Grade 12 Introduction to Calculus (45S)

A Course for Independent Study

Field Validation Version

Manitoba

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Grade 12 Introduction to Calculus (45S)

Introduction

Introduction

Overview

Welcome to *Grade 12 Introduction to Calculus*! This course is a continuation of the concepts you have studied in previous years, as well as an introduction to new topics. It builds upon the pre-calculus topics you were introduced to in *Grade 12 Pre-Calculus Mathematics*. Many of the skills that you have already learned will be put to use as you solve problems and learn new skills along the way. This course helps you develop the skills, ideas, and confidence you will need to continue studying mathematics in the future.

As a student enrolled in a distance learning course, you have taken on a dual role—that of a student and a teacher. As a student, you are responsible for mastering the lessons and completing the learning activities and assignments. As a teacher, you are responsible for checking your work carefully, noting areas in which you need to improve and motivating yourself to succeed.

What Will You Learn in This Course?

In this course, problem solving, communication, reasoning, and mental math are some of the themes you will experience in each module. You will engage in a variety of activities that promote the connections between symbolic math ideas and the world around you.

How Is This Course Organized?

This course consists of the following four modules:

- Module 1: Limits
- Module 2: Derivatives
- Module 3: Applications of Derivatives
- Module 4: Integration

Each module in this course consists of several lessons, which contain the following components:

- **Lesson Focus:** The Lesson Focus at the beginning of each lesson identifies one or more specific learning outcomes (SLOs) that are addressed in the lesson. The SLOs identify the knowledge and skills you should have achieved by the end of the lesson.
- **Introduction:** Each lesson begins by outlining what you will be learning in that lesson.
- **Lesson:** The main body of the lesson consists of the content and processes that you need to learn. It contains information, explanations, diagrams, and completed examples.
- **Learning Activities:** Each lesson has a learning activity that focuses on the lesson content. Your responses to the questions in the learning activities will help you to practise or review what you have just learned. Once you have completed a learning activity, check your responses with those provided in the Learning Activity Answer Key found at the end of the applicable module. Do not send your learning activities to the Distance Learning Unit for assessment.
- **Assignments:** Assignments are found throughout each module within this course. At the end of each module, you will mail or electronically submit all your completed assignments from that module to the Distance Learning Unit for assessment. All assignments combined will be worth a total of 55 percent of your final mark in this course.
- **Lesson Summary:** Each lesson ends with a brief review of what you just learned.

What Resources Will You Need for This Course?

Please note that you do not need a textbook to complete this course. All of the content is included with this package.

Required Resources

The **only** required resources for this course are a scientific calculator and graph paper. Graph paper is available as one of the online resources available in the learning management system (LMS).

Optional Resources

- **A graphing calculator:** This includes a graphing calculator, a freeware graphing program, or a graphing calculator app. However, none of these are required or allowed when writing your final examination.
- **A computer with spreadsheet and graphing capabilities:** Access to a computer with spreadsheet software and graphing capabilities may be helpful to you for exploration and checking your understanding. However, none of these are required or allowed when writing your final examination.
- **A computer with Internet access:** Use of the Internet may be suggested as a resource in some places, but if you do not have access to an online computer you can still complete the related learning activities and assignments without it.
- **A photocopier:** With access to a photocopier/scanner, you could make a copy of your assignments before submitting them so that if your tutor/marker wants to discuss an assignment with you over the phone, each of you will have a copy. It would also allow you to continue studying or to complete further lessons while your original work is with the tutor/marker. Photocopying or scanning your assignments will also ensure that you keep a copy in case the originals are lost.

Who Can Help You with This Course?

Taking an independent study course is different from taking a course in a classroom. Instead of relying on the teacher to tell you to complete a learning activity or an assignment, you must tell yourself to be responsible for your learning and for meeting deadlines. There are, however, two people who can help you be successful in this course: your tutor/marker and your learning partner.

Your Tutor/Marker

Tutor/markers are experienced educators who tutor Independent Study Option (ISO) students and mark assignments and examinations. When you are having difficulty with something in this course, contact your tutor/marker, who is there to help you. Your tutor/marker's name and contact information were sent to you with this course. You can also obtain this information in the learning management system (LMS).

Your Learning Partner

A learning partner is someone **you choose** who will help you learn. It may be someone who knows something about calculus, but it doesn't have to be. A learning partner could be someone else who is taking this course, a teacher, a parent or guardian, a sibling, a friend, or anybody else who can help you. Most importantly, a learning partner should be someone with whom you feel comfortable and who will support you as you work through this course.

Your learning partner can help you keep on schedule with your coursework, read the course with you, check your work, look at and respond to your learning activities, or help you make sense of assignments. You may even study for your examination with your learning partner. If you and your learning partner are taking the same course, however, your assignment work should not be identical.

One of the best ways that your learning partner can help you is by reviewing your final practice examination with you. It is found in the learning management system (LMS), along with their answer key. Your learning partner can administer your practice examination, check your answers with you, and then help you learn the things that you missed.

How Will You Know How Well You Are Learning?

You will know how well you are learning in this course by how well you complete the learning activities, assignments, and examination.

Learning Activities

The learning activities in this course will help you to review and practise what you have learned in the lessons. You will not submit the completed learning activities to the Distance Learning Unit. Instead, you will complete the learning activities and compare your responses to those provided in the Learning Activity Answer Key found at the end of each module.

Each learning activity has two parts—Part A has BrainPower questions and Part B has questions related to the content in the lesson

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for you before trying the other questions. Each question should be completed quickly and without using a calculator, and most should be completed without using pencil and paper to write out the steps. Some of the questions will relate directly to content of the course. Some of the questions will review content from previous courses—content that you need to be able to answer efficiently.

Being able to do these questions in a few minutes will be helpful to you as you continue with your studies in calculus. If you are finding it is taking you longer to do the questions, you can try one of the following:

- work with your learning partner to find more efficient strategies for completing the questions
- ask your tutor/marker for help with the questions
- search online for websites that help you practice the computations so you can become more efficient at completing the questions

The assignment and examination questions do not require you to do the calculations quickly or without a calculaor: a graphing and/or scientific calculator is allowed for the assignments and a scientific calculator is allowed for the examination. However, being able to complete the BrainPower exercises quickly and without the use of a calculator will help build your confidence in mathematics. They are like the warm-up to a sporting event.

Part B: Course Content Questions

One of the easiest and fastest ways to find out how much you have learned is to complete Part B of the learning activities. These have been designed to let you assess yourself by comparing your answers with the answer keys at the end of each module. There is at least one learning activity in each lesson. You will need a notebook or loose-leaf pages to write your answers.

Make sure you complete the learning activities. Doing so will not only help you to practise what you have learned, but will also prepare you to complete your assignments and the examination successfully. Many of the questions on the examination will be similar to the questions in the learning activities. Remember that you **will not submit learning activities to the Distance Learning Unit**.

Assignments

Lesson are located throughout the modules, and include questions similar to the questions in the learning activities of previous lessons. The assignments have space provided for you to write your answers on the question sheets. **You need to show all your steps as you work out your solutions, and make sure your answers are clear (include units, where appropriate).**

Once you have completed all the assignments in a module, you will submit them to the Distance Learning Unit for assessment. The assignments are worth a total of 55 percent of your final course mark. You must complete each assignment in order to receive a final mark in this course. **You will mail or electronically submit these assignments to the Distance Learning Unit along with the appropriate cover page once you complete each module.**

The tutor/marker will mark your assignments and return them to you. Remember to keep all marked assignments until you have finished the course so that you can use them to study for your examination.

Final Examination

This course contains a final examination.

- The **final examination** is based on Modules 1 to 4 and is worth 45 percent of your final course mark. You will write the final examination when you have completed Module 4.

In order to do well on the examination, you should review all of the work that you have completed from Modules 1 to 4 for your final examination, including all learning activities and assignments. You will be required to bring the following supplies when you write your final examination: pencils (2 or 3 of each), blank paper, a ruler, and a scientific calculator.

You will write your examination under supervision.

Practice Examination and Answer Key

To help you succeed in your examination, you will have an opportunity to complete a Final Practice Examination. This examination, along with the answer key, is found in the learning management system (LMS). If you do not have access to the Internet, contact the Distance Learning Unit at 1-800-465-9915 to obtain a copy of the practice examination.

This practice examination is similar to the actual examination you will be writing. The answer key enables you to check your answers. This will give you the confidence you need to do well on your examination.

Requesting Your Examination

You are responsible for making arrangements to have the examination sent to your proctor from the Distance Learning Unit. Please make arrangements before you finish Module 4 to write the final examination.

To write your examination, you need to make the following arrangements:

- **If you are attending school,** your examination will be sent to your school as soon as all the applicable assignments have been submitted. You should make arrangements with your school's ISO school facilitator to determine a date, time, and location to write the examination.

- **If you are not attending school,** check the Examination Request Form for options available to you. Examination Request Forms can be found on the Distance Learning Unit's website, or look for information in the learning management system (LMS). Two weeks before you are ready to write the examination, fill in the Examination Request Form and mail, fax, or email it to

  Distance Learning Unit
  500–555 Main Street
  PO Box 2020
  Winkler MB R6W 4B8
  Fax: 204-325-1719
  Toll-Free Telephone: 1-800-465-9915
  Email: distance.learning@gov.mb.ca

How Much Time Will You Need to Complete This Course?

Learning through independent study has several advantages over learning in the classroom. You are in charge of how you learn and you can choose how quickly you will complete the course. You can read as many lessons as you wish in a single session. You do not have to wait for your teacher or classmates.

From the date of your registration, you have a maximum of **12 months** to complete the course, but the pace at which you proceed is up to you. Read the following suggestions on how to pace yourself.

Chart A: Semester 1

If you want to start this course in September and complete it in January, you can follow the timeline suggested below.

Module	Completion Date
Module 1	End of September
Module 2	End of October
Module 3	End of November
Module 4	Middle of January
Final Examination	End of January

Chart B: Semester 2

If you want to start the course in February and complete it in May, you can follow the timeline suggested below.

Module	Completion Date
Module 1	Middle of February
Module 2	Middle of March
Module 3	Middle of April
Module 4	Beginning of May
Final Examination	Middle May

Chart C: Full School Year (Not Semestered)

If you want to start the course in September and complete it in May, you can follow the timeline suggested below.

Module	Completion Date
Module 1	End of October
Module 2	Middle of January
Module 3	Middle of March
Module 4	Middle of April
Final Examination	Middle of May

Timelines

Do not wait until the last minute to complete your work, since your tutor/marker may not be available to mark it immediately. It may take a few weeks for your tutor/marker to assess your work and return it to you or your school.

If you need this course to graduate this school year, all coursework must be received by the Distance Learning Unit on or before the first Friday in May, and all examinations must be received by the Distance Learning Unit on or before the last Friday in May. Any coursework or examinations received after these deadlines may not be processed in time for a June graduation. Assignments or examinations submitted after these recommended deadlines will be processed and marked as they are received.

When and How Will You Submit Completed Assignments?

When to Submit Assignments

While working on this course, you will submit completed assignments to the Distance Learning Unit four times. The following chart shows you exactly what assignments you will be submitting at the end of each module.

Submission of Assignments	
Submission	Assignments You Will Submit
1	**Module 1: Limits** Module 1 Cover Sheet Assignment 1.1: Limits Assignment 1.2: Limit Theorems Assignment 1.3: Solving the Indeterminate Form of Limits Assignment 1.4: Exploring One-Sided Limits Assignment 1.5: Determining the Asymptotes of a Graph Assignment 1.6: Continuity
2	**Module 2: Derivatives** Module 2 Cover Sheet Assignment 2.1: Slope of the Tangent Line to a Curve Assignment 2.2: Definition of the Derivative Assignment 2.3: Basic Differentiation Rules Assignment 2.4: Differentiation with Product and Quotient Rules Assignment 2.5: Differentiation with the Chain Rule and Higher Order Derivatives Assignment 2.6: Implicit Differentiation
3	**Module 3: Applications of Derivatives** Module 3 Cover Sheet Assignment 3.1: Solving Inequalities Assignment 3.2: Particle Motion Problems Assignment 3.3: First Derivative Applications Assignment 3.4: Optimization Problems Assignment 3.5: Concavity and Sketching Polynomial Functions Assignment 3.6: Related Rates
4	**Module 4: Integration** Module 4 Cover Sheet Assignment 4.1: Antidifferentiation and Integration Assignment 4.2: Differential Equations Assignment 4.3: Definite Integral Assignment 4.4: Area under a Curve Assignment 4.5: Area between Two Functions

How to Submit Assignments

In this course, you have the choice of submitting your assignments either by mail or electronically.

- **Mail:** Each time you **mail** something, you must include the print version of the applicable Cover Sheet (found at the end of this Introduction). Complete the information at the top of each Cover Sheet before submitting it along with your assignments.
- **Electronic submission:** You do not need to include a cover sheet when submitting assignments electronically.

Submitting Your Assignments by Mail

If you choose to mail your completed assignments, please photocopy/scan all the materials first so that you will have a copy of your work in case your package goes missing. You will need to place the applicable module Cover Sheet and assignment(s) in an envelope, and address it to

Distance Learning Unit
500–555 Main Street
PO Box 2020
Winkler MB R6W 4B8

Your tutor/marker will mark your work and return it to you by mail.

Submitting Your Assignments Electronically

Assignment submission options vary by course. Sometimes assignments can be submitted electronically and sometimes they must be submitted by mail. Specific instructions on how to submit assignments were sent to you with this course. In addition, this information is available in the learning management system (LMS).

If you are submitting assignments electronically, make sure you have saved copies of them before you send them. That way, you can refer to your assignments when you discuss them with your tutor/marker. Also, if the original hand-in assignments are lost, you are able to resubmit them.

Your tutor/marker will mark your work and return it to you electronically.

> The Distance Learning Unit does not provide technical support for hardware-related issues. If troubleshooting is required, consult a professional computer technician.

What Are the Guide Graphics For?

Guide graphics are used throughout this course to identify and guide you in specific tasks. Each graphic has a specific purpose, as described below.

Lesson Introduction: The introduction sets the stage for the lesson. It may draw upon prior knowledge or briefly describe the organization of the lesson. It also lists the learning outcomes for the lesson. Learning outcomes describe what you will learn.

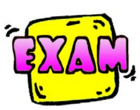
Learning Partner: Ask your learning partner to help you with this task.

Learning Activity: Complete a learning activity. This will help you to review or practise what you have learned and prepare you for an assignment or an examination. You will not submit learning activities to the Distance Learning Unit. Instead, you will compare your responses to those provided in the Learning Activity Answer Key found at the end of the applicable module.

Assignment: Complete an assignment. You will submit your completed assignments to the Distance Learning Unit for assessment at the end of a given module.

Mail or Electronic Submission: Mail or electronically submit your completed assignments to the Distance Learning Unit for assessment.

Phone or Email: Telephone or email your tutor/marker.



Examination: Write your final examination at this time.

Note: Take note of and remember this important information or reminder.

Notes

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Module 1: Limits
Cover Sheet

Please complete this sheet and place it on top of your assignments to assist in proper recording of your work. Submit the package to:

Drop-off/Courier Address
Distance Learning Unit
555 Main Street
Winkler MB R6W 1C4

Mailing Address
Distance Learning Unit
500–555 Main Street
PO Box 2020
Winkler MB R6W 4B8

Contact Information

Legal Name: ___________________________ Preferred Name: ___________________________

Phone: ___________________________ Email: ___________________________

Mailing Address: ____

City/Town: ___________________________ Postal Code: _______________

Attending School: ❑ No ❑ Yes

School Name: ____

Has your contact information changed since you registered for this course? ❑ No ❑ Yes

Note: Please keep a copy of your assignments so that you can refer to them when you discuss them with your tutor/marker.

For Student Use	For Office Use Only	
Module 1 Assignments Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	**Attempt 1** __________ Date Received	**Attempt 2** __________ Date Received
❑ Assignment 1.1: Limits	________ /18	________ /18
❑ Assignment 1.2: Limit Theorems	________ /13	________ /13
❑ Assignment 1.3: Solving the Indeterminate Forms of Limits	________ /13	________ /13
❑ Assignment 1.4: Exploring One-Sided Limits	________ /28	________ /28
❑ Assignment 1.5: Determining the Asymptotes of a Graph	________ /33	________ /33
❑ Assignment 1.6: Continuity	________ /10	________ /10
	Total: ____ **/115**	**Total:** ____ **/115**

For Tutor/Marker Use
Remarks:

Grade 12 Introduction to Calculus (45S)

Module 2: Derivatives
Cover Sheet

Please complete this sheet and place it on top of your assignments to assist in proper recording of your work. Submit the package to:

Drop-off/Courier Address

Distance Learning Unit
555 Main Street
Winkler MB R6W 1C4

Mailing Address

Distance Learning Unit
500–555 Main Street
PO Box 2020
Winkler MB R6W 4B8

Contact Information

Legal Name: ___________________________ Preferred Name: _________________________

Phone: ________________________________ Email: ____________________________________

Mailing Address: ____

City/Town: ____ Postal Code: _______________

Attending School: ❑ No ❑ Yes

School Name: ____

Has your contact information changed since you registered for this course? ❑ No ❑ Yes

Note: Please keep a copy of your assignments so that you can refer to them when you discuss them with your tutor/marker.

For Student Use	For Office Use Only	
Module 2 Assignments Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	**Attempt 1** __________ Date Received	**Attempt 2** __________ Date Received
❑ Assignment 2.1: Slope of the Tangent Line to a Curve	________ /18	________ /18
❑ Assignment 2.2: Definition of the Derivative	________ /15	________ /15
❑ Assignment 2.3: Basic Differentiation Rules	________ /10	________ /10
❑ Assignment 2.4: Differentiation with Product and Quotient Rules	________ /27	________ /27
❑ Assignment 2.5: Differentiation with the Chain Rule and Higher Order Derivatives	________ /17	________ /17
❑ Assignment 2.6: Implicit Differentiation	________ /19	________ /19
	Total: ____ **/106**	**Total:** ____ **/106**

For Tutor/Marker Use
Remarks:

Grade 12 Introduction to Calculus (45S)

Module 3: Applications of Derivatives
Cover Sheet

Please complete this sheet and place it on top of your assignments to assist in proper recording of your work. Submit the package to:

Drop-off/Courier Address

Distance Learning Unit
555 Main Street
Winkler MB R6W 1C4

Mailing Address

Distance Learning Unit
500–555 Main Street
PO Box 2020
Winkler MB R6W 4B8

Contact Information

Legal Name: ____________________ Preferred Name: ____________________

Phone: ____________________ Email: ____________________

Mailing Address: ____

City/Town: ____________________ Postal Code: ____________

Attending School: ❑ No ❑ Yes

School Name: ____

Has your contact information changed since you registered for this course? ❑ No ❑ Yes

Note: Please keep a copy of your assignments so that you can refer to them when you discuss them with your tutor/marker.

For Student Use	For Office Use Only	
Module 3 Assignments Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	**Attempt 1** __________ Date Received	**Attempt 2** __________ Date Received
❑ Assignment 3.1: Solving Inequalities	_______ /8	_______ /8
❑ Assignment 3.2: Particle Motion Problems	_______ /10	_______ /10
❑ Assignment 3.3: First Derivative Applications	_______ /13	_______ /13
❑ Assignment 3.4: Optimization Problems	_______ /20	_______ /20
❑ Assignment 3.5: Concavity and Sketching Polynomial Functions	_______ /22	_______ /22
❑ Assignment 3.6: Related Rates	_______ /20	_______ /20
	Total: ____ /93	**Total: ____ /93**

For Tutor/Marker Use
Remarks:

Grade 12 Introduction to Calculus (45S)

Module 4: Integration
Cover Sheet

Please complete this sheet and place it on top of your assignments to assist in proper recording of your work. Submit the package to:

Drop-off/Courier Address
Distance Learning Unit
555 Main Street
Winkler MB R6W 1C4

Mailing Address
Distance Learning Unit
500–555 Main Street
PO Box 2020
Winkler MB R6W 4B8

Contact Information

Legal Name: ___________________________ Preferred Name: _________________________

Phone: ________________________________ Email: ___________________________________

Mailing Address: ____

City/Town: ____ Postal Code: _______________

Attending School: ❑ No ❑ Yes

School Name: ____

Has your contact information changed since you registered for this course? ❑ No ❑ Yes

Note: Please keep a copy of your assignments so that you can refer to them when you discuss them with your tutor/marker.

For Student Use	For Office Use Only	
Module 4 Assignments Which of the following are completed and enclosed? Please check (✓) all applicable boxes below.	**Attempt 1** ________ Date Received	**Attempt 2** ________ Date Received
❑ Assignment 4.1: Antidifferentiation and Integration	______ /11	______ /11
❑ Assignment 4.2: Differential Equations	______ /35	______ /35
❑ Assignment 4.3: Definite Integral	______ /18	______ /18
❑ Assignment 4.4: Area under a Curve	______ /23	______ /23
❑ Assignment 4.5: Area between Two Functions	______ /20	______ /20
	Total: ____ **/107**	**Total:** ____ **/107**

For Tutor/Marker Use
Remarks:

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Module 1
Limits

Module 1: Limits

Introduction

Upon successful completion of this module, you will develop an understanding of the limit concept and how it applies to graphical and analytical functions. You will also be exposed to contextual applications to provide meaning behind this abstract mathematical concept.

Assignments in Module 1

When you have completed the assignments for Module 1, submit your completed assignments to the Distance Learning Unit either by mail or electronically through the learning management system (LMS). The staff will forward your work to your tutor/marker.

Lesson	Assignment Number	Assignment Title
2	Assignment 1.1	Limits
3	Assignment 1.2	Limit Theorems
4	Assignment 1.3	Solving the Indeterminate Form of Limits
5	Assignment 1.4	Exploring One-Sided Limits
6	Assignment 1.5	Determining the Asymptotes of a Graph
7	Assignment 1.6	Continuity

Important Note about Assessment

All the assignments and the final examination will be evaluated under three categories:

- concept—correct demonstration of the use of the concept
- procedure—logical steps taken to arrive at a conclusion
- notation—mathematical language communicated correctly

Often students will describe how they are frustrated with little mistakes they have made but they usually are referring to errors in procedure and notation. Although it is important to understand the concept, it is equally important to communicate your understanding logically with correct mathematical language.

Lesson 1: Welcome to Calculus

Lesson Focus

In this lesson, you will

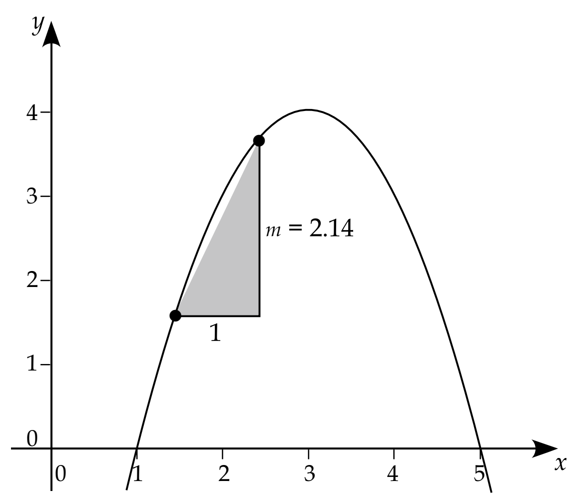
- ❑ identify connections between pre-calculus and calculus concepts
- ❑ describe applications of calculus

Lesson Introduction

Why Study Calculus?

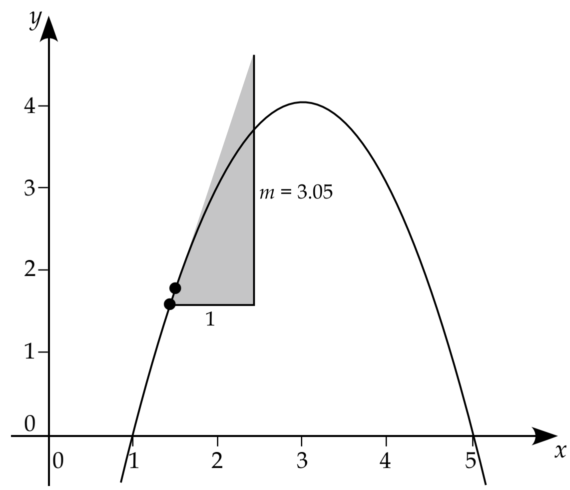
You have spent the last three years studying pre-calculus mathematics, so a natural progression would be to study calculus. Calculus is beautiful, empowering, and fascinating. It allows you to solve more complex problems than you would be able to solve with the algebra of pre-calculus mathematics alone. Calculus is used in many occupations by professionals such as engineers, scientists, and economists. Anyone planning to study mathematics or science at the university level would benefit by getting a head start from taking a calculus course at high school. Although this calculus course does not cover the topics to the same depth as a first-year university calculus course, it covers the fundamental concepts.

What Is Calculus?



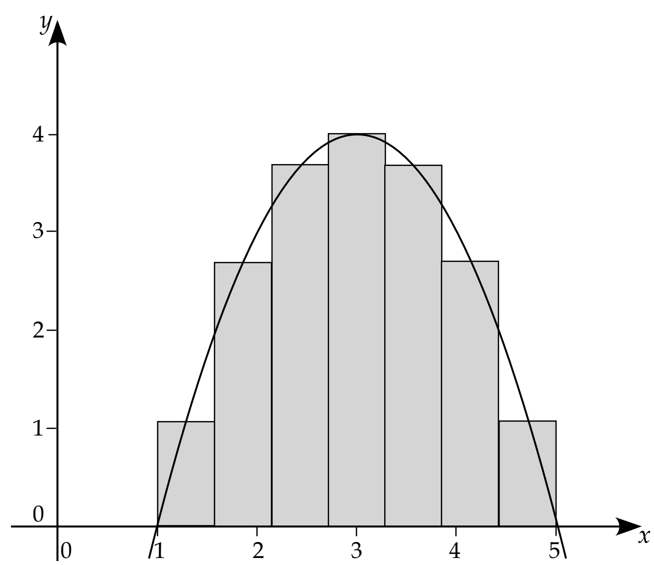
In pre-calculus courses, you learned about a wide variety of **functions**. In mathematics, functions are useful for modelling real-world phenomena, such as weather patterns, planetary motion, business trends, and population growth. Here are some examples. **Trigonometric functions** can be used to represent repeating or cyclical patterns, such as the energy in a heartbeat. **Exponential and logarithmic functions** can be used to represent population growth and radioactive decay. **Polynomial functions** can be used as approximations to many other functions in a restricted domain.

Calculus was developed to perform further analysis on these familiar functions, including an analysis of the **slope** of and the **area under a function curve**. For example, the slope of a line on a distance-time graph indicates the rate of change of distance over time. In this case, the slope has a specific meaning, which is the speed or velocity of the object in question. In reality, objects do not often move at a constant speed and the distance-time graph may be a complex curve rather than a line.



Without calculus, you can approximate the rate of change over a small time interval and estimate the slope. Then, you can zoom in on the function over an increasingly smaller time interval. You will learn the concept of limits to formalize the "zooming in" process on a function curve. Then, you will learn to use a branch of calculus called *differential calculus* to determine the exact rate of change of a function at an instant in time.

More analysis can be done when you learn to use a branch of calculus called *integral calculus* that involves the area under a function curve. Without calculus, you can approximate the area under an interval of a function by drawing rectangles under the curve and using the sum of the rectangle areas as an estimate. You can use the concept of *limits* to formalize the process of adding the areas of infinitely narrow rectangles. When you learn *integral calculus*, you can determine the exact area under a function curve.



Applications of Calculus

Calculus is a branch of mathematics that studies the rate at which quantities change. Although many mathematicians were involved in its development, the discovery of calculus is attributed to two very famous individuals from the 17th century: Isaac Newton, a British scientist; and Gottfried Leibniz, a German mathematician. Although it has long been a point of controversy over who should take credit for inventing calculus, both men independently made discoveries that led to what we know now as calculus.

The analysis of the slope at a point on and the area under a function curve forms the basis of the study of calculus. In the late 1600s, the mathematics concepts of calculus were developed in part as a tool to aid in geometry and physics work on the motions of celestial bodies. As with many developments in mathematics, calculus was developed with one area in mind but it has been found to be crucial to the study of many other fields. For instance, calculus is used in economics to determine marginal cost and marginal profit. The marginal cost is the extra cost for a manufacturer to produce one more item; similarly, marginal profit is the profit expected from selling one more item.

Calculus is also used in the study of electronics. For example, the PID electronic control-loop feedback system that is used in automatic ship steering or heat controllers requires both integral and differential calculus. The electronics in vehicles and airplanes also use calculus. Calculus is also used to analyze the graphs of seismic surveys and gravity and magnetic data that are collected as part of the process of oil exploration. Even many everyday items require calculus, such as the way a cube root is calculated in a scientific calculator.

One of the primary purposes of the pre-calculus courses you have taken is to give you the skills and knowledge that are essential to learning calculus. For example, you have learned how to factor polynomials, which is a foundational skill that you will use in each of the modules of this course. You have also learned about several different function types and how to sketch them. You will be using your knowledge of functions throughout the course, including sketching more complicated functions with the help of calculus. In pre-calculus, you learned how to calculate the sum of infinite geometric series that are convergent. This concept and the concept of infinity in general are directly related to the main concept you will study in Module 1: Limits. After studying calculus, you will be able to solve real-world problems such as:

1. Finding the radius and height of a cylinder to minimize the material needed to make an aluminum pop can that has a volume of 355 mL.
2. Finding the time it takes a ball to return to the ground if it is thrown upward with an initial velocity of 10 m/s.

You will encounter many other connections between what you have learned in previous math courses and what you are learning as you study calculus.

Learning Activity 1.1

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module. Questions 1 to 3 relate to the topic of limits that you will study in Module 1. Question 4 relates to the topic of derivatives that you will study in Module 2. Question 5 relates to the topic of integration that you will study in Module 4.

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. What happens to the value of $\frac{1}{x}$ as x increases and gets very large, approaching infinity?
2. What happens to the value of $\frac{1}{x}$ as x decreases and gets very close to zero?
3. Given the geometric series $8 + 4 + 2 + 1 + \frac{1}{2} + \ldots$
   a) Find the sum of the first 5 terms of the series.
   b) Find the sum of the first 8 terms of the series.
   c) Find the sum of the first 10 terms of the series.
   d) Keep adding more terms. As the number of terms in the series gets larger (approaching infinity), what happens to the sum of the series?

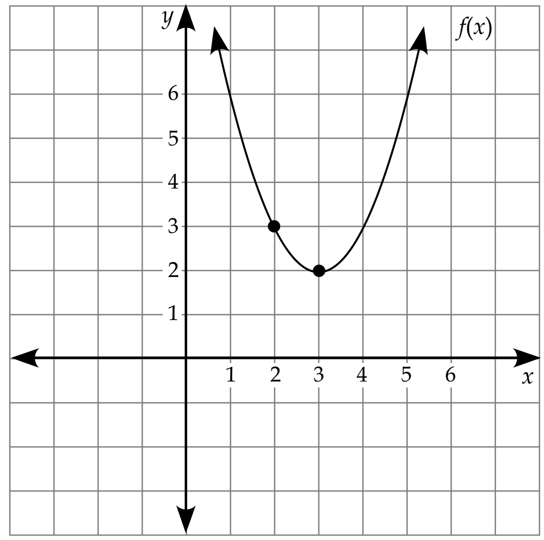
continued

Learning Activity 1.1 (continued)

4. Estimate the slope of $f(x)$ at the point:

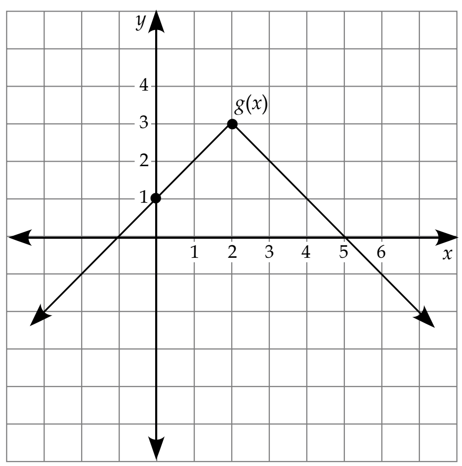
   a) where $x = 3$

   b) where $x = 2$

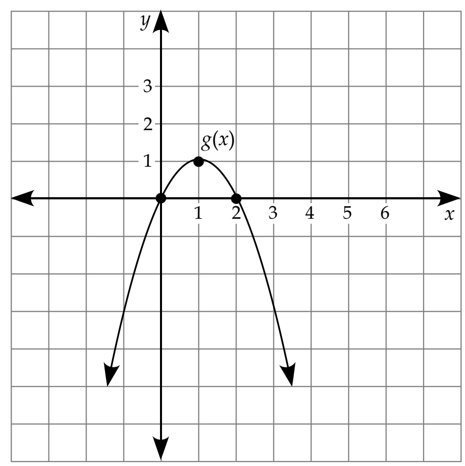


5. Estimate the area above the x-axis and below $g(x)$.

   a)



   b)



Notes

Lesson 2: Understanding the Concept of the Limit

Lesson Focus

In this lesson, you will

- ❑ analyze a function’s graph and table of values to explore the concept of limits
- ❑ express a function’s limit at a specific point

Lesson Introduction

A tiny bug is 16 m from a wall and walks halfway toward the wall in 1 hour. The bug is then 8 m from the wall and walks halfway toward the wall in another hour. Now the bug is 4 m from the wall and it continues to walk halfway toward the wall each hour.

How far will the bug get, and how long will it take?

This question illustrates the concept of limit. The bug always walks halfway to the wall, so it never quite gets there. The bug can keep walking halfway and get as close as it wants to the wall that was once 16 m away. Theoretically, the bug could keep doing this forever.

In this lesson, you will learn about the graphical and numerical representations of the limit. Connections between a function’s table of values and graph will help demonstrate the limit concept. In addition, you will practice using proper limit notation when representing your solutions to limit problems.

You know that $6 \div 3 = 2$ and that means $2 \times 3 = 6$. You cannot evaluate $6 \div 0$ as a number, since there is no number times 0 that will result in 6.

Limits were developed to mathematically get around the fact that division by zero cannot happen. Instead, use limits to consider what happens when you divide by a value that is very small and is close to, but not equal, to zero.

Limits are the building blocks of many concepts in calculus. They are used to prove theorems in derivatives and integrals for later concepts.

Exploring the Limit

You will study two functions, $f(x)$ and $g(x)$, around a specific point; $f(x)$ is defined at the specific point but $g(x)$ does not exist at the specific point. You will make three observations by analyzing the graph, the x-values to the left of the point and the x-values to the right of the point.

Part A

You will begin by analyzing the graph and table of values of the **radical function** $f(x) = \sqrt{x-2}$ around $x = 6$.

1. The table below demonstrates the values of $x < 6$ but very close to $x = 6$.

x	4	5	5.5	5.7	5.9	5.99	6
$f(x)$	1.414	1.732	1.871	1.924	1.975	1.998	2

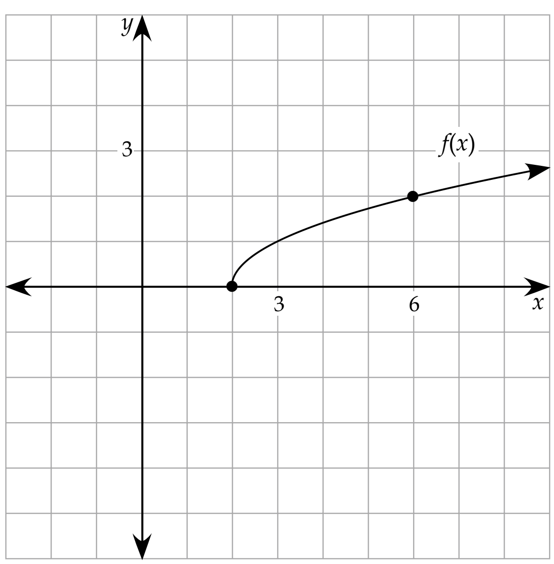
According to the above table, as x approaches 6 from the left, $f(x)$ approaches 2.

2. The second table demonstrates the values of $x > 6$ but very close to $x = 6$. Read the table from right to left.

x	6	6.01	6.1	6.3	6.5	7	8
$f(x)$	2	2.003	2.025	2.074	2.121	2.236	2.450

According to the above table, as x approaches 6 from the right, $f(x)$ approaches 2.

3. After studying the graph of $f(x)$ below, you should notice that the function value exists at $x = 6$ and $f(6) = 2$.



After reviewing your observations about $f(x)$, you can determine that as x approaches 6 from values less than or greater than 6, the value of $f(x)$ approaches 2. Thus, the function value of 2 is called the limit of the function as x approaches 6.

Special Note: In this function, the limit is also the function value. However, in many limit questions, the function value is either not defined or is a different value from the limit value.

Part B

Now you will analyze the graph and table of values of the **rational function** $g(x) = \dfrac{x+1}{x^2+3x+2}$ around $x = -1$.

1. The table below demonstrates the values of $x < -1$ but very close to $x = -1$.

x	−1.9	−1.5	−1.2	−1.1	−1.01	−1
$g(x)$	10	2	1.25	1.111	1.010	DNE (does not exist)

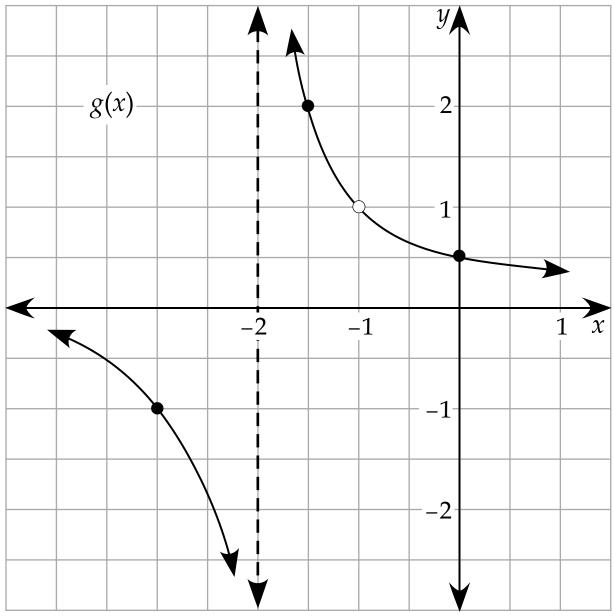
   The above table shows that as x approaches −1 from the left, $g(x)$ approaches 1.

2. The second table demonstrates the values of $x > -1$ but very close to $x = -1$. Read the table from right to left.

x	−1	−0.99	−0.9	−0.8	−0.5	0
$g(x)$	DNE	0.990	0.909	0.833	0.667	0.5

   The above table shows that as x approaches −1 from the right, $g(x)$ approaches 1.

3. After studying the graph of $g(x)$ below, you should notice that $g(-1)$ does not exist because there is a point of discontinuity (hole) in the graph. In addition, when the value is substituted into its equation, $g(-1) = \dfrac{-1+1}{(-1)^2+3(-1)+2} = \dfrac{0}{0}$, so the equation has no solution since you cannot divide by zero.

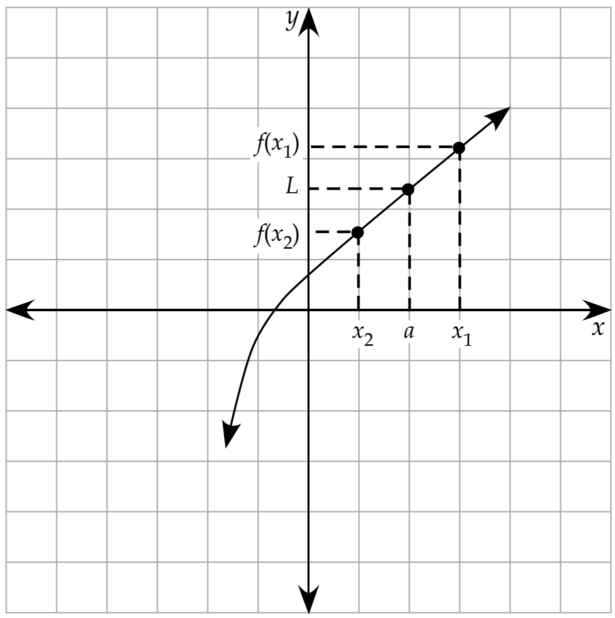


After reviewing your observations about $g(x)$, you can determine that as x approaches −1 from values less than −1 or greater than −1, the value of $g(x)$ approaches 1. Thus, the limit of $g(x)$ as x approaches −1 is 1. Even though the function value approaches 1, the function value does not exist. Thus, there is a hole on the graph at this location. This is consistent with your understanding of rational functions because $x = -1$ is a non-permissible value that makes the denominator equal to zero. You cannot divide by zero, so $g(x)$ is not defined at $x = -1, -2$.

Special note: For this function, the limit exists at the point where $x = -1$ but the function **does not exist** at the point. A limit does not need to equal the function value because, in this case, it describes the value of $f(x)$ as x approaches, *but does not equal*, −1.

Limit Definition and Notation

The limit of a function is best described visually with a graph such as the one below.



As x_1 and x_2 approach "*a*," the function values of $f(x_1)$ and $f(x_2)$ approach "*L*," regardless if the function exists at $x = a$.

$$\lim_{x \to a} f(x) = L$$

Essentially, as x approaches a specific number, a, from either the left or the right, the limit exists if the function value, $f(x)$ (y-value), approaches a specific number L.

> The limit of any function can be written in the form
>
> $$\lim_{x \to a} f(x) = L$$
>
> and we read it as "the limit of $f(x)$ as x approaches a is equal to L."

Example 1

The limit of the function in Part A, $f(x) = \sqrt{x-2}$, discussed earlier can be written as $\lim_{x \to 6} f(x) = 2$ because as the x-value approaches 6, the function value approaches 2. This is different than saying $f(6) = 2$, since the limit describes the value of $f(x)$ when x is close to, but not equal to, 6.

Example 2

The limit of function in Part B, $g(x) = \dfrac{x+1}{x^2+3x+2}$, discussed earlier can be written as $\lim\limits_{x \to -1} g(x) = 1$ because as the x-value approaches −1, the function value approaches 1. $f(-1)$ does not exist since $x = -1$ is a non-permissible value. However, the limit exists because it is describing the value of $f(x)$ when x is close to, but not equal to, −1.

Learning Activity 1.2

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. What is the non-permissible value of the expression $\dfrac{5}{x-6}$?
2. Evaluate: $\dfrac{3^2 - 4}{3^2 + 4^2}$
3. What is the domain of $g(x) = \dfrac{5}{x-6}$?
4. What is the y-intercept of $g(x) = \dfrac{5}{x-6}$?
5. If $g(x) = \dfrac{5}{x-6}$, determine $g(1)$.
6. Does the function value have to exist if the limit exists?
7. Determine the domain of $f(x) = \sqrt{x+5}$.
8. If $f(x) = \sqrt{x+5}$, determine $f(-4)$.

continued

Learning Activity 1.2 (continued)

Part B: Limits

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Given the function $f(x) = \dfrac{x^2 - 4}{x + 2}$

   a) Complete its table of values below.

x	-2.5	-2.1	-2.01	-2	-1.99	-1.9	-1.5
$f(x)$							

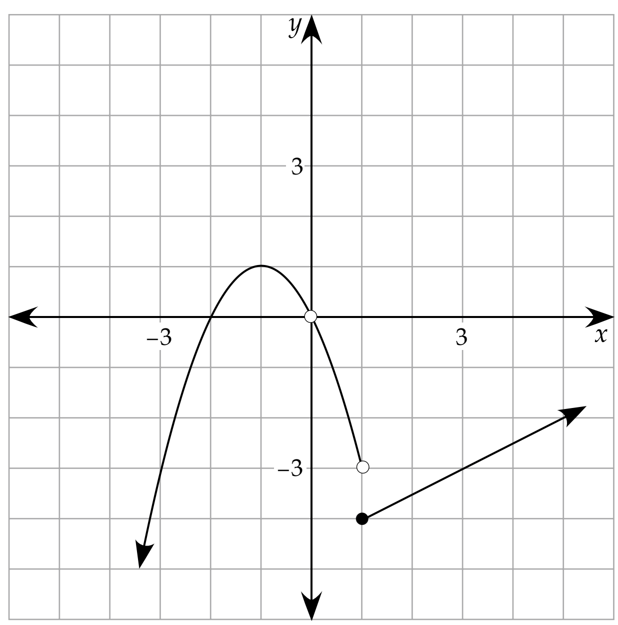
   b) What function value does $f(x)$ approach as x approaches -2? Explain how you arrived at this answer.

   c) Graph the function by plotting the points and sketching the curve.

   d) What is the domain and range for the function? How are the domain's restrictions expressed on the graph?

   e) Use the graph and the table of values to explain why $\lim\limits_{x \to -2} f(x) = -4$, but $f(-2) \neq -4$.

2. Given the graph of $g(x)$ below, complete the chart that follows.



continued

Learning Activity 1.2 (continued)

Determine the following limits and function values:	Solution
a) $\lim_{x \to -1} g(x)$	
b) $\lim_{x \to 0} g(x)$	
c) $\lim_{x \to 1} g(x)$	
d) $g(-1)$	
e) $g(0)$	
f) $g(1)$	

Lesson Summary

The table of values and the graph of a function allowed you to visualize the limit concept. However, you have not used the equation of the function to determine the limit analytically. In the remainder of the module, you will explore the many algebraic strategies used to determine the limit of a function analytically.

Assignment 1.1

Limits

Total: 18 marks

1. Given the function $f(x) = \dfrac{x^2 - 1}{x + 1}$

   a) Complete the table of values for the function. *(2 marks)*

x	−2	−1.5	−1.1	−1.01	−1	−0.99	−0.9	−0.5	0
$f(x)$									

x	0.5	0.9	0.99	1	1.01	1.1	1.5	2
$f(x)$								

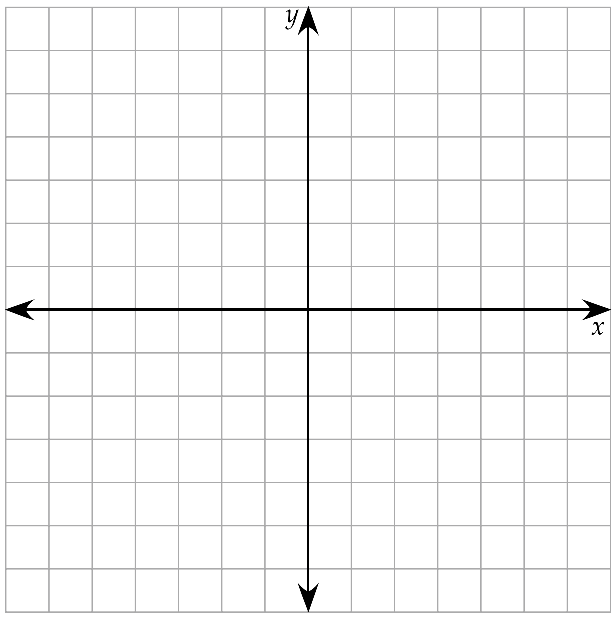
   b) What function value does $f(x)$ approach as x approaches −1? Explain how you arrived at this answer. *(2 marks)*

   c) What function value does $f(x)$ approach as x approaches 1? Explain how you arrived at this answer. *(2 marks)*

continued

Assignment 1.1: Limits (continued)

d) Graph the function by plotting the points and sketching the curve. *(2 marks)*



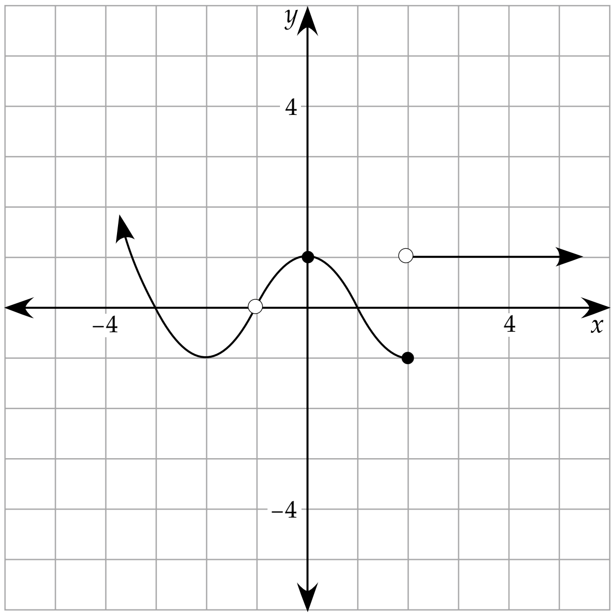
e) What is the domain and range for the function? How are the domain limitations expressed on the graph? *(2 marks)*

f) How does the graph and the table of values show that $\lim_{x \to -1} f(x) = -2$ but $f(-1) \neq -2$? *(1 mark)*

g) Why does $\lim_{x \to 1} f(x) = 0$ and $f(1) = 0$? *(1 mark)*

continued

2. Use the graph of $g(x)$ below to answer the questions that follow.



Determine the following limits and function values, if possible. *(6 marks)*

a) $\lim_{x \to -1} g(x)$

b) $\lim_{x \to -2} g(x)$

c) $\lim_{x \to 2} g(x)$

d) $g(-1)$

continued

Assignment 1.1: Limits (continued)

e) $g(-2)$

f) $g(2)$

Lesson 3: Limit Theorems and Direct Substitution

Lesson Focus

In this lesson, you will

- ❑ verify the limit theorems
- ❑ determine the limit of functions by direct substitution by utilizing the limit theorems

Lesson Introduction

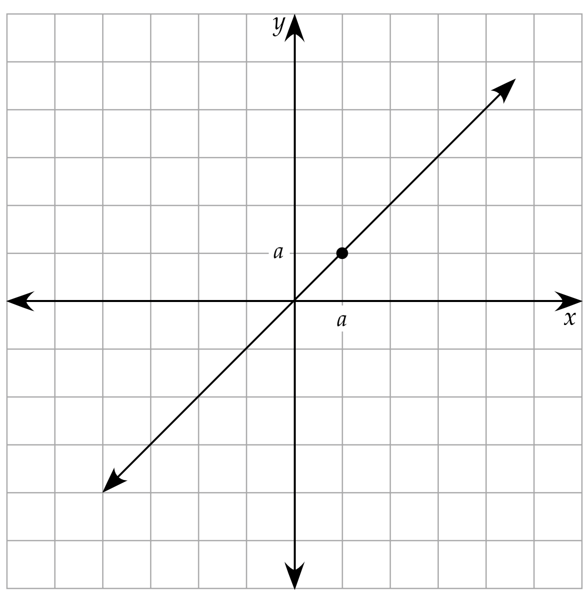
In this lesson, you will begin to learn how to determine the limit of a function analytically using direct substitution with the limit theorems.

Limit Theorems

There are seven limit theorems that are useful in calculating limits analytically. You are not required to prove these theorems in this course. The theorems will be verified with examples showcasing their usefulness.

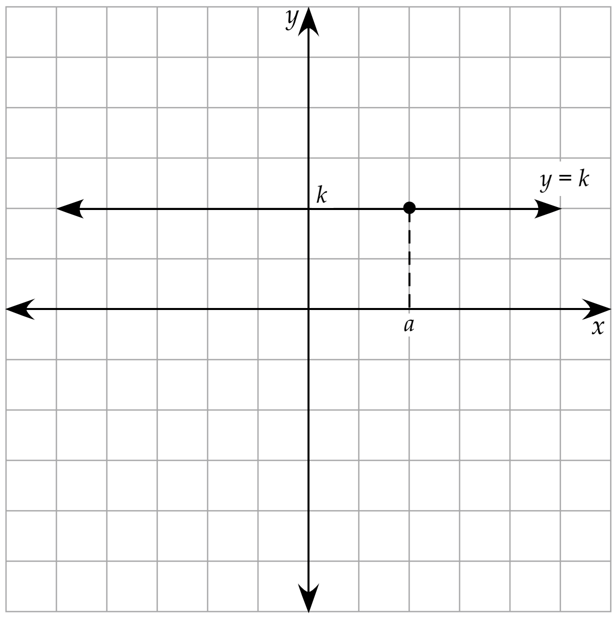
1. Limit of the identity function

$$\lim_{x \to a} x = a$$



2. Limit of the constant function

$$\lim_{x \to a} k = k$$



By applying the two basic theorems above, you can calculate the limits of all polynomial functions. The following properties are used to calculate other unfamiliar limits.

3. Constant times a Function Theorem

$$\lim_{x \to a} \left[k \cdot f(x)\right] = k \cdot \lim_{x \to a} f(x)$$

4. Sum/Difference Theorem

$$\lim_{x \to a}\left[f(x) \pm g(x)\right] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

5. Product Theorem

$$\lim_{x \to a} \left[f(x) \cdot g(x)\right] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

6. Quotient Theorem

$$\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

Special Note: In the quotient theorem, the denominator cannot equal zero.

7. Power Theorem

$$\lim_{x \to a} \left[f(x)^n \right] = \left(\lim_{x \to a} f(x) \right)^n$$

Special Note: Brackets are very important in limit notation because they change the meaning of the answer. For example,

$$\lim_{x \to 1} \left((x+2)^2 + x \right) \neq \lim_{x \to 1} (x+2)^2 + x$$

Left Side **(limit applies to the whole sum)**	**Right Side** **(limit only applies to the first part)**
$\lim_{x \to 1} \left((x+2)^2 + x \right)$ $= \lim_{x \to 1} \left(x^2 + 4x + 4 + x \right) = \lim_{x \to 1} \left(x^2 + 5x + 4 \right)$ $= (1)^2 + 5(1) + 5 = 1 + 5 + 4$ $= 10$	$\lim_{x \to 1} (x+2)^2 + x$ $= \lim_{x \to 1} \left(x^2 + 4x + 4 \right) + x$ $= (1)^2 + 4(1) + 4 + x = 1 + 4 + 4 + x$ $= 9 + x$
LS ≠ RS	

Verifying the Theorems

Example 1

Use the power theorem to evaluate:

$$\lim_{x \to 3} x^5$$

Solution

$$\lim_{x \to 3} x^5 = \left(\lim_{x \to 3} x \right)^5 = 3^5$$

This theorem states that direct substitution into a power function is all that is required for evaluating the limit.

Example 2:

Use the constant theorem to evaluate:

$$\lim_{x \to 3} 5$$

Solution

$$\lim_{x \to 3} 5 = 5$$

This theorem states that the limit of a constant is the constant itself.

Example 3

Verify the constant times the function theorem:

$$\lim_{x\to 3}\left[5\cdot(x+2)\right] = 5\cdot\lim_{x\to 3}(x+2)$$

Solution

Left Side	Right Side
$\lim_{x\to 3}\left[5(x+2)\right]$	$5\cdot\lim_{x\to 3}(x+2)$
$=\lim_{x\to 3}\left[5x+10\right]$	$=5\cdot(3+2)$
$=5(3)+10$	$=5\cdot(5)$
$=25$	$=25$

LS = RS

This theorem states that the limit of a constant times a function is the constant times the limit of the function.

Example 4

Verify the sum/difference theorem:

$$\lim_{x\to 3}\left[(x+2)+(3x-1)\right]=\lim_{x\to 3}(x+2)+\lim_{x\to 3}(3x-1)$$

Solution

Left Side	Right Side
$\lim_{x\to 3}\left[(x+2)+(3x-1)\right]$	$\lim_{x\to 3}(x+2)+\lim_{x\to 3}(3x-1)$
$=\lim_{x\to 3}(x+2+3x-1)$	$=(3+2)+\left[3(3)-1\right]$
$=\lim_{x\to 3}(4x+1)$	$=5+8$
$=4(3)+1$	$=13$
$=13$	

LS = RS

This theorem states that the limit of a sum or difference of two functions is the sum or difference of the limits of the two functions.

Example 5

Verify the product theorem:

$$\lim_{x\to 3}\left[(x+2)\cdot(3x-1)\right]=\lim_{x\to 3}(x+2)\cdot\lim_{x\to 3}(3x-1)$$

Solution

Left Side	Right Side
$\lim_{x\to 3}\left[(x+2)\cdot(3x-1)\right]$	$\lim_{x\to 3}(x+2)\cdot\lim_{x\to 3}(3x-1)$
$=\lim_{x\to 3}\left(3x^2+5x-2\right)$	$=(3+2)\cdot\left[3(3)-1\right]$
$=3(3)^2+5(3)-2$	$=(5)\cdot(8)$
$=40$	$=40$

LS = RS

This theorem states that the limit of a product of two functions is the product of the limits of the two functions.

Example 6

Verify the quotient theorem:

$$\lim_{x\to 3}\left[\frac{x+2}{3x-1}\right]=\frac{\lim_{x\to 3}(x+2)}{\lim_{x\to 3}(3x-1)}$$

Solution

Left Side	Right Side
$\lim_{x\to 3}\left[\frac{x+2}{3x-1}\right]$	$\frac{\lim_{x\to 3}(x+2)}{\lim_{x\to 3}(3x-1)}$
$=\left[\frac{3+2}{3(3)-1}\right]$	$=\frac{\left[3+2\right]}{\left[3(3)-1\right]}$
$=\frac{5}{8}$	$=\frac{5}{8}$

LS = RS

This theorem states that the limit of a quotient of two functions is the quotient of the limits of two functions.

Direct Substitution

You can now evaluate limits analytically by directly substituting the value into the variable in the function. However, substitution is not always possible. You will explore this in the next lesson.

Let's verify direct substitution using the limit theorems.

Example 1

Evaluate $\lim\limits_{x \to 2}\left(2x^3 - 3x + 5\right)$.

Solution

Solution showcasing each limit theorem:

$$\lim_{x \to 2}\left(2x^3 - 3x + 5\right)$$

$= \lim\limits_{x \to 2} 2x^3 - \lim\limits_{x \to 2} 3x + \lim\limits_{x \to 2} 5$	Sum/difference theorem.
$= 2 \cdot \lim\limits_{x \to 2} x^3 - 3 \cdot \lim\limits_{x \to 2} x + \lim\limits_{x \to 2} 5$	Constant times a function theorem.
$= 2 \cdot 2^3 - 3 \cdot 2 + 5$	Substitution.
$16 - 6 + 5$	Simplify.
$= 15$	

Special Note: Ensure that the limit notation remains until the value given in the limit notation is substituted for the variable in the function.

Based on the limit theorems, it is perfectly valid to find the solution with direct substitution into the function:

$$\lim_{x \to 2}\left(2x^3 - 3x + 5\right)$$

$$= 2(2)^3 - 3(2) + 5$$

$$= 16 - 6 + 5$$

$$= 15$$

Direct substitution applies the limit theorems simultaneously to arrive at the same limit of 15 with less work.

Example 2

Evaluate $\lim\limits_{x\to 2}\left[\dfrac{3x+5}{x-1}\right]$.

Solution

Solution showcasing each limit theorem:

$$\lim_{x\to 2}\left[\frac{3x+5}{x-1}\right]$$

$$=\frac{\lim\limits_{x\to 2}(3x+5)}{\lim\limits_{x\to 2}(x-1)} \qquad \text{Quotient theorem.}$$

$$=\frac{\lim\limits_{x\to 2}3x+\lim\limits_{x\to 2}5}{\lim\limits_{x\to 2}x-\lim\limits_{x\to 2}1} \qquad \text{Sum/difference theorem.}$$

$$=\frac{3\cdot\lim\limits_{x\to 2}x+\lim\limits_{x\to 2}5}{\lim\limits_{x\to 2}x-\lim\limits_{x\to 2}1} \qquad \text{Constant times a function.}$$

$$=\frac{3(2)+5}{2-1} \qquad \text{Substitution.}$$

$$=\frac{11}{1}=1 \qquad \text{Simplify.}$$

Again, it is valid due to the limit theorems to find the solution with direct substitution into the function:

$$\lim_{x\to 2}\left[\frac{3x+5}{x-1}\right]=\frac{3(2)+5}{2-1}=\frac{11}{1}=11$$

Again, direct substitution yielded the same limit of 11 with a lot less work. However, direct substitution is not always sufficient to evaluate a limit question. For example, the limits below cannot be solved by direct substitution alone. Can you see why?

$$\lim_{x\to 2}\left(\frac{3x+5}{x-2}\right) \qquad \text{OR} \qquad \lim_{x\to 1}\left[\frac{x^2-1}{x-1}\right]$$

You will study other strategies for solving limits analytically in later lessons.

Learning Activity 1.3

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Determine the non-permissible value(s) for $\dfrac{\sqrt{x}+3}{x+9}$.

2. Determine the non-permissible value(s) for $\dfrac{x+2}{\sqrt{x}-2}$.

3. Determine the greatest common factor for 6 and 15.

4. Develop and simplify: $(2x + 1)^2$

5. Factor: $4x^2 - 1$

6. Simplify: $(4x^2 - x) - (2x^2 + 3x)$

7. Simplify: $(2x - 3)(2x + 3)$

8. What is the greatest common factor between $4x^2 - 1$ and $2x + 1$?

continued

Learning Activity 1.3 (continued)

Part B: Limit Theorems

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Verify the following limit theorems by evaluating each side.

   a) $\lim_{x \to 2}\left[(4x+1)-(2x+3)\right] = \lim_{x \to 2}(4x+1) - \lim_{x \to 2}(2x+3)$

   b) $\lim_{x \to 1}\left[\dfrac{4x+1}{2x+3}\right] = \dfrac{\lim_{x \to 1}(4x+1)}{\lim_{x \to 1}(2x+3)}$

2. Evaluate the following with direct substitution:

   a) $\lim_{x \to 2}(-x^4 + 3x^3 - 7x^2 + 1)$

   b) $\lim_{x \to 9}\left[\dfrac{\sqrt{x}+3}{x+9}\right]$

3. Use direct substitution to find an equivalent expression to represent the limit:

   a) $\lim_{a \to 3}\left[(x+a)^2\right]$

   b) $\lim_{a \to 2}\left[\dfrac{x^2-a^2}{x-a}\right]$

Lesson Summary

In this lesson, you learned the limit theorems, verified them, and evaluated limits with direct substitution. You will now learn about limits that cannot be evaluated with direct substitution alone.

Notes

Assignment 1.2

Limit Theorems

Total: 13 marks

1. Verify the following by evaluating each side:

   a) $\lim_{x \to -1} \left[5 \cdot \left(x^2 + 3 \right) \right] = 5 \cdot \lim_{x \to -1} \left(x^2 + 3 \right)$ *(4 marks)*

   b) $\lim_{x \to 0} \left[\left(6x - 8 \right) \cdot \left(9x + 2 \right) \right] = \lim_{x \to 0} \left(6x - 8 \right) \cdot \lim_{x \to 0} \left(9x + 2 \right)$ *(4 marks)*

continued

Assignment 1.2: Limit Theorems (continued)

2. Evaluate the following using direct substitution.

$$\lim_{x \to 2} \left(\frac{\frac{1}{x} + \frac{1}{2}}{x + 2} \right) \qquad \textit{(2 marks)}$$

3. Use direct substitution to find an equivalent expression to represent the limit.

$$\lim_{h \to 1} \left(\frac{(x + h)^2 - x^2}{2h} \right) \qquad \textit{(3 marks)}$$

Lesson 4: Evaluating the $\frac{0}{0}$ Indeterminate Form of Limits

Lesson Focus

In this lesson, you will

- ❑ explain why $\frac{0}{0}$ is called an indetermination
- ❑ solve limits of indeterminate form by algebraic manipulation

Lesson Introduction

When graphing a rational function, such as $f(x) = \frac{x^2 + 2x + 1}{x + 1}$, there will be a point of discontinuity (a hole) when $x = -1$. The value of $x = -1$ is a non-permissible value because it makes the denominator of the rational function zero. Furthermore, in factored form, you can see that there is an $(x + 1)$ factor in the numerator as well.

$$f(x) = \frac{x^2 + 2x + 1}{x + 1}$$

$$f(x) = \frac{(x + 1)(x + 1)}{x + 1}$$

When you substitute $x = -1$ into the function, the result is $f(0) = \frac{0}{0}$.

In this lesson, you will learn how to solve limits that, upon direct substitution, result in the indeterminate form $\frac{0}{0}$. Algebraically manipulating the original function will allow the limit to be solved by direct substitution.

This lesson describes the first of many ways to analytically solve a limit question by algebraically manipulating the function.

Indeterminate Form $\frac{0}{0}$ of Limits

When evaluating the limit of a function by **direct substitution** results in zeros in both the numerator and denominator, the expression is said to be in **indeterminate form**. This means that the limit value cannot be determined in this manner and further investigation is required.

The $\frac{0}{0}$, or indeterminate form, is problematic because it unclear if the answer should be zero, one, or infinity. When the numerator is zero, it is usually expected that the answer will be zero. In all other cases, when two numbers are divided by themselves, the fraction is reduced and one is the result. Furthermore, when you divide by a really small number, you expect the answer to approach a really larger number; as the denominator approaches zero the fraction value approaches infinity. This whole problem arises when you try to divide by zero. **You cannot divide by zero!** Limits were developed by mathematicians to describe what is happening to the value of a function as it approaches this indeterminate value of $\frac{0}{0}$.

Thus, the form may require algebraic manipulation of the original function to eliminate the zero in the denominator. While you try to eliminate the **zero-producing factor** in the denominator, it is important to note that the function is not defined at the value in question since it is a non-permissible value, which has resulted in the indeterminate form. However, the function could be converging on a value, which is why the limit could still exist. A limit is the value of the function as *x approaches* a specific value even if the function is undefined when *x* equals the value.

Three examples will showcase common types of questions that require algebraic manipulation.

Example 1

Determine $\lim_{x \to 3} \left[\frac{x^2 - 9}{x - 3} \right]$.

Solution

$$\lim_{x \to 3} \left[\frac{x^2 - 9}{x - 3} \right] = \frac{9 - 9}{3 - 3} = \frac{0}{0}$$

After direct substitution, the solution is in **indeterminate form (or I.F.)**. So, you need to try to algebraically manipulate the function.

In this function, *factoring the numerator*, which is a **difference of squares**, will allow you to cancel the binomial in the denominator. Your goal is to eliminate the factor that is posing the problem, in this case the $x - 3$ produces a zero in the denominator.

$\lim\limits_{x \to 3} \left[\dfrac{x^2 - 9}{x - 3} \right]$	Start from the beginning.
$= \lim\limits_{x \to 3} \left[\dfrac{(x - 3)(x + 3)}{x - 3} \right]$	Factor the numerator.
$= \lim\limits_{x \to 3} (x + 3)$	Simplify the rational expression.
$= 3 + 3 = 6$	Solve by direct substitution.

Special Note: Ensure that the limit notation remains until the value is substituted into the function.

After factoring the numerator and reducing, you were able to solve for the limit of the function as $x \to 3$. However, it is important to note that the function value at $x = 3$ is still undefined, since 3 is a non-permissible value.

The value of the limit describes what is happening to the function value as x approaches 3 but is not equal to 3.

Example 2

Determine $\lim\limits_{x \to 4} \left[\dfrac{\sqrt{x} - 2}{x - 4} \right]$.

Solution

$$\lim_{x \to 4} \left[\frac{\sqrt{x} - 2}{x - 4} \right] = \frac{\sqrt{4} - 2}{4 - 4} = \frac{2 - 2}{4 - 4} = \frac{0}{0}$$

After direct substitution, the solution is in **indeterminate form**. So you need to try to algebraically manipulate the function.

When there is a difference between two terms involving a radical, you can use an algebraic technique called *rationalizing the numerator*. Multiplying the numerator and the denominator by the **conjugate** of the numerator may allow you to cancel the binomial in the denominator. Your goal is to eliminate the factor that is posing the problem; in this case, the difference between two terms, $x - 4$, in the denominator is eliminated, leaving the sum of two terms, $\sqrt{x} + 2$.

$\lim_{x \to 4}\left[\dfrac{\sqrt{x}-2}{x-4}\right]$ Start from the beginning.

$= \lim_{x \to 4}\left[\dfrac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)}\right]$ Multiply both numerator and denominator by the conjugate $\sqrt{x}+2$.

$= \lim_{x \to 4}\left[\dfrac{x-4}{(x-4)(\sqrt{x}+2)}\right]$ **Simplify the numerator only.** Leave the denominator in factored form.

$= \lim_{x \to 4}\left(\dfrac{1}{\sqrt{x}+2}\right)$ Simplify.

$= \dfrac{1}{\sqrt{4}+2} = \dfrac{1}{2+2} = \dfrac{1}{4}$ Solve by direct substitution.

After rationalizing the numerator and reducing, you were able to solve for the limit of the function as $x \to 4$. However, it is important to note that the function value at $x = 4$ is still undefined, even though the limit as x approaches 4 is $\dfrac{1}{4}$.

Example 3

Determine $\lim_{x \to 2}\left(\dfrac{\frac{1}{x}-\frac{1}{2}}{x-2}\right)$.

Solution

$$\lim_{x \to 2}\left(\frac{\frac{1}{x}-\frac{1}{2}}{x-2}\right) = \left(\frac{\frac{1}{2}-\frac{1}{2}}{2-2}\right) = \frac{0}{0}$$

After direct substitution, the solution is in **indeterminate form**. So you need to try to algebraically manipulate the function.

In this function, *determining the lowest common denominator in the numerator* will allow you to cancel the binomial in the denominator. Your goal is to eliminate the factor that is posing the problem—in this case, the $x - 2$ in the denominator.

$$\lim_{x \to 2} \left(\frac{\frac{1}{x} - \frac{1}{2}}{x - 2} \right)$$

Start from the beginning.

$$= \lim_{x \to 2} \left(\frac{\frac{1}{x} - \frac{1}{2}}{x - 2} \right) \cdot \left(\frac{2x}{2x} \right)$$

Multiply both numerator and denominator by the lowest common denominator of the fractions in the numerator, $2x$.

$$= \lim_{x \to 2} \left(\frac{2 - x}{(x - 2) \cdot (2x)} \right)$$

Simplify the numerator only. Leave the denominator in factored form.

$$= \lim_{x \to 2} \left(\frac{-(x - 2)}{(x - 2) \cdot (2x)} \right)$$

Factor -1 out of the numerator so that you will be able to cancel common factors.
Note: $2 - x$ is the **additive inverse** of $x - 2$.

$$= \lim_{x \to 2} \left(\frac{-1}{2x} \right)$$

$$= \frac{-1}{2(2)} = -\frac{1}{4}$$

Solve by direct substitution.

After simplifying complex fractions in the numerator and cancelling it with part of the denominator, you were able to solve for the limit of the function as $x \to 2$. However, it is important to note that the function value at $x = 2$ is still undefined, even though the limit as x approaches 2 is $-\frac{1}{4}$.

Using Functional Notation

You can use functional notation when solving limits. This will become very useful when you begin the discussion of rate of change in Module 2: Derivatives.

Example 4

If $f(x) = 2x^2 - 5x + 1$, determine $\lim_{h \to 0} \left[\frac{f(1+h) - f(1)}{h} \right]$.

Solution

$$\lim_{h \to 0} \left[\frac{f(1+h) - f(1)}{h} \right] = \left[\frac{f(1) - f(1)}{0} \right] = \frac{0}{0}$$

After direct substitution, the solution is in **indeterminate form**. So you need to algebraically manipulate the function.

When working with functional notation in limits, typically you would simplify the expression first before factoring the zero-producing factor from the denominator. Your goal is to eliminate the factor that is posing the problem—in this case, the *h* from the denominator.

$$\lim_{h \to 0} \left[\frac{f(1+h) - f(1)}{h} \right] \text{ where } f(x) = 2x^2 - 5x + 1$$

Start at the beginning. Use $f(1 + h) = 2(1 + h)^2 - 5(1 + h) + 1$ below.

$$= \lim_{h \to 0} \left[\frac{\left(2(1+h)^2 - 5(1+h) + 1\right) - \left(2(1)^2 - 5(1) + 1\right)}{h} \right]$$

Replace the functional notation with the appropriate expression.

$$= \lim_{h \to 0} \left[\frac{\left(2(1 + 2h + h^2) - 5 - 5h + 1\right) - (2 - 5 + 1)}{h} \right]$$

Simplify expressions in the numerator.

$$= \lim_{h \to 0} \left[\frac{\left(2 + 4h + 2h^2 - 5 - 5h + 1\right) - (2 - 5 + 1)}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{\left(2h^2 - h - 2\right) - (-2)}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{\left(2h^2 - h - 2 + 2\right)}{h} \right] = \lim_{h \to 0} \left[\frac{\left(2h^2 - h\right)}{h} \right]$$

$= \lim_{h \to 0} \left[\frac{h \cdot (2h - 1)}{h} \right]$ Factor out the h from the numerator.

$= \lim_{h \to 0} (2h - 1)$ Cancel the common factor h.

$= 2(0) - 1 = -1$ Solve by direct substitution.

After simplifying and factoring the expression, you were able to solve for the limit of the expression as $h \to 0$. As you will see when you get to Module 2, using function notation in limits in this way tests your algebraic skill and is crucial in the development of derivatives in calculus.

Learning Activity 1.4

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Factor: $x^2 - x - 6$
2. Evaluate the limit, $\lim_{x \to 2} \left(x^2 + 5x \right)$.
3. Rationalize the numerator: $\frac{2 - \sqrt{x}}{4 - x}$
4. Simplify by finding the common denominator: $\frac{3}{x} + \frac{2}{3}$
5. What is the additive inverse of $x - 5$?
6. What is the conjugate of $\sqrt{x + 3} - 5$?
7. Factor: $x^2 - 16$
8. Simplify: $\sqrt{3}\left(2\sqrt{3} + 1\right)$

continued

Learning Activity 1.4 (continued)

Part B: Solving the Indeterminate Form of Limits

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Determine $\lim\limits_{x \to 1}\left[\dfrac{x^2 - 1}{x - 1}\right]$.

2. Determine $\lim\limits_{x \to 9}\left[\dfrac{\sqrt{x} - 3}{x - 9}\right]$.

3. Determine $\lim\limits_{x \to 3}\left[\dfrac{x^2 - x - 6}{x - 3}\right]$.

4. Determine $\lim\limits_{x \to 1}\left[\dfrac{\sqrt{x + 3} - 2}{x - 1}\right]$.

5. Determine $\lim\limits_{x \to 3}\left(\dfrac{\frac{1}{x} - \frac{1}{3}}{x - 3}\right)$.

6. If $f(x) = 3x - 8$, determine $\lim\limits_{h \to 0}\left[\dfrac{f(2 + h) - f(2)}{h}\right]$.

7. If $f(x) = \dfrac{2}{\sqrt{x}}$, determine $\lim\limits_{h \to 0}\left[\dfrac{f(4 + h) - f(4)}{h}\right]$.

Lesson Summary

In this lesson, you learned to recognize the indeterminate form $\frac{0}{0}$, which is undefined but whose limit can sometimes be found. Furthermore, you learned to algebraically manipulate some functions to eliminate the zero-producing factor in the denominators so that you could evaluate the limits with direct substitution. You used three strategies of algebraic manipulation to eliminate the zero-producing factor in the denominator—factoring, rationalizing the numerator, and simplifying complex fractions. You also solved limit questions in functional notation.

You will now continue to work with other indeterminate forms when evaluating limits.

Notes

Assignment 1.3

Solving the Indeterminate Form of Limits

Total: 13 marks

1. Determine $\lim_{x \to 2} \left[\frac{x^2 - 4}{x - 2} \right]$. *(2 marks)*

2. Determine $\lim_{x \to 2} \left[\frac{\sqrt{x + 2} - 2}{x - 2} \right]$. *(4 marks)*

continued

Assignment 1.3: Solving the Indeterminate Form of Limits (continued)

3. Determine $\lim_{x \to -10} \left(\dfrac{\dfrac{5}{x} + \dfrac{1}{2}}{x + 10} \right)$. *(3 marks)*

4. If $f(x) = 2 - x^2$, determine $\lim_{h \to 0} \left[\dfrac{f(2 + h) - f(2)}{h} \right]$. *(4 marks)*

Lesson 5: Exploring One-Sided Limits

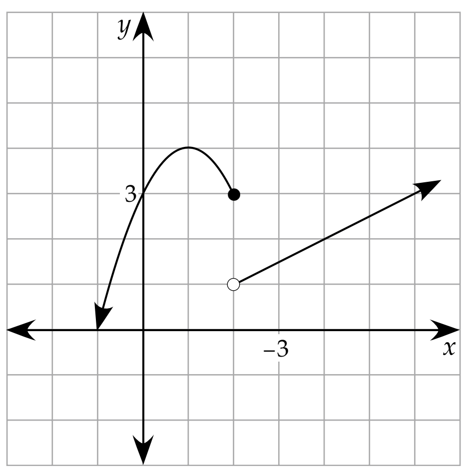
Lesson Focus

In this lesson, you will

- ❑ determine the value of limits approaching a domain value from either the left or the right
- ❑ explain what is meant by one-sided limits
- ❑ determine one-sided limits of piecewise functions
- ❑ explain the behaviour of limits in the form $\lim_{x \to 0} \frac{\text{number}}{x}$

Lesson Introduction

The piecewise function, $f(x)$, is shown below. As x approaches 2 from the left side, the value of $f(x)$ approaches 3. However, as x approaches 2 from the right side, the value of $f(x)$ approaches 1.



