4)

The following information is from the TI-84 Stat, Calc, ExpReg function.



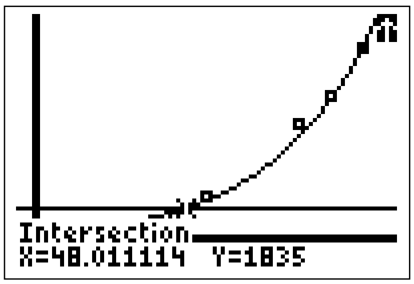
$y = 410.99(1.0317)^x$

Name: ___________________________________

c) Winkler was officially incorporated as a town when the population was approximately 1835 inhabitants. According to this function model, during what year did this happen? *(3 marks)*

Answer: (Module 1, Lesson 4)

The following information is from the TI-84. The line "y = 1835" was graphed. Then the 2nd–Calc, Intersect function was used to find the x-value of the regression function when y = 1835.



The population of Winkler was approximately 1835 during its 48th year, or in 1954.

d) Using the exponential regression equation or technology, determine the approximate population of the city of Winkler during its centennial anniversary in 2006. *(2 marks)*

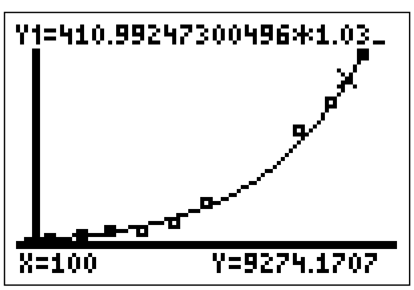
Answer: (Module 1, Lesson 4)

$y = 410.99 \times 1.0317^x$

$y = 410.99 \times 1.0317^{100}$

$y = 9314.8$

Or



The population was approximately 9300 inhabitants.

5. Match each the following equations with its corresponding graph. Write the letter of the equation below the correct graph. *(6 marks)*

   a) $y = \log(x) + 5$

   b) $y = 5^x$

   c) $y = \ln x$

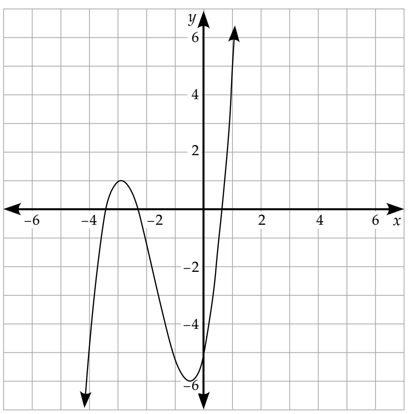
   d) $y = e^x$

   e) $y = 0.5(x^2 + x - 6)$

   f) $y = x^3 + 5x^2 + 4x - 5.1$

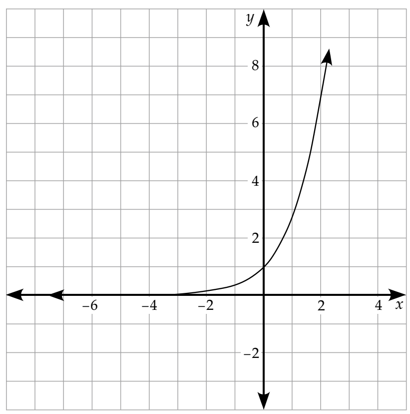
Answers: (Module 1, Lessons 1, 3, and 5)

A



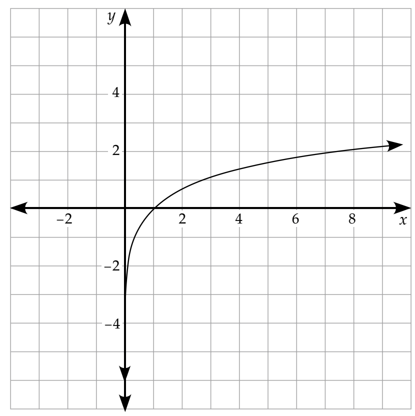
f) $y = x^3 + 5x^2 + 4x - 5.1$

B



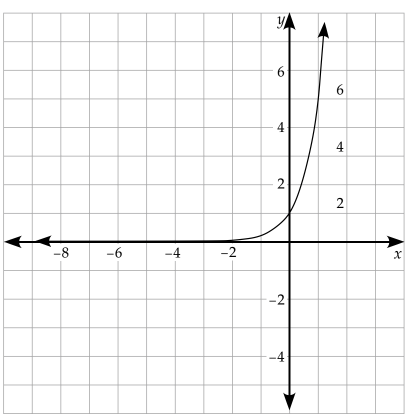
d) $y = e^x$

C



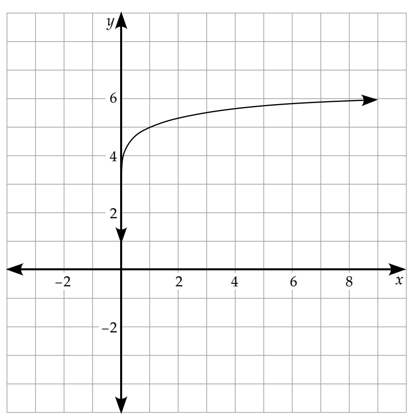
c) $y = \ln x$

D



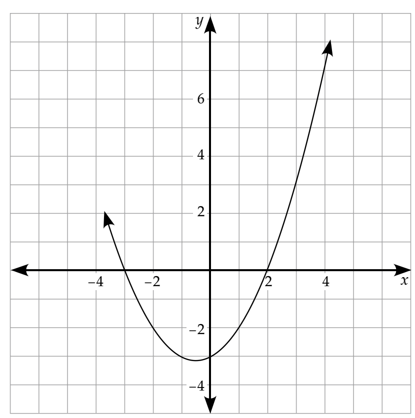
b) $y = 5^x$

E



a) $y = \log(x) + 5$

F

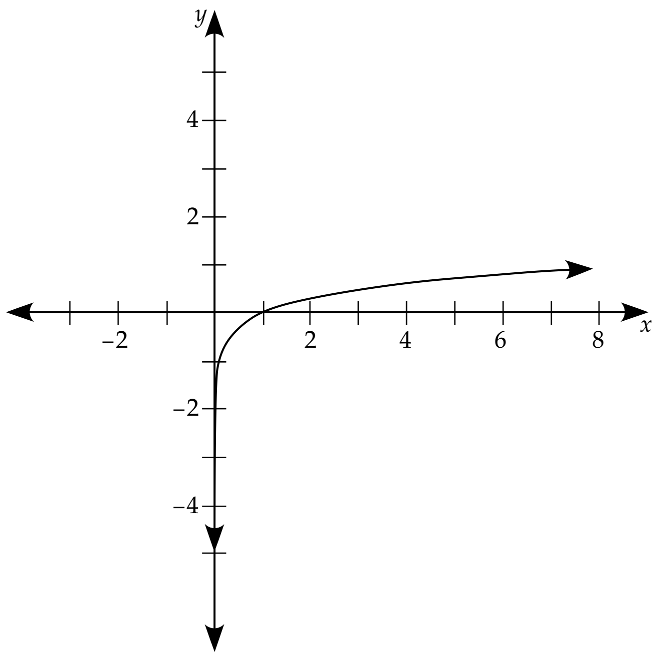


e) $y = 0.5(x^2 + x - 6)$

Name: ______________________________________

6. Use graphing technology to sketch the graph of $y = \log(x)$ and complete the table of information. *(5 marks)*

Answer: (Module 1, Lesson 5)



Domain	$\{x > 0\}$
Range	$\{y \in \Re\}$
x-intercept	1
Equation of asymptote	$x = 0$

Part B: Mathematics Research Project *(5 marks)*

7. List **five** things to consider when assessing the accuracy, reliability, and relevance of data and information. *(5 marks)*

   Answer: (Module 2, Lesson 2)

   Answers will vary. Possible answers include:

   Is the data consistent with multiple reputable sources? Do qualified experts agree with the data and information? Is the data primary or secondary? Has the data been taken from a sample that was an appropriately selected random and representative cross-section of the population? Is the data free from errors, carefully collected, and recorded?

   Are the sources used by the author trustworthy and do they adhere to high standards, or are they opinionated and far-fetched? When was the page published or last updated? Why was the information published? Is the site trying to sell something, to inform others, to teach, to persuade? Who is the intended audience?

   Find out who actually authored the information you are considering and check their credentials. Do they speak with authority? Are they working for a reputable organization? Is the author's name and contact information given? Who holds the copyright to the information published?

   Advertising or sponsorship on the site may indicate that you need to be suspicious of the content of that page. Look for biased wording or examples of opinion rather than the use of facts. Determine if underlying assumptions are affecting the reliability of the data.

Name: ______________________________________

Part C: Logical Reasoning *(22 marks)*

8. Create a Venn diagram to represent the following information and answer the question below. *(5 marks)*

   Students at a private dance studio may take ballet, hip hop, or tap dance classes.

   3 students at the studio take all three.

   7 students take tap and hip hop.

   5 are in ballet and hip hop classes.

   All students who take tap also take either ballet or hip hop.

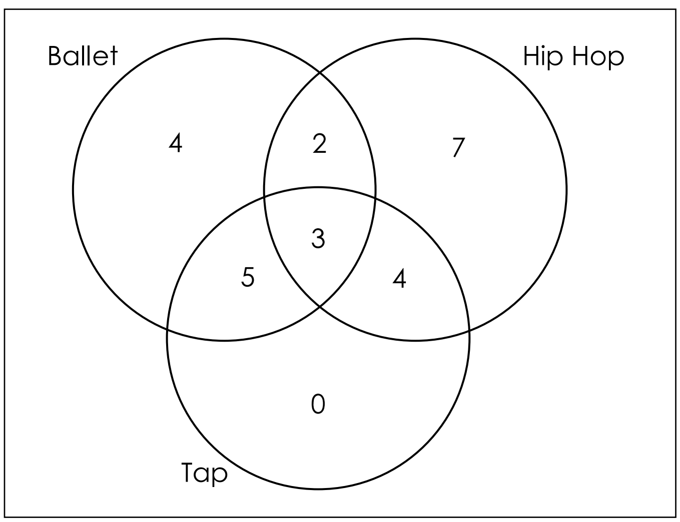
   21 of the students take hip hop or tap.

   14 are in ballet classes.

   12 students take tap.

   How many students attend the studio? ____________

   Answer: (Module 3, Lesson 3)



How many students attend the studio? 25

9. In a class of 25 students, 8 are on student council, 5 are on the volleyball team, and 2 students are on both the council and the team. How many students are neither on the council nor on the team? Justify your answer. *(3 marks)*

Answer: (Module 3, Lesson 3)

There are 2 students on both the student council and the volleyball team. This means that there are 6 students who are only on the student council and 3 students who are only on the volleyball team. That leaves 14 students who are on neither $(25 - 2 - 6 - 3)$.

10. Sets A, B, and C are defined as:

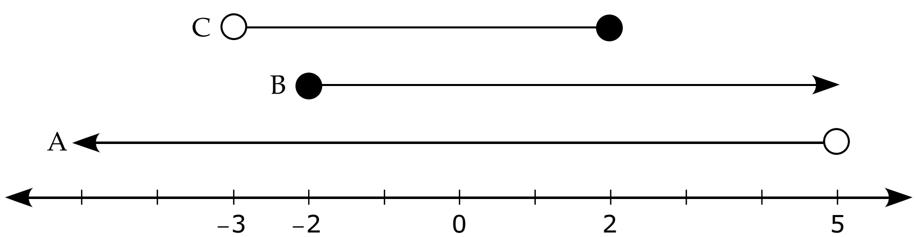
$A = \{x \mid x < 5, x \in \Re\}$

$B = \{x \mid x \geq -2, x \in \Re\}$

$C = \{x \mid -3 < x \leq 2, x \in \Re\}$

a) Graph sets A, B, and C using the number line below. *(3 marks)*

Answer: (Module 3, Lesson 1)



b) Using set notation, define the following: *(3 marks)*

i) $A \cap B =$

ii) $B \cup C =$

iii) $A \cap B' =$

Answers: (Module 3, Lesson 1)

i) $A \cap B = \{x \mid -2 \leq x < 5, x \in \Re\}$

ii) $B \cup C = \{x \mid x > -3, x \in \Re\}$

iii) $A \cap B' = \{x \mid x < -2, x \in \Re\}$

Name: ____

11. Answer the following questions based on the conditional statement, "If a polygon is an octagon, then it has 8 sides." *(8 marks)*

a) State the hypothesis and the conclusion of this statement. *(1 mark)*

Answer: (Module 3, Lesson 2)

Hypothesis: The polygon is an octagon.

Conclusion: The polygon has 8 sides.

b) Complete the following truth table for the conditional statement. Justify your results by describing the possible states of the hypothesis and conclusion for each of the four possible cases. *(5 marks)*

Answer: (Module 3, Lesson 2)

Hypothesis	**Conclusion**	**Conditional Statement**
T	T	T
T	F	F
F	T	F
F	F	T

Case 1: The polygon is an octagon. The polygon has 8 sides. True

Case 2: The polygon is an octagon. The polygon does not have 8 sides. False

Case 3: The polygon is not an octagon. The polygon has 8 sides. False

Case 4: The polygon is not an octagon. The polygon does not have 8 sides. True

c) Write the converse of the given statement. *(1 mark)*

Answer: (Module 3, Lesson 2)

"If the polygon has 8 sides, then it is an octagon."

d) Is the given statement biconditional? Explain. *(1 mark)*

Answer: (Module 3, Lesson 2)

The statement is biconditional because both the statement and the converse are true.

Part D: Probability *(29 marks)*

12. An event consists of tossing three coins and noting if they land Heads, H, or Tails, T. Write the sample space, S, of this event. *(1 mark)*

    Answer: (Module 4, Lesson 1)

    S = {HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}

13. Each of the letters in the words PRACTICE EXAM are written on individual cards and placed in a hat. *(5 marks)*

    a) One card is randomly drawn from the hat. Determine the probability of drawing a C. Write the final answer as a reduced fraction. *(1 mark)*

    Answer: (Module 4, Lesson 1)

    $$P(\text{C}) = \frac{2}{12} = \frac{1}{6}$$

    b) One card is randomly drawn from the hat. Determine the probability of drawing a vowel. Write the final answer as a percentage. *(1 mark)*

    Answer: (Module 4, Lesson 1)

    $$P(\text{vowel}) = \frac{5}{12} = 41.7\%$$

    c) One card is drawn, the letter is noted, and the card is replaced. A second card is drawn. Determine the probability of drawing the P and then the M. Write the final answer as a fraction. *(1 mark)*

    Answer: (Module 4, Lesson 1)

    $$P(\text{P, M}) = \frac{1}{12} \times \frac{1}{12} = \frac{1}{144}$$

    d) One card is drawn, the letter is noted, and then a second card is drawn without replacing the first card. What is the probability of drawing two consonants? Write the final answer as a decimal to the nearest thousandth. *(1 mark)*

    Answer: (Module 4, Lesson 1)

    $$P(\text{consonant, consonant}) = \frac{7}{12} \times \frac{6}{11}$$

    $$P(\text{consonant, consonant}) = \frac{42}{132} = \frac{7}{22} = 0.3\overline{18}$$

Name: ____

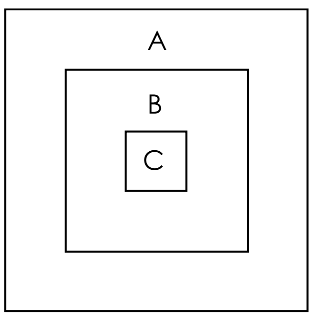
e) One card is drawn at random from the hat. Determine the probability that it is not the X. *(1 mark)*

Answer: (Module 4, Lesson 1)

$$P(X') = 1 - P(X) = 1 - \frac{1}{12}$$

$$P(X') = \frac{11}{12} \text{ or } 0.917$$

14. A dart is thrown randomly and lands on a square dartboard with three areas, as pictured below. Note that the sides of square A are 5 times as long as the sides of square C, and the sides of square B are 3 times as long as the sides of square C.

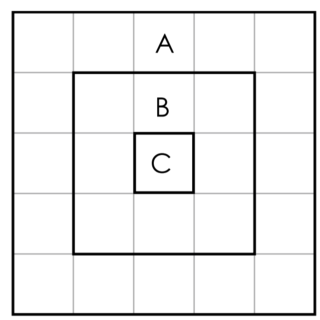


Calculate: *(4 marks)*

a) Probability that a dart lands in area A

Answer: (Module 4, Lesson 1)

$$A = \frac{16}{25}$$



b) Probability that a dart lands in area B

Answer: (Module 4, Lesson 1)

$$B = \frac{8}{25}$$

c) Odds in favour of a dart landing in area B

Answer: (Module 4, Lesson 2)

$B = 8:17$

d) Odds against a dart landing in area C

Answer: (Module 4, Lesson 2)

$C = 24:1$

15. An experiment consists of randomly drawing a card from a deck of 50 cards numbered from 1 to 50. What is the probability of drawing a card that is an even number or a multiple of 5? *(4 marks)*

Answer: (Module 4, Lesson 3)

$$P(\text{even or multiple of five}) = P(\text{even}) + P(\text{multiple of five}) - P(\text{even and multiple of five})$$

$$P(\text{even or multiple of five}) = \frac{25}{50} + \frac{10}{50} - \frac{5}{50}$$

$$P(\text{even or multiple of five}) = \frac{30}{50} = \frac{3}{5}$$

16. Three bags contain marbles:

Bag 1	Bag 2	Bag 3
3 green	2 green	4 green
2 red	1 red	3 white
1 white	5 black	2 black

Answer the following questions regarding one or more of the bags. *(11 marks)*

a) Using Bag 1, determine the probability of drawing a green marble. *(1 mark)*

Answer: (Module 4, Lesson 2)

$$P(\text{green, Bag 1}) = \frac{3}{6} = \frac{1}{2}$$

b) Using Bag 2, determine the odds in favour of drawing a green marble. *(1 mark)*

Answer: (Module 4, Lesson 2)