In this lesson, you will learn how to determine one-sided limits of functions from their graphs and equations. You will discover applications of one-sided limits to piecewise functions. The undefined limit will be explored further with one-sided limits.

One-Sided Limits

The limit of $f(x)$, as x approaches a using values of $x < a$, is called the **left-hand limit** and is written:

$\lim\limits_{x \to a^-} f(x)$ [read, "the limit of $f(x)$ as x approaches a from the left]

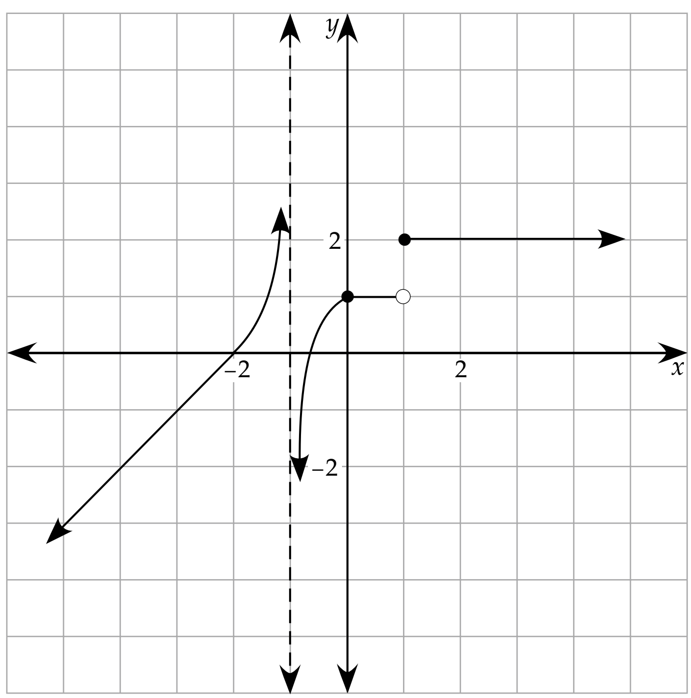
The limit of $f(x)$, as x approaches a using values of $x > a$, is called the **right-hand limit** and is written:

$\lim\limits_{x \to a^+} f(x)$ [read, "the limit of $f(x)$ as x approaches a from the right]

One-sided limits can be used to explore the limit behaviour in more detail. It has applications with piecewise functions and asymptotic behaviour, as you will learn later in this course.

Analyzing the Limit Graphically

For a limit of a point to exist, the value of the limit as you approach the point from the left must be the ***same*** as the value of the limit as you approach the point from the right. Use the graph of the function $g(x)$ below to determine the following limits.



Determine	Solution	Explanation
a) $\lim_{x \to -1} g(x)$	No solution	No specific value approached from both the left and right.
b) $\lim_{x \to 0} g(x)$	1	Function approaches the same value from both the left and right.
c) $\lim_{x \to 1} g(x)$	No solution	No specific value approached from both the left and right.
d) $\lim_{x \to 2} g(x)$	2	Function approaches the same value from both the left and right.

If you now explore the one-sided limits, you can determine more specific information about the function's limit behaviour.

Determine	Solution	Explanation
a) $\lim_{x \to -1^-} g(x)$	∞	As $x \to -1$ from the left (for values $x < -1$), then $g(x) \to \infty$.
b) $\lim_{x \to -1^+} g(x)$	$-\infty$	As $x \to -1$ from the right (for values $x > -1$), then $g(x) \to -\infty$.
c) $\lim_{x \to 0^-} g(x)$	1	As $x \to 0$ from the left (for values $x < 0$), then $g(x) \to 1$.
d) $\lim_{x \to 0^+} g(x)$	1	As $x \to 0$ from the right (for values $x > 0$), then $g(x) \to 1$.
e) $\lim_{x \to 1^-} g(x)$	1	As $x \to 1$ from the left (for values $x < 1$), then $g(x) \to 1$.
f) $\lim_{x \to 1^+} g(x)$	2	As $x \to 1$ from the right (for values $x > 1$), then $g(x) \to 2$.
g) $\lim_{x \to 2^-} g(x)$	2	As $x \to 2$ from the left (for values $x < 2$), then $g(x) \to 2$.
h) $\lim_{x \to 2^+} g(x)$	2	As $x \to 2$ from the right (for values $x > 2$), then $g(x) \to 2$.

Upon exploration of the one-sided limits, you may have noticed that if the left- and right-hand limits are not equal, then the limit does not exist at that value; but if the left- and right-hand limits are equal, then the limit exists. Unless the superscript "+" or "−" is indicated in the limit notation, it is assumed that the limit refers to two sides.

Definition of One-Sided Limits

If $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x) = L$.

However:

If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x)$ does not exist.

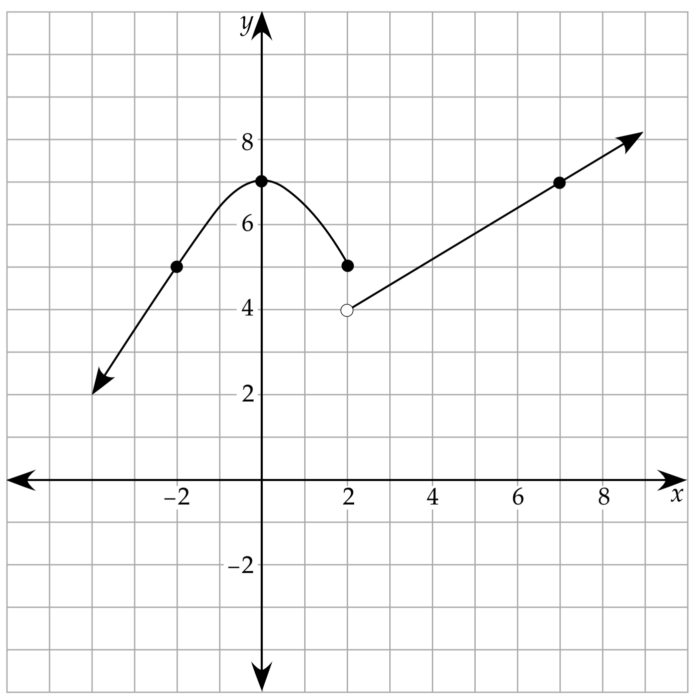
In other words, if the left-hand and the right-hand limits are equal, then the limit exists.

However, if the left-hand and the right-hand limits are not equal, then the limit does not exist.

You can now apply this knowledge to an example.

Example 1:

The graph of $g(x)$ is shown below.



From the graph above, you can determine the following.

Determine	Solution
a) $\lim_{x \to -2^-} g(x)$	5
b) $\lim_{x \to -2^+} g(x)$	5
c) $\lim_{x \to -2} g(x)$	5, because the left-hand and right-hand limits at $x = -2$ are equal.
d) $\lim_{x \to 2^-} g(x)$	5
e) $\lim_{x \to 2^+} g(x)$	4
f) $\lim_{x \to 2} g(x)$	Does not exist because the left-hand and right-hand limits are not equal.
g) $\lim_{x \to 7^-} g(x)$	7
h) $\lim_{x \to 7^+} g(x)$	7
i) $\lim_{x \to 7} g(x)$	7, because the left-hand and right-hand limits are equal.

Now that you know how to find one-sided limits graphically, you can learn how to determine one-sided limits analytically.

Piecewise Functions

A **piecewise function** is a function with more than one function definition on its domain. These functions are very useful for the discussion with one-sided limits. You will be able to discuss one-sided limits analytically in addition to graphically.

Let's explore the following function:

$$f(x) = \begin{cases} 3x^2 - 4 & \text{if } x \le 2 \\ x + 6 & \text{if } x > 2 \end{cases}$$

Remember that a piecewise function can be interpreted as having more than one definition on different parts of the domain. This function has two definitions on its domain.

If $x \le 2$, then use the definition $3x^2 - 4$.

If $x > 2$, then use the definition $x + 6$.

Example 1

Determine the following limits analytically for the function:

$$f(x)=\begin{cases} 3x^2-4 & \text{if } x \le 2 \quad \text{(that is, at or to the left of 2)} \\ x+6 & \text{if } x > 2 \quad \text{(that is, to the right of 2)} \end{cases}$$

Question	Solution
1. $\lim\limits_{x\to 2^-} f(x)$	$\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} \left(3x^2-4\right) = 3(2)^2-4 = 3(4)-4 = 12-4 = 8$
2. $\lim\limits_{x\to 2^+} f(x)$	$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} \left(x+6\right) = 2+6 = 8$
3. $\lim\limits_{x\to 2} f(x)$	Since $\lim\limits_{x\to 2^-} f(x) = 8 = \lim\limits_{x\to 2^+} f(x)$, then $\lim\limits_{x\to 2} f(x) = 8$.

So, for this piecewise function, both pieces approach the same value as x approaches 2, so the limit as $x \to 2$ exists and its value is 8.

One-sided limits are transparent when working with piecewise functions. However, they can sometimes not be as clear with functions that have only one definition. For example, functions with absolute-value expressions are piecewise functions in disguise (two different linear equations). Let's analyze a function with absolute values.

If you algebraically manipulate the function $f(x) = \dfrac{|2x-8|}{x-4}$, you see that it can be rewritten as a piecewise function that will be easier to solve in limits.

Remember that if $x < 0$, then $|x| = -x$, but if $x > 0$, then $|x| = x$.

So, when $x < 4$, you can simplify the expression by replacing $|x-4|$ with $-(x-4)$.

$$\frac{|2x-8|}{x-4} = \frac{|2(x-4)|}{x-4} = \frac{2\cdot|x-4|}{x-4} = 2\cdot\frac{|x-4|}{x-4} = 2\cdot\left(\frac{-(x-4)}{x-4}\right) = 2(-1) = -2$$

Similarly, when $x > 4$, you can simplify the expression by replacing $|x-4|$ with $(x-4)$

$$\frac{|2x-8|}{x-4} = \frac{|2(x-4)|}{x-4} = \frac{2\cdot|x-4|}{x-4} = 2\cdot\frac{|x-4|}{x-4} = 2\cdot\left(\frac{x-4}{x-4}\right) = 2(1) = +2$$

The new definition of $f(x)$ is shown below:

$$f(x)=\begin{cases}-2 & \text{if } x<4\\ +2 & \text{if } x>4\end{cases}$$

Note that $f(x)$ is not defined at $x = 4$ because $f(4)=\frac{0}{0}$.

Example 2

Determine the following, given $f(x)=\frac{|2x-8|}{x-4}$ or $f(x)=\begin{cases}-2 & \text{if } x<4\\ +2 & \text{if } x>4\end{cases}$.

Question	Solution
1. $\lim_{x\to 4^-} f(x)$	$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-}(-2) = -2$
2. $\lim_{x\to 4^+} f(x)$	$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+}(+2) = +2$
3. $\lim_{x\to 4} f(x)$	Does not exist because the one-sided limits are not equal.

Remember that if the function $f(x)=\frac{|2x-8|}{x-4}$ has not been rewritten as a piecewise function, you will be required to do so in order to solve the limit question.

One-sided limits of polynomial functions that are not piecewise are very straightforward because the definition of the function is the same for both the left-hand and right-hand limits.

Example 3

Determine the following limits for $g(x) = -4x^4 + 5$.

Question	Solution
1. $\lim_{x \to -1^-} g(x)$	$\lim_{x \to -1^-} g(x) = \lim_{x \to -1^-} \left(-4x^4 + 5\right) = -4(-1)^4 + 5 = -4 + 5 = 1$
2. $\lim_{x \to -1^+} g(x)$	$\lim_{x \to -1^+} g(x) = \lim_{x \to -1^+} \left(-4x^4 + 5\right) = -4(-1)^4 + 5 = -4 + 5 = 1$
3. $\lim_{x \to -1} g(x)$	1, since $\lim_{x \to -1^-} g(x) = 1 = \lim_{x \to -1^+} g(x)$

Since the definition of the function is the same for $x \geq -1$ and $x < -1$, then the one-sided limits were not necessary to solve the two-sided limit. However, potentially undefined limits make one-sided limits necessary to determine if a limit is truly undefined.

Undefined Limits

When direct substitution results in a non-zero number divided by zero, you have a potentially undefined limit, as shown in the example below.

Example 1

Determine $\lim_{x \to 2} \left(\frac{4x - 1}{x - 2} \right)$.

Solution

$$\lim_{x \to 2} \left(\frac{4x - 1}{x - 2} \right) = \frac{8 - 1}{2 - 2} = \frac{7}{0}$$

This limit is undefined because you cannot divide by zero or it could be described as approaching positive or negative infinity. Let's investigate further by using one-sided limits.

The Left-Hand Limit

First of all, what happens to the function value when $x < 2$ and as x approaches values from the left increasingly closer to 2?

x	1.9	1.99	1.999	1.9999	2
$\frac{4x-1}{x-2}$	$\frac{6.6}{-0.1}=-66$	$\frac{6.96}{-0.01}=-696$	$\frac{6.996}{-0.001}=-6996$	$\frac{6.9996}{-0.0001}=-69\ 996$	$-\infty$

You can also use the limit notation to show what happens when x is a number just a little less than 2 (such as 1.99999 . . .).

$$\lim_{x \to 2^-}\left(\frac{4x-1}{x-2}\right)=\frac{7}{\text{very small negative number}} \text{ approaches } -\infty$$

Essentially, when $x < 2$ and $x \to 2$, the function value approaches $-\infty$.

The Right-Hand Limit

Now, what happens to the function value when $x > 2$ and x approaches 2 for values from the right increasingly closer to 2?

x	2.1	2.01	2.001	2.0001	2
$\frac{4x-1}{x-2}$	$\frac{7.4}{0.1}=74$	$\frac{7.04}{0.01}=704$	$\frac{7.004}{0.001}=7004$	$\frac{7.0004}{0.0001}=70\ 004$	$+\infty$

You can also use the limit notation to show what happens when x is a number just a little more than 2 (such as 2.000001).

$$\lim_{x \to 2^+}\left(\frac{4x-1}{x-2}\right)=\frac{7}{\text{very small positive number}} \text{ approaches } +\infty$$

Essentially, when $x > 2$ and $x \to 2$, the function value approaches $+\infty$.

Now, you can conclusively say that the limit does not exist when $x \to 2$ because the one-sided limits are not equal.

$$\text{If } \lim_{x \to 2^-}\left(\frac{4x-1}{x-2}\right) \neq \lim_{x \to 2^+}\left(\frac{4x-1}{x-2}\right), \text{ then } \lim_{x \to 2}\left(\frac{4x-1}{x-2}\right) \text{ does not exist.}$$

However, the following limit in Example 2 is an indeterminate form but the limit is defined. Why?

Example 2

Determine $\lim_{x \to 2} \frac{1}{(x-2)^2}$.

Solution

$$\lim_{x \to 2} \frac{1}{(x-2)^2} = \frac{1}{(2-2)^2} = \frac{1}{0} \quad \text{(undefined)}$$

The Left-Hand Limit

$$\lim_{x \to 2^-} \frac{1}{(x-2)^2} = \frac{1}{\underbrace{\text{very small positive number}}_{\text{since denominator is squared}}} \text{ approaches } +\infty$$

The Right-Hand Limit

$$\lim_{x \to 2^+} \frac{1}{(x-2)^2} = \frac{1}{\text{very small positive number}} \text{ approaches } +\infty$$

In Example 1, the unequal one-sided limits proved that the two-sided limit did not exist. Example 2 proved the opposite.

The limit exists because the one-sided limits are equal.

More specifically:

$$\lim_{x \to 2} \frac{1}{(x-2)^2} = +\infty \text{ because } \lim_{x \to 2^-} \frac{1}{(x-2)^2} = \lim_{x \to 2^+} \frac{1}{(x-2)^2}$$

In summary, if upon direct substitution a limit yields a non-zero number divided by zero, then the limit approaches positive infinity or negative infinity. Further investigation is required with one-side limits. If upon further investigation the one-sided limits are not equal, then the limit does not exist; but if the limits are equal, then the limit is defined.

Learning Activity 1.5

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Evaluate: $\frac{5}{0.1}$

2. Evaluate: $\frac{-8}{0.001}$

3. Evaluate: $10.001 - 10$

4. Evaluate: $3.99 - 4$

5. Simplify: $\left(3x^2 + 4x - 5\right) \cdot \left(\frac{1}{x^2}\right)$

6. Determine the non-permissible value of $\frac{5}{1-x}$.

For Questions 7 and 8, given $h(x) = \begin{cases} 7x & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$

7. Evaluate: $h(-2)$

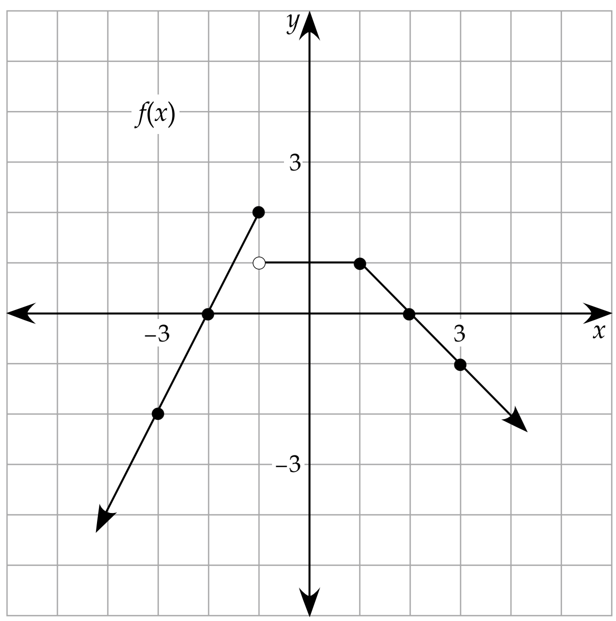
8. Evaluate: $h(2)$

continued

Part B: Evaluating One-Sided Limits

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Use the graph of $f(x)$ below to answer the following questions.



a) $\lim_{x \to -1^-} f(x)$ b) $\lim_{x \to -1^+} f(x)$

c) $\lim_{x \to -1} f(x)$ d) $\lim_{x \to 1^-} f(x)$

e) $\lim_{x \to 1^+} f(x)$ f) $\lim_{x \to 1} f(x)$

2. Determine the following limits:

a) $\lim_{x \to 2^-} \left(2x^3 + 4x^2 - 7x - 1\right)$

b) $\lim_{x \to 2^+} \left(2x^3 + 4x^2 - 7x - 1\right)$

c) $\lim_{x \to 2} \left(2x^3 + 4x^2 - 7x - 1\right)$

continued

Learning Activity 1.5 (continued)

3. Use the function below to answer the following questions:

$$h(x)=\begin{cases}7x & \text{if } x\le -1\\ x^2 & \text{if } x>-1\end{cases}$$

a) $\lim\limits_{x\to -1^-} h(x)$

b) $\lim\limits_{x\to -1^+} h(x)$

c) $\lim\limits_{x\to -1} h(x)$

4. Use the function below to answer the following questions:

$$m(x)=\begin{cases}-2 & \text{if } x\le 0\\ x-2 & \text{if } x>0\end{cases}$$

a) $\lim\limits_{x\to 0^-} m(x)$

b) $\lim\limits_{x\to 0^+} m(x)$

c) $\lim\limits_{x\to 0} m(x)$

5. Determine the following limits:

a) $\lim\limits_{x\to 3^-} \dfrac{|3x-9|}{x-3}$

b) $\lim\limits_{x\to 3^+} \dfrac{|3x-9|}{x-3}$

c) $\lim\limits_{x\to 3} \dfrac{|3x-9|}{x-3}$

continued

Learning Activity 1.5 (continued)

6. Determine the following limits if they exist:

a) $\lim\limits_{x \to 3^-} \dfrac{2x}{x-3}$

b) $\lim\limits_{x \to 3^+} \dfrac{2x}{x-3}$

c) $\lim\limits_{x \to 3} \dfrac{2x}{x-3}$

d) $\lim\limits_{x \to -2^-} \dfrac{2}{(x+2)^2}$

e) $\lim\limits_{x \to -2^+} \dfrac{2}{(x+2)^2}$

f) $\lim\limits_{x \to -2} \dfrac{2}{(x+2)^2}$

Lesson Summary

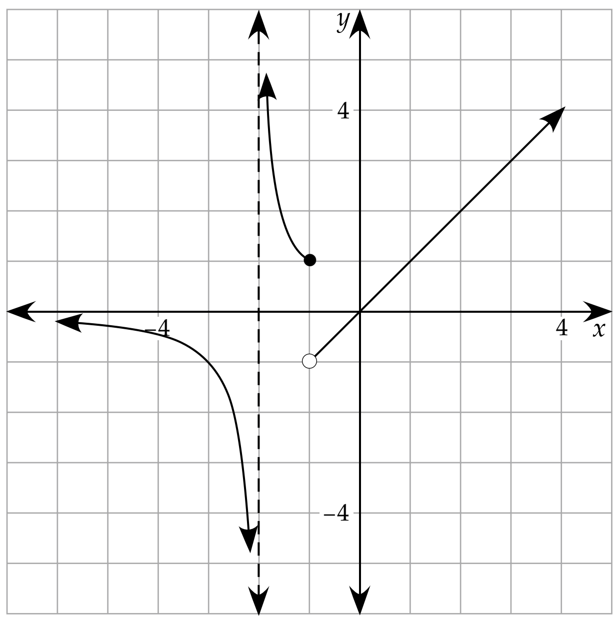
In this lesson, you explored one-sided limits graphically, in piecewise functions, and with potentially undefined limits. You learned that for a limit to exist, its one-sided limits must be equal. You were able to show this graphically and analytically. In the next two lessons, you will continue to use one-sided limits to explore asymptotes and continuity.

Assignment 1.4

Exploring One-Sided Limits

Total: 28 marks

1. Use the graph of $g(x)$, shown below, to answer the following questions. *(7 marks)*



a) $\lim_{x \to -2^-} g(x)$

b) $\lim_{x \to -2^+} g(x)$

continued

Assignment 1.4: Exploring One-Sided Limits (continued)

c) $\lim_{x \to -2} g(x)$

d) $\lim_{x \to -1^-} g(x)$

e) $\lim_{x \to -1^+} g(x)$

f) $\lim_{x \to -1} g(x)$

g) $\lim_{x \to 3} g(x)$

continued

Assignment 1.4: Exploring One-Sided Limits (continued)

2. Determine the following limits: *(3 marks)*

   a) $\lim_{x \to 1^-} \left(7x^3 + 4x\right)$

   b) $\lim_{x \to 1^+} \left(7x^3 + 4x\right)$

   c) $\lim_{x \to 1} \left(7x^3 + 4x\right)$

3. Use the function below to answer the questions that follow. *(3 marks)*

$$h(x) = \begin{cases} 6x - 1 & \text{if } x \le 0 \\ \sqrt{x+1} & \text{if } x > 0 \end{cases}$$

   a) $\lim_{x \to 0^-} h(x)$

   b) $\lim_{x \to 0^+} h(x)$

   c) $\lim_{x \to 0} h(x)$

continued

Assignment 1.4: Exploring One-Sided Limits (continued)

4. Determine the following limits:

a) $\lim_{x \to \frac{1}{2}^-} \frac{|4x - 2|}{2x - 1}$ *(2 marks)*

b) $\lim_{x \to \frac{1}{2}^+} \frac{|4x - 2|}{2x - 1}$ *(2 marks)*

c) $\lim_{x \to \frac{1}{2}} \frac{|4x - 2|}{2x - 1}$ *(1 mark)*

continued

Assignment 1.4: Exploring One-Sided Limits (continued)

5. Determine whether the one-sided undefined limits approach negative or positive infinity (also describe the two-sided limit if possible).

   a) $\lim_{x \to -3^-} \frac{8}{x+3}$ *(2 marks)*

   b) $\lim_{x \to -3^+} \frac{8}{x+3}$ *(2 marks)*

   c) $\lim_{x \to -3} \frac{8}{x+3}$ *(1 mark)*

continued

Assignment 1.4: Exploring One-Sided Limits (continued)

d) $\lim\limits_{x \to -1^-} \frac{5x}{(x+1)^2}$ *(2 marks)*

e) $\lim\limits_{x \to -1^+} \frac{5x}{(x+1)^2}$ *(2 marks)*

f) $\lim\limits_{x \to -1} \frac{5x}{(x+1)^2}$ *(1 mark)*

Lesson 6: Using Limits to Determine the Asymptotes of a Graph

Lesson Focus

In this lesson, you will
- ❑ determine one-side limits and vertical asymptotes
- ❑ determine limits at infinity
- ❑ determine horizontal asymptotes

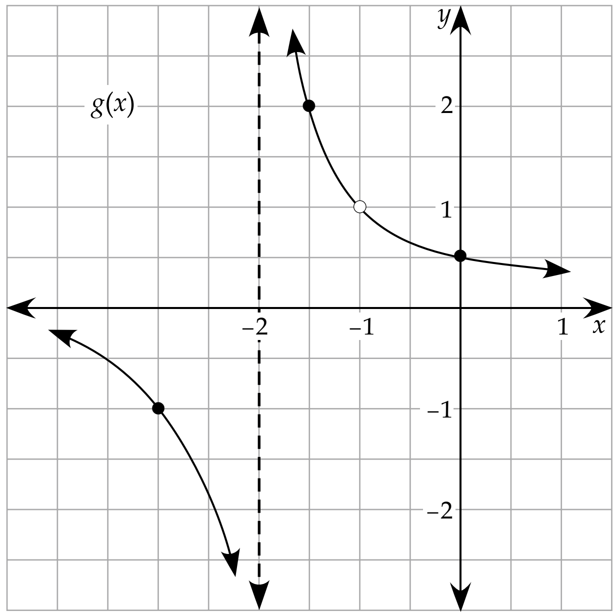
Lesson Introduction

In this lesson, you will learn how to determine vertical asymptotes with one-side limits. You will also learn how to determine limits as x approaches positive and negative infinity, which you know as horizontal asymptotes.

Vertical Asymptotes

Vertical asymptotes are invisible barriers on a graph that display a non-permissible value where the function is not defined, typically found in **rational functions**. Points of discontinuity or holes on a graph are also values where the function is undefined. However, the holes on the graph of a function approach numerical limit values, whereas the left and right limits around vertical asymptotes approach positive or negative infinity. Ultimately, the conversation begins with determining the x-values where the function is undefined before the exploration can proceed. For rational functions, the x-values or non-permissible values that make the denominator zero are where the function is undefined.

Let's explore the graph of $g(x)$ that you studied in Lesson 2.



At $x = -2$, you have a vertical asymptote because as x approaches 2, the function is approaching infinity (either negative or positive).

However, at $x = -1$, you have a hole in the graph and the function is approaching 1 as x approaches -1.

Let's now study the function's equation $g(x) = \dfrac{x+1}{x^2+3x+2} = \dfrac{x+1}{(x+2)(x+1)}$

by exploring the limit behaviour around the two x-values above, which are the non-permissible values of the rational function.

At $x = -2$, the two-sided limit is $\lim\limits_{x \to -2} \dfrac{x+1}{x^2+3x+2} = \dfrac{-1}{0}$ (undefined).

So, you need to analyze the one-sided limits to investigate further.

$$\lim_{x \to -2^-}\left(\frac{x+1}{x^2+3x+2}\right) = \lim_{x \to -2^-}\left(\frac{x+1}{(x+1)(x+2)}\right) = \lim_{x \to -2^-}\left(\frac{1}{x+2}\right)$$

$$= \frac{1}{\text{very small negative number}} \text{ and approaches } -\infty$$

$$\lim_{x \to -2^+}\left(\frac{x+1}{x^2+3x+2}\right) = \lim_{x \to -2^+}\left(\frac{1}{x+2}\right)$$

$$= \frac{1}{\text{very small positive number}} \text{ and approaches } +\infty$$

As you may have noticed by analyzing the graph, the two-sided limit does not exist at $x = -2$, but the one-sided limits approach positive and negative infinity.

Now at $x = -1$, the two-sided limit is $\lim_{x \to -1} \frac{x+1}{x^2 + 3x + 2} = \frac{0}{0}$ I.F.

So, you need to algebraically manipulate the function.

$$\lim_{x \to -1}\left(\frac{x+1}{x^2+3x+2}\right) = \lim_{x \to -1}\left(\frac{x+1}{(x+1)(x+2)}\right) = \lim_{x \to -1}\left(\frac{1}{x+2}\right) = \frac{1}{1} = 1$$

You discover that this two-sided limit exists and is equal to 1.

From your explorations above, you can expect to find a vertical asymptote of a function for an x-value at which its associated function value is undefined but the limit of the function approaches positive or negative infinity.

More specifically, $x = h$ is a vertical asymptote of $f(x)$ if $f(h)$ and one of the following criteria is met:

a) $\lim_{x \to h^-} f(x) = +\infty$

b) $\lim_{x \to h^-} f(x) = -\infty$

c) $\lim_{x \to h^+} f(x) = +\infty$

d) $\lim_{x \to h^+} f(x) = -\infty$

Example 1

Determine the equation of the vertical asymptote of the function $f(x) = \frac{2x-2}{x+3}$.

Solution

$$f(-3) = \frac{2(-3)-2}{-3+3} = \frac{-8}{0} \text{ (undefined)}$$

Therefore, $x = -3$ is a potential vertical asymptote. You now determine the one-sided limits around this value.

$$\lim_{x \to -3^-} \left(\frac{2x-2}{x+3} \right) = \frac{-8}{0} \quad \text{(undefined)}$$

$$\lim_{x \to -3^-} \left(\frac{2x-2}{x+3} \right) = \frac{-8}{\text{very small negative number}} \text{ and approaches } -\infty$$

$$\lim_{x \to -3^+} \left(\frac{2x-2}{x+3} \right) = \frac{-8}{\text{very small positive number}} \text{ and approaches } +\infty$$

The vertical asymptote of $f(x)=\frac{2x-2}{x+3}$ is $x = -3$ because $f(-3)$ is undefined and $\lim_{x \to c^-} f(x) = -\infty$ and $\lim_{x \to c^+} f(x) = +\infty$.

Note if there is more than one non-permissible value (npv) for a function, then it is possible for the function to have more than one vertical asymptote or a hole. Remember, npvs of x graph as holes if the function simplifies to $\frac{0}{0}$ for a value of x and npvs of x graph as vertical asymptotes if the function simplifies to a number divided by zero for a value of x.

Limits at Infinity

You will first explore the limit as x goes to infinity of the rational function $f(x) = \frac{1}{x}$.

Example 1

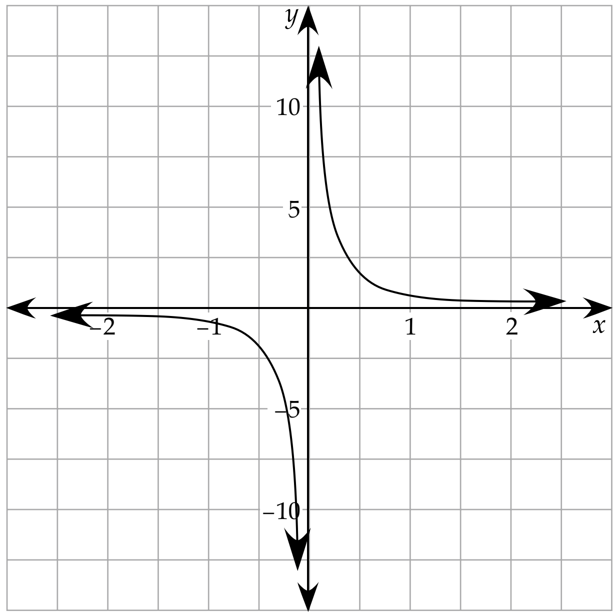
Determine $\lim_{x \to \infty} \frac{1}{x}$.

Solution

$$\lim_{x \to \infty} \frac{1}{x} = \frac{1}{\infty} \text{ (undefined)}$$

This limit cannot be solved by simple direct substitution because $x \to \infty$ means that as x gets larger and larger; it has no bound and will continue indefinitely. Thus, you will need to explore it graphically and its table of values to make sense of the limit analytically.

The graph of $f(x)=\dfrac{1}{x}$ is seen below. According to the graph, how does the function behave as $x \to \infty$?



The graph demonstrates that as $x \to \infty$, the value of the function is decreasing and appears to approach zero.

If you now explore the table of values of the function as $x \to \infty$, do you arrive at the same conclusion?

x	1	10	100	1000	10 000
$\dfrac{1}{x}$	1	0.1	0.01	0.001	0.0001

As x increases ($x \to \infty$), the function value decreases and approaches zero $\left(\dfrac{1}{x} \to 0\right)$.

You can see graphically and with the table of values that:

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

Since zero does not have a sign, with a similar analysis, you can also determine that:

$$\lim_{x\to-\infty} \frac{1}{x} = 0$$

These two simple limits will allow us to solve most limits as x approaches negative or positive infinity. In addition, you can use the limit properties to show the following extensions are also true:

a) $\lim_{x\to\infty} \frac{k}{x} = k \cdot \lim_{x\to\infty} \frac{1}{x} = k \cdot 0 = 0$, if k is constant

b) $\lim_{x\to\infty} \frac{1}{x^n} = \lim_{x\to\infty} \left(\frac{1}{x}\right)^n = 0^n = 0$, if n is a positive constant

Example 2

Determine $\lim_{x\to\infty} \left(\frac{2}{7x-5}\right)$.

Solution

$$\lim_{x\to\infty} \left(\frac{2}{7x-5}\right) = \frac{2}{7(\infty)-5} \quad \text{(undefined)}$$

You will need to algebraically manipulate the function to solve this limit.

$$\lim_{x\to\infty} \left(\frac{2}{7x-5}\right)$$

Start from the beginning.

$$= \lim_{x\to\infty} \left[\left(\frac{2}{7x-5}\right) \cdot \left(\frac{\frac{1}{x}}{\frac{1}{x}}\right)\right] = \lim_{x\to\infty} \left(\frac{2 \cdot \frac{1}{x}}{7x \cdot \frac{1}{x} - 5 \cdot \frac{1}{x}}\right)$$

Multiply the numerator and denominator by $\frac{1}{x}$, which is the reciprocal of the highest power of x in the denominator.

$$= \lim_{x\to\infty} \left(\frac{\frac{2}{x}}{7 - \frac{5}{x}}\right)$$

Simplify.

$$= \frac{\lim_{x\to\infty} \frac{2}{x}}{\lim_{x\to\infty} 7 - \lim_{x\to\infty} \frac{5}{x}}$$

Remember that $\lim_{x\to\infty} \frac{2}{x} = 0$ and $\lim_{x\to\infty} \frac{5}{x} = 0$.

$$= \frac{0}{7-0} = \frac{0}{7} = 0$$

Solution.

Example 3

Determine $\lim_{x\to\infty}\left(\frac{2x^2+3x-4}{5x^2-7x+8}\right)$.

Solution

$$\lim_{x\to\infty}\left(\frac{2x^2+3x-4}{5x^2-7x+8}\right)=\frac{2(\infty)^2+3(\infty)-4}{5(\infty)^2-7(\infty)+8} \text{ which approaches } \frac{\infty}{\infty} \text{ I.F.}$$

$$\lim_{x\to\infty}\left(\frac{2x^2+3x-4}{5x^2-7x+8}\right)$$

Start at the beginning.

$$=\lim_{x\to\infty}\left[\left(\frac{2x^2+3x-4}{5x^2-7x+8}\right)\cdot\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right)\right]$$

Multiply both numerator and denominator by the reciprocal of the highest power of x in the denominator, which is $\frac{1}{x^2}$.

$$=\lim_{x\to\infty}\left[\frac{2x^2\cdot\frac{1}{x^2}+3x\cdot\frac{1}{x^2}-4\cdot\frac{1}{x^2}}{5x^2\cdot\frac{1}{x^2}-7x\cdot\frac{1}{x^2}+8\cdot\frac{1}{x^2}}\right]$$

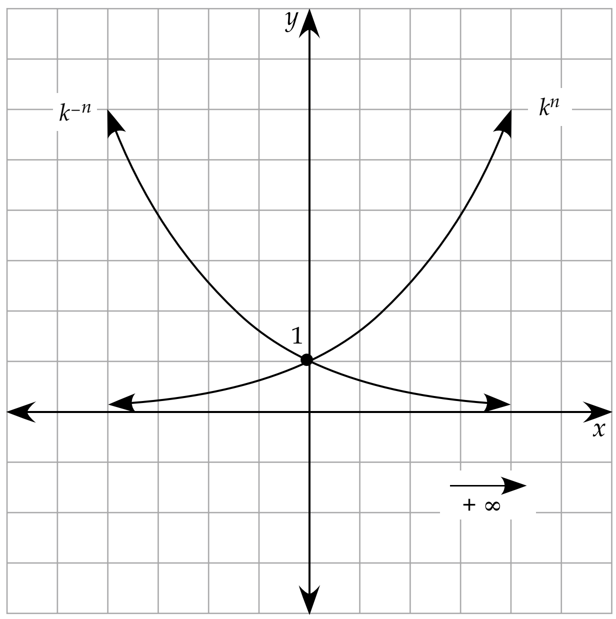
Simplify.

$$=\lim_{x\to\infty}\left(\frac{2+\frac{3}{x}-\frac{4}{x^2}}{5-\frac{7}{x}+\frac{8}{x^2}}\right)$$

$$=\frac{2+0+0}{5-0+0}=\frac{2}{5}$$

Use direct substitution with $\lim_{x\to\infty}\frac{3}{x}=0$, $\lim_{x\to\infty}\frac{7}{x}=0$, $\lim_{x\to\infty}\frac{4}{x^2}=0$, $\lim_{x\to\infty}\frac{8}{x^2}=0$.

In future calculus courses, you will explore $\lim_{n \to \infty} k^n = \infty$, where k is a positive constant, $k > 1$, using your knowledge of exponential functions. However, $\lim_{n \to \infty} k^{-n} = \lim_{n \to \infty} \left(\frac{1}{k^n} \right) = 0$, where k is a positive constant, $k > 1$.



Horizontal Asymptotes

Limits at infinity can be used to determine the equations of horizontal asymptotes of a function. Unlike vertical asymptotes, horizontal asymptotes are not found with one-sided limits approaching a numerical value, but rather they are found with limits approaching infinity. A horizontal asymptote describes the behaviour of the function way off to the right (as $x \to \infty$) or way off to the left (as $x \to -\infty$). They are typically seen in rational functions but not exclusively. It is prudent to assume that if a function has a vertical asymptote that it could also have a horizontal asymptote. However, exponential functions only have a horizontal asymptote and not a vertical asymptote.

The horizontal line $y = k$ is a horizontal asymptote for the function $f(x)$ if one or both of the following is true:

$$\lim_{x \to +\infty} f(x) = k \text{ or } \lim_{x \to -\infty} f(x) = k$$

Example 1:

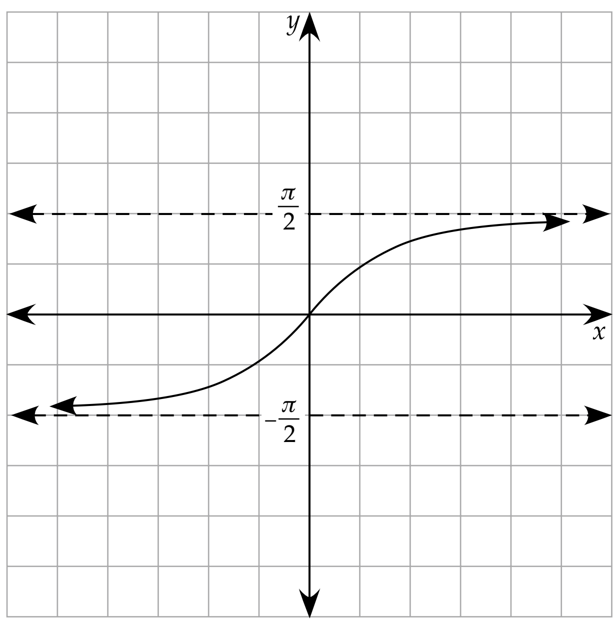
Determine the equation of the horizontal asymptote of $f(x)=\dfrac{2}{7x-5}$.

Solution

$\lim\limits_{x\to\infty}\left(\dfrac{2}{7x-5}\right)=0$ according to an example earlier in the lesson.

Thus, $y = 0$ is the horizontal asymptote as you go way off to the right for the function $f(x)=\dfrac{2}{7x-5}$. The limit of $f(x)$ as x approaches negative infinity is also $y = 0$ and is the horizontal asymptote as you go way off to the left.

Note that it is unusual for a function to have more than one horizontal asymptote but it is possible. A function may have two horizontal asymptotes if it approaches one value as you go right (as $x \to \infty$) and a different value as you go left (as $x \to -\infty$). One such function is the graph of the inverse tangent function $y = \tan^{-1} x$.



Example 2

Determine the equation of the horizontal asymptote of $g(x) = \dfrac{2x^2 + 3x - 4}{5x^2 - 7x + 8}$.

Solution

$\lim\limits_{x \to \infty} \left(\dfrac{2x^2 + 3x - 4}{5x^2 - 7x + 8} \right) = \dfrac{2}{5}$ according to an example earlier in the lesson.

Thus, $y = \dfrac{2}{5}$ is the equation of the horizontal asymptote for

$g(x) = \dfrac{2x^2 + 3x - 4}{5x^2 - 7x + 8}$.

Finding Horizontal Asymptotes of Rational Functions

Note that horizontal asymptotes can be determined by finding the limit as $x \to \infty$ and $x \to -\infty$. Find the limit by dividing all terms by the highest power of x in the denominator. However, the horizontal asymptotes of rational functions that are ratios of polynomial functions can be quicker to find. If $r(x) = \dfrac{p(x)}{q(x)}$, where $q(x) \neq 0$, then the equation of the horizontal asymptote of $r(x)$ can be found by using the **degree** of each of the polynomials, $p(x)$ and $q(x)$.

1. If the degree of $p(x)$ is *less than* the degree of $q(x)$, then the denominator increases more quickly than the numerator and the ratio $\dfrac{p(x)}{q(x)}$ goes to zero. So, the horizontal asymptote is $y = 0$.
2. If the degree of $p(x)$ is *equal to* the degree of $q(x)$, then the horizontal asymptote is a ratio of the **leading coefficients** of each polynomial $y = \dfrac{a}{b}$.
3. If the degree of $p(x)$ is *more than* the degree of $q(x)$, then the numerator increases more quickly than the denominator and the ratio $\dfrac{p(x)}{q(x)}$ goes off to infinity. So, there is no horizontal asymptote.

Now that you know how to use one-sided limits to determine vertical asymptotes and limits at infinity to determine horizontal asymptotes, it is time to practice.

Learning Activity 1.6

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Factor: $x^2 - 4x - 12$

2. Factor: $x^2 + 7x + 10$

3. Simplify: $\dfrac{x^2 - 25}{x^2 - 2x - 15}$

4. What are the non-permissible values of $\dfrac{x^2 - 25}{x^2 - 2x - 15}$?

5. What is the domain of $f(x) = \dfrac{x^2 - 25}{x^2 - 2x - 15}$?

For Questions 6, 7, and 8, given $g(x) = \dfrac{5}{\sqrt{x+1}}$

6. Determine the non-permissible values of $g(x)$.

7. Evaluate: $g(3)$

8. Determine an expression for $m(x)$ if $m(x) = g(x) \cdot g(x)$.

continued

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

Part B: Using Limits to Determine Asymptotes of a Graph

Consider the function $f(x) = \dfrac{2x-1}{x+4}$ for the Questions 1 to 3.

1. Determine the following limits:

   a) $\lim\limits_{x \to -4^-} \left(\dfrac{2x-1}{x+4} \right)$

   b) $\lim\limits_{x \to -4^+} \left(\dfrac{2x-1}{x+4} \right)$

   c) $\lim\limits_{x \to -\infty} \left(\dfrac{2x-1}{x+4} \right)$

   d) $\lim\limits_{x \to +\infty} \left(\dfrac{2x-1}{x+4} \right)$

2. Determine the equation of the vertical asymptote of $f(x) = \dfrac{2x-1}{x+4}$. Explain your answer using the information from Question 1.

3. Determine the equation of the horizontal asymptote of $f(x) = \dfrac{2x-1}{x+4}$. Explain your answer using the information from Question 1.

Consider the function $f(x) = \dfrac{x^2-25}{x^2-2x-15}$ for Questions 4 to 6.

4. Determine the following limits:

   a) $\lim\limits_{x \to 5} \left(\dfrac{x^2-25}{x^2-2x-15} \right)$

   b) $\lim\limits_{x \to -3} \left(\dfrac{x^2-25}{x^2-2x-15} \right)$

continued

Learning Activity 1.6 (continued)

5. Determine the equation of the vertical asymptote of $f(x) = \dfrac{x^2 - 25}{x^2 - 2x - 15}$.

   Explain your answer using the information from Question 4.

6. Determine the equation of the horizontal asymptote of $f(x) = \dfrac{x^2 - 25}{x^2 - 2x - 15}$.

   Explain your answer using the information from Question 4.

7. Determine the equation of the vertical asymptote of $h(x) = \dfrac{3x}{\sqrt{x-2}}$.

8. Determine the equation of the horizontal asymptote of $h(x) = \dfrac{3x}{\sqrt{x-2}}$, if it exists.

Lesson Summary

In this lesson, you learned how to find asymptotes using limits. One-sided limits that approached positive or negative infinity were used to determine vertical asymptotes; while limits as x approaches $\pm\infty$ were used to determine horizontal asymptotes. In addition, you learned how to algebraically manipulate a rational expression to determine infinite limits. In the next lesson, you will explore the concept of continuity of a function, which can involve the function's limit at a point.

Notes

Assignment 1.5

Determining the Asymptotes of a Graph

Total: 33 marks

Consider the function $f(x) = \dfrac{3x - 1}{x + 2}$ for the Questions 1 to 3.

1. Determine the following limits:

   a) $\lim\limits_{x \to -2^-} \left(\dfrac{3x - 1}{x + 2} \right)$ *(2 marks)*

   b) $\lim\limits_{x \to -2^+} \left(\dfrac{3x - 1}{x + 2} \right)$ *(2 marks)*

continued

c) $\lim_{x \to -\infty} \left(\frac{3x - 1}{x + 2} \right)$ *(3 marks)*

d) $\lim_{x \to +\infty} \left(\frac{3x - 1}{x + 2} \right)$ *(3 marks)*

2. Determine the equation of the vertical asymptote of $f(x) = \frac{3x - 1}{x + 2}$. Explain your answer using the information from Question 1. *(2 marks)*

3. Determine the equation of the horizontal asymptote of $f(x) = \frac{3x - 1}{x + 2}$. Explain your answer using the information from Question 1. *(2 marks)*

continued

Assignment 1.5: Determining the Asymptotes of a Graph (continued)

Consider the function $f(x) = \dfrac{x^2 - 16}{x^2 + 6x + 8}$ for Questions 4 to 6.

4. Determine the following limits:

   a) $\lim\limits_{x \to -2} \left(\dfrac{x^2 - 16}{x^2 + 6x + 8} \right)$ *(4 marks)*

   b) $\lim\limits_{x \to -4} \left(\dfrac{x^2 - 16}{x^2 + 6x + 8} \right)$ *(3 marks)*

continued

Assignment 1.5: Determining the Asymptotes of a Graph (continued)

5. Determine the equation of the vertical asymptote of $f(x) = \dfrac{x^2 - 16}{x^2 + 6x + 8}$. Explain your answer using the information from Question 4. *(3 marks)*

6. Determine the equation of the horizontal asymptote of $f(x) = \dfrac{x^2 - 16}{x^2 + 6x + 8}$. Explain your answer using the information from Question 4. *(3 marks)*

continued

Assignment 1.5: Determining the Asymptotes of a Graph (continued)

7. Determine whether the following function has a vertical asymptote and/or a hole.

   $h(x) = \dfrac{x-1}{\sqrt{x-1}}$ *(3 marks)*

8. Determine the equation of the horizontal asymptote of $h(x) = \dfrac{x-1}{\sqrt{x-1}}$, if it exists. *(3 marks)*

Notes

Lesson 7: Understanding the Concept of Continuity

Lesson Focus

In this lesson, you will

- ❑ determine whether a function is continuous graphically
- ❑ determine whether a function is continuous analytically

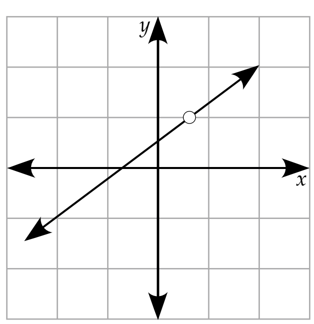
Lesson Introduction

In this lesson, you will learn how to recognize a graph that is continuous at a specific value of x. You will define the continuity of a function at a specific value of x using limits. You will determine analytically if a function is continuous at a specific value of x using limits.

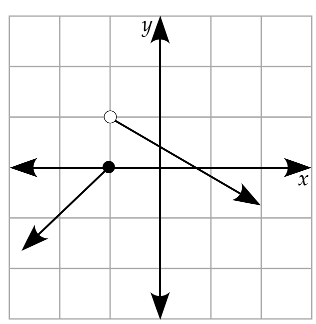
What Is Continuity?

A function is said to be **continuous** if its graph has no hole or breaks in it. An intuitive description is that you can draw the entire graph of a continuous function without lifting your pencil from the paper. You can study the graphs of functions to determine if they are continuous. Some examples of continuous functions include those that you have studied before: linear functions, polynomial functions, radical functions, sinusoidal functions, exponential functions, logarithmic functions, and absolute value functions.

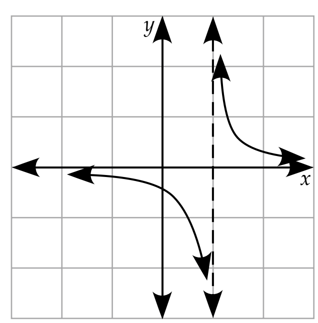
Below are examples of functions with holes and breaks in their graphs that result in their discontinuity.



hole



break



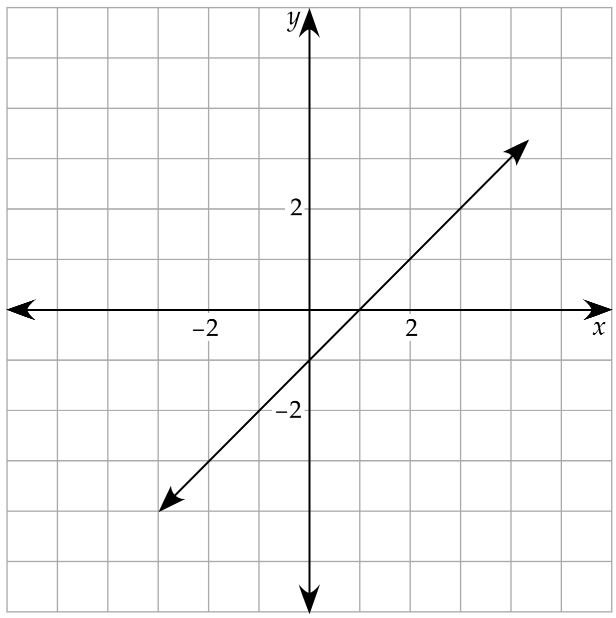
infinity break

Determining Continuity Graphically

Polynomial Functions

Example 1:

Is $f(x) = x - 2$ continuous?

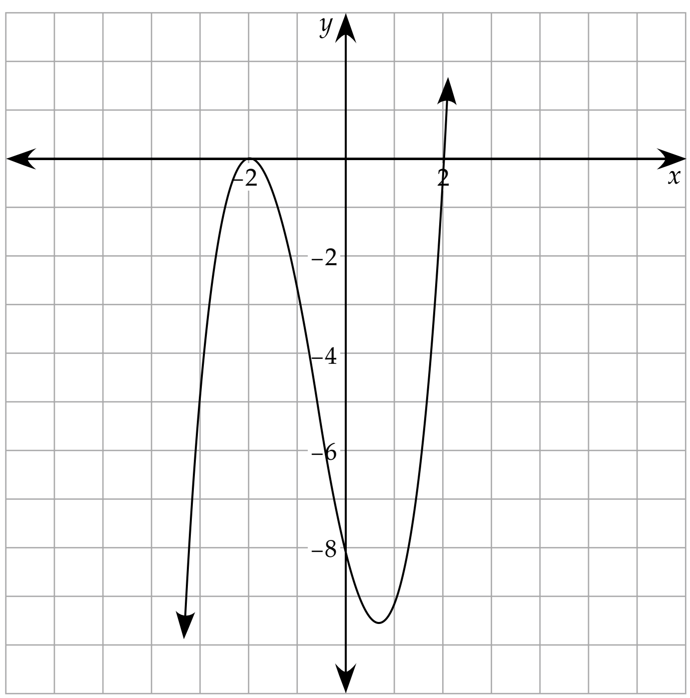


Solution

The graph of $f(x) = x - 2$ has no holes or breaks. Therefore, the function is continuous.

Example 2

Is $g(x) = x^3 + 2x^2 - 4x - 8$ continuous?



Solution

The graph of $g(x) = x^3 + 2x^2 - 4x - 8$ has no holes or breaks in its graph. Therefore, this function is continuous.

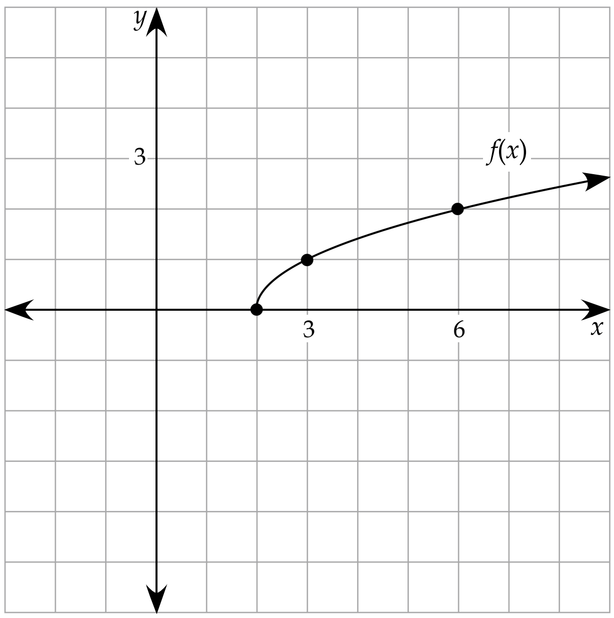
In fact, all **polynomial functions** are continuous because their graphs never have holes or breaks in them.

Non-polynomial Functions

Example 3:

Is $f(x) = \sqrt{x-2}$ continuous?

Solution



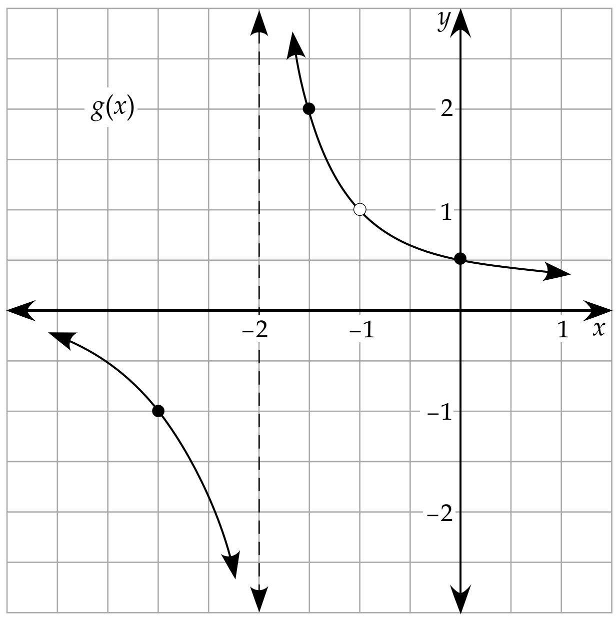
The graph of $f(x) = \sqrt{x-2}$ does not have any holes or breaks; therefore, the function is continuous.

Note that the function does not have to be defined for all real numbers. When determining continuity, you only check if the function is continuous on its domain.

Example 4

Is $g(x) = \dfrac{x+1}{x^2 + 3x + 2}$ continuous?

Solution



The graph of $g(x) = \dfrac{x+1}{x^2 + 3x + 2}$ has one hole at $x = -1$ a break at $x = -2$. The function is discontinuous on any interval including -1 and -2 because of these two points of discontinuity.

You studied $g(x) = \dfrac{x+1}{x^2 + 3x + 2}$ earlier in this module and noticed that this function was undefined at $x = -1$ and at $x = -2$. However, its limit behaviour at these values was different.

Determining Continuity Analytically

A. When you revisit $f(x) = \sqrt{x-2}$ at $x = 6$, you will notice the following:

1. $f(6) = \sqrt{6-2} = \sqrt{4} = 2$ (function value exists)
2. $\lim\limits_{x \to 6} f(x) = \lim\limits_{x \to 6} \sqrt{x-2} = \sqrt{6-2} = \sqrt{4} = 2$ (limit value exists)
3. The graph of $f(x)$ is continuous at $x = 6$, as shown earlier in this lesson.

B. When you revisit $g(x) = \dfrac{x+1}{x^2+3x+2}$ at $x = -1$, you will notice the following:

1. $g(-1)$ is undefined because $g(-1) = \dfrac{-1+1}{(-1)^2+3(-1)+2} = \dfrac{0}{0}$

   (function value does not exist)

2. $\lim_{x\to -1} g(x) = 1$ because

   $$\lim_{x\to -1}\left(\frac{x+1}{x^2+3x+2}\right) = \lim_{x\to -1}\left(\frac{x+1}{(x+1)(x+2)}\right) = \lim_{x\to -1}\left(\frac{1}{x+2}\right) = \frac{1}{-1+2} = \frac{1}{1} = 1$$

   (limit value exists)

3. The graph of $g(x)$ is discontinuous at $x = -1$

C. Continuing with $g(x) = \dfrac{x+1}{x^2+3x+2}$ at $x = -2$, you will notice the following:

1. $g(-2)$ is undefined because $g(-2) = \dfrac{-2+1}{(-2)^2+3(-2)+2} = \dfrac{-1}{0}$

   (function value does not exist)

2. $\lim_{x\to -2} g(x)$ does not exist because the one-sided limits were not equal.

   $$\lim_{x\to -2^-}\left(\frac{1}{x+2}\right) = \frac{1}{\text{very small negative number}} = -\infty$$

   $$\lim_{x\to -2^+}\left(\frac{1}{x+2}\right) = \frac{1}{\text{very small positive number}} = +\infty$$

   $$\lim_{x\to -2^-}\left(\frac{1}{x+2}\right) \neq \lim_{x\to -2^+}\left(\frac{1}{x+2}\right)$$

   (limit value does not exist)

3. The graph of $g(x)$ is discontinuous at $x = -2$

Definition of Continuity at a Point

A function, $f(x)$, is continuous at $x = a$ if ALL of the following criteria is met.

1. $f(a)$ is defined (x = a is part of the domain of $f(x)$)
2. $\lim_{x \to a} f(x)$ exists (the one-sided limits are equal)
3. $f(a) = \lim_{x \to a} f(x)$ (the function value equals the limit value)

The above definition confirms your observations. A function is only continuous at a specific point if its function value equals its limit value. A function is not continuous at a point of discontinuity (a hole) or at a value approaching infinity (a vertical asymptote), since the function is not defined at those x-values.

A. The function $f(x) = \sqrt{x-2}$ is continuous at $x = 6$ because it satisfies the following three criteria:

   1. $f(6)$ is defined; that is, $f(6) = 2$
   2. $\lim_{x \to 6} f(x)$ exists; that is, $\lim_{x \to 6} f(x) = 2$
   3. $\lim_{x \to 6} f(x) = 2 = f(6)$, the function value equals the limit value

B. The function $g(x) = \dfrac{x+1}{x^2+3x+2}$ is discontinuous at $x = -1$ because its function value is undefined even though its limit value is defined:

   1. $g(-1)$ is undefined because $g(-1) = \dfrac{0}{0}$
   2. $\lim_{x \to -1} g(x) = 1$
   3. Function value cannot equal the limit value because the function value is undefined.

C. The function $g(x) = \dfrac{x+1}{x^2+3x+2}$ is discontinuous at $x = -2$ because its function value and limit value are both undefined:

1. $g(-2)$ is undefined because $g(-2) = \dfrac{-1}{0}$

2. $\lim\limits_{x \to -2} g(x)$ does not exist because the one-sided limits are not equal.

   $$\lim_{x \to -2^-} \left(\frac{1}{x+2} \right) = -\infty \text{ but } \lim_{x \to -2^+} \left(\frac{1}{x+2} \right) = +\infty$$

   $$\text{so } \lim_{x \to -2^-} \left(\frac{1}{x+2} \right) \neq \lim_{x \to -2^+} \left(\frac{1}{x+2} \right)$$

3. Function value cannot equal the limit value because neither the function value nor limit value exist.

Note: To prove continuity at a point, you have to meet all three criteria of the definition of continuity but to disprove continuity only one of the criteria has to fail.

Let's now try a few more examples.

Example 1

Is $f(x) = \dfrac{x^3+x^2-x-1}{x+1}$ continuous at $x = -1$?

Solution

Let's check the criteria for continuity.

1. The function value is undefined because

   $$f(-1) = \frac{(-1)^3 + (-1)^2 - (-1) - 1}{-1+1} = \frac{0}{0}$$

Since the first criterion failed, the function is discontinuous at $x = -1$.

Example 2

Is $f(x)$ continuous at $x = -1$, given $f(x) = \begin{cases} \dfrac{x^3 + x^2 - x - 1}{x + 1}, & \text{when } x \neq -1 \\ 2 & \text{when } x = -1 \end{cases}$

Solution

Let's check the three criteria:

1. The function value is defined because $f(-1) = 2$.
2. The limit value is defined because the one-sided limits are equal.

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \left(\frac{x^3 + x^2 - x - 1}{x + 1} \right) = \lim_{x \to 1^-} \left(\frac{(x + 1)(x^2 - 1)}{x + 1} \right)$$

$$= \lim_{x \to 1^-} \left(x^2 - 1 \right) = (1)^2 - 1 = 0$$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \left(\frac{x^3 + x^2 - x - 1}{x + 1} \right) = \lim_{x \to 1^+} \left(\frac{(x + 1)(x^2 - 1)}{x + 1} \right)$$

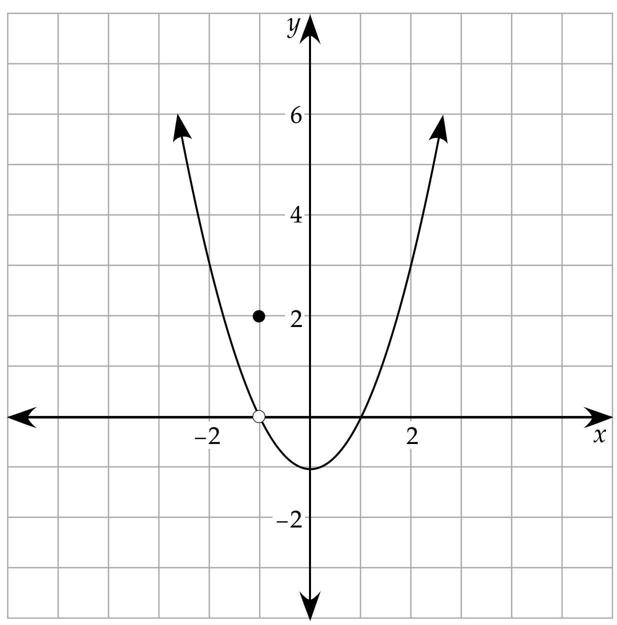
$$= \lim_{x \to 1^+} \left(x^2 - 1 \right) = (1)^2 - 1 = 0$$

$$\lim_{x \to -1^-} f(x) = 0 = \lim_{x \to -1^+} f(x)) = \lim_{x \to -1} f(x) = 0$$

3. However, the function value is not equal to the limit value, since $f(-1) = 2$ and $\lim\limits_{x \to -1} = 0$.

Therefore, the function is discontinuous at $x = -1$.

The graph of the function, shown on the right, also confirms that the function is discontinuous because there is a hole at $x = -1$.



Note: This type of discontinuity is sometimes called a jump discontinuity.

Example 3

Is $g(x)$ continuous at $x = -1$, given $g(x) = \begin{cases} (x+2)^2 + 2, & \text{when } x \le -1 \\ 3 & \text{when } x > -1 \end{cases}$?

Solution

Check the three criteria from the continuity definition.

1. The function value is defined because g(−1) = 3.
2. The limit value is defined because

$$\lim_{x \to -1^-} g(x) = \lim_{x \to -1^-} \left((x+2)^2 + 2\right) = (-1+2)^2 + 2 = 1^2 + 2 = 3$$

$$\lim_{x \to -1^+} g(x) = \lim_{x \to -1^+} \left((x+2)^2 + 2\right) = (-1+2)^2 + 2 = 1^2 + 2 = 3$$

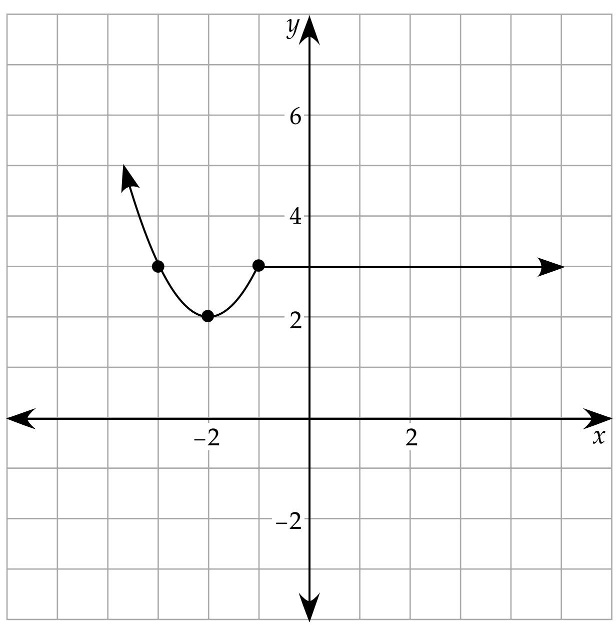
$$\lim_{x \to -1^-} g(x) = 3 = \lim_{x \to -1^+} g(x)$$

so $\lim_{x \to -1} g(x) = 3$

3. The function value equals the limit value—that is, $g(-1) = 3 = \lim_{x \to -1} g(x)$.

Therefore, the function is continuous at $x = -1$ because all three criteria were met.

The graph of the function below confirms the continuity claim because there are no holes or breaks in the graph.



Now that you know the graphical and analytical meaning of continuity, you should be able to apply this concept through this course.

Learning Activity 1.7

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

Given $f(x) = \begin{cases} (x+1)^2, & \text{when } x \le 0 \\ 1 & \text{when } x > 0 \end{cases}$

1. Evaluate: $f(-2)$
2. Evaluate: $f(0)$
3. Evaluate: $f(2)$
4. What is the domain of $f(x)$?

Given $g(x) = \begin{cases} -1, & \text{when } x \le 0 \\ x-2 & \text{when } x > 0 \end{cases}$

5. Evaluate: $g(-2)$
6. Evaluate: $g(0)$
7. Evaluate: $g(2)$
8. What is the domain of $g(x)$?

continued

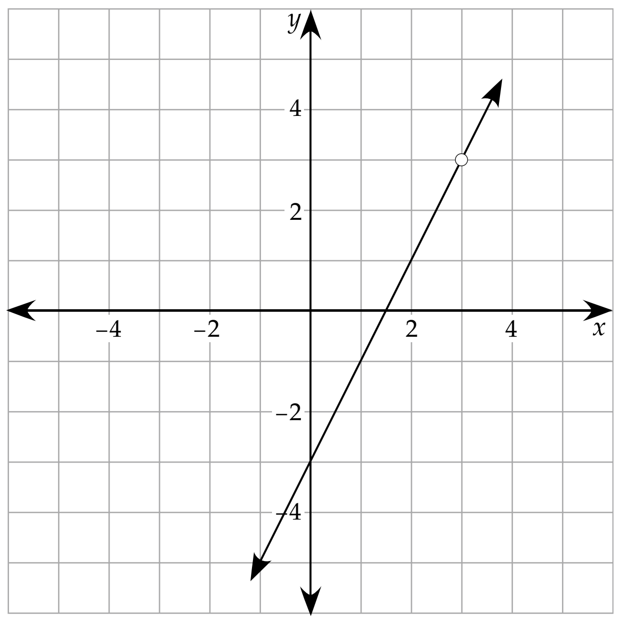
Learning Activity 1.7 (continued)

Part B: Continuity

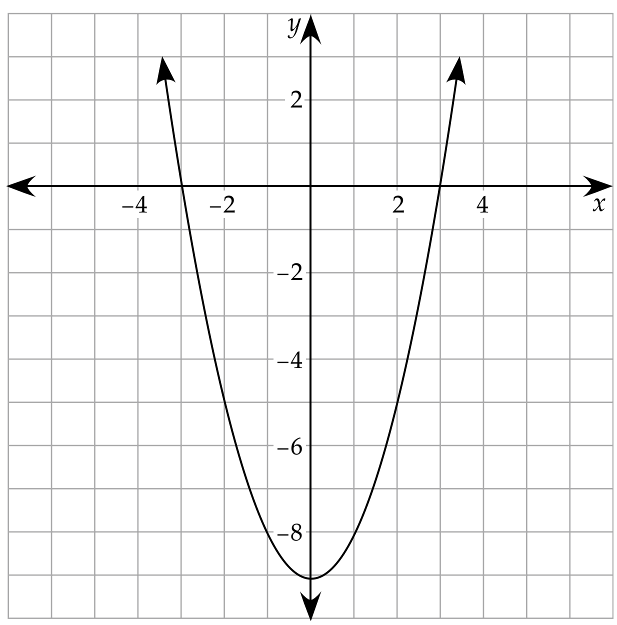
Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Determine whether each of the following functions is continuous using their graphs. If a function is discontinuous, then state the point of discontinuity.

a)

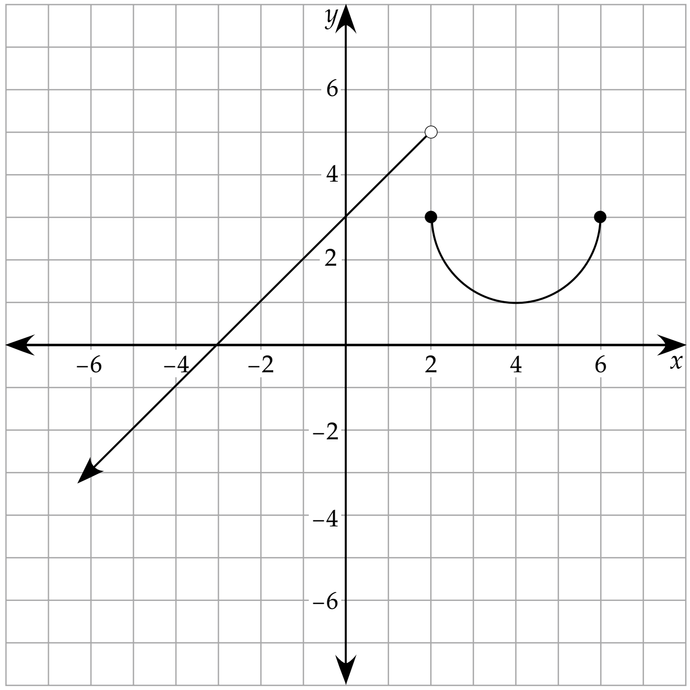


b)



continued

c)



2. Is $f(x)$ continuous at $x = 0$, given $f(x) = \begin{cases} (x+1)^2, & \text{when } x \le 0 \\ 1 & \text{when } x > 0 \end{cases}$?

3. Is $g(x)$ continuous at $x = 0$, given $g(x) = \begin{cases} -1, & \text{when } x \le 0 \\ x - 2 & \text{when } x > 0 \end{cases}$?

Lesson Summary

In this lesson, you learned how to determine continuity graphically by looking for holes or breaks in the graph, and analytically by ensuring the function value and limit values at a point were equal.

Limits are used as a tool for proofs of some calculus topics.

In the next module, you will use concepts of continuity and limits to define rates of change and, more specifically, derivatives. Derivatives are one of the two key concepts of calculus that you will study in this course.

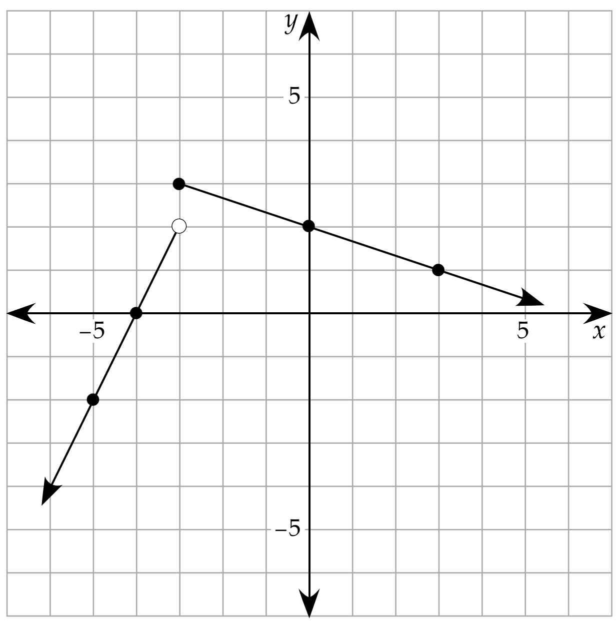
Assignment 1.6

Continuity

Total: 10 marks

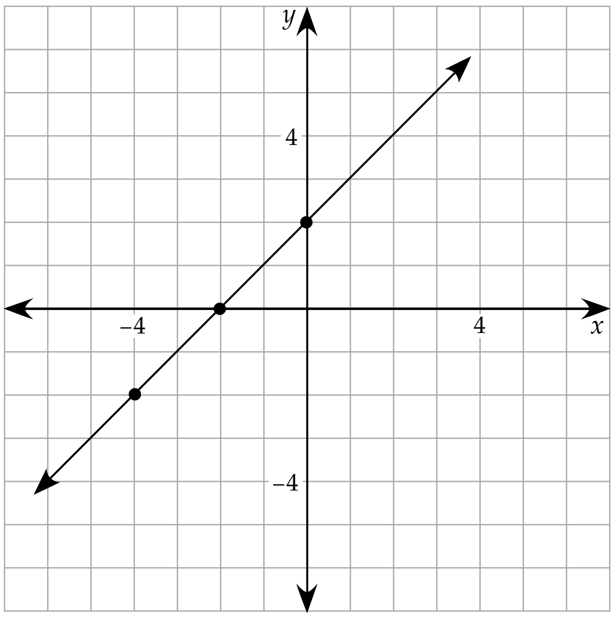
1. Determine whether each of the following functions is continuous using their graphs. If a function is discontinuous, then state the point of discontinuity.

   a) *(2 marks)*



continued

b) *(2 marks)*



2. Is $f(x)$ continuous at $x = -3$, given $f(x) = \begin{cases} 2x + 8, & \text{when } x \le -3 \\ -\frac{1}{3}x + 2, & \text{when } x > -3 \end{cases}$? *(3 marks)*

continued

Assignment 1.6: Continuity (continued)

3. Is $g(x)$ continuous at $x = -4$, given $g(x) = \begin{cases} \dfrac{x^2 + 6x + 8}{x + 4}, & \text{when } x \neq -4 \\ -2, & \text{when } x = -4 \end{cases}$? *(3 marks)*

Notes

Module 1 Summary

Congratulations, you have finished the first module in the course.