$2:6$ or $1:3$

c) Using Bag 3, determine the odds against drawing a black marble. *(1 mark)*

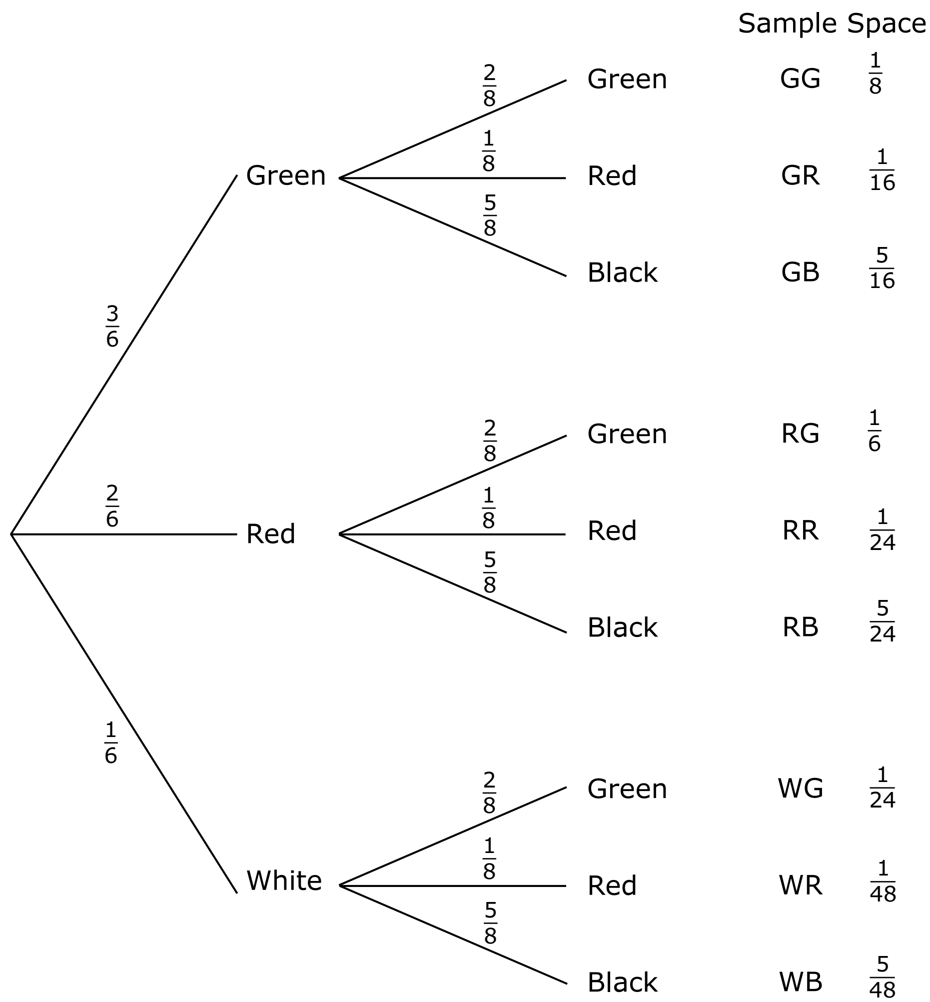
Answer: (Module 4, Lesson 2)

$7:2$

Name: ____________________________________

d) Create a tree diagram to represent the sample space for the event of drawing one marble from Bag 1 and then one marble from Bag 2. Include the probability of drawing each colour along each branch. *(5 marks)*

Answer: (Module 4, Lesson 1)



e) When choosing one marble from each of Bag 1 and Bag 2, what is the probability of choosing two marbles of the same colour? *(2 marks)*

Answer: (Module 4, Lesson 1)

$$P(\text{GG or RR}) = \frac{3}{6}\cdot\frac{2}{8} + \frac{2}{6}\cdot\frac{1}{8} = \frac{6}{48} + \frac{2}{48} = \frac{8}{48} = \frac{1}{6}$$

f) When choosing one marble from each of Bag 1 and Bag 2, what is the probability of choosing two marbles of different colours? *(1 mark)*

Answer: (Module 4, Lesson 1)

$$1 - P(\text{GG or RR}) = 1 - \frac{1}{6} = \frac{5}{6}$$

17. A customer enters a restaurant. The probability that the customer orders one of either a steak or a salad is $\frac{8}{11}$. The probability that the customer orders steak is $\frac{2}{11}$, while the probability of ordering salad is $\frac{7}{11}$. What is the probability that the customer orders both a steak and a salad? *(4 marks)*

Answer: (Module 4, Lesson 3)

P(Steak or Salad) = P(Steak) + P(Salad) − P(Steak and Salad)

$$\frac{8}{11} = \frac{2}{11} + \frac{7}{11} - P(\text{Steak and Salad})$$

$$\frac{8}{11} - \frac{2}{11} - \frac{7}{11} = -P(\text{Steak and Salad})$$

$$\frac{-1}{11} = -P(\text{Steak and Salad})$$

$$\frac{1}{11} = P(\text{Steak and Salad})$$

The probability the customer orders both a steak and a salad is $\frac{1}{11}$.

Grade 12 Applied Mathematics (40S)

Final Practice Examination

GRADE 12 APPLIED MATHEMATICS

Final Practice Examination

Name: ______________________________________

Student Number: ______________________________

Attending ❑ Non-Attending ❑

Phone Number: _______________________________

Address: ____________________________________

____

____

For Marker's Use Only

Date: _________________________________

Final Mark: ________ /100 = ________ %

Comments:

Instructions

The final examination is based on Modules 5 to 8 of the Grade 12 Applied Mathematics course. It is worth 25% of your final mark in this course.

Time

You will have a maximum of **3.0 hours** to complete the final examination.

Format

The format of the examination will be as follows:

Part A: Games and Numbers	4 marks
Part B: Financial Mathematics	42 marks
Part C: Techniques of Counting	18 marks
Part D: Sinusoidal Functions	18 marks
Part E: Design and Measurement	18 marks
Total	100 marks

(see over)

Notes:

You are allowed to bring the following to the examination: pens/pencils (2 or 3 of each), metric and imperial rulers, a graphing and/or scientific calculator, and your Final Exam Resource Sheet. Your Final Exam Resource Sheet must be handed in with the examination.

Graphing and financial applications technology (either computer software or a graphing calculator) **are required** to complete this examination.

Show all calculations and formulas used. Use all decimal places in your calculations and round the final answers to the correct number of decimal places. Include units where appropriate. Clearly state your final answer. Final answers without supporting calculations or explanations will **not** be awarded full marks. Indicate equations and/or keystrokes used in calculations.

When using graphing technology, include a screenshot or printout of graphs **or** sketch the image and indicate the window settings (maximum and minimum *x*- and *y*-values), increments, and axis labels, including units. When using a financial TVM solver, state all input values used (N, I%, PV, PMT, FV, P/Y, C/Y) and the results of calculations.

Name: ______________________________________

Answer all questions to the best of your ability. Show all your work.

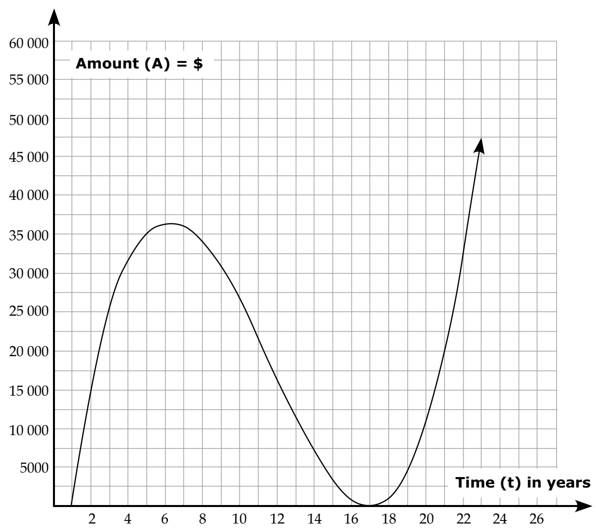
Part A: Games and Numbers *(4 marks)*

1. A quarter is worth \$0.25, a dime is worth \$0.10, and a nickel is worth \$0.05. Ava has coins in her pocket worth \$3.70. She tells you she has 1 dime and equal numbers of quarters and nickels. How many coins does she have in her pocket? *(4 marks)*

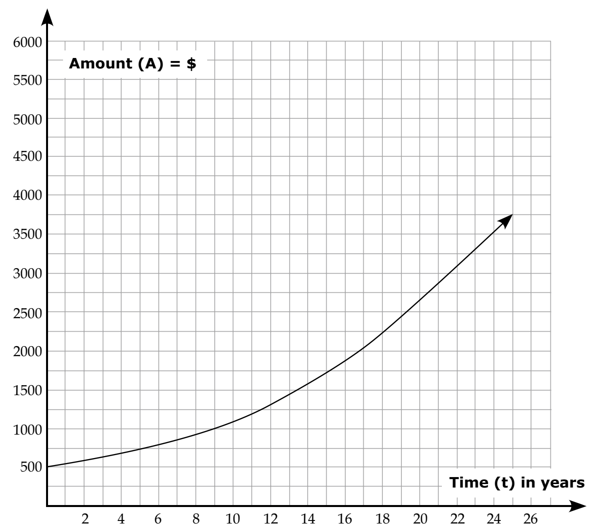
Part B: Financial Mathematics *(42 marks)*

1. Circle the graph below that best represents the dollar value of an investment earning compound interest over a period of years. *(1 mark)*