Submitting Your Assignments

It is now time for you to submit Assignments 1.1 to 1.6 to the Distance Learning Unit so that you can receive some feedback on how you are doing in this course. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 1 assignments and organize your material in the following order:

- ❑ Module 1 Cover Sheet (found at the end of the course Introduction)
- ❑ Assignment 1.1: Limits
- ❑ Assignment 1.2: Limit Theorems
- ❑ Assignment 1.3: Solving the Intermediate Form of Limits
- ❑ Assignment 1.4: Exploring One-Sided Limits
- ❑ Assignment 1.5: Determining the Asymptotes of a Graph
- ❑ Assignment 1.6: Continuity

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Notes

Grade 12 Introduction to Calculus (45S)

Graph Paper

Grade 12 Introduction to Calculus (45S)

Module 1
Limits

Learning Activity Answer Keys

MODULE 1: LIMITS

Learning Activity 1.1

Questions 1 to 3 relate to the topic of limits that you will study in Module 1. Question 4 relates to the topic of derivations that you will study in Module 2. Question 5 relates to the topic of integration that you will study in Module 4.

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. What happens to the value of $\frac{1}{x}$ as x increases and gets very large approaching infinity?

   Answer:

   When x increases and gets very large, $\frac{1}{x}$ gets very small and close to zero because the reciprocal of a really large number is a really small number.

2. What happens to the value of $\frac{1}{x}$ as x decreases and gets very close to zero?

   Answer:

   When x decreases and gets very close to zero, $\frac{1}{x}$ increases and gets very large because the reciprocal of a really small number is a really large number.

3. Given the geometric series $8 + 4 + 2 + 1 + \frac{1}{2} + \ldots$

   a) Find the sum of the first 5 terms of the series.

   Answer:

   $$9 + 4 + 2 + 1 + \frac{1}{2} = 15.5$$

   b) Find the sum of the first 8 terms of the series.

   Answer:

   $$8 + 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 15.9375$$

c) Find the sum of the first 10 terms of the series.

Answer:

$$15.9375 + \frac{1}{32} + \frac{1}{64} = 15.984375$$

d) Keep adding more terms. As the number of terms in the series gets larger (approaching infinity), what happens to the sum of the series?

Answer:

As the number of terms approaches infinity, the sum of the series approaches 16.

4. Estimate the slope of $f(x)$ at the point:

a) where $x = 3$

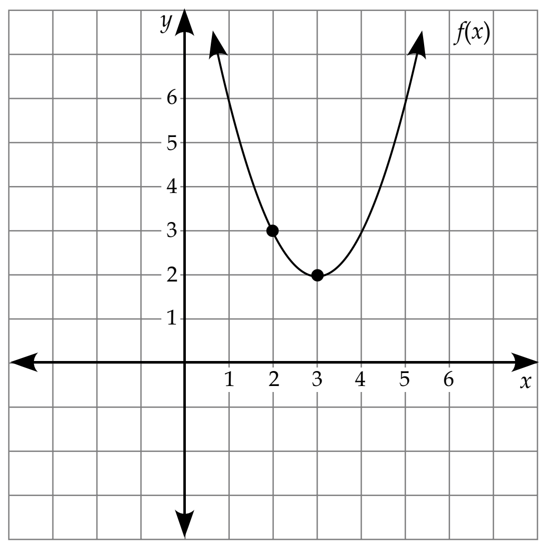
Answer:

Slope is zero.

b) where $x = 2$

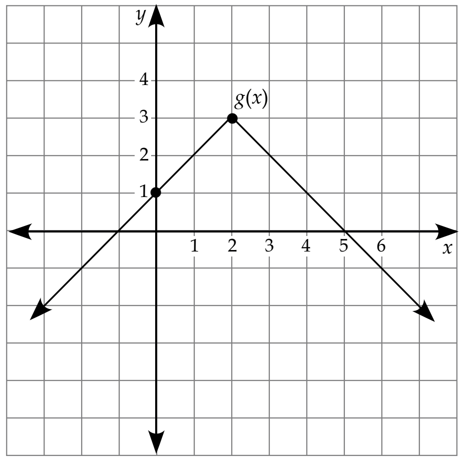
Answer:

Slope is ≈ -2.



5. Estimate the area above the x-axis and below $g(x)$.

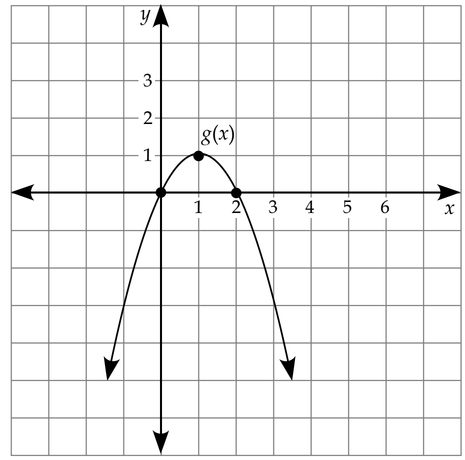
a)



Answer:

Area is 9 units^2.

b)



Answer:

Area is a little more than 1 unit^2.

Actual area using calculus is $\frac{4}{3}$.

Learning Activity 1.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. What is the non-permissible value of the expression $\frac{5}{x-6}$?
2. Evaluate: $\frac{3^2 - 4}{3^2 + 4^2}$
3. What is the domain of $g(x) = \frac{5}{x-6}$?
4. What is the y-intercept of $g(x) = \frac{5}{x-6}$?
5. If $g(x) = \frac{5}{x-6}$, determine $g(1)$.
6. Does the function value have to exist if the limit exists?
7. Determine the domain of $f(x) = \sqrt{x+5}$.
8. If $f(x) = \sqrt{x+5}$, determine $f(-4)$.

Answers:

1. $x = 6$
2. $\frac{1}{5}\left(\frac{9-4}{9+16} = \frac{5}{25}\right)$
3. $\{x \in \Re, x \neq 6\}$
4. $y = \frac{-5}{6}$ (let $x = 0$)
5. $g(1) = -1 \left(\frac{5}{1-6}\right)$
6. No (The curve could be approaching a point of discontinuity (a hole) but the limit could still exist.)
7. $\{x \in \Re, x \geq -5\}$
8. $f(-4) = 1 \left(\sqrt{-4+5}\right)$

Part B: Limits

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Given the function $f(x) = \dfrac{x^2 - 4}{x + 2}$

   a) Complete its table of values below.

   Answer:

x	−2.5	−2.1	−2.01	−2	−1.99	−1.9	−1.5
$f(x)$	−4.5	−4.1	−4.01	undefined	−3.99	−3.9	−3.5

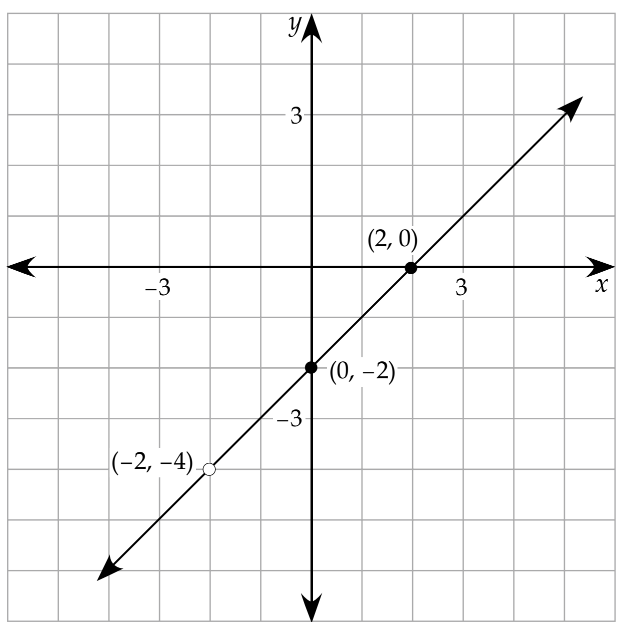
   b) What function value does $f(x)$ approach as x approaches –2? Explain how you arrived at this answer.

   Answer:

   As x approaches −2, $f(x)$ approaches −4. According to the table of values, as the x-value approaches −2 from the left or the right, the $f(x)$-value approaches −4 even though the function value is undefined when $x = -2$.

   c) Graph the function by plotting the points and sketching the curve.

   Answer:



In general, when sketching a function, the graph requires correctly labelled axes and increment values. The axes and the curve require appropriate arrowheads or end points where applicable. Holes or asymptotes should be clear.

d) What is the domain and range for the function? How are the domain's restrictions expressed on the graph?

Answer:

Domain: $\{x \in \Re, x \neq -2\}$; this is demonstrated by the point of discontinuity (or hole) in the graph.

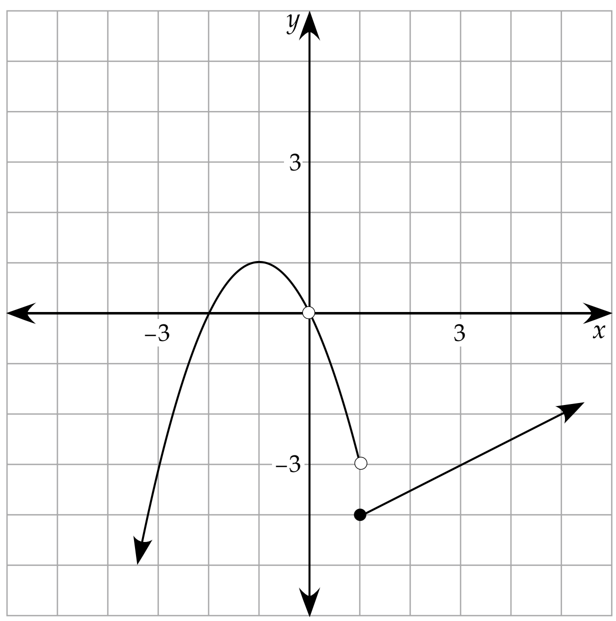
Range: $\{y \in \Re, y \neq -4\}$

e) Use the graph and the table of values to explain why $\lim_{x \to -2} f(x) = -4$, but $f(-2) \neq -4$.

Answer:

The limit exists because the hole on the graph demonstrates that as x approaches –2, the curve approaches −4 but the function does not exist at $x = -2$, so $f(-2) \neq -4$.

2. Given the graph of $g(x)$ below, complete the chart that follows.



Check what is happening to the value $g(x)$ as x approaches the indicated value.

Determine the following limits and function values:	Solution
a) $\lim_{x \to -1} g(x)$	1
b) $\lim_{x \to 0} g(x)$	0
c) $\lim_{x \to 1} g(x)$	"does not exist" since the value of $g(x)$ approaches −3 from the left of $x = 1$, but approaches −4 from the right of $x = 1$
d) $g(-1)$	1
e) $g(0)$	"does not exist" since it is a point of discontinuity
f) $g(1)$	−4 since the function value is represented by the solid dot

Learning Activity 1.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Determine the non-permissible value(s) for $\frac{\sqrt{x}+3}{x+9}$.
2. Determine the non-permissible value(s) for $\frac{x+2}{\sqrt{x}-2}$.
3. Determine the greatest common factor for 6 and 15.
4. Develop and simplify: $(2x + 1)^2$
5. Factor: $4x^2 - 1$
6. Simplify: $(4x^2 - x) - (2x^2 + 3x)$
7. Simplify: $(2x - 3)(2x + 3)$
8. What is the greatest common factor between $4x^2 - 1$ and $2x + 1$?

Answers:

1. npv is $x = -9$
2. npvs are $x = 4$ and $x < 0$
3. 3 $(6 = 3 \times 2, 15 = 3 \times 5)$
4. $4x^2 + 4x + 1$
5. $(2x - 1)(2x + 1)$
6. $2x^2 - 4x$
7. $4x^2 - 9$
8. $2x + 1$ $(4x^2 - 1 = (2x + 1)(2x - 1))$

Part B: Limit Theorems

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Verify the following limit theorems by evaluating each side.

   a) $\lim\limits_{x \to 2}\left[(4x+1)-(2x+3)\right]=\lim\limits_{x \to 2}(4x+1)-\lim\limits_{x \to 2}(2x+3)$

   Answer:

Left Side	Right Side
$\lim\limits_{x \to 2}\left[(4x+1)-(2x+3)\right]$	$\lim\limits_{x \to 2}(4x+1)-\lim\limits_{x \to 2}(2x+3)$
$=\lim\limits_{x \to 2}(4x+1-2x-3)$	$=(4(2)+1)-(2(2)+3)$
$=\lim\limits_{x \to 2}(2x-2)$	$=(8+1)-(4+3)$
$=2(2)-2=4-2$	$=9-7$
$=2$	$=2$

LS = RS

   b) $\lim\limits_{x \to 1}\left[\dfrac{4x+1}{2x+3}\right]=\dfrac{\lim\limits_{x \to 1}(4x+1)}{\lim\limits_{x \to 1}(2x+3)}$

   Answer:

Left Side	Right Side
$\lim\limits_{x \to 1}\left[\dfrac{4x+1}{2x+3}\right]$	$\dfrac{\lim\limits_{x \to 1}(4x+1)}{\lim\limits_{x \to 1}(2x+3)}$
$=\dfrac{4(1)+1}{2(1)+3}=\dfrac{5}{5}$	$=\dfrac{(4(1)+1)}{(2(1)+3)}=\dfrac{(5)}{(5)}$
$=1$	$=1$

LS = RS

2. Evaluate the following with direct substitution:

a) $\lim_{x \to 2} (-x^4 + 3x^3 - 7x^2 + 1)$

Answer:

$$\lim_{x \to 2} \left(-x^4 + 3x^3 - 7x^2 + 1\right)$$

$$= -(2)^4 + 3(2)^3 - 7(2)^2 + 1$$

$$= -16 + 24 - 28 + 1 = -19$$

b) $\lim_{x \to 9} \left[\frac{\sqrt{x} + 3}{x + 9} \right]$

Answer:

$$\lim_{x \to 9} \left[\frac{\sqrt{x} + 3}{x + 9} \right]$$

$$= \frac{\sqrt{9} + 3}{9 + 9} = \frac{3 + 3}{18} = \frac{6}{18} = \frac{1}{3}$$

3. Use direct substitution to find an equivalent expression to represent the limit:

a) $\lim_{a \to 3} \left[(x + a)^2 \right]$

Answer:

$(x + 3)^2$ or $x^2 + 6x + 9$

b) $\lim_{a \to 2} \left[\frac{x^2 - a^2}{x - a} \right]$

Answer:

$$\lim_{a \to 2} \left[\frac{(x - a)(x + a)}{x - a} \right]$$

$$= \lim_{a \to 2} [x + a]$$

$$= x + 2$$

Learning Activity 1.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Factor: $x^2 - x - 6$
2. Evaluate the limit, $\lim_{x \to 2}\left(x^2 + 5x\right)$.
3. Rationalize the numerator: $\frac{2-\sqrt{x}}{4-x}$
4. Simplify by finding the common denominator: $\frac{3}{x} + \frac{2}{3}$
5. What is the additive inverse of $x - 5$?
6. What is the conjugate of $\sqrt{x+3} - 5$?
7. Factor: $x^2 - 16$
8. Simplify: $\sqrt{3}\left(2\sqrt{3} + 1\right)$

Answers:

1. $(x - 3)(x + 2)$
2. 14 $(2^2 + 5(2))$
3. $\frac{1}{2+\sqrt{x}}\left(\frac{2-\sqrt{x}}{4-x} \cdot \frac{2+\sqrt{x}}{2+\sqrt{x}} = \frac{4-x}{(4-x)\left(2+\sqrt{x}\right)}\right)$
4. $\frac{9+2x}{3x}$
5. $-x + 5$ or $5 - x$
6. $\sqrt{x+3} + 5$
7. $(x + 4)(x - 4)$
8. $6 + \sqrt{3}$ $\left(2\sqrt{3}\sqrt{3} + \sqrt{3}\right)$

Part B: Solving the Indeterminate Form of Limits

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Determine $\lim_{x \to 1}\left[\frac{x^2 - 1}{x - 1}\right]$.

   Answer:

   $$\lim_{x \to 1}\left[\frac{x^2 - 1}{x - 1}\right] = \frac{1^2 - 1}{1 - 1} = \frac{0}{0} \quad \text{I.F.}$$

   Solve by factoring.

   $$\lim_{x \to 1}\left[\frac{x^2 - 1}{x - 1}\right] = \lim_{x \to 1}\left[\frac{(x - 1)(x + 1)}{x - 1}\right] = \lim_{x \to 1}(x + 1)$$

   $= 1 + 1 = 2$ (**Note:** Keep limit notation until substituting 1 for x.)

2. Determine $\lim_{x \to 9}\left[\frac{\sqrt{x} - 3}{x - 9}\right]$.

   Answer:

   $$\lim_{x \to 9}\left[\frac{\sqrt{x} - 3}{x - 9}\right] = \frac{\sqrt{9} - 3}{9 - 9} = \frac{3 - 3}{0} = \frac{0}{0} \quad \text{I.F.}$$

   Solve by rationalizing the numerator.

   $$= \lim_{x \to 9}\left[\frac{\sqrt{x} - 3}{x - 9}\right] = \lim_{x \to 9}\left[\frac{\left(\sqrt{x} - 3\right)\left(\sqrt{x} + 3\right)}{(x - 9)\left(\sqrt{x} + 3\right)}\right] = \lim_{x \to 9}\left[\frac{x - 9}{(x - 9)\left(\sqrt{x} + 3\right)}\right]$$

   $$= \lim_{x \to 9}\left(\frac{1}{\sqrt{x} + 3}\right)$$

   $$= \frac{1}{\sqrt{9} + 3} = \frac{1}{3 + 3} = \frac{1}{6}$$

3. Determine $\lim_{x \to 3} \left[\frac{x^2 - x - 6}{x - 3} \right]$.

Answer:

$$\lim_{x \to 3} \left[\frac{x^2 - x - 6}{x - 3} \right] = \frac{(3)^2 - (3) - 6}{3 - 3} = \frac{9 - 3 - 6}{0} = \frac{0}{0} \quad \text{I.F.}$$

Solve by factoring.

$$\lim_{x \to 3} \left[\frac{x^2 - x - 6}{x - 3} \right] = \lim_{x \to 3} \left[\frac{(x - 3)(x + 2)}{x - 3} \right] = \lim_{x \to 3} (x + 2)$$

$= 3 + 2 = 5$ (**Note:** Keep limit notation until substituting 3 for x.)

4. Determine $\lim_{x \to 1} \left[\frac{\sqrt{x + 3} - 2}{x - 1} \right]$.

Answer:

$$\lim_{x \to 1} \left[\frac{\sqrt{x + 3} - 2}{x - 1} \right] = \frac{\sqrt{1 + 3} - 2}{1 - 1} = \frac{\sqrt{4} - 2}{0} = \frac{2 - 2}{0} = \frac{0}{0} \quad \text{I.F.}$$

Solve by rationalizing the numerator.

$$= \lim_{x \to 1} \left[\frac{\sqrt{x + 3} - 2}{x - 1} \right] = \lim_{x \to 1} \left[\frac{\left(\sqrt{x + 3} - 2\right)\left(\sqrt{x + 3} + 2\right)}{(x - 1)\left(\sqrt{x + 3} - 2\right)} \right]$$

$$= \lim_{x \to 1} \left[\frac{(x + 3) - 4}{(x - 1)\left(\sqrt{x + 3} + 2\right)} \right]$$

$$= \lim_{x \to 1} \left[\frac{x - 1}{(x - 1)\left(\sqrt{x + 3} + 2\right)} \right] = \lim_{x \to 1} \left(\frac{1}{\sqrt{x + 3} + 2} \right)$$

$$= \frac{1}{\sqrt{1 + 3} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{2 + 2} = \frac{1}{4}$$

5. Determine $\lim_{x\to 3}\left(\dfrac{\frac{1}{x}-\frac{1}{3}}{x-3}\right)$.

Answer:

$$\lim_{x\to 3}\left(\frac{\frac{1}{x}-\frac{1}{3}}{x-3}\right)=\frac{\frac{1}{3}-\frac{1}{3}}{3-3}=\frac{0}{0}\ \text{I.F.}$$

Solve by simplifying the complex fraction.

$$\lim_{x\to 3}\left(\frac{\frac{1}{x}-\frac{1}{3}}{x-3}\right)=\lim_{x\to 3}\left(\frac{\frac{1}{x}-\frac{1}{3}}{x-3}\right)\cdot\left(\frac{3x}{3x}\right)=\lim_{x\to 3}\left(\frac{3-x}{(x-3)(3x)}\right)$$

$$=\lim_{x\to 3}\left(\frac{-(x-3)}{(x-3)(3x)}\right)=\lim_{x\to 3}\left(\frac{-1}{3x}\right)$$

$$=\frac{-1}{3(3)}=-\frac{1}{9}$$

6. If $f(x) = 3x - 8$, determine $\lim_{h\to 0}\left[\dfrac{f(2+h)-f(2)}{h}\right]$.

Answer:

$$\lim_{h\to 0}\left[\frac{f(2+h)-f(2)}{h}\right]=\frac{f(2+0)-f(2)}{0}=\frac{f(2)-f(2)}{0}=\frac{0}{0}\ \text{I.F.}$$

Convert from function notation and simplify.

$$\lim_{h\to 0}\left[\frac{f(2+h)-f(2)}{h}\right]=\lim_{h\to 0}\left[\frac{(3(2+h)-8)-(3(2)-8)}{h}\right]$$

$$=\lim_{h\to 0}\left[\frac{6+3h-8-6+8}{h}\right]$$

$$=\lim_{h\to 0}\left(\frac{3h}{h}\right)=\lim_{h\to 0}(3)$$

$$=3$$

7. If $f(x)=\dfrac{2}{\sqrt{x}}$, determine $\lim\limits_{h\to 0}\left[\dfrac{f(4+h)-f(4)}{h}\right]$.

Answer:

$$\lim_{h\to 0}\left[\frac{f(4+h)-f(4)}{h}\right]=\frac{f(4+0)-f(4)}{0}=\frac{f(4)-f(4)}{0}=\frac{0}{0}\quad \text{I.F.}$$

Convert from function notation.

$$\lim_{h\to 0}\left[\frac{f(4+h)-f(4)}{h}\right]=\lim_{h\to 0}\left(\frac{\frac{2}{\sqrt{4+h}}-\frac{2}{\sqrt{4}}}{h}\right)=\lim_{h\to 0}\left(\frac{\frac{2}{\sqrt{4+h}}-\frac{2}{2}}{h}\right)$$

Simplify the complex fraction.

$$=\lim_{h\to 0}\left[\left(\frac{\frac{2}{\sqrt{4+h}}-1}{h}\right)\cdot\left(\frac{\sqrt{4+h}}{\sqrt{4+h}}\right)\right]$$

$$=\lim_{h\to 0}\left(\frac{2-\sqrt{4+h}}{h\cdot\left(\sqrt{4+h}\right)}\right)=\lim_{h\to 0}\left[\left(\frac{2-\sqrt{4+h}}{h\cdot\left(\sqrt{4+h}\right)}\right)\left(\frac{2+\sqrt{4+h}}{2+\sqrt{4+h}}\right)\right]$$

$$=\lim_{h\to 0}\left(\frac{4-(4+h)}{h\left(\sqrt{4+h}\right)\left(2+\sqrt{4+h}\right)}\right)$$

$$=\lim_{h\to 0}\left(\frac{-h}{h\left(\sqrt{4+h}\right)\left(2+\sqrt{4+h}\right)}\right)=\lim_{h\to 0}\left[\left(\frac{h}{h}\right)\left(\frac{-1}{\left(\sqrt{4+h}\right)\left(2+\sqrt{4+h}\right)}\right)\right]$$

$$=\lim_{h\to 0}\left(\frac{-1}{2\sqrt{4+h}+4+h}\right)$$

$$=\frac{-1}{2\sqrt{4+0}+4+0}=\frac{-1}{2\sqrt{4}+4}=\frac{-1}{2(2)+4}=-\frac{1}{8}$$

Learning Activity 1.5

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Evaluate: $\frac{5}{0.1}$
2. Evaluate: $\frac{-8}{0.001}$
3. Evaluate: $10.001 - 10$
4. Evaluate: $3.99 - 4$
5. Simplify: $\left(3x^2 + 4x - 5\right) \cdot \left(\frac{1}{x^2}\right)$
6. Determine the non-permissible value of $\frac{5}{1-x}$.

For questions 7 and 8, given $h(x) = \begin{cases} 7x & \text{if } x \le -1 \\ x^2 & \text{if } x > -1 \end{cases}$

7. Evaluate: $h(-2)$
8. Evaluate: $h(2)$

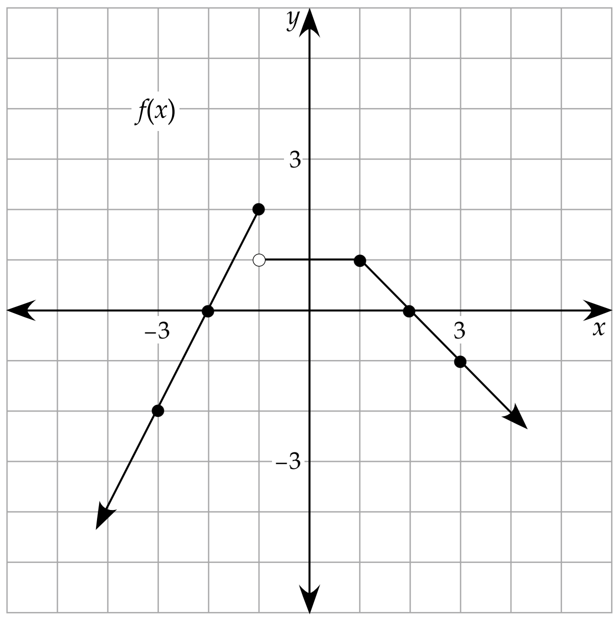
Answers:

1. $50 \left(5 \div \frac{1}{10}\right)$
2. $-8000 \left(-8 \div \frac{1}{1000}\right)$
3. 0.001
4. -0.01
5. $3 + \frac{4}{x} - \frac{5}{x^2}$
6. $x = 1$
7. -14 (since $-2 \le -1$, then $7(-2)$)
8. 4 (since $2 > -1$, then $(2)^2$)

Part B: Evaluating One-Sided Limits

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Use the graph of $f(x)$ below to answer the following questions.



a) $\lim_{x \to -1^-} f(x)$

Answer:

2 (As $x \to -1$ from the left, then $f(x) \to 2$.)

b) $\lim_{x \to -1^+} f(x)$

Answer:

1 (As $x \to -1$ from the right, then $f(x) \to 1$.)

c) $\lim_{x \to -1} f(x)$

Answer:

Does not exist. (The one-sided limits are not equal.)

d) $\lim_{x \to 1^-} f(x)$

Answer:

1 (As $x \to 1$ from the left, then $f(x) \to 1$.)

e) $\lim_{x \to 1^+} f(x)$

Answer:

1 (As $x \to 1$ from the right, then $f(x) \to 1$.)

f) $\lim_{x \to 1} f(x)$

Answer:

1 (The one-sided limits both equal 1.)

2. Determine the following limits:

a) $\lim_{x \to 2^-} \left(2x^3 + 4x^2 - 7x - 1\right)$

Answer:

$2(2)^3 + 4(2)^2 - 7(2) - 1 = 16 + 16 - 14 - 1 = 17$

b) $\lim_{x \to 2^+} \left(2x^3 + 4x^2 - 7x - 1\right)$

Answer:

$2(2)^3 + 4(2)^2 - 7(2) - 1 = 16 + 16 - 14 - 1 = 17$

c) $\lim_{x \to 2} \left(2x^3 + 4x^2 - 7x - 1\right)$

Answer:

$2(2)^3 + 4(2)^2 - 7(2) - 1 = 16 + 16 - 14 - 1 = 17$

Since the polynomial function is not a piecewise function, all three of these limits approach the same value.

3. Use the function below to answer the following questions:

$$h(x) = \begin{cases} 7x & \text{if } x \le -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

a) $\lim\limits_{x \to -1^-} h(x)$

Answer:

$$\lim_{x \to -1^-} h(x) = \lim_{x \to -1^-} (7x) = 7(-1) = -7$$

b) $\lim\limits_{x \to -1^+} h(x)$

Answer:

$$\lim_{x \to -1^+} h(x) = \lim_{x \to -1^+} \left(x^2\right) = (-1)^2 = 1$$

c) $\lim\limits_{x \to -1} h(x)$

Answer:

Does not exist because the one-sided limits are not equal

4. Use the function below to answer the following questions:

$$m(x) = \begin{cases} -2 & \text{if } x \le 0 \\ x - 2 & \text{if } x > 0 \end{cases}$$

a) $\lim\limits_{x \to 0^-} m(x)$

Answer:

$$\lim_{x \to 0^-} m(x) = \lim_{x \to 0^-} (-2) = -2$$

b) $\lim\limits_{x \to 0^+} m(x)$

Answer:

$$\lim_{x \to 0^+} m(x) = \lim_{x \to 0^+} (x - 2) = 0 - 2 = -2$$

c) $\lim_{x \to 0} m(x)$

Answer:

-2 (The one-sided limits both equal -2.)

5. Determine the following limits:

a) $\lim_{x \to 3^-} \frac{|3x-9|}{x-3}$

Answer:

$$\lim_{x \to 3^-} \frac{|3x-9|}{x-3} = \frac{|3(3)-9|}{3-3} = \frac{0}{0} \quad \text{I.F.}$$

The absolute value function can be written as a piecewise function.

$$\lim_{x \to 3^-} \frac{|3x-9|}{x-3} = \lim_{x \to 3^-} \frac{|3(x-3)|}{x-3} = 3 \cdot \lim_{x \to 3^-} \frac{|x-3|}{x-3}$$

$$= 3 \cdot \lim_{x \to 3^-} \frac{-(x-3)}{x-3} = 3 \cdot \lim_{x \to 3^-} (-1)$$

$$= 3(-1) = -3$$

b) $\lim_{x \to 3^+} \frac{|3x-9|}{x-3}$

Answer:

$$\lim_{x \to 3^+} \frac{|3x-9|}{x-3} = \lim_{x \to 3^+} \frac{|3(x-3)|}{x-3} = 3 \cdot \lim_{x \to 3^-} \frac{|x-3|}{x-3}$$

$$= 3 \cdot \lim_{x \to 3^-} \frac{(x-3)}{x-3} = 3 \cdot \lim_{x \to 3^-} (1)$$

$$= 3(1) = 3$$

c) $\lim_{x \to 3} \frac{|3x-9|}{x-3}$

Answer:

Does not exist because the one-sided limits are not equal.

6. Determine the following limits if they exist:

a) $\lim_{x\to 3^-} \frac{2x}{x-3}$

Answer:

$$\lim_{x\to 3^-} \frac{2x}{x-3} = \frac{6}{0} \quad \text{(undefined)}$$

$$\lim_{x\to 3^-} \frac{2x}{x-3} = \frac{6}{\text{very small negative number}} \text{ approaches } -\infty$$

b) $\lim_{x\to 3^+} \frac{2x}{x-3}$

Answer:

$$\lim_{x\to 3^+} \frac{2x}{x-3} = \frac{6}{0} \quad \text{(undefined)}$$

$$\lim_{x\to 3^+} \frac{2x}{x-3} = \frac{6}{\text{very small positive number}} \text{ approaches } +\infty$$

c) $\lim_{x\to 3} \frac{2x}{x-3}$

Answer:

Does not exist because the one-sided limits are not equal.

d) $\lim_{x\to -2^-} \frac{2}{(x+2)^2}$

Answer:

$$\lim_{x\to -2^-} \frac{2}{(x+2)^2} = \frac{2}{0} \quad \text{(undefined)}$$

$$\lim_{x\to -2^-} \frac{2}{(x+2)^2} = \frac{2}{\text{very small positive number}} \text{ approaches } +\infty$$

e) $\lim_{x \to -2^+} \frac{2}{(x+2)^2}$

Answer:

$$\lim_{x \to -2^+} \frac{2}{(x+2)^2} = \frac{2}{0} \quad \text{(undefined)}$$

$$\lim_{x \to -2^+} \frac{2}{(x+2)^2} = \frac{2}{\text{very small positive number}} \text{ approaches } +\infty$$

f) $\lim_{x \to -2} \frac{2}{(x+2)^2}$

Answer:

$+\infty$ because the one-sided limits are equal.

Learning Activity 1.6

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Factor: $x^2 - 4x - 12$
2. Factor: $x^2 + 7x + 10$
3. Simplify: $\dfrac{x^2 - 25}{x^2 - 2x - 15}$
4. What are the non-permissible values of $\dfrac{x^2 - 25}{x^2 - 2x - 15}$?
5. What is the domain of $f(x) = \dfrac{x^2 - 25}{x^2 - 2x - 15}$?

For questions 6, 7, and 8, given $g(x) = \dfrac{5}{\sqrt{x+1}}$

6. Determine the non-permissible values of $g(x)$.
7. Evaluate: $g(3)$
8. Determine an expression for $m(x)$ if $m(x) = g(x) \cdot g(x)$.

Answers:

1. $(x - 6)(x + 2)$
2. $(x + 2)(x + 5)$
3. $\dfrac{x+5}{x+3}$, $x \neq 5$ $\left(\dfrac{(x+5)(x-5)}{(x-5)(x+3)}\right)$
4. $x = 5$ and $x = -3$ are non-permissible values (npvs)
5. $\{x \in \Re, x \neq 5 \text{ and } x \neq -3\}$
6. $\{x < -1, x \in \Re\}$ are non-permissible values (npvs)
7. $\dfrac{5}{2}$ $\left(g(3) = \dfrac{5}{\sqrt{3+1}} = \dfrac{5}{\sqrt{4}} = \dfrac{5}{2}\right)$
8. $m(x) = \dfrac{25}{x+1}$ $\left(\left(\dfrac{5}{\sqrt{x+1}}\right) \cdot \left(\dfrac{5}{\sqrt{x+1}}\right)\right)$

Part B: Using Limits to Determine Asymptotes of a Graph

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

Consider the function $f(x) = \dfrac{2x-1}{x+4}$ for the Questions 1 to 3.

1. Determine the following limits:

a) $\lim\limits_{x \to -4^-} \left(\dfrac{2x-1}{x+4} \right)$

Answer:

$$\lim_{x \to -4^-} \left(\frac{2x-1}{x+4} \right) = \frac{-9}{0} \text{ (undefined)}$$

$$\lim_{x \to -4^-} \left(\frac{2x-1}{x+4} \right) = \frac{-9}{\text{very small negative number}} = +\infty$$

b) $\lim\limits_{x \to -4^+} \left(\dfrac{2x-1}{x+4} \right)$

Answer:

$$\lim_{x \to -4^+} \left(\frac{2x-1}{x+4} \right) = \frac{-9}{0} \text{ (undefined)}$$

$$\lim_{x \to -4^+} \left(\frac{2x-1}{x+4} \right) = \frac{-9}{\text{very small positive number}} = -\infty$$

c) $\lim_{x \to -\infty} \left(\frac{2x - 1}{x + 4} \right)$

Answer:

$$\lim_{x \to -\infty} \left(\frac{2x - 1}{x + 4} \right) = \frac{-\infty}{-\infty} \text{ I.F.}$$

$$\lim_{x \to -\infty} \left(\frac{2x - 1}{x + 4} \right) = \lim_{x \to -\infty} \left[\left(\frac{2x - 1}{x + 4} \right) \cdot \left(\frac{\frac{1}{x}}{\frac{1}{x}} \right) \right] = \lim_{x \to -\infty} \left(\frac{2 - \frac{1}{x}}{1 + \frac{4}{x}} \right)$$

$$= \frac{2 - 0}{1 + 0} = \frac{2}{1} = 2$$

d) $\lim_{x \to +\infty} \left(\frac{2x - 1}{x + 4} \right)$

Answer:

$$\lim_{x \to \infty} \left(\frac{2x - 1}{x + 4} \right) = \frac{\infty}{\infty} \text{ I.F.}$$

$$\lim_{x \to \infty} \left(\frac{2x - 1}{x + 4} \right) = \lim_{x \to \infty} \left[\left(\frac{2x - 1}{x + 4} \right) \cdot \left(\frac{\frac{1}{x}}{\frac{1}{x}} \right) \right] = \lim_{x \to \infty} \left(\frac{2 - \frac{1}{x}}{1 + \frac{4}{x}} \right)$$

$$= \frac{2 - 0}{1 + 0} = \frac{2}{1} = 2$$

2. Determine the equation of the vertical asymptote of $f(x) = \frac{2x - 1}{x + 4}$. Explain your answer using the information from Question 1.

   Answer:

   The equation of the vertical asymptote is $x = -4$ because the one-side limits around $x = -4$ approach ∞ and $-\infty$.

3. Determine the equation of the horizontal asymptote of $f(x) = \dfrac{2x - 1}{x + 4}$.

Explain your answer using the information from Question 1.

Answer:

The equation of the horizontal asymptote is $y = 2$ because the limits approach 2 as x approaches $-\infty$ and as x approaches $+\infty$.

Consider the function $f(x) = \dfrac{x^2 - 25}{x^2 - 2x - 15}$ for Questions 4 to 6.

4. Determine the following limits:

a) $\lim\limits_{x \to 5} \left(\dfrac{x^2 - 25}{x^2 - 2x - 15} \right)$

Answer:

$$\lim_{x \to 5} \left(\frac{x^2 - 25}{x^2 - 2x - 15} \right) = \frac{5^2 - 25}{5^2 - 2(5) - 15} = \frac{0}{0} \text{ I.F.}$$

$$\lim_{x \to 5} \left(\frac{x^2 - 25}{x^2 - 2x - 15} \right) = \lim_{x \to 5} \left(\frac{(x - 5)(x + 5)}{(x - 5)(x + 3)} \right)$$

$$= \lim_{x \to 5} \left(\frac{x + 5}{x + 3} \right)$$

$$= \frac{10}{8} = \frac{5}{4}$$

b) $\lim\limits_{x \to -3} \left(\dfrac{x^2 - 25}{x^2 - 2x - 15} \right)$

Answer:

$$\lim_{x \to -3} \left(\frac{x^2 - 25}{x^2 - 2x - 15} \right) = \frac{(-3)^2 - 25}{(-3)^2 - 2(-3) - 15} = \frac{-16}{0} \text{ I.F.}$$

$$\lim_{x \to -3} \left(\frac{x^2 - 25}{x^2 - 2x - 15} \right) = \lim_{x \to -3} \left(\frac{(x - 5)(x + 5)}{(x - 5)(x + 3)} \right) = \lim_{x \to -3} \left(\frac{x + 5}{x + 3} \right)$$

$$= \frac{2}{0} \text{ (undefined)}$$

So you need to analyze the one-side limits for this question.

$$\lim_{x \to -3^-} \left(\frac{x+5}{x+3} \right) = \frac{2}{\text{very small negative number}} = -\infty$$

$$\lim_{x \to -3^+} \left(\frac{x+5}{x+3} \right) = \frac{2}{\text{very small positive number}} = +\infty$$

The limit does not exist because one-sided limits are not equal.

5. Determine the equation of the vertical asymptote of $f(x) = \dfrac{x^2 - 25}{x^2 - 2x - 15}$.

   Explain your answer using the information from Question 4.

   Answer:

   The equation of the vertical asymptote is $x = -3$ because the one-sided limits on either side of $x = -3$ approach ∞ and $-\infty$. Although the value $x = 5$ is undefined, it is not the equation of a vertical asymptote because the two-sided limit at $x = 5$ is not ∞ but a numerical value. Thus, there is a point of discontinuity (a hole) on the graph of this function at $\left(5, \frac{5}{4}\right)$.

6. Determine the equation of the horizontal asymptote of $f(x) = \dfrac{x^2 - 25}{x^2 - 2x - 15}$.

   Explain your answer using the information from Question 4.

   Answer:

   The equation of the horizontal asymptote is $y = 1$ because the rational function is a ratio of two polynomials with the same degree. The ratio of the leading coefficients of the two polynomials is $\frac{1}{1} = 1$. This question could also be answered using infinite limits.

$$\lim_{x\to-\infty}\left(\frac{x^2-25}{x^2-2x-15}\right)=\frac{-\infty}{-\infty}\ \text{I.F.}$$

$$\lim_{x\to-\infty}\left(\frac{x^2-25}{x^2-2x-15}\right)=\lim_{x\to-\infty}\left[\left(\frac{x^2-25}{x^2-2x-15}\right)\cdot\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right)\right]$$

$$=\lim_{x\to-\infty}\left(\frac{1-\frac{25}{x^2}}{1-\frac{2}{x}-\frac{15}{x^2}}\right)=\frac{1-0}{1-0-0}=\frac{1}{1}=1$$

$$\lim_{x\to\infty}\left(\frac{x^2-25}{x^2-2x-15}\right)=\frac{\infty}{\infty}\ \text{I.F.}$$

$$\lim_{x\to\infty}\left(\frac{x^2-25}{x^2-2x-15}\right)=\lim_{x\to\infty}\left[\left(\frac{x^2-25}{x^2-2x-15}\right)\cdot\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right)\right]$$

$$=\lim_{x\to\infty}\left(\frac{1-\frac{25}{x^2}}{1-\frac{2}{x}-\frac{15}{x^2}}\right)=\frac{1-0}{1-0-0}=\frac{1}{1}=1$$

7. Determine the equation of the vertical asymptote of $h(x)=\frac{3x}{\sqrt{x-2}}$.

Answer:

The undefined value for this function is $x = 2$, so determine the limit as x approaches 2.

$$\lim_{x\to2}\left(\frac{3x}{\sqrt{x-2}}\right)=\frac{6}{0}\quad\text{(undefined)}$$

$$\lim_{x\to2^-}\left(\frac{3x}{\sqrt{x-2}}\right)=\frac{6}{\sqrt{\text{very small negative number}}}=\text{undefined}$$

$$\lim_{x\to2^+}\left(\frac{3x}{\sqrt{x-2}}\right)=\frac{6}{\sqrt{\text{very small positive number}}}=+\infty$$

Although only the right-hand limit approaches ∞, the function still has a vertical asymptote at $x = 2$.

8. Determine the equation of the horizontal asymptote of $h(x) = \dfrac{3x}{\sqrt{x-2}}$, if it exists.

Answer:

To determine the equation of the horizontal asymptote, solve for the infinite limit.

$$\lim_{x\to\infty}\left(\frac{3x}{\sqrt{x-2}}\right) = \frac{\infty}{\infty} \quad \text{I.F.}$$

$$\lim_{x\to\infty}\left(\frac{3x}{\sqrt{x-2}}\right) = \lim_{x\to\infty}\left[\left(\frac{3x}{\sqrt{x-2}}\right)\cdot\left(\frac{\frac{1}{x}}{\frac{1}{\sqrt{x^2}}}\right)\right] = \lim_{x\to\infty}\left(\frac{3}{\sqrt{\frac{1}{x}-\frac{2}{x^2}}}\right) = \frac{3}{\sqrt{0-0}}$$

$$= \frac{3}{\text{small positive number}} \; (\rightarrow \text{approaches } \infty)$$

Since the infinite limit does not approach a value, but increases without bounds, then there is no horizontal asymptote. Remember that if the degree of the numerator is higher than that of the denominator then there is no horizontal asymptote. You only needed to consider x approaching positive infinity since, due to the radical part, the domain is limited to $x \geq 2$ so the function does not exist as x approaches negative infinity.

Learning Activity 1.7

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

Given $f(x) = \begin{cases} (x+1)^2, & \text{when } x \le 0 \\ 1 & \text{when } x > 0 \end{cases}$

1. Evaluate: $f(-2)$
2. Evaluate: $f(0)$
3. Evaluate: $f(2)$
4. What is the domain of $f(x)$?

Given $g(x) = \begin{cases} -1, & \text{when } x \le 0 \\ x - 2 & \text{when } x > 0 \end{cases}$

5. Evaluate: $g(-2)$
6. Evaluate: $g(0)$
7. Evaluate: $g(2)$
8. What is the domain of $g(x)$?

Answers:

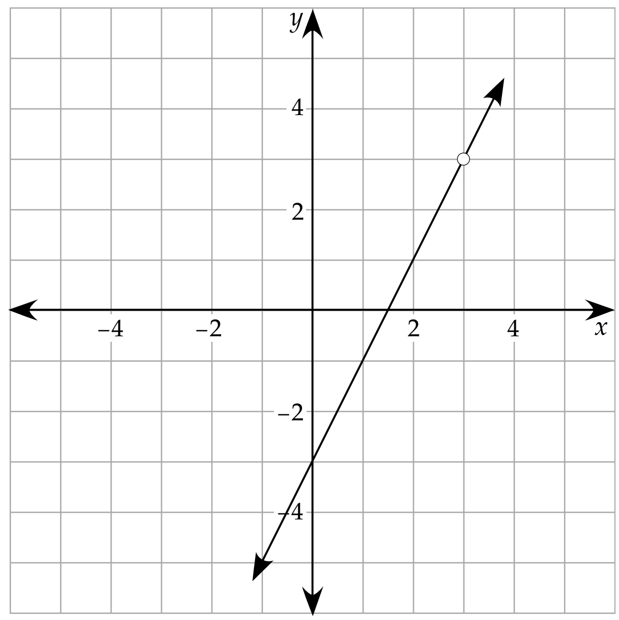
1. 1 ($f(-2) = (-2 + 1)^2 = (-1)^2 = 1$)
2. 1 ($f(0) = (0 + 1)^2 = (1)^2 = 1$)
3. 1 ($f(2) = 1$, since $x > 0$)
4. The domain is all real numbers.
5. −1($g(-2) = -1$, since $x \le 0$)
6. −1 ($g(0) = -1$, since $x \le 0$)
7. 0 ($g(2) = 2 - 2 = 0$)
8. The domain is all real numbers.

Part B: Continuity

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Determine whether each of the following functions is continuous using their graphs. If a function is discontinuous, then state the point of discontinuity.

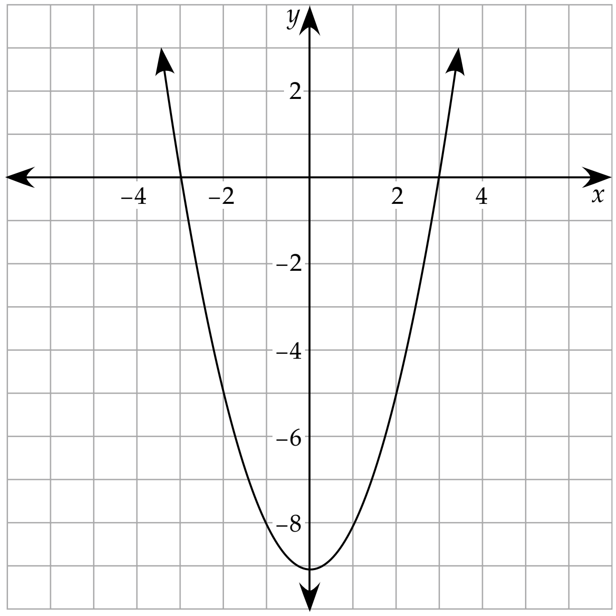
a)



Answer:

The function is discontinuous because there is a hole (a point of discontinuity) at $x = 3$.

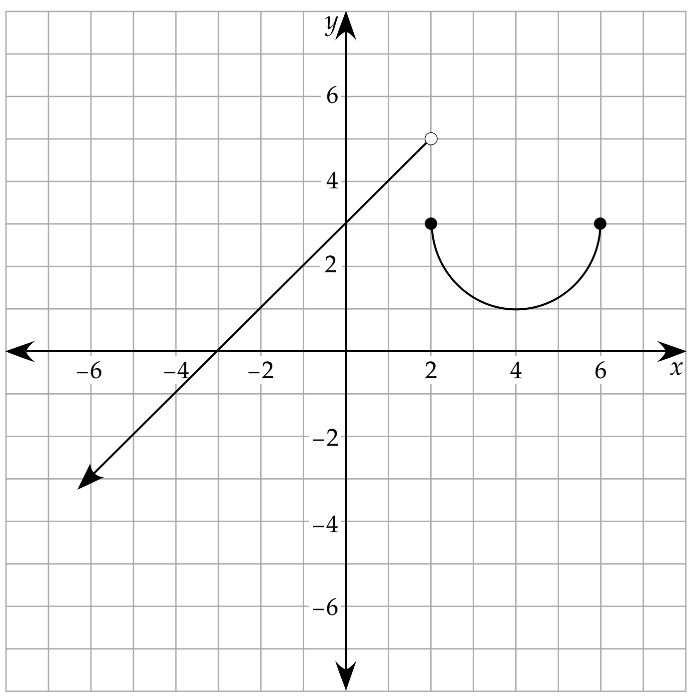
b)



Answer:

The function is continuous because there are no holes or breaks in its graph.

c)



Answer:

The function is discontinuous because there is a break in the graph (a point of discontinuity) at $x = 2$.

2. Is $f(x)$ continuous at $x = 0$, given $f(x) = \begin{cases} (x+1)^2, & \text{when } x \le 0 \\ 1 & \text{when } x > 0 \end{cases}$?

Answer:

To check continuity, you need to determine whether the function and limit values exist at $x = 0$ and that they are equal.

a) The function value exists; that is, $f(0) = (0 + 1)^2 = 1^2 = 1$.

b) The limit value exists because the one-sided limits are equal.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x+1)^2 = (0+1)^2 = 1^2 = 1$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 = 1$$

$$\lim_{x \to 0^-} f(x) = 1 = \lim_{x \to 0^+} f(x)$$

$$\lim_{x \to 0} f(x) = 1$$

c) The function and the limit values are equal.

$$f(0) = 1 = \lim_{x \to 0} f(x)$$

Therefore, the function is continuous at $x = 0$.

3. Is $g(x)$ continuous at $x = 0$, given $g(x) = \begin{cases} -1, & \text{when } x \le 0 \\ x - 2 & \text{when } x > 0 \end{cases}$?

Answer:

To check continuity, you need to determine whether the function and limit values exist at $x = 0$ and that they are equal.

a) The function value exists; that is, $g(0) = -1$.

b) The limit value doesn't exist because the one-sided limits aren't equal.

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (-1) = -1$$

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x - 2) = 0 - 2 = -2$$

$$\lim_{x \to 0^-} g(x) \ne \lim_{x \to 0^+} g(x)$$

c) The function value cannot equal the limit value, because the limit value does not exist.

The function is discontinuous $x = 0$, because the limit value does not exist.

Grade 12 Introduction to Calculus (45S)

Module 2
Derivatives

Module 2: Derivatives

Introduction

The slope of a line is the same everywhere on the line. The slope of a non-linear function curve is different on different parts of the curve. In this module, you will explore the changing slope of a curve. By using your knowledge of the slope of a line and the tangent line to a curve, you will be able to comment on the slope of a curve. Your understanding of limits will be extended to introduce one of the two main calculus concepts of this course: derivatives.

Assignments in Module 2

When you have completed the assignments for Module 2, submit your completed assignments to the Distance Learning Unit either by mail or electronically through the learning management system (LMS). The staff will forward your work to your tutor/marker.

Lesson	Assignment Number	Assignment Title
1	Assignment 2.1	Slope of the Tangent Line to a Curve
2	Assignment 2.2	Definition of the Derivative
3	Assignment 2.3	Basic Differentiation Rules
4	Assignment 2.4	Differentiation with Product and Quotient Rules
5	Assignment 2.5	Differentiation with the Chain Rule and Higher Order Derivatives
6	Assignment 2.6	Implicit Differentiation

Notes

Lesson 1: The Slope of a Curve

Lesson Focus

In this lesson, you will

- ❑ determine the equation of a line
- ❑ explain how slopes of secant lines can approximate the slope of a tangent line
- ❑ determine the slope of the tangent line to a curve

Lesson Introduction

In this lesson, you will review how to find the slope of a line and use it to determine the equation of the line. Then, you will explore secant and tangent lines to curves. In your exploration of slopes of secant lines, you will find that the shorter the secant line, the closer the slope is to the slope of the tangent line at a point. In other words, you will use the concept of limit as the length of the secant line approaches zero.

The Slope and the y-intercept of a Line

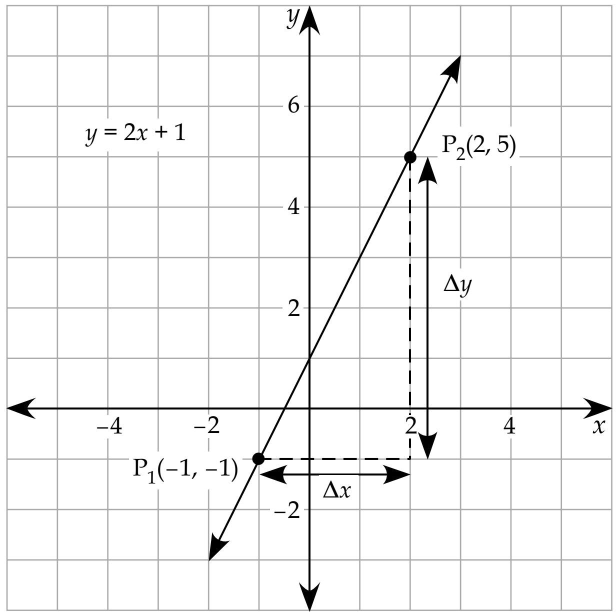
Before beginning to study the slopes of function curves, you will review and practise some concepts you first learned in pre-calculus courses.

Slope

The **slope of a line** or any line segment represents the ratio of the "change of y" with respect to the "change of x." Slope is often represented by the variable m. You can use the coordinates of the two points shown on the graph to determine Δy (change in y) and Δx (change in x) with the following mathematical relationship for slope.

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}}$$

Now let's study the line $y = 2x + 1$, whose graph is shown below.



You can determine the slope of the line by using the coordinates of the points P_1 and P_2. Obtained from the graph, the points are $P_1(-1, -1)$ and $P_2(2, 5)$. You can determine the slope of line segment P_1P_2 using the previously mentioned formula.

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - (-1)}{2 - (-1)} = \frac{6}{3} = 2$$

Example 1

Determine the slope of the line passing through the two points A(−2, 3) and B(−4, −3).

Solution

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3 - (3)}{-4 - (-2)} = \frac{-6}{-2} = 3$$

Special note: A non-zero slope represents an oblique line (diagonal line) but a slope of zero represents a horizontal line. In addition, an infinitely large, undefined (zero denominator) slope represents a vertical line.

y-intercept

The ***y*-intercept** of the line is the intersection of the line with the y-axis where $x = 0$. The y-intercept is often represented by the letter ***b***. The y-intercept can also be calculated by substituting $x = 0$ into an equation and solving for y.

When you look at your previous equation, $y = 2x + 1$, and its graph to determine its y-intercept, what do you notice?

1. The intersection of the line with the y-axis is 1.
2. If you substitute $x = 0$ into $y = 2x + 1$, you also get 1 (as shown)

    $y = 2(0) + 1 = 0 + 1 = 1$

Example 2

Determine the y-intercept of the line $y = 3x + 2$.

Solution

Substitute $x = 0$ into the equation and solve for y.

$y = 3(0) + 2 = 0 + 2 = 2$

Thus, the y-intercept, $b = 2$.

The Equation of a Line

The equation of a line can be written in the **slope-intercept** form

$y = mx + b$

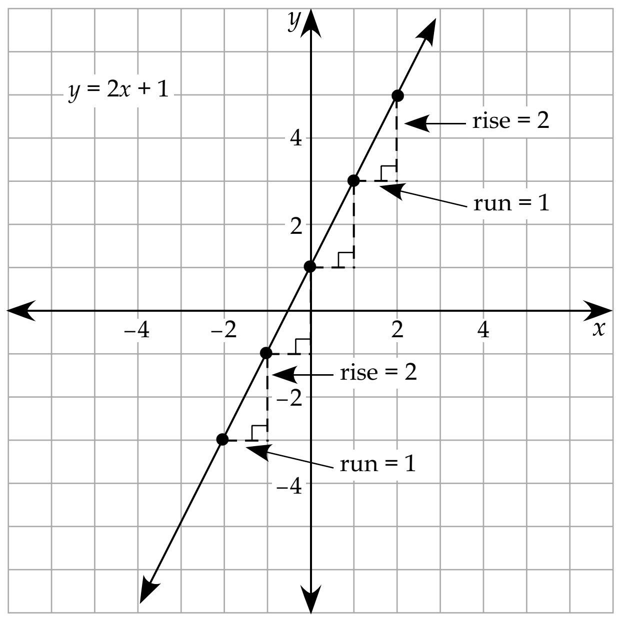
Recall that m represents the slope and b represents the y-intercept.

Special note: The equation of a horizontal line is $y = b$ since $m = 0$; and the equation of a vertical line is $x = a$ if the x-intercept is a.

When you recall the line mentioned previously with the equation $y = 2x + 1$, you notice two things:

1. The slope of the graph of $m = 2$ is represented by the coefficient of the term with the x in the equation.
2. The y-intercept of $b = 1$ on its graph is represented by the constant in its equation.

You can use the equation of a line to quickly determine its slope and y-intercept, and, conversely, you can use the slope and y-intercept of the line to determine its equation. Unlike most functions, **linear functions** have a **constant slope**, m, and are graphically represented by a straight line. See the graph below.



constant slope

$m = 2 = \frac{2}{1} = \frac{\text{rise}}{\text{run}}$

Example 3

Determine the slope and *y*-intercept of the line $y = -3x + 5$.

Solution

The slope is $m = -3$. It is the coefficient on the term with the *x*, so every change of 1 unit in *x* results in a change of -3 units in *y*. The *y*-intercept is $b = 5$. It is the constant in the equation and the value of *y* when *x* is zero.

Example 4

Determine the equation of the line with a slope of -2 and *y*-intercept of -3.

Solution

Since $m = -2$ and $b = 3$, then the equation of the line is $y = -2x - 3$.

Determining the Equation of a Line

The equation of a line can also be determined given graphical information such as

1. one point on the line and the slope of the line
2. two points on the line

Example 5

Determine the equation of the line passing through the point (6, 7) with a slope of 2.

Solution

Since you already know the slope, all you need to do is solve for the y-intercept.

$$\begin{aligned} y &= mx + b \\ 7 &= 2(6) + b \\ 7 &= 12 + b \\ 7 - 12 &= b \\ b &= -5 \end{aligned}$$

Substitute $m = 2$ and $(x, y) = (6, 7)$ into the equation $y = mx + b$.

Now you can state the equation of the line as $y = 2x - 5$.

Example 6

Determine the equation of the line passing through the points (−1, 2) and (4, −3).

Solution

$$m = \frac{-3 - 2}{4 - (-1)} = \frac{-5}{5} = -1$$

First, determine the slope of the line using the two points (−1, 2) and (4, −3).

$$\begin{aligned} y &= mx + b \\ -3 &= -1(4) + b \\ -3 &= -4 + b \\ -3 + 4 &= b \\ b &= +1 \end{aligned}$$

Second, solve for the y-intercept by using the slope and either of the given points. Let's use the point (4, −3).

Finally, you can state the equation of the line as $y = -x + 1$.

Determining the Equation of a Line Using Parallel and Perpendicular Lines

The slopes of parallel and perpendicular lines are related, allowing you to use their respective slopes to determine the slope of your line. More specifically, the slopes of parallel lines are equal. However, the slopes of perpendicular lines are negative reciprocals of one another. Thus, if you know the slope of a parallel or a perpendicular line, you can use it to find the slope of your line.

Example 7

Use the equation of the line $y = 2x - 5$ to determine the slope of a line that is

a) parallel

b) perpendicular

Solution

a) The slope of the given line is $m = 2$, so the slope of a parallel is also $m = 2$.

b) The slope of the given line is $m = 2$, then the slope of a perpendicular line is the negative reciprocal, which is $m = -\frac{1}{2}$.

Example 8

Determine the equation of a line that passes through the point $(-3, -4)$ and is perpendicular to the line $y = -\frac{2}{3}x + 1$.

Solution

$$m = -\frac{2}{3} \rightarrow \text{perpendicular } m = \frac{3}{2}$$

The slope of the perpendicular line is the negative reciprocal of the slope of the given line.

$$y = mx + b$$

$$-4 = \frac{3}{2}(-3) + b$$

$$-4 = -\frac{9}{2} + b$$

$$-4 + \frac{9}{2} = b$$

$$-\frac{8}{2} + \frac{9}{2} = b$$

$$b = \frac{1}{2}$$

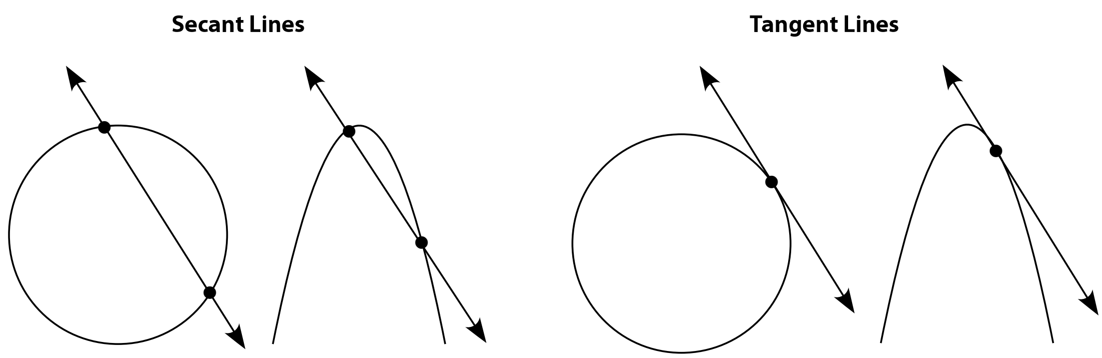
Now that you know the slope and a given point, you can solve for the y-intercept.

Therefore, the equation of the line is $y = \frac{3}{2}x + \frac{1}{2}$.

From your pre-calculus courses, you know that linear functions are modelled by lines with a constant slope. You also know how to determine the slope of a line and how it relates to its equation. However, most functions are non-linear and are not modelled by lines but rather by curves. In this calculus course, you will learn how to determine the slope of these curves. Although a curve does not have a constant slope like that of a line, you can use lines called secants to approximate the slope of a curve.

Secant and Tangent Lines to a Curve

You may remember hearing the terms **tangent** and **secant** in relation to trigonometric functions. The function names originate from the geometric meanings of the terms. A secant line is a line that intersects or passes through a circle or small section of a function curve at two distinct points. A tangent line is a line that touches a circle once or touches a small section of a function curve at two distinct points.

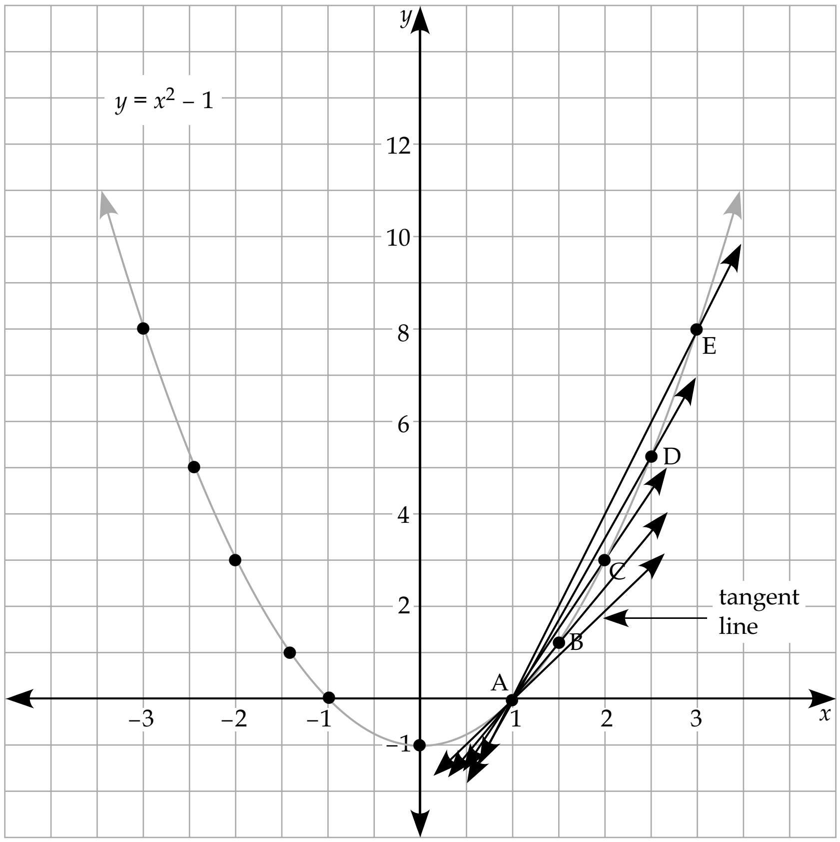


The slope of a secant line can easily be found using the coordinates of the two intersecting points. However, something else must be done to determine the slope of a tangent line, since you only have the coordinates at one point where it touches the curve. For example, you may recall that the slope of the tangent to a circle can be found by using the coordinates of the centre and point of tangency, since the radius segment is perpendicular to the tangent.

The slope of the tangent line at a point of tangency on a function curve can be estimated using a second point and the slope of the resulting secant line. The closer the second point of the secant line is to the point of tangency, the shorter the secant segment gets and the better the slope of the secant segment is as an estimate for the slope of the tangent line.

You will now investigate how the slope of secant lines to a curve change as the second point approaches the first point and the secant line slope is used as an estimate for the tangent line slope.

Let's investigate the slope of $y = x^2 - 1$ at A(1, 0) by using the slopes of secant lines.



Note: As the second point approaches the first point (at A), the slope of the secant line approaches the slope of the tangent line.

The slopes of the secant lines AD, AC, and AB are calculated below.

Points	Calculations	Slopes
A(1, 0) and E(3, 8)	$m = \frac{8-0}{3-1} = \frac{8}{2} = 4$	Slope of secant line AE is 4
A(1, 0) and D(2.5, 5.25)	$m = \frac{5.25-0}{2.5-1} = \frac{5.25}{1.5} = 3.5$	Slope of secant line AD is 3.5
A(1, 0) and C(2, 3)	$m = \frac{3-0}{2-1} = \frac{3}{1} = 3$	Slope of secant line AC is 3
A(1, 0) and B(1.5, 1.25)	$m = \frac{1.25-0}{1.5-1} = \frac{1.25}{0.5} = 2.5$	Slope of secant line AB is 2.5

According to the above calculations, the slopes of secant lines AE, AD, AC, and AB appear to be converging on a number. Graphically, the secant lines appear to be approaching the tangent line. Let's get even closer by approaching A from the left and right.

What happens to the slope of $y = x^2 - 1$ when we approach A(1, 0), using the second point of a secant line from the left?

(0.75, −0.4375)	(0.9, −0.19)	(0.99, −0.0199)	(0.999, −0.001999)	(0.9999, −0.00019999)
$m = \dfrac{-0.4375 - 0}{0.75 - 1}$ $= \dfrac{-0.43755}{-0.25}$ $= 1.75$	$m = \dfrac{-0.19 - 0}{0.9 - 1}$ $= \dfrac{-0.19}{-0.1}$ $= 1.9$	$m = \dfrac{-0.0199 - 0}{0.99 - 1}$ $= \dfrac{-0.0199}{-0.01}$ $= 1.99$	$m = \dfrac{-0.001999 - 0}{0.999 - 1}$ $= \dfrac{-0.001999}{-0.001}$ $= 1.999$	$m = \dfrac{-0.00019999 - 0}{0.9999 - 1}$ $= \dfrac{-0.00019999}{-0.0001}$ $= 1.9999$

What happens to the slope when we approach A(1, 0), using the second point of a secant line from the right?

(1.25, 0.5625)	(1.1, 0.21)	(1.01, 0.0201)	(1.001, 0.002001)	(1.0001, 0.00020001)
$m = \dfrac{0.5625 - 0}{1.25 - 1}$ $= \dfrac{0.5625}{0.25}$ $= 2.25$	$m = \dfrac{0.21 - 0}{1.1 - 1}$ $= \dfrac{0.21}{0.1}$ $= 2.1$	$m = \dfrac{0.0201 - 0}{1.01 - 1}$ $= \dfrac{0.0201}{0.01}$ $= 2.01$	$m = \dfrac{0.002001 - 0}{1.001 - 1}$ $= \dfrac{0.002001}{0.001}$ $= 2.001$	$m = \dfrac{0.00020001 - 0}{1.0001 - 1}$ $= \dfrac{0.00020001}{0.0001}$ $= 2.0001$

As the second point approaches A(1,0) from the left and right, the slopes appear to converge on the number 2. As the second point of the secant line approaches the tangent line at point A, their respective slopes approach the slope of the tangent line. You can use a limit calculation on the slopes of the secant lines to determine the slope of the tangent line by using P as your hypothetical second point.

If A(1, 0) and P(x, $x^2 - 1$), then

$$\lim_{x\to 1} m = \lim_{x\to 1}\left(\frac{y_2 - y_1}{x_2 - x_1}\right) = \lim_{x\to 1}\left(\frac{\left(x^2 - 1\right) - 0}{x - 1}\right) = \lim_{x\to 1}\left(\frac{x^2 - 1}{x - 1}\right)$$

$$= \lim_{x\to 1}\left(\frac{(x-1)(x+1)}{(x-1)}\right) = \lim_{x\to 1}(x+1) = 1 + 1 = 2$$

As predicted, the slopes are converging on the number 2. Thus, the slope of the tangent line is 2.

The slope of the tangent line can be used to determine the slope of the curve because the points on the secant lines are also points on the curve. Using the slope of the tangent line to determine the slope of the curve at a specific point is very important for many branches of science that analyze rates of change at an instant. Slope is the ratio comparing the change in both variables, often called **rate of change**.

Learning Activity 2.1

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Evaluate: $\dfrac{2-(-5)}{-3-4}$

2. Evaluate: $\dfrac{-2-5}{3-(-4)}$

3. Simplify: $\dfrac{x^2-4}{x-2}$

4. Simplify: $\dfrac{x^2-9}{x+3}$

5. Determine the slope of a line that is perpendicular to a line with a slope of $-\dfrac{5}{4}$.

6. What are the slope and y-intercept of $y = -3x + 4$?

7. What is the slope of a line parallel to $y = -3x + 4$?

8. What is the slope of a line perpendicular to $y = -3x + 4$?

continued

Learning Activity 2.1 (continued)

Part B: The Slope of the Tangent to a Curve

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Determine the equation of the line passing through the points (−2, 3) and (4, −1).
2. Determine the equation of the line passing through (0, 2) and perpendicular to the line $y = -3x + 4$.
3. The point M(1, 0) lies on the curve $y = -x^2 + 1$.
   a) If $Q(x, -x^2 + 1)$ is a point on the curve, determine the slope of the secant lines MQ from (1, 0) to the points, using the following values of x:

x	$Q(x, -x^2 + 1)$	Slope of MQ
2		
1.5		
1.1		
1.01		
1.001		

   b) Use the slopes of the secant lines to estimate the slope of the tangent line at M(1, 0).
   c) Sketch the curve, two secant lines, and the tangent line to M(1, 0).
   d) Use limits to calculate the slope of the tangent line at M(1, 0).

Lesson Summary

In this lesson, you learned how to calculate the slope of a line from its graph using points on the line and how to determine the slope from the equation of the line. The relationship between the slopes of parallel and perpendicular lines was also discussed. More importantly for calculus, you learned how to use the limit of the slopes of secant lines to determine the slope of a tangent line at a point on a curve. Ultimately, the focus of the remainder of this module is on determining the slope of the tangent line—first using limits and later using other algebraic methods, which form a foundation for the study of calculus.

Notes

Assignment 2.1

Slope of the Tangent Line to a Curve

Total: 18 marks

1. Determine the equation of the line passing through the points $(-1, 3)$ and $(-4, -2)$. *(3 marks)*

2. Determine the equation of the line passing through $(1, 2)$ and perpendicular to the line $y = 3x - 4$. *(3 marks)*

continued

Assignment 2.1: Slope of the Tangent Line to a Curve (continued)

3. The point M(1, 2) lies on the curve $y = x^2 + 1$.

   a) If $Q(x, x^2 + 1)$ is a point on the curve, determine the slope of the secant lines MQ for the following values of x: *(5 marks)*

x	$Q(x, x^2 + 1)$	**Slope of MQ**
2		
1.5		
1.1		
1.01		
1.001		

   b) Use the slopes of the secant lines to estimate the slope of the tangent line at M(1, 2). *(1 mark)*

continued

Assignment 2.1: Slope of the Tangent Line to a Curve (continued)

c) Sketch the curve, two secant lines, and the tangent line to M(1, 2). Label your lines as secant or tangent. *(3 marks)*

d) Use limits to calculate the slope of the tangent line at M(1, 2). *(3 marks)*

Notes

Lesson 2: The Definition of the Derivative

Lesson Focus

In this lesson, you will

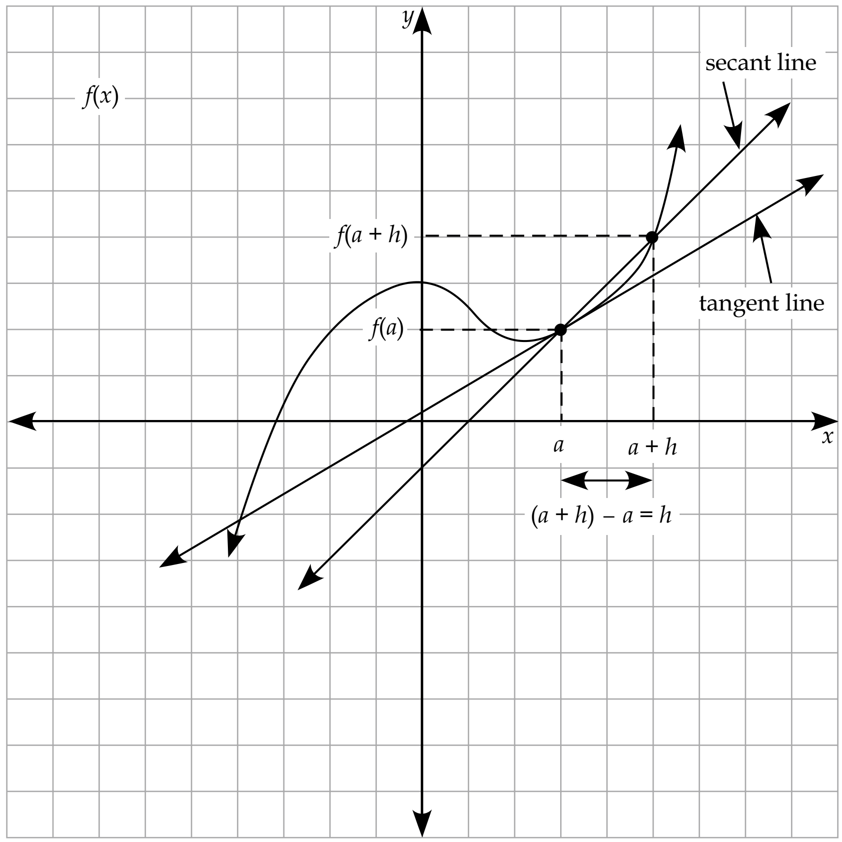
- ❑ determine the limit of the difference quotient as the value of the slope of the tangent line at a point
- ❑ define a derivative function, $f'(x)$, as a function that describes the slope of all points of a function, $f(x)$

Lesson Introduction

In this lesson, you will learn of a specific expression called the **difference quotient**, whose limit determines the slope of the tangent line to a curve at a point using the limit of the slopes of the secant lines. You will also learn of the derivative function, which is a function that describes the slope of a function, $f(x)$, at every point in its domain.

Slope of the Tangent Line

As explored in the previous lesson, the limit of the slopes of the secant lines to a curve is the slope of the tangent line to the curve at the point of tangency. We can generalize this idea to come up with an equation to represent the slope of the tangent line at any point where $x = a$. More specifically, the following graph of $f(x)$ shows that as the value of h decreases (that is, the distance of the second point from the first point decreases), the slope of the secant line approaches the slope of the tangent line.



The slope of $f(x)$ when $x = a$ can be estimated using a second point where $x = a + h$ for a very small value of h.

The slope of this secant line, m_s, can be found with the following calculation:

$$m_S = \frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$$

The expression is often used in calculus and is given the name **difference quotient**.

Thus, the slope of the tangent line is found by decreasing the distance between the two points of $f(x)$. In other words, as you decrease the value of h, the slope of the secant line approaches the slope of the tangent line. More specifically, you can take the limit of the difference quotient as h approaches 0.

Slope of the tangent line where $x = a$ is:

$$m_T = \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right)$$

You can use the limit equation above to find the slope of the tangent line to a function, $f(x)$, at any point where $x = a$. Here is an example.

Example 1

Determine the slope of the tangent line to the function $f(x) = 3x^2 - 1$ at $x = 2$.

Solution

Recall functional notation from Module 1 when evaluating this limit.

$$m_T = \lim_{h \to 0} \left(\frac{f(2+h) - f(2)}{h} \right)$$

Write the limit equation of the slope of the tangent line to $f(x)$ at $x = 2$.

$$= \lim_{h \to 0} \left(\frac{\left(3(2+h)^2 - 1\right) - \left(3(2)^2 - 1\right)}{h} \right)$$

Substitute the input values into:

$f(x) = 3x^2 - 1$

$f(2+h) = 3(2+h)^2 - 1$

$f(2) = 3(2)^2 - 1$

$$= \lim_{h \to 0} \left(\frac{3\left(4 + 4h + h^2\right) - 1 - \left(3(4) - 1\right)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{\left(12 + 12h + 3h^2\right) - 1 - 11}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{12 + 12h + 3h^2 - 12}{h} \right) = \lim_{h \to 0} \left(\frac{12h + 3h^2}{h} \right)$$

Simplify the numerator of the limit by first expanding the binomial and then grouping like terms.

$$= \lim_{h \to 0} \left(\frac{3h(4+h)}{h} \right)$$

$$= \lim_{h \to 0} \left(3(4+h)\right)$$

Factor out the greatest common factor $3h$ from the numerator in order to reduce the rational expression so the denominator is 1.

$$= 3(4+0) = 12$$

Evaluate the limit.

$$m_T = 12$$

The slope of the tangent line at $x = 2$ is 12.

The slope of the tangent line can be generalized into a function for the *x-values on its domain of f(x)* using the notation below:

$$\text{Slope of the tangent line} = m_T = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right).$$

The value of this limit at x is called the *derivative of y with respect to x* because it is derived from the original function. Written in this form, you can go beyond finding the slope of $f(x)$ at a point by defining a function that describes the slope of $f(x)$ everywhere on its domain.

The Definition of the Derivative

The **derivative** of a function $f(x)$ is another function called $f'(x)$ (read "f prime"), whose value at any point, x, on its domain is the slope of the tangent line to $f(x)$ at any point.

$$\text{The derivative of } f(x) = f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right).$$

If $y = f(x)$ is a function, its derivative with respect to x can be denoted in a variety of ways that all mean the same thing:

$$f'(x)\ ,\ y'\ ,\ \frac{dy}{dx},\ \frac{df(x)}{dx},\ \frac{df}{dx},\ D_x y$$

In this course, you will primarily use the first three notations for the derivative of a function with respect to x.

Since $f'(x)$ represents the first time the derivative of $f(x)$ has been taken, it is called the **first derivative of $f(x)$**, whereas $f''(x)$ represents the **second derivative of $f(x)$** or the first derivative of $f'(x)$. The first derivative represents the rate of change of y with respect to x, which is the slope. The second derivative represents the rate of change of the slope with respect to x. The second derivative will be explored in Lesson 5.

In conclusion, you can say that *the first derivative is a function representing the slope of the tangent line to another function curve*.

Example 2

Use the definition of the derivative to determine the derivative of $f(x) = x^2$.

Solution

Sometimes it is easier to evaluate parts of the difference quotient before determining the definition of the derivative.

$f(x) = x^2$ and

$f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$ — State $f(x)$ and simplify $f(x + h)$ first.

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

Write the limit of the difference quotient.

$$= \lim_{h \to 0} \left(\frac{x^2 + 2xh + h^2 - x^2}{h} \right)$$

Substitute $f(x)$ and $f(x + h)$ into the limit of the difference quotient.

$$= \lim_{h \to 0} \left(\frac{2xh + h^2}{h} \right)$$

Simplify the numerator.

$$= \lim_{h \to 0} \left(\frac{h\,(2x + h)}{h} \right)$$

Factor the h from the numerator.

$$= \lim_{h \to 0} (2x + h)$$

Reduce.

$$= 2x + 0$$

Evaluate the limit at $h = 0$.

$$= 2x$$

The derivative of $f(x) = x^2$ is $f'(x) = 2x$.

Example 3

Use the definition of the derivative to determine the derivative of $f(x) = \sqrt{x-3}$.

Solution

Sometimes it is easier to evaluate parts of the difference quotient before determining the definition of the derivative.

$f(x) = \sqrt{x-3}$ and

$f(x+h) = \sqrt{(x+h)-3} = \sqrt{x+h-3}$

State $f(x)$ and simplify $f(x+h)$ first.

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{\sqrt{x+h-3} - \sqrt{x-3}}{h} \right)$$

Substitute $f(x)$ and $f(x+h)$ into the limit of the difference quotient.

$$= \lim_{h \to 0} \left(\left(\frac{\sqrt{x+h-3} - \sqrt{x-3}}{h} \right) \cdot \left(\frac{\sqrt{x+h-3} + \sqrt{x-3}}{\sqrt{x+h-3} + \sqrt{x-3}} \right) \right)$$

Multiply numerator and denominator by the conjugate $\sqrt{x+h-3} + \sqrt{x-3}$.

$$= \lim_{h \to 0} \left(\frac{(x+h-3) - (x-3)}{h\left(\sqrt{x+h-3} + \sqrt{x-3}\right)} \right)$$

$$= \lim_{h \to 0} \left(\frac{x+h-3-x+3}{h\left(\sqrt{x+h-3} + \sqrt{x-3}\right)} \right)$$

$$= \lim_{h \to 0} \left(\frac{h}{h\left(\sqrt{x+h-3} + \sqrt{x-3}\right)} \right)$$

Simplify the numerator but do not simplify the denominator, so the h can easily be cancelled later.

$$= \lim_{h \to 0} \left(\frac{1}{\left(\sqrt{x+h-3} + \sqrt{x-3}\right)} \right)$$

Reduce.

$$= \frac{1}{\sqrt{x+0-3} + \sqrt{x-3}}$$

Evaluate limit at $h = 0$.

$$= \frac{1}{\sqrt{x-3} + \sqrt{x-3}}$$

Simplify.

$$= \frac{1}{2\sqrt{x-3}}$$

The derivative of $f(x) = \sqrt{x-3}$ is $f'(x) = \dfrac{1}{2\sqrt{x-3}}$.

Example 4

Use the definition of the derivative to find the slope of $g(x) = \dfrac{x+1}{x}$,

at $x = -2$.

Solution

First, determine the first derivative function and then substitute $x = -2$.

$$g(x) = \frac{x+1}{x} \text{ and}$$

$$g(x+h) = \frac{(x+h)+1}{(x+h)} = \frac{x+h+1}{x+h}$$

State $g(x)$ and simplify $g(x + h)$ first.

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\frac{x+h+1}{x+h}\right) - \frac{x+1}{x}}{h}$$

Substitute $g(x)$ and $g(x + h)$ into the limit of the difference quotient.

$$= \lim_{h \to 0} \left[\frac{\left(\frac{x+h+1}{x+h}\right) - \frac{x+1}{x}}{h}\right] \cdot \left(\frac{x(x+h)}{x(x+h)}\right)$$

Use the LCD, $x(x + h)$, from the numerator's fractions to simplify the whole fraction by multiplying the numerator and denominator by the LCD.

$$= \lim_{h \to 0} \left[\frac{\left(\frac{x+h+1}{x+h}\right) \cdot x(x+h) - \left(\frac{x+1}{x}\right) \cdot x(x+h)}{h \cdot x(x+h)}\right]$$

$$= \lim_{h \to 0} \left(\frac{\frac{(x+h+1)x\cancel{(x+h)}}{\cancel{(x+h)}} - \frac{(x+1)\cancel{x}(x+h)}{\cancel{x}}}{h \cdot x(x+h)}\right)$$

$$= \lim_{h \to 0} \left(\frac{(x+h+1)x - (x+1)(x+h)}{h \cdot x(x+h)} \right)$$

Simplify and reduce each fraction. Then collect like terms.

$$= \lim_{h \to 0} \left(\frac{\left(x^2 + xh + x\right) - \left(x^2 + xh + x + h\right)}{h \cdot x(x+h)} \right)$$

$$= \lim_{h \to 0} \left(\frac{x^2 + xh + x - x^2 - xh - x - h}{h \cdot x(x+h)} \right)$$

$$= \lim_{h \to 0} \left(\frac{-h}{h \cdot x(x+h)} \right)$$

$$= \lim_{h \to 0} \left(\frac{-1}{x(x+h)} \right)$$

Cancel out the h.

$$= \frac{-1}{x(x+0)}$$

Evaluate the limit at $h = 0$.

$$= \frac{-1}{x^2}$$

The derivative of $g(x) = \dfrac{x+1}{x}$ is $g'(x) = \dfrac{-1}{x^2}$.

$$g'(-2) = \frac{-1}{(-2)^2}$$

$$g'(-2) = -\frac{1}{4}$$

Substitute $x = -2$ into $g'(x) = \dfrac{-1}{x^2}$.

The slope of $g(x) = \dfrac{x+1}{x}$ at $x = -2$ is $-\dfrac{1}{4}$.

The derivative of a function is a function describing the slope of the tangent line anywhere in its domain. Using the limit definition of the derivative allows you to find the slope of the tangent at all x-values or at a specific value of $x = a$. Although the limit definition of the derivative is elegant, its solution can sometimes be difficult to simplify. In the next lesson, you will learn more efficient rules for finding the derivative of some types of functions.

Learning Activity 2.2

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

Given: $f(x) = x^2 + 3x$ and $g(x) = \frac{1}{x}$

1. Evaluate: $f(2)$
2. Evaluate: $g(2)$
3. Evaluate: $f(2 + h)$
4. Evaluate: $g(2 + h)$
5. Simplify: $f(x + h)$
6. Simplify: $g(x + h)$
7. Simplify: $\frac{(x+h)^2 - x^2}{h}$
8. Rationalize numerator: $\frac{\sqrt{x+h} - \sqrt{x}}{h}$

continued

Learning Activity 2.2 (continued)

Part B: The Definition of the Derivative

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Determine the slope of the tangent line to the function $f(x) = x^2 + 3x$ at $x = 2$.
2. Use the limit of the difference quotient to determine the derivative function, $f'(x)$, of $f(x) = \dfrac{1}{x+1}$.
3. Use the limit of the difference quotient to determine the derivative of $f(x) = \sqrt{x-1}$ and the slope of $f(x)$ at $x = 5$.

Lesson Summary

In this lesson, you learned how to determine the slope of the tangent line to a function $f(x)$ at a point, $x = a$, using a limit definition. This limit definition was extended to find the function representing the slope of the tangent lines for all x-values, called the derivative. You also learned how to determine the derivative at a specific x-value using the definition of the derivative. In the next lesson, you will explore how to determine the derivative using differentiation rules derived using limits and some pre-calculus algebra.

Assignment 2.2

Definition of the Derivative

Total: 15 marks

1. Using the limit of the difference quotient, determine the slope of the tangent line to the function $f(x) = 2x^2 + x$ at $x = 1$. *(4 marks)*

continued

Assignment 2.2: Definition of the Derivative (continued)

2. Use the limit of the difference quotient to determine the derivative function, $f'(x)$, of $f(x) = \dfrac{1}{x-1}$. *(5 marks)*

continued

Assignment 2.2: Definition of the Derivative (continued)

3. Use the limit of the difference quotient to determine the derivative of $f(x) = \sqrt{x+1}$, and the slope of $f(x)$ at $x = 3$. *(6 marks)*

Notes

LESSON 3: BASIC DIFFERENTIATION RULES

Lesson Focus

In this lesson, you will

- ❑ prove the power rule for differentiation of terms with integral exponents
- ❑ prove the rule for the derivative of a constant times $f(x)$
- ❑ prove the rule for the derivative of the sum and difference of two functions
- ❑ determine the derivative of functions by applying the power rule

Lesson Introduction

In this lesson, you will learn how to apply some of the basic rules of differentiation to determine the derivative of a function as an alternative to the limit method of determining the derivative learned in the previous lesson. With the use of the power rule, you will be able to determine the derivative of polynomial functions quickly and accurately.

Differentiation

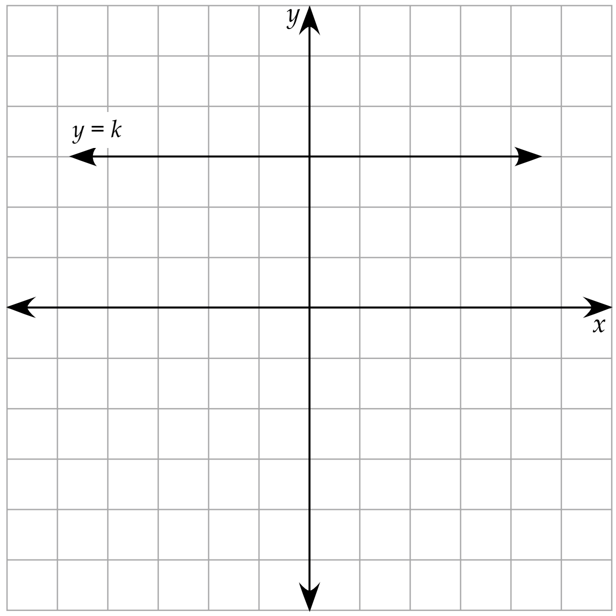
The process of determining the derivative of a function is called **differentiation**.

In the previous lesson, you learned how to determine the derivative by using the limit definition of the derivative, but it was time-consuming. In this lesson, you will learn how to use some differentiation rules to streamline the process.

Derivative of a Constant

If $f(x) = k$, where k is a real number, then $f'(x) = 0$.

You can verify this rule by using the graph of the function $f(x) = k$.



Notice that the constant function $f(x) = k$ is the horizontal line $y = k$. Further, the slope of any horizontal line is zero. When you recall that the derivative of the function describes the slope of $f(x)$, then it is consistent that for $f(x) = k$, $f'(x) = 0$. In words, the derivative of a constant is zero.

Formal proof:

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$ Definition of the derivative.

$$= \lim_{h \to 0} \left(\frac{k - k}{h} \right)$$ Both $f(x)$ and $f(x + h)$ are equal to k.

$$= \lim_{h \to 0} \left(\frac{0}{h} \right)$$ Upon direct substitution, the derivative of any constant is zero.

$$= \lim_{h \to 0} (0)$$

$$= 0$$

Remember, the limit describes the value of the function when h is close to, but not equal to, zero.

Example 1

If $f(x) = 27$, determine $f'(x)$.

Solution

According to our first differentiation rule for the derivative of a constant, $f'(x) = 0$.

Example 2

Differentiate $y = -9$.

Solution

The derivative of y is $y' = 0$.

The Power Rule for Derivatives

If $f(x) = x^n$, where n is a rational number, then $f'(x) = nx^{n-1}$.

You can verify this rule by determining the derivative of $f(x) = x^2$ using the definition of the derivative. According to the power rule, $f'(x) = 2x$!

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) = \lim_{h\to 0}\left(\frac{(x+h)^2-x^2}{h}\right) = \lim_{h\to 0}\left(\frac{x^2+2xh+h^2-x^2}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{2xh+h^2}{h}\right) = \lim_{h\to 0}\left(\frac{h(2x+h)}{h}\right) = \lim_{h\to 0}(2x+h) = 2x$$

Formal Proof:

This proof requires that you know how to expand using the binomial theorem. Do you remember the binomial expansion theorem from pre-calculus? You will not be required to reproduce proofs, but simply follow each step in them.

This is how to expand the binomial:

$$(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ldots + nxh^{n-1} + h^n$$

Before you differentiate the function using the definition of the derivative, simplify the terms.

If $f(x) = x^n$, then

$$f(x+h) = (x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ldots + nxh^{n-1} + h.$$

Now, substitute into the limit of the difference quotient.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)\\
&= \lim_{h\to 0}\left(\frac{\left(\cancel{x^n} + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ldots + nxh^{n-1} + h\right) - \cancel{x^n}}{h}\right)\\
&= \lim_{h\to 0}\left(\frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ldots + nxh^{n-1} + h}{h}\right)\\
&= \lim_{h\to 0}\left(\frac{h\left(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \ldots + nxh^{n-2} + h^{n-1}\right)}{h}\right)\\
&= \lim_{h\to 0}\left(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \ldots + nxh^{n-2} + h^{n-1}\right)\\
&= \left(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}(0)^2 + \ldots + nx(0)^{n-2} + (0)^{n-1}\right)\\
&= nx^{n-1}
\end{aligned}
$$

Here are two examples of finding $f'(x)$ using the power rule rather than limits:

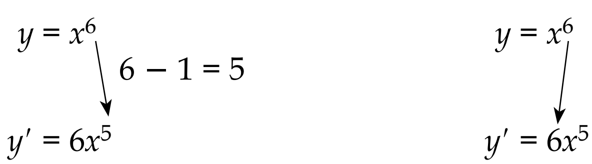
$$f(x) = x^2 \rightarrow f'(x) = 2x^{2-1} \rightarrow f'(x) = 2x^1$$

$$f(x) = x^3 \rightarrow f'(x) = 3x^{3-1} \rightarrow f'(x) = 3x^2$$

The power rule for differentiation shows two important patterns.

1. The **exponent** of the derivative is one less than the exponent of the original function.
2. The **coefficient** of the derivative is the exponent of the original function.

See the illustration that follows.



Example 3

Determine the derivative of $f(x) = x^4$ using the power rule.

Solution

The power rule states that $f'(x) = nx^{n-1}$. In this example, $n = 4$.

So $f'(x) = 4x^{4-1} = 4x^3$.

Notice that the power rule requires considerably less work than the limit definition in order to determine the derivative of a power.

The power rule also applies to functions with exponents that are negative numbers or non-integers, as shown in the next few examples. The proof of the power rule for non-integer exponents can be found online but will not be formally presented as part of this course.

Example 4

Differentiate $y = x^{-3}$.

Solution

You can use the power rule to determine the derivative with $n = -3$.

$$y' = --3x^{-3-1} = -3x^{-4}$$

Notice that -4 is "one less" than -3.

Note: A common mistake when using the power rule with negative exponents is adding 1 to the exponent instead of subtracting 1.

Example 5

Differentiate $y = x^{\frac{1}{3}}$.

Solution

You can use the power rule to determine the derivative with $n = \frac{1}{3}$.

$$\frac{dy}{dx} = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{\frac{1}{3}-\frac{3}{3}} = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}}$$

Notice that $-\frac{2}{3}$ is "one less" than $\frac{1}{3}$.

Example 6

Find the derivative of $g(x) = \dfrac{1}{x^4}$.

Solution

You can use the power rule to determine the derivative, but you have to rearrange the function. Write it as a power in the numerator using $n = -4$.

Recall that $\dfrac{1}{x^n} = x^{-n}$.

$$g(x) = \frac{1}{x^4} = x^{-4}$$

$$g'(x) = -4x^{-4-1} = -4x^{-5} = \frac{-4}{x^5}$$

Note: The derivative a reciprocal function can be found using the power rule if the function is first rewritten as a power using a negative exponent.

Example 7

Differentiate $y = \sqrt{x}$.

Solution

You can use the power rule to determine the derivative, but we have to rearrange the function as a power using $n = \dfrac{1}{2}$. Recall that $\sqrt[n]{x} = x^{\frac{1}{n}}$.

$$y = \sqrt{x} = x^{\frac{1}{2}}$$

$$y' = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{\frac{1}{2}-\frac{2}{2}} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$$

Note: The derivative of a radical function can be found using the power rule if the function is first rewritten using a rational exponent.

The application of the power rule can be expanded considerably by investigating two derivative properties: a constant times a function and the sum and difference of two functions.

Derivative of a Constant Times a Function

If $f(x) = k \cdot g(x)$, where k is a real number, than $f'(x) = k \cdot g'(x)$.

In words, this property states that the *derivative of a constant times a function is equal to the constant times the derivative of the function*.

Let's use the function $y = 4x^3$ to verify this rule. The rule states that the derivative of a constant times a function is equal to the constant times the derivative of the function, as seen below.

$$\frac{d}{dx}\left(4x^3\right) = 4 \cdot \frac{d}{dx}\left(x^3\right)$$

You can verify this rule by evaluating the **left side** using the definition of the derivative.

If $f(x) = 4x^3$,

then $f(x+h) = 4(x+h)^3 = 4\left(x^3 + 3x^2h + 3xh^2 + h\right)^3$

$= 4x^3 + 12x^2h + 12xh^2 + 4h^3$.

$$f'(x) = \frac{d}{dx}\left(4x^3\right) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$ Use the definition of the derivative.

$$f'(x) = \lim_{h \to 0}\left(\frac{\left(4x^3 + 12x^2h + 12xh^2 + 4h^3\right) - 4x^3}{h}\right)$$ Substitute the previously simplified expressions.

$$f'(x) = \lim_{h \to 0}\left(\frac{12x^2h + 12xh^2 + 4h^3}{h}\right)$$

$$= \lim_{h \to 0}\left(\frac{h\left(12x^2 + 12xh + 4h^2\right)}{h}\right)$$

$$= \lim_{h \to 0}\left(12x^2 + 12xh + 4h^2\right)$$

$$= 12x^2 + 12x(0) + 4(0)^2$$ Simplify the limit.

$$f'(x) = 12x^2$$ The left side.

You can evaluate the **right side** using the power rule.

$$4 \cdot \frac{d}{dx}\left(x^3\right) = 4 \cdot \left(3x^2\right) = 12x^2$$

Notice that it is much easier finding $f'(4x^3) = 12x^2$ using the power rule than it is using limits.

Since the left side equals the right side, the rule is verified with your example. Although this verification helps you understand the derivative of a constant times a function rule, it is not a formal proof. A proof follows.

Proof:

If $f(x) = k \cdot g(x)$, where k is a real number, then $f'(x) = k \cdot g'(x)$.

$$f'(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

Use the definition of the derivative to prove this rule.

$$f'(x) = \lim_{h \to 0}\left(\frac{k \cdot g(x+h) - k \cdot g(x)}{h}\right)$$

$$= \lim_{h \to 0}\left(\frac{k\left(g(x+h) - g(x)\right)}{h}\right)$$

$$= \lim_{h \to 0}\left(k \cdot \frac{\left(g(x+h) - g(x)\right)}{h}\right)$$

When you can, simplify the limit using your limit rules from Module 1. The limit of a product equals the product of the limits.

$$f'(x) = \left(\lim_{h \to 0} k\right) \cdot \lim_{h \to 0}\left(\frac{g(x+h) - g(x)}{h}\right)$$

$$= k \cdot g'(x)$$

The limit of the constant k is equal to k and $g'(x) = \lim_{h \to 0}\left(\frac{g(x+h) - g(x)}{h}\right)$.

Example 8

Differentiate $y = -2x^{-3}$.

Solution

$$\frac{d}{dx}(y) = -2 \cdot \frac{d}{dx}\left(x^{-3}\right)$$

$$\frac{dy}{dx} = -2 \cdot \left(-3x^{-4}\right)$$

$$\frac{dy}{dx} = 6x^{-4}$$

Derivative of the Sum or Difference of Two Functions

If f and g are differentiable functions, then:

a) $(f + g)'(x) = f'(x) + g'(x)$

b) $(f - g)'(x) = f'(x) - g'(x)$

In words, this derivative property is stated as the derivative of a sum of two functions is equal to the sum of the derivatives, and the derivative of a difference of two functions is equal to the difference of the derivatives.

Proof:

If f and g are differentiable functions, show that $(f + g)'(x) = f'(x) + g'(x)$.

Let $F(x) = (f + g)(x) = f(x) + g(x)$, then $F(x + \text{h}) = f(x + h) + g(x + h)$.

$$F'(x) = \lim_{h \to 0} \left(\frac{F(x+h) - F(x)}{h} \right)$$

Begin by using the definition of the derivative.

$$F'(x) = \lim_{h \to 0} \left(\frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} \right)$$

Substitute.

$$F'(x) = \lim_{h \to 0} \left(\frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} \right)$$

$$F'(x) = \lim_{h \to 0} \left(\frac{[f(x+h) - f(x)] + [g(x+h) - g(x)]}{h} \right)$$

$$F'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right)$$

Simplify the fraction by grouping $f(x)$ and $g(x)$ functions into two separate fractions.

$$F'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right) + \lim_{h \to 0} \left(\frac{g(x+h) - g(x)}{h} \right)$$

The limit of a sum is the sum of the limits.

$F'(x) = f'(x) + g'(x)$

Substitute:

$$\lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right) = f'(x)$$

and

$$\lim_{h \to 0} \left(\frac{g(x+h) - g(x)}{h} \right) = g'(x)$$

The properties for limits were used to prove the properties for derivatives.

Thus, you can say that the derivative of a sum of two functions is the sum of the derivatives. You can prove the derivative of a difference of two functions is the difference of the derivatives in a very similar manner. Using these properties for derivatives, you can apply the power rule on functions with polynomial terms and radical terms. Here are some examples.

Example 9

Differentiate the following:

a) $y = 2x^3 + 4x^4$

b) $y = 3x^{-4} - 5x^2 + \dfrac{2}{\sqrt{x}}$

Solution

a) $y = 2x^3 + 4x^4$ — The given function.

$\dfrac{d}{dx}(y) = \dfrac{d}{dx}\left(2x^3 + 4x^4\right) = \dfrac{d}{dx}\left(2x^3\right) + \dfrac{d}{dx}\left(4x^4\right)$ — Differentiate using the derivative of a sum rule.

$\dfrac{dy}{dx} = 2 \cdot \dfrac{d}{dx}\left(x^3\right) + 4 \cdot \dfrac{d}{dx}\left(x^4\right)$

$= 2 \cdot \left(3x^2\right) + 4 \cdot \left(4x^3\right)$ — Simplify the derivative using the derivative of a constant times a function rule, then the power rule.

$\dfrac{dy}{dx} = 6x^2 + 16x^3$ — The derivative.

Can you see how you might do the steps mentally to go from the original function, y, to its derivative, y', without requiring that any work be shown? If not yet, then you will soon.

b) $y = 3x^{-4} - 5x^2 + \frac{2}{\sqrt{x}}$ The given function.

$\frac{d}{dx}(y) = \frac{d}{dx}\left(3x^{-4} - 5x^2 + \frac{2}{\sqrt{x}}\right)$ Differentiate using the derivative of a sum rule.

$\frac{dy}{dx} = \frac{d}{dx}\left(3x^{-4}\right) - \frac{d}{dx}\left(5x^2\right) + \frac{d}{dx}\left(\frac{2}{\sqrt{x}}\right)$

$= 3 \cdot \frac{d}{dx}\left(x^{-4}\right) - 5 \cdot \frac{d}{dx}\left(x^2\right) + 2 \cdot \frac{d}{dx}\left(x^{-\frac{1}{2}}\right)$ Simplify the derivative using the derivative of a constant times a function rule, then the power rule.

$= 3 \cdot \left(-4x^{-5}\right) - 5 \cdot (2x) + 2 \cdot \left(-\frac{1}{2}x^{-\frac{3}{2}}\right)$

$\frac{dy}{dx} = -12x^{-5} - 10x - x^{-\frac{3}{2}}$ The derivative.

When you become comfortable in differentiation, you will be able to go straight from the given function to the derivative. The derivative can also be written as:

$$\frac{dy}{dx} = \frac{-12}{x^5} - 10x - \frac{1}{x^{\frac{3}{2}}}$$

or

$$\frac{dy}{dx} = \frac{-12}{x^5} - 10x - \frac{1}{\sqrt{x^3}}$$

Learning Activity 2.3

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

For Questions 1 to 4, write the monomials using rational exponents.

1. $\sqrt{x^3}$

2. $\sqrt[4]{x}$

3. $\left(3\sqrt[3]{x}\right)^2$

4. $-2\left(\sqrt[4]{x}\right)^3$

For Questions 5 to 8, rewrite the expression using negative exponents.

5. $\dfrac{-5}{x}$

6. $\dfrac{7}{2x^5}$

7. $\dfrac{1}{2\sqrt{x}}$

8. $\dfrac{3}{4\sqrt[3]{x}}$

continued

Learning Activity 2.3 (continued)

Part B: Basic Differentiation Rules

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Differentiate the following using the power rule and derivative properties:

   a) $y = 5x + 7$

   b) $y = 4x^2 - 10$

   c) $y = \frac{1}{3}x^3 - 3x^2 - \frac{1}{2}x + 2$

   d) $y = -2x^{\frac{3}{2}} + 2\sqrt{x}$

   e) $y = \frac{1}{x} + x^{\frac{3}{4}} + 2\sqrt[3]{x}$

   f) $y = \frac{2}{\sqrt{x}} + 4\sqrt{x^3}$

Lesson Summary

Now that you know how to determine the derivative of some functions using derivative properties and the power rule for differentiation, you will discover that there are many types of functions and differentiation rules. In the next few lessons, you learn about more complex differentiation rules developed to make differentiation easier and that allow you to determine the derivative of most of the types of functions that you know.

Notes

Assignment 2.3

Basic Differentiation Rules

Total: 10 marks

1. Differentiate the following using the power rule and derivative properties:

   a) $y = -2x + 1$ *(2 marks)*

   b) $y = 7x^3 + 8$ *(2 marks)*

   c) $y = \frac{1}{4}x^2 - 2x^3 + \frac{1}{3}x - 5$ *(2 marks)*

continued

Assignment 2.3: Basic Differentiation Rules (continued)

d) $y = 9x^{\frac{2}{3}} - 2\sqrt[3]{x}$ *(2 marks)*

e) $y = \dfrac{1}{x^2} + x^{\frac{1}{4}} - 2\sqrt[4]{x^3}$ *(2 marks)*

Lesson 4: Differentiation with Product and Quotient Rules

Lesson Focus

In this lesson, you will

- ❑ prove the derivative formulas for the derivative of a product and a quotient
- ❑ apply the product and quotient rules for derivatives to differentiate more complex expressions
- ❑ apply the derivative rules to determine the equation of a tangent line at a point given a function equation and a point on the function

Lesson Introduction

Now that you have proven the power law and some properties for derivatives, you can differentiate functions that would be difficult to differentiate using a limit of the difference quotient and algebraic techniques. One of the goals of those who developed calculus in the past was to establish other formulas (or rules) to make the differentiation of complex functions easier.

In this lesson, you will learn formulas (or rules) to aid in the differentiation of functions that are written as the product of two functions and as the quotient of two functions. These two rules will greatly expand the types of functions for which you can find the derivative. Remember, the derivative function describes the slope of the original function at every point in the domain. So, you will be able to apply the derivative rules to find the equation of tangent lines of a wide variety of functions.

Differentiation Continued

The need for additional differentiation rules arose when derivatives were required for more complex functions such as the one below:

$$f(x) = \left(x^3 + 1\right)\left(2 - x^4\right)$$

The derivative of $f(x)$ cannot be determined using your current differentiation rules unless you multiply the product to create one function. Although this method of simplification of the function is manageable for polynomial functions, it is not efficient for non-polynomial functions. Thus, the product rule was developed.

Derivative of a Product of Two Functions

If both f and g are differentiable functions, then so is $f \cdot g$ and:

$$(f \cdot g)'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x) \text{ or}$$

$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

In Leibniz notation:

$$\frac{d}{dx}\left[f(x) \cdot g(x)\right] = f(x) \cdot \frac{d}{dx}g(x) + g(x) \cdot \frac{d}{dx}f(x) \text{ or}$$

$$\frac{d}{dx}\left[f(x) \cdot g(x)\right] = \left(\frac{d}{dx}f(x)\right) \cdot g(x) + f(x) \cdot \left(\frac{d}{dx}g(x)\right)$$

In words, the product rule states:

> The derivative of a product of two functions is equal to the derivative of the first function times the second function plus the first function times the derivative of the second function.

Now let's compare the two methods of determining its derivative using the aforementioned function.

$$f(x) = \left(x^3 + 1\right)\left(2 - x^4\right)$$

In method one, you will simplify the function before differentiating; whereas in method two you will use the product differentiation rule.

Method 1

$f(x) = (x^3 + 1)(2 - x^4)$

$= x^3 \cdot 2 + x^3 \cdot (-x^4) + 1 \cdot 2 + 1 \cdot (-x^4)$

$= 2x^3 - x^7 + 2 - x^4$

Simplify the function using the distribution property.

$f'(x) = 6x^2 - 7x^6 + 0 - 4x^3$

$= -7x^6 - 4x^3 + 6x^2$

Differentiate using the power rule and differentiation properties.

Method 2

Let $f(x) = g(x) \cdot h(x)$

$g(x) = x^3 + 1$ and $h(x) = 2 - x^4$

Set up the original function as a product of two functions.

$f'(x) = g'(x) \cdot h(x) + g(x) \cdot h'(x)$

$= (3x^2)(2 - x^4) + (x^3 + 1)(-4x^3)$

Use the product rule: "derivative of first times second plus first times derivative of second."

$f'(x) = 6x^2 - 3x^6 - 4x^6 - 4x^3$

$= -7x^6 - 4x^3 + 6x^2$

Simplify the derivative.

In the comparison of both methods, the product rule does not appear to be more efficient in this case. However, if the two functions were trinomials or radicals, the simplification process would be more onerous. In addition, if the derivative did not need to be simplified, then you could eliminate one more step in Method 2, making it even more efficient. Ultimately, if the goal is to find the derivative, then the more direct process is always favoured.

In order to appreciate the elegance of the product rule, you need to review its proof. In post-secondary calculus courses, it is often required that students be familiar with and be able to reproduce this proof. However, for this course you should follow and understand each step, but you do not need to memorize or reproduce the steps. Let's review the proof of the product rule.

Proof of the Product Rule for Derivatives

If $k(x) = f(x) \cdot g(x)$, show that $k'(x) = f(x) \cdot g'(x) + f'(x) \cdot g(x)$.

Use the limit definition of the derivative to prove this rule.

Express the derivative of the original function, $k(x)$, using the limit definition of the derivative and as a product of the two functions, $f(x)$ and $g(x)$.

$$k'(x) = \lim_{h \to 0} \left(\frac{k(x+h) - k(x)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h} \right)$$

Add and subtract $f(x + h) \cdot g(x)$ in the numerator.

$$k'(x) = \lim_{h \to 0} \left(\frac{f(x+h) \cdot g(x+h) - f(x+h) \cdot g(x) + f(x+h) \cdot g(x) - f(x) \cdot g(x)}{h} \right)$$

Factor like terms.

$$k'(x) = \lim_{h \to 0} \left(f(x+h) \cdot \left(\frac{g(x+h) - g(x)}{h} \right) + g(x) \cdot \left(\frac{f(x+h) - f(x)}{h} \right) \right)$$

Simplify using limit properties.

$$k'(x) = \lim_{h \to 0} f(x+h) \cdot \lim_{h \to 0} \left(\frac{g(x+h) - g(x)}{h} \right) + \lim_{h \to 0} g(x) \cdot \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

Evaluate limits $\lim_{h \to 0} f(x+h) = f(x)$ and $\lim_{h \to 0} g(x) = g(x)$.

$$k'(x) = f(x) \cdot \lim_{h \to 0} \left(\frac{g(x+h) - g(x)}{h} \right) + g(x) \cdot \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

The limits of the difference quotients are derivatives, as previously defined.

$$k'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

Example 1

Determine the derivative of the following using the product rule. Do not simplify.

a) $h(x) = \left(2x^3 - 6x + 1\right) \cdot \left(\sqrt{x} - 1\right)$

b) $j(x) = \left(-4x^{-3} + 7x - 1\right) \cdot \left(\sqrt[3]{x} + x\right)$

Solutions

a) $h(x) = \left(2x^3 - 6x + 1\right) \cdot \left(\sqrt{x} - 1\right)$

"derivative of first times second plus first times derivative of second"

$$h'(x) = \left(6x^2 - 6\right) \cdot \left(\sqrt{x} - 1\right) + \left(2x^3 - 6x + 1\right) \cdot \left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$

b) $j(x) = \left(-4x^{-3} + 7x - 1\right) \cdot \left(\sqrt[3]{x} + x\right)$

"derivative of first times second plus first times derivative of second"

$$j'(x) = \left(12x^{-4} + 7\right)\left(\sqrt[3]{x} + x\right) + \left(-4x^{-3} + 7x - 1\right)\left(\frac{1}{3}x^{-\frac{2}{3}} + 1\right)$$

Now that we know how to determine the derivative of a product of two functions, let's take a look at the derivative of a quotient of two functions.

Derivative of a Quotient of Two Functions

If both f and g are differentiable functions, then so is $\frac{f}{g}$ and:

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g^2(x)}, \text{ where } g(x) \neq 0$$

In Leibniz notation:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \cdot \frac{d}{dx} f(x) - f(x) \cdot \frac{d}{dx} g(x)}{g^2(x)}$$

In words, the quotient rules states:

> The derivative of a quotient of two functions is equal to the denominator function times the derivative of the numerator function minus the numerator function times the derivative of the denominator function all divided by the square of the denominator function.

"Low D(High) minus High D(Low), over the denominator squared—we go."

Does this little rhyme for the quotient rule make sense to you? The denominator function is "Low" and the numerator function is "High." If you can make sense of the rhyme, it may help you remember the quotient rule.

The quotient rule is very valuable in differentiation of a rational function. As with the product rule, read and understand the steps of the proof of this new rule before you proceed with examples.

Proof of the Quotient Rule for Derivatives

If $h(x) = \dfrac{f(x)}{g(x)}$, show that $h'(x) = \dfrac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g^2(x)}$.

Use the product rule to prove the quotient rule.

Since $h(x) = \dfrac{f(x)}{g(x)}$, then $f(x) = h(x) \cdot g(x)$.	Rewrite the quotient as a product equation with $h(x)$ so that we can use the product rule.
$f'(x) = h(x) \cdot g'(x) + h'(x) \cdot g(x)$	Apply the product rule.
$h'(x) \cdot g(x) = f'(x) - h(x) \cdot g'(x)$ $h'(x) = \dfrac{f'(x) - h(x) \cdot g'(x)}{g(x)}$	Isolate $h'(x)$.
$h'(x) = \dfrac{f'(x) - \dfrac{f(x)}{g(x)} \cdot g'(x)}{g(x)}$	Substitute $h(x) = \dfrac{f(x)}{g(x)}$.
$h'(x) = \left(\dfrac{f'(x) - \dfrac{f(x)}{g(x)} \cdot g'(x)}{g(x)} \right) \cdot \dfrac{g(x)}{g(x)}$	Simplify the complex fraction.
$h'(x) = \dfrac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g^2(x)}$	Rewrite the limit as the derivative.

Now, practice using the quotient rule.

Example 2

Differentiate using the quotient rule, but do not simplify.

$$f(x) = \frac{3x^2 - 4x + 7}{-2x^3 + 9x}$$

Solution

$$f(x) = \frac{3x^2 - 4x + 7}{-2x^3 + 9x}$$

"Low D(High) minus High D(Low) over the denominator squared—we go."

$$f'(x) = \frac{\left(-2x^3 + 9x\right) \cdot \frac{d}{dx}\left(3x^2 - 4x + 7\right) - \left(3x^2 - 4x + 7\right) \cdot \frac{d}{dx}\left(-2x^3 + 9x\right)}{\left(-2x^3 + 9x\right)^2}$$

$$= \frac{\left(-2x^3 + 9x\right) \cdot (6x - 4) - \left(3x^2 - 4x + 7\right) \cdot \left(-6x^2 + 9\right)}{\left(-2x^3 + 9x\right)^2}$$

Sometimes it is advantageous to simplify the derivative, especially if it will be used to determine the slope of the tangent line.

Example 3

Differentiate using the quotient rule and simplify, $g(x) = \frac{x^3 + 2}{x^2 - 1}$.

Solution

$$g'(x) = \frac{\left(x^2 - 1\right) \cdot \frac{d}{dx}\left(x^3 + 2\right) - \left(x^3 + 2\right) \cdot \frac{d}{dx}\left(x^2 - 1\right)}{\left(x^2 - 1\right)^2}$$

$$g'(x) = \frac{\left(x^2 - 1\right)\left(3x^2\right) - \left(x^3 + 2\right)(2x)}{\left(x^2 - 1\right)^2}$$

$$g'(x) = \frac{3x^4 - 3x^2 - 2x^4 - 4x}{\left(x^2 - 1\right)^2}$$

$$g'(x) = \frac{x^4 - 3x^2 - 4x}{\left(x^2 - 1\right)^2}$$

This simplified version of $g'(x)$ would be an easier expression to use to determine the slope of $g(x)$ at a point.

Note: It is acceptable to leave the denominator factored and squared in simplified form.

Example 4

Differentiate $g(x) = \dfrac{\sqrt{x}}{1+\sqrt[3]{x}}$ and simplify the resulting derivative expression.

Solution

$$g(x) = \frac{\sqrt{x}}{1+\sqrt[3]{x}}$$

Differentiate using the quotient rule.

$$g'(x) = \frac{\left(1+\sqrt[3]{x}\right)\cdot\frac{d}{dx}\left(\sqrt{x}\right) - \left(\sqrt{x}\right)\cdot\frac{d}{dx}\left(1+\sqrt[3]{x}\right)}{\left(1+\sqrt[3]{x}\right)^2}$$

$$= \frac{\left(1+x^{\frac{1}{3}}\right)\cdot\frac{1}{2}(x)^{-\frac{1}{2}} - (x)^{\frac{1}{2}}\cdot\frac{1}{3}(x)^{-\frac{2}{3}}}{\left(1+x^{\frac{1}{3}}\right)^2}$$

Simplify numerator.

$$= \frac{\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{6}} - \frac{1}{3}x^{-\frac{1}{6}}}{\left(1+x^{\frac{1}{3}}\right)^2}$$

$$= \frac{\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{6}x^{-\frac{1}{6}}}{\left(1+x^{\frac{1}{3}}\right)^2}$$

$$g'(x) = \frac{\frac{1}{2\sqrt{x}} + \frac{1}{6\sqrt[6]{x}}}{\left(1+\sqrt[3]{x}\right)^2}$$

The simplified derivative.

As you can see, some of this simplification work tests the algebra skills you learned in pre-calculus classes.

Example 5

Determine the equation of the tangent line to $f(x) = \dfrac{3x^3 + 4}{1 - 2x^3}$ at (1, −7).

Solution

$$f(x) = \frac{3x^3 + 4}{1 - 2x^3}$$

Determine the derivative of $f(x)$.

$$f'(x) = \frac{\left(1 - 2x^3\right) \cdot \frac{d}{dx}\left(3x^3 + 4\right) - \left(3x^3 + 4\right) \cdot \frac{d}{dx}\left(1 - 2x^3\right)}{\left(1 - 2x^3\right)^2}$$

$$= \frac{\left(1 - 2x^3\right) \cdot \left(9x^2\right) - \left(3x^3 + 4\right) \cdot \left(-6x^2\right)}{\left(1 - 2x^3\right)^2}$$

Recall that the slope of the tangent line to $f(x)$ at $x = a$ is $m = f'(a)$. However, substitution into the unsimplified derivative is lengthy. You can simplify the derivative before substitution.

$$f'(x) = \frac{\left(1 - 2x^3\right) \cdot \left(9x^2\right) - \left(3x^3 + 4\right) \cdot \left(-6x^2\right)}{\left(1 - 2x^3\right)^2}$$

$$= \frac{9x^2 - 18x^5 - \left(-18x^5 - 24x^2\right)}{\left(1 - 2x^3\right)^2}$$

$$= \frac{9x^2 - 18x^5 + 18x^5 + 24x^2}{\left(1 - 2x^3\right)^2}$$

$$f'(x) = \frac{33x^2}{\left(1 - 2x^3\right)^2}$$

Find the slope at $x = 1$.

$$m = f'(1) = \frac{33(1)^2}{\left(1 - 2(1)^3\right)^2} = \frac{33(1)}{(1 - 2)^2} = 33$$

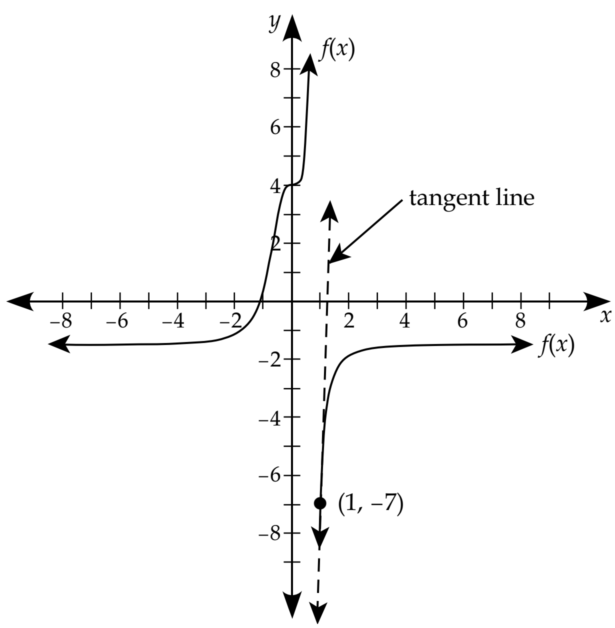
Substitute the point and slope into $y = mx + b$ to determine the y-intercept.

$$y = mx + b$$
$$-7 = (33)(1) + b$$
$$-7 = 33 + b$$
$$b = -7 - 33 = -40$$

The equation of the tangent line is $y = 33x - 40$.

Rather than using the slope-intercept form, $y = mx + b$, to find the equation of the tangent line, you can use the point-slope form, $y - y_1 = m(x - x_1)$, since you know the coordinates of one point, (x_1, y_1). Using the point $(1, -7)$ and the slope, m, of 33, the equation of the tangent line is $y + 7 = 33(x - 1)$. It can be left in this form.

The graph of the function and the tangent line through the point $(1, -7)$ are shown below.



Learning Activity 2.4

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

For Questions 1 to 4, simplify into a monomial with positive rational exponents.

1. $\sqrt{x} \cdot \sqrt[3]{x}$

2. $\left(x^{-2}\right)\left(5x^{3}\right)$

3. $\left(\dfrac{x^{3}}{\sqrt[4]{x}}\right)^{2}$

4. $\dfrac{\sqrt[3]{x^{5}}}{\sqrt{x}}$

For Questions 5 to 8, evaluate without the use of a calculator.

5. $\sqrt[3]{(27)^{2}}$

6. $\dfrac{\sqrt{50}}{\sqrt{2}}$

7. $\left(1-2^{3}\right)^{2}$

8. $\dfrac{3^{2}-1}{\sqrt[4]{16}}$

continued

Learning Activity 2.4 (continued)

Part B: Differentiation with Product and Quotient Rules

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Differentiate the following without simplifying the derivative:
   a) $f(x) = \left(3x^3 + 4\right)\left(1 - 2x^3\right)$
   b) $g(x) = \left(x - \sqrt{x}\right)\left(x^2 + \sqrt{x}\right)$
   c) $h(x) = \dfrac{6 + x^{-2}}{8x^{10} - 5x^3}$
   d) $m(x) = \dfrac{x^2 + \sqrt{x}}{1 - 2x^3}$

2. Differentiate the following and simplify the derivative:
   a) $f(x) = \left(2x^4 + 4x\right)\left(x^{-1} - 3x^2\right)$
   b) $f(x) = \dfrac{3\sqrt{x} + 4}{1 - 2x^3}$

3. Determine the equation of the tangent line to $h(x) = \left(1 + x^{-2}\right)\left(8x^{10} - 5x^3\right)$ at $(1, 6)$.

Lesson Summary

After completing this lesson, you can appreciate the necessity for the product and quotient rules in determining the derivative of a more complex function and, ultimately, the slope of the tangent line to a function. You probably have discovered the importance of meticulous algebraic manipulation of exponents and rational expressions when working with derivatives, especially those that involve product and quotient rules. Later in this module, you will learn even more rules to expand on the kinds of functions you can differentiate.

Notes

Assignment 2.4

Differentiation with Product and Quotient Rules

Total: 27 marks

1. Differentiate the following without simplifying the derivative:

   a) $f(x) = \left(-x^4 + 2\right)\left(1 - 3x^2\right)$ *(3 marks)*

   b) $g(x) = \left(1 - \sqrt{x}\right)\left(x^2 - \sqrt{x}\right)$ *(3 marks)*

   c) $h(x) = \dfrac{6 + x^{-1}}{3x^7 - 5x^4}$ *(4 marks)*

   d) $m(x) = \dfrac{-x^2 + \sqrt{x}}{1 - 3x^3}$ *(4 marks)*

continued

Assignment 2.4: Differentiation with Product and Quotient Rules (continued)

2. Differentiate the following and simplify the derivative:

   a) $f(x) = \left(2x^3 - x\right)\left(x^{-2} + 4x^2\right)$ *(4 marks)*

   b) $f(x) = \dfrac{6x - 1}{1 + x^3}$ *(4 marks)*

continued

Assignment 2.4: Differentiation with Product and Quotient Rules (continued)

3. Determine the equation of the tangent line to $h(x) = (3 + x^{-1})(2x^3 - x)$ at $(1, 4)$. *(5 marks)*

Notes

Lesson 5: Differentiation with the Chain Rule and Higher Order Derivatives

Lesson Focus

In this lesson, you will

- ❑ apply the chain rule to differentiate a composition of functions
- ❑ describe the first derivative of a displacement function as a velocity function
- ❑ describe the meaning of the second derivative of a displacement function as an acceleration function

Lesson Introduction

In this lesson, you will expand the types of functions for which you can find a derivative to include compositions of functions. As you may recall from your work in pre-calculus math courses, a composition of functions can be written in the form $y = f(g(x))$, where the range of g becomes the domain of f. The method of differentiation for compositions of functions that you will learn is called the **chain rule**.

In this lesson, you will also learn about higher order derivatives. As you know, the first derivative of a function describes the slope of the function, which is a rate of change (of y with respect to x). It may be apparent to you that the second derivative of a function describes the rate of change of the slope of the function. As you will learn, a common application of first and second derivatives is in the study of the motion of objects.

Composite Functions Reviewed

A **composite function** is a function that consists of one function within another.

Example 1

Given $f(x) = x^2$ and $g(x) = 2x - 1$,

a) determine an expression for $h(x) = f(g(x))$.

b) evaluate $f(g(-1))$.

Solution

a) Substitute for the x-variable of $f(x)$ with the expression for the y-value or function of $g(x)$ to arrive at the new function $h(x)$.

$$h(x) = f(g(x)) = f(2x - 1) = (2x - 1)^2$$

b) Substitute $x = -1$ into $g(x)$.

Evaluate the inside of the composition, $g(-1) = 2(-1) - 1 = -2 - 1 = -3$.

Then substitute the y-value of $g(-1) = -3$ into $f(x)$.

$$h(-1) = f(g(-1)) = f(-3) = (-3)^2 = 9$$

Alternatively, you could have substituted $x = -1$ into our $h(x)$ function from part (a).

$$h(-1) = (2(-1) - 1)^2 = (-2 - 1)^2 = (-3)^2 = 9$$

Some composite functions are rational, and particular attention is required when working with the domain since the domain of rational functions cannot include values of x that make the denominator zero.

Example 2

Given $f(x) = \frac{1}{x^2}$ and $g(x) = 2x - 1$,

a) determine an expression for $h(x) = f(g(x))$.

b) evaluate $h(-1)$.

Solution

a) Substitute for the x-variable of $f(x)$ with the expression for the y-value or function of $g(x)$ to arrive at the new function $h(x)$.

$$h(x) = f(g(x)) = f(2x - 1) = \frac{1}{(2x - 1)^2}$$

Special note: The domain of $h(x)$ is $x \neq \frac{1}{2}$ because the denominator cannot be equal to zero

b) Substitute $x = -1$ into $g(x)$.

Evaluate the inside of the composition, $g(-1) = 2(-1) - 1 = -2 - 1 = -3$.

Then substitute the y-value of $g(-1) = -3$ into $f(x)$.

$$h(-1) = f\left(g(-1)\right) = f(-3) = \frac{1}{(-3)^2} = \frac{1}{9}$$

Alternatively, we could have substituted $x = -1$ into our $h(x)$ function from part (a).

$$h(-1) = \frac{1}{\left(2(-1) - 1\right)^2} = \frac{1}{(-2 - 1)^2} = \frac{1}{(-3)^2} = \frac{1}{9}$$

Writing a Function as a Composition of Functions

Many functions that are familiar to you could be written as the composition of two functions. When finding the derivative of a composition, you will find that you need to be able to split functions in this way. Here are some examples.

Example 3

Write the given function in terms of the composition of an inside function, $g(x)$, and an outside function, $f(x)$, so that it is equivalent to $y = f(g(x))$.

a) Define $f(x)$ and $g(x)$ to express $y = \sqrt{x - 1}$ as a composition in the form $y = f(g(x))$.

b) Define $f(x)$ and $g(x)$ to express $y = (2x + 1)^3$ as a composition in the form $y = f(g(x))$.

Solution

a) The inside function is defined as $g(x) = x - 1$.

The outside function is defined as $f(x) = \sqrt{x}$.

Then, $y = \sqrt{x - 1}$ is $y = f\left(g(x)\right)$.

b) The inside function is defined as $g(x) = 2x + 1$.

The outside function is defined as $f(x) = x^3$.

Then, $y = (2x + 1)^3$ is $y = f(g(x))$.

Differentiation of Composite Functions

The derivative of a composite function can be solved by either simplifying the original function before differentiation or by applying a differentiation technique called the chain rule on the original form of the function.

For example, if you needed to determine the derivative of $f(x) = (3x^2 + 1)^3$, you could either simplify the function before differentiating or you could apply the chain rule. Before comparing the methods, let's first define the chain rule.

The Chain Rule

If f, g, and h are differentiable functions, and $h(x) = f(g(x))$,

then $h'(x) = f'(g(x)) \cdot g'(x)$.

In Leibniz notation:

If $y = f(u)$ and $u = g(x)$, and f and g are differentiable,

then $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$.

The chain rule allows you to differentiate functions that are compositions (functions within one another). Often the chain rule is called the *"outside-inside rule."* In words, to determine the derivative of a composite function, you first differentiate the outside function evaluated with respect to the inside function and then multiply that by the derivative of the inside function.

It will take several examples and a lot of practice to use the chain rule with confidence.

Example 4

Differentiate $h(x) = (3x^2 + 1)^3$ using two methods and compare the two processes.

Solution

Method 1:

Simplifying first, then differentiating.

$h(x) = (3x^2 + 1)^3$	
$h(x) = (3x^2 + 1)(3x^2 + 1)(3x^2 + 1)$	Expand the function.
$h(x) = (9x^4 + 6x^2 + 1)(3x^2 + 1)$	Simplify.
$= 27x^6 + 18x^4 + 3x^2 + 9x^4 + 6x^2 + 1$	
$= 27x^6 + 27x^4 + 9x^2 + 1$	
$h'(x) = 162x^5 + 108x^3 + 18x$	Differentiate using the power rule.

Method 2:

Differentiating with the chain rule. First define the inside function as $u = g(x) = 3x^2 + 1$ and the outside function as $f(u) = u^3$.

$h(x) = (3x^2 + 1)^3$	Let $h(x) = f(u)$, $f(u) = (u)^3$, and $u = 3x^2 + 1$.
$h'(x) = f'(g(x)) \cdot g'(x)$	
$h'(x) = f'(u) \cdot u'$	Chain rule.
$f'(u) = 3u^2$ and $u' = 6x$	Differentiate pieces using the power rule where $f(u) = u^3$ and $u = 3x^2 + 1$.
$h'(x) = 3u^2 \cdot 6x$	
$= 3(3x^2 + 1)^2 \cdot 6x$	Put it all together and substitute $u = 3x^2 + 1$.
$f'(x) = 3(9x^4 + 6x^2 + 1) \cdot 6x$	Simplify the derivative.
$= 18x \cdot (9x^4 + 6x^2 + 1)$	
$= 162x^5 + 108x^3 + 18x$	

The simplification steps required in Method 1 can be very challenging, whereas the chain rule allows you to keep the number of terms manageable. In addition, if the derivative does not need to be simplified, then chain rule is even faster.

Example 5

Differentiate $y = (6x - 1)^5$ using the chain rule. Do not simplify the derivative.

Solution

Label the inside and outside functions that make up this composite function.

Let the outside function be $f(u) = u^5$ and let the inside function be $u = 6x - 1$.

Differentiate using the power rule.

So, $\frac{d}{du}\left(f(u)\right) = 5u^4$ and $\frac{du}{dx} = 6$.

By the chain rule:

$$\frac{dy}{dx} = f'(u) \cdot u'$$

$$\frac{dy}{dx} = 5u^4 \cdot 6 \quad (\text{now write } u \text{ in terms of } x)$$

$$\frac{dy}{dx} = 5(6x - 1)^4 \cdot 6$$

$$\frac{dy}{dx} = 30(6x - 1)^4$$

Example 6

Differentiate $y = \frac{1}{(3x - 2)^3}$ using the chain rule.

Solution

The quickest way to differentiate this function is to rewrite it with a negative exponent; otherwise, you could use the quotient rule.

$$y = \frac{1}{(3x - 2)^3} = (3x - 2)^{-3}$$

Now label the inside and outside functions that make up this composite function.

Let the outside function be $f(u) = u^{-3}$ and let the inside function be $u = 3x - 2$.

The chain rule states that $y' = f'(u) \cdot (u)'$.

So you need to find:

Derivative of outside function evaluated at inside function	*times*	*Derivative of inside function*
$f'(u) = -3u^{-4}$	$\cdot$	$u' = 3$

Now write the product.

$$y' = f'(u) \cdot u' = -3u^{-4} \cdot 3 = -9u^{-4}.$$

You need to write this in terms of x, so substitute $u = 3x - 2$.

$$y' = -9(3x-2)^{-4} = -\frac{9}{(3x-2)^4}$$

As your confidence builds, the last few lines are sufficient for showing your calculation of the derivative using the chain rule.

Example 7

Find $\frac{dy}{dt}$, given $y = (5t + 6)^4$.

Solution

This example demonstrates the minimal steps that need to be shown when finding the derivative of compositions using the chain rule. Only this minimum amount of work will be shown in the examples of the next lessons in this course.

$$\frac{dy}{dt} = 4(inside)^3 \cdot inside' \qquad [\text{where } inside = 5t + 6]$$

$$\frac{dy}{dt} = 4(5t+6)^3 \cdot 5$$

$$\frac{dy}{dt} = 20(5t+6)^3$$

Chain rule questions can get more complicated when you are asked to combine product rule or quotient rule. Let's explore this with the next few examples.

Example 8

Differentiate the following:

a) $y = \left(3x^2 + 2\right)^5 \left(2x^3 + 5\right)$

b) $y = \dfrac{\left(3x^4 + 2\right)^3}{2x^2 + 4}$

Solution

a) The question is a product of two functions, but only the first function will require the use of chain rule (since it is a composition of functions). Differentiate each function separately before applying the product rule.

Let $f(x) = \left(3x^2 + 2\right)^5$

$f'(x) = 5\left(3x^2 + 2\right)^4 (6x)$

Differentiate the composition using "outside-inside" strategy of the chain rule.

Let $g(x) = 2x^3 + 5$

$g'(x) = 6x^2$

Differentiate using the power rule.

Now, since:

$y = \left(3x^2 + 2\right)^5 \left(2x^3 + 5\right) = f(x) \cdot g(x)$

Then, $y' = f(x) \cdot g'(x) + f'(x) \cdot g(x)$

$y' = \left(3x^2 + 2\right)^5 \cdot 6x^2 + 5\left(3x^2 + 2\right)^4 (6x) \cdot \left(2x^3 + 5\right)$

Substitute the four components into the product rule.

It looks like a complicated expression but since the question did not ask you to simplify the derivative, stop here.

b) The question is a quotient of two functions, but only the numerator function will require the use of the chain rule since it is a composition of functions. Differentiate each function separately before applying the quotient rule.

Let $f(x) = \left(3x^4 + 2\right)^3$

$f'(x) = 3\left(3x^4 + 2\right)^2\left(12x^3\right)$

Differentiate the composition using the "outside-inside" strategy of the chain rule.

Let $g(x) = 2x^2 + 4$

$g'(x) = 4x^1$

Differentiate using the power rule.

Now, since:

$$y = \frac{\left(3x^4 + 2\right)^3}{2x^2 + 4} = \frac{f(x)}{g(x)}$$

Then, $y' = \dfrac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g^2(x)}$

$$y' = \frac{\left(2x^2 + 4\right) \cdot 3\left(3x^4 + 2\right)^2\left(12x^3\right) - \left(3x^4 + 2\right)^3 \cdot 4x}{\left(2x^2 + 4\right)^2}$$

Substitute the four components into the product rule.

From these last two examples, you can see why differentiating them separately can make your life easier. However, it is valid if you differentiate the function simultaneously while you are applying either the product rule or quotient rules.

Differentiating Higher Order Derivatives

The second derivative represents the slope of the first derivative function. The first derivative describes the slope of a function; the second derivative describes how quickly the slope of a function is changing. The second derivative can be written:

$$f''(x) = \frac{d}{dx} f'(x)$$

And the third derivative is the derivative of the second derivative, and so on. Higher order derivatives have applications in physics with displacement, velocity, acceleration, and jerk, all of which are related as rates of change.

Essentially if the rate of the change of the function is equal to the first derivative, then, by the same logic, the rate of change of the first derivative is equal to the second derivative.

In physics, the rate of change of displacement is called velocity, and the rate of change of velocity is called acceleration. The rate of change of acceleration is called jerk. If you have ever driven with someone who is learning to drive a car with a standard transmission, you will have experienced jerk since the acceleration is not changing smoothly. If you differentiate the displacement function twice, you would have the acceleration function. If you find the third derivative of displacement, you will have a function describing jerk.

Let $s(t)$ = displacement.

Then, $s'(t) = \frac{ds}{dt} = v(t) = \text{velocity}$.

And, $v'(t) = \frac{dv}{dt} = a(t) = \text{acceleration}$.

Or, $s''(t) = \frac{d}{dt}s'(t) = \frac{d^2s}{dt^2} = \text{acceleration}$.

In this course, you will often determine higher order derivatives, which are not in a particular context.

Example 9

The displacement of an object in motion is given by the function $f(x) = (4x^2 - 3)^3$. Determine functions to describe velocity, acceleration, and jerk.

Solution

velocity:	$f'(x) = 3(4x^2 - 3)^2(8x)$	Use the chain rule to determine first order derivative.
velocity:	$f'(x) = (4x^2 - 3)^2(24x)$	Simplify first derivative before determining second order derivative.
acceleration:	$f''(x) = (4x^2 - 3)^2 \cdot (24) + 2(4x^2 - 3)(8x) \cdot (24x)$	Determine the second order derivative using the product rule and the chain rule.
acceleration:	$f''(x) = 24(4x^2 - 3)^2 + (4x^2 - 3)(384x^2)$	Simplify second order derivative before determining third order derivative.
jerk:	$f'''(x) = 24 \cdot 2(4x^2 - 3)(8x) + (4x^2 - 3)(768x) + (8x)(384x^2)$	Determine the third order derivative using the product rule and the chain rule.

Higher order derivatives can be very challenging because the differentiation process requires a lot of simplification between steps.

Learning Activity 2.5

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

Given: $f(x) = \sqrt[3]{x}$ and $g(x) = \dfrac{1}{x-2}$

1. Evaluate: $g\left(f(27)\right)$

2. Evaluate: $f\left(g(10)\right)$

3. Determine: $f\left(g(x)\right)$

4. Determine: $g\left(f(x)\right)$

5. Determine the domain of $f(x) = \sqrt[3]{x}$.

6. Determine the domain of $g(x) = \dfrac{1}{x-2}$.

Define each function given in the following two questions in terms of the composition of an inside function, $g(x)$, and an outside function, $f(x)$, so that it is equivalent to $y = f(g(x))$.

7. $y = \sqrt{x^3 - 2}$

8. $y = (5x+1)^5$

continued

Learning Activity 2.5 (continued)

Part B: Differentiation with the Chain Rule and Higher Order Derivatives

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Differentiate the following using the chain rule (no need to simplify):

   a) $y = \left(6 - x^2\right)^4$

   b) $y = \sqrt{3x^2 + 6x - 7}$

2. Differentiate the following using the product rule and the chain rule (no need to simplify).

   a) $y = (4x - 3)^4 \cdot (5x + 2)$

   b) $y = \sqrt{3x - 7} \cdot \left(8 - x^2\right)$

3. Differentiate the following using the quotient rule and the chain rule (no need to simplify).

   a) $y = \dfrac{\sqrt{3x - 1}}{2x + 3}$

   b) $y = \dfrac{3x - 2}{\left(4x^2 - 5\right)^6}$

4. Determine the first and second order derivative of $f(x) = \sqrt{4x^2 - 3}$.

Lesson Summary

In this lesson, you learned how to differentiate composite functions using the chain rule. You have seen how the ability to differentiate with minimal simplification is efficient and strategic. The chain rule and the other differentiation rules will be applied frequently throughout this course.
In addition, you learned about some of the applications of higher order derivatives. The last differentiation topic that you will explore in this course is implicit differentiation, which is relevant for relations and functions in which the function equation may not be known but its derivative can be found.

Assignment 2.5

Differentiation with the Chain Rule and Higher Order Derivatives

Total: 17 marks

1. Differentiate the following compositions using the chain rule (no need to simplify):

   a) $y = \left(x - x^2\right)^3$ *(2 marks)*

   b) $y = \sqrt[4]{-x^2 + 2x - 7}$ *(2 marks)*

2. Differentiate $y = (4x + 1)^2 \cdot \left(8 - 3x^2\right)$ using the product rule and the chain rule as appropriate (no need to simplify). *(4 marks)*

continued

Assignment 2.5: Differentiation with the Chain Rule and Higher Order Derivatives (continued)

3. Differentiate $y = \dfrac{6x + 2}{\sqrt{7x^3 - 1}}$ using the quotient rule and the chain rule as appropriate (no need to simplify). *(4 marks)*

4. Determine the first and second order derivative of $f(x) = \sqrt[3]{3x^2 - 4}$. *(5 marks)*

LESSON 6: IMPLICIT DIFFERENTIATION

Lesson Focus

In this lesson, you will

- ❑ differentiate a relation implicitly
- ❑ determine the equation of a tangent line to a relation given a point
- ❑ determine higher order derivatives using implicit differentiation

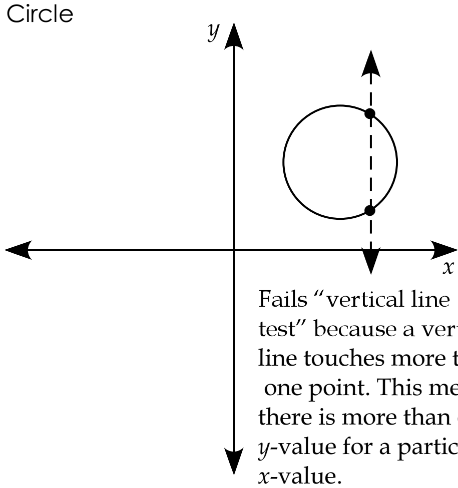
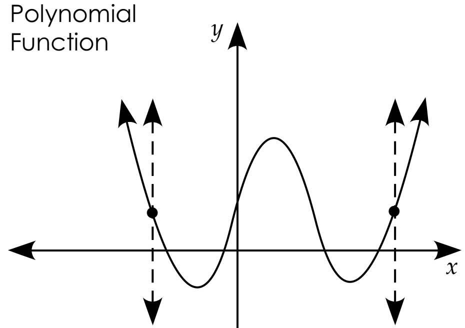
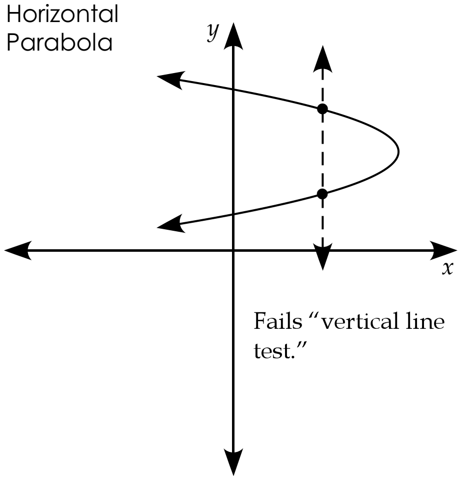
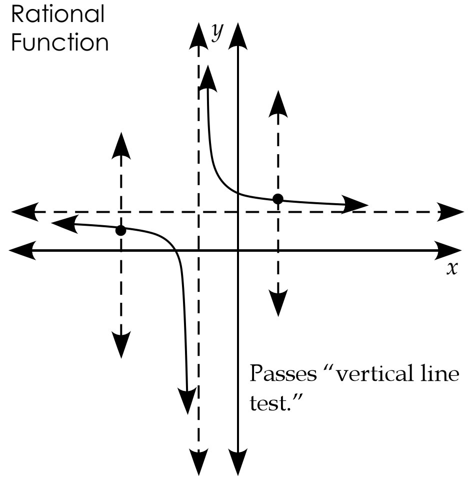
Lesson Introduction

In this lesson, you will learn how to use implicit differentiation when a relation is not explicitly defined. In other words, sometimes a relation does not have y isolated, so we cannot determine y' by differentiating explicitly using the rules you have learned. You will explore relations that are explicitly and implicitly defined and the implications on differentiation. In addition, you will learn how to determine the equation of a tangent line to a relation at a point after determining its slope function with implicit differentiation. Further, higher order derivatives will be determined using implicit differentiation techniques.

Relations

A **relation** describes the relationship between two variables, typically x and y. A relation is often defined in the form of an equation. A relation need not have a variable isolated in its equation. More specifically, a **function** is a relation that has one unique y for every x, and, as a result, it passes the **vertical line test**. Not all relations are functions but all functions are relations.

Below are some examples of relations and functions:

Relations	Functions
Circle  Fails "vertical line test" because a vertical line touches more than one point. This means there is more than one y-value for a particular x-value.	Polynomial Function  Passes "vertical line test" because any vertical line only touches one point. So, there is only one y-value for each x-value.
Horizontal Parabola  Fails "vertical line test."	Rational Function  Passes "vertical line test."

Implicit Versus Explicit Definitions

An **implicit** definition of a relation is an indirect definition for the value of y *in terms of* x; essentially, the dependent variable is not isolated to one side. For example, $x^2 + y^2 = 1$.

An **explicit** definition of a relation is a direct definition for the value of y *in terms of* x; particularly, the dependent variable, y, is isolated on one side of the equation directly stating what it equals to in terms of x. For example, $y = x^2 + 1$.

Relations are often defined implicitly because they represent curves such as circles or horizontal parabolas that have squared y-values. However, functions can sometimes be defined implicitly but then rewritten in explicit form.

	Implicit Definition	Explicit Definition
Function	$x^2 - 2x + 3y = 4$	$y = -\frac{1}{3}x^2 + \frac{2}{3}x + \frac{4}{3}$
Relation	$4x^2 + 4y^2 = 16$	$y = \pm\sqrt{4 - x^2}$ Note that y has more than one value for each x.

If you are asked to differentiate a relation that is defined implicitly, you would have to redefine it explicitly in terms of x, to use the differentiation rules as you have learned them, as shown in the example below.

Example 1

Differentiate $x^2 - 2x + y = 4$ with respect to x.

Solution

This relation is not explicitly defined in terms of x, so you need to isolate the y variable to define it explicitly in terms of x.

$$x^2 - 2x + y = 4 \longrightarrow y = -x^2 + 2x + 4$$

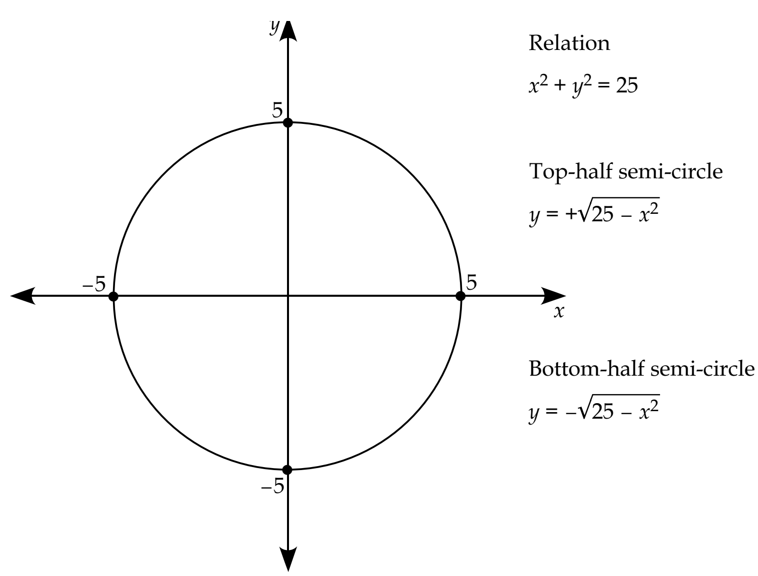
When the relation is explicitly defined in terms of x, you can differentiate y with respect to x using the power rule.

$$y = -x^2 + 2x + 4$$

$$\frac{dy}{dx} = -2x + 2$$

However, what happens when the relation cannot be explicitly defined as one unique function in terms of x, such as the relation below?

$$x^2 + y^2 = 25 \longrightarrow y^2 = 25 - x^2 \text{ or } y = \pm\sqrt{25 - x^2}$$



The graph of $x^2 + y^2 = 25$ is a circle, which is clearly not a function because it fails the vertical line test. However, when you rewrite the relation explicitly you get two functions: one for the top semi-circle, $y = +\sqrt{25 - x^2}$, and one for the bottom semi-circle, $y = -\sqrt{25 - x^2}$.

A dilemma then arises when you have to decide which of these to differentiate. Unless, you know the point of tangency, you don't know which to use. So this is why it is advantageous to have a strategy that allows you to differentiate the relation in its original implicitly defined form.

Fortunately, you can differentiate a relation implicitly without having to rewrite it. Implicit differentiation relies on your understanding of the chain rule. Leibniz notation, $\frac{dy}{dx}$, allows us to organize and display our work clearly but we could also use y' notation.

Implicit Differentiation

1. Differentiating **an expression** with respect to x.

$$\frac{d}{dx}(u) = \frac{du}{dy} \cdot \frac{dy}{dx}$$

2. Differentiate both sides of **an equation** with respect to x.

Use your differentiation rules,

then, isolate y' or $\frac{dy}{dx}$.

In the following example, you will see how the chain rule is used to differentiate implicitly with respect to x.

Example 2

Differentiate the following with respect to x where y is a function of x, but you don't have the $y = f(x)$ equation.

a) y^3

b) $3y^4$

c) $-2\sqrt{y}$

Solution

a) Since y is a function of x, use the chain rule to differentiate with respect to x.

$$\frac{d}{dx}\left(y^3\right)$$

$$= \frac{d}{dy}\left(y^3\right) \cdot \frac{dy}{dx}$$

$$= \left(3y^2\right) \cdot \frac{dy}{dx}$$

$$= 3y^2 \cdot y'$$

Use the chain rule with Leibniz notation to expand and let $u = y^3$:

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}$$

b) Since y is a function of x, use the chain rule to differentiate with respect to x.

$$\frac{d}{dx}\left(3y^4\right)$$

$$= \frac{d}{dy}\left(3y^4\right) \cdot \frac{dy}{dx}$$

$$= \left(12y^3\right) \cdot \frac{dy}{dx}$$

$$= 12y^3 \cdot y'$$

Use the chain rule with Leibniz notation to expand and let $u = 3y^4$:

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}$$

c) Rewrite the radical expression using a rational exponent and then use the chain rule to differentiate with respect to x.

$$\frac{d}{dx}\left(-2\sqrt{y}\right) = \frac{d}{dx}\left(-2(y)^{\frac{1}{2}}\right)$$

$$= \frac{d}{dy}\left(-2(y)^{\frac{1}{2}}\right) \cdot \frac{dy}{dx}$$

$$= -2\left(\frac{1}{2}y^{-\frac{1}{2}}\right) \cdot \frac{dy}{dx}$$

$$= -y^{-\frac{1}{2}} \cdot y'$$

Use the chain rule with Leibniz notation to expand and let $u = -2y^{\frac{1}{2}}$:

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}$$

Let's now practice using implicit differentiation with an equation and use the resulting expression to determine the equation of a tangent line.

Example 3

a) If $x^2 + y^2 = 25$, determine $\frac{dy}{dx}$.

b) Determine the equation of the tangent line to the circle $x^2 + y^2 = 25$ at $(-4, 3)$.

Solution

a) $\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(25)$

Differentiate each term on both sides with respect to x.

The derivative of the first term:

$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}\left(x^2\right) = 2x$$

The derivative of the second term:

To differentiate y^2, use chain rule since y is a function of x even though you haven't written it in the form "$y = \ldots$"

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dy}\left(y^2\right) \cdot \frac{dy}{dx} = 2y \cdot \frac{dy}{dx}$$

The derivative of the term on the right side:

$$\frac{d}{dx}(25) = 0$$

Therefore, put it all together:

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

$$2y \cdot \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

Notice that you have found a relation describing the slope of the original relation, $x^2 + y^2 = 25$, without ever writing the function in the form, $y = \ldots$ The resulting derivative is in terms of both x and y.

b) If $m_T = \frac{dy}{dx} = -\frac{x}{y}$ and $(-4, 3)$.