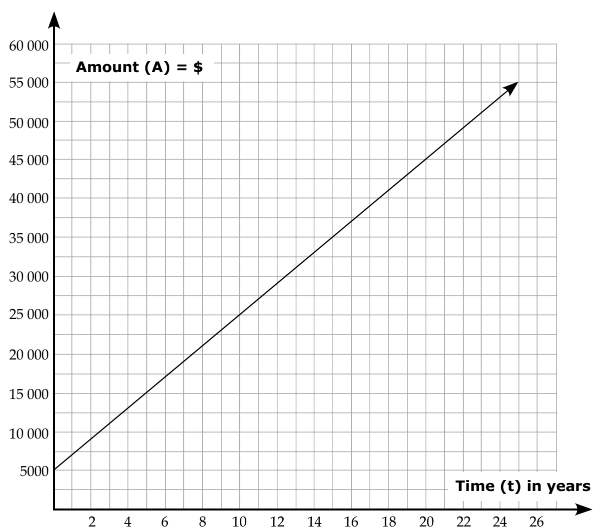
A



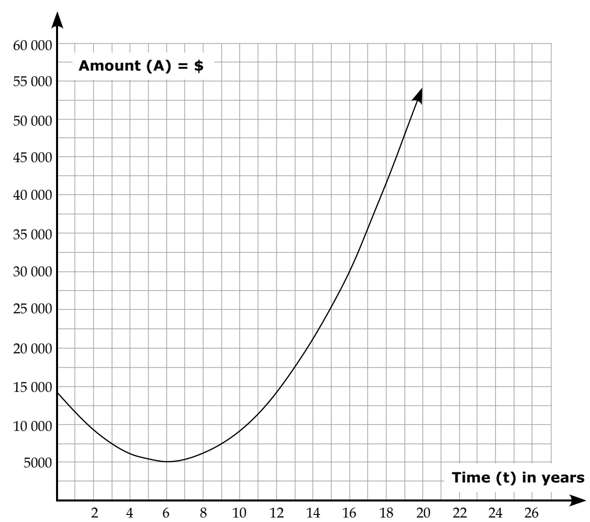
B



C



D



Name: ____

2. Name a situation in which compound interest is earned or paid. *(1 mark)*

3. Mireille invests $1000 in a term deposit at 3%, compounded annually for 5 years. Nathanael invests $1000 in a term deposit at 3%, compounded weekly for 5 years.

   a) Who will earn more in interest? Explain. *(1 mark)*

   b) How much are these investments worth after 5 years? *(1 mark)*

4. Keith takes out a car loan from his bank for $33 999. He negotiates a 5-year term at 3.75%, compounded semi-annually and paid monthly.

   a) Calculate the amount of his monthly payment. *(2 marks)*

b) Determine the amount of interest he will pay over the term of this loan. *(2 marks)*

5. KaranVeer is negotiating the terms of a mortgage with his bank. The house he would like to purchase is $210,000. He has a down payment of $42,000 available. The bank offers him a 25-year term at 3%, compounded monthly.

   a) Determine his monthly payment amount and the total interest paid if he accepts these terms. *(2 marks)*

Name: ____

b) If KaranVeer divided the monthly payment in half and paid that amount every two weeks instead, how many payments would be required to pay off the mortgage? *(1 mark)*

c) If KaranVeer makes his payments every two weeks, how much interest will he have saved by the end of the mortgage? *(1 mark)*

d) Suggest two other specific things KaranVeer could change in the terms of his mortgage to reduce the total amount of interest paid. *(1 mark)*

6. Thomas considers purchasing new furniture worth $999 (including taxes) from a store that offers a "Buy now, pay later" promotion. He reads the fine print: A 15% deposit of total sale (including taxes) and a processing fee of $79.95 are due at the time of purchase. Balance is due 12 months from the date of purchase. Outstanding balances are subject to 29% annual interest, compounded monthly, from the date of purchase.

   Thomas pays the appropriate amount at the time of purchase but is one day late in paying his balance after the 12 months. What is the total amount he will pay for the furniture? *(4 marks)*

7. You compare the offers from a dealership to either buy or lease a car. The price for the vehicle is $24,999 plus taxes. You have a $5000 down payment for either option.

   The lease is over 4 years and payments are $300 plus 13% tax per month. The residual value is set at 45%. You would take the option to purchase it after the four years and pay for it outright (include 13% taxes). There is a lease acquisition fee of $649.

   To finance the car with monthly payments over 48 months, the bank offers you a loan with an interest rate of 6.5%, compounded monthly.

   a) Find the monthly payment if you finance the car. *(2 marks)*

   b) Find the total amount of interest you pay over the loan period. *(1 mark)*

Name: ________________________________

c) Find the total cost to lease the car and buy it out at the end of the term. *(3 marks)*

d) How much do you save by purchasing instead of leasing and then buying it out? *(2 marks)*

e) Describe two situations when leasing might be a better option than buying a depreciating asset such as a car. *(2 marks)*

8. Approximately how long will it take for an investment to double in value if it is invested at 8%, compounded interest? *(1 mark)*

9. Tanya likes to buy a coffee and muffin each morning. However, this year she is training for a marathon and decides to forgo this daily routine and puts the $4.95 she saves each day into a Growth Fund account. The account is compounded daily at 4.5%. How much will she have saved after one year? *(2 marks)*

Name: ____

10. Naomi purchases 75 shares in a certain stock. The purchase price is $44.13 per share. Her broker charges $25 plus $0.06 per share each time she buys or sells shares. If she sells her shares three years later for $52.60 per share, what is the rate of return on her investment? *(3 marks)*

11. Scott thinks he can afford to pay $1000 per month for a mortgage payment for a property that has annual property taxes of $2400 and heating costs estimated at $62 per month. His gross monthly income is $3450. Based on this information, should he expect the bank to lend him the money to buy the house? Justify your answer. *(4 marks)*

12. Mehrit and Yacob are saving for a down payment on a home. Mehrit suggests they invest \$200 every two weeks for 3 years in a term deposit earning 5.4%, compounded bi-weekly. Yacob suggests they invest a lump sum of \$4800 each year for three years in a term deposit at 5.4%, compounded annually. Determine whose investment strategy will result in larger savings for a down payment. Justify your answer. *(5 marks)*

Name: ____

Part C: Techniques of Counting (18 marks)

1. a) How many ten-digit telephone numbers can be created if they must begin with area code 204? *(2 marks)*

   b) What assumptions are you making? *(1 mark)*

2. Represent the following situation with a graphic organizer such as a tree diagram or table to illustrate all the ways in which you can both choose one marble from a box containing blue, red, and green marbles, as well as flip a coin and have it land either heads or tails. *(3 marks)*

3. In how many ways can five sets of twins be arranged in a row for a photo if each set must be seated together? State your answer in factorial notation and solve. *(2 marks)*

4. In how many ways can the letters in the word EXAMINATION be arranged? Show your work. *(3 marks)*

5. Lillia is arranging flowers into a bouquet for her grandmother. If she has 7 different coloured daisies and 8 different types of roses, in how many ways can she make a bouquet containing four daisies and three roses? Show your work. *(3 marks)*

6. Rani's locker code consists of three different numbers, each of which is from one to twenty-nine. What is the probability her locker code uses only single-digit numbers? *(4 marks)*

Name: ______________________________

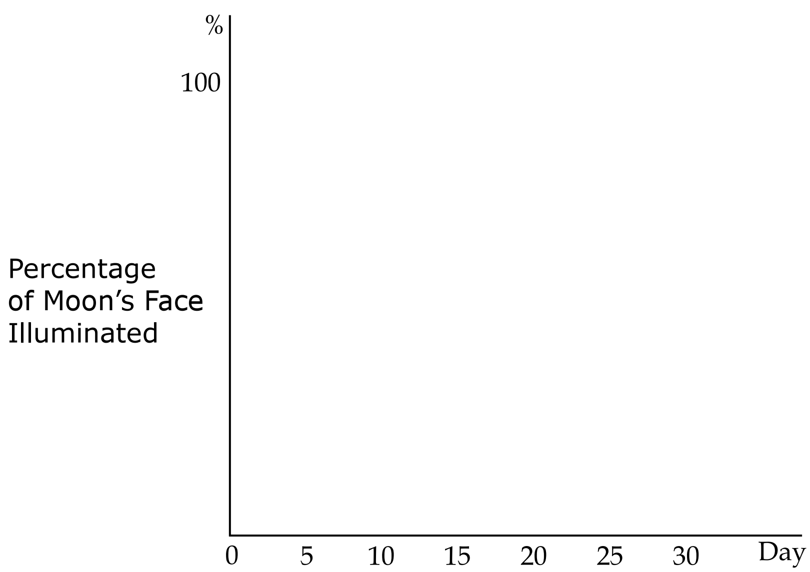
Part D: Sinusoidal Functions (*18 marks*)

1. The phases of the moon cycle between new moon, first quarter, full moon, and last quarter in the period of a lunar month. During a full moon, 100% of the moon's visible surface is illuminated, while 50% is visible on the first and last quarters, and 0% of the moon's visible surface is illuminated at new moon.

   Keith observes the moon through his telescope on various nights during the month of January and calculates the approximate percentage of the moon's visible surface that is illuminated. He records his data in a table.