You can determine the slope of the tangent line by substituting the point of tangency into the derivative.

then $m_T = -\left(\frac{-4}{3}\right) = \frac{4}{3}$

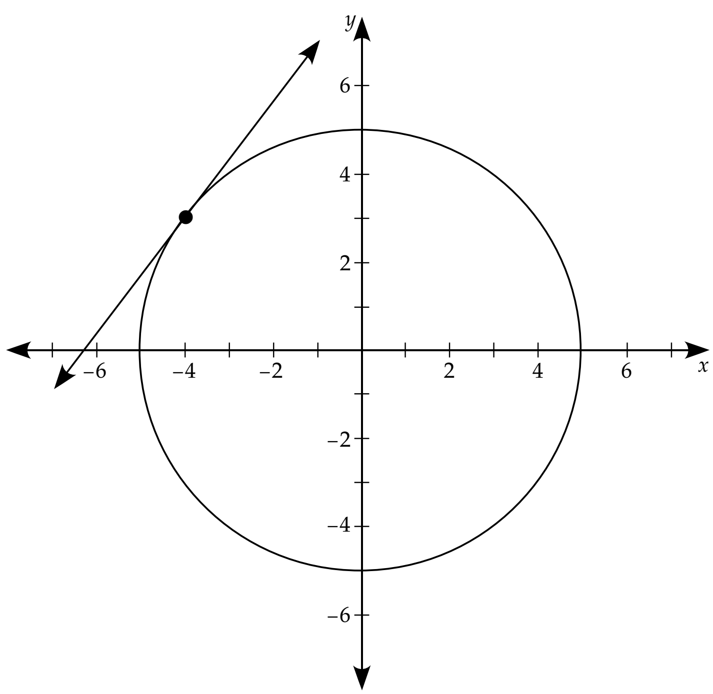
$$y = mx + b$$

$$3 = \frac{4}{3}(-4) + b$$

$$b = 3 + \frac{16}{3} = \frac{9}{3} + \frac{16}{3} = \frac{25}{3}$$

$$y = \frac{4}{3}x + \frac{25}{3}$$

Then to determine the equation of the tangent line, use the slope and the point of tangency to solve for the y-intercept = b.



Special Note: In order to differentiate the previous example explicitly, you would have had to rewrite the equation of the circle with y in terms of x. Then, choose which definition of the semi-circle to differentiate (positive or negative) in order to determine the equation of the tangent line. This is why implicit differentiation is a more efficient and preferred strategy.

What do you need to do if, when you are differentiating implicitly, there is more than one term with $y' = \frac{dy}{dx}$? You would need to collect like terms and isolate $y' = \frac{dy}{dx}$, as shown in the next example.

Example 4

Determine $y' = \frac{dy}{dx}$ for $x^2 + 3y^2 + x - 4y = 1$.

Solution

Determine the derivative of each term in the equation with respect to x. Use the chain rule to differentiate $3y^2$ with respect to x. That is,

$$3\left(2(inside)^1\right) \cdot inside' \text{ where } inside = y.$$

Term	x^2	$3y^2$	x	$-4y$	1
Derivative	$2x$	$6yy'$	1	$-4y'$	0

$$2x + 6yy' + 1 - 4y' = 0 \qquad \text{Put it all together.}$$

$$6yy' - 4y' = -2x - 1$$

$$y'(6y - 4) = -2x - 1 \qquad \text{Isolate } y'.$$

$$y' = \frac{-2x - 1}{6y - 4}$$

Let's explore implicit differentiation with the product rule in the next example.

Example 5

Use implicit differentiation to find y' for $3xy^2 - 4x - 2y^3 = 2$.

Solution

Determine the derivative of each term in the equation with respect to x.

Note: The product rule is required for the first term since both the $3x$ and the y^2 factors are explicit or implicit functions of x.

To differentiate $3xy^2$ with respect to x:

let $u = 3x$ and $v = y^2$

$$\frac{du}{dx} = 3 \text{ and } \frac{dv}{dx} = 2y \cdot y'$$

Now, use the product rule.

$u \cdot v' + u' \cdot v$

Term	$3xy^2$	$-4x$	$-2y^3$	2
Derivative	$3x \cdot 2yy' + 3 \cdot y^2$	-4	$-6y^2y'$	0

$$3x \cdot 2yy' + 3 \cdot y^2 - 4 - 6y^2y' = 0$$

$$6x\,yy' + 3y^2 - 4 - 6y^2y' = 0 \qquad \text{Put it all together.}$$

$$6x\,y\,y' - 6y^2y' = 4 - 3y^2$$

$$y'\left(6xy - 6y^2\right) = 4 - 3y^2 \qquad \text{Isolate } y'.$$

$$y' = \frac{4 - 3y^2}{6xy - 6y^2}$$

Relations can be differentiated implicitly using all the previous differentiation rules. In addition, we can explore higher order derivatives with implicit differentiation.

Higher Order Derivatives with Implicit Differentiation

The key to determining higher order derivatives is to simplify the first derivative before determining the second derivative, and so on.

Example 6

Determine y'' for $x^4 + y^3 = 6$ using implicit differentiation.

Solution

$$\frac{d}{dx}\left(x^4 + y^3 = 6\right)$$

$$4x^3 + 3y^2y' = 0$$ Differentiate implicitly with respect to x.

$$3y^2y' = -4x^3$$

$$y' = \frac{-4x^3}{3y^2}$$ Isolate the first derivative, y'.

$$\frac{d}{dx}\left(y' = \frac{-4x^3}{3y^2}\right)$$

$$y'' = \frac{\left(3y^2\right)\frac{d}{dx}\left(-4x^3\right) - \left(-4x^3\right)\frac{d}{dx}\left(3y^2\right)}{\left(3y^2\right)^2}$$

$$y'' = \frac{\left(3y^2\right)\left(-12x^2\right) - \left(-4x^3\right)\left(6yy'\right)}{\left(3y^2\right)^2}$$ Differentiate implicitly again with respect to x, using the quotient rule.

$$y'' = \frac{-36x^2y^2 + 24x^3yy'}{9y^4}$$ Simplify.

$$y'' = \frac{-36x^2y^2 + 24x^3y \cdot \left(\frac{-4x^3}{3y^2}\right)}{9y^4}$$

$$y'' = \frac{-36x^2y^2 - \frac{32x^6}{y}}{9y^4}$$

Substitute $y' = \frac{-4x^3}{3y^2}$ so that the second derivative is only in terms of x and y.

$$y'' = \left(\frac{-36x^2y^2 - \frac{32x^6}{y}}{9y^4}\right) \cdot \frac{y}{y}$$

Substitute the complex fraction by multiplying both numerator and denominator by y.

$$y'' = \frac{-36x^2y^3 - 32x^6}{9y^5}$$

The second derivative.

If you wanted to evaluate the second derivative at a specific point, the final simplified equation in the previous example would be ready for substitution.

Learning Activity 2.6

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

For Questions 1 to 4, differentiate with respect to x.

1. y^5

2. $9y^{-1}$

3. $\sqrt[3]{y}$

4. $-2\sqrt{y^3}$

For Questions 5 to 8, evaluate y' at the point $(-1, 2)$.

5. $y' = \dfrac{x}{y}$

6. $y' = \dfrac{-2x}{y^2}$

7. $y' = \dfrac{x + y}{2y}$

8. $y' = \dfrac{-y + 1}{x^2}$

continued

Learning Activity 2.6 (continued)

Part B: Implicit Differentiation

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Determine y' for the following using implicit differentiation.
   a) $x^2y^2 + 3y^3 + x^2 = 8$
   b) $(x + y)^2 - 3x = 2$
2. Determine the equation of the tangent line to $x^2 + xy + y^2 + x = 0$ at $(-1, 0)$.
3. Determine the second derivative of $x^2 + y^2 = 10x$ with respect to x in terms of x and y.

Lesson Summary

In this lesson, you learned about the differences between relations and functions. When relations were defined implicitly, you compared the difference of rewriting a relation explicitly and then differentiating to just differentiating implicitly. You explored various implicitly defined relations that required the use of other differentiation rules, such as the chain rule and product rule. You determined the equation of a tangent line to a relation at a specific point. Finally, you explored higher order differentiation with implicitly defined relations.

Notes

Assignment 2.6

Implicit Differentiation

Total: 19 marks

1. Determine y' for the following using implicit differentiation.

   a) $xy^2 - 4x^2 + 3xy = 5$ *(5 marks)*

   b) $y^2 = 100x$ *(2 marks)*

continued

Assignment 2.6: Implicit Differentiation (continued)

2. Determine the equation of the tangent line to $x^5y^3 + x^2y^8 = 0$ at $(1, -1)$. *(6 marks)*

continued

Assignment 2.6: Implicit Differentiation (continued)

3. Determine the second derivative of $2x^5 + 3y^8 = 0$ with respect to x using implicit differentiation in terms of x and y. *(6 marks)*

Notes

MODULE 2 SUMMARY

Congratulations, you have finished the second module in the course.

Submitting Your Assignments

It is now time for you to submit Assignments 2.1 to 2.6 to the Distance Learning Unit so that you can receive some feedback on how you are doing in this course. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 2 assignments and organize your material in the following order:

- ❑ Module 2 Cover Sheet (found at the end of the course Introduction)
- ❑ Assignment 2.1: Slope of the Tangent Line to a Curve
- ❑ Assignment 2.2: Definition of the Derivative
- ❑ Assignment 2.3: Basic Differentiation Rules
- ❑ Assignment 2.4: Differentiation with Product and Quotient Rules
- ❑ Assignment 2.5: Differentiation with the Chain Rule and Higher Order Derivatives
- ❑ Assignment 2.6: Implicit Differentiation

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Notes

Grade 12 Introduction to Calculus (45S)

Module 2
Derivatives

Learning Activity Answer Keys

Module 2: Derivatives

Learning Activity 2.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Evaluate: $\dfrac{2-(-5)}{-3-4}$

2. Evaluate: $\dfrac{-2-5}{3-(-4)}$

3. Simplify: $\dfrac{x^2-4}{x-2}$

4. Simplify: $\dfrac{x^2-9}{x+3}$

5. Determine the slope of a line that is perpendicular to a line with a slope of $-\dfrac{5}{4}$.

6. What are the slope and y-intercept of $y = -3x + 4$?
7. What is the slope of a line parallel to $y = -3x + 4$?
8. What is the slope of a line perpendicular to $y = -3x + 4$?

Answers:

1. $-1 \left(\dfrac{2+5}{-7} = \dfrac{7}{-7}\right)$

2. $-1 \left(\dfrac{-7}{7}\right)$

3. $x + 2 \; \left(\dfrac{(x-2)(x+2)}{x-2}\right)$

4. $x - 3 \; \left(\dfrac{(x-3)(x+3)}{x+3}\right)$

5. $\frac{4}{5}$

6. Since the equation of a line is written $y = mx + b$, then slope, m, is −3 and y-intercept, b, is 4.

7. The slopes of parallel lines are equal. Since the slope of $y = -3x + 4$ is $m = -3$, then the slope of a line parallel to it would also have a slope of $m = -3$.

8. The slopes of perpendicular lines are negative reciprocals of one another. Since the slope of $y = -3x + 4$ is $m = -3$, then the slope of a line perpendicular to it would have a slope of $m = +\frac{1}{3}$.

Part B: The Slope of the Tangent to a Curve

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Determine the equation of the line passing through the points (−2, 3) and (4, −1).

   Answer:

   In order to write the equation of a line as $y = mx + b$, you need to calculate the slope, m, and the y-intercept, b, using these two points.

   First, find the slope.

$$m = \frac{-1-3}{4-(-2)} = \frac{-4}{6} = -\frac{2}{3}$$

   Then, use the value of m and one of the (x, y) coordinates to find b.

$$\begin{aligned} y &= mx + b \\ -1 &= -\frac{2}{3}(4) + b \\ -1 &= \frac{-8}{3} + b \\ -1 + \frac{8}{3} &= b \\ -\frac{3}{3} + \frac{8}{3} &= b \\ b &= \frac{5}{3} \end{aligned}$$

So, $y = -\frac{2}{3}x + \frac{5}{3}$.

2. Determine the equation of the line passing through (0, 2) and perpendicular to the line $y = -3x + 4$.

   Answer:

   In order to determine the equation of the line we need to find the slope of the perpendicular line and then substitute the given point and perpendicular slope into the equation for b.

   Since the slope of $y = -3x + 4$ is $m = -3$, then the slope of a line perpendicular to it would be $+\frac{1}{3}$. Now, you need to find the value of b given the point (0, 2). The y-intercept, b, is 2.

   The equation of the line is $y = \frac{1}{3}x + 2$.

3. The point M(1, 0) lies on the curve $y = -x^2 + 1$.

   a) If $Q(x, -x^2 + 1)$ is a point on the curve, determine the slope of the secant lines MQ from (1, 0) to the points using the following values of x:

   Answer:

x	$Q(x, -x^2 + 1)$	Slope of MQ
2	(2, −3)	$m = \frac{0-(-3)}{1-2} = \frac{3}{-1} = -3$
1.5	(1.5, −1.25)	$m = \frac{0-(-1.25)}{1-1.5} = \frac{1.25}{-0.5} = -2.5$
1.1	(1.1, −0.21)	$m = \frac{0-(-0.21)}{1-1.1} = \frac{0.21}{-0.1} = -2.1$
1.01	(1.01, −0.0201)	$m = \frac{0-(-0.0201)}{1-1.01} = \frac{0.0201}{-0.01}$ $= -2.01$
1.001	(1.001, −0.002001)	$m = \frac{0-(-0.002001)}{1-1.001} = \frac{0.002001}{-0.001}$ $= -2.001$

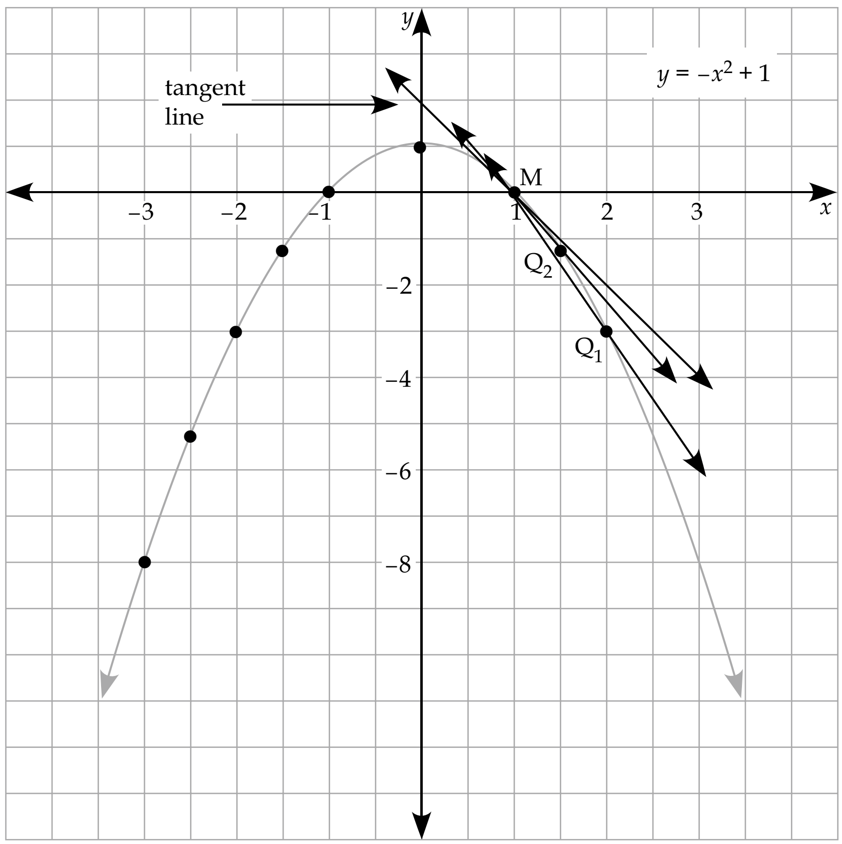
b) Use the slopes of the secant lines to estimate the slope of the tangent line at M(1, 0).

Answer:

As the x-values get closer to $x = 1$, the slope of the secant lines are approaching -2.

c) Sketch the curve, two secant lines, and the tangent line to M(1, 0).

Answer:



d) Use limits to calculate the slope of the tangent line at M(1, 0).

Answer:

$$\lim_{x \to 1} m = \lim_{x \to 1}\left(\frac{y_2 - y_1}{x_2 - x_1}\right) = \lim_{x \to 1}\left(\frac{\left(-x^2 + 1\right) - 0}{x - 1}\right) = \lim_{x \to 1}\left(\frac{-x^2 + 1}{x - 1}\right)$$

$$= \lim_{x \to 1}\left(\frac{(-x + 1)(x + 1)}{x - 1}\right) = \lim_{x \to 1}\left(\frac{-1(x - 1)(x + 1)}{x - 1}\right)$$

$$= \lim_{x \to 1}\left(-1(x + 1)\right) = -1(1 + 1) = -1(2) = -2$$

Learning Activity 2.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

Given: $f(x) = x^2 + 3x$ and $g(x) = \frac{1}{x}$

1. Evaluate: $f(2)$
2. Evaluate: $g(2)$
3. Evaluate: $f(2 + h)$
4. Evaluate: $g(2 + h)$
5. Simplify: $f(x + h)$
6. Simplify: $g(x + h)$
7. Simplify: $\frac{(x+h)^2 - x^2}{h}$
8. Rationalize numerator: $\frac{\sqrt{x+h} - \sqrt{x}}{h}$

Answers:

1. 10 (substitute $x = 2$ into $f(x)$, so $f(2) = (2)^2 + 3(2) = 4 + 6$)
2. $\frac{1}{2}$ (substitute $x = 2$ into $g(x)$, so $g(2) = \frac{1}{2}$)
3. $10 + 7h + h^2$ (substitute $x = 2 + h$ into $f(x)$, so $f(2+h) = (2+h)^2 + 3(2+h) = 4 + 4h + h^2 + 6 + 3h$)
4. $\frac{1}{2+h}$ (substitute $x = 2 + h$ into $g(x)$, so $g(x+h) = \frac{1}{2+h}$)
5. $x^2 + 2xh + h^2 + 3x + 3h$ (substitute $x = x + h$ into $f(x)$, so $f(x+h) = (x+h)^2 + 3(x+h)$)
6. $\frac{1}{x+h}$ (substitute $x = x + h = g(x)$, so $g(x+h) = \frac{1}{x+h}$)

7. $2x + h \left(\dfrac{x^2 + 2xh + h^2 - x^2}{h} = \dfrac{2xh + h^2}{h} = \dfrac{h(2x + h)}{h} \right)$

8. $\dfrac{1}{\sqrt{x+h} + \sqrt{x}} \left(\dfrac{\sqrt{x+h} - \sqrt{x}}{h} = \dfrac{\left(\sqrt{x+h} - \sqrt{x}\right)}{h} \cdot \dfrac{\left(\sqrt{x+h} + \sqrt{x}\right)}{\left(\sqrt{x+h} + \sqrt{x}\right)} \right.$

$$\left. = \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{1}{\sqrt{x+h} + \sqrt{x}} \right)$$

Part B: The Definition of the Derivative

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Determine the slope of the tangent line to the function $f(x) = x^2 + 3x$ at $x = 2$.

   Answer:

   To determine the slope of the tangent line, use $a = 2$ in

   $$m_T = \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right).$$

   $$m_T = \lim_{h \to 0} \left(\frac{f(2+h) - f(2)}{h} \right) = \lim_{h \to 0} \left(\frac{\left[(2+h)^2 + 3(2+h) \right] - \left[(2)^2 + 3(2) \right]}{h} \right)$$

   $$= \lim_{h \to 0} \left(\frac{[4 + 4h + h^2 + 6 + 3h] - [4 + 6]}{h} \right)$$

   $$= \lim_{h \to 0} \left(\frac{h^2 + 7h + 10 - 10}{h} \right) = \lim_{h \to 0} \left(\frac{h^2 + 7h}{h} \right)$$

   $$= \lim_{h \to 0} \left(\frac{h(h+7)}{h} \right) = \lim_{h \to 0} (h + 7)$$

   $$= 0 + 7 = 7$$

   The slope of $f(x) = x^2 + 3x$ at $x = 2$ is 7.

2. Use the limit of the difference quotient to determine the derivative function, $f'(x)$, of $f(x) = \frac{1}{x+1}$.

Answer:

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

Determine parts of the difference quotient first.

$$f(x) = \frac{1}{x+1} \text{ and } f(x+h) = \frac{1}{x+h+1}$$

Substitute.

$$f'(x) = \lim_{h \to 0} \left(\frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} \right) = \frac{0}{0} = \text{I.F.}$$

Limit needs to be simplified because it is in the indeterminate form.

Multiply the numerator and denominator by the LCD in order to simplify the limit ratio. LCD = $(x + h + 1)(x + 1)$

$$f'(x) = \lim_{h \to 0} \left(\frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} \right) \cdot \frac{(x+h+1)(x+1)}{(x+h+1)(x+1)}$$

$$= \lim_{h \to 0} \left(\frac{(x+h+1)(x+1)\left(\frac{1}{x+h+1}\right) - (x+h+1)(x+1)\left(\frac{1}{x+1}\right)}{h(x+h+1)(x+1)} \right)$$

$$= \lim_{h \to 0} \left(\frac{(x+1) - (x+h+1)}{h(x+h+1)(x+1)} \right)$$

Simplify the complex fraction by cancelling out like factors in the numerator.

$$= \lim_{h \to 0} \left(\frac{x+1-x-h-1}{h(x+h+1)(x+1)} \right)$$

$$= \lim_{h \to 0} \left(\frac{-h}{h(x+h+1)(x+1)} \right)$$

Simplify the numerator and leave the denominator unsimplified to be ready to cancel h in the denominator.

$$f'(x) = \lim_{h \to 0} \left(\frac{-1}{(x+h+1)(x+1)} \right)$$

Reduce.

$$f'(x) = \frac{-1}{(x+0+1)(x+1)} = \frac{-1}{(x+1)(x+1)} = \frac{-1}{(x+1)^2}$$

Evaluate the limit at $h = 0$.

The derivative of $f(x)$ is $f'(x) = \frac{1}{(x+1)^2}$.

3. Use the limit of the difference quotient to determine the derivative of $f(x) = \sqrt{x-1}$ and the slope of $f(x)$ at $x = 5$.

Answer:

First determine the derivative and then substitute $x = 5$ into it.

$f(x) = \sqrt{x-1}$ and

$f(x+h) = \sqrt{(x+h)-1} = \sqrt{x+h-1}$

State $f(x)$ and simplify $f(x + h)$ first.

$$f'(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

$$= \lim_{h \to 0}\left(\frac{\sqrt{x+h-1} - \sqrt{x-1}}{h}\right)$$

Substitute $f(x)$ and $f(x + h)$ into the definition of the derivative.

$$= \lim_{h \to 0}\left(\left(\frac{\sqrt{x+h-1} - \sqrt{x-1}}{h}\right) \cdot \left(\frac{\sqrt{x+h-1} + \sqrt{x-1}}{\sqrt{x+h-1} + \sqrt{x-1}}\right)\right)$$

Multiply numerator and denominator by the conjugate $\sqrt{x+h-1} + \sqrt{x-1}$.

$$= \lim_{h \to 0}\left(\frac{(x+h-1) - (x-1)}{h\left(\sqrt{x+h-1} + \sqrt{x-1}\right)}\right)$$

$$= \lim_{h \to 0}\left(\frac{x+h-1-x+1}{h\left(\sqrt{x+h-1} + \sqrt{x-1}\right)}\right)$$

Simplify the numerator but do not simplify the denominator.

$$= \lim_{h \to 0}\left(\frac{h}{h\left(\sqrt{x+h-1} + \sqrt{x-1}\right)}\right)$$

$$= \lim_{h \to 0}\left(\frac{1}{\left(\sqrt{x+h-1} + \sqrt{x-1}\right)}\right)$$

Reduce.

$$f'(x) = \frac{1}{\sqrt{x+0-1} + \sqrt{x-1}}$$

$$= \frac{1}{\sqrt{x-1} + \sqrt{x-1}}$$

$$= \frac{1}{2\sqrt{x-1}}$$

Evaluate the limit at $h = 0$.

$$f'(5) = \frac{1}{2\sqrt{5-1}} = \frac{1}{2\sqrt{4}} = \frac{1}{2(2)} = \frac{1}{4}$$

Substitute $x = 5$ into the derivative function.

The slope of $f(x)$ at $x = 5$ is $\frac{1}{4}$.

Learning Activity 2.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

For Questions 1 to 4, write the monomials using rational exponents.

1. $\sqrt{x^3}$
2. $\sqrt[4]{x}$
3. $\left(3\sqrt[3]{x}\right)^2$
4. $-2\left(\sqrt[4]{x}\right)^3$

For Questions 5 to 8, rewrite the expression using negative exponents.

5. $\dfrac{-5}{x}$
6. $\dfrac{7}{2x^5}$
7. $\dfrac{1}{2\sqrt{x}}$
8. $\dfrac{3}{4\sqrt[3]{x}}$

Answers:

1. $x^{\frac{3}{2}}$ $\left(\left(x^3\right)^{\frac{1}{2}} = x^{3\cdot\frac{1}{2}} = x^{\frac{3}{2}}\right)$
2. $x^{\frac{1}{4}}$
3. $9x^{\frac{2}{3}}$ $\left(\left(3\cdot x^{\frac{1}{3}}\right)\cdot\left(3\cdot x^{\frac{1}{3}}\right) = 9\cdot x^{\frac{1}{3}+\frac{1}{3}}\right)$
4. $-2x^{\frac{3}{4}}$ $\left(-2\cdot\left(x^{\frac{1}{4}}\right)^3 = -2x^{\frac{3}{4}}\right)$
5. $-5x^{-1}$ $\left(-5\cdot x^{-1}\right)$
6. $\dfrac{7}{2}x^{-5}$ $\left(\dfrac{7}{2x^5} = \dfrac{7}{2}\cdot x^{-5}\right)$
7. $\dfrac{1}{2}x^{-\frac{1}{2}}$ $\left(\dfrac{1}{2x^{\frac{1}{2}}} = \dfrac{1}{2}x^{-\frac{1}{2}}\right)$

8. $\frac{3}{4}x^{-\frac{1}{3}} \left(\frac{3}{4x^{\frac{1}{3}}} = \frac{3}{4}x^{-\frac{1}{3}} \right)$

Part B: Basic Differentiation Rules

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Differentiate the following using the power rule and derivative properties:

Note: For these questions, all of these steps do not need to be shown. Also, the notation used can be $\frac{dy}{dx}$ or y'.

a) $y = 5x + 7$

Answer:

$$\frac{dy}{dx} = \frac{d}{dx}(5x + 7) = \frac{d}{dx}(5x) + \frac{d}{dx}(7)$$

$$= 5 \cdot \frac{d}{dx}\left(x^1\right) + 0 = 5 \cdot \left(1x^{1-1}\right) = 5 \cdot x^0 = 5 \cdot 1 = 5$$

b) $y = 4x^2 - 10$

Answer:

$$\frac{dy}{dx} = \frac{d}{dx}\left(4x^2 - 10\right) = 4 \cdot (2x) - 0 = 8x$$

c) $y = \frac{1}{3}x^3 - 3x^2 - \frac{1}{2}x + 2$

Answer:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{3}x^3 - 3x^2 - \frac{1}{2}x + 2\right)$$

$$\frac{dy}{dx} = \frac{1}{3} \cdot \left(3x^2\right) - 3 \cdot (2x) - \frac{1}{2} \cdot (1) + 0 = x^2 - 6x - \frac{1}{2}$$

d) $y = -2x^{\frac{3}{2}} + 2\sqrt{x}$

Answer:

$$y = -2x^{\frac{3}{2}} + 2\sqrt{x} = -2x^{\frac{3}{2}} + 2(x)^{\frac{1}{2}}$$

$$y' = -2 \cdot \left(\frac{3}{2}x^{\frac{1}{2}}\right) + 2 \cdot \left(\frac{1}{2}x^{-\frac{1}{2}}\right) = -3x^{\frac{1}{2}} + x^{-\frac{1}{2}}$$

or

$$y' = -3\sqrt{x} + \frac{1}{\sqrt{x}}$$

e) $y = \frac{1}{x} + x^{\frac{3}{4}} + 2\sqrt[3]{x}$

Answer:

$$y = \frac{1}{x} + x^{\frac{3}{4}} + 2\sqrt[3]{x} = x^{-1} + x^{\frac{3}{4}} + 2(x)^{\frac{1}{3}}$$

$$y' = -1x^{-2} + \frac{3}{4}x^{-\frac{1}{4}} + 2 \cdot \left(\frac{1}{3}x^{-\frac{2}{3}}\right) = -x^{-2} + \frac{3}{4}x^{-\frac{1}{4}} + \frac{2}{3}x^{-\frac{2}{3}}$$

or

$$y' = \frac{-1}{x^2} + \frac{3}{4\sqrt[4]{x}} + \frac{2}{3\sqrt[3]{x^2}}$$

f) $y = \frac{2}{\sqrt{x}} + 4\sqrt{x^3}$

Answer:

$$y = 2x^{-\frac{1}{2}} + 4x^{\frac{3}{2}}$$

$$y' = 2 \cdot \left(\frac{-1}{2}x^{-\frac{3}{2}}\right) + 4 \cdot \left(\frac{3}{2}x^{\frac{1}{2}}\right)$$

$$y' = -x^{-\frac{3}{2}} + 6x^{\frac{1}{2}}$$

or

$$y' = -\frac{1}{\sqrt{x^3}} + 6\sqrt{x}$$

Learning Activity 2.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

For Questions 1 to 4, simplify into a monomial with positive rational exponents.

1. $\sqrt{x} \cdot \sqrt[3]{x}$
2. $(x^{-2})(5x^3)$
3. $\left(\dfrac{x^3}{\sqrt[4]{x}}\right)^2$
4. $\dfrac{\sqrt[3]{x^5}}{\sqrt{x}}$

For Questions 5 to 8, evaluate without the use of a calculator.

5. $\sqrt[3]{(27)^2}$
6. $\dfrac{\sqrt{50}}{\sqrt{2}}$
7. $(1 - 2^3)^2$
8. $\dfrac{3^2 - 1}{\sqrt[4]{16}}$

Answers:

1. $x^{\frac{5}{6}} \left(x^{\frac{1}{2}} \cdot x^{\frac{1}{3}} = x^{\frac{1}{2}+\frac{1}{3}} = x^{\frac{3}{6}+\frac{2}{6}} = x^{\frac{5}{6}} \right)$

2. $5x \; \left(5 \cdot x^{-2+3} = 5x^1\right)$

3. $x^{\frac{11}{2}} \left(\left(\dfrac{x^3}{\sqrt[4]{x}}\right)^2 = \dfrac{x^{3\times 2}}{x^{\frac{1}{4}\times 2}} = x^{6-\frac{1}{2}} = x^{\frac{12}{2}-\frac{1}{2}} = x^{\frac{11}{2}} \right)$

4. $x^{\frac{7}{6}} \left(\dfrac{\sqrt[3]{x^5}}{\sqrt{x}} = \dfrac{x^{\frac{5}{3}}}{x^{\frac{1}{2}}} = x^{\frac{5}{3}-\frac{1}{2}} = x^{\frac{10}{6}-\frac{3}{6}} = x^{\frac{7}{6}} \right)$

5. $9 \; \left(\sqrt[3]{(27)^2} = \left(\sqrt[3]{27}\right)^2 = 3^2 \right)$

6. $5 \left(\dfrac{\sqrt{50}}{\sqrt{2}} = \sqrt{\dfrac{50}{2}} = \sqrt{25} \right)$

7. $49 \left(\left(1 - 2^3\right)^2 = (1-8)^2 = (-7)^2 = +49 \right)$

8. $4 \left(\frac{3^2 - 1}{\sqrt[4]{16}} = \frac{9-1}{2} = \frac{8}{2} = 4 \right)$

Part B: Differentiation with Product and Quotient Rules

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Differentiate the following without simplifying the derivative:

   a) $f(x) = \left(3x^3 + 4\right)\left(1 - 2x^3\right)$

   Answer:

$$\begin{aligned} f'(x) &= \left(3x^3 + 4\right) \cdot \frac{d}{dx}\left(1 - 2x^3\right) + \frac{d}{dx}\left(3x^3 + 4\right) \cdot \left(1 - 2x^3\right) \\ &= \left(3x^3 + 4\right) \cdot \left(-6x^2\right) + \left(9x^2\right) \cdot \left(1 - 2x^3\right) \end{aligned}$$

   b) $g(x) = \left(x - \sqrt{x}\right)\left(x^2 + \sqrt{x}\right)$

   Answer:

$$\begin{aligned} g'(x) &= \left(x - \sqrt{x}\right) \cdot \frac{d}{dx}\left(x^2 + \sqrt{x}\right) + \frac{d}{dx}\left(x - \sqrt{x}\right) \cdot \left(x^2 + \sqrt{x}\right) \\ &= \left(x - \sqrt{x}\right) \cdot \left(2x + \frac{1}{2}x^{-\frac{1}{2}}\right) + \left(1 - \frac{1}{2}x^{-\frac{1}{2}}\right) \cdot \left(x^2 + \sqrt{x}\right) \end{aligned}$$

c) $h(x) = \dfrac{6 + x^{-2}}{8x^{10} - 5x^3}$

Answer:

$$h'(x) = \frac{\left(8x^{10} - 5x^3\right) \cdot \frac{d}{dx}\left(6 + x^{-2}\right) - \left(6 + x^{-2}\right) \cdot \frac{d}{dx}\left(8x^{10} - 5x^3\right)}{\left(8x^{10} - 5x^3\right)^2}$$

$$= \frac{\left(8x^{10} - 5x^3\right) \cdot \left(-2x^{-3}\right) - \left(6 + x^{-2}\right) \cdot \left(80x^9 - 15x^2\right)}{\left(8x^{10} - 5x^3\right)^2}$$

d) $m(x) = \dfrac{x^2 + \sqrt{x}}{1 - 2x^3}$

Answer:

$$m'(x) = \frac{\left(1 - 2x^3\right) \cdot \frac{d}{dx}\left(x^2 + \sqrt{x}\right) - \left(x^2 + \sqrt{x}\right) \cdot \frac{d}{dx}\left(1 - 2x^3\right)}{\left(1 - 2x^3\right)^2}$$

$$= \frac{\left(1 - 2x^3\right) \cdot \left(2x + \frac{1}{2}x^{-\frac{1}{2}}\right) - \left(x^2 + \sqrt{x}\right) \cdot \left(-6x^2\right)}{\left(1 - 2x^3\right)^2}$$

2. Differentiate the following and simplify the derivative:

a) $f(x) = \left(2x^4 + 4x\right)\left(x^{-1} - 3x^2\right)$

Answer:

$$f'(x) = \left(2x^4 + 4x\right) \cdot \frac{d}{dx}\left(x^{-1} - 3x^2\right) + \frac{d}{dx}\left(2x^4 + 4x\right) \cdot \left(x^{-1} - 3x^2\right)$$

$$= \left(2x^4 + 4x\right) \cdot \left(-x^{-2} - 6x\right) + \left(8x^3 + 4\right) \cdot \left(x^{-1} - 3x^2\right)$$

$$= -2x^2 - 12x^5 - 4x^{-1} - 24x^2 + 8x^2 - 24x^5 + 4x^{-1} - 12x^2$$

$$= -36x^5 - 30x^2$$

b) $f(x) = \dfrac{3\sqrt{x} + 4}{1 - 2x^3}$

Answer:

$$f'(x) = \frac{\left(1 - 2x^3\right) \cdot \frac{d}{dx}\left(3\sqrt{x} + 4\right) - \left(3\sqrt{x} + 4\right) \cdot \frac{d}{dx}\left(1 - 2x^3\right)}{\left(1 - 2x^3\right)^2}$$

$$= \frac{\left(1 - 2x^3\right) \cdot \left(\frac{3}{2}x^{-\frac{1}{2}}\right) - \left(3\sqrt{x} + 4\right) \cdot \left(-6x^2\right)}{\left(1 - 2x^3\right)^2}$$

$$= \frac{\frac{3}{2}x^{-\frac{1}{2}} - 3x^{-\frac{1}{2}+3} - \left(-18x^{\frac{1}{2}+2} - 24x^2\right)}{\left(1 - 2x^3\right)^2} = \frac{\frac{3}{2}x^{-\frac{1}{2}} - 3x^{\frac{5}{2}} + 18x^{\frac{5}{2}} + 24x^2}{\left(1 - 2x^3\right)^2}$$

$$f'(x) = \frac{\frac{3}{2}x^{-\frac{1}{2}} + 15x^{\frac{5}{2}} + 24x^2}{\left(1 - 2x^3\right)^2}$$

3. Determine the equation of the tangent line to $h(x) = \left(1 + x^{-2}\right)\left(8x^{10} - 5x^3\right)$ at $(1, 6)$.

Answer:

Differentiate using the product rule.

$$h'(x) = \left(1 + x^{-2}\right) \cdot \frac{d}{dx}\left(8x^{10} - 5x^3\right) + \frac{d}{dx}\left(1 + x^{-2}\right) \cdot \left(8x^{10} - 5x^3\right)$$

$$= \left(1 + x^{-2}\right) \cdot \left(80x^9 - 15x^2\right) + \left(-2x^{-3}\right) \cdot \left(8x^{10} - 5x^3\right)$$

Determine the slope of the tangent using $m_T = h'(1)$.

$$h'(1) = \left(1 + (1)^{-2}\right) \cdot \left(80(1)^9 - 15(1)^2\right) + \left(-2(1)^{-3}\right) \cdot \left(8(1)^{10} - 5(1)^3\right)$$

$$= (1 + 1) \cdot (80 - 15) + (-2) \cdot (8 - 5) = (2)(65) + (-2)(3) = 130 - 6$$

$$m_T = h'(1) = 124$$

Determine the equation of the tangent line through (1, 6) with slope $m = 128$, using the point-slope form: $y - y_1 = m(x - x_1)$.

The equation of the tangent line is:

$$y - 6 = 124(x - 1)$$

Alternatively, you could use the slope-intercept form:

$$y = mx + b$$

Determine the y-intercept of the tangent line using the slope and the given point.

$$y = mx + b$$

$$6 = 124(1) + b$$

$$b = 6 - 124 = -118$$

Equation of the tangent line is:

$$y = 124x - 118$$

Either form of the equation is acceptable.

Learning Activity 2.5

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

Given: $f(x) = \sqrt[3]{x}$ and $g(x) = \dfrac{1}{x-2}$

1. Evaluate: $g(f(27))$
2. Evaluate: $f(g(10))$
3. Determine: $f(g(x))$
4. Determine: $g(f(x))$
5. Determine the domain of $f(x) = \sqrt[3]{x}$.
6. Determine the domain of $g(x) = \dfrac{1}{x-2}$.

Define each function given in the following two questions in terms of the composition of an inside function, $g(x)$, and an outside function, $f(x)$, so that it is equivalent to $y = f(g(x))$.

7. $y = \sqrt{x^3 - 2}$
8. $y = (5x+1)^5$

Answers:

1. $1 \left(g\left(\sqrt[3]{27}\right) = g(3) = \dfrac{1}{3-2} = \dfrac{1}{1} \right)$
2. $\dfrac{1}{2} \left(f\left(\dfrac{1}{8}\right) = \sqrt[3]{\dfrac{1}{8}} = \dfrac{1}{2} \right)$
3. $\sqrt[3]{\dfrac{1}{x-2}} \left(f\left(\dfrac{1}{x-2}\right) = \sqrt[3]{\dfrac{1}{x-2}} \right)$

4. $\frac{1}{\sqrt[3]{x}-2}$ $\left(g\left(\sqrt[3]{x}\right)=\frac{1}{\sqrt[3]{x}-2}\right)$

5. Since we can cube root positive and negative numbers, then the domain is all real numbers.

6. Since we cannot divide by the zero, the non-permissible value is $x = 2$, and the domain is $\{x \mid x \neq 2, x \in \Re\}$.

7. $g(x) = x^3 - 2$

   $f(x) = \sqrt{x}$

8. $g(x) = 5x + 1$

   $f(x) = x^5$

Part B: Differentiation with the Chain Rule and Higher Order Derivatives

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Differentiate the following compositions using the chain rule (no need to simplify).

   a) $y = \left(6 - x^2\right)^4$

   Answer:

   Use the outside-inside strategy when using the chain rule.

   $y' = 4(inside)^3 \cdot inside'$

   $y' = 4\left(6 - x^2\right)^3(-2x)$

b) $y = \sqrt{3x^2 + 6x - 7}$

Answer:

Use the outside-inside strategy when using the chain rule.

$$y = \left(3x^2 + 6x - 7\right)^{\frac{1}{2}}$$

$$y' = \frac{1}{2}(inside)^{-\frac{1}{2}} \cdot inside'$$

$$y' = \frac{1}{2}\left(3x^2 + 6x - 7\right)^{-\frac{1}{2}} \cdot (6x + 6)$$

2. Differentiate the following using the product rule and the chain rule as appropriate (no need to simplify).

a) $y = (4x - 3)^4 \cdot (5x + 2)$

Answer:

Use the product rule and chain rule to differentiate.

Let $y = u \cdot v$, $u = (4x - 3)^4$, and $v = (5x + 2)$.	Rewrite as a product of two functions.
$u' = 4(4x - 3)^3(4)$	Differentiate composition $u = (4x - 3)^4$ using the chain rule.
$v' = 5$	Differentiate $v = (5x + 2)$ using the power rule.
$y' = u \cdot v' + u' \cdot v$	The product rule.
$y' = (4x - 3)^4 \cdot 5 + 4(4x - 3)^3(4) \cdot (5x + 2)$	The derivative.

b) $y = \sqrt{3x - 7} \cdot \left(8 - x^2\right)$

Answer:

Use the product rule and the chain rule to differentiate.

Let $y = u \cdot v$, $u = \sqrt{3x - 7}$, and $v = \left(8 - x^2\right)$.	Rewrite as a product of two functions.
$u = (3x - 7)^{\frac{1}{2}}$ $u' = \frac{1}{2}(3x - 7)^{-\frac{1}{2}}(3)$	Differentiate composition $u = \sqrt{3x - 7}$ using the chain rule.
$v' = -2x$	Differentiate $v = (8 - x^2)$ using the power rule.
$y' = u \cdot v' + u' \cdot v$	The product rule.
$y' = (3x - 7)^{\frac{1}{2}} \cdot (-2x) + \frac{1}{2}(3x - 7)^{-\frac{1}{2}}(3) \cdot \left(8 - x^2\right)$	The derivative.

3. Differentiate the following using the quotient rule and the chain rule as appropriate (no need to simplify),

a) $y = \dfrac{\sqrt{3x - 1}}{2x + 3}$

Answer:

Use the quotient rule and the chain rule to differentiate.

Let $y = \dfrac{u}{v}$, $u = \sqrt{3x - 1}$, and $v = 2x + 3$.	Rewrite as a quotient of two functions.
$u = (3x - 1)^{\frac{1}{2}}$ $u' = \dfrac{1}{2}(3x - 1)^{-\frac{1}{2}}(3)$	Differentiate $u = \sqrt{3x - 1}$ using the chain rule.
$v' = 2$	Differentiate $v = 2x + 3$ using the power rule.
$y' = \dfrac{v \cdot u' - u \cdot v'}{v^2}$	The quotient rule.
$y' = \dfrac{(2x + 3) \cdot \frac{1}{2}(3x - 1)^{-\frac{1}{2}}(3) - (3x - 1)^{\frac{1}{2}} \cdot 2}{(2x + 3)^2}$	The derivative.

b) $y = \dfrac{3x - 2}{\left(4x^2 - 5\right)^6}$

Answer:

Use the quotient rule and chain rule to differentiate.

Let $y = \dfrac{u}{v}$, $u = 3x - 2$, and $v = \left(4x^2 - 5\right)^6$.	Rewrite as a quotient of two functions.
$u' = 3$	Differentiate $u = 3x - 2$ using the chain rule.
$v' = 6\left(4x^2 - 5\right)^5(8x)$	Differentiate $v = (4x^2 - 5)^6$ using the power rule.
$y' = \dfrac{v \cdot u' - u \cdot v'}{v^2}$	The quotient rule.
$y' = \dfrac{\left(4x^2 - 5\right)^6 \cdot 3 - (3x - 2) \cdot 6\left(4x^2 - 5\right)^5(8x)}{\left(\left(4x^2 - 5\right)^6\right)^2}$	The derivative.

4. Determine the first and second order derivative of $f(x) = \sqrt{4x^2 - 3}$.

Answer:

$$f(x) = \sqrt{4x^2 - 3} = \left(4x^2 - 3\right)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}\left(4x^2 - 3\right)^{-\frac{1}{2}}(8x)$$

Use the chain rule to determine the first order derivative.

$$f'(x) = \left(4x^2 - 3\right)^{-\frac{1}{2}}(4x)$$

Simplify the first derivative before determining the second order derivative.

$$f''(x) = \left(4x^2 - 3\right)^{-\frac{1}{2}} \cdot (4) + \left(-\frac{1}{2}\right)\left(4x^2 - 3\right)^{-\frac{3}{2}}(8x) \cdot (4x)$$

Determine the second order derivative using the product rule and chain rule.

Learning Activity 2.6

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

For Questions 1 to 4, differentiate with respect to x.

1. y^5
2. $9y^{-1}$
3. $\sqrt[3]{y}$
4. $-2\sqrt{y^3}$

For Questions 5 to 8, evaluate y' at the point (–1, 2).

5. $y' = \dfrac{x}{y}$
6. $y' = \dfrac{-2x}{y^2}$
7. $y' = \dfrac{x+y}{2y}$
8. $y' = \dfrac{-y+1}{x^2}$

Answers:

1. $5y^4 \cdot y' \quad \left(\dfrac{d}{dx}\left(y^5\right) = \dfrac{d}{dy}\left(y^5\right)\cdot\dfrac{dy}{dx} = 5y^4\cdot\dfrac{dy}{dx}\right)$

2. $-9y^{-2} \cdot y' \quad \left(\dfrac{d}{dx}\left(9y^{-1}\right) = \dfrac{d}{dy}\left(9y^{-1}\right)\cdot\dfrac{dy}{dx} = -9y^{-2}\cdot\dfrac{dy}{dx}\right)$

3. $\dfrac{1}{3}y^{-\frac{2}{3}} \cdot y' \quad \left(\dfrac{d}{dx}\left(y^{\frac{1}{3}}\right) = \dfrac{1}{3}y^{-\frac{2}{3}}y'\right)$

4. $-3\sqrt{y} \cdot y' \quad \left(\dfrac{d}{dx}\left(-2\sqrt{y^3}\right) = \dfrac{d}{dx}\left(-2y^{\frac{3}{2}}\right) = -2\left(\dfrac{3}{2}\right)y^{\frac{1}{2}}y' = -3y^{1/2}y'\right)$

5. $-\dfrac{1}{2} \quad \left(y' = \dfrac{-1}{2}\right)$

6. $\dfrac{1}{2} \quad \left(y' = \dfrac{-2(-1)}{(2)^2} = \dfrac{2}{4}\right)$

7. $\dfrac{1}{4} \quad \left(y' = \dfrac{-1+2}{2(2)}\right)$

8. $-1\left(y' = \dfrac{-2+1}{(-1)^2} = \dfrac{-1}{1}\right)$

Part B: Implicit Differentiation

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Determine y' for the following using implicit differentiation.

   a) $x^2y^2 + 3y^3 + x^2 = 8$

   Answer:

Term	x^2y^2	$3y^2$	x^2	8
Derivative	$x^2 \cdot 2yy' + 2x \cdot y^2$	$9y^2y'$	$2x$	0

$$x^2 \cdot 2yy' + 2x \cdot y^2 + 9y^2y' + 2x = 0$$
$$x^2 \cdot 2yy' + 9y^2y' = -2x - 2xy^2$$
$$y'\left(x^2 \cdot 2y + 9y^2\right) = -2x - 2xy^2$$
$$y' = \frac{-2x - 2xy^2}{2x^2y + 9y^2}$$

   b) $(x+y)^2 - 3x = 2$

   Answer:

Term	$(x+y)^2$	$-3x$	2
Derivative	$2(x+y)(1+y')$	-3	0

$$2(x+y)(1+y') - 3 = 0$$
$$(2x+2y)(1+y') = 3$$
$$2x + 2xy' + 2y + 2yy' = 3$$
$$2xy' + 2yy' = 3 - 2x - 2y$$
$$y'(2x+2y) = 3 - 2x - 2y$$
$$y' = \frac{3 - 2x - 2y}{2x + 2y}$$

2. Determine the equation of the tangent line to $x^2 + xy + y^2 + x = 0$ at $(-1, 0)$.

Answer:

Determine $\frac{dy}{dx}$ for this relation.

$$\frac{d}{dx}\left(x^2 + xy + y^2 + x = 0\right)$$

$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(xy\right) + \frac{d}{dx}\left(y^2\right) + \frac{d}{dx}(x) = \frac{d}{dx}(0)$$

$$2x + \left(x \cdot y' + 1 \cdot y\right) + 2yy' + 1 = 0$$

$$2x + xy' + y + 2yy' + 1 = 0$$

$$xy' + 2yy' = -1 - 2x - y$$

$$y'\left(x + 2y\right) = -1 - 2x - y$$

$$y' = \frac{-1 - 2x - y}{x + 2y}$$

Now determine the slope of the tangent line by substituting the point $(-1, 0)$ into the derivative.

$$m_T = y' = \frac{-1 - 2(-1) - (0)}{(-1) + 2(0)}$$

$$m_T = \frac{-1 + 2 - 0}{-1 + 0} = \frac{1}{-1} = -1$$

Then, substitute the point and slope into $y = mx + b$.

$$0 = (-1)(-1) + b$$

$$0 = 1 + b$$

$$b = -1$$

The equation of the tangent line is $y = -x - 1$.

or

Substitute the point $(-1, 0)$ into $y - y_1 = m(x - x_1)$.

$$y - 0 = -1(x + 1)$$

$$y = -x - 1$$

3. Determine the second derivative of $x^2 + y^2 = 10x$ with respect to x in terms of x and y.

Answer:

First, determine the first derivative implicitly.

$$\frac{d}{dx}\left(x^2 + y^2 = 10x\right)$$
$$2x + 2yy' = 10$$

Then, isolate y'.

$$2yy' = 10 - 2x$$
$$y' = \frac{10 - 2x}{2y}$$

Then, determine the second derivative implicitly.

$$\frac{d}{dx}\left(y' = \frac{10 - 2x}{2y}\right)$$
$$y'' = \frac{(2y)\frac{d}{dx}(10 - 2x) - (10 - 2x)\frac{d}{dx}(2y)}{(2y)^2}$$
$$y'' = \frac{(2y)(-2) - (10 - 2x)(2y')}{(2y)^2}$$

Next, simplify and substitute y' into the second derivative.

$$y'' = \frac{-4y - 20y' + 4xy'}{4y^2}$$
$$y'' = \frac{-4y - 20\left(\frac{10 - 2x}{2y}\right) + 4x\left(\frac{10 - 2x}{2y}\right)}{4y^2}$$

Finally, you could simplify the complex fraction by multiplying both the numerator and denominator by $2y$. The result would be:

$$y'' = \frac{-8y^2 - 8x^2 + 80x - 200}{8y^3}$$

Grade 12 Introduction to Calculus (45S)

Module 3
Applications of Derivatives

Module 3: Applications of Derivatives

Introduction

In Module 2, you learned to calculate the derivative of a function using limits, and you learned to use several derivative rules to differentiate some complex functions. Ultimately, calculus concepts are used to solve problems in a wide variety of applications.

In this module, you will apply your understanding of the derivative to four specific categories: particle motion problems, optimization problems, related rate problems, and curve sketching. In each of these applications, you will need to recall and use all of the differentiation rules and, most importantly, apply your understanding that the derivative is the rate of change of one variable with respect to another.

Assignments in Module 3

When you have completed the assignments for Module 3, submit your completed assignments to the Distance Learning Unit either by mail or electronically through the learning management system (LMS). The staff will forward your work to your tutor/marker.

Lesson	Assignment Number	Assignment Title
1	Assignment 3.1	Solving Inequalities
2	Assignment 3.2	Particle Motion Problems
3	Assignment 3.3	First Derivative Applications
4	Assignment 3.4	Optimization Problems
5	Assignment 3.5	Concavity and Sketching Polynomial Functions
6	Assignment 3.6	Related Rates

Notes

Lesson 1: Solving Inequalities

Lesson Focus

In this lesson, you will

- ❑ describe a domain using interval notation, set notation, and number line graphs
- ❑ solve linear and non-linear inequalities

Lesson Introduction

When you were working on questions from Modules 1 and 2, you may have noticed that you were using many of the skills and a lot of the knowledge that you had gained in your pre-calculus classes. The topic of this lesson may also be familiar to you.

In this lesson, you will solve linear and non-linear inequalities using sign diagrams. You will also state intervals using interval notation and set notation, and then graph the solution on a number line graph. These skills will be required for analysis of some of the calculus applications you will study in this module.

Interpreting Inequalities

An **inequality** is a mathematical statement that relates two unequal expressions to each other using the following symbols:

Inequality Statement	Translation
$a \neq b$	a is not equal to b
$a \leq b$	a is less than or equal to b
$a < b$	a is less than b
$a \geq b$	a is greater than or equal to b
$a > b$	a is greater than b

Unlike some equations, inequalities usually have many solutions, and often they have an infinite number of solutions. Let's review how to state a set of domain values using interval notation and set notation. You will also graph the corresponding number line graphs to represent the intervals.

Interval versus Set Notation

The following chart shows inequality statements and the corresponding brackets used in interval notation.

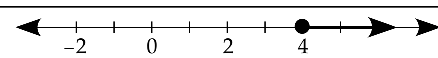
Note: The square bracket, [ ], is used to show the boundary value is included and the round bracket, ( ), indicates the boundary value is not included in the interval. Notice that the round bracket is always used next to the infinity symbol, since infinity can never be included.

Set Notation	Interval Notation
$x < b$	$(-\infty, b)$
$x \leq b$	$(-\infty, b]$
$a < x < b$	(a, b)
$a \leq x \leq b$	$[a, b]$
$x \geq b$	$[b, \infty)$
$x > b$	(b, ∞)

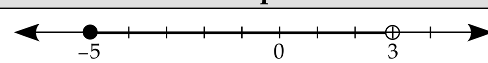
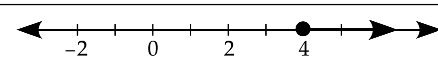
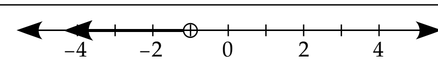
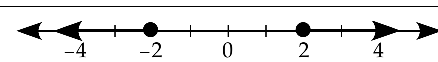
Example 1

Complete the following chart expressing an interval in three forms: interval notation, set notation, and then graphically on a number line.

Note: Holes on the line graphs are used for boundary points that are not included in the solution but are the limits. Also note that the braces, { }, mean "set of," the vertical bar, |, means "given that" or "such that," and the Greek letter, ∈, means "is an element of." So, the notation $\{x \mid x > 0, x \in \Re\}$ can be read, "The set of all x such that x is greater than zero and x is an element of the real number system."

Interval Notation	Set Notation	Graph
$[-5, 3)$		
	$\{x \mid x \geq 4, x \in \Re\}$	
		
$(-\infty, 2] \cup [2, \infty)$		

Solution

Interval Notation	Set Notation	Graph
$[-5, 3)$	$\{x \mid -5 \leq x < 3, x \in \Re\}$	
$[4, \infty)$	$\{x \mid x \geq 4, x \in \Re\}$	
$(-\infty, 3)$	$\{x \mid x < 3, x \in \Re\}$	
$(-\infty, 2] \cup [2, \infty)$	$\{x \mid x \leq 2, x \in \Re\}$ or $\{x \mid x \geq 2, x \in \Re\}$	

Example 2

Determine which of the following numbers are in the interval $[-5, 3)$.

a) $x = -6$

b) $x = -5$

c) $x = 1$

d) $x = 3$

e) $x = 4$

Solution

Since $[-5, 3)$ can also be expressed in set notation as an inequality $-5 \le x < 3$, you can substitute our value into the inequality to see if it is a solution.

1. Substitute $x = -6$ into $-5 \le x < 3$ and you get $-5 \le -6 < 3$.

   Although $-6 < 3$ is true, $-5 \le -6$ is false, so $x = -6$ **is not a solution**.

2. Substitute $x = -5$ into $-5 \le x < 3$ and you get $-5 \le -5 < 3$.

   Both $-5 < 3$ and $-5 \le -5$ are true, so $x = -5$ **is a solution**.

3. Substitute $x = 1$ into $-5 \le x < 3$ and you get $-5 \le 1 < 3$.

   Both $1 < 3$ and $-5 \le 1$ are true, so $x = 1$ **is a solution**.

4. Substitute $x = 3$ into $-5 \le x < 3$ and you get $-5 \le 3 < 3$.

   Although $-5 \le 3$ is true, $3 < 3$ is false, so $x = 3$ **is not a solution**.

5. Substitute $x = 4$ into $-5 \le x < 3$ and you get $-5 \le 4 < 3$.

   Although $-5 \le 4$ is true, $4 < 3$ is false, so $x = 4$ **is not a solution**.

Solving Linear Inequalities

A **linear inequality** is an inequality with constant terms and a variable to the exponent of one, similar to linear equations. You solve linear inequalities similarly to solving linear equations; however, when you multiply or divide both sides by a negative number or take the reciprocal of both sides, the inequality symbol changes direction. The rules of simplifying inequalities are reviewed below.

Rules for Simplifying Inequalities

1. If $a < b$, then $a + c < b + c$.
2. If $a < b$ and $c < d$, then $a + c < b + d$.
3. If $a < b$ and $c > 0$, then $ac < bc$.
4. If $a < b$ and $c < 0$, then $ac > bc$.
5. If $0 < a < b$, then $\frac{1}{a} > \frac{1}{b}$.

To illustrate, $3 < 5$, but when we multiply both sides by -1, you must change the direction of the inequality, $-3 > -5$.

Example 3

Solve the linear inequality $1 + x < 6x - 4$.

Solution

$1 + x < 6x - 4$	The original inequality that requires its variable to be isolated.
$x < 6x - 5$	Add -1 to both sides to cancel the $+1$ from the left side.
$-5x < -5$	Add $-6x$ to both sides to cancel the $6x$ on the right side.
$x > 1$	Divide both sides by -5 to determine the inequality statement for x. Note that we switch the direction of the inequality when multiplying or dividing by a negative number.

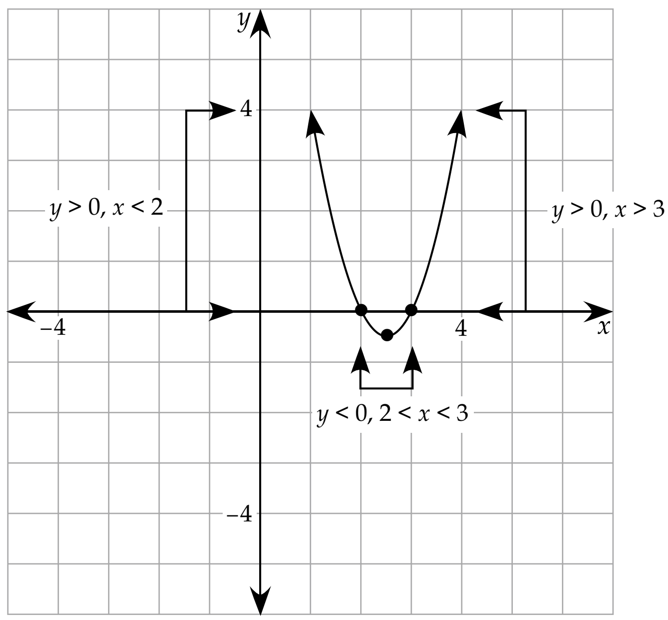
Note: The solution to the above inequality is an interval (i.e., a set of x-values), not just one or two values of x but an infinite number of real numbers that satisfy the inequality.

Solving Non-linear Inequalities

Solving inequalities becomes more complicated when we solve non-linear inequalities. **Non-linear inequalities** can include any polynomial or rational expressions in an inequality statement. Non-linear inequalities require a more extensive strategy to solve them because the solution can consist of more than one interval. Often the graph of a corresponding function helps us better understand the solution to a non-linear inequality.

Let's look at $f(x) = x^2 - 5x + 6$ and explore where the function values are positive and negative—that is,

- when is $f(x) > 0$ or $x^2 - 5x + 6 > 0$?
- when is $f(x) < 0$ or $x^2 - 5x + 6 < 0$?



From the above graph, we notice:

a) $f(x) > 0$ or $x^2 - 5x + 6 > 0$ on the x-axis intervals $x < 2$ or $x > 3$

b) $f(x) < 0$ or $x^2 - 5x + 6 < 0$ on the interval $2 < x < 3$

Furthermore, we notice that the function values switch from positive to negative and vice versa at the x-intercepts where $f(x) = 0$. Recall how to determine the x-intercepts of the original function using its equation.

$$x^2 - 5x + 6 = 0$$
$$(x - 3)(x - 2) = 0$$
$$x - 3 = 0 \text{ or } x - 2 = 0$$
$$x = 3 \text{ or } x = 2$$

Factor the polynomial, and then use the zero product theorem to determine the root(s). The roots to the equation are the x-intercepts.

The roots form the boundary point where the function may change from positive to negative or negative to positive.

Upon further analysis, you don't need to rely on the graph of the related function; you can use the factored form of the inequality to determine where the expression is positive and negative. Recall that the product of two negative or two positive numbers is always positive, but the product of a negative and positive number, regardless of order, is always negative.

$x^2 - 5x + 6 > 0$ $(x - 2)(x - 3) > 0$	True when both $x - 2 > 0$ and $x - 3 > 0$, or when both $x -- 2 < 0$ and $x - 3 < 0$.	If the linear factors are the same sign, then their product is positive.
$x^2 - 5x + 6 < 0$ $(x - 2)(x - 3) < 0$	True when either $x - 2 > 0$ and $x - 3 < 0$, or when $x - 2 < 0$ and $x - 3 > 0$.	If the linear factors are the opposite sign, then their product is negative.

You will use the above facts along with a sign diagram to determine the solution intervals to non-linear inequalities.

Example 4

Solve the inequality $x^2 - 4x - 12 < 0$.

Solution

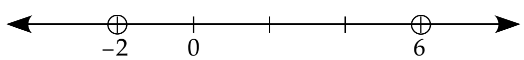
$x^2 - 4x - 12 < 0$ — You are trying to determine x-values that satisfy this inequality.

$x^2 - 4x - 12 = 0$ — Determine the roots to the corresponding quadratic equation to get the interval boundary points.

$(x - 6)(x + 2) = 0$

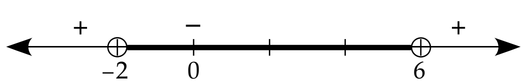
$x = 6, -2$

Define the testing intervals using these roots. The two roots divide the number line into three sections: left of –2, from –2 to 6, and right of 6.



Use a test point to determine the sign of each linear factor on each interval. Then determine the sign of the product on each interval and mark the interval with a "+" or "–".

- Test $x = -3$ on $(x - 6)(x + 2)$. It is $(-9)(-1)$, which is greater than 0 so mark a "+".
- Test $x = 0$ on $(x - 6)(x + 2)$. It is $(-6)(+2)$, which is less than 0 so mark a "−".
- Test $x = 7$ on $(x - 6)(x + 2)$. It is $(1)(9)$, which is greater than 0 so mark a "+".



The solution to the inequality is indicated on the sign diagram above and can be written in set notation as shown below.

$x^2 - 4x - 12 < 0$ when $\{x \mid -2 < x < 6, x \in \Re\}$

Example 5

Solve the inequality $(x - 1)(x + 3)(x - 2) \geq 0$.

Solution

$(x-1)(x+3)(x-2) \geq 0$

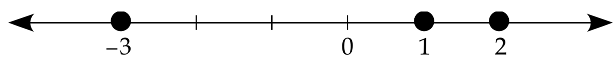
You are trying to determine x-values that satisfy this inequality.

$(x-1)(x+3)(x-2) = 0$

$x = 1, -3, 2$

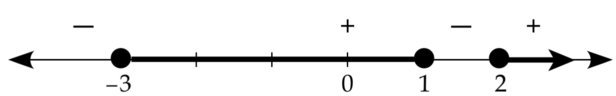
Determine the roots to the corresponding quadratic equation to get the interval boundary points.

Define the testing intervals using these roots. The three roots divide the number line into four sections, as shown.



Use a test point on each interval to determine the sign of each linear factor. Then, determine the sign of the product on each interval and mark the interval with a "+" or "−".

- Test $x = -4$ on $(x - 1)(x + 3)(x - 2)$. It is $(-5)(-1)(-6)$, which is less than 0 so mark a "−".
- Test $x = 0$ on $(x - 1)(x + 3)(x - 2)$. It is $(-1)(3)(-2)$, which is greater than 0 so mark a "+".
- Test $x = 1.5$ on $(x - 1)(x + 3)(x - 2)$. It is $(0.5)(4.5)(-0.5)$, which is less than 0 so mark a "−".
- Test $x = 3$ on $(x - 1)(x + 3)(x - 2)$. It is $(2)(6)(1)$, which is greater than 0 so mark a "+".



The solution to the inequality is indicated on the sign diagram and can be written in set notation, as shown below.

$(x - 1)(x + 3)(x - 2) \geq 0$ where $\{x \mid -3 \leq x \leq 1 \text{ or } x \geq 2, x \in \Re\}$

Note: An open circle is used on the number line to indicate a boundary point that is not included, and a shaded circle is used to indicate a boundary point that is included.

The ability to solve inequalities will help us solve application of derivative questions later on in this module.

Learning Activity 3.1

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

For Questions 1 to 4, determine if each number is a solution to $x \leq -2$.

1. 3
2. −2
3. −3
4. 0

For Questions 5 to 9, determine if the number is on the interval $[-1, 6)$.

5. −2
6. −1
7. 4
8. 6
9. 7

Part B: Solving Inequalities

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Express the interval $(-1, 7]$ in set notation and graph it on a number line graph.
2. Solve the inequality $x^2 + 7x + 10 \geq 0$.
3. Solve the inequality $(x + 1)(x - 3)(x + 3) > 0$.

Lesson Summary

In this lesson, you reviewed the three forms of expressing a solution domain with interval notation, set notation, and a number line graph. You also reviewed solving linear and non-linear inequalities using roots of the corresponding equation, and a sign diagram. Being able to determine solutions to inequalities is useful to your study of the applications of derivatives later in this module when you will need to know where a function is positive or negative.

Notes

Assignment 3.1

Solving Inequalities

Total: 8 marks

1. Express the interval [2, 5) in set notation and graph it on a number line graph. *(2 marks)*

2. Solve the inequality $x^2 - 8x + 12 \leq 0$. *(3 marks)*

continued

Assignment 3.1: Solving Inequalities (continued)

3. Solve the inequality $(x - 7)(x - 1)(x + 4) > 0$. *(3 marks)*

Lesson 2: Particle Motion Problems

Lesson Focus

In this lesson, you will

- ❑ describe the meaning of position function
- ❑ determine average and instantaneous velocity given a position function
- ❑ determine average and instantaneous acceleration given a position function
- ❑ solve particle motion problems

Lesson Introduction

One of the original uses of calculus concepts was the study of the motion of planets by Sir Isaac Newton. Today, calculus is used extensively in the analysis of the motion of objects in space and on Earth.

In this lesson, you will explore the definition of position function. You will learn how calculus is used to relate the position, velocity, and acceleration of an object. You will solve particle motion problems using your knowledge of derivatives.

Derivative as a Rate of Change

In Module 2, you determined the slope of a curve using the concept of the derivative. A further understanding of the concept or a different interpretation is to view it as a rate of change between two measurements, such as distance and time.

Consider an object moving in a straight line. The distance changes as the time changes. The rate of change of the distance travelled to the time elapsed is called velocity. If you use Δy to represent change in distance and Δt to represent change in time, then velocity is represented by $\frac{\Delta y}{\Delta t}$. When $\Delta t \to 0$, $\frac{\Delta y}{\Delta t}$ becomes the instantaneous velocity at a specific time. You probably recognize this as a limit and it is also the derivative of y with respect to t.

This topic is further study into the relationship that exists among the three concepts—position, velocity, and acceleration—and the role of the derivative to connect them.

The Position Function

To study the position function, you first need to compare and contrast the terms *distance* and *displacement*.

Distance is the measure of length between two points. This measurement is always positive in sign regardless of the direction it takes. Distance is called a **scalar quantity** because it has magnitude, but not direction, associated with it. For example, the distance from Winnipeg to Calgary is approximately 1200 kilometres. Notice that there is no mention of the direction you would have to travel to make the trip.

Displacement is a measure of the length between two points with a given direction. Displacement is the straight line distance between the initial position and the final position of an object including the direction of travel. To correctly describe a displacement, you must always state the magnitude of the distance between the two objects and the direction travelled. A quantity whose measure requires a magnitude and a direction is called a **vector quantity**. In calculus and other mathematics courses, you use positive and negative signs to display direction along a number line. Essentially, the absolute value of displacement is the distance.

Motion in One Dimension

The application of the derivative to motion that you will study is limited to one-dimension problems, even though it is applicable to the solution of motion problems in two and three dimensions. Since distance is the measure of the length between two points, it will always be positive in sign regardless of the direction in which it is measured. Whereas, since displacement is the measure of the length between two points in a specific direction, displacement to the right is considered positive and displacement to the left is considered negative along a horizontal axis (or number line).

Speed is the distance travelled per unit time. **Average speed** can be defined as the total distance an object travelled in changing its position, divided by the total elapsed time. Since no consideration is given to the direction the object moved, average speed is a scalar quantity.

Velocity is the displacement travelled per unit time. **Average velocity** can be defined as the total change in position or displacement divided by the total elapsed time. Since displacement is dependent on direction, average velocity is also dependent on direction and, therefore, it is a vector quantity.

It is important to note that the average velocity is calculated using the total displacement from the initial position to the final position. Whereas, the average speed is calculated using the total distance travelled from the initial position to the final position.

To illustrate, Doug travelled from Winnipeg to Brandon, a distance of 230 km, and back to Winnipeg.

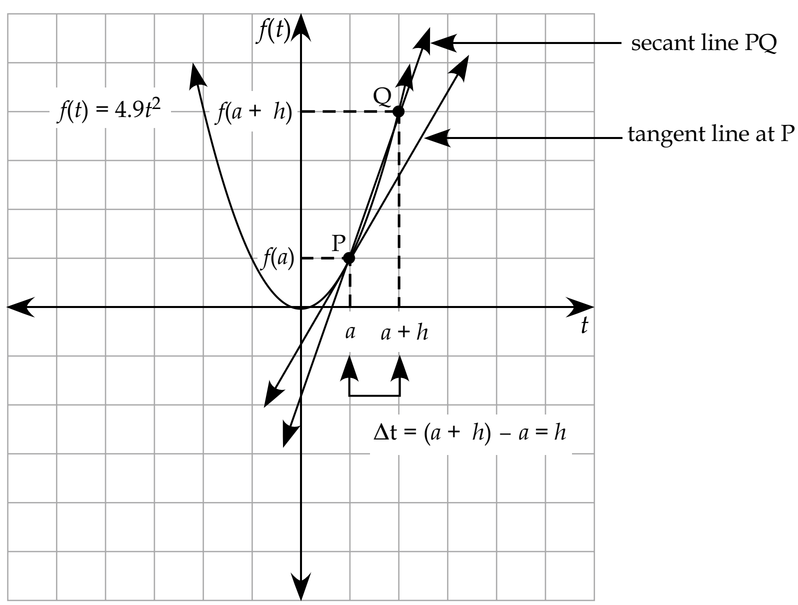
- His total distance travelled is 230 + 230 = 460 km.
- His total displacement travelled is +230 − 230 = 0 km.
- His average speed is not the same number as his average velocity.

For your purposes, particle motion will be treated in one dimension only and in two directions (left or right along a line). Therefore, velocity will also be in one dimension only and can be in two directions (positive and negative).

Average Velocity Versus Instantaneous Velocity

Average velocity can be calculated using the slope of the secant line to the curve that represents the motion of a particle travelled in a specified elapsed time. The rise value is the change in y, which is the total displacement in the time interval. Since the velocity of the object is constantly changing, the slope of the secant does not represent the instantaneous velocity at any point in that time interval.

Below is a graph of a position function, $f(t) = 4.9t^2$.



The slope of the secant PQ provides us with the average velocity of the object during the time interval $a \le t \le a + h$ seconds (Δt) and that the slope of the secant is represented by $\frac{\Delta y}{\Delta t}\left(\frac{\text{rise}}{\text{run}}\right)$.

The following formula for average velocity in a time period from $t = a$ to $t = a + h$ is determined:

$$\text{Slope of PQ} = V_{\text{average}}$$

$$\text{Slope of PQ} = \frac{\Delta y}{\Delta t}$$

$$\text{So, } V_{\text{average}} = \frac{\Delta y}{\Delta t} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$$

If you decrease the time interval Δt by allowing point Q to move closer to point P, the slope of the secant line will approach the slope of the tangent line at point P. Since the slope of the secant line represents the average velocity as h gets smaller, the value of the average velocity converges to the velocity of the object at the instant that it passes through the point P. The velocity at an instant is called the **instantaneous velocity**. The corresponding formula for instantaneous velocity is expressed below using our knowledge of limits and derivatives.

$$\lim_{\Delta t \to 0}(\text{slope of PQ}) = \text{slope at P}$$

$$\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = V_{\text{instantaneous}}$$

$$\text{So, } V_{\text{instantaneous}} = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \lim_{h \to 0}\left[\frac{f(a+h) - f(a)}{h}\right]$$

$$V_{\text{instantaneous}} = \frac{dy}{dt} \text{ or } y'$$

Thus, instantaneous velocity is the derivative of the displacement function of a particle with respect to time.

Example 1

The displacement of a particle moving along a straight line from a fixed point, measured in metres, is determined by the position function $s(t) = 32 - 2t^2 + t^3$, $t \ge 0$.

a) Determine the average velocity during the time interval from $t = 1$ second to $t = 4$ seconds.

b) Determine the time when the instantaneous velocity is zero.

Solution

a) $s(4) = 32 - 2(4)^2 + (4)^3 = 32 - 32 + 64 = 64$ metres

$s(1) = 32 - 2(1)^2 + (1)^3 = 32 - 2 + 1 = 31$ metres

$$V_{\text{average}} = \frac{\Delta s}{\Delta t} = \frac{s(4) - s(1)}{4 - 1} = \frac{64 - 31}{3} = \frac{33 \text{ m}}{3 \text{ s}} = 11 \text{ m/s}$$

b) $V_{\text{instantaneous}} = v(t) = s'(t)$

Find the derivative of $s(t) = 32 - 2t^2 + t^3$, which is $v(t) = -4t + 3t^2$.

Set velocity equation to zero to determine the time when the instantaneous velocity is zero.

$$0 = v(t) = -4t + 3t^2$$

$$0 = t(-4 + 3t)$$

$$t = 0 \quad \text{or} \quad -4 + 3t = 0$$

$$3t = 4$$

$$t = \frac{4}{3}$$

The velocity is zero at 0 and $\frac{4}{3}$ seconds.

Example 2

The displacement, y, in centimetres, of a particle at any time, t, in seconds, is given by $y = 2t^2 - 10t + 12$, $t \geq 0$.

a) Determine the displacement for the times 0, 1, 2, 3, and 5 seconds.

b) What is the average velocity after three seconds?

c) What is the instantaneous velocity when $t = 5$ seconds?

d) When is the velocity zero?

e) When is the velocity negative and when is it positive?

f) When does the particle reverse direction?

g) What is the displacement when the velocity is zero?

Solutions

a) Determine the displacement by substituting the times into the function.