Date	% Illuminated
Jan. 3	12
Jan. 6	39
Jan. 9	70
Jan. 11	87
Jan. 15	100
Jan. 19	80
Jan. 24	29
Jan. 26	12
Jan. 27	6

   a) Sketch a graph of this data. You may use technology and print a copy of the graph created, or sketch it below. *(3 marks)*



b) What is the range of y-values in this situation? Write it in set notation. *(2 marks)*

c) State the maximum and minimum y-values possible. What phase of the moon do these values represent? *(2 marks)*

d) What is the amplitude in this situation? *(1 mark)*

e) What is the median value in this situation? *(1 mark)*

Name: ____

f) Use technology to determine a sinusoidal regression equation that models this data. (**Note:** The next full moon Keith observed was on February 13th.) *(5 marks)*

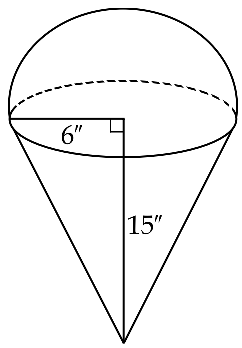
g) Use technology or the sinusoidal regression equation to determine the approximate date in January of a first-quarter moon. This occurs when 50% of the moon's visible surface is illuminated, and the phases are increasing towards a full moon. *(2 marks)*

h) If there are two full moons in one given month, the second full moon that occurs is called a "Blue Moon." Are there any months in the year during which it would be impossible for a "Blue Moon" to occur? Explain your answer. *(2 marks)*

Name: ________________________________

Part E: Design and Measurement (*18 marks*)

1. The three-dimensional solid (shown below) is to be constructed out of plastic, which costs $1.87 per cubic foot, including taxes.



a) Determine the cost to produce the solid. *(6 marks)*

b) If it costs 0.8¢ per square inch, including taxes, to apply a spray finish to the outside surface of the object, determine the cost of finishing. *(6 marks)*

Name: ____

2. Denis has $50 to create a flower garden for his mother. He must put down 4 inches of topsoil, add fertilizer, and plant the flowers. The topsoil costs $1.79 per cubic foot. The fertilizer costs $0.58 per square foot, and flowers are $0.79 each. He would need three flowers per square foot. All costs already include taxes.

   a) Determine the maximum size of garden he can create within his $50 budget. Show your work. *(5 marks)*

b) Sketch a diagram showing the shape and dimensions of a potential garden within his budget. *(1 mark)*

Grade 12 Applied Mathematics (40S)

Final Practice Examination Answer Key

Grade 12 Applied Mathematics

Final Practice Examination
Answer Key

Name: ______________________________

Student Number: ______________________________

Attending ❑ Non-Attending ❑

Phone Number: ______________________________

Address: ______________________________

____

____

For Marker's Use Only

Date: ______________________________

Final Mark: ________ /100 = ________ %

Comments:

Instructions

The final examination is based on Modules 5 to 8 of the Grade 12 Applied Mathematics course. It is worth 25% of your final mark in this course.

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The format of the examination will be as follows:

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Part D: Sinusoidal Functions	18 marks
Part E: Design and Measurement	18 marks
Total	100 marks

(see over)

Notes:

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Name: ____

Answer all questions to the best of your ability. Show all your work.

Part A: Games and Numbers *(4 marks)*

1. A quarter is worth \$0.25, a dime is worth \$0.10, and a nickel is worth \$0.05. Ava has coins in her pocket worth \$3.70. She tells you she has 1 dime and equal numbers of quarters and nickels. How many coins does she have in her pocket? *(4 marks)*

   Answer: (Modules 1 to 8, Cover Assignments)

   Strategies may vary. For example, you may use reasoning, model the situation algebraically, or make a systematic list.

   Method 1:

   Since there is 1 dime, the value of the quarters and nickels is \$3.60. Four quarters and four nickels are worth \$1.20. We want \$3.60 so we need 3 such groups of quarters and nickels. Therefore, she has 12 quarters, 12 nickels, and 1 dime for a total of 25 coins.

   Method 2:

   $$0.10 + 5(x) + 25(x) = 3.70$$
   $$30x = 3.60$$
   $$x = 12$$

   She has 12 nickels, 12 quarters, and 1 dime for a total of 25 coins.

   Method 3:

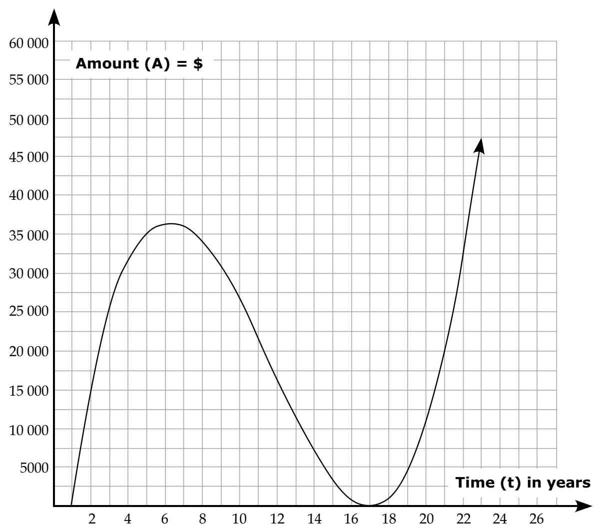
Number of Dimes	Number of Nickels	Number of Quarters	Value
1	1	1	0.10 + 0.05 + 0.25 = 0.40
1	10	10	0.10 + 0.50 + 2.50 = 3.10
1	11	11	0.10 + 0.55 + 2.75 = 3.40
1	12	12	0.10 + 0.60 + 3.00 = 3.70

   She has 1 dime, 12 nickels, and 12 quarters for a total of 25 coins

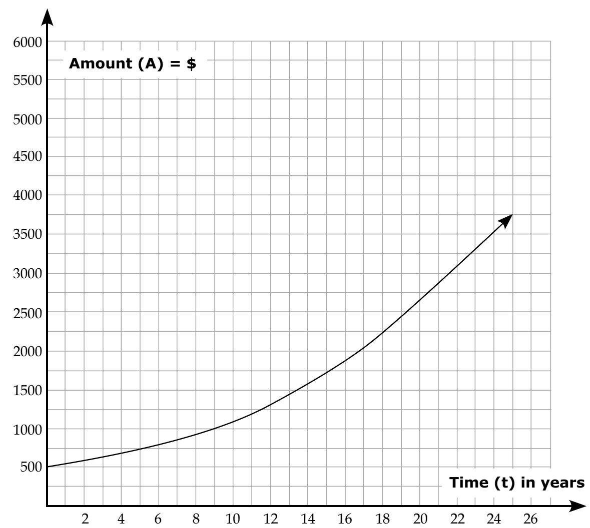
Part B: Financial Mathematics *(42 marks)*

1. Circle the graph below that best represents the dollar value of an investment earning compound interest over a period of years. *(1 mark)*

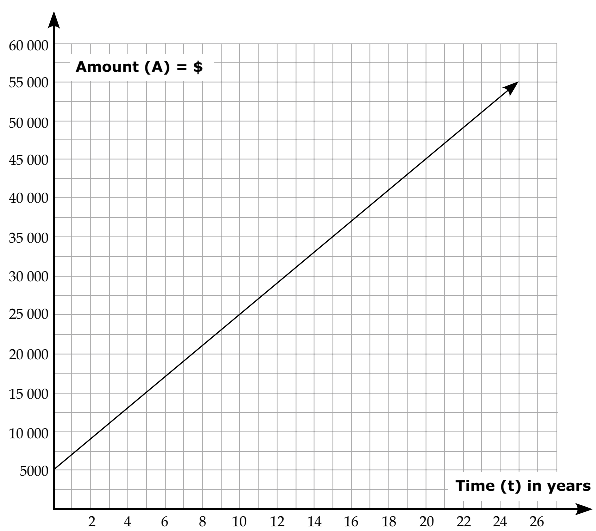
A



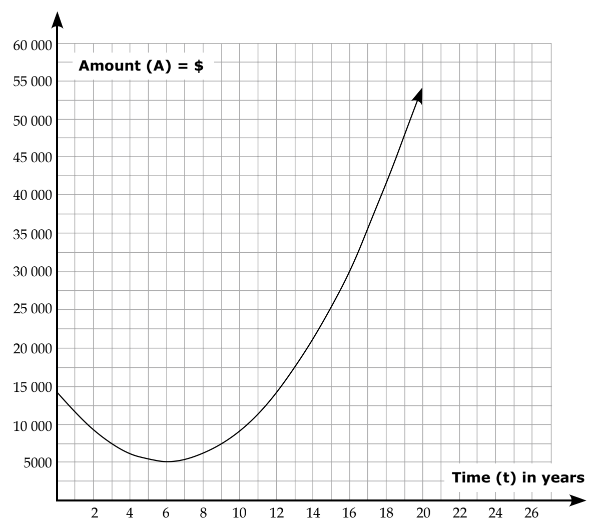
B



C



D



Answer: (Module 5, Lesson 1)

Graph B.

Name: ______________________________

2. Name a situation in which compound interest is earned or paid. *(1 mark)*

   Answer: (Module 5, Lesson 1)

   Answers may vary. Possible solutions include mortgages, loans, credit cards, and investments.

3. Mireille invests $1000 in a term deposit at 3%, compounded annually for 5 years. Nathanael invests $1000 in a term deposit at 3%, compounded weekly for 5 years.

   a) Who will earn more in interest? Explain. *(1 mark)*

   Answer: (Module 5, Lesson 1)

   Nathanael. His investment is compounded more often.

   b) How much are these investments worth after 5 years? *(1 mark)*

   Answer: (Module 5, Lesson 1)

   Mireille: A = $1159.27

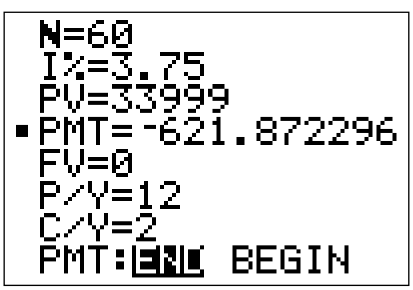
   Nathanael: A = $1161.78

4. Keith takes out a car loan from his bank for $33 999. He negotiates a 5-year term at 3.75%, compounded semi-annually and paid monthly.

   a) Calculate the amount of his monthly payment. *(2 marks)*

   Answer: (Module 5, Lesson 3)

   The image below shows the values using a TVM solver to calculate the payment.