When $t = 0$, that is,

$y = 2(0)^2 - 10(0) + 12 = 0 - 0 + 12 = 12$ and

When $t = 2$, that is,

$y = 2(2)^2 - 10(2) + 12 = 8 - 20 + 12 = 0$

Time (s)	0	1	2	3	5
Displacement (cm)	12	4	0	0	12

b) $$V_{\text{average}} = \frac{\Delta y}{\Delta t} = \frac{y(3) - y(0)}{3 - 0} = \frac{0 - 12}{3} = \frac{-12 \text{ cm}}{3 \text{ s}} = -4 \text{ cm/s}$$

This means, on average, the particle is moving 4 cm/s to the left in the time interval [0, 3].

c) $$V_{\text{instantaneous}} = v(t) = y'(t)$$

Find the derivative of:

$y = 2t^2 - 10t + 12$

$y' = 4t - 10$

Substitute $t = 5$ into $v(t) = 4t - 10$.

$$v(5) = y'(5) = 4(5) - 10 = 20 - 10 = 10 \text{ cm/s}$$

At $t = 5$ seconds, the particle is moving at a rate of 10 m/s to the right.

d) Set $v(t) = 4t - 10$ to zero and solve.

$$0 = v(t) = 4t - 10$$

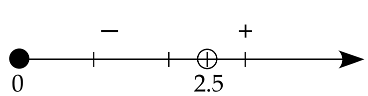
$$10 = 4t$$

$$t = \frac{10}{4} = \frac{5}{2}$$

So velocity is zero at 2.5 seconds.

e) Use a sign diagram to indicate when the velocity is positive and negative. The boundary point is $t = 2.5$ seconds.

Test $v = 4t - 10$.



The velocity is negative when $0 < t < 2.5$ but the velocity is positive when $t > 2.5$ seconds.

f) Since the sign on the velocity value determines direction, the particle changes direction when the sign of the velocity changes. As shown above, velocity changes sign at 2.5 seconds. So the particle changes direction at 2.5 seconds.

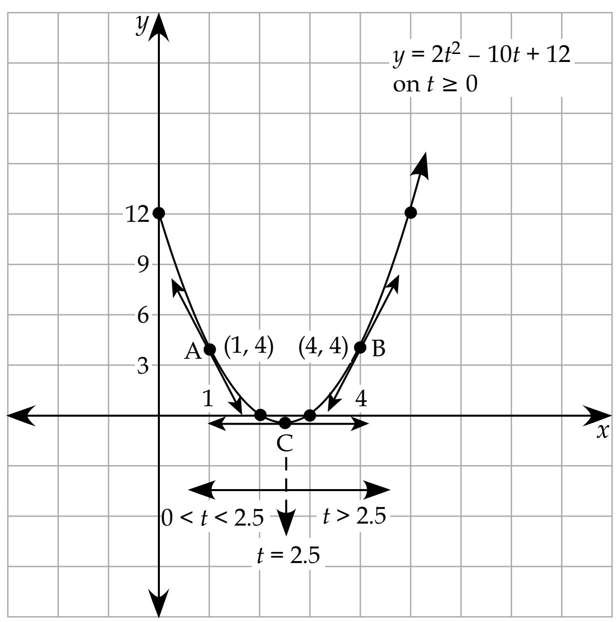
g) To determine the displacement when velocity is zero, substitute $t = 2.5$ into $y(t) = 2t^2 - 10t + 12$.

$$y(2.5) = 2(2.5)^2 - 10(2.5) + 12 = 2(6.25) - 25 + 12$$

$$= 12.5 - 25 + 12 = -0.5 \text{ cm}$$

The displacement when the velocity is zero is −0.5 cm, which is 0.5 cm left of 0.

Compare the following graph of the position function from Example 2 to the calculated values.



1. The displacements at 1, 2, and 3 seconds are 4 m, 0 m, and 0 m respectively. These are the same answers we calculated in (a).
2. The slope of the curve in each of the following areas.
   a) Negative slope for $0 < t < 2.5$ seconds because the tangent is down and to the right, which is consistent with our calculated negative velocity, since the velocity is represented by the slope of the position function.
   b) Positive slope for $t > 2.5$ seconds because the tangent is up and to the right, which is consistent with our calculated positive velocity.

c) Zero slope for t = 2.5 seconds because the tangent is a horizontal line, which is consistent with our zero velocity calculation. The particle is moving in a negative direction (since the slope is negative, the displacement is decreasing) before 2.5 seconds, and then it is moving in a positive direction (since the slope is positive, the displacement is increasing) after 2.5 seconds.

Note: Given a graph of a position function, note that the displacement is negative only when the function is below the x-axis. The velocity is negative when the slope of the function is negative.

Average Versus Instantaneous Acceleration

To review, velocity is defined as the change in displacement per unit time $\left(\frac{\Delta y}{\Delta t}\right)$. Acceleration is defined as the change in velocity per unit time $\left(\frac{\Delta v}{\Delta t}\right)$.

Note that both definitions of acceleration and velocity are rates of change with respect to time.

For example, if a car travels at a constant velocity of 100 km/h, then as each successive hour passes, the car will be another 100 km closer to the destination. However, if a car has varying velocities, you can analyze how its velocity changes with respect to time.

If the same car can reach a velocity of 100 km/h in 10 seconds from rest, then its change in velocity (Δv) is 100 km/h during a 10-second period of time (Δt). So its average change in velocity, or **average acceleration**, can be calculated:

100 km/h ÷ 10 s

The acceleration of the car is 10 km/h/s. The units for acceleration can be expressed more neatly as m/s/s or m/s^2. Therefore,

10 km/h/s = 10 × 1000 ÷ 3600 = 2.8 m/s^2

The acceleration of the car is 2.8 m/s^2.

What does this mean? It means that during each successive second that the car is accelerating, its velocity increases by 2.8 m/s. At the end of the first second the velocity of the car is 2.8 m/s. At the end of the second second it is 5.6 m/s, and at the end of the third second it is 8.4 m/s. As a formula, the average acceleration of a particle can be found by:

$$\text{Average acceleration} = \frac{\text{change in velocity}}{\text{change in time}} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta v}{\Delta t}$$

Imagine that Δt is changing steadily in very small increments. As Δt approaches zero, the acceleration approaches the acceleration at a specific time. We can use a limit to determine the value of the acceleration at a specific instant of time, called **instantaneous acceleration**. Instantaneous acceleration is often denoted by the variable a, as shown in the formula below:

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

Since $\lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$ is the derivative of v with respect to t, you can say that instantaneous acceleration is equivalent to the derivative of instantaneous velocity with respect to time:

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} \text{ or } v'$$

In addition, recall that:

$$V_{\text{instantaneous}} = \frac{dy}{dt} \text{ or } y$$

Then:

$$a = \frac{dv}{dt} = \frac{d}{dt}(v) = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2} \text{ or } y''$$

Essentially, acceleration is the first derivative of the velocity function and the second derivative of the position function with respect to time.

Since acceleration is also a vector, it has direction and magnitude. The direction is represented by its sign. A negative acceleration means that the velocity is decreasing, but a positive acceleration means that the velocity is increasing. So when the acceleration changes from negative to positive or from positive to negative, the velocity changes from decreasing to increasing or from increasing to decreasing, respectively.

Let's continue the study of position functions using the additional acceleration information.

Example 3

The position function that represents a particle travelling along a straight line is $s = 2t^3 - 4t^2 - 12t + 24$, $t \geq 0$, where distance is measured in m and time in s.

a) Determine the velocity function at any time t.

b) Determine the acceleration function at any time t.

c) Determine the acceleration at $t = 2$ seconds.

d) Determine at what time the acceleration is zero.

e) Determine when velocity changes from decreasing to increasing.

Solution

a) You can determine the velocity function by differentiating the position function. If $s = 2t^3 - 4t^2 - 12t + 24$, then $v = s' = 6t^2 - 8t - 12$.

b) You can determine the acceleration function by differentiating the velocity function. If $v = s' = 6t^2 - 8t - 12$, then $a = v' = s'' = 12t - 8$.

c) You can substitute $t = 2$ seconds into the acceleration function.

$$a(2) = 12(2) - 8 = 24 - 8 = 16 \text{ m/s}^2$$

The acceleration is 16 m/s^2.

d) You can determine the time when acceleration is zero by setting the equation to zero.

$$0 = a = 12t - 8$$

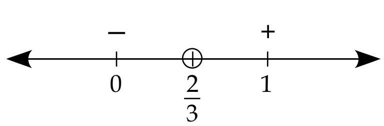
$$8 = 12t$$

$$t = \frac{8}{12} = \frac{2}{3} \text{ s}$$

The acceleration is zero at $\frac{2}{3}$ seconds.

e) Use a sign diagram to determine positive and negative acceleration.

Test $a = 12t - 8$ on each side of the boundary point $t = \frac{2}{3}$.



Velocity is increasing when acceleration is positive and velocity is decreasing when acceleration is negative. So, velocity changes from decreasing to increasing when acceleration changes sign at $\frac{2}{3}$ seconds.

You can now solve many particle motion problems using the displacement, velocity, and acceleration formulas but, most importantly, you know how the formulas relate using derivatives.

Learning Activity 3.2

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Factor: $x^2 - 5x + 6$
2. Factor: $x^2 - x - 12$
3. Factor: $2x^2 - 9x - 5$
4. Factor: $3x^2 + 8x + 4$
5. Solve: $(x - 1)(x + 4) = 0$
6. Solve: $(1 - x)(x - 3) = 0$
7. Solve: $x(2x - 5) = 0$
8. Solve: $x^2 - 36 = 0$

continued

Learning Activity 3.2 (continued)

Part B: Particle Motion Problems

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. The displacement in metres of a particle from a fixed point is given by the position function $s = 8t - t^2 - t^3$, $t \geq 0$.
   a) Determine the velocity function at any time t.
   b) Evaluate the displacement at the following times.

Time (s)	0	1	2	3
Displacement (m)				

   c) Calculate the exact time at which the velocity is zero.
   d) Does the particle change direction? Justify your answer.
   e) Determine the acceleration function for any time t?
   f) Does the acceleration reach zero on the interval $t \geq 0$? Justify your answer.

2. A rocket travels upward a distance $s = t^3$ in metres for the first t seconds after takeoff. Find its velocity when it reaches a height of 1000 metres.

3. An automobile travelling at 30 m/s is brought to a halt by steadily increasing braking force. The function, $s = 30t - 0.1t^3$, $t \geq 0$, represents the braking distance in metres for a time in seconds until the automobile stops.
   a) Determine the velocity and acceleration functions in terms of t.
   b) Determine when the vehicle stops.
   c) Determine the stopping distance.
   d) Determine the deceleration at the time when the vehicle stops.

Lesson Summary

In this lesson, you learned that the position function is defined using displacement rather than distance. You also learned that velocity is not just a scalar quality with magnitude, but requires direction expressed with a positive or negative sign. The limit of the average velocity over a time interval as the length of the interval approaches zero was found to be the instantaneous velocity at a specific time. Furthermore, the derivative of the position function is the velocity function, and the derivative of the velocity function is the acceleration function.

You learned that the particle is stopped and may change direction when the velocity function is zero. In addition, the sign on the velocity function was used to determine the direction of the particle's movement (whether the displacement is increasing or decreasing), and the sign on the acceleration function was used to determine where the particle's velocity was increasing or decreasing. These new functions allow you to solve many particle motion problems. You will continue to learn about other derivative applications throughout this module.

Notes

Assignment 3.2

Particle Motion Problems

Total: 10 marks

1. The displacement in metres of a particle from a fixed point is given by the position function $s = t^3 - 3t^2 + 3t$, $t \geq 0$, where time is in minutes.

   a) Determine the velocity function at any time t. *(1 mark)*

   b) Evaluate the displacement and velocity at the following times. *(2 marks)*

Time (minutes)	0	1	2	3
Displacement (m)				
Velocity (m/minute)				

   c) Calculate the exact time(s) at which the velocity is zero. *(2 marks)*

continued

Assignment 3.2: Particle Motion Problems (continued)

d) Does the particle change direction? Justify your answer. *(2 marks)*

e) Determine the acceleration function for any time *t*. *(1 mark)*

f) Does velocity change from increasing to decreasing or vice versa? Justify your answer. *(2 marks)*

Lesson 3: First Derivative Applications

Lesson Focus

In this lesson, you will

- ❑ determine relative extremes and absolute extremes graphically
- ❑ determine the critical values of a function
- ❑ determine the intervals where the function is increasing and decreasing
- ❑ determine relative extremes algebraically

Lesson Introduction

When functions are used to model phenomena in our world, sometimes the analysis of the function involves finding maximum or minimum values over an interval. For example, a function model might be used to determine the radius and height of a can to be sure the minimum amount of metal is used. The maximum area of an enclosure could be found using a function model that relates the area to the length of fencing on hand. When analyzing economic trends over time, calculus can be used for analysis to determine the time when the rate of change of growth begins to decrease.

In this lesson, you will learn how the first and second derivatives of a function are utilized to predict some graphical and algebraic features of a function. You will learn how to use the first derivative test and the second derivative test to determine the relative extremes of a function algebraically.

Extreme Values

The **extreme values** of a function are the maximum and minimum function values or y-values. Some extreme values are absolute over the whole domain of the function, while others are relative extremes over a defined interval of the function. Below are the distinctions between each.

Absolute Maximum

A continuous function, f, has an **absolute maximum** at c if $f(c) \geq f(x)$ for all x in the domain of the function and the number, $f(c)$, is called ***the*** maximum value of the function.

Absolute Minimum

A continuous function, f, has an **absolute minimum** at c if $f(c) \leq f(x)$ for all x in the domain of the function and the number, $f(c)$, is called ***the*** minimum value of the function.

Essentially, an absolute maximum is the *one and only largest y*-value of a function, whereas, an absolute minimum is the *one and only smallest y*-value of a function over its whole domain.

Relative Maximum

A continuous function, f, has a **relative maximum** at c if $f(c) \geq f(x)$ when x is close to c and $f(c)$ is called ***a*** relative maximum value of the function.

Relative Minimum

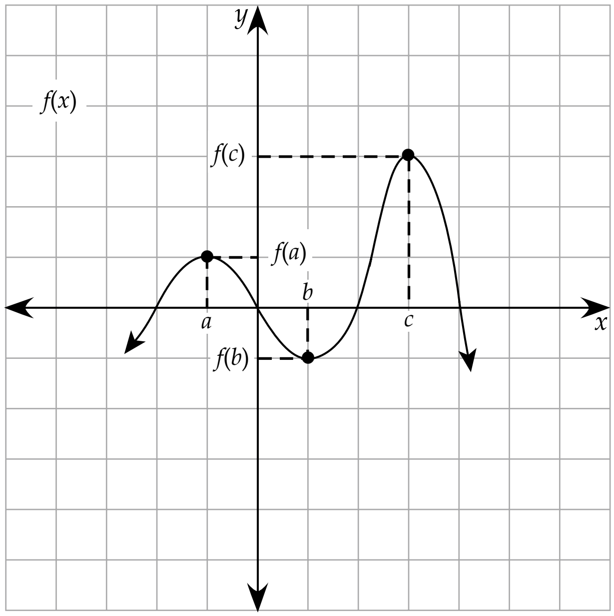
A continuous function, f, has a **relative minimum** at c if $f(c) \leq f(x)$ when x is close to c and $f(c)$ is called ***a*** relative minimum value of the function.

Essentially, a relative maximum is the *largest y*-value of a function on a specific interval and a relative minimum is the *smallest y*-value of a function on a specific interval.

Let's use these definitions of the extreme values to determine the extreme values for the following function.

Example 1

Given the graph of $f(x)$ below, determine its absolute extreme values and its relative extreme values.



Solution

x-value	y-value	Type	Reason
a	$f(a)$	Relative maximum	$f(a) \geq f(x)$ for values close to a
b	$f(b)$	Relative minimum	$f(b) \leq f(x)$ for values close to b
c	$f(c)$	Relative maximum and **absolute maximum**	$f(c) \geq f(x)$ for values close to x and for all values in the domain of $f(x)$

Note: In this example, there is no absolute minimum since the function goes down to $-\infty$.

Intervals of Increase and Decrease

With your study of slope in Module 2, you observed that graphs of functions rise and fall, which is reflected in their slope or rate of change. A function is said to be increasing on an interval when it is rising from left to right and decreasing on an interval when it is falling from left to right. A more formal definition of increasing and decreasing intervals is given below.

Rising

A function is **increasing** on an interval if $f(x_1) < f(x_2)$, when $x_1 < x_2$ for all x-values on the interval.

Falling

A function is **decreasing** on an interval if $f(x_1) > f(x_2)$, when $x_1 < x_2$ for all x-values on the interval.

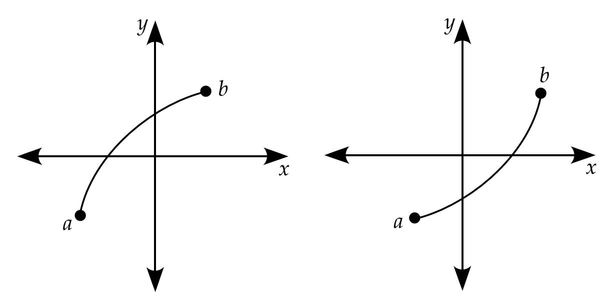
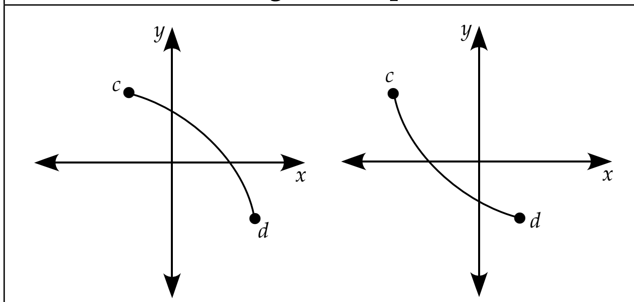
Example 2

Determine the intervals on the graph of $f(x)$ from Example 1 that are increasing and decreasing.

Solution

$(-\infty, a)$	(a, b)	(b, c)	(c, ∞)
increasing	decreasing	increasing	decreasing

The function is increasing on $\{(-\infty, a) \cup (b, c)\}$ and decreasing on $\{(a, b) \cup (c, \infty)\}$, Upon reviewing slope in Module 2, you learned that positive slopes are for rising lines, negative slopes are for falling lines, and zero slopes are horizontal lines. In Module 2, you also discovered that the slope of a function at a point is the slope of the tangent line and ultimately the function's derivative at that point. Let's analyze the slope of a function on specific intervals and compare it to increasing and decreasing intervals.

Positive Slope		
	$f'(x) > 0$	**Increasing** interval on (a, b)
Negative Slope		
	$f'(x) < 0$	**Decreasing** interval on (c, d)

Essentially, the first derivative of a function can be used to determine the intervals in which a function is increasing and decreasing.

Testing for increasing and decreasing intervals

1. If $f'(x) > 0$ for all values of x on an interval (a, b), then the function $f(x)$ is said to be **increasing** on (a, b).
2. If $f'(x) < 0$ for all values of x on an interval (c, d), then the function $f(x)$ is said to be **decreasing** on (c, d).

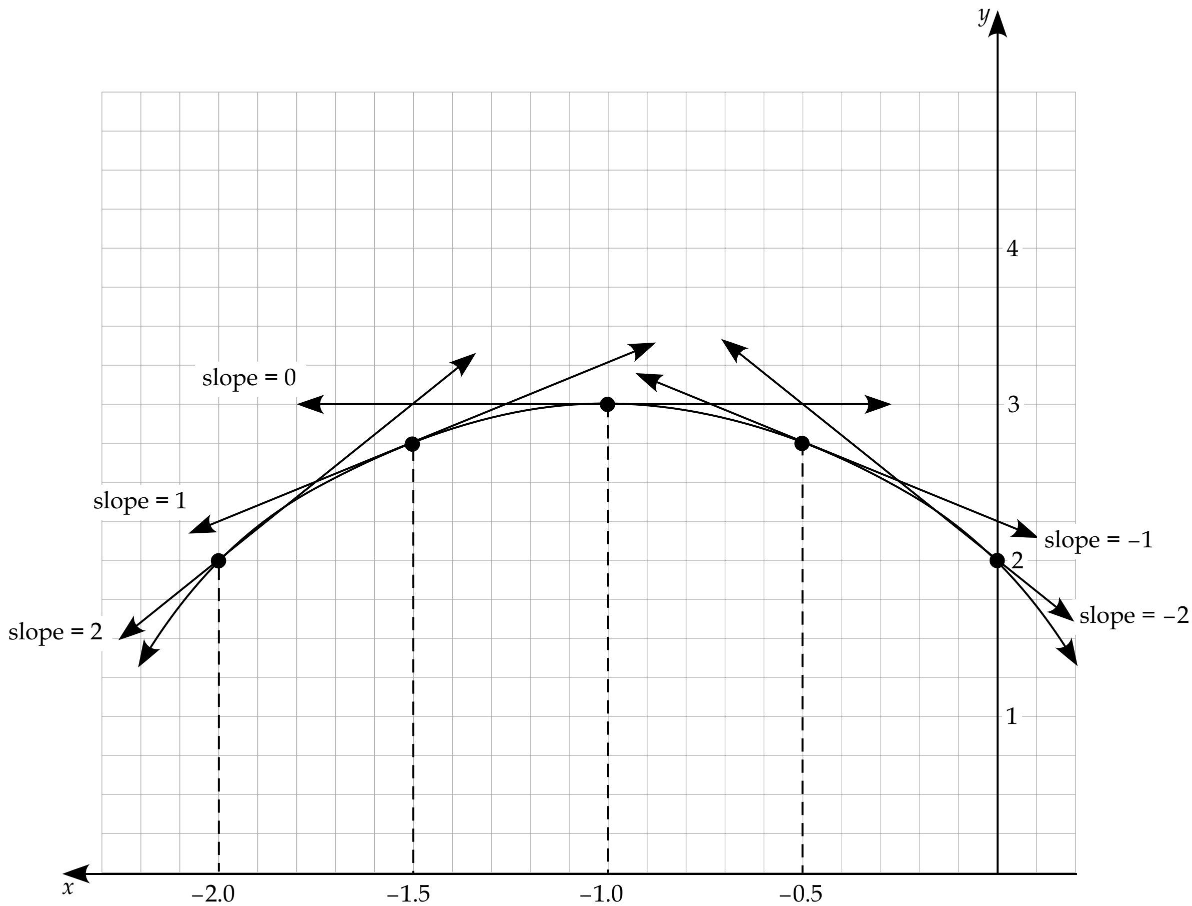
Now, how do you determine the interval boundaries using only the function's equation?

You need an algebraic procedure that allows you to determine these critical values that define the interval boundaries. You will notice that the function $f(x)$ changes from increasing to decreasing and vice versa at the relative extremes. Let's analyze this further.

Critical Values

Let’s analyze the slope of the tangent lines of the two functions below as they approach their relative extremes.

1. The graph of $g(x) = -x^2 - 2x + 2$ below shows a relative and absolute maximum at (−1, 3). Let’s analyze the slopes of the tangent lines as they get close to $x = -1$.



The slope of the tangent line to a curve at a specific point is equal to the derivative of the function at that point. The derivative of $g(x) = -x^2 - 2x + 2$ is $m_T = g'(x) = -2x - 2$.

x-value	−2	−1.5	−1.1	−1	−0.9	−0.5	0
slope	2	1	0.2	0	−0.2	−1	−2

Observations:

a) The slope of that tangent line at $x = -1$ is zero because it is a horizontal line.

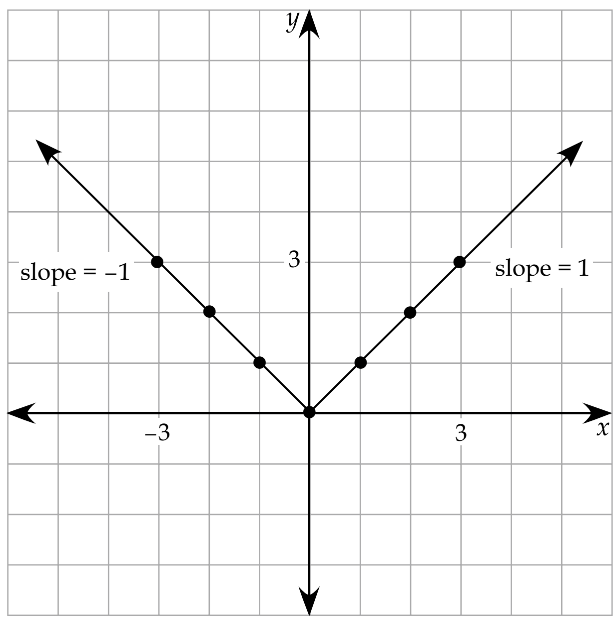
b) The slopes of the tangent lines gradually decrease as the value of x moves from left to right.

c) The slopes of the tangent lines approach zero from both the left and right as x approaches -1.

d) The slopes of the tangent lines change sign when they cross over $x = -1$.

e) There is a relative maximum at $x = -1$ and the slope at that point is zero.

2. The graph of $h(x) = | x |$ below shows a relative and absolute minimum at (0, 0). Let's analyze the slopes of the tangent lines as they get close to $x = 0$.



The slope of the tangent line to a curve at a specific point is equal to the derivative of the function at that point.

Note: The absolute-value function can be defined as two oblique lines whose definition is split at $x = 0$. The piecewise definition:

$$h(x) = |x| \text{ is } h(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Thus, the derivative of $h(x) = |x|$ is $m_T = h'(x) = \begin{cases} +1, & x > 0 \\ -1, & x < 0 \end{cases}$.

x-value	−1	−0.5	−0.1	0	0.1	0.5	1
slope	−1	−1	−1	undefined	1	1	1

Observations:

a) The slope of that tangent line at $x = 0$ does not exist because there isn't a gradual change in the curve; there is a sharp turn or corner at $x = 0$.

b) The slopes do not approach the same number from the left and right as x approaches 0, so the limit of the slope at $x = 0$ is undefined.

c) The slopes of the tangent lines change sign when they cross over $x = 0$.

d) There is a relative minimum at $x = 0$ but the slope at $x = 0$ is undefined.

Fermat's Theorem

If a function $f(x)$ has a relative maximum or minimum at $x = c$, then $f'(c) = 0$ or $f'(c)$ does not exist.

It is important to note that the converse of Fermat's theorem is not true (i.e., just because $f'(c) = 0$ or $f'(c)$ = does not exist at $x = c$ does not guarantee a relative extreme). However, a potential relative extreme exists at $x = c$, which is why we call $x = c$ a critical number. The critical numbers require further investigation.

A **critical number** $x = c$ is a number on the domain of a function $f(x)$ such that either $f'(c) = 0$ or $f'(c)$ = does not exist.

These critical numbers are important since they are the x-values where you will need to look for relative extremes (maxima or minima). Let's first practice finding critical values.

Example 3

Determine the critical values (x-values) of the following two functions.

a) $f(x) = x^3 + 6x^2 + 9x + 2$

b) $g(x) = \sqrt[3]{x}$

Solution

a) In order to determine the critical values of a function we first need to determine its derivative, then set the derivative equation equal to zero and solve.

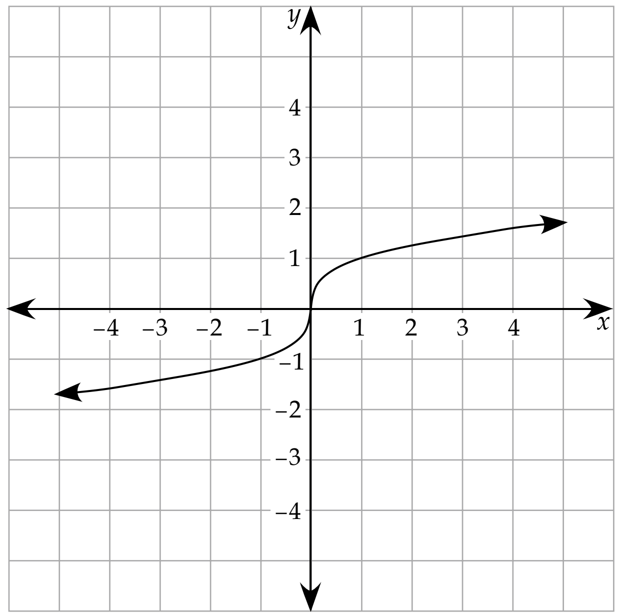
$f(x) = x^3 + 6x^2 + 9x + 2$ $f'(x) = 3x^2 + 12x + 9$	The derivative.
$0 = 3x^2 + 12x + 9$ $0 = 3\left(x^2 + 4x + 3\right)$ $0 = 3(x+1)(x+3)$	Set the derivative equal to zero and solve by factoring.
$x = -1,\ x = -3$	The roots of the derivative equation.

Since there are no corners in a cubic function and its slope is always defined, then there are no values where $f'(x)$ does not exist. Furthermore, since $f'(-1) = 0$ and $f'(-3) = 0$, then the only **critical values** of the function are $x = -1, -3$.

b) In order to determine the critical values of a function we first need to determine its derivative, then determine the root of the derivative equation equal to zero.

$g(x) = \sqrt[3]{x} = (x)^{\frac{1}{3}}$ $g'(x) = \frac{1}{3}(x)^{-\frac{2}{3}} = -\frac{1}{3\sqrt[3]{x^2}}$	The derivative.
$0 = -\frac{1}{3\sqrt[3]{x^2}}$	Set the derivative equal to zero.
$x = 0$	The non-permissible value of $g'(x)$.

There is no x-value that will make $g'(x)$ equal zero. However, there is a non-permissible x-value that will make $g'(x)$ undefined. Notice the slope of the tangent is undefined at $x = 0$, since it is a vertical line.



Since $f'(x)$ is never zero and since $f'(x)$ does not exist when $x = 0$, then the only **critical value** of the function is $x = 0$.

Determining Relative Extremes Algebraically

Since knowing that an x-value is a critical value is insufficient to determine relative extremes, another step is required. From the previous analysis of increasing and decreasing intervals, you noticed that functions changed their increasing and decreasing behavior at the relative extremes. So if you combine your knowledge of increasing and decreasing intervals with that of critical values, you can determine relative extremes algebraically using the first derivative, as shown below.

First Derivative Test

Let $x = c$ be a critical number of a continuous function $f(x)$.

- If $f'(x)$ changes from positive to negative—in other words, $f(x)$ changes from increasing to decreasing as it crosses $x = c$—then $f(c)$ is a *relative maximum*.
- If $f'(x)$ changes from negative to positive—in other words, $f(x)$ changes from decreasing to increasing as it crosses $x = c$—then $f(c)$ is a **relative minimum**.
- If $f'(x)$ does not change—in other words, $f(x)$ does not change its increasing or decreasing behaviour as it crosses $x = c$—then $f(c)$ is **not a relative extreme**.

Example 4

Use the first derivative test to determine the relative extremes of $y = x^3 - 3x - 1$.

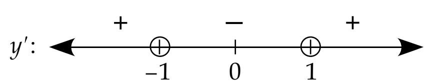
Solution

You will first determine the critical values of the function, then its increasing and decreasing behaviour using a sign diagram. The curve is smooth and its slope is defined everywhere on the domain of a cubic function. You will find critical values where $f'(x) = 0$.

$y = x^3 - 3x - 1$	Determine the derivative.
$y' = 3x^2 - 3$	
$0 = 3x^2 - 3$	Determine the roots of the derivative equation equal to zero in order to find the critical values.
$0 = 3\left(x^2 - 1\right)$	
$0 = 3(x - 1)(x + 1)$	
$x = \pm 1$	

Use the critical values, −1 and 1, to determine the interval boundaries for the sign diagram.

Complete a sign diagram to determine the intervals of increasing and decreasing around the critical values.



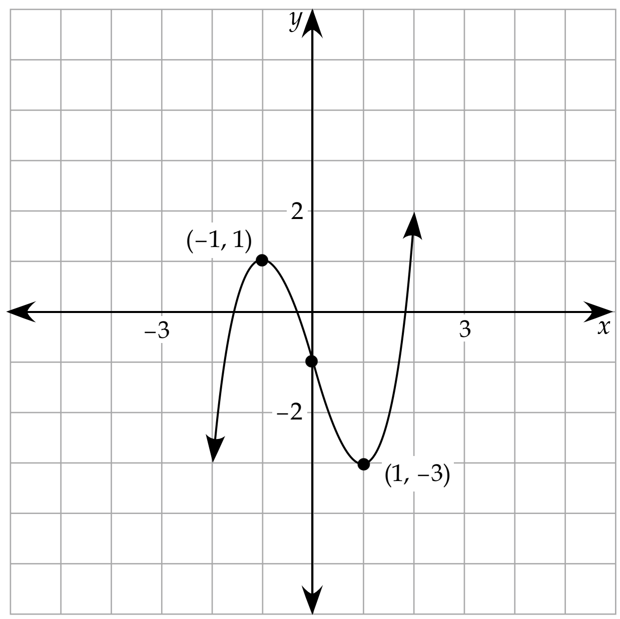
The sign diagram tells us that $y' > 0$, which means the function is increasing on $(-\infty, -1) \cup (1, \infty)$, and $y' < 0$, which means the function is decreasing on $(-1, 1)$.

According to the first derivative test, the function has a **relative maximum** when $x = -1$ because the function changes from increasing to decreasing (or f' goes from + to −) as it crosses $x = -1$. Similarly, the function has a **relative minimum** at $x = 1$ because the function changes from decreasing to increasing (or f' goes from − to +) as it crosses $x = 1$.

The function values at the critical values are the relative extremes.

- At $x = -1$, the function value is $y = (-1)^3 - 3(-1) - 1 = -1 + 3 - 1 = 1$.
- At $x = 1$, the function value is $y = (1)^3 - 3(1) - 1 = -3$.

The graph of the function will confirm the relative extremes that you determined algebraically. There is a relative maximum at (−1, 1) and a relative minimum at (1, −3). The y-intercept of $y = x^3 - 3x - 1$ is −1.



Example 5

Determine the relative extremes of $h(x) = -2x^3 - 2$ using the first derivative test.

Solution

First determine the critical values of the function, and then its increasing and decreasing behaviour using a sign diagram. Since $h'(x)$ is defined everywhere for a cubic function, find the critical values when $h'(x) = 0$.

$$h(x) = -2x^3 - 3$$
$$h'(x) = -6x^2$$

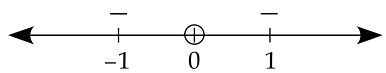
Determine the derivative.

$$0 = -6x^2$$
$$x^2 = 0$$
$$x = 0$$

Determine the root of the derivative equation equal to zero in order to find the critical value.

Use the critical value, $x = 0$, to determine the interval boundaries for the sign diagram.

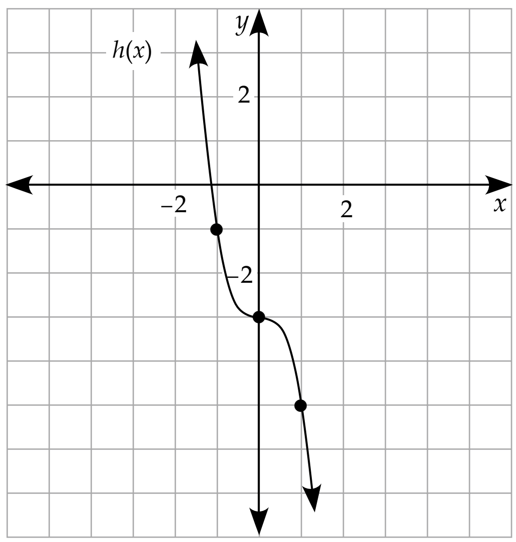
Complete a sign diagram to determine the intervals where the function is increasing and decreasing around the critical value.



The sign diagram tells us $h'(x) < 0$ or the function is decreasing on $(-\infty, 0) \cup (0, \infty)$.

According to the first derivative test, the function does **not** have a relative extreme on its domain because the function is decreasing on either side of its critical value $x = 0$.

The graph of the function below confirms that the slope of $h(x)$ is zero at $x = 0$ but there is no relative extreme.



As you witnessed in this example, knowing that $f'(x) = 0$ at a particular value of x does not guarantee that there is a maximum or minimum function value there. You need to do further analysis using a sign diagram to determine if the function goes from increasing (slope is positive) to decreasing (slope is negative) or if the function goes from decreasing (slope is negative) to increasing (slope is positive).

Learning Activity 3.3

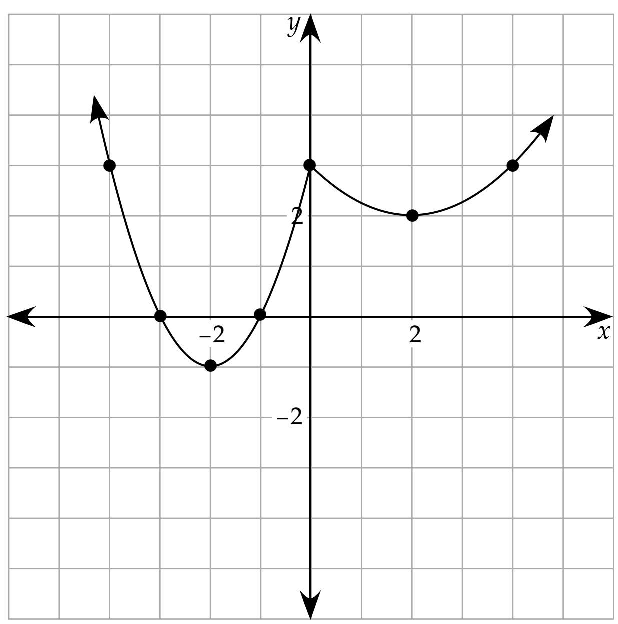
Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Is $x = 0$ a critical value of $y = 2x^2$?
2. Factor: $4x - 12x^2$
3. What sign is the product of an odd number of negative numbers?
4. Differentiate: $y = 3x^4 - 7x + 2$

Use the graph of $f(x)$ below to answer Questions 5 to 8.



5 Determine the critical values where $f'(x) = 0$.

6. Determine the critical values where $f'(x)$ is undefined.

7. What is the range of $f(x)$?

8. Is there an absolute maximum for $f(x)$?

continued

Learning Activity 3.3 (continued)

Part B: First Derivative Applications

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Use the graph of $f(x)$ in Part A to:
   a) determine its absolute extreme values and its relative extreme values.
   b) determine the intervals on the graph of $f(x)$ where the function is increasing and decreasing.
2. Determine the critical values of the following two functions.
   a) $f(x) = 2x^3 - 5x^2 + 4x + 1$
   b) $g(x) = -3x^{-2}$
3. Use the first derivative test to determine the nature and coordinates of the relative extremes of $h(x) = -3x^3 + 6x^2 - 7$.

Lesson Summary

In this lesson, you learned how to recognize and determine the absolute and relative extremes of a function using its graph. In addition, you applied your knowledge of slope of a function as a measure of rate of change from Module 2 to determine where the graph of a function was either increasing or decreasing. Lastly, you learned how to use the first derivative test to determine the relative extremes for a function using critical values, increasing and decreasing intervals, and the sign diagram. In the next lesson, you will apply the first derivative test to solve problems.

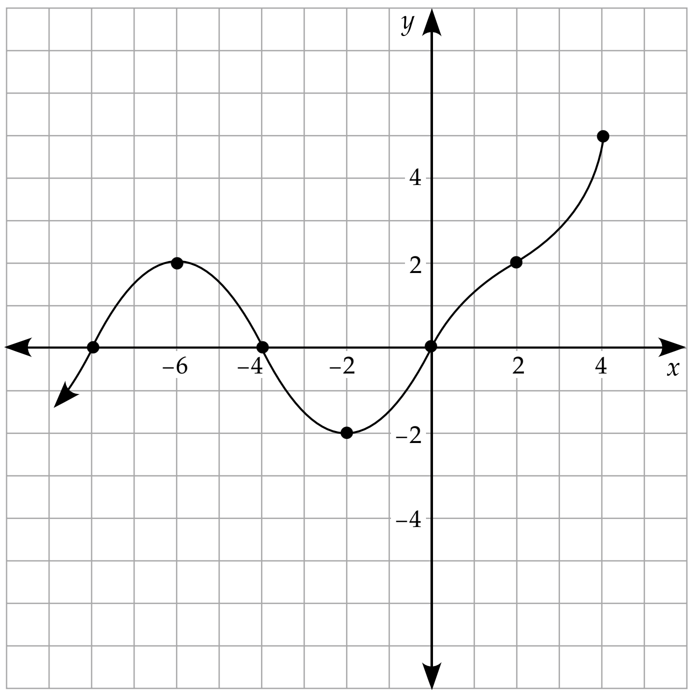
Notes

Assignment 3.3

First Derivative Applications

Total: 13 marks

1. Given the graph of $f(x)$ below, answer the questions that follow.



a) Determine the absolute extreme values and the relative extreme values of $f(x)$. *(3 marks)*

b) Determine the increasing and decreasing intervals on the graph of $f(x)$. *(3 marks)*

continued

Assignment 3.3: First Derivative Applications (continued)

2. Use the first derivative test to determine the relative extreme function values of $f(x) = 4x^3 + 9x^2 - 12x - 10$. *(7 marks)*

Lesson 4: Optimization Problems

Lesson Focus

In this lesson, you will

- ❑ apply the second derivative test to check for extreme values
- ❑ solve optimization problems using derivatives

Lesson Introduction

In previous pre-calculus courses, you learned to solve optimization problems involving quadratic functions. You learned that the maximum of a quadratic function pointing down is at the vertex and the minimum of a quadratic function pointing up is at the vertex. Now, using what you know about calculus, you can find maximum and minimum values with derivatives and solve optimization problems that go beyond quadratic functions.

In this lesson, you will use the perimeter, area, and volume formulas that you know. In addition, you will use your knowledge of the first derivative test and combine it with the second derivative test to solve optimization problems.

The Second Derivative Test

You learned in Module 2 that the second derivative of a given function is the derivative of the first derivative. For example, the function $y = x^3 - 3x - 1$ has a first derivative of $y' = 3x^2 - 3$ and a second derivative of $y'' = 6x$. A few reminders on notation:

	First Derivative	**Second Derivative**
Functional Notation	$f'(x)$	$f''(x)$
Leibniz Notation	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$

Now, you will learn that the second derivative of a function can be used to determine whether a critical value is at a relative maximum or a minimum without exploring the function's increasing and decreasing behaviour.

Let's investigate this concept by analyzing the rate of change of the slopes of the tangent lines on the function $y = x^3 - 3x - 1$ (from Example 4 in the previous lesson). In other words, how does the derivative of the derivative change around its critical values, where $y' = 0$ or y' is undefined.

1. How does the first derivative and second derivative behave around $x = -1$?

x	−1.5	−1.1	−1	−0.9	−0.5
$y' = 3x^2 - 3$	+3.75	0.63	0	−0.57	−2.25
$y'' = 6x$	−9	−6.6	−6	−5.4	−3

Observations:

a) In the previous lesson, you learned that when $y' > 0$, then the function is increasing; and similarly, when $y' < 0$, then the function is decreasing.

b) According to the first derivative test, a relative maximum exists at a critical value if the function changes from increasing to decreasing when it crosses the critical value of x.

c) Looking at the table values for y', you can see the first derivative values are decreasing from $x = -1.5$ to $x = -0.5$.

d) Not coincidentally, the second derivative of the function is negative everywhere between −1.5 and −0.5, which means that the first derivative is decreasing.

Since the second derivative represents the rate of change of the first derivative, the second derivative will be negative in the region of a relative maximum since the slope of the function is decreasing. For example, just before a maximum, the slope of a curve (the first derivative) is positive and as the slope changes, it decreases in value and eventually becomes zero. Then, the slope decreases even more and becomes negative.

2. How do the first derivative and second derivative behave around the critical value, $x = +1$?

x	0.5	0.9	1	1.1	1.5
$y' = 3x^2 - 3$	−2.25	−0.57	0	+0.63	+3.75
$y'' = 6x$	+3	+5.4	+6	+6.6	+9

Observations:

a) In the previous lesson, you learned that when $y' > 0$, then the function is increasing; and similarly, when $y' < 0$, then the function is decreasing.

b) According to the first derivative test, a relative minimum exists at a critical value if the function changes from decreasing to increasing when it crosses the critical value of x .

c) The table values show that y' (the first derivative) values are increasing from $x = 0.5$ to $x = 1.5$.

d) It is consistent that the second derivative of the function is positive everywhere between 0.5 and 1.5, which means that the first derivative is increasing.

Since the second derivative represents the rate of change of the first derivative, the second derivative will be positive in the region of a relative minimum since the slope of the function is increasing. For example, just before a minimum, the slope of a curve (the first derivative) is negative and, as the slope changes, it increases in value and eventually becomes zero. Then, the slope increases more and becomes positive.

Second Derivative Test

If $f'(c) = 0$ and $f''(c) > 0$, then $f(x)$ has a relative minimum at c.

If $f'(c) = 0$ and $f''(c) < 0$, then $f(x)$ has a relative maximum at c.

In words, if the second derivative is positive when evaluated at a critical point, that critical point is at a relative minimum. If the second derivative is negative when evaluated at a critical point, that critical point is at a relative maximum.

The second derivative test can be used as an alternative to the use of a sign diagram of f'.

Example 1

Determine the relative extremes of $f(x) = x^3 + 6x^2 + 9x + 2$ using the second derivative test.

Solution

Determine the critical values by determining the zeroes of the first derivative function.

$$f'(x) = 3x^2 + 12x + 9$$

$$f'(x) = 3\left(x^2 + 4x + 3\right)$$

$$f'(x) = 3(x+1)(x+3)$$

$$f'(x) = 0 \text{ when } x = -1 \text{ and } x = -3$$

The critical values are $x = -1$ and $x = -3$.

Then, determine the value of the second derivative at each critical value.

$$f''(x) = 6x + 12 \longrightarrow \begin{aligned} f''(-1) &= 6(-1) + 12 \\ &= -6 + 12 = +6 \end{aligned} \longrightarrow \begin{aligned} f''(-3) &= 6(-3) + 12 \\ &= -18 + 12 = -6 \end{aligned}$$

According to the second derivative test, there is a **relative minimum** at $f(-1) = -2$ because the second derivative is positive at $x = -1$; and there is a **relative maximum** at $f(-3) = 2$ because the second derivative is negative at $x = -3$.

When is the second derivative test insufficient? If the critical value is a non-permissible value, then you cannot use the second derivative test because a second derivative value does not exist at that point. In addition, what happens if the second derivative is zero at the critical value? Let's review Example 5 from the previous lesson.

Example 2

Determine the relative extremes of $h(x) = -2x^3 - 2$ using the second derivative test.

Solution

Determine the critical values by determining the zeroes of the first derivative function.

$$h'(x) = -6x^2 \qquad \begin{aligned} 0 &= h'(x) = -6x^2 \\ 0 &= x^2 \end{aligned} \qquad x = 0$$

The critical value is $x = 0$.

Then, determine the value of the second derivative at each critical value.

$$h''(x) = -12x \qquad h''(0) = -12(0) = 0$$

Since the second derivative is neither positive nor negative, then there is no relative extreme at $h(0) = -3$, as it was previously shown in Lesson 3 using the first derivative test.

The usefulness of the second derivative is felt when you do not need to know how the whole function behaves but just if there is a maximum or minimum at the critical value of x.

Optimization Problems

Whenever you use words such as *largest*, *most*, *best*, *least*, and *smallest*, you can easily translate them into mathematical language in terms of maxima or minima. For example, business people want to know how they can earn the most profit, manufacturers want to produce products at the lowest possible cost, engineers want to know how to run an engine at maximum efficiently, and apartment management firms want to know what rent should be charged to gain them the most profit. Calculus can help you solve these kinds of questions.

The application of calculus to real-life situations requires careful analysis to discover a mathematical model that accurately describes the problem you need to solve.

General Steps to Solving Optimization Problems

1. Read the problem very carefully. Take note of any given values and conditions.
2. Assign variables to the known and unknown quantities. Identify which quantity needs to be optimized. If possible, express a relationship between the quantity to be optimized and the other variables.
3. If possible, draw a diagram and identify the variables.
4. If necessary, express all the relationships between the variables in one equation to be optimized in the problem.
5. Express the quantity to be optimized as a function of one variable. Determine the possible domain of the variable.
6. With the quantity to be optimized as a function of one variable, you can now apply the first or second derivative tests to determine whether or not any maxima or minima exist for this problem.
7. Finally, express your final answer in clear and concise terms, which can be easily understood by someone else reading your solution. Make certain that your answer makes sense with respect to the original statement of the problem.

Example 3

Find two natural numbers whose sum is 8 such that their product is a maximum.

Solution

Remember that you are trying to maximize the product (so you need a function to represent the product using only one variable).

$a + b = 8$	Define the variables.
$a = 8 - b$	Let a, b be natural numbers and P be their product.
$P = a \cdot b$	You need an expression to represent the product, since you are to find its maximum
$P = (8 - b) \cdot b$ $P = 8b - b^2$	Combine the two equations so that you have a function defined with only two variables.
$P' = 8 - 2b$	Determine the first derivative with respect to b.
$0 = P' = 8 - 2b$	
$2b = 8$	Determine the critical value(s).
$b = 4$	The critical value of b to make $P' = 0$ is $b = 4$.
$P'' = -2$	Determine the second derivative (using $P' = 8 - 2b$).
$P'' < 0$	Evaluate the second derivative at the critical value. (The second derivative is -2 everywhere on the domain of P.)
Relative maximum	According to the second derivative test, there is a relative maximum at $b = 4$ because the second derivative is a negative.
$a = 4$ and $b = 4$	The two numbers whose sum is 8 with a maximum product.

The next few examples involving measurement of 2-D and 3-D objects demonstrate the need for diagrams to help you solve the problems.

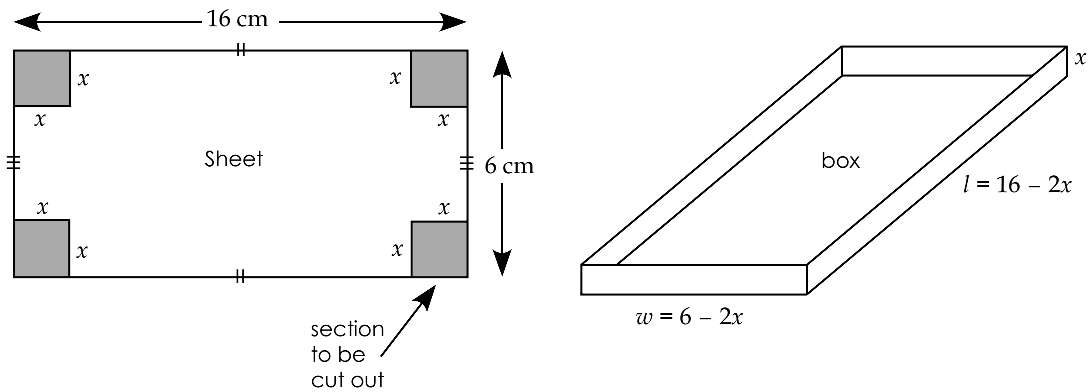
Example 4

A rectangular metal box without a lid is made from a sheet of tin 16 cm by 6 cm. Equal square corners are cut from each of the four corners of the sheet of tin. The edges are turned up and soldered (joined together) to form a box. Find the dimensions of the box having maximum volume and determine the maximum volume.

Solution

Remember your goal is to maximize the volume of the box (so you need a function representing volume in one variable).

In this case, a diagram can be extremely helpful.



$V = l \cdot w \cdot h$	$h = x$ $l = 16 - 2x$ $w = 6 - 2x$	Define the variables. Let $h = x$ be the height of the box, w and l be the width and length of the box, and V be the volume of the box.

$$V = (16 - 2x)(6 - 2x)x$$

$$V = \left(96 - 44x + 4x^2\right)x$$

State the combined function.

$$V = 96x - 44x^2 + 4x^3$$

Consider the domain in this context where each dimension must be positive. Solve the inequalities to determine the domain of the function.

$$h = x > 0$$

$$l = 16 - 2x > 0$$

$$-2x > -16$$

$$x < 8$$

$$w = 6 - 2x > 0$$

$$-2x > -6$$

$$x < 3$$

So, the domain is $0 < x < 3$.

Determine the first derivative of V with respect to x.

$$V' = 96 - 88x + 12x^2$$

Determine the critical value(s) that are valid within the domain.

$$V' = 12x^2 - 88x + 96$$

$$V' = 4\left(3x^2 - 22x + 24\right)$$

$$V' = (3x - 4)(x - 6)$$

Solve $V' = 0$ to find the critical values $x = c$.

$$c = \frac{4}{3} \text{ and } c = 6$$

Note that $x = 6$ is not in the domain so $x = \frac{4}{3}$ is the only critical value.

Determine the second derivative.

$$V'' = -88 + 24x$$

Evaluate the second derivative at the critical value, $x = \frac{4}{3}$.

$$V'' = -88 + 24\left(\frac{4}{3}\right) = -88 + 32 = -56$$

$$V'' < 0$$

According to the second derivative test since V'' is negative at $x = \frac{4}{3}$, so there is a relative maximum there.

$$h = \frac{4}{3} \approx 1.3 \text{ cm}$$

$$w = 6 - 2\left(\frac{4}{3}\right) = 6 - \frac{8}{3} = \frac{18}{3} - \frac{8}{3} = \frac{10}{3}$$

$$\approx 3.3 \text{ cm}$$

$$l = 16 - 2\left(\frac{4}{3}\right) = \frac{48}{3} - \frac{8}{3} = \frac{40}{3}$$

$$\approx 13.3 \text{ cm}$$

$$V = \left(\frac{40}{3}\right)\left(\frac{4}{3}\right)\left(\frac{10}{3}\right) = \frac{1600}{27} \approx 59.3 \text{ cm}^3$$

The dimensions that produce the maximum volume are $\frac{4}{3}$, $\frac{10}{3}$, and $\frac{40}{3}$. The maximum volume is $\frac{1600}{27} \approx 59.3 \text{ cm}^3$.

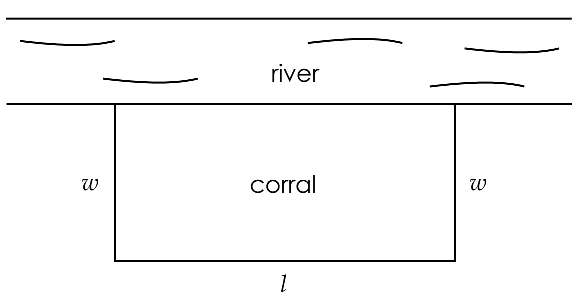
Example 5

You have 160 m of fencing to construct a fence around a corral with a maximum area. If one side of the corral borders a river, find the largest rectangular area that can be closed by 160 m of fencing, if no fence is needed along the river.

Solution

The goal is to determine the maximum area (so find a function for area in one variable).

In this case, a diagram can be extremely helpful.



Define the variables.

Let l, w, and A be length, width, and area respectively.

$$l + 2w = 160$$

$$l = 160 - 2w$$

$$A = l \cdot w$$

You need an expression for area, A, to use to find the maximum area.

$$A = (160 - 2w) \cdot w$$ Combine the equations to form one function with two variables.

$$A = 160w - 2w^2$$

Each dimension must be greater than zero.

Solve the inequalities to determine the domain of the function.

$$w > 0$$

$$l = 160 - 2w > 0$$

$$-2w > -160$$

$$w < 80$$

So, the domain of the function is $0 < w < 80$.

$$A' = 160 - 4w$$ Determine the derivative.

$$0 = 160 - 4w$$

$$4w = 160$$ Determine the critical value(s) where $A' = 0$.

$$w = 40$$

The critical value is at $w = 80$.

$A'' = -4$ Determine the second derivative.

$A'' < 0$ Evaluate the second derivative at the critical value. The value of A'' is -4 everywhere on the domain.

Since $A'' < 0$, then there is a relative maximum at $w = 40$.

$$w = 40 \text{ cm}$$

$$l = 160 - 2(40) = 160 - 80 = 80 \text{ cm}$$

$$A = (40)(80) = 3200 \text{ cm}^2$$

The maximum area is 3200 cm^2.

Example 6

If 2700 cm^2 of material is available to make a box with a square base and an open top, determine the largest possible volume of the box. (**Hint:** The surface area can be found using $SA = b^2 + 4bh$.)

Solution

Remember that the goal is to determine the maximum volume, so you need an expression for volume in terms of b or h.

$$V = b^2h$$

Define the variables.

Let b represent a side length of the square base, and SA, h, and V represent surface area, height, and volume, respectively.

$$SA = b^2 + 4bh = 2700$$

$$4bh = 2700 - b^2$$

$$h = \frac{2700 - b^2}{4b}$$

Combine the equations to form one function for volume in terms of b.

$$V = b^2\left(\frac{2700 - b^2}{4b}\right) = \frac{2700b}{4} - \frac{b^3}{4}$$

Determine the domain for this scenario.

Each dimension should be positive.

$$b > 0$$

$$h > 0$$

$$\frac{2700 - b^2}{4b} > 0$$

$$2700 - b^2 > 0$$

$$-b^2 > -2700$$

$$b^2 < 2700$$

So, $b < \sqrt{2700}$.

$$b < 51.96 \text{ cm}$$

Determine the first derivative of V in terms of b.

$$V' = \frac{2700}{4} - \frac{3b^2}{4}$$

Solve for b when $V' = 0$ to determine the critical value of b.

$$\frac{2700}{4} - \frac{3b^2}{4}$$

$$0 = V' = \frac{2700}{4} - \frac{3b^2}{4}$$

$$\frac{3b^2}{4} = \frac{2700}{4}$$

$$b^2 = \frac{2700}{4} \cdot \frac{4}{3} = 900$$

$$b = \pm 30$$

The critical value of b is 30 cm.

Determine the second derivative.

$$V'' = -\frac{6b}{4} = -1.5b$$

Evaluate the second derivative at the critical value.

$$V'' = -1.5(30) = -45$$

According to the second derivative test, since $V'' < 0$ when b = 30, then there is a relative maximum there.

Find the maximum volume.

$$h = \frac{2700 - (30)^2}{4(30)}$$

$$h = \frac{2700 - 900}{120} = \frac{1800}{120} = 15$$

$$V = (30)^2(15) = 900 \cdot 15 = 13\,500$$

The maximum volume is 13 500 cm^3.

Solving optimization problems can also be helpful to business owners. Sometimes they can best be explored as a table of values first when trying to determine the equation.

Example 7

The cost of running an aircraft ($/h) cruising at an altitude of *h* metres at an airspeed of 500 km/h is given by the function:

$$C(h) = 2500 + \frac{h}{10} + \frac{10\,000\,000}{h}$$

Find the optimum height when the cost of operation is the least—in other words, the cruising altitude that yields the minimum cost.

Solution

Remember that the goal is to minimize the cost.

Define the variables.

Let h be the altitude of the aircraft and C be the cost of operating the aircraft.

$$C(h) = 2500 + \frac{h}{10} + \frac{10\,000\,000}{h}$$

Determine the domain of the function.

$$h > 0$$

The altitude must be a positive number.

Determine the first derivative.

$$C(h) = 2500 + \frac{1}{10}h + 10\,000\,000h^{-1}$$

$$C'(h) = \frac{1}{10} - 10\,000\,000h^{-2}$$

Determine the critical value(s) when $C'(h) = 0$.

$$0 = \frac{1}{10} - 10\,000\,000h^{-2}$$

$$0 = \frac{1}{10} - \frac{10\,000\,000}{h^2}$$

$$0 = \frac{h^2 - 100\,000\,000}{10h^2}$$

$$0 = h^2 - 100\,000\,000$$

$$h^2 = 100\,000\,000$$

$$h = \pm 10\,000$$

Determine the critical value(s) when $C'(h)$ does not exist.

$$C'(h) = \frac{1}{10} - \frac{10\,000\,000}{h^2}$$

$h = 0$ is a non-permissible value

The critical values of h are 0, 10 000, and −10 000.

Notice that only one of the three potential critical values is within the domain. Therefore, $h = 10\,000$ is the only critical value to consider.

Determine the second derivative.

$$C''(h) = 20\,000\,000h^{-3}$$

Evaluate the second derivative at the critical value, $h = 10\,000$.

$$C''(10\,000) = 20\,000\,000(10\,000)^{-3}$$
$$= 0.00002$$

$$C''(10\,000) > 0$$

Since $C''(10\,000)$ is positive, according to the second derivative test, there is a relative minimum at $h = 10\,000$ m.

The optimum altitude that yields the least cost is 10 000 m.

Example 8

If a farmer digs potatoes on July 1, his or her crop will be 1200 kg, which will sell as new potatoes at \$0.30/kg. If the crop is allowed to mature further, it will increase at 60 kg per week. However, the price will drop \$0.01/kg per week. When should he or she dig the crop for maximum revenue?

Solution

The goal is to determine the week that will generate maximum revenue. You need a function model to represent revenue in one variable. Revenue is calculated as crop yield × selling price or $R = C \times P$.

The table below describes how the revenue changes per week.

Week (#)	**Crop Yield (kg)**	**Selling Price ($/kg)**	**Revenue ($)**
0	1200	0.30	1200 × 0.30 = 360
1	1200 + 60 = 1260	0.30 − 0.01 = 0.29	365.40
2	1260 + 60 = 1320 1200 + 2(60) = 1320	0.29 − 0.01 = 0.28 0.30 − 2(0.01) = 0.28	369.60
3	1320 + 60 = 1380 1200 + 3(60) = 1380	0.28 − 0.01 = 0.27 0.30 − 3(0.01) = 0.27	372.60
4	1200 + 4(60) = 1440	0.30 − 4(0.01) = 0.26	374.40
n	$1200 + n(60)$	$0.30 - n(0.01)$	$(1200 + 60n)(0.30 - 0.01n)$

Define the variables.

Let C, P, and R be crop yield, selling price, and revenue, respectively. In addition, n, represents the number of weeks after July 1st.

$$C = 1200 + 60n$$

$$P = 0.30 - 0.01n$$

$$R = C \cdot P$$

Combine the equations to form one function defined with only two variables.

$$R = (1200 + 60n)(0.30 - 0.01n)$$

$$R = 360 + 6n - 0.6n^2$$

Determine the domain for this scenario.

Both the crop yield and selling price must be positive. Solve the inequalities to determine the domain of the function.

$$C > 0$$

$$1200 + 60n > 0$$

$$60n > -1200$$

$$n > -20$$

$$P > 0$$

$$0.30 - 0.01n > 0$$

$$-0.01n > 0.30$$

$$n < 30$$

But the number of weeks needs to be positive and $n > 0$.

So, the domain of the function is $0 < n < 20$.

Determine the first derivative.

$$R = -0.6n^2 + 6n + 360$$

$$R' = -1.2n + 6$$

Solve for $R' = 0$ to find the critical value.

$$0 = -1.2n + 6$$

$$6 = 1.2n$$

$$n = 5$$

The critical value is $n = 5$.

Determine the second derivative.

$$R'' = -1.2$$

Evaluate the second derivative at the critical value. R'' is negative everywhere.

$$R'' < 0$$

According to the second derivative test, since R'' is negative, there is a relative maximum at $n = 5$.

$$C = 1200 + 60(5) = 1200 + 300 = 1500 \text{ kg}$$

$$P = 0.30 - 0.01(5) = 0.30 - 0.05 = \$0.25/\text{kg}$$

$$R = 1500 \times 0.25 = \$375$$

The farmer should dig his or her potatoes **five weeks** after July 1st to yield the maximum revenue of $375.

All these examples of optimization problems showcase how efficient the second derivative test is in determining the nature of the extreme values of a problem. It is important to note that you could have found the nature of the extreme values using a sign diagram of the first derivative of the function.

Learning Activity 3.4

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Determine the first and second derivative for $y = 5x^3 - 3x^2 + 2$.
2. Determine the first and second derivative for $f(x) = 5x + \frac{2}{x}$.
3. Determine the perimeter of a rectangle that has a width of 5 cm and a length of 7 cm.
4. Determine the area of a rectangle that has a width of 12 cm and a length of 7 cm.
5 Determine the volume of a box with a square base, 9 cm on each side, and a height of 5 cm.
6. Factor: $2x^2 - 3x - 2$
7. Solve: $15 - 3x > 0$
8. Solve: $0 = 3x^2 - 6$

continued

Learning Activity 3.4 (continued)

Part B: Optimization Problems

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Determine the relative extremes of $y = 5x^3 - 3x^2 + 2$, using the second derivative test.
2. Find two natural numbers such that their product is 9 and their sum is a minimum.
3. A box with an open top is to be constructed from a square piece of cardboard, 2 metres wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume of such a box.
4. A retailer finds he or she can sell 20 CDs per month at $25 each for a gross revenue of $500. For each $1 reduction in price, the retailer can sell 2 more CDs per month. What selling price will produce the maximum revenue per month, and how many CDs would he or she sell at this price?

Lesson Summary

In this lesson, you reviewed the measurement formulas and how to determine the second derivative. Then, you learned how to use the second derivative test to determine the relative extremes of a function. The second derivative test was then applied to solve many types of optimization problems. In order to correctly solve the optimization problems, you had to recall how to determine the domain of a function by solving linear inequalities. In the next lesson, you will revisit your knowledge of how to determine the characteristics of the graph of a function from Lesson 3 and learn about a new characteristic using the second derivative.

Assignment 3.4

Optimization Problems

Total: 20 marks

1. Find two numbers whose difference is 150 and whose product is a minimum. *(6 marks)*

continued

Assignment 3.4: Optimization Problems (continued)

2. The lifeguard at a public beach has 400 metres of rope to lay out a restricted swimming area using a straight shoreline as one side of the rectangle. If she wants to maximize the swimming area, what should be the dimensions of the rectangle? *(7 marks)*

continued

Assignment 3.4: Optimization Problems (continued)

3. A hockey team plays in an arena that holds 16 000 spectators. Average attendance to the games is 13 000 when tickets are $35 each. When tickets are lowered by $1, the attendance increases by 500 people. If this pattern continues, what ticket price should the team set to ensure maximum revenue? *(7 marks)*

Notes

Lesson 5: Concavity and Sketching Polynomial Functions

Lesson Focus

In this lesson, you will

- ❑ determine intervals where the graph of a function is concave up and concave down
- ❑ determine the coordinates of points of inflection
- ❑ accurately sketch a polynomial function using its characteristics, including intercepts, domain, range, maxima, minima, points of inflection, and concavity

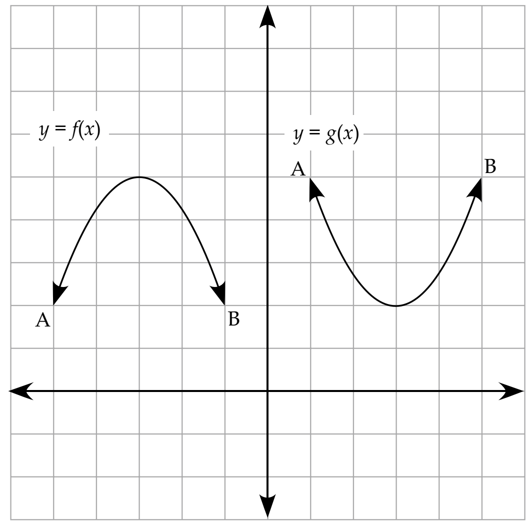
Lesson Introduction

With technology and graphing software, you can quickly and accurately sketch a function or a relation on a coordinate plane. In pre-calculus math classes, you learned to accurately sketch linear and quadratic functions without the use of graphing technology. You even expanded your graphing repertoire to include exponential, logarithmic, and trigonometric functions. In this lesson, you will build on what you know about graphing polynomial functions to include coordinates of relative maxima and minima. Furthermore, you will learn to find the coordinates of points where the rate of change of slope goes from decreasing to increasing and vice versa. Until the onset of technology, accurately graphing functions and relations was made possible by applying calculus concepts. Today, calculus is not needed for accurate graphing but it is still necessary for the analysis of complex function and relation curves.

In this lesson, you will learn the meaning of concavity and what it means for a curve to be concave up or concave down. In addition, you will learn to sketch a polynomial function using all of its features: x- and y-intercepts, domain, range, relative extrema, concavity, and points of inflection.

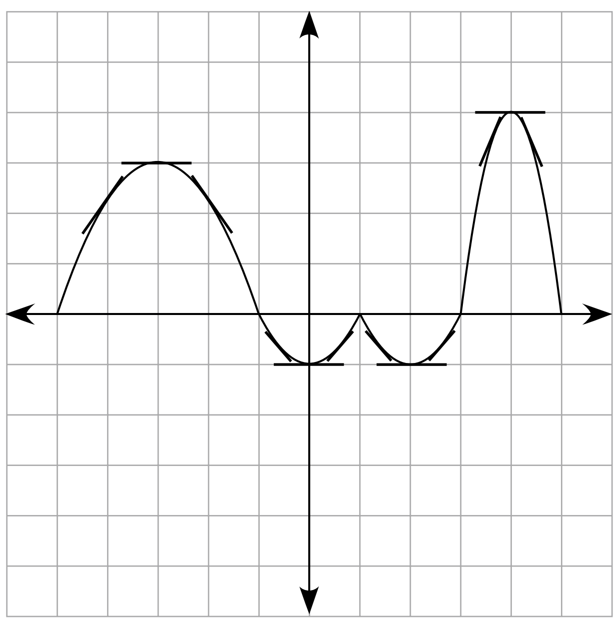
Concavity

When sketching the graphs of functions, you will need to know the appearance of a curve that is considered to be either concave up or concave down. To help you to get an idea of the concept of concavity, study the graphs of two functions, $f(x)$ and $g(x)$.



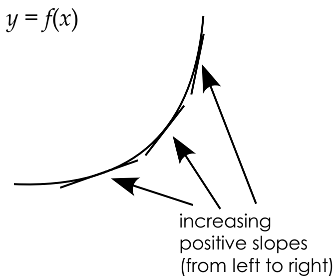
The graph of $f(x)$ is said to be concave down between A and B. The graph of $g(x)$ is said to be concave up between A and B.

Graphically, one way to determine whether a function is concave up or concave down on a particular interval is to draw tangent lines to the curve (see the graph of $h(x)$ below).

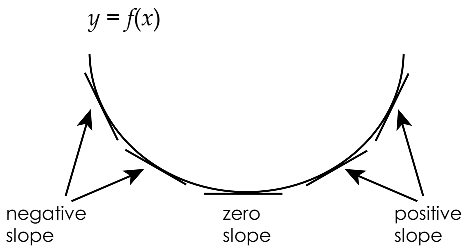


If the curve lies **below** the tangent lines, it is **concave down**. If the curve lies **above** the tangent lines, it is **concave up**.

Let's analyze a function $f(x)$ when it is concave up. Study the illustrations below, each showing an interval of a curve that is concave up.



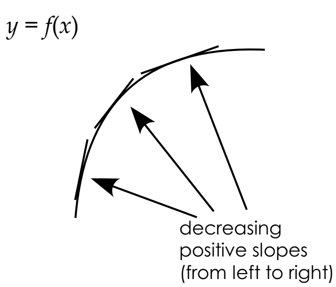
Note that the slopes of the tangent lines increase from left to right as the tangent lines change from horizontal to vertical. This indicates that the first derivative is increasing on the interval shown. An increasing first derivative means that the rate of change of the slope is positive and the second derivative is positive.



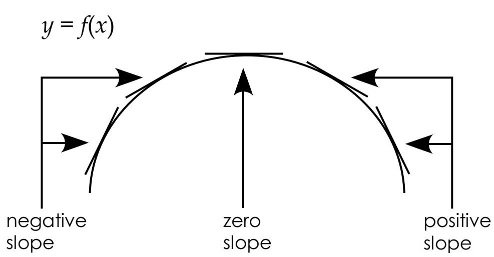
In the diagram at the left, you notice that the slopes of the tangent lines are also increasing from left to right because the slopes of the five tangent lines shown are $-2, -1, 0, 1$, and 2 from left to right. The slopes of the tangent lines changed from negative to zero to positive. Again, an increasing first derivative means that the second derivative is positive.

The graph of $f(x)$ is said to be concave up on an interval if $f'(x)$, the slope of $f(x)$, is increasing throughout that interval, which means that $f''(x) > 0$.

Now, let's analyze a function $f(x)$ when it is concave down. Study the illustrations below, each showing an interval of a curve that is concave down.



Although the slopes of the tangent lines are positive, they decrease from left to right as the tangent lines change from vertical to horizontal. This indicates that the first derivative is decreasing on the interval shown. A decreasing first derivative represents a negative rate of change and a negative second derivative.



In the diagram on the left, we notice that the slopes of the tangent lines are decreasing from left to right because the slopes of the five tangent lines shown are $2, 1, 0, -1$, and -2 from left to right. The slopes of the tangent lines changed from positive to zero to negative. Again, a decreasing first derivative represents a negative second derivative.

The graph of $f(x)$ is said to be concave down on an interval if $f'(x)$, the slope of $f(x)$, is decreasing throughout that interval, which means $f''(x) < 0$.

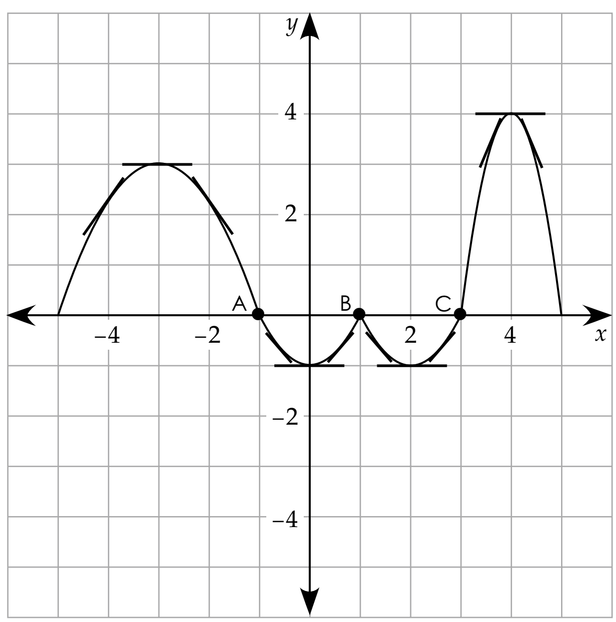
Below is a summary of what you learned about the relationship between concavity and the second derivative.

Concavity

1. If $f''(x) > 0$ for all x on the interval I, then the graph of the function $f(x)$ is **concave up** on I.
2. If $f''(x) < 0$ for all x on the interval I, then the graph of the function $f(x)$ is **concave down** on I.

The next question you need to address is how to determine the intervals of concavity or, more importantly, how do you identify where concavity changes. Recall the usefulness of critical values of the first derivative to separate the increasing and decreasing intervals of a function. More specifically, these critical values occurred where the first derivative changed sign, where the first derivative was zero, or was undefined. Thus, it is fair to extrapolate that the zeroes of the second derivative will be critical in determining when a the concavity of a function changes. The point at which a function changes concavity is called the **point of inflection**.

To assist in understanding the meaning of the term **point of inflection**, use the same function, $h(x)$, shown earlier in this lesson.



Interval	(−5, −1)	(−1, 1)	(1, 3)	(3, 5)
Concavity	Down	Up	Up	Down

Remember that a point on a curve is called a **point of inflection** if the curve changes from concave down to concave up, or from concave up to concave down at that point. Since the curve changes concavity at points A and C, then the points A and C are inflection points. The curve does not change concavity at point B, so B is not a point of inflection.

Definition of a Point of Inflection

A point $(c, f(c))$ is a point of inflection on the graph of $f(x)$ if:

1. $f(c)$ is defined (the point must lie in the domain of the curve).
2. $f''(x)$ changes sign as x increases through $x = c$.

In words, there will be a point of inflection at any point on a function where the sign of the second derivative changes (assuming the point exists in the domain). That is, the function curve changes from concave up to concave down or from concave down to concave up at a point of inflection.

Example 1

Determine the concave up and concave down intervals and any points of inflection for the function $f(x) = 2x^3 + 9x^2 + 7x - 6$.

Solution

A point of inflection occurs where the second derivative, $f''(x)$, changes signs. Consequently, your first task is to find the second derivative of the given function.

$$f'(x) = 6x^2 + 18x + 7$$

$$f''(x) = 12x + 18$$

Now, find potential points of inflection by setting $f''(x) = 0$ and solving for x.

$$0 = 12x + 18$$

$$-12x = 18$$

$$x = -1.5$$

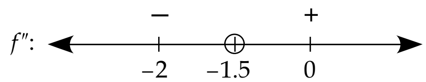
and

$$f(-1.5) = 2(-1.5)^3 + 9(-1.5)^2 + 7(-1.5) - 6 = -3$$

Therefore, a possible inflection point is (−1.5, 3). Now you need to check that f'' changes sign at $x = -1.5$.

You need to determine the sign of the second derivative for values of x less than -1.5 and also for values of x greater than -1.5 using a sign diagram.