   His monthly payment will be $621.87.

b) Determine the amount of interest he will pay over the term of this loan. *(2 marks)*

Answer: (Module 5, Lesson 3)

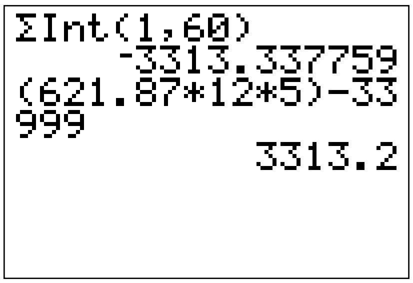
Method 1:

Add up all payments and subtract the principal.

621.87 × (5 × 12) − 33 999 = $3313.20

Method 2:

The first two lines in the image below show the value using the TVM solver "sum of interest" command for payments 1 to 60. The other calculations show the total amount paid in 60 payments, minus the principal, as shown in Method 1.



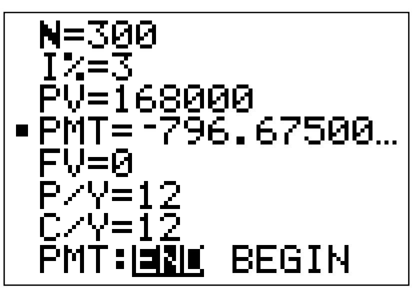
Using the TVM solver: $3313.34

5. KaranVeer is negotiating the terms of a mortgage with his bank. The house he would like to purchase is $210,000. He has a down payment of $42,000 available. The bank offers him a 25-year term at 3%, compounded monthly.

a) Determine his monthly payment amount and the total interest paid if he accepts these terms. *(2 marks)*

Answer: (Module 5, Lesson 3)

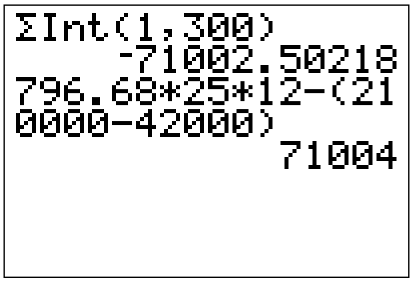
Use the TVM solver to calculate the payment.



His monthly payment would be $796.68.

Name: ______________________________

The total interest paid can be calculated using the TVM solver "sum of interest" command, as shown in the first two lines of the image below. Alternately, the interest can be calculated by adding up all payments and subtracting the principal, as shown in the next three lines of the image below.



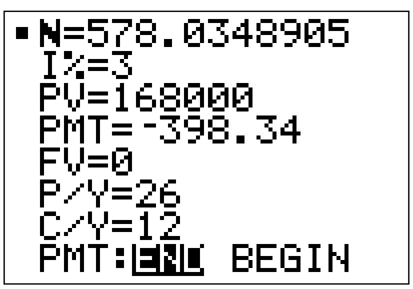
Depending on the method used, the total interest paid would be about $71,002.50 or $71,004.00.

b) If KaranVeer divided the monthly payment in half and paid that amount every two weeks instead, how many payments would be required to pay off the mortgage? *(1 mark)*

Answer: (Module 5, Lesson 3)

Use the TVM solver to calculate the number of payments after calculating the value of the bi-weekly payment.

796.68 ÷ 2 =3 98.34

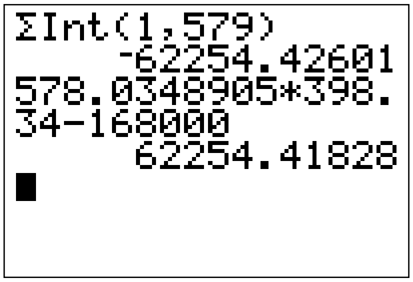


579 payments are required (or 578 full payments and a 579th partial payment).

c) If KaranVeer makes his payments every two weeks, how much interest will he have saved by the end of the mortgage? *(1 mark)*

Answer: (Module 5, Lesson 3)

Use the TVM solver to calculate the interest earned over 579 payments, as shown in the first two lines of the image below. Alternately, calculate the interest by adding up all payments and subtracting the principal.



The amount of interest saved is the difference between the answers in parts (a) and (c).

He would save about $8748.08.

d) Suggest two other specific things KaranVeer could change in the terms of his mortgage to reduce the total amount of interest paid. *(1 mark)*

Answer: (Module 5, Lesson 3)

Answers may vary. Possible solutions may include the following:

- Shorten the length of the mortgage to 15 or 20 years instead of 25.
- Increase his monthly payment amount to $800 or $850.
- Increase his payment frequency to $400 semi-monthly, $400 biweekly, or $200 weekly.
- Make annual lump-sum contributions to reduce the principal.
- Negotiate a lower interest rate.
- Increase the amount of his down payment.

6. Thomas considers purchasing new furniture worth \$999 (including taxes) from a store that offers a "Buy now, pay later" promotion. He reads the fine print: A 15% deposit of total sale (including taxes) and a processing fee of \$79.95 are due at the time of purchase. Balance is due 12 months from the date of purchase. Outstanding balances are subject to 29% annual interest, compounded monthly, from the date of purchase.

   Thomas pays the appropriate amount at the time of purchase but is one day late in paying his balance after the 12 months. What is the total amount he will pay for the furniture? *(4 marks)*

   Answer: (Module 5, Lesson 2)

   He will pay 999 × 0.15 = 149.85 + processing fee. 149.85 + 79.95 = \$229.80 at the time of purchase and the remaining 999 × 0.85 = \$849.15 is due within 12 months. Since he is one day late in paying the balance, he is charged 29% interest on this amount for the 12 months and owes \$1130.92.

   Method 1:

   Use the TVM solver to calculate the future value, FV.

```
 N=1
 I%=29
 PV=-849.15
 PMT=0
▪FV=1130.920447
 P/Y=1
 C/Y=12
 PMT:END BEGIN
```

   The furniture costs him 229.80 + 1130.92 = \$1360.72.

   Method 2:

   Use the compound interest formula to calculate the future amount, A.

$$A = 849.15\left(1 + \frac{0.29}{12}\right)^{1\times 12}$$

$$A = 1130.92$$

   The furniture costs him 229.80 + 1130.92 = \$1360.72.

7. You compare the offers from a dealership to either buy or lease a car. The price for the vehicle is $24,999 plus taxes. You have a $5000 down payment for either option.

The lease is over 4 years and payments are $300 plus 13% tax per month. The residual value is set at 45%. You would take the option to purchase it after the four years and pay for it outright (include 13% taxes). There is a lease acquisition fee of $649.

To finance the car with monthly payments over 48 months, the bank offers you a loan with an interest rate of 6.5%, compounded monthly.

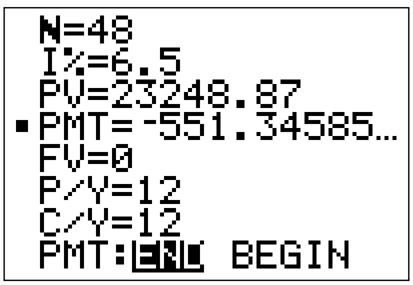
a) Find the monthly payment if you finance the car. *(2 marks)*

Answer: (Module 5, Lesson 6)

Add taxes to the total cost and find the principal of the loan after the down payment.

$24\,999 \times 1.13 - 5000 = 23\,248.87$

Use the TVM solver to calculate the payment.

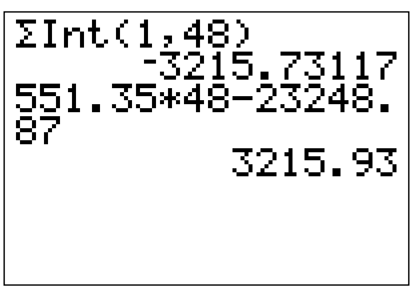


The monthly payment would be $551.35 per month.

b) Find the total amount of interest you pay over the loan period. *(1 mark)*

Answer: (Module 5, Lesson 6)

Depending on the method used, the total interest paid is $3215.73 or $3215.93.



Name: ____________________________________

c) Find the total cost to lease the car and buy it out at the end of the term. *(3 marks)*

Answer: (Module 5, Lesson 6)

48 payments + down payment + acquisition fee + residual value = total cost

$48 \times (300 \times 1.13) + 5000 + 649 + (24\,999 \times 0.45 \times 1.13) = 34\,632.99$

The total cost is $34 632.99.

d) How much do you save by purchasing instead of leasing and then buying it out? *(2 marks)*

Answer: (Module 5, Lesson 6)

To purchase the car with financing: $551.35 \times 48 + 5000 = 31\,464.80$

Total cost to purchase is $31 464.80, a savings of $3168.19 over leasing.

e) Describe two situations when leasing might be a better option than buying a depreciating asset such as a car. *(2 marks)*

Answer: (Module 5, Lesson 6)

Possible answers include 2 of the following:

- when you want to upgrade every few years and always drive a new car
- when you always want full warranty coverage
- when you can write off a portion of the lease payments as a business expense
- when you want lower payments than typically given for loan payments
- when you cannot get a loan
- when you don't want to worry about having to sell the car in a couple of years

8. Approximately how long will it take for an investment to double in value if it is invested at 8%, compounded interest? *(1 mark)*

Answer: (Module 5, Lesson 1)

$$\frac{72}{\text{interest rate}} = \text{approximate years to double}$$

$$\frac{72}{8} = 9$$

It will take about 9 years to double.