$f''(x) = 12x + 18$



From the sign diagram, you can determine that:

a) $f(x)$ is concave down on $(-\infty, -1.5)$, because $f''(x) < 0$ there,

b) $f(x)$ is concave up on $(-1.5, \infty)$, because $f''(x) > 0$ there.

The sign of the second derivative changes, from negative to positive, at the point $(-1.5, -3)$.

In summary, the two requirements for an inflection point have been met.

a) $f(-1.5)$ is defined.

b) The sign of $f''(x)$ changes at $x = -1.5$.

Thus, the point $(-1.5, -3)$ is a point of inflection.

Sketching Polynomial Graphs

In previous math courses, you could only draw rough sketches of polynomial functions because there was no easy way to determine the locations of the relative maximum and minimum values. Now, with the aid of your calculus knowledge, you can find maxima, minima, and inflection points and determine concavity so you have all that you need to accurately graph every feature of any polynomial function. You will use all the characteristics of a polynomial function to sketch its graph.

Determining Intercepts

The ***y*-intercept** is found by substituting $x = 0$ into the function and solving for $y = f(0)$.

The y-intercept of the function $f(x) = 2x^3 + 9x^2 + 7x - 6$ is:

$$f(0) = 2(0)^3 + 9(0)^2 + 7(0) - 6 = -6$$

The ***x*-intercept(s)** is/are found by substituting $y = 0$ and solving for x.

a) You can solve linear or quadratic equations by factoring or using the quadratic formula.

b) To solve other equations, we can use the factor theorem, and long or synthetic division.

Example 2

Determine the x-intercept(s) of $f(x) = 2x^3 + 9x^2 + 7x - 6$.

Solution

Use the factor theorem to determine one of the factors of
$f(x) = 2x^3 + 9x^2 + 7x - 6$.

You may remember from pre-calculus courses that the **factor theorem** states: "If $f(a) = 0$, then $x - a$ is a factor of $f(x)$."

Use the factors of the constant in the polynomial: ±1, ±2, ±3, ±6

Check $x = 1$.

$$f(1) = 2(1)^3 + 9(1)^2 + 7(1) - 6 = 2 + 9 + 7 - 6 = 12 \neq 0$$

So, $x - 1$ is not a factor.

Try a negative value such as $x = -2$.

$$f(-2) = 2(-2)^3 + 9(-2)^2 + 7(-2) - 6 = -16 + 36 - 14 - 6 = 0$$

So, $x + 2$ is a factor.

Now to determine the remaining factors, use long or synthetic division.

Note: If you need a refresher on this strategy, see your Grade 12 pre-calculus textbook or similar resources.

divisor → −2	2	9	7	−6	← dividend
	↓	−4	−10	6	
	2	5	−3	0	← remainder

(2, 5, −3: quotient)

The last line provides us with the quotient $2x^2 + 5x - 3$ and the remainder of zero.

So $f(x) = 2x^3 + 9x^2 + 7x - 6 = (x + 2)(2x^2 + 5x - 3)$.

We can continue to factor the quadratic.

$$2x^2 + 5x - 3$$

$$(2x - 1)(x + 3)$$

So $f(x) = (x + 2)(2x^2 + 5x - 3) = (x + 2)(2x - 1)(x + 3)$.

Now that the function is factored, you can set $f(x) = 0$ and solve for x to find the x-intercepts.

$$0 = f(x) = (x + 2)(2x - 1)(x + 3)$$

$$x + 2 = 0 \quad 2x - 1 = 0 \quad x + 3 = 0$$

$$x = -2 \quad\quad x = 0.5 \quad\quad x = -3$$

The y-intercept is always easy to find. The x-intercepts are sometimes difficult. You can draw a good sketch of a polynomial function without the precise locations of the x-intercepts.

Domain and Range

The **domain** of a polynomial function is always $x \in \Re$ unless otherwise stated.

The range of a polynomial function depends on its degree. If the polynomial's degree is odd (such as a line or a cubic), then it has no absolute extremes and its range is always $y \in \Re$ unless otherwise stated. However, if its degree is even (such as a parabola or quartic), then the range is dependent on the absolute extremes.

- An even degree polynomial with a positive leading coefficient opens up, so it will have an absolute minimum at one of its relative minimums.

  For example, $y = x^4 + 2x^3 - 3x + 1$ is an even degree polynomial with a positive leading coefficient, so it will have an absolute minimum and a limited range.

- An even degree polynomial with a negative leading coefficient opens down, so it will have an absolute maximum at one of its relative minimums.

  For example, $y = -3x^4 - 5x^3 + 7x - 8$ is an even degree polynomial with a negative leading coefficient, so it will have an absolute maximum and a limited range.

You can use calculus to determine the range by finding the absolute and relative extreme values.

Sketching Polynomials Using Its Characteristics

To sketch a polynomial function, plot its x- and y-intercepts, its relative extremes, and its points of inflection. Then, connect the points using the concavity information. Normally, you determine the domain and range after the sketch is made.

Example 3

Sketch the graph of an odd degree polynomial function that has the following information:

Relative minimum: $(-0.5, -7.5)$

Relative maximum: $(-2.5, 1.5)$

y-intercept: -6

x-intercepts: -2, 0.5, and -3

Inflection point: $(-1.5, -3)$

Concave down on: $(-\infty, -1.5)$

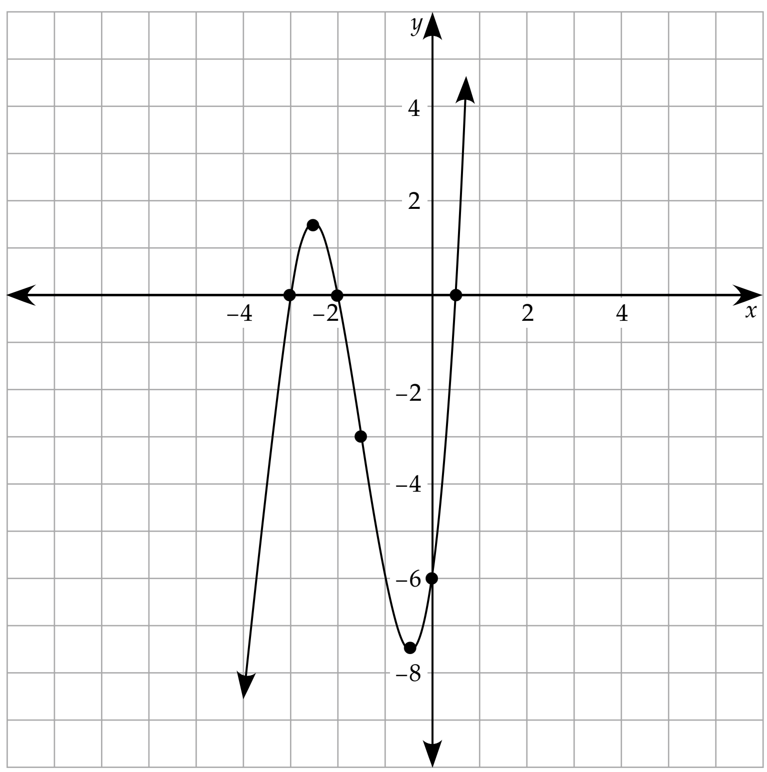
Concave up on: $(-1.5, \infty)$

Domain: $x \in \Re$

Range: $y \in \Re$

Solution

1. Start the sketch by plotting all the points.
2. Connect the points using the concavity information.
3. Use the domain and range to confirm the end behaviour.



What if the characteristics are not already stated? Before you could sketch the graph, you would need to determine the x- and y-intercepts of the function, its relative extremes, its concavity, and its points of inflection, and then the domain and range. You don't always need to find x-intercepts or the y-intercept to accurately sketch the function. However, if they are not too difficult to find, they can be helpful points.

Example 4

Sketch the graph of $f(x) = x^4 - 5x^2 + 4$, after identifying its x- and y-intercepts, relative extremes, its concavity and points of inflection, and then its domain and range.

Solution

Determine its y-intercept by substituting $x = 0$ into the function.

$$f(0) = (0)^4 - 5(0)^2 + 4 = 4$$

y-intercept: 4

Determine its x-intercepts by using the factor theorem or factor.

Test $x = 1$.

$$f(1) = (1)^4 - 5(1)^2 + 4 = 1 - 5 + 4 = 0$$

So $(x - 1)$ is a factor.

Similarly $x = -1, -2$, and 2 are factors.

$$f(-1) = (-1)^4 - 5(-1)^2 + 4 = 1 - 5 + 4 = 0$$

$$f(2) = (2)^4 - 5(2)^2 + 4 = 16 - 20 + 4 = 0$$

$$f(-2) = (-2)^4 - 5(-2)^2 + 4 = 16 - 20 + 4 = 0$$

$$f(x) = (x - 1)(x + 1)(x - 2)(x + 2)$$

Or, determine its x-intercepts by factoring the trinomial directly.

$$f(x) = x^4 - 5x^2 + 4$$

$$f(x) = \left(x^2 - 4\right)\left(x^2 - 1\right)$$

$$f(x) = (x + 2)(x - 2)(x + 1)(x - 1)$$

Find the x-intercepts by solving $f(x) = 0$.

The x-intercepts are -2, -1, 1, and 2.

Determine its first derivative and critical values.

$$f'(x) = 4x^3 - 10x$$

$$f'(x) = 2x\left(2x^2 - 5\right)$$

Solve $f'(x) = 0$.

$$2x = 0 \quad \text{or} \quad 2x^2 - 5 = 0$$

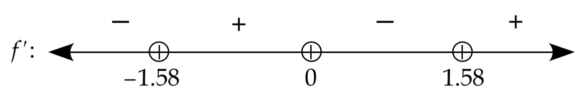
$$x = 0 \qquad\qquad x^2 = \frac{5}{2}$$

$$x = \pm\sqrt{2.5}$$

$$x = \pm 1.58$$

So the critical values are −1.58, 0, 1.58.

Create a sign diagram for $f' = 4x^3 - 10x$.



The f' sign diagram shows the function is decreasing on $(-\infty, -1.58)$ or $(0, 1.58)$ and it is increasing on $(-1.58, 0)$ or $(1.58, \infty)$.

Determine the function values at the critical values.

$$f\left(-\sqrt{2.5}\right) = \left(-\sqrt{2.5}\right)^4 - 5\left(-\sqrt{2.5}\right)^2 + 4$$
$$= -2.25$$

$$f\left(\sqrt{2.5}\right) = ((\sqrt{2.5})^4 - 5\left(\sqrt{2.5}\right)^2 + 4$$
$$= -2.25$$

$$f(0) = (0)^4 - 5(0)^2 + 4$$
$$= 0 - 0 + 4 = 4$$

The coordinates at the critical values are (−1.58, −2.25), (0, 4), and (1.58, −2.25). There is a minimum at (−1.58, −2.25) and (1.58, −2.25) since f' goes negative to positive. There is a minimum at (0, 4) since f' goes positive to negative.

Determine the second derivative.

$$f'(x) = 4x^3 - 10x$$

$$f''(x) = 12x^2 - 10$$

Solve $f''(x) = 0$ to find potential points of inflection.

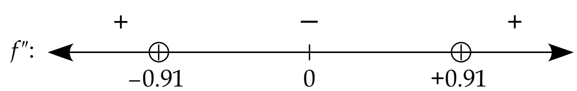
$$0 = 12x^2 - 10$$

$$12x^2 = 10$$

$$x^2 = 0.8\dot{3}$$

$$x = \pm\sqrt{0.8\dot{3}} \approx \pm 0.91$$

Create a sign diagram for $f'' = 12x^2 - 10$.



The f'' sign diagram shows there is a point of inflection at −0.91 and +0.91 since f'' changes sign there. The f'' sign diagram also shows $f(x)$ is concave down on the interval (−0.91, 0.91) and concave up on (−∞, −0.91) or (0.91, ∞).

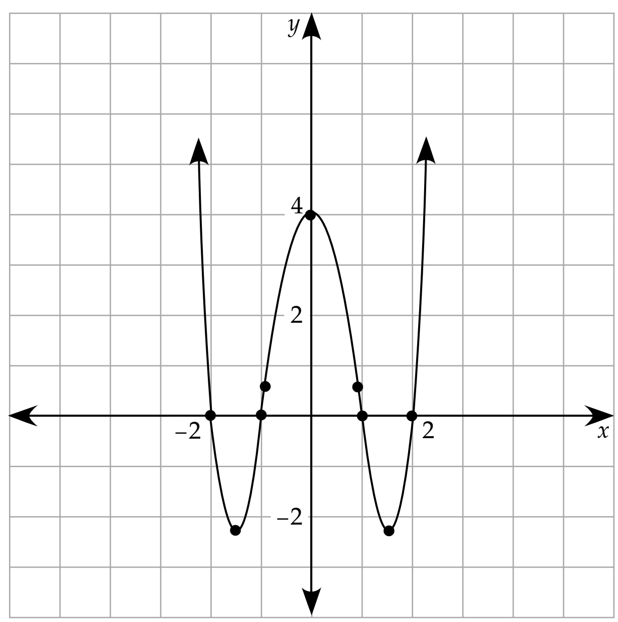
Determine the coordinates of point(s) of inflection.

$$f\left(\sqrt{0.8\dot{3}}\right) = \left(\sqrt{0.8\dot{3}}\right)^4 - 5\left(\sqrt{0.8\dot{3}}\right)^2 + 4 \approx 0.54$$

$$f\left(-\sqrt{0.8\dot{3}}\right) = \left(-\sqrt{0.8\dot{3}}\right)^4 - 5\left(-\sqrt{0.8\dot{3}}\right)^2 + 4 \approx 0.54$$

The points of inflection are at (−0.91, 0.54) and (0.91, 0.54).

Use all of your information to sketch the curve. Begin by placing all the coordinate points you know. Then use the concavity information to sketch the curve. Confirm that your function is increasing and decreasing in the required intervals.



$x \in \Re$

State the domain.

Since the polynomial has an even degree and a positive leading coefficient, then it opens up with an absolute minimum at −2.25, and the range is $\{y \mid y \geq -2.25, y \in \Re\}$.

State the range.

You can now use your knowledge of finding the key characteristics of a function in order to sketch its graph.

Learning Activity 3.5

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Determine the x-intercept(s) of $f(x) = (x - 2)(x + 3)(x - 1)$.
2. Determine the y-intercept of $f(x) = (x - 2)(x + 3)(x - 1)$.
3. Determine the critical values when $f'(x) = 12x - 6$.
4. What is the concavity of $f(x)$ on an interval where $f'(x) > 0$ and $f''(x) > 0$?
5. What is the concavity of $f(x)$ on an interval where $f'(x) > 0$ and $f''(x) < 0$?
6. Draw a curve that is increasing and concave up.
7. Draw a curve that is increasing and concave down.
8. Is $x - 1$ a factor of $f(x) = 2x^3 + 9x^2 - 4x - 7$?

continued

Part B: Concavity and Sketching Polynomial Functions

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Find the point(s) of inflection and determine where the graph is concave up and concave down for the function $f(x) = x^3 - 5x^2 - 2x + 24$.

2. Sketch the graph of an odd degree polynomial function that has the following information:

   Relative minimum: $(-1.5, -3)$

   Relative maximum: $(1, -1)$

   y-intercept: -1.5

   x-intercept: -3

   Inflection point: $(--0.5, -2)$

   Concave down on: $(-0.5, \infty)$

   Concave up on: $(-\infty, -0.5)$

   Domain: $x \in \Re$

   Range: $y \in \Re$

3. Sketch the graph of $f(x) = x^4 - 2x^3 + x - 2$, after identifying its x- and y-intercepts, relative extremes, its concavity, its points of inflection, and then its domain and range.

Lesson Summary

In this lesson, you learned how to determine points of inflection and concavity using the second derivative. You also reviewed how to determine the intercepts, domain, and range of a polynomial function. You used all your knowledge about the characteristics of a function to sketch its graph. In the last lesson of this module, you will learn how to apply calculus concepts to problems involving related rates.

Assignment 3.5

Concavity and Sketching Polynomial Functions

Total: 22 marks

1. Find the point(s) of inflection and determine where the graph is concave up and concave down for the function $f(x) = x^4 - x^3 - 6x$. *(6 marks)*

continued

Assignment 3.5: Concavity and Sketching Polynomial Functions (continued)

2. Sketch the graph of an even degree polynomial function that has the following information: *(4 marks)*

   Relative minimum: $(0, 1)$

   Relative maximums: $(-2, 3)$ and $(3, 5)$

   y-intercept: 1

   x-intercepts: -4 and 6

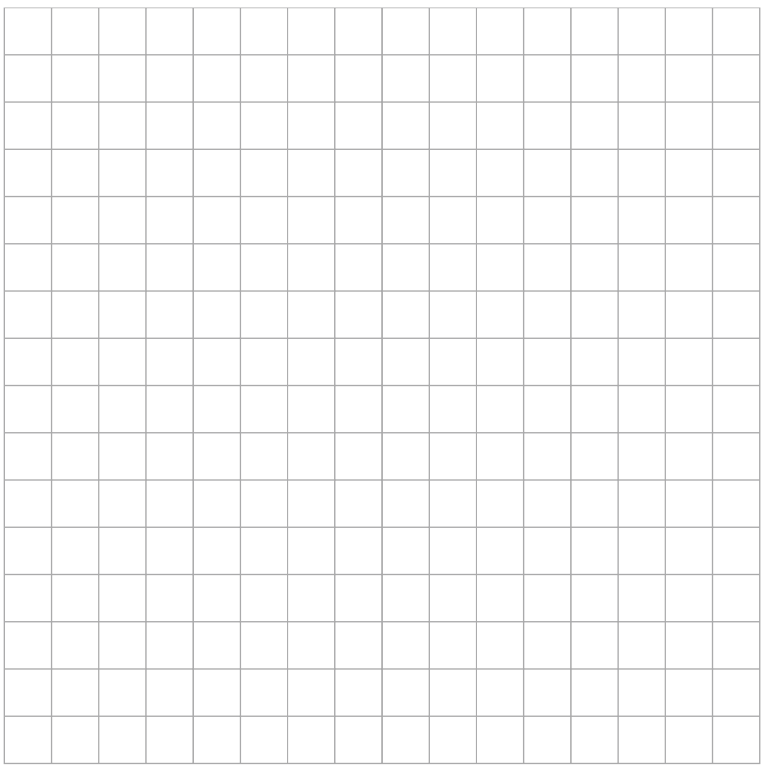
   Inflection points: $(-1, 2)$ and $(1, 2)$

   Concave down on: $(-\infty, -1) \cup (1, \infty)$

   Concave up on: $(-1, 1)$

   Domain: $x \in \Re$

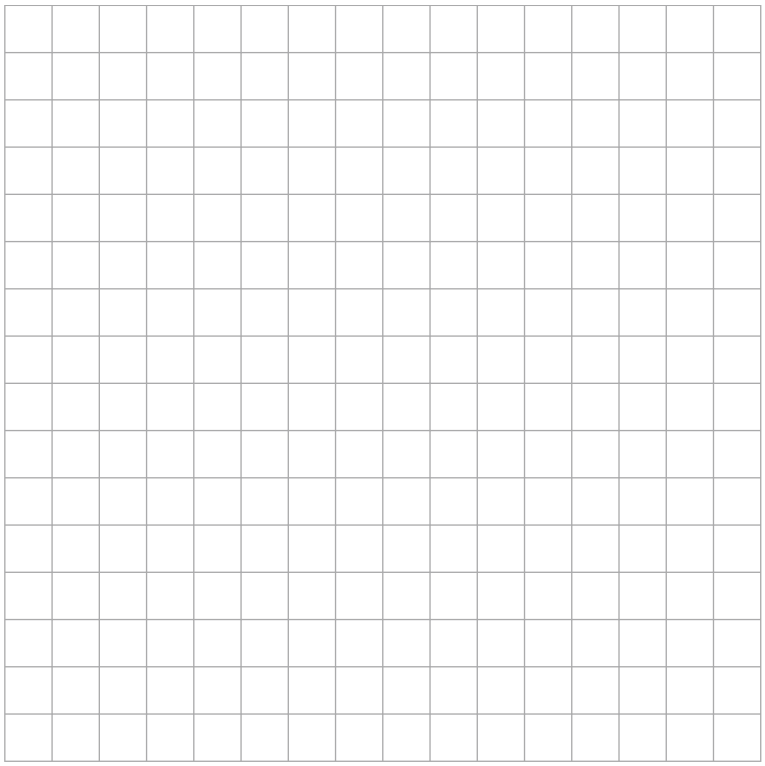
   Range: $\{y \mid y \leq 5, y \in \Re$



continued

Assignment 3.5: Concavity and Sketching Polynomial Functions (continued)

3. Sketch the graph of $f(x) = x^3 - 2x^2 - 5x + 6$, after identifying its x- and y-intercepts, relative extremes, its concavity, its points of inflection, and then its domain and range. *(12 marks)*



Notes

Lesson 6: Related Rate Problems

Lesson Focus

In this lesson, you will

❑ apply the chain rule and implicit differentiation to determine rates of change

❑ solve problems involving related rates

Lesson Introduction

Have you ever watched as water is steadily poured and fills a container that is shaped like a cone? If you have, you may have noticed that the change in height of the water in the cone is increasing (speeds up) over time, but since the water is steadily poured, the change in volume of water in the cone is constant. The rate of change of volume and the rate of change of height of the water are related. They are examples of related rates. In this lesson, you will review the chain rule and implicit differentiation to segue into rates of change. You will discover that rates of change with respect to time can help solve related rates problems. You will learn how to solve these related rates problems in varying contexts using implicit differentiation.

Rates of Change

Recall that the derivative of a function is not only referred to as the slope of the tangent line but also as a **rate of change**. If y is a function of x, then $\frac{dy}{dx}$ is the rate of change of y with respect to x.

Examples of rates:

- If a rocket's position, x, is given as a function of time, then the derivative is the rate of change of the rocket's position with respect to time (speed or velocity in km/h).
- If the hydraulic pressure in a brake line is increasing by 5 kp every second, then the rate of increase in pressure is 5 kp/s and the value of the first derivative of the pressure as a function of time is 5 kp/s.
- Designers of airplanes need to be aware of the change of air pressure (in Pascals) with respect to altitude (in metres). A function describing this rate of change is the first derivative in units of Pa/m.

The idea of the first derivative as a rate of change can be applied to algebraic models of real-world phenomena. A modern technological society would be unable to survive without it as it helps engineers, economists, actuaries, statisticians, and many other scientists solve complex problems.

Calculus is used in the analysis of many different fields of study because rates of change are not limited to measures with respect to time. For example, the volume of a liquid (such as mercury) is related to the temperature of the liquid. In this case, the rate of change of volume could be measured in mm^3 per °C. When manufacturing widgets, the cost to produce one more widget may depend on the total number of widgets produced. The rate of change of the cost would be measured in dollars per widget. Although rates are not always with respect to time, you will analyze most of your rates in this course with respect to time.

Rate of Change with Respect to Time

If a quantity, x, changes with respect to time, then in general any quantity, y, that depends on x also changes with time. When y is related to x, the rate at which y changes is related to the rate at which x changes. They are called **related rates**.

If $y = f(x)$, then according to the chain rule, $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$.

For example, when a balloon expands, its radius increases. The volume and surface area of the balloon are related to its radius. The rate of change of the surface area and volume are related to the rate of change of the radius. The radius of the balloon cannot change without affecting the surface area and volume.

We know the formula for the surface area of a sphere is $A = 4\pi r^2$.

For a spherical balloon, if the radius changes as a function of time, the surface area also changes as a function of time. The rate of change of surface area with respect to time can be determined by differentiating both sides of the surface area equation with respect to time.

$$A = 4\pi r^2$$

$$\frac{dA}{dt} = \frac{d}{dt}\left(4\pi r^2\right) = 4\pi \frac{d}{dt}\left(r^2\right) = 4\pi\left(2r\frac{dr}{dt}\right) = 8\pi r\frac{dr}{dt}$$

Remember the chain rule is used for finding the derivative of a composition of functions. Area is a function of radius (since $A = 4\pi r^2$). If you were to differentiate area with respect to radius, you would get $\frac{dA}{dr} = 4\pi\left(2r\right) = 8\pi r$.

However, you are differentiating area with respect to time and the value of the radius is a function of time. You differentiate r with respect to time using implicit differentiation and the chain rule because r is changing as a function of time.

The best way to learn to solve related rate problems is to work through several examples. It may require a concerted effort on your part to carefully follow the discussion given for each step.

Example 1

Given two variables that are functions with respect to time, $x(t)$ and $y(t)$, where $x^4 - y^4 = 65$ and $\frac{dy}{dt} = 3$, determine $\frac{dx}{dt}$ at the instant when $y = 2$.

Solution

Notice that both the values of x and y are actually functions of time, since you are given $\frac{dy}{dt}$ and you are asked to find $\frac{dx}{dt}$. So the derivatives of x and y will require the chain rule. The derivative of the constant, 65, is zero.

State the given information. (It is important to note that you will not use the $y = 2$ until after you have done the implicit differentiation because y is not a constant. It is a function of time.)

$x^4 - y^4 = 65$

$$\frac{dy}{dt} = 3$$

What are you looking for?

$$\frac{dx}{dt} = ?$$

Use implicit differentiation with respect to time.

$$\frac{d}{dt}\left(x^4 - y^4 = 65\right)$$

$$\frac{d}{dt}\left(x^4\right) - \frac{d}{dt}\left(y^4\right) = \frac{d}{dt}(65)$$

$$\frac{d\left(x^4\right)}{dx} \cdot \frac{dx}{dt} - \frac{d\left(y^4\right)}{dt} \cdot \frac{dy}{dt} = 0$$

$$4x^3 \cdot \frac{dx}{dt} - 4y^3 \cdot \frac{dy}{dt} = 0$$ ⟵ Equation with respect to time.

Determine the value of x at the instant when $y = 2$, using the original equation so you can substitute for all variables except $\frac{dx}{dt}$.

$$x^4 - (2)^4 = 65$$

$$x^4 - 16 = 65$$

$$x^4 = 65 + 16 = 81$$

$$x = \pm 3$$

Substitute the given information into the equation with respect to time for the instant $y = 2$ and $x = 3$.

If $x = 3$,

$$4(3)^3 \cdot \frac{dx}{dt} - 4(2)^3 \cdot (3) = 0$$

$$108 \cdot \frac{dx}{dt} - 96 = 0$$

$$108 \cdot \frac{dx}{dt} = 96$$

$$\frac{dx}{dt} = \frac{96}{108} \approx 0.89$$

Substitute the given information into the equation with respect to time at the instant when $y = 2$ and $x = -3$.

If $x = -3$,

$$4(-3)^3 \cdot \frac{dx}{dt} - 4(2)^3 \cdot (-3) = 0$$

$$-108 \cdot \frac{dx}{dt} + 96 = 0$$

$$-108 \cdot \frac{dx}{dt} = -96$$

$$\frac{dx}{dt} = \frac{96}{108} \approx 0.89$$

The rate $\frac{dx}{dt} = \frac{96}{108} \approx 0.89$.

The rate of change, $\frac{dx}{dt} \approx 0.89$, when $y = 2$.

Related Rates Problems

Solving these problems requires organization and clarity on what information is being sought. If you need some help with organization, the table below should guide you in solving these types of problems.

General Steps to Solving Related Rate Problems

1. Read the problem over carefully. Assign variable names to the given and unknowns. Use derivative $\left(\frac{dy}{dx}\right)$ notation to describe any rates that are given.
2. If possible, draw a diagram and label it with the given and unknowns.
3. Write the equation that relates the various quantities in the problem which, when differentiated with respect to time, will provide the related rate equation necessary.
4. If you have more than two variables changing with time, you may have to use the geometry of the situation or systems of equations to eliminate one of the variables through substitution.
5. Use implicit differentiation and the chain rule to differentiate both sides of the equation with respect to time, t.
6. Substitute the necessary given information into the resulting equation and solve for the required rate.
7. Finally, express your answer in clear and concise terms as a conclusion statement, ensuring you are answering the question. Make certain that your answer and the units of your answer make sense with respect to the original statement of the problem.

Example 2

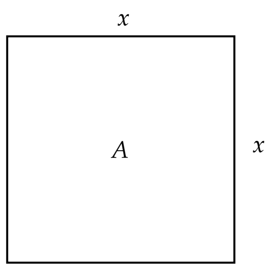
Knowing that the sides of a square are increasing at the rate of 4 cm/s, how fast is the area increasing at the instant when each side of the square is 20 cm?

Solution

Define the variables.

Let x be the length of the side of the square, A be the area of the square, and t be the time in seconds.

$$\frac{dx}{dt} = 4 \text{ cm/s}$$



$$\frac{dA}{dt} = ?$$

Write an equation relating A and x.

Differentiate the area equation with respect to time using your knowledge of the chain rule and implicit differentiation.

$$A = x^2$$

$$\frac{d}{dt}(A) = \frac{d}{dt}\left(x^2\right)$$

$$\frac{dA}{dt} = \frac{d}{dx}\left(x^2\right) \cdot \frac{dx}{dt}$$

$$\frac{dA}{dt} = 2x \cdot \frac{dx}{dt}$$ ← Equation with respect to time.

Substitute the given information and evaluate the rate of change of the area with respect to time at the instant when $x = 20$.

$$\frac{dA}{dt} = 2(20) \cdot (4) = 160$$

The area of the square is increasing at a rate of 160 cm^2/s at the instant the square is 20 cm on a side.

Example 3

While using a ripple tank to do an experiment in physics, a student sets up a circular wave front that travels outward at the rate of 12 cm/s. At what rate is the area inside the wave increasing?

a) When the radius is 6 cm?

b) When the area is 100π cm^2?

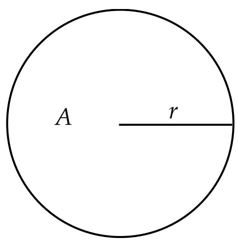
c) When 5 seconds have elapsed?

Solution

Define the variables.

Let r represent the radius of the wave, A the area of the wave, and t the time elapsed.

$$\frac{dr}{dt} = 12 \text{ cm/s}$$



$$\frac{dA}{dt} = ?$$

Write an equation relating A and r.

Differentiate with respect to time.

$$A = \pi r^2$$

$$\frac{d}{dt}(A) = \frac{d}{dt}\left(\pi r^2\right)$$

$$\frac{dA}{dt} = \frac{d\left(\pi r^2\right)}{dr} \cdot \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$ ⟵ Equation with respect to time.

You will use the equation with respect to time for all parts of this question.

a) Substitute the given information and evaluate the rate of change of the area with respect to time at the instant when $r = 6$.

$$\frac{dA}{dt} = 2\pi(6)\cdot(12)$$

$$= 144\pi \text{ cm}^2/\text{s}$$

The area of the wave is increasing at a rate of 144π cm^2/s when the radius is 6 cm.

b) What are you looking for? Find the radius at the instant when the area is 100π and then use it to determine $\frac{dA}{dt}$.

$$A = \pi r^2$$

$$100\pi = \pi r^2$$

$$100 = r^2$$

$$r = 10$$

Substitute the given information into the equation with respect to time and evaluate the rate of change of the area with respect to time when $A = 100\pi$ and $r = 10$.

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi(10)\cdot(12)$$

$$= 240\pi \text{ cm}^2/\text{s}$$

The area of the wave is increasing at a rate of 240π cm^2/s when the area is 100π cm^2.

c) What are you looking for? You need to find the radius at the instant when 5 seconds has elapsed. Note in this context that the initial radius is zero. Use the time and the rate, $\frac{dr}{dt} = 12$ cm/s, to determine r.

The radius starts at $r = 0$ and changes at 12 cm/s for 5 seconds. So, the radius is 60 cm when 5 seconds have elapsed.

Substitute the given information and evaluate the rate of change of the area with respect to time when 5 seconds have elapsed.

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi (60) \cdot (12)$$

$$= 1440\pi \text{ cm}^2/\text{s}$$

The area of the wave is increasing at a rate of 1440π cm^2/s when 5 seconds have elapsed.

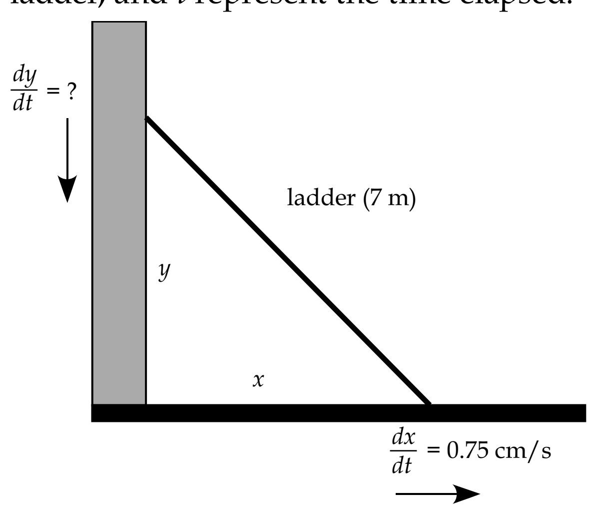
Example 4

A ladder, 7 m long, has been placed against a wall. Not realizing that the bottom of the ladder was on a fairly smooth surface, a man started to climb the ladder. To his dismay, the bottom of the ladder slipped horizontally (away from the wall) at the rate of 0.75 m/s. How fast is the top of the ladder falling (vertically downward) when the bottom of the ladder is 5 m from the wall?

Solution

Define the variables.

Let x represent the horizontal distance from the wall at the bottom of the ladder, y represent the vertical distance from the ground to the top of the ladder, and t represent the time elapsed.



Draw a diagram to model the situation.

$$x^2 + y^2 = 7^2$$

Write an equation relating x and y.

Differentiate with respect to time.

$$\frac{d}{dt}\left(x^2 + y^2\right) = \frac{d}{dt}\left(7^2\right)$$

$$\frac{d}{dt}\left(x^2\right) + \frac{d}{dt}\left(y^2\right) = 0$$

$$\frac{d\left(x^2\right)}{dx} \cdot \frac{dx}{dt} + \frac{d\left(y^2\right)}{dt} \cdot \frac{dy}{dt} = 0$$

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$$ ← Equation with respect to time.

Determine the value of y at the instant when $x = 5$. Note the y-value must be positive.

$$5^2 + y^2 = 49$$

$$y^2 = 49 - 25 = 24$$

$$y = \sqrt{24}$$

Substitute the given information and evaluate the rate of change of the vertical distance of the ladder with respect to time when $x = 5$ m.

$$2(5)(0.75) + 2\left(\sqrt{24}\right) \cdot \frac{dy}{dt} = 0$$

$$2\left(\sqrt{24}\right) \cdot \frac{dy}{dt} = -7.5$$

$$\frac{dy}{dt} = \frac{-7.5}{2\left(\sqrt{24}\right)} \approx -0.77 \text{ cm/s}$$

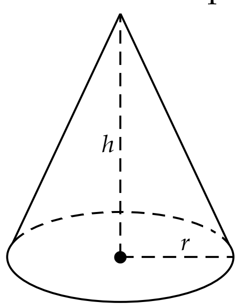
The vertical distance of the ladder above the ground is decreasing at a rate of 0.77 cm/s when $x = 5$ m.

Example 5

The radius of a conical stalagmite (rising upward from a cave floor) is always 8% of its height. Its volume is increasing at the rate of 156π cm^3/century. What is the rate of change of the radius when the stalagmite is 60 cm high?

Solution

Define the variables. Let r, h, and V represent the radius, height, and volume of the cone respectively.



$$\frac{dV}{dt} = 156\pi \text{ cm}^3/\text{century}$$

$$r = 0.08h$$

$$\frac{dr}{dt} = ?$$

The equation relating the volume of a cone to its radius and height is:

$$V = \frac{1}{3}\pi r^2 h$$

You can reduce the equation to one variable using the fact that $r = 0.08h$.

$$V = \frac{1}{3}\pi (0.08h)^2 h$$

$$V = \frac{0.0064}{3}\pi h^3$$

Differentiate both sides with respect to t, since the rates of change of volume and height with respect to time are related rates.

$$\frac{dV}{dt} = \frac{0.0064}{3}\pi \cdot \frac{d}{dt}\left(h^3\right)$$

$$\frac{dV}{dt} = \frac{0.0064}{3}\pi \cdot 3h^2 \frac{dh}{dt} \longleftarrow \text{Equation with respect to } t.$$

Substitute the given information to evaluate the rate of change of the height at the instant the height is 60 cm.

$$156\pi = \frac{0.0064}{3}\pi \cdot 3(60)^2 \frac{dh}{dt}$$

$$6.7708 = \frac{dh}{dt}$$

In the equation for volume, V, above, you could have eliminated r instead of h to find $\frac{dr}{dt}$, but that is not necessary since you know that $r = 0.08h$. Now you can use the rate of change of h to find the rate of change of r. Differentiate the equation with respect to t on both sides to get $\frac{dr}{dt} = 0.08\frac{dh}{dt}$.

So, $\frac{dr}{dt} = 0.08(6.7708) = 0.5417$.

The radius of the stalagmite is increasing at a rate of 0.54 cm/century at the instant the height is 60 cm.

Example 6

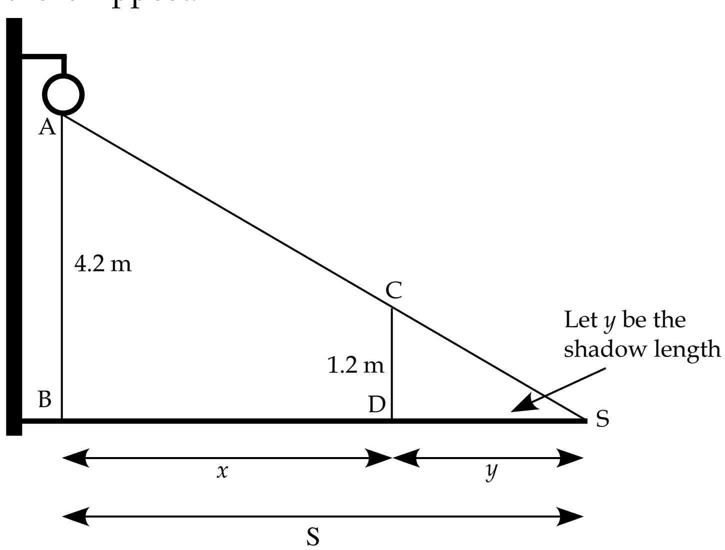
A boy, 1.2 m tall, runs toward a lamppost at the rate of 2.2 m/s. The light bulb is hanging 4.2 m above the ground.

a) At what rate is the end of his shadow moving when he is 2.5 m from the post?

b) At what rate is the length of his shadow changing when he is 2.5 m from the post?

Solution

Define the variables. Let x represent the boy's distance from the lamppost, y, the length of the boy's shadow, and s, the distance of the end of the shadow to the lamppost.



The diagram demonstrates the similar triangles (assume the bulb and post are in the same vertical plane).

$\frac{dx}{dt} = 2.2$ m/s

a) $\frac{ds}{dt} = ?$

b) $\frac{dy}{dt} = ?$

Relate y and s using the similar triangles.

$$\frac{y}{1.2} = \frac{s}{4.2}$$

$$4.2y = 1.2s$$

$$y = \frac{1.2}{4.2}s$$

$$y = \frac{2}{7}s$$

a) Since you are given $\frac{dx}{dt}$ and you want to find $\frac{ds}{dt}$, you need to find an equation that relates s and x.

$$s = x + y$$

$$s = x + \frac{2}{7}s \quad \text{substitute } y = \frac{2}{7}s$$

$$x = s - \frac{2}{7}s$$

$$x = \frac{7}{7}s - \frac{2}{7}s$$

$$x = \frac{5}{7}s$$

$$s = 1.4x$$

Differentiate both sides of the equation with respect to time since both s and x are functions that change with time.

$$\frac{ds}{dt} = 1.4\frac{dx}{dt}$$

Substitute the given value to determine the rate of change of the end of his shadow.

$$\frac{ds}{dt} = 1.4(2.2) = 3.08 \text{ m/s}$$

The end of the shadow to the lamppost is moving at a rate of 3.08 m/s. The shadow tip is approaching the pole at a constant rate (faster than the boy is approaching), but it is a constant rate that does not depend on the boy's distance from the pole. The distance of 2.5 m from the post is not relevant.

b) To find the rate of change of y, you can use one of the equations, $s = x + y$ or $y = \frac{2}{7}s$.

$$s = x + y$$

$$\frac{ds}{dt} = \frac{dx}{dt} + \frac{dy}{dt}$$

$$3.08 = 2.2 + \frac{dy}{dt}$$

$$0.88 = \frac{dy}{dt}$$

The shadow is changing length at a rate of 0.88 m/s. Since the boy is walking at a constant rate, the shadow length is also changing at a constant rate and does not depend on his distance from the pole. The distance of 2.5 m from the post is not relevant.

You will have noticed that the substitution step was very near the end. Be sure not to substitute the given information before differentiating or you may not arrive at the correct answer. If you substitute the constant before differentiating, you will arrive at an incorrect solution since the derivative of a constant is zero and the derivative of the variable with respect to time is not zero. Now it is time for you to try some related rates problems on your own.

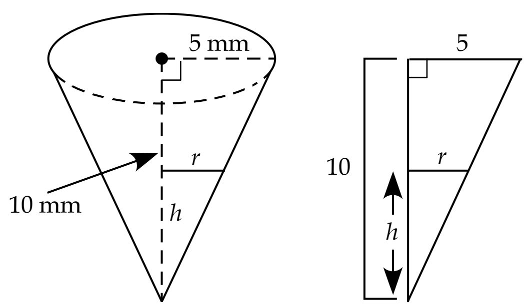
Learning Activity 3.6

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Differentiate $4x^3$ with respect to time knowing that x is a function of time.
2. If the base of a square increases at 5 cm/s, what is the length of the side after 5 seconds? (Assume it starts at zero.)
3. What is the area of the square at the 5 second mark in Question 2?
4. If a right cylinder has a radius of 5 mm and a vertical height of 10 mm, write a relation that represents the height of the cone with respect to the radius at any height.



5. Differentiate $y = \sqrt{3x^2 + 4x}$ using the chain rule.
6. Write an expression for V in terms of h using the following two equations: $r = 2h$ and $V = \frac{1}{3}\pi r^2 h$
7. Simplify $\frac{28\pi}{70\pi}$ by reducing the fraction into lowest terms.
8. Solve for y in $x^3 + y^2 = 1$ when $x = -1$.

continued

Learning Activity 3.6 (continued)

Part B: Related Rates

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. How fast is the volume of a spherical balloon increasing at the instant when the radius is 8 cm, if the radius is increasing at the rate of 10 cm per second?
2. One end of a 6 m ladder leans against a wall and the other rests on the ground. If the lower end of the ladder slips horizontally along the ground at the rate of 0.3 m/s, at what rate is the top of the ladder moving toward the ground at the instant when the foot of the ladder is 4 m from the wall?
3. An overhead projector lights up a screen 5 m away. A student standing in front of the projector entertains his or her classmates by using a 20 cm long hand to cast a shadow of a barking dog on the screen. If the student walks to the screen at a speed of 0.8 m/s, how fast is the size of the dog's shadow decreasing at the instant when the student is 2 m from the projector?

Lesson Summary

In this lesson, you learned how to solve problems involving related rates in a variety of contexts by using the chain rule and implicit differentiation.

In the next module, you will look at antidifferentiation and its connection to the area under the curve.

Notes

Assignment 3.6

Related Rates

Total: 20 marks

1. Given that x and y are functions of t, if $x^3 + y^3 = 9$ and $\frac{dx}{dt} = 4$, find $\frac{dy}{dt}$ when $x = 2$. *(5 marks)*

2. A stone is dropped into a lake, creating a circular wave whose area increases at a rate of 5 cm^2/min. Determine the rate at which the radius increases when the radius is 4 cm. *(5 marks)*

continued

3. A ladder 4 m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a speed of 0.30 m/s, how quickly is the top of the ladder sliding down the wall when the bottom of the ladder is 2 m from the wall? *(5 marks)*

continued

Assignment 3.6: Related Rates (continued)

4. A water tank is built in the shape of a circular cone with the height 5 m and the diameter 6 m at the top. If water is being pumped into the tank at a rate of 1.6 m^3/min, at what rate is the water level rising when the water is 2 m deep? *(5 marks)*

Notes

Module 3 Summary

Congratulations, you have finished the third module in the course.

Submitting Your Assignments

It is now time for you to submit Assignments 3.1 to 3.6 to the Distance Learning Unit so that you can receive some feedback on how you are doing in this course. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 3 assignments and organize your material in the following order:

- ❑ Module 3 Cover Sheet (found at the end of the course Introduction)
- ❑ Assignment 3.1: Solving Inequalities
- ❑ Assignment 3.2: Particle Motion Problems
- ❑ Assignment 3.3: First Derivative Applications
- ❑ Assignment 3.4: Optimization Problems
- ❑ Assignment 3.5: Concavity and Sketching Polynomial Functions
- ❑ Assignment 3.6: Related Rates

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Notes

Grade 12 Introduction to Calculus (45S)

Module 3
Applications of Derivatives

Learning Activity Answer Keys

MODULE 3: APPLICATIONS OF DERIVATIVES

Learning Activity 3.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

For Questions 1 to 4, determine if each number is a solution to $x \le -2$.

1. 3
2. −2
3. −3
4. 0

For Questions 5 to 9, determine if the number is on the interval $[-1, 6)$.

5. −2
6. −1
7. 4
8. 6
9. 7

Answers:

1. No (Since $3 \le -2$ is false.)
2. Yes (Since $-2 \le -2$ is true.)
3. Yes (Since $-3 \le -2$ is true.)
4. No (Since $0 \le -2$ is false.)
5. No (Since $-1 \le -2$ is false.)
6. Yes (Since $-1 \le -1 < 6$ is true.)
7. Yes (Since $-1 \le 4 < 6$ is true.)
8. No (Since $6 < 6$ is false.)
9. No (Since $7 < 6$ is false.)

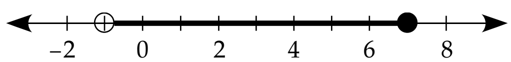
Part B: Solving Inequalities

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Express the interval $(-1, 7]$ in set notation and graph it on a number line graph.

   Answer:

   $\{x \mid -1 < x \leq 7, x \in \Re\}$



2. Solve the inequality $x^2 + 7x + 10 \geq 0$.

   Answer:

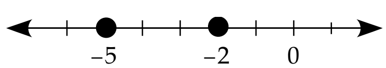
   Determine the roots to the corresponding quadratic equation to determine the interval boundary points, and use test points to determine which x-values make the product negative.

$$x^2 + 7x + 10 = 0$$

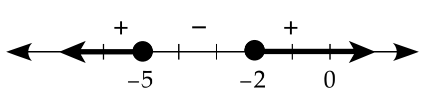
$$(x + 5)(x + 2) = 0$$

$$x = -5, x = -2$$

These two boundary points divide the number line into three sections where the boundary points are included, since the inequality includes "equal to."



- Test $x = -6$ on $(x + 5)(x + 2)$. It is $(-1)(-4)$, which is positive.
- Test $x = -3$ on $(x + 5)(x + 2)$. It is $(2)(-1)$, which is negative.
- Test $x = 0$ on $(x + 5)(x + 2)$. It is $(5)(2)$, which is positive.



The solution to $x^2 + 7x + 10 \geq 0$ is $\{x \mid x \leq -5 \text{ or } x \geq -2, x \in \Re\}$.

3. Solve the inequality $(x + 1)(x - 3)(x + 3) > 0$.

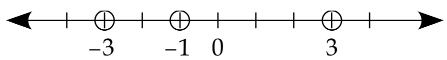
Answer:

Determine the roots to the corresponding polynomial equation and use the intervals to determine which x-values make the product positive or equal to zero.

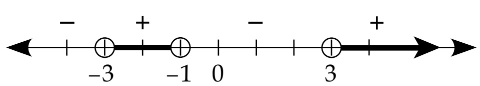
$$(x + 1)(x - 3)(x + 3) = 0$$

$$x = -1, x = 3, x = -3$$

The three roots divide the number line into four sections where the boundary points are not included, since the inequality does not include "equal to."



- Test $x = -4$ on $(x + 1)(x - 3)(x + 3)$. It is $(-3)(-7)(-1)$, which is negative.
- Test $x = -2$ on $(x + 1)(x - 3)(x + 3)$. It is $(-1)(-5)(1)$, which is positive.
- Test $x = 0$ on $(x + 1)(x - 3)(x + 3)$. It is $(1)(-3)(3)$, which is negative.
- Test $x = 4$ on $(x + 1)(x - 3)(x + 3)$. It is $(5)(1)(7)$, which is positive.



The solution to $(x + 1)(x - 3)(x + 3) > 0$ is $\{x \mid -3 < x < -1 \text{ or } x > 3, x \in \Re\}$.

Learning Activity 3.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Factor: $x^2 - 5x + 6$
2. Factor: $x^2 - x - 12$
3. Factor: $2x^2 - 9x - 5$
4. Factor: $3x^2 + 8x + 4$
5. Solve: $(x - 1)(x + 4) = 0$
6. Solve: $(1 - x)(x - 3) = 0$
7. Solve: $x(2x - 5) = 0$
8. Solve: $x^2 - 36 = 0$

Answers:

1. $(x - 2)(x - 3)$
2. $(x - 4)(x + 3)$
3. $(2x + 1)(x - 5)$ $\begin{pmatrix} 2x^2 - 10x + x - 5 \\ 2x(x-5) + 1(x-5) \\ (2x+1)(x-5) \end{pmatrix}$
4. $(x + 2)(3x + 2)$ $\begin{pmatrix} 3x^2 + 2x + 6x + 4 \\ x(3x+2) + 2(3x+2) \\ (x+2)(3x+2) \end{pmatrix}$
5. $x = 1, x = -4$
6. $x = 1, x = 3$
7. $x = 0,\ x = \frac{5}{2}$
8. $x = \pm 6$

Part B: Particle Motion Problems

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. The displacement in metres of a particle from a fixed point is given by the position function $s = 8t - t^2 - t^3$, $t \geq 0$.

   a) Determine the velocity function at any time t.

   Answer:

   Differentiate the position function to determine the velocity function.

   $s = 8t - t^2 - t^3$

   $v = s' = 8 - 2t - 3t^2$

   b) Evaluate the displacement at the following times.

   Answer:

Time (s)	0	1	2	3
Displacement (m)	0	$8(1) - 1^2 - 1^3 = 6$	$16 - 4 - 8 = 4$	$24 - 9 - 27 = -12$

   c) Calculate the exact time at which the velocity is zero.

   Answer:

   $0 = v = 8 - 2t - 3t^2$ — Set the velocity function to zero to determine the time.

   $0 = 8 - 2t - 3t^2$

   $3t^2 + 2t - 8 = 0$

   $(3t - 4)(t + 2) = 0$ — Multiply both sides by −1 so that the quadratic form is positive, and then factor the equation.

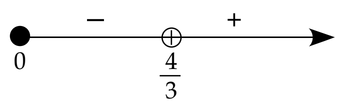
   $t = -2,\ t = \frac{4}{3}$ — Solve for the roots.

   $t = \frac{4}{3}$ — But the function is only defined for positive time so the velocity is zero at $\frac{4}{3}$ seconds.

d) Does the particle change direction? Justify your answer.

Answer:

We can determine if the particle changes direction by checking what happens on each side of the time when the velocity is zero. This can be done by creating a sign diagram with a starting point at 0 and a boundary point at $\frac{4}{3}$. Test $t = 1$ on $(3t - 4)(t + 2)$. It is $(-1)(3)$, which is negative. Test $t = 2$ on $(3t - 4)(t + 2)$. It is $(2)(4)$, which is positive.



The particle changes direction at $\frac{4}{3}$ seconds because the velocity changes from negative to positive at $\frac{4}{3}$ seconds. You could also say the displacement goes from decreasing to increasing when $t = \frac{4}{3}$.

e) Determine the acceleration function for any time t?

Answer:

We can determine the acceleration function by differentiating the velocity function.

If $v = 8 - 2t - 3t^2$, then $a = v' = s'' = -2 - 6t$.

f) Does the acceleration reach zero on the interval $t \geq 0$? Justify your answer.

Answer:

Set the acceleration function equal to zero.

$$0 = a = -2 - 6t$$

$$6t = -2$$

$$t = -\frac{1}{3}$$

Since this time is not positive, the acceleration never reaches zero.

2. A rocket travels upward a distance $s = t^3$ in metres for the first t seconds after takeoff. Find its velocity when it reaches a height of 1000 metres.

   Answer:

   Determine the velocity function by finding the derivative of the position function.

   If $s = t^3$, then $v = s' = 3t^2$.

   Determine the time it takes the rocket to reach 1000 metres by substituting 1000 m into the position function and solving for time.

   $$1000 = t^3$$

   $$\sqrt[3]{1000} = \sqrt[3]{t^3}$$

   $$10 = t$$

   The rocket reaches the height of 1000 metres after 10 seconds.

   The velocity of the rocket at 1000 metres can be determined by substituting 10 seconds into the velocity function.

   $$v = 3(10)^2$$

   $$v = 3(100) = 300$$

   The velocity of the rocket at 1000 metres is 300 m/s.

3. An automobile travelling at 30 m/s is brought to a halt by steadily increasing braking force. The function, $s = 30t - 0.1t^3$, $t \geq 0$, represents the braking distance in metres for a time in seconds until the automobile stops.

   a) Determine the velocity and acceleration functions in terms of t.

   Answer:

   The velocity function is the derivative of the position function.

   If $s = 30t - 0.1t^3$, then $v = s' = 30 - 0.3t^2$.

   The acceleration function is the derivative of the velocity function.

   If $v = 30 - 0.3t^2$, then $a = -0.6t$.

b) Determine when the vehicle stops.

Answer:

The vehicle stops when the velocity function is zero. Determine the time when velocity is equal to zero.

$$0 = v = 30 - 0.3t^2$$

$$0.3t^2 = 30$$

$$t^2 = \frac{30}{0.3} = 100$$

$$t = \pm 10$$

You do not include the negative time as part of the solution because it is out of the domain. Thus, the velocity is zero at 10 seconds. In other words, the vehicle stops after 10 seconds.

c) Determine the stopping distance.

Answer:

If you know that it takes the vehicle 10 seconds to stop, you can then substitute that time into the position function to solve for the distance.

$$s = 30(10) - 0.1(10)^3$$

$$s = 300 - 0.1(1000)$$

$$s = 300 - 100 = 200$$

The stopping distance is 200 metres.

d) Determine the deceleration at the time when the vehicle stops.

Answer:

The deceleration when the vehicle stops can be found by substituting the stopping time, 10 seconds, into the acceleration function.

$$a = -0.6(10)$$

$$a = -6$$

The acceleration is negative because the vehicle is decelerating. The deceleration when the vehicle stops is -6 m/s^2.

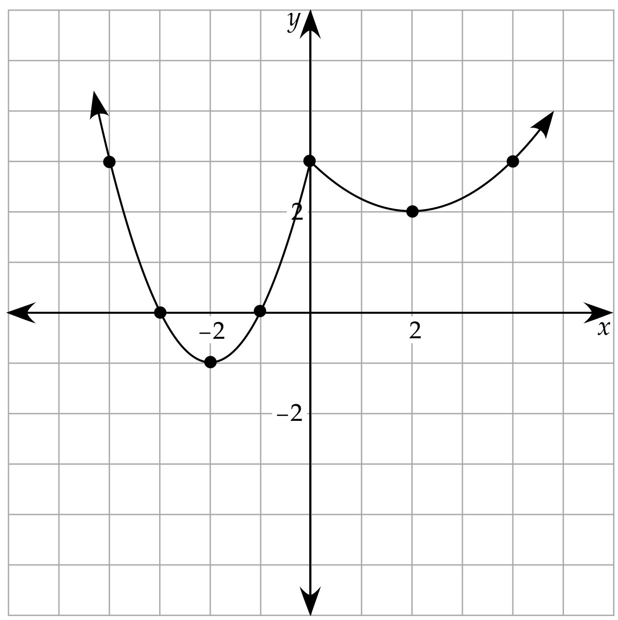
Learning Activity 3.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Is $x = 0$ a critical value of $y = 2x^2$?
2. Factor: $4x - 12x^2$
3. What sign is the product of an odd number of negative numbers?
4. Differentiate: $y = 3x^4 - 7x + 2$

Use the graph of $f(x)$ below to answer Questions 5 to 8.



5 Determine the critical values where $f'(x) = 0$.
6. Determine the critical values where $f'(x)$ is undefined.
7. What is the range of $f(x)$?
8. Is there an absolute maximum for $f(x)$?

Answers:

1. Yes (The minimum of the quadratic function occurs at $x = 0$.)
2. $4x(1 - 3x)$
3. Negative
4. $y' = 12x^3 - 7$

5. $x = -2$ and $x = 2$

6. $x = 0$ (since the slope to the left is different than the slope to the right)

7. $\{y \mid y \geq -1, y \in \Re\}$

8. No, because there is no limit on how high the function values can reach as expressed in the range in the function.

Part B: First Derivative Applications

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Use the graph of $f(x)$ in Part A to:

   a) determine its absolute extreme values and its relative extreme values.

   Answer:

x-value	y-value	Type	Reason
−2	−1	Relative minimum	$-1 \leq f(x)$ for x-values close to -2
		Absolute minimum	$-1 \leq f(x)$ for all values of x
0	3	Relative maximum	$3 \geq f(x)$ for x-values close to 0
2	2	Relative minimum	$2 \leq f(x)$ for x-values close to 2

   b) determine the intervals on the graph of $f(x)$ where the function is increasing and decreasing.

   Answer:

	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
Curve Behaviour	falling	rising	falling	rising
	decreasing	increasing	decreasing	increasing

   The function is decreasing on $\{(-\infty, -2) \cup (0, 2)\}$ and increasing on $\{(-2, 0) \cup (2, \infty)\}$.

2. Determine the critical values of the following two functions.

   a) $f(x) = 2x^3 - 5x^2 + 4x + 1$

   Answer:

   Remember that the critical values of a function are the zeros and non-permissible values of the derivative function.

   $f'(x) = 6x^2 - 10x + 4$

   $0 = 6x^2 - 10x + 4$

   $0 = 2\left(3x^2 - 5x + 2\right)$

   $0 = 2(3x - 2)(x - 1)$

   Critical values at $x = 1$ and $x = \frac{2}{3}$.

   The critical values of $f(x)$ are the zeros of the derivative function. There are no non-permissible values for the derivative function.

   b) $g(x) = -3x^{-2}$

   Answer:

   Remember that the critical values of a function are the zeros and non-permissible values of the derivative function.

   $g'(x) = 6x^{-3}$

   $0 = \frac{6}{x^3}$

   $x^3 \neq 0$

   The critical value of $g(x)$ is the non-permissible value, $x = 0$, of the derivative function. There are no zeros for this function

3. Use the first derivative test to determine the nature and coordinates of the relative extremes of $h(x) = -3x^3 + 6x^2 - 7$.

Answer:

$h'(x) = -9x^2 + 12x$ — Determine the derivative.

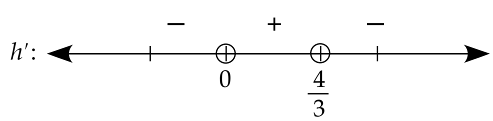
$0 = h'(x) = -9x^2 + 12x$

$0 = -3x(3x - 4)$ — Determine the critical values. These form the boundaries on the intervals and are potential relative extremes.

$0 = -3x \qquad 0 = 3x - 4$

$x = 0, \ x = \dfrac{4}{3}$

The critical values are $x = 0$ and $x = \dfrac{4}{3}$.



Since $h(x)$ is decreasing when $h'(x) < 0$, $h(x)$ is decreasing on $\left\{(-\infty, 0) \cup \left(\dfrac{4}{3}, \infty\right)\right\}$.

Since $h(x)$ is increasing when $h'(x) > 0$, $h(x)$ is increasing on $\left(0, \dfrac{4}{3}\right)$.

Find the function values at the critical points.

When $x = 0$:

$$h(0) = -3(0)^3 + 6(0)^2 - 7 = 0 + 0 - 7$$

$$h(0) = -7$$

When $x = \dfrac{4}{3}$:

$$h\left(\frac{4}{3}\right) = -3\left(\frac{4}{3}\right)^3 + 6\left(\frac{4}{3}\right)^2 - 7 = -3\left(\frac{64}{27}\right) + 6\left(\frac{16}{9}\right) - 7$$

$$= -\left(\frac{64}{9}\right) + \left(\frac{96}{9}\right) - \left(\frac{63}{9}\right) = -\left(\frac{127}{9}\right) + \left(\frac{96}{9}\right) = -\frac{31}{9} \approx -3.4$$

$$h\left(\frac{4}{3}\right) = -\frac{31}{9} \approx -3.4$$

According to the first derivative test:

- $h(0) = -7$ is a **relative minimum** because, as the sign diagram indicates, the function changes from decreasing to increasing as it crosses $x = 0$.
- $h\left(\frac{4}{3}\right) = -\frac{31}{9} \approx -3.4$ is a **relative maximum** because the function changes from increasing to decreasing as it crosses $x = \frac{4}{3}$.

Learning Activity 3.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Determine the first and second derivative for $y = 5x^3 - 3x^2 + 2$.
2. Determine the first and second derivative for $f(x) = 5x + \frac{2}{x}$.
3. Determine the perimeter of a rectangle that has a width of 5 cm and a length of 7 cm.
4. Determine the area of a rectangle that has a width of 12 cm and a length of 7 cm.
5 Determine the volume of a box with a square base, 9 cm on each side, and a height of 5 cm.
6. Factor: $2x^2 - 3x - 2$
7. Solve: $15 - 3x > 0$
8. Solve: $0 = 3x^2 - 6$

Answers:

1. $y' = 15x^2 - 6x$ and $y'' = 30x - 6$
2. $f'(x) = 5 - 2x^{-2}$ and $f''(x) = 4x^{-3}$ (initially think $f(x) = 5x + 2x^{-1}$)
3. 24 cm ($P = 2l + 2w = 2(7) + 2(5) = 24$ cm)
4. 84 cm^2 ($A = l \times w = 7 \times 10 + 7 \times 2 = 70 + 14$)
5. 405 cm^3 ($V = l \times w \times h = 9 \times 9 \times 5 = 81 \times 5 = 400 + 5$)
6. $(2x + 1)(x - 2)$
7. $x < 5$ ($-3x > -15$)
8. $x = \pm\sqrt{2}$ $\left(\begin{array}{l} 0 = 3x^2 - 6 \\ 3x^2 = 6 \\ x^2 = 2 \end{array}\right)$

Part B: Optimization Problems

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Determine the relative extremes of $y = 5x^3 - 3x^2 + 2$, using the second derivative test.

   Answer:

   Determine the first derivative.

   $$y' = 15x^2 - 6x$$

   Determine the critical values by solving for $y' = 0$.

   $$0 = 15x^2 - 6x$$

   $$0 = 3x(5x - 2)$$

   $$x = 0,\ x = \frac{2}{5}$$

   The critical values where $y' = 0$ are $x = 0$ and $x = \dfrac{2}{5}$.

   Determine the second derivative.

   $$y'' = 30x - 6$$

   Evaluate the second derivative at the critical values.

   At $x = 0$

   $$y'' = 30(0) - 6 = -6$$

   $$y'' < 0$$

   At $x = \dfrac{2}{5}$

   $$y'' = 30\left(\frac{2}{5}\right) - 6 = 12 - 6 = 6$$

   $$y'' > 0$$

Determine the function value of the critical values.

At $x = 0$

$$y = 5(0)^3 - 3(0)^2 + 2$$
$$y = 2$$

At $x = \frac{2}{5}$

$$y = 5\left(\frac{2}{5}\right)^3 - 3\left(\frac{2}{5}\right)^2 + 2$$
$$y = 5\left(\frac{8}{125}\right) - 3\left(\frac{4}{25}\right) + 2$$
$$y = \frac{40}{125} - \frac{60}{125} + \frac{250}{125} = \frac{230}{125} = \frac{46}{25}$$
$$y = 1.84$$

According to the second derivative test, since $y''(0)$ is negative, there is a relative maximum at (0, 2) and, since $y''(0.4)$ is positive, there is a relative minimum at (0.4, 1.84).

2. Find two natural numbers such that their product is 9 and their sum is a minimum.

Answer:

Remember that the goal is to minimize the sum of two natural numbers. So, you need an expression for the sum and then differentiate to find the minimum.

Define the variables.

$$x \cdot y = 9$$
$$y = \frac{9}{x}$$
$$S = x + y$$

Combine the equations into one function with two variables.

$$S = x + \frac{9}{x} = x + 9x^{-1}$$

Define the domain.

$$x > 0$$

Determine the first derivative.

$$S' = 1 - 9x^{-2} = 1 - \frac{9}{x^2} = \frac{x^2 - 9}{x^2}$$

Determine the critical values (i.e., when $S' = 0$ and S' does not exist).

$$0 = S' = \frac{x^2 - 9}{x^2}$$

$$0 = x^2 - 9$$

$$x^2 = 9$$

$$x = \pm 3$$

The non-permissible value for S' is $x = 0$, so it is also a critical point.

Notice that only one of the three potential critical values is within the domain, $x = 3$.

Determine the second derivative.

$$S'' = 18x^{-3} = \frac{18}{x^3}$$

Evaluate the second derivative at the critical value.

$$S''(3) = \frac{18}{(3)^3} = \frac{18}{27} = \frac{2}{3}, \text{ so } S'' > 0$$

According to the second derivative test, there is a relative minimum at $x = 3$, since $S''(3)$ is positive.

$$x = 3$$

$$y = \frac{9}{3} = 3$$

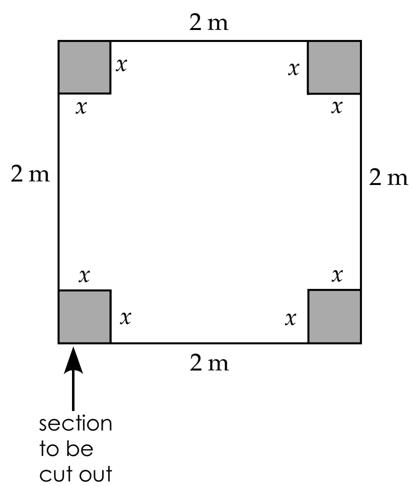
$$S = 3 + 3 = 6$$

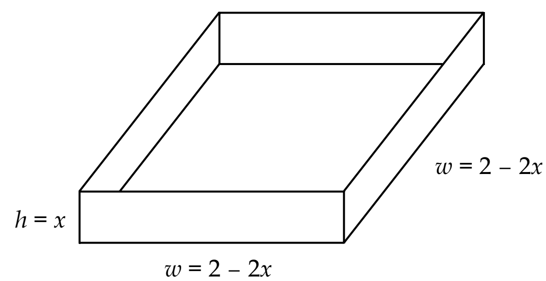
The two natural numbers are both 3 and their minimum sum is 6.

3. A box with an open top is to be constructed from a square piece of cardboard, 2 metres wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume of such a box.

Answer:

Remember that the goal is to maximize the volume of the box. In this case, a diagram may be helpful.





$$V = l \cdot w \cdot h \qquad h = x \qquad l = 2 - 2x \qquad w = 2 - 2x$$

Define the variables.
Let $h = x$ be the height of the box, w and l be the width and length of the box, and V be the volume of the box.

State the combined function for volume in terms of height, x.

$$V = (2 - 2x)(2 - 2x)x$$

$$V = \left(4 - 8x + 4x^2\right)x$$

$$V = 4x - 8x^2 + 4x^3$$

Each dimension must be positive. Solve the inequalities to determine the domain of the function.

$$h = x > 0$$

$$l = 2 - 2x > 0$$

$$-2x > -2$$

$$x < 1$$

$$w = 2 - 2x > 0$$

$$-2x > -2$$

$$x < 1$$

So, $0 < x < 1$.

Determine the first derivative.

$$V' = 4 - 16x + 12x^2$$

Find the critical values by solving $V' = 0$ and finding where V' does not exist.

$$0 = \left(4 - 16x + 12x^2\right)$$

$$0 = 4\left(3x^2 - 4x + 1\right)$$

$$0 = 4(3x - 1)(x - 1)$$

$$x = \frac{1}{3} \text{ and } x = 1$$

Determine the critical value(s) that are valid within the domain. V' exists everywhere, so the only critical point in the domain is where $V' = 0$ at $x = \frac{1}{3}$.

Determine the second derivative.

$$V'' = -16 + 24x$$

Evaluate the second derivative at the critical value $x = \frac{1}{3}$.

$$V'' = -16 + 24\left(\frac{1}{3}\right) = -16 + 8 = -8$$

$$V'' < 0$$

According to the second derivative test, there is a relative maximum at $x = \frac{1}{3}$, since V'' is negative.

Find the dimensions and the volume.

$$h = \frac{1}{3} \approx 0.3 \text{ m}$$

$$w = 2 - 2\left(\frac{1}{3}\right) = \frac{6}{3} - \frac{2}{3} = \frac{4}{3} \approx 1.3 \text{ m}$$

$$l = 2 - 2\left(\frac{1}{3}\right) = \frac{6}{3} - \frac{2}{3} = \frac{4}{3} \approx 1.3 \text{ m}$$

$$V = \left(\frac{1}{3}\right)\left(\frac{4}{3}\right)\left(\frac{4}{3}\right) = \left(\frac{16}{27}\right) \approx 0.59 \text{ m}^3$$

The dimensions that produce the maximum volume are $\frac{1}{3}$ by $\frac{4}{3}$ by $\frac{4}{3}$ m.

The maximum volume is $\frac{16}{27} \approx 0.59\ \text{m}^3$.

4. A retailer finds he or she can sell 20 CDs per month at $25 each for a gross revenue of $500. For each $1 reduction in price, the retailer can sell 2 more CDs per month. What selling price will produce the maximum revenue per month, and how many CDs would he or she sell at this price?

   Answer:

   The goal is to maximize revenue and determine the selling price and number of CDs sold at this price. You need a function for revenue in one variable so you can differentiate to find the maximum.

   Revenue = CDs × price

   The table below describes how the gross financial sales change.

Increase #	CDs #	Selling Price ($/CD)	Gross Financial Revenue ($)
0	20	25	$20 \times 25 = 500$
1	$20 + 2 = 22$	$25 - 1 = 24$	$22 \times 24 = 528$
2	$22 + 2 = 24$ $20 + 2(2) = 24$	$24 - 1 = 23$ $25 - 2(1) = 23$	$24 \times 23 = 552$
3	$24 + 2 = 26$ $20 + 3(2) = 26$	$23 - 1 = 22$ $25 - 3(1) = 22$	$26 \times 22 = 572$
n	$20 + n(2)$	$25 - n(1)$	$(20 + 2n)(25 - n)$

   Define the variables.

   Let C, P, and R be number of CDs, selling price, and revenue, respectively. In addition, n, represents the number of reductions.

$$C = 20 + 2n$$

$$P = 25 - n$$

$$R = C \cdot P$$

   Combine the equations to form one function defined with only two variables.

$$R = (20 + 2n)(25 - n)$$

$$R = 500 + 30n - 2n^2$$

Determine the domain.

Both the number of CDs and selling price must be positive. Solve the inequalities to determine the domain of the function.

$C > 0$

$20 + 2n > 0$

$2n > -20$

$n > -10$

$n > 0$

$P > 0$

$25 - n > 0$

$-n > -25$

$n < 25$

So, the domain of the function is $0 < n < 25$.

Determine the first derivative of R.

$R = 500 + 30n - 2n^2$

$R' = 30 - 4n$

Determine the critical value by solving $R' = 0$.

$0 = 30 - 4n$

$4n = 30$

$n = 7.5$

The critical value of n is 7.5.

Determine the second derivative.

$R'' = -4$

Evaluate the second derivative at the critical value.

$R'' < 0$

According to the second derivative test, since $R'' < 0$, then there is a maximum at $n = 7.5$.

$C = 20 + 2(7.5) = 20 + 15 = 35$ CDs

$P = 25 - 1(7.5) = 25 - 7.50 = \$17.50/\text{CD}$

$R = 35 \times 17.50 = \$612.50$

The retailer can sell 35 CDs at $17.50 each to attain a maximum revenue of $612.50.

Learning Activity 3.5

Part A: BrainPower

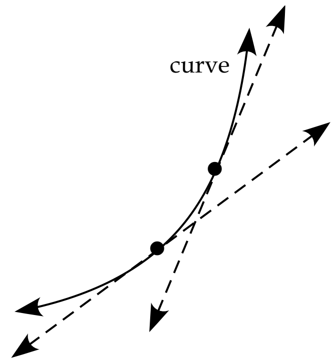
The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Determine the x-intercept(s) of $f(x) = (x - 2)(x + 3)(x - 1)$.
2. Determine the y-intercept of $f(x) = (x - 2)(x + 3)(x - 1)$.
3. Determine the critical values when $f'(x) = 12x - 6$.
4. What is the concavity of $f(x)$ on an interval where $f'(x) > 0$ and $f''(x) > 0$?
5. What is the concavity of $f(x)$ on an interval where $f'(x) > 0$ and $f''(x) < 0$?
6. Draw a curve that is increasing and concave up.
7. Draw a curve that is increasing and concave down.
8. Is $x - 1$ a factor of $f(x) = 2x^3 + 9x^2 - 4x - 7$?

Answers:

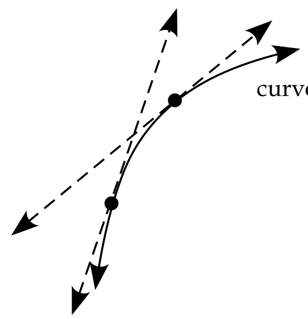
1. The x-intercepts are 2, −3, and 1. (Set the function to zero.)
2. 6 (Set $x = 0$ and solve for the function value $f(0) = (0 - 2)(0 + 3)(0 - 1) = (-2)(3)(-1) = 6$.)
3. $x = 0.5$ (Determine the zeros of the derivative function, $f'(x)$. Solve $0 = 12x - 6$.)
4. Concave up (The concavity is determined by the sign of the second derivative regardless of the sign of the first derivative, so if the second derivative is positive then the function is concave up on the interval.)
5. Concave down (The concavity is determined by the sign of the second derivative regardless of the sign of the first derivative, so if the second derivative is negative then the function is concave down on the interval.)

6.



positive slope ($f'(x) > 0$)
increasing slopes ($f''(x) > 0$)

7.



positive slope ($f'(x) > 0$)
decreasing slopes ($f''(x) < 0$)

8. Yes (According to the factor theorem, if $f(1) = 0$, then $x - 1$ is a factor. Since $f(1) = 2(1)^3 + 9(1)^2 - 4(1) - 7 = 2 + 9 - 4 - 7 = 0$, then $x - 1$ is a factor.)

Part B: Concavity and Sketching Polynomial Functions

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Find the point(s) of inflection and determine where the graph is concave up and concave down for the function $f(x) = x^3 - 5x^2 - 2x + 24$.

   Answer:

$$f(x) = x^3 - 5x^2 - 2x + 24$$
$$f'(x) = 3x^2 - 10x - 2$$
$$f''(x) = 6x - 10$$

Determine the potential inflection point(s) by determining the zero(s) of the second derivative.

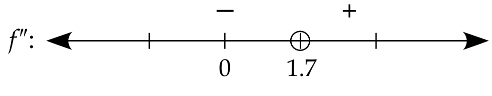
$$0 = 6x - 10$$
$$6x = 10$$
$$x = \frac{5}{3} \approx 1.7$$

$$f\left(\frac{5}{3}\right) = \left(\frac{5}{3}\right)^3 - 5\left(\frac{5}{3}\right)^2 - 2\left(\frac{5}{3}\right) + 24$$
$$= \frac{125}{27} - \frac{125}{9} - \frac{10}{3} + 24 = \frac{125}{27} - \frac{375}{27} - \frac{90}{27} + \frac{648}{27}$$
$$= \frac{308}{27} \approx 11.4$$

The potential point of inflection is (1.7, 11.4).

Use a sign diagram with the second derivative to determine concavity.

$$f'' = 6x - 10$$



The function is concave up on (1.7, ∞) and concave down on (–∞, 1.7).

The point of inflection at (1.7, 11.4) is confirmed because the concavity of the function changes sign at $x = 1.7$.