9. Tanya likes to buy a coffee and muffin each morning. However, this year she is training for a marathon and decides to forgo this daily routine and puts the $4.95 she saves each day into a Growth Fund account. The account is compounded daily at 4.5%. How much will she have saved after one year? *(2 marks)*

Answer: (Module 5, Lesson 4)

```
 N=365
 I%=4.5
 PV=0
 PMT=4.95
▪FV=-1847.902084
 P/Y=365
 C/Y=365
 PMT:END BEGIN
```

After one year, she will have saved $1847.90 by investing her coffee and muffin money.

Name: ___________________________________

10. Naomi purchases 75 shares in a certain stock. The purchase price is $44.13 per share. Her broker charges $25 plus $0.06 per share each time she buys or sells shares. If she sells her shares three years later for $52.60 per share, what is the rate of return on her investment? *(3 marks)*

Answer: (Module 5, Lesson 5)

$75 \times 44.13 = 3309.75$	Amount invested
$75 \times 52.60 = 3945.00$	Value of shares when sold
$3945 - 3309.75 = 635.25$	Amount earned
$75 \times 0.06 = 4.50$	
$4.50 + 25 = 29.50$	
$29.50 \times 2 = 59$	Broker fees
$635.25 - 59 = 576.25$	Profit

$$\text{ROR} = \frac{576.25}{3309.75}$$

$$\text{ROR} = 0.1741$$

Her rate of return is 17.4%.

11. Scott thinks he can afford to pay $1000 per month for a mortgage payment for a property that has annual property taxes of $2400 and heating costs estimated at $62 per month. His gross monthly income is $3450. Based on this information, should he expect the bank to lend him the money to buy the house? Justify your answer. *(4 marks)*

Answer: (Module 5, Lesson 5)

Calculate the Gross Debt Service (GDS) ratio. It must be less than 32%.

$$\text{GDS ratio} = \frac{\begin{pmatrix} \text{monthly} & & \text{monthly} & & \text{monthly} \\ \text{mortgage} & + & \text{property} & + & \text{heating} \\ \text{payment} & & \text{taxes} & & \text{costs} \end{pmatrix}}{\text{Gross Monthly Income}}$$

$$\text{GDS ratio} = \frac{1000 + \left(\frac{2400}{12}\right) + 62}{3450}$$

$$\text{GDS ratio} = \frac{1262}{3450} \approx 0.3657 \approx 37\%$$

Since the Gross Debt Service ratio is 37%, which is greater than 32%, the bank is not likely to loan him the money.

12. Mehrit and Yacob are saving for a down payment on a home. Mehrit suggests they invest $200 every two weeks for 3 years in a term deposit earning 5.4%, compounded bi-weekly. Yacob suggests they invest a lump sum of $4800 each year for three years in a term deposit at 5.4%, compounded annually. Determine whose investment strategy will result in larger savings for a down payment. Justify your answer. *(5 marks)*

Answer: (Module 5, Lesson 4)

Mehrit's plan sees them investing a total of $15 600 in $200 contributions 26 times per year. Yacob would invest $14 400 over the three years in three payments. This is the equivalent of $400 per month, but since Mehrit invests bi-weekly, she contributes $5200 per year.

Mehrit

```
 N=78
 I%=5.4
 PV=0
 PMT=200
▪FV=-16915.6689
 P/Y=26
 C/Y=26
 PMT:END BEGIN
```

Yacob

```
 N=3
 I%=5.4
 PV=0
 PMT=4800
▪FV=-15191.5968
 P/Y=1
 C/Y=1
 PMT:END BEGIN
```

The difference in the investments is: 16915.67 − 15191.60.

Following Mehrit's suggestion, they would earn $1724.07 more for their down payment after 3 years.

Name: ____

Part C: Techniques of Counting (18 marks)

1. a) How many ten-digit telephone numbers can be created if they must begin with area code 204? *(2 marks)*

   Answer: (Module 6, Lesson 1)

   $(\underline{\ 1\ } \times \underline{\ 1\ } \times \underline{\ 1\ }) \times \underline{\ 10\ } \times \underline{\ 10\ } \times \underline{\ 10\ } \times \underline{\ 10\ } \times \underline{\ 10\ } \times \underline{\ 10\ } \times \underline{\ 10\ } = 10^7$

   b) What assumptions are you making? *(1 mark)*

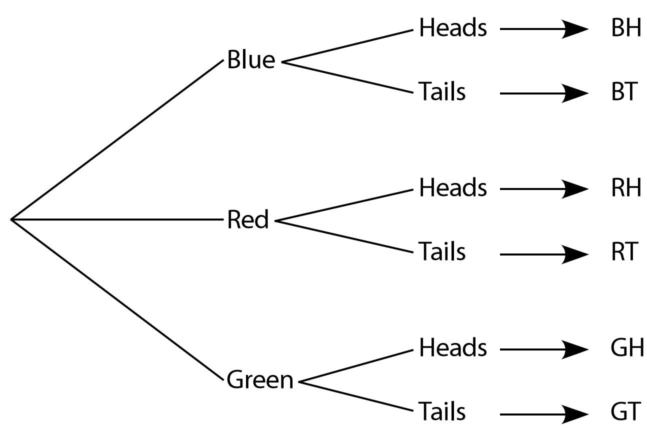
   Answer: (Module 6, Lesson 1)

   Use all digits from 0 to 9 with repetitions, no restrictions

   Other assumptions could lead to other answers.

2. Represent the following situation with a graphic organizer such as a tree diagram or table to illustrate all the ways in which you can both choose one marble from a box containing blue, red, and green marbles, as well as flip a coin and have it land either heads or tails. *(3 marks)*

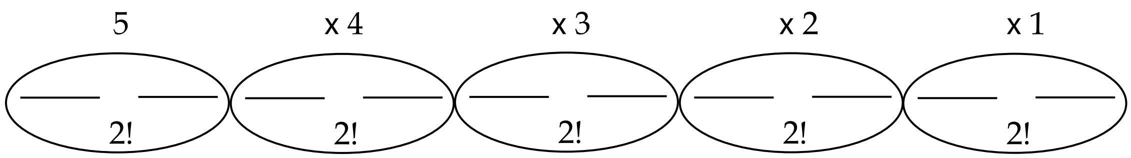
   Answer: (Module 6, Lesson 1)



3. In how many ways can five sets of twins be arranged in a row for a photo if each set must be seated together? State your answer in factorial notation and solve. *(2 marks)*

Answer: (Module 6, Lesson 2)

Group the five different pairs of twins together.



$5!2!2!2!2!2! = 3840$

4. In how many ways can the letters in the word EXAMINATION be arranged? Show your work. *(3 marks)*

Answer: (Module 6, Lesson 2)

Because there are duplicate As, Is, and Ns, 11! must be divided by 2!2!2! to account for the identical arrangements of the repeated letters.

$$\frac{11!}{2!2!2!} = 4\ 989\ 600$$

5. Lillia is arranging flowers into a bouquet for her grandmother. If she has 7 different coloured daisies and 8 different types of roses, in how many ways can she make a bouquet containing four daisies and three roses? Show your work. *(3 marks)*

Answer: (Module 6, Lesson 4)

$$\underbrace{{}_7C_4}_{\text{Daisies}} \times \underbrace{{}_8C_3}_{\text{Roses}} = 1960$$

6. Rani's locker code consists of three different numbers, each of which is from one to twenty-nine. What is the probability her locker code uses only single-digit numbers? *(4 marks)*

Answer: (Module 6, Lesson 3)

The total number of arrangements using any of the numbers is:

$${}_{29}P_3 = 21924$$

There are 9 single-digit numbers and the number of arrangements is:

$${}_9P_3 = 504$$

The probability that the code uses only single-digit numbers is:

$$\frac{504}{21924} = 0.0229$$

Name: ___________________________________

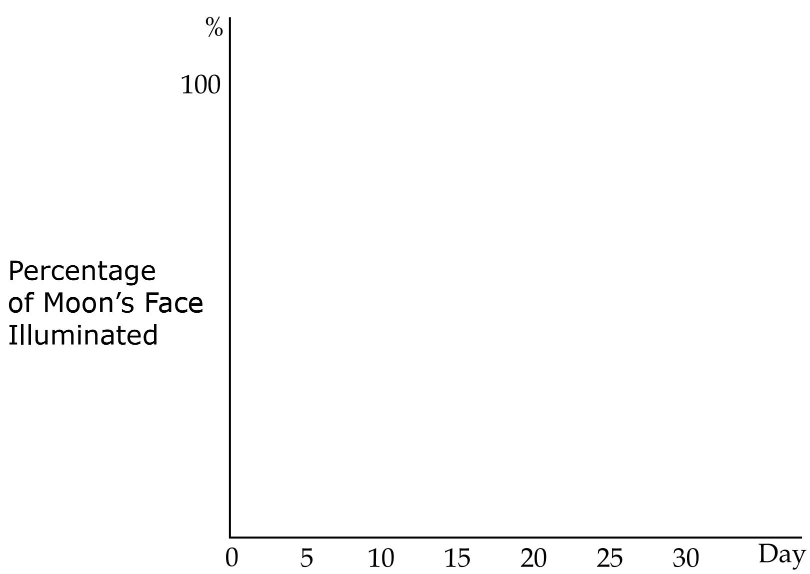
Part D: Sinusoidal Functions (*18 marks*)

1. The phases of the moon cycle between new moon, first quarter, full moon, and last quarter in the period of a lunar month. During a full moon, 100% of the moon's visible surface is illuminated, while 50% is visible on the first and last quarters, and 0% of the moon's visible surface is illuminated at new moon.

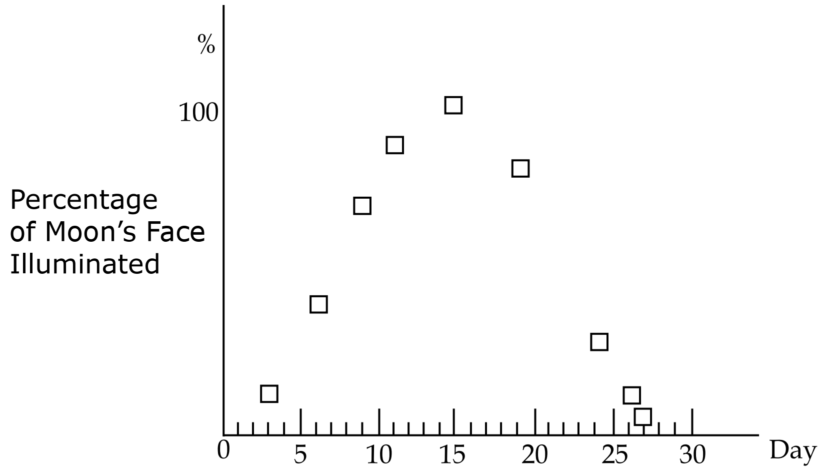
   Keith observes the moon through his telescope on various nights during the month of January and calculates the approximate percentage of the moon's visible surface that is illuminated. He records his data in a table.

Date	% Illuminated
Jan. 3	12
Jan. 6	39
Jan. 9	70
Jan. 11	87
Jan. 15	100
Jan. 19	80
Jan. 24	29
Jan. 26	12
Jan. 27	6

   a) Sketch a graph of this data. You may use technology and print a copy of the graph created, or sketch it below. *(3 marks)*



Answer: (Module 7, Lesson 1)



b) What is the range of *y*-values in this situation? Write it in set notation. *(2 marks)*

Answer: (Module 7, Lesson 1)

$\{y \mid 0 \leq y \leq 100, y \in \Re\}$

c) State the maximum and minimum *y*-values possible. What phase of the moon do these values represent? *(2 marks)*

Answer: (Module 7, Lesson 1)

The minimum value is 0%. This is at the new moon phase.

The maximum value is 100%. This is a full moon.

d) What is the amplitude in this situation? *(1 mark)*

Answer: (Module 7, Lesson 1)

The amplitude is 50.

e) What is the median value in this situation? *(1 mark)*

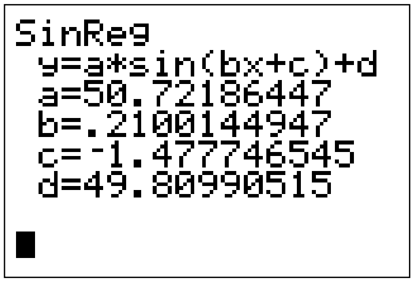
Answer: (Module 7, Lesson 1)

The median value is 50%.

Name: ____________________________________

f) Use technology to determine a sinusoidal regression equation that models this data. (**Note:** The next full moon Keith observed was on February 13th.) *(5 marks)*

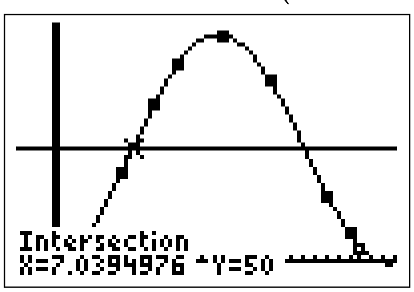
Answer: (Module 7, Lesson 2)



$y = 50.7\sin(0.21x - 1.5) + 49.8$

g) Use technology or the sinusoidal regression equation to determine the approximate date in January of a first-quarter moon. This occurs when 50% of the moon's visible surface is illuminated, and the phases are increasing towards a full moon. *(2 marks)*

Answer: (Module 7, Lesson 2)



On approximately January 7th.

$$\text{period} = \frac{2\pi}{b}$$

$$b = \frac{2\pi}{29}$$

$$b = 0.217$$

$$\text{Phase shift} = \frac{-c}{b} \qquad y = 50\sin(0.217x - 1.5) + 50$$

$$7 = \frac{-c}{0.217}$$

$$c = -1.5$$

$$a = 50$$

$$d = 50$$

If the next full moon is observed on Feb 13th, the period of this function is 29 days.

h) If there are two full moons in one given month, the second full moon that occurs is called a "Blue Moon." Are there any months in the year during which it would be impossible for a "Blue Moon" to occur? Explain your answer. *(2 marks)*

Answer: (Module 7, Lesson 2)

A Blue Moon could not occur in February, unless it is a leap year, since the period of the lunar cycle is about 29 or 30 days and February usually only has 28 days.

Period calculated from information given: 29 days

Period calculated using equation: 30 days

$$\text{period} = \frac{2\pi}{b}$$

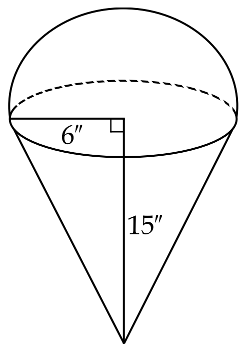
$$\text{period} = \frac{2\pi}{0.21}$$

$$\text{period} = 29.9$$

Name: ______________________________

Part E: Design and Measurement (*18 marks*)

1. The three-dimensional solid (shown below) is to be constructed out of plastic, which costs \$1.87 per cubic foot, including taxes.



a) Determine the cost to produce the solid. *(6 marks)*

Answer: (Module 8, Lesson 1)

$$V_{\text{sphere}} = \frac{4}{3}\pi r^3$$

$$V_{\text{sphere}} = \frac{4}{3}\pi 6^3$$

$$V_{\text{sphere}} = 904.778$$

Volume of the half sphere is 452.389 in.^3

$$V_{\text{cone}} = \frac{1}{3}\pi r^2 h$$

$$V_{\text{cone}} = \frac{1}{3}\pi 6^2 (15)$$

$$V_{\text{cone}} = 565.487 \text{ in.}^3$$

Total volume = 452.389 + 565.487 = 1017.876 in.^3

Convert from cubic inches to cubic feet.

$$\frac{1 \text{ ft.}^3}{(12 \text{ in.})^3} = \frac{x \text{ ft.}^3}{(1017.876 \text{ in.})^3}$$

$$\frac{1}{1728} = \frac{x}{1017.876}$$

$$x = 0.589 \text{ ft.}^3$$

Cost = 0.589 ft.^3 × \$1.87 \$/ft.^3 = \$1.10

b) If it costs 0.8¢ per square inch, including taxes, to apply a spray finish to the outside surface of the object, determine the cost of finishing. *(6 marks)*

Answer: (Module 8, Lesson 2)

$SA_{\text{sphere}} = 4\pi r^2$

$SA_{\text{sphere}} = 4\pi \times 6^2$

$SA_{\text{sphere}} = 452.389$

Surface area of the half sphere is 226.195 in.^{2}.

Slant height = $\sqrt{15^2 + 6^2}$

$s = 16.15549$

$SA_{\text{cone}} = \pi rs$

$SA_{\text{cone}} = \pi 6(16.15549)$

$SA_{\text{cone}} = 304.524$ in.^{2}

Total surface area = 226.195 + 304.524 = 530.719 in.^{2}

Cost = 530.719 × 0.8

Cost = 424.575 cents or $4.25

Name: ______________________________

2. Denis has \$50 to create a flower garden for his mother. He must put down 4 inches of topsoil, add fertilizer, and plant the flowers. The topsoil costs \$1.79 per cubic foot. The fertilizer costs \$0.58 per square foot, and flowers are \$0.79 each. He would need three flowers per square foot. All costs already include taxes.

   a) Determine the maximum size of garden he can create within his \$50 budget. Show your work. *(5 marks)*

   Answer: (Module 8, Lesson 3)

   Calculate the cost per square foot of garden area.

   Soil: 12″ × 12″ × 4″ = 576 in.^{3}

   Since there are 1728 in.^{3} in a ft.^{3}, he would need 0.3333 ft.^{3} of soil for each square foot of garden area.

   Cost of this topsoil: 0.3333 × 1.79 = \$0.60 per square foot

   Flowers: 0.79 × 3 = \$2.37 per square foot

   Fertilizer: 0.58 per square foot

   Cost per square foot of garden: 0.60 + 2.37 + 0.58 = 3.55

   With \$50, he can create $\dfrac{50}{3.55}$ = 14.0845 sq. ft. of garden.

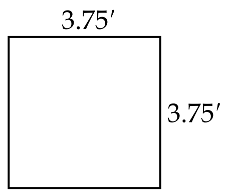
   He can create a garden with a maximum area of approximately 14 ft.^{2}.

b) Sketch a diagram showing the shape and dimensions of a potential garden within his budget. *(1 mark)*

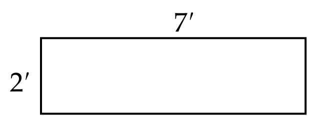
Answer: (Module 8, Lesson 3)

Answers will vary. Possible diagrams are shown below.

3.75 ft. square



or, 2 ft. by 7 ft. rectangle



or, circle with radius of 2.1 ft.