2. Sketch the graph of an odd degree polynomial function that has the following information:

Relative minimum: $(-1.5, -3)$

Relative maximum: $(1, -1)$

y-intercept: -1.5

x-intercept: -3

Inflection point: $(-0.5, -2)$

Concave down on: $(-0.5, \infty)$

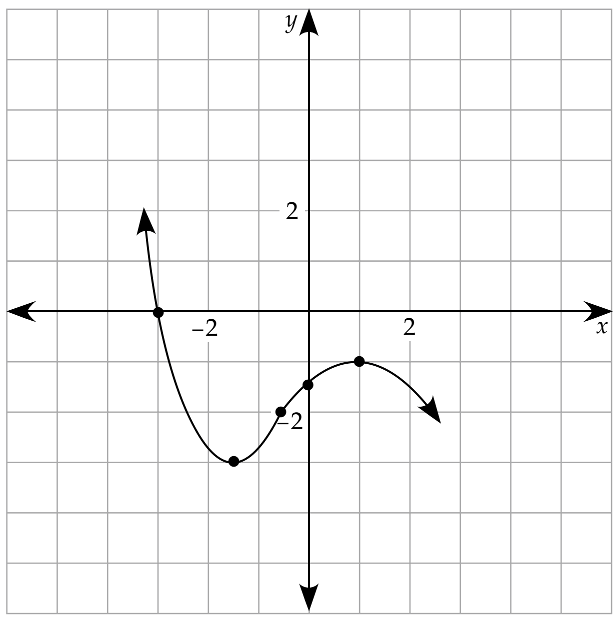
Concave up on: $(-\infty, -0.5)$

Domain: $x \in \Re$

Range: $y \in \Re$

Answer:

Plot the points first and then connect them using the concavity.



3. Sketch the graph of $f(x) = x^4 - 2x^3 + x - 2$, after identifying its x- and y-intercepts, relative extremes, its concavity, its points of inflection, and then its domain and range.

Answer:

Determine the y-intercept by setting $x = 0$.

$$f(0) = (0)^4 - 2(0)^3 + (0) - 2$$

$$f(0) = -2$$

$$y\text{-intercept} = -2$$

Determine the x-intercepts using the factor theorem and synthetic division.

$$\begin{aligned} f(1) &= 1^4 - 2(1)^3 + 1 - 2 \\ &= 1 - 2 + 1 - 2 \\ &= -2 \end{aligned}$$

So, $(x - 1)$ is not a factor.

$$\begin{aligned} f(-1) &= (-1)^4 = 2(-1)^3 - 1 - 2 \\ &= 1 + 2 - 1 - 2 = 0 \end{aligned}$$

So, $(x + 1)$ is a factor.

$$\begin{aligned} f(2) &= 2^4 - 2(2)^3 + 2 - 2 \\ &= 16 - 16 + 2 - 2 = 0 \end{aligned}$$

So, $(x - 2)$ is a factor.

$$\begin{array}{r|rrrrr} -1 & 1 & -2 & 0 & 1 & -2 \\ & & -1 & 3 & -3 & 2 \\ \hline & 1 & -3 & 3 & -2 & 0 \end{array}$$

$$\begin{array}{r|rrrr} 2 & 1 & -3 & 3 & -2 \\ & & 2 & -2 & 2 \\ \hline & 1 & -1 & 1 & 0 \end{array}$$

$$\underbrace{1 \quad -1 \quad 1}_{x^2 - x + 1}$$

Try solving $x^2 - x + 1 = 0$ using the quadratic formula and you will find there are no real roots. You could just check the discriminant ($b^2 - 4ac$).

$$(-1)^2 - 4(1)(1) = 1 - 4 = -3$$

Since the discriminant is negative, there are no more real roots.

So, $y = x^2 - x + 1$ has no x-intercepts.

The x-intercepts of $f(x)$ are -1 and 2.

Find the critical values for maxima and minima using the first derivative.

$$f(x) = x^4 - 2x^3 + x - 2$$

$$f'(x) = 4x^3 - 6x^2 + 1$$

Use the factor theorem to factor $f'(x)$.

$$f'(1) = 4(1)^3 - 6(1)^2 + 1 = 4 - 6 + 1 = -1$$

$$f'(-1) = 4(-1)^3 - 6(-1)^2 + 1 = -4 - 6 + 1 = -9$$

So $(x + 1)$ is not a factor of f'.

$$f'\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 - 6\left(\frac{1}{2}\right)^2 + 1 = \frac{4}{8} - \frac{6}{4} + 1$$

$$= \frac{1}{2} - \frac{3}{2} + 1 = 0$$

So $2x - 1$ is a factor of f'.

$$f(x) = x^4 - 2x^3 + x - 2$$

$$f'(x) = 4x^3 - 6x^2 + 1$$

Use synthetic or long division to write $f'(x)$ as a product of factors.

$$f'(x) = (2x - 1)\left(2x^2 - 2x - 1\right)$$

Use the quadratic formula to determine the other zeros of $f'(x)$.

$$2x^2 - 2x - 1 = 0$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-1)}}{2(2)}$$

$$x = \frac{2 \pm \sqrt{4+8}}{4}$$

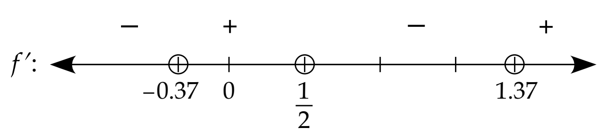
$$x = \frac{2 \pm \sqrt{12}}{4}$$

$$x_1 = 1.37$$

$$x_2 = -0.37$$

$$f'(x) = 0 \text{ when } x = \frac{1}{2}, 1.37, -0.37$$

Use a sign diagram of $f'(x) = (2x - 1)(x - 1.37)(x + 0.37)$.



There is a minimum at $x = -0.37$ and $x = 1.37$, since f' goes negative to positive. There is a maximum at $x = \frac{1}{2}$, since f' goes positive to negative.

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^4 - 2\left(\frac{1}{2}\right)^3 + \frac{1}{2} - 2 = \frac{-27}{16}$$

At $x = \frac{1}{2}$, there is a maximum of $\frac{-27}{16} = -1.6875$.

$$f(1.37) = (1.37)^4 - 2(1.37)^3 + 1.37 - 2 = -2.25$$

At $x = 1.37$, there is a minimum of -2.25.

$$f(-0.37) = (-0.37)^4 - 2(-0.37)^3 - 0.37 - 2 = -2.25$$

At $x = -0.37$, there is a minimum of -2.25.

Determine the second derivative.

knowing: $f'(x) = 4x^3 - 6x^2 + 1$

then: $f''(x) = 12x^2 - 12x$

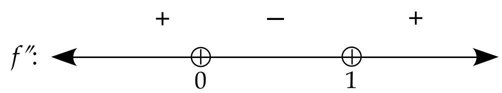
Determine the critical values of the second derivative.

$$0 = f''(x)$$

$$0 = 12x(x - 1)$$

$$x = 0, 1$$

Create a sign diagram to determine concavity.



$f(x)$ is concave up on $(-\infty, 0) \cup (1, \infty)$ because $f''(x) > 0$.

$f(x)$ is concave down on $(0, 1)$ because $f''(x) < 0$.

Determine points of inflection. Since the concavity changes sign at both $x = 0$ and $x = 1$, then there are points of inflection at both x-values.

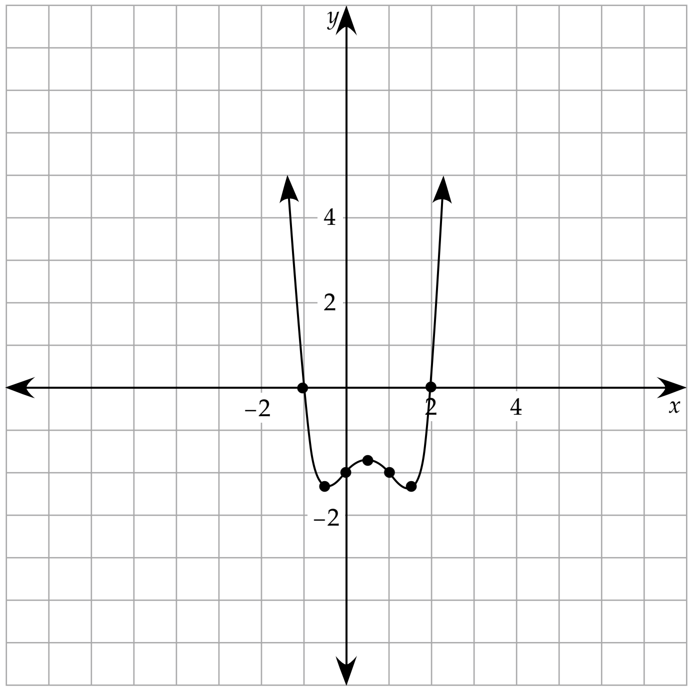
$$f(0) = -2$$

$$f(1) = 1^4 - 2(1)^3 + 1 - 2$$

$$= 1 - 2 + 1 - 2 = -2$$

$(0, -2)$ and $(1, -2)$ are points of inflection.

Plot the intercepts, relative extremes, and points of inflection. Use the domain and range to sketch.



State domain and range of the even degree polynomial function with a positive leading coefficient.

D: $\left(-\infty, \infty\right)$

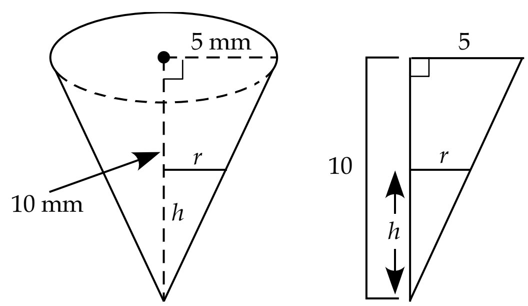
R: $\left\{y \mid y \geq -2.25,\ y \in \Re\right\}$

Learning Activity 3.6

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Differentiate $4x^3$ with respect to time knowing that x is a function of time.
2. If the base of a square increases at 5 cm/s, what is the length of the side after 5 seconds? (Assume it starts at zero.)
3. What is the area of the square at the 5 second mark in Question 2?
4. If a right cylinder has a radius of 5 mm and a vertical height of 10 mm, write a relation that represents the height of the cone with respect to the radius at any height.



5. Differentiate $y = \sqrt{3x^2 + 4x}$ using the chain rule.
6. Write an expression for V in terms of h using the following two equations: $r = 2h$ and $V = \dfrac{1}{3}\pi r^2 h$
7. Simplify $\dfrac{28\pi}{70\pi}$ by reducing the fraction into lowest terms.
8. Solve for y in $x^3 + y^2 = 1$ when $x = -1$.

Answers:

1. $12x^2 \cdot \dfrac{dx}{dt}$ $\left(\dfrac{d}{dt}\left(4x^3\right) = 4 \cdot \dfrac{d\left(x^3\right)}{dx} \cdot \dfrac{dx}{dt} = 4 \cdot 3x^2 \cdot \dfrac{dx}{dt} = 12x^2 \cdot \dfrac{dx}{dt}\right)$
2. 25 cm (Since the base increases at a rate of 5 cm/s, then the length of the base after 5 seconds is $5 \times 5 = 25$ cm.
3. 625 cm^2 ($A = s^2 = 25^2 = 625$ cm^2)

4. $h = 2r$ (According to the similar triangles in the diagram $\frac{r}{5} = \frac{h}{10}$. Isolate the height to find the relationship.)

5. $\frac{3x + 2}{\sqrt{3x^2 + 4x}}$ (You need to use the chain rule to differentiate this function, but first we have to rewrite the original equation in exponential form.)

$$y = \sqrt{3x^2 + 4x} = \left(3x^2 + 4x\right)^{\frac{1}{2}}$$

$$y' = \frac{1}{2}\left(3x^2 + 4x\right)^{-\frac{1}{2}}(6x + 4) = (3x + 2)\left(3x^2 + 4x\right)^{-\frac{1}{2}}\Bigg)$$

6. $V = \frac{4}{3}\pi h^3 \left(V = \frac{1}{3}\pi\,(2h)^2\,h = \frac{1}{3}\pi\,4h^3 = \frac{4}{3}\pi h^3\right)$

7. $\frac{2}{5}$ (Factor the numerator and denominator looking for the highest common factor. Then cancel the common factors.)

8. $y = \pm\sqrt{2}$ (Substitute $x = -1$ into $x^3 + y^2 = 1$ and then solve for y.

$$(-1)^3 + y^2 = 1$$

$$-1 + y^2 = 1$$

$$y^2 = 2$$

$$y = \pm\sqrt{2}$$

Don't forget that the answer can be both positive and negative.)

Part B: Related Rates

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. How fast is the volume of a spherical balloon increasing at the instant when the radius is 8 cm, if the radius is increasing at the rate of 10 cm per second?

   Answer:

   Define the variables.

   Let r and V represent the radius and the volume of the sphere, respectively.

   $$\frac{dr}{dt} = 10 \text{ cm/s}$$

   $$\frac{dV}{dt} = ?$$

   $$V = \frac{4}{3}\pi r^3$$

   Differentiate with respect to time.

   $$\frac{d}{dt}\left(V = \frac{4}{3}\pi r^3\right)$$

   $$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

   $$\frac{dV}{dt} = \frac{4}{3}\pi \cdot \frac{d\left(r^3\right)}{dr} \cdot \frac{dr}{dt}$$

   $$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt}$$

   $$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

   Substitute the given information at the instant when $r = 8$ cm.

   $$\frac{dV}{dt} = 4\pi\,(8)^2 \cdot (10)$$

   $$= 2560\pi \text{ m}^3/\text{s}$$

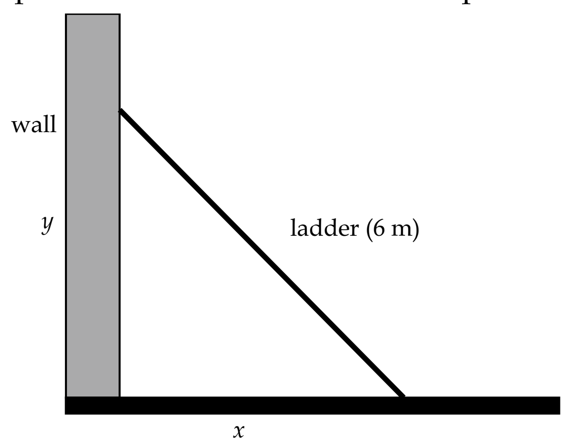
   The volume of the balloon is increasing at a rate of 2560 cm^3/s at the instant when the radius is 8 cm.

2. One end of a 6 m ladder leans against a wall and the other rests on the ground. If the lower end of the ladder slips horizontally along the ground at the rate of 0.3 m/s, at what rate is the top of the ladder moving toward the ground at the instant when the foot of the ladder is 4 m from the wall?

   Answer:

   Define the variables.

   Let x represent the distance from the bottom of the ladder and the wall, and y represent the distance from the top of the ladder and the ground.



$$\frac{dx}{dt} = 0.3 \text{ cm/s}$$

$$\frac{dy}{dt} = ?$$

$$x^2 + y^2 = 6^2$$

   Differentiate with respect to time.

$$\frac{d}{dt}\left(x^2 + y^2 = 6^2\right)$$

$$\frac{d}{dt}\left(x^2\right) + \frac{d}{dt}\left(y^2\right) = 0$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$y\frac{dy}{dt} = -x\frac{dx}{dt}$$

$$\frac{dy}{dt} = -\frac{x}{y} \cdot \frac{dx}{dt}$$

Solve for the y in the equation at the instant x is 4 m. Remember that distances are positive.

$$4^2 + y^2 = 6^2$$

$$y^2 = 36 - 16 = 20$$

$$y = \pm\sqrt{20}$$

$$y = \sqrt{20 \text{ m}}$$

Substitute the given information.

$$\frac{dy}{dt} = -\frac{4}{\sqrt{20}} \cdot (0.30) \approx -0.27 \text{ m/s}$$

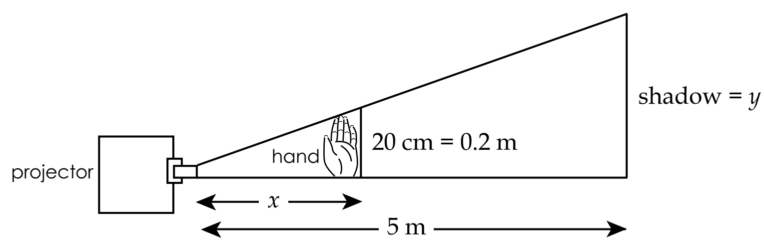
The ladder is **falling** at a rate of 0.27 m/s.

3. An overhead projector lights up a screen 5 m away. A student standing in front of the projector entertains his or her classmates by using a 20 cm long hand to cast a shadow of a barking dog on the screen. If the student walks to the screen at a speed of 0.8 m/s, how fast is the size of the dog's shadow decreasing at the instant when the student is 2 m from the projector?

   Answer:

   Define the variables.

   Let x represent the distance of the student from the projector, and y represent the height of the shadow. Use similar triangles to relate the variables.



$$\frac{y}{5} = \frac{0.2}{x}$$

$$\frac{dx}{dt} = 0.8 \text{ cm/s}$$

$$\frac{dy}{dt} = ? \quad \text{(at the instant when } x = 2 \text{ m)}$$

Write an expression for y in terms of x.

$$\frac{y}{5} = \frac{0.2}{x}$$

$$y = \frac{1}{x} = x^{-1}$$

Differentiate with respect to time.

$$\frac{dy}{dt} = -x^{-2} \cdot \frac{dx}{dt}$$

Substitute the given information at the instant $x = 2$ m.

$$\frac{dy}{dt} = -(2)^{-2} \cdot (0.8) = (-0.25) \cdot (0.8)$$

$$\frac{dy}{dt} = -0.2 \text{ m/s}$$

The dog's shadow is decreasing at 0.2 m/s at the instant the student is 2 m from the projector.

Grade 12 Introduction to Calculus (45S)

Module 4
Integration

MODULE 4: INTEGRATION

Introduction

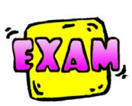
In Module 4, you will learn the inverse operation of differentiation called antidifferentiation, which, as you will see, is related to a process called integration. In some applications of calculus, it will be necessary to use antidifferentiation to find the original function when the derivative is known. In other applications, integration will be used to determine the area under the curve of a function.

Assignments in Module 4

When you have completed the assignments for Module 4, submit your completed assignments to the Distance Learning Unit either by mail or electronically through the learning management system (LMS). The staff will forward your work to your tutor/marker.

Lesson	Assignment Number	Assignment Title
1	Assignment 4.1	Antidifferentiation and Integration
2	Assignment 4.2	Differential Equations
3	Assignment 4.3	Definite Integral
4	Assignment 4.4	Area under a Curve
5	Assignment 4.5	Area between Two Functions

Writing Your Final Examination



You will write the final examination when you have completed Module 4 of this course. The final examination is based on Modules 1 to 4, and is worth 45 percent of your final mark in the course. To do well on the final examination, you should review all the work you complete in Modules 1 to 4, including all the learning activities and assignments. You will write the final examination under supervision.

Notes

LESSON 1: ANTIDIFFERENTIATION AND INTEGRATION

Lesson Focus

In this lesson, you will

- ❑ describe antidifferentiation as the inverse operation of differentiation
- ❑ determine the antiderivative family of functions given the derivative of a function
- ❑ define integration in terms of the area bounded by a function curve and the x-axis
- ❑ relate antidifferentiation and integration with the Fundamental Theorem of Calculus

Lesson Introduction

In the last three modules, the focus of what you have learned was on finding the derivative of a function and its applications. In this lesson, you will work backwards and determine the antiderivative of a function when given a polynomial function, $f'(x)$. You learned that the derivative of a function, $f'(x)$, is a function that describes the slope of $f(x)$ at every value in the function's domain. What does the antiderivative describe in terms of the original function, $f(x)$? Answering this question will be the focus of this lesson.

Antiderivative

A function, $F(x)$, is called an **antiderivative** of a function, $f(x)$, if the derivative of $F(x)$ is $f(x)$; essentially, $\frac{d}{dx}F(x) = f(x)$.

For example, the following functions are antiderivatives of the function, $f(x) = x^2$.

$$F(x) = \frac{1}{3}x^3$$

$$F(x) = \frac{1}{3}x^3 + 5$$

$$F(x) = \frac{1}{3}x^3 - 2.7$$

Since the derivative of a constant is zero, the derivative of each of the three functions above is x^2.

These examples suggest that the antiderivative of a function is *not unique*, in other words, a function may have more than one antiderivative. It is as if some piece of information is lost in the differentiation process. *Do you know what information is lost when you differentiate?*

What is the same in the three functions below?

$$F(x) = \frac{1}{3}x^3$$

$$F(x) = \frac{1}{3}x^3 + 5$$

$$F(x) = \frac{1}{3}x^3 - 2.7$$

The terms with variables are common, $\frac{1}{3}x^3$, but the constants are not. Thus, if $F(x)$ is any antiderivative of a function $f(x)$ and C is any constant, then $F(x) + C$ is also an antiderivative of $f(x)$, as shown below:

$$\frac{d}{dx}\left(F(x) + C\right) = \frac{d}{dx}\left(F(x)\right) + \frac{d}{dx}(C) = f(x) + 0 = f(x)$$

On any interval, essentially every antiderivative of $f(x)$, that is, the family of antiderivatives, is expressible in the form $F(x)$ plus a constant, $F(x) + C$.

Example 1

Determine three possible antiderivatives for $f(x) = 3x^2 + 2x$.

Solution

For $F(x)$ to be an antiderivative of $f(x) = 3x^2 + 2x$, $\frac{d}{dx}F(x) = f(x)$ must be true.

Let's guess and check.

The antiderivative of $3x^2$ is x^3, since $\frac{d}{dx}\left(x^3\right) = 3x^2$.

The antiderivative of $2x$ is x^2, since $\frac{d}{dx}\left(x^2\right) = 2x$.

$$\frac{d}{dx}\left(x^3 + x^2\right) = 3x^2 + 2x \quad \text{but} \quad \frac{d}{dx}\left(x^3 + x^2 + 6\right) = 3x^2 + 2x$$

So $F(x) = x^3 + x^2 + 6$, $F(x) = x^3 + x^2$, and $F(x) = x^3 + x^2 - 3$ are three antiderivative examples that have the derivative, $f(x) = 3x^2 + 2x$.

In general, the antiderivative of $f(x)$ could be written as $F(x) = x^3 + x^2 + C$.

From the above example, you confirmed that the antiderivative is not one specific function, but is a family of functions.

Example 2

Determine an antiderivative of the function $g(t) = t^4 + 3t - 5$.

Solution

You can find an antiderivative of each term of $g(t)$ separately by thinking about the inverse of the differentiation of polynomial terms.

The antiderivative of the first term, t^4, is $\frac{t^5}{5}$. You can check by finding the derivative:

$$\frac{d}{dt}\left(\frac{1}{5}t^5\right) = \frac{1}{5}\left(5t^4\right) = t^4$$

The antiderivative of the second term, $3t$, is $\frac{3t^2}{2}$. You can check by finding the derivative:

$$\frac{d}{dt}\left(\frac{3}{2}t^2\right) = \frac{3}{2}\left(2t^1\right) = 3t$$

The antiderivative of the third term, −5, is −5*t*. You can check by finding the derivative:

$$\frac{d}{dt}(-5t) = -5\left(1t^0\right) = -5$$

Therefore an antiderivative of $g(t) = t^4 + 3t - 5$ is $G(t) = \frac{t^5}{5} + \frac{3t^2}{2} - 5t$.

Note: This is just one of an infinite number of the related family of functions that are antiderivatives. Since the derivative of a constant is zero, another antiderivative could be written with any constant added, such as $G(t) = \frac{t^5}{5} + \frac{3t^2}{2} - 5t + 7$.

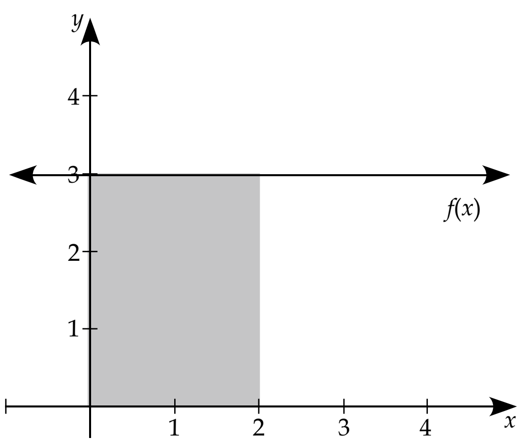
Integration

Now we will move to a process called integration, which is the process of finding the area under a function curve. It is easy to find the area under some function curves over specified intervals such as:

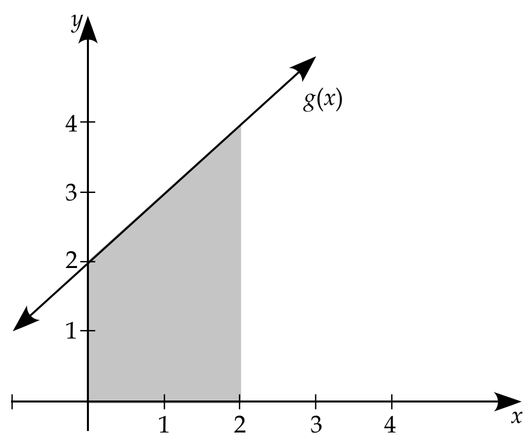
$$f(x) = 3$$

$$g(x) = x + 2$$

Find the area above the *x*-axis under each of the curves over the interval from $x = 0$ to $x = 2$. You can do this by sketching the graphs of $f(x)$ and $g(x)$ and shading the required areas.



The area under $f(x)$ from 0 to 2 is 6 units².



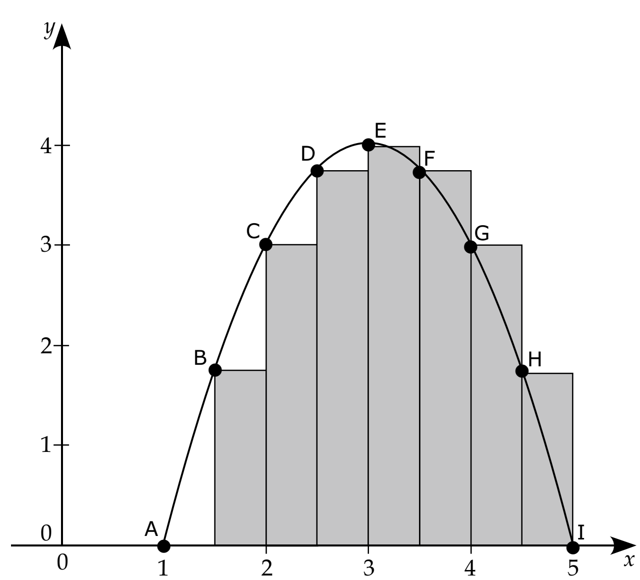
The area under $g(x)$ from 0 to 2 is also 6 units².

Integration is the process of finding the area under a function curve—using the calculus term, the **integral** of $f(x)$ as x goes from 0 to 2 is 6 and the integral of $g(x)$ as x goes from 0 to 2 is 6. As you will see, finding the integral for more "curvy" functions is more challenging. A process was developed to approximate the area under a function curve by adding up the areas of several thin rectangles. Since integration involves finding this area sum, the symbol used to represent the integral of $f(x)$ is an elongated S. The area under a function curve, the integral, can be written as $\int f(x)\,dx$, and is read "the integral of $f(x)$," "D," "x." The integral of $g(x)$ as x goes from 0 to 2 is the area and is written as $\text{Area} = \int_0^2 g(x)dx$. Therefore, you could write

$$\int_0^2 (x+2)\,dx = 6.$$

The following example illustrates the process of approximating the area under a quadratic function curve from $x = 1$ to $x = 5$. The quadratic function shown has the equation $h(x) = -x^2 + 6x - 5$.

The top-left side of each of the rectangles touches the function curve. The rectangles are all drawn to have widths, Δx, of 0.5 units. The first "rectangle" at point A has a height of zero and has no area. The other heights can be found evaluating the function equation for $h(x)$ at the x-value on the left of each rectangle. Find the area of each rectangle by multiplying the rectangle height, which is the function value, $h(x)$, by the base which is Δx ("delta x"). To calculate the approximate area, find the sum of the areas of the rectangles. This process can be represented using summation notation; the approximate area is $\sum_{x=1}^{5} h(x) \cdot \Delta x$.

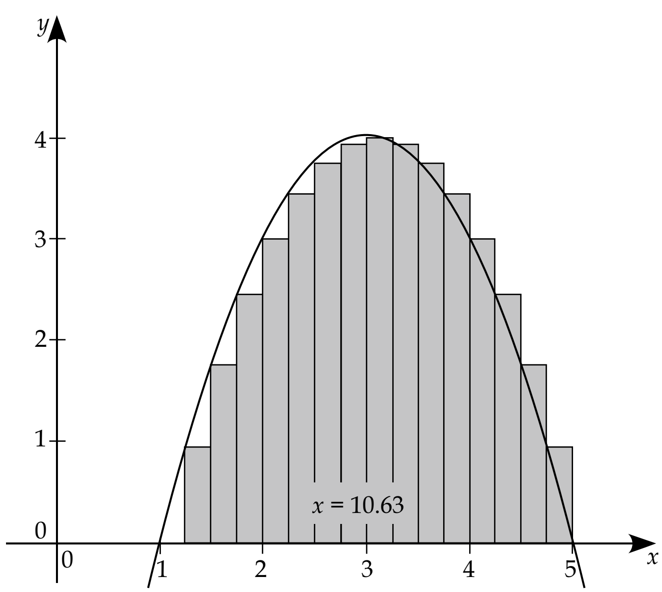


Compare this with the integral notation for the area, $\int_1^5 h(x)\,dx$. The "dx" is called the **differential** and is used to indicate that the integral is with respect to the domain variable, x. The differential, dx, can be considered to be an infinitesimally small Δx value. The areas of the rectangles are shown in the table where $\Delta x = 0.5$.

Top Left Point	Width	Height	Area
A (1,0)	0.5	0	0
B (1.5, 1.75)	0.5	1.75	0.875
C (2.0, 3)	0.5	3	1.5
D (2.5, 3.75)	0.5	3.75	1.875
E (3.0, 4)	0.5	4	2
F (3.5, 3.75)	0.5	3.75	1.875
G (4.0, 3)	0.5	3	1.5
H (4.5, 1.75)	0.5	1.75	0.875

The total area of all rectangles is 10.5 square units. So, the approximate area of $h(x)$ from 1 to 5 is 10.5, which can be written, $\int_1^5 h(x)\,dx \approx 10.5$.

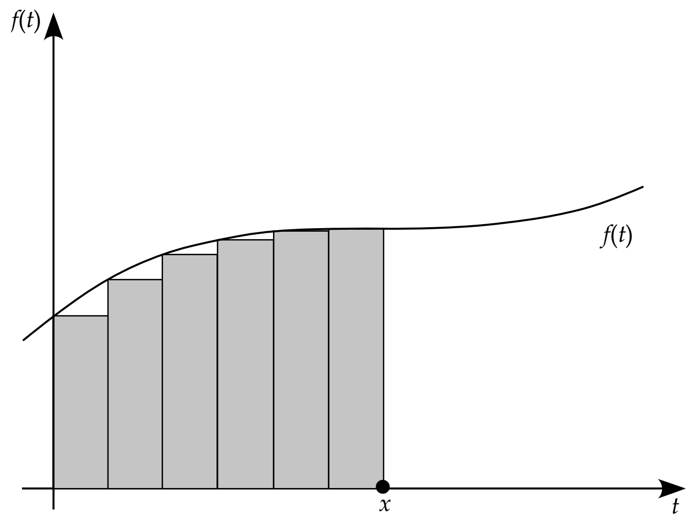
You can get a better approximation by using more rectangles. There are twice as many rectangles shown in the following diagram where each rectangle width is 0.25 units.

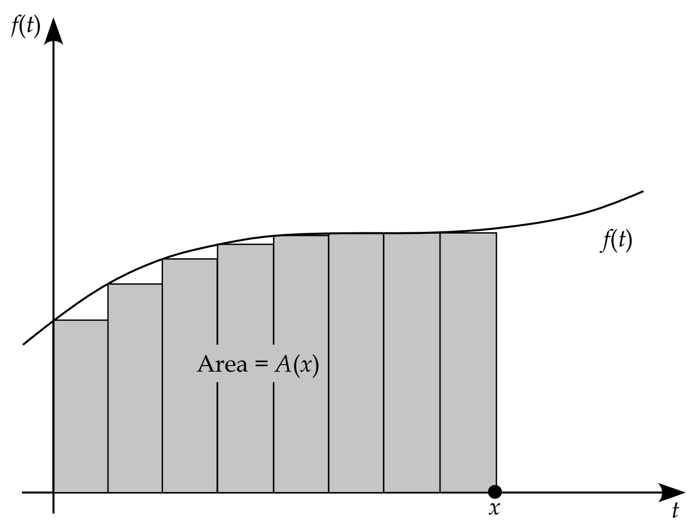


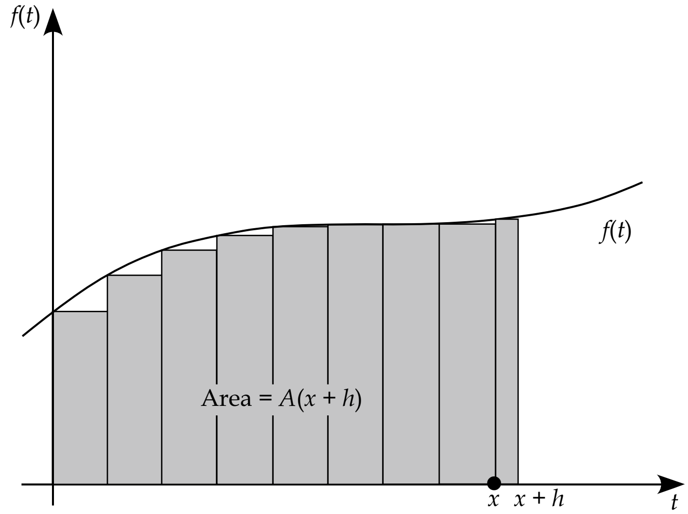
The approximation has improved to 10.63 units.

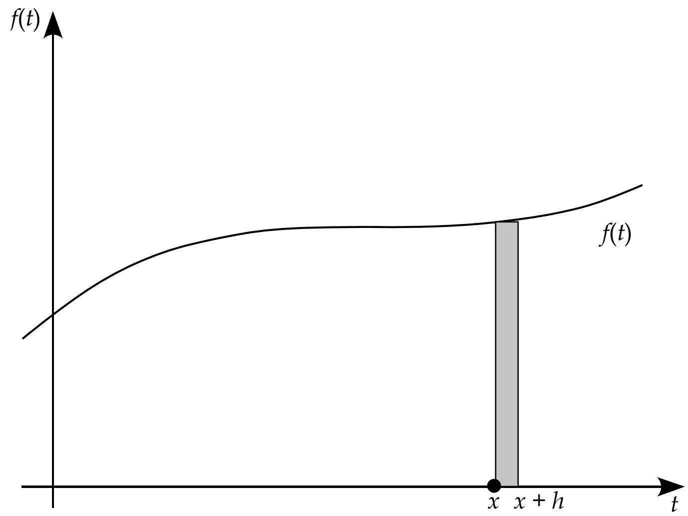
As you will see, there is a direct link between antidifferentiation of a function and the integral of (area under) its function curve. You will not be exposed to a rigorous proof of the relationship between the area under a curve and antidifferentiation; that will be left for a more in-depth calculus course that you may decide to take in the future. Nevertheless, the following illustration will give you a sense of the connection.

The following graph shows a function, $f(t)$, with an interval beginning at the origin and ending at a variable value, x. The size of the area under the curve shown will increase as the value of x increases. That is, the area is a function of x, written, $A(x)$.









Expressions to represent the area of the single thin rectangle added to the right of $A(x)$ can be made in two ways.

First, the area of the thin rectangle can be found by multiplying base times height.

Area = base $\cdot$ height

Area = $h \cdot f(x)$

Also, the area of the thin rectangle is the difference between the two shaded areas, $A(x + h)$ and $A(x)$.

Area = $A(x + h) - A(x)$

Therefore, you can equate the two expressions for area and rearrange the equation to solve for $f(x)$:

$$h \cdot f(x) = A(x+h) - A(x)$$

$$f(x) = \frac{A(x+h) - A(x)}{h}$$

The right side of this equation is the difference quotient for the area function, $A(x)$. This can be written as a limit as h approaches zero. The limit on the left side can just be written as $f(x)$ because h approaching zero doesn't affect the value of $f(x)$. The limit on the right side meets the definition of the derivative of the area function, $A'(x)$.

$$\lim_{h \to 0}\left(f(x)\right) = \lim_{h \to 0}\left(\frac{A(x+h) - A(x)}{h}\right)$$

$$f(x) = A'(x)$$

This means that the function, $f(x)$, can be written as the derivative of the area function. Saying this another way, the antiderivative of $f(x)$ is the same as the function describing the area under the curve. That is, the antiderivative of $f(x)$ is the same as the integral of $f(x)$!

This relationship between the area under a function curve and the antiderivative of the function is a very important result and is called the **Fundamental Theorem of Calculus** (FTC). The Fundamental Theorem of Calculus comes in two parts (the second part will be dealt with in a future lesson) and the first part is written as:

If $F(x) = \int_a^x f(t)\,dt$,

then $F'(x) = f(x)$.

In words, the derivative of the integral of a function is equal to the original function. $F(x)$ is the area under the curve of $f(x)$ and $F(x)$ is also the antiderivative of $f(x)$. The process of finding the area under the function curve, the integral, is the inverse operation to finding the derivative of the function.

The following table shows the use of integral notation with an **integrand** and differential, as well as the related general antiderivative function. The function to be integrated, the integrand, is always placed between the integral sign and the differential.

Integral Notation	Integrand	Differential	General Antiderivative
$\int x^2\,dx$	x^2	dx	$\frac{1}{3}x^3 + C$
$\int \left(4x^3 + 1\right)dx$	$4x^3 + 1$	dx	$x^4 + x + C$
$\int 5\,dx$	5	dx	$5x + C$

You can approach integration from the point of view of the **differential equation**, which is an equation involving $\frac{dy}{dx}$. If you solve the differential for y, you must use integration.

$$\frac{dy}{dx} = f(x)$$

$$dy = f(x)\,dx$$

Take the integral of both sides:

$$\int 1\,dy = \int f(x)\,dx$$

$$y = \int f(x)\,dx$$

But more specifically, the Fundamental Theorem of Calculus allows you to write:

$$\int f(x)\,dx = F(x) + C$$

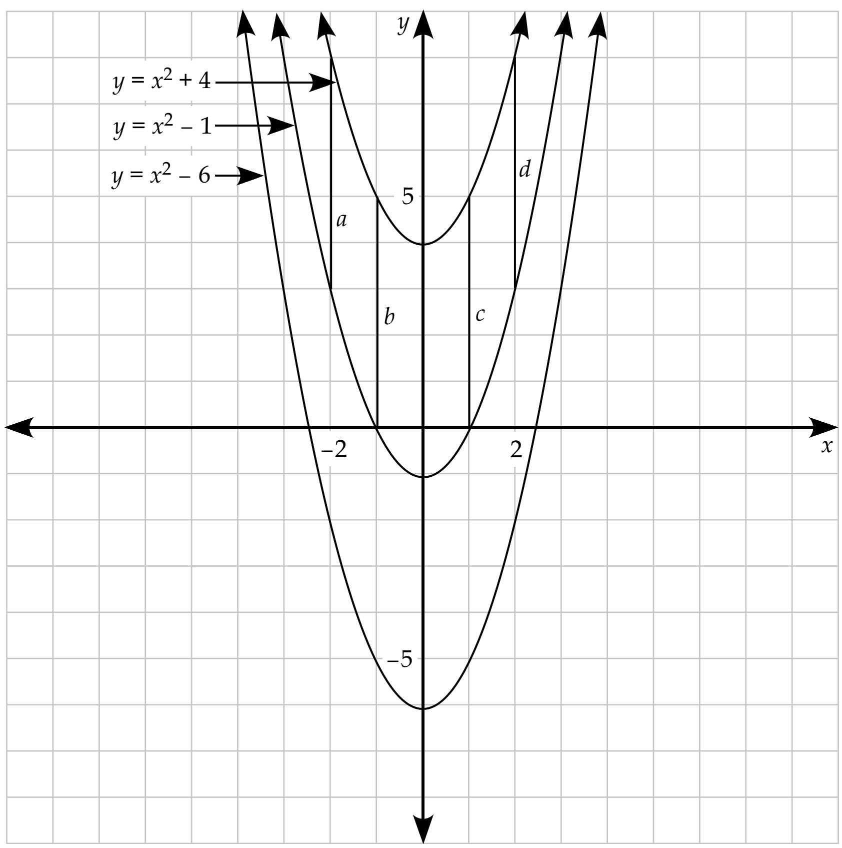
In the equation above, $F(x)$ is the antiderivative of $f(x)$ and C is any constant called a **constant of integration**. Notice that the right-hand side of the equation above does not represent a definite function because the constant is ambiguous in the integration process. Therefore, the equation is read as "the indefinite integral of $f(x)$ equals $F(x) + C$."

Indefinite Integral

The **indefinite integral** of a function $f(x)$ is written as:

$$\int f(x)\,dx = F(x) + C$$

The solution of $\int f(x)\,dx$ is a function described by the general antiderivative $F(x) + C$. The addition of a constant to a function in its equation causes a vertical shift of C to the graph of the function. More specifically, the integral $F(x) + C$ represents a family of curves, all of which have exactly the same shape. Each curve in the family of curves is parallel to any other member of the same family; that is, the curves are always separated by the same vertical distance. For any given x-coordinate, each curve has exactly the same slope and, therefore, the same derivative.



In each curve, you will notice that the constant C gives the y-intercept; and the vertical distances a, b, c, and d are equidistant. Thus, the curves are parallel and have the same slope at any specific x-coordinate.

Integration as the Inverse of Differentiation

$$\text{If } \int f(x)\,dx = F(x) + C,$$

$$\text{then } \frac{d}{dx}\big(F(x) + C\big) = f(x) \text{ or } \frac{d}{dx}\Big(\int f(x)\,dx\Big) = f(x).$$

The above mathematical statement demonstrates the relationship between the derivative and the integral; namely, they are inverse operations. This allows you to establish the integration rules in terms of what you already know about differentiation. These rules formalize what you have already done with antidifferentiation.

Power Rule for Integration

If n is any rational number except –1, then

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C.$$

Note

Remember that $\frac{d}{dx}\big(F(x) + C\big) = f(x)$, where $F(x)$ is an antiderivative of $f(x)$.

Show that the derivative of the function on the right is the integrand of the function on the left.

$$\frac{d}{dx}\left(\frac{1}{n+1}x^{n+1} + C\right) = \frac{d}{dx}\left(\frac{1}{n+1}x^{n+1}\right) + \frac{d}{dx}C = \frac{1}{n+1}\big((n+1)x^n\big) + 0 = x^n$$

Constant Times a Power Rule for Integration

The integral of a constant, k, times a function is equal to the constant times the integral of the function.

$$\int k \cdot f(x)\,dx = k \cdot \int f(x)\,dx$$

Note

$$\frac{d}{dx}\Big(k \cdot \int f(x)\,dx\Big) = k \cdot \frac{d}{dx}\Big(\int f(x)\,dx\Big) = k \cdot f(x)$$

The Sum Rule for Integration

The antiderivative of a sum is the sum of the antiderivatives.

$$\int \left(f(x) + g(x)\right) dx = \int f(x)\,dx + \int g(x)\,dx$$

Note

$$\frac{d}{dx}\left(\int f(x)\,dx + \int g(x)\,dx\right) = \frac{d}{dx}\int f(x)\,dx + \frac{d}{dx}\int g(x)\,dx = f(x) + g(x)$$

Example 3

Determine the indefinite integral (the general antiderivative function) for the following functions:

a) $f(x) = x^6$

b) $f(x) = x^{-27}$

c) $f(x) = 7x^4$

d) $f(x) = x^5 + x^3$

Solution

a) Use power rule:

$$\int x^6\,dx = \frac{x^{6+1}}{6+1} + C = \frac{1}{7}x^7 + C$$

b) Use power rule with negative exponent:

$$\int x^{-27}\,dx = \frac{x^{-27+1}}{-27+1} + C = -\frac{1}{26}x^{-26} + C$$

c) Use constant times a power rule:

$$\int 7x^4\,dx = 7 \cdot \frac{x^{4+1}}{4+1} + C = \frac{7}{5}x^5 + C$$

d) Use the sum and power rules:

$$\int \left(x^5 + x^3\right)dx = \frac{x^{5+1}}{5+1} + \frac{x^{3+1}}{3+1} + C = \frac{1}{6}x^6 + \frac{1}{4}x^4 + C$$

Learning Activity 4.1

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Predict which of the following functions have the same derivative and then find the derivatives.
   a) $f(x) = 7x^4 - 1$
   b) $f(x) = x^{-27} + 8$
   c) $f(x) = 7x^4 + 3$
   d) $f(x) = x^4 - 5$

2. Simplify: $8 \cdot \dfrac{x^{-5+1}}{-5+1}$

3. Simplify: $-2 \cdot \dfrac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$

4. Evaluate: $-3(4-2)^3$

5. Evaluate: $\dfrac{5+3}{4-2} + 2$

continued

Learning Activity 4.1 (continued)

Part B: Antidifferentiation and Integration

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Match each function with a possible antiderivative function.

	Functions
A	$2x + 1$
B	$4x^3 + 2$
C	$-2x^{-3} + 1$
D	$-x^{-2} + x$

Matching Letter	Antiderivatives
	$x^{-2} + x + 100$
	$x^{-1} + \frac{1}{2}x^2 + 1$
	$x^2 + x - 1$
	$x^4 + 2x - 3$

2. Determine the indefinite integral (the general antiderivative function) for the following functions:
   a) $f(x) = x^6 - x^{-2}$
   b) $f(x) = 2x^3 + 9x^2 - 3$
   c) $f(x) = x^{-1}$

3. Write an expression to represent each integral.
   a) $\int x\,dx$
   b) $\int 1\,dx$
   c) $\int dy$

Lesson Summary

In this lesson, you learned about the antiderivative. You learned that integration is the calculus term used to describe the process of finding the area under a function curve. You learned that the Fundamental Theorem of Calculus relates the integral and the antiderivative of a function. You determined the general antiderivative or indefinite integral using integration rules. In particular, you learned about the importance of the integration constant when determining the general antiderivative or the indefinite integral. The family of curves that represent the general antiderivative will be explored further in another lesson in this module.

Notes

Assignment 4.1

Antidifferentiation and Integration

Total: 11 marks

1. Match each function with a possible antiderivative function. *(4 marks)*

	Functions
A	$3x + 1$
B	$4x^3 + 2x$
C	$6x^{-4} + 1$
D	$x^{-2} + 3x^2$

Matching Letter	**Antiderivatives**
	$-2x^{-3} + x + 100$
	$\frac{3}{2}x^2 + x - 1$
	$-1x^{-1} + x^3 + 1$
	$x^4 + x^2 - 3$

continued

Assignment 4.1: Antidifferentiation and Integration (continued)

2. Determine the indefinite integral (the general antiderivative function) for the following functions.

   a) $f(x) = x^{-2} + 3x$ *(3 marks)*

   b) $f(x) = 2x^2 - x + 4$ *(4 marks)*

Lesson 2: Differential Equations

Lesson Focus

In this lesson, you will

- ❑ determine a specific antiderivative given the derivative function and the coordinates of a point
- ❑ apply integration to particle motion

Lesson Introduction

A differential equation is the name given to an equation involving the derivative or, in other words, the differentials *dy* and *dx*. As you will see in this lesson, you can solve differential equations using integration. You have learned how particle motion can be interpreted using differentiation. In this lesson, you will learn the role that integration plays in the study of particle motion. Furthermore, you will learn to solve a differential equation by finding the specific antiderivative function that is defined by the integral of the function and the coordinates of one function point.

Differential Equation

You know that the indefinite integral or the general antiderivative of a function represents a family of curves:

$$\int f(x)\,dx = F(x) + C$$

The value of the constant, C, determines which specific antiderivative or member of the family of curves is represented. A **differential equation** is an equation involving a derivative. To solve a differential equation, you will use integration as shown below.

$$\frac{dy}{dx} = f'(x)$$

$$dy = f'(x)\,dx$$

$$\int dy = \int f'(x)\,dx$$

$$y + C_1 = f(x) + C_2$$

$$y = f(x) + C$$

Notice that the constants, C_1 and C_2, were combined into a single constant, C. If you are given one point on the function, $f(x)$, you can then solve for the constant, C. This is demonstrated in the example below.

Example 1

Determine the equation of the function, $f(x)$, passing through the origin (0, 0) and for which $f'(x) = 2x - 1$.

Solution

$f'(x) = 2x - 1$	Given differential equation.
$\int f'(x)\,dx = \int (2x - 1)\,dx$	Integrate to find the function $f(x)$.
$f(x) = \int 2x\,dx - \int 1\,dx$	Sum rule for integration.
$= x^2 - x + C$	Power rule for integration (to find antiderivatives).
$f(x) = x^2 - x + C$	The family of functions that have slope described by $f'(x) = 2x - 1$, but an unknown value for the constant, C.

Use the given point (0, 0) to determine the value of C.

$f(0) = 0$ $f(0) = (0)^2 - (0) + C = 0$ $C = 0$	Substitute the coordinates into the equation and solve for the constant.

Write the specific function for $f(x)$ using the calculated value of C.

$f(x) = x^2 - x + 0$ $f(x) = x^2 - x$	The specific equation at the given condition.

Example 2

Determine $g(x)$, if $g'(x) = 3(x + 2)^2$ and $g(-1) = 3$.

Solution

$$g'(x) = 3(x+2)^2$$
$$= 3\left(x^2 + 4x + 4\right)$$
$$= 3x^2 + 12x + 12$$

Simplify given differential equation.

$$\int g'(x)\,dx = \int \left(3x^2 + 12x + 12\right)dx$$

Integrate to find the function $g(x)$.

$$g(x) = \int 3x^2\,dx + \int 12x\,dx + \int 12\,dx$$

Sum rule for integration.

$$= 3\int x^2\,dx + 12\int x\,dx + \int 12\,dx$$

Constant times a power rule for integration.

$$= 3\left(\frac{x^{2+1}}{2+1}\right) + 12\left(\frac{x^{1+1}}{1+1}\right) + 12\left(\frac{x^{0+1}}{0+1}\right) + C$$
$$= \frac{3x^3}{3} + \frac{12x^2}{2} + \frac{12x}{1} + C$$
$$g(x) = x^3 + 6x^2 + 12x + C$$

Power rule for integration. **Special note:** If possible, you can do all the integration rules mentally and you do not need to show all your thinking.

$$g(x) = x^3 + 6x^2 + 12x + C$$

The family of functions that have the same slope described by $g'(x) = 3(x + 2)^2$ but an unknown value for the constant, C.

Use the given point $(-1, 3)$ to determine the value of C.

$$g(-1) = 3$$
$$(-1)^3 + 6(-1)^2 + 12(-1) + C = 3$$
$$-1 + 6 - 12 + C = 3$$
$$-7 + C = 3$$
$$C = 10$$

Since the function has the given condition $g(-1) = 3$, substitute the values into the equation and solve for the constant, C.

Write the specific function for $g(x)$.

$$g(x) = x^3 + 6x^2 + 12x + 10$$

Next, you will explore how differential equations can be used to solve particle motion problems.

Particle Motion

In Module 3, you learned that differentiation could be used to solve particle motion problems. Velocity was defined as the rate of change of position (or displacement) with respect to time, and acceleration was defined as the rate of change of velocity with respect to time.

$$s(t) \quad \rightarrow \quad v = \frac{ds}{dt} \quad \rightarrow \quad a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Working backwards, if an object has acceleration, it has a change in velocity; if it has velocity, it has a change in position. Working in reverse, knowing the value of the acceleration of an object permits you to express the velocity and position of that object. This can be accomplished using integration and knowledge of the initial position and initial velocity.

If you studied the motion of objects in a physics class, you will have learned several formulas that relate displacement, velocity, and acceleration. You can solve the following differential equation to get an expression for velocity.

Write an equation representing velocity given, $\frac{dv}{dt} = a$.

$dv = a\ dt$ (separate the differentials)

$\int dv = \int a\ dt$ (integrate both sides)

$v = at + C$ (constants for both integrals are combined on the right)

The notation often used for the velocity when $t = 0$ is v_0 or sometimes v_i (for initial velocity). Therefore, the initial condition for velocity at $t = 0$ can be given as coordinates $(0, v_0)$. Substitute the coordinates to find an expression for C.

$$v_0 = a(0) + C$$

$$C = v_0$$

Therefore, given a constant acceleration, a, and an initial velocity, v_0, the equation for velocity in terms of time is:

$$v = at + v_0$$

If you know the values of the acceleration and the initial velocity, you can write an explicit expression for velocity in terms of time.

Example 3

The acceleration of a particle is $a = -9.8$ m/s^2.

a) Determine an expression for the velocity (m/s) at any time (second) given the initial conditions of $v_0 = 64$ m/s.

b) Determine the velocity when $t = 3$ s.

c) Determine an expression for the displacement of the particle at any time if $s = 16$ m at $t = 0$ s.

d) Determine the displacement for $t = 3$ s.

Solution

a) Since acceleration is the derivative of velocity, you need to integrate acceleration to find velocity.

$v' = a = -9.8$	Acceleration written as a differential equation.
$v = \int a\,dt = \int (-9.8)\,dt$	Integrate acceleration to find velocity.
$v = -9.8t + C$	Indefinite integral.
$64 = (-9.8)(0) + C$	Substitute the initial conditions to find C.
$C = 64$	
$v = -9.8t + 64$	Specific equation for velocity at any time.

b) Substitute $t = 3$ s into the velocity equation above.

$$v = -9.8t + 64$$

$$v = -9.8(3) + 64 = -29.4 + 64$$

$$v = 34.6 \text{ m/s}$$

c) Since velocity is the derivative of displacement, you need to integrate velocity to find displacement.

$s' = v = -9.8t + 64$	Velocity written as a differential equation.
$s = \int v\,dt = \int(-9.8t + 64)$	Integrate velocity to find velocity.
$s = \frac{-9.8t^2}{2} + 64t + C$ $s = -4.9t^2 + 64t + C$	Indefinite integral.
$16 = -4.9(0)^2 + 64(0) + C$ $16 = C$	Substitute the initial conditions $s = 16$ m at $t = 0$ s to solve for the constant.
$s = -4.9t^2 + 64t + 16$	Specific equation for displacement at any time.

d) Substitute $t = 3$ s into the displacement equation above.

$$s = -4.9t^2 + 64t + 16$$

$$s = -4.9(3)^2 + 64(3) + 16 = -4.9(9) + 192 + 16 = -44.1 + 192 + 16$$

$$s = 163.9 \text{ m}$$

Now, you can solve particle motion problems using integration as well as differentiation. You may have learned a lot of formulas in physics class related to motion. Now that you know calculus, you have a way to solve these problems without remembering those formulas. Instead you can either take the first and second derivative of displacement to find velocity and acceleration. Going the other way, you can use given initial conditions and take the integral of acceleration to find velocity and the integral of velocity to find displacement.

Example 4

A ball is thrown vertically upward at 3 m/s from a height of 2 m with the constant rate of acceleration -9.8 m/s^2 due to gravity.

a) Find expressions for the velocity, v, and height, s, at any time, t, if height is measured upward from the point of projection.

b) When does the ball hit the ground?

Solution

a) Since acceleration is the derivative of velocity, the velocity is the indefinite integral of acceleration. Similarly, since velocity is the derivative of displacement (height), then height is the indefinite integral of velocity.

$v' = a = -9.8$	Acceleration written as a differential equation.
$v = \int a\, dt = \int (-9.8)\, dt$	Integrate acceleration to find velocity.
$v = -9.8t + C$	Indefinite integral.
$3 = (-9.8)(0) + C$	Substitute the initial conditions to find C.
$C = 3$	

Specific equation for velocity is $v = -9.8t + 3$.

Now find an equation for displacement.

$s' = v = -9.8t + 3$	Velocity written as a differential equation.
$s = \int v\, dt = \int (-9.8t + 3)\, dt$	Integrate velocity to find velocity.
$s = \frac{-9.8t^2}{2} + 3t + C$ $s = -4.9t^2 + 3t + C$	Indefinite integral.
$2 = -4.9(0)^2 + 3(0) + C$ $2 = C$	Substitute the initial conditions s = 2 m at $t = 0$ s to solve for the constant

Specific equation for displacement at any time is $s = -4.9t^2 + 3t + 2$.

b) When the ball hits the ground, $s = 0$. You can, therefore, solve the displacement function above for t by setting $s = 0$.

$$s = -4.9t^2 + 3t + 2$$

$$0 = -4.9t^2 + 3t + 2$$

Solve this quadratic equation using the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$$t = \frac{-3 \pm \sqrt{3^2 - 4(-4.9)(2)}}{2(-4.9)}$$

$$t = 1.0145 \text{ and } t = -0.4023$$

Negative time values are invalid in this context; the ball hits the ground in 1.01 seconds.

Differential equations can be applied to particle motion problems and to problems involving rates of change in other contexts. You now know how to solve differential equations using the process of integration and information about one ordered pair of coordinates.

Learning Activity 4.2

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

Questions 1 to 3

Evaluate if $f(x) = 3x^2 - 2x - 1$.

1. $f(-1)$
2. $f(0)$
3. $f(2)$

Questions 4 to 6

If $g(x) = -2x^2 + C$, determine the value of C for each condition that follows.

4. $g(0) = -1$
5. $g(0) = 4$
6. $g(-2) = 3$
7. If $h'(x) = 1$, determine two possible antiderivatives that are 5 units apart shifted vertically.
8. If $h'(x) = 2x$, determine two possible antiderivatives that are 5 units apart shifted vertically.

continued

Learning Activity 4.2 (continued)

Part B: Differential Equations

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Find the equation of the curve $f(x)$, which passes through the point $(-1, 1)$, for each of the differential equations below.
   a) $f'(x) = \frac{2}{3}$
   b) $f'(x) = x^3 + 5x - 1$
   c) $f'(x) = \frac{1}{8x^2}$
2. Find the formula for distance s at any time t, using the given velocity function and specific information regarding s and t:
   $v = 1 + 2t$ and $s = 3$ m, when $t = 0$ s
3. Find the formula for velocity at any time t and displacement at any time t, using the given acceleration function and specific information regarding v, s, and t:
   $a = 2 + 36t$ and $v_0 = 3$ m/s, $s = 0$ m, when $t = 0$ s
4. An object is shot upward from the ground with an initial velocity of 30 m/s. The acceleration of gravity is -9.8 m/s^2.
   a) Calculate the elapsed time from the initial shot until the object returned to the ground.
   b) Calculate the maximum height to which the object would rise.

Lesson Summary

In this lesson, you learned how to solve a differential equation using integration and a given condition in order to determine the original function. Velocity and acceleration functions are examples of differential equations that originated from the derivative of displacement and the derivative of velocity functions, respectively. You learned how to integrate velocity and acceleration functions to determine the displacement and velocity functions at given conditions. In the next lesson, you will further explore integration over a specific interval and its relation to area under the curve.

Assignment 4.2

Differential Equations

Total: 35 marks

1. Find the equation of the curve $f(x)$, which passes through the point (2, −1), for each of the differential equations below.

   a) $f'(x) = 5$ through $(2,-1)$ *(3 marks)*

continued

b) $f'(x) = x^4 - 2x^2 + 2$ through $(2, -1)$ *(5 marks)*

continued

c) $f'(x) = \frac{1}{3x^3}$ through $(2,-1)$ *(4 marks)*

continued

Assignment 4.2: Differential Equations (continued)

2. Find the formula for displacement s at any time t, using the given velocity function and specific information regarding s and t: *(5 marks)*

   $v = t^2 - 2t + 3$ and $s = 8$ m when $t = 1$ s

3. Find the formula for velocity at any time t and displacement at any time t, using the given acceleration function and specific information regarding v, s, and t: *(8 marks)*

   $a = 4 - 2t + 3t^2$ and $v = 10$ m/s and $s = 4$ m when $t = 1$ s

continued

Assignment 4.2: Differential Equations (continued)

4. A ball is shot upward from the ground against gravity with an initial velocity of 12 m/s. The acceleration due to gravity is -9.8 m/s^2.

   a) Find an equation for its velocity at any time t. *(2 marks)*

   b) Find an equation for its height at any time t. *(3 marks)*

   c) Find its maximum height. *(3 marks)*

   d) Find the time it takes to return to the ground. *(2 marks)*

Notes

Lesson 3: Definite Integral

Lesson Focus

In this lesson, you will

- ❑ calculate the area under an interval of a function curve using integration
- ❑ define the definite integral as a numerical value equal to an area
- ❑ evaluate definite integrals using antiderivatives and the Fundamental Theorem of Calculus
- ❑ evaluate definite integrals geometrically by calculating area
- ❑ determine that definite integrals of functions below the x-axis are negative

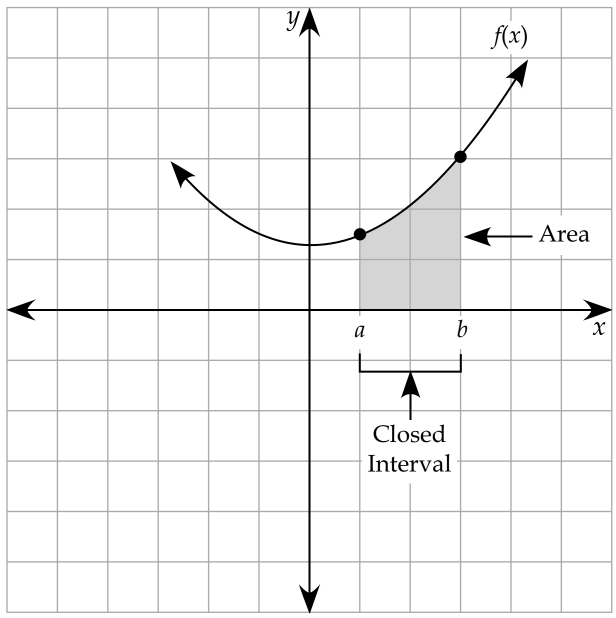
Lesson Introduction

You have learned how the area under a function curve and the antiderivative of the function curve are related. In the last lesson, you found the indefinite integral of the derivative of a function to produce a family of functions that all have the same derivative. In this lesson, you will learn of the definite integral that uses integration to determine a value representing the area under the function curve over a specified interval. The second part of the Fundamental Theorem of Calculus relates the integral over an interval and values of the antiderivative to calculate the area under the curve.

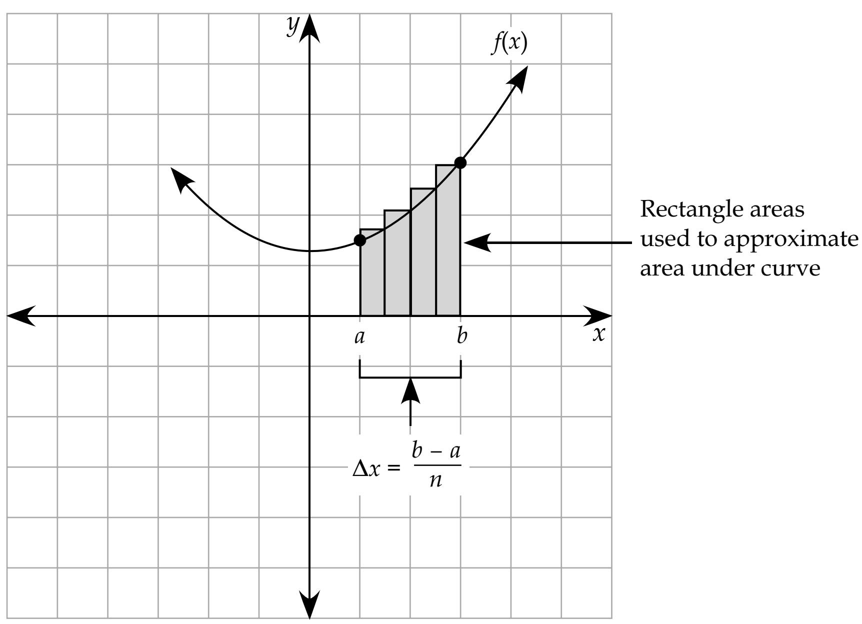
The area under a curve is widely used in applications involving rates of change. For example, the definite integral over a time interval of a velocity function allows you to find the change in displacement of the travelling object. The definite integral over an interval of a function involving oil consumption in litres per hour will yield the number of litres used over the time interval. In this lesson. you will learn to evaluate definite integrals geometrically and algebraically

Area Under the Curve

An important purpose of integration is to determine the area under a curve on an interval. You can determine the area under a curve, $f(x)$, bounded by the x-axis on an interval, $[a, b]$, in different ways: geometrically using rectangle approximations or algebraically using antiderivatives.



On the graph above, you see an irregular shaped area that could be approximated geometrically with smaller rectangles. As the number of rectangles increases, the width of these rectangles can be minimized to achieve a more accurate value of the area under the curve. In this case, the right side of each of the four rectangles touches the function curve.



The widths of the rectangles are all the same and can be found by calculating $\Delta x = \frac{b-a}{n}$, where n represents the number of rectangles. The area under the curve is the sum of these rectangular areas. As their width becomes infinitely small, $n \to \infty$.

More specifically, if $f(x)$ is a continuous function on $[a, b]$,
where the closed interval is portioned into equal lengths $\Delta x = \frac{b-a}{n}$,
then the area under the curve is $\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k) \cdot \Delta x$,
where each c_k is, in this case, the x-value of the right side
of each rectangle for the k^{th} subinterval.

As you saw in Lesson 1 of this module, this sum of the areas under the curve is equivalent to evaluating the integral of the function on the closed interval. The integral symbol ∫ is an elongated S, representing the sum of the rectangular areas.

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k) \cdot \Delta x = \int_a^b f(x)\,dx$$

The slope of a function is represented by the derivative of the function. The area under a function curve is represented by the integral of the function. As you learned in Lesson 1 of this module, the Fundamental Theorem of Calculus states that the integral is related to the antiderivative of the function. In case the awesomeness of this sentence passed you by, in their development of their development of "The Calculus," Leibniz and Newton proved these two seemingly disconnected concepts, the slope of a curve and the area under the curve, are related in that they are inverse functions of one another.

Definite Integral

Unlike an indefinite integral, which describes a function (or family of functions), the **definite integral** is a numerical value. This numerical value represents the area under the function's curve bounded by the *x*-axis on a closed interval. A wide variety of applications including physical science, engineering, and commerce are expressed and evaluated with definite integrals.

The definite integral of a continuous function, $f(x)$, on the closed interval $[a, b]$, is expressed as $\int_a^b f(x)\,dx$ and read as "the integral from *a* to *b* of $f(x)$ with respect to *x*." In this notation, $f(x)$ is the integrand, and the lower bound *a* and the upper bound *b* are called the limits of integration.

Fundamental Theorem of Calculus (Part 2)

The Fundamental Theorem of Calculus (FTC) appears in two parts. As you learned previously, Part 1 describes the inverse relationship between differentiation and integration. Part 2 outlines how to evaluate the definite integral on a closed interval using antiderivatives, as shown below.

If $f(x)$ is continuous at every point on the closed interval $[a, b]$, and $F(x)$ is the antiderivative of $f(x)$ on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

The definite integral is determined in two main steps. First, determine the antiderivative function and place it inside square brackets with the bounds shown to its right, $\left[F(x) + C\right]_a^b$. Then, substitute the upper and lower bounds into $F(x)$ and subtract.

$$\int_a^b f(x)\,dx = \left[F(x) + C\right]_a^b = \left[F(b) + C\right] - \left[F(a) + C\right] = F(b) - F(a)$$

Note: For definite integrals, the constant of integration is not necessary because the constant will always cancel out in the subtraction.

Let's now practice evaluating definite integrals.

Example 1

Evaluate the definite integral $\int_{3}^{7} 5\,dx$.

Solution

$$\int_{3}^{7} 5\,dx = [5x]_{3}^{7}$$

The antiderivative, $F(x)$, with the lower bound of 3 and an upper bound of 7.

$$= F(7) - F(3)$$

$$= 5(7) - 5(3)$$

$$= 35 - 15 = 20$$

The integral evaluated using the FTC.

Example 2

Evaluate the definite integral $\int_{-1}^{2} \left(x^3 + 3x^2 - x\right) dx$.

Solution

$$\int_{-1}^{2} \left(x^3 + 3x^2 - x\right) dx = \left[\frac{x^4}{4} + x^3 - \frac{x^2}{2}\right]_{-1}^{2}$$

The antiderivative, $F(x)$, with the lower bound of −1 and an upper bound of 2.

$$= F(2) - F(1)$$

$$= \left[\frac{(2)^4}{4} + (2)^3 - \frac{(2)^2}{2}\right] - \left[\frac{(-1)^4}{4} + (-1)^3 - \frac{(-1)^2}{2}\right]$$

$$= [4 + 8 - 2] - \left[\frac{1}{4} - 1 - \frac{1}{2}\right]$$

$$= 10 - \left(-\frac{5}{4}\right) = 10 + \frac{5}{4} = \frac{45}{4}$$

The integral evaluated using the FTC.

Example 3

Evaluate the definite integral $\int_1^4 \frac{2}{\sqrt{x^3}}\,dx$.

Solution

$$\int_1^4 \frac{2}{\sqrt{x^3}}\,dx = \int_1^4 \left(2x^{-\frac{3}{2}}\right)dx = \left[-4x^{-\frac{1}{2}}\right]_1^4$$

The antiderivative, $F(x)$, with the lower bound of 1 and an upper bound of 4.

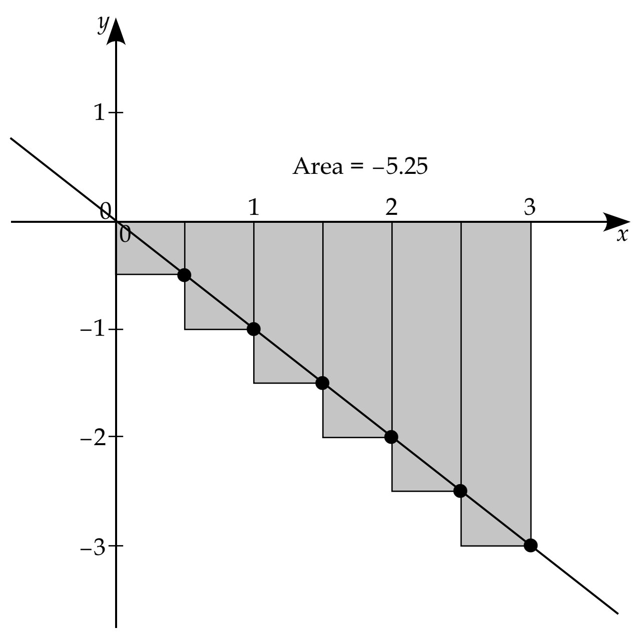
$$= F(4) - F(1)$$

$$= \left[-4(4)^{-\frac{1}{2}}\right] - \left[-4(1)^{-\frac{1}{2}}\right]$$

$$= -4 \cdot \frac{1}{2} - (-4) = -2 + 4 = 2$$

The integral evaluated using the FTC.

The process of finding the definite integral involves finding the sum of the areas of rectangles each with a base of Δx and a height of $f(x)$, at several values of x. To estimate the area under the function, several rectangles are drawn. In this case, the rectangles are drawn so the right side of each rectangle touches the function (as shown). Since the $f(x)$ values are negative, the sum of the "areas" will be negative.



As a result, the definite integral over an interval of a function that is below the x-axis results in a value that is negative. The diagram shows an estimate of the definite integral of the function, $f(x) = -x$, using 6 rectangles from $x = 0$ to $x = 3$. The estimated integral is -5.25.

In general, the areas above the x-axis are positive and the areas below the x-axis are negative.

Example 4

Evaluate the definite integral $\int_0^3 -x \, dx$.

Solution

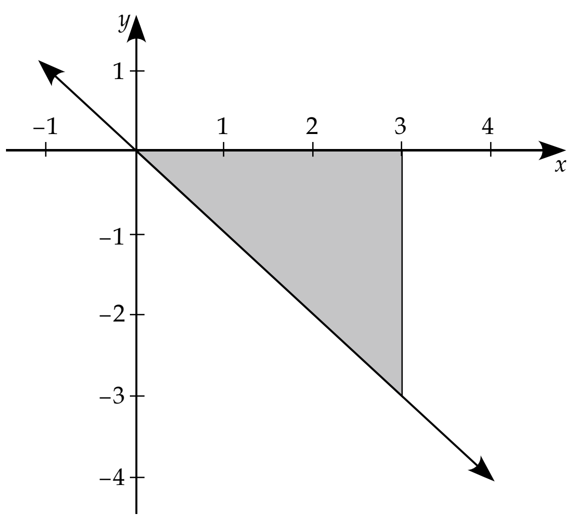
$\int_0^3 -x \, dx = \left[-\frac{x^2}{2}\right]_0^3$ The antiderivative, $F(x)$.

$= F(3) - F(0)$

$= -\frac{3^2}{2} - \frac{0^2}{2}$

$= -\frac{9}{2} = -4.5$

Alternatively, you could find the definite integral, $\int_0^3 -x \, dx$, geometrically by graphing the function and finding the area between the function and the x-axis (as shaded). The shaded triangle has a base of 3 and a height of 3, so Area = (3 × 3) ÷ 2 = 4.5. Since the area is below the x-axis, the value of the definite integral is negative. So, $\int_0^3 -x \, dx = -4.5$.



Example 5

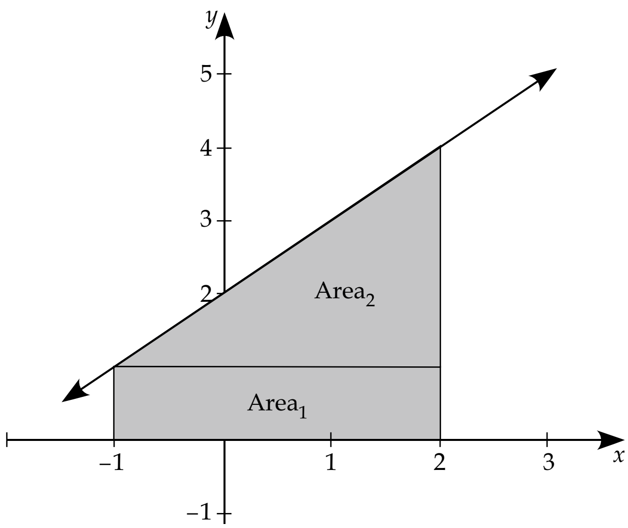
Sketch the function $f(x) = x + 2$, and use the graph to evaluate the definite integral $\int_{-1}^{2} (x + 2)\, dx$, geometrically.

Solution

Evaluate the definite integral by finding the area under the function from $x = -1$ to $x = 2$.

$\text{Area}_2 = (3 \times 3) \div 2 = 4.5$

$\text{Area}_1 = 3 \times 1 = 3$



The whole area is above the x-axis so it is positive. Therefore,

$\int_{-1}^{2} (x + 2)\, dx = 7.5.$

Note: You could check this result using the antiderivative, $F(x)$, and the Fundamental Theorem of Calculus, where $\int_{-1}^{2} (x + 2)\, dx = F(2) - F(-1)$.

Learning Activity 4.3

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Determine an antiderivative of $5x$.
2. Given $F(x) = 3x - 1$, evaluate $F(3) - F(1)$.
3. Given $F(x) = 3x^3 + 4x$, evaluate $F(1) - F(-1)$.
4. Given $F(x) = \sqrt{x}$, evaluate $F(4) - F(0)$.
5. Simplify $\dfrac{x^3 + x^2}{x^2}$.
6. Simplify $5(x + 1)^2$ by expanding.
7. Simplify $\left(1 + \sqrt{x}\right)^2$ by expanding.
8. Simplify $\dfrac{1}{\sqrt[3]{x^4}}$ with negative rational exponents.

continued

Part B: Definite Integral

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Evaluate the definite integral $\int_{1}^{4} \left(4x^3\right) dx$.

2. a) Evaluate the definite integral $\int_{-2}^{2} \left(x + x^3\right) dx$.

   b) Sketch $y = x + x^3$ and explain why the definite integral on the interval from −2 to 2 can be equal to zero.

3. Simplify the integrand before evaluating the following definite integral:

   a) $\int_{0}^{1} 4x(x+1)^2\, dx$

   b) $\int_{1}^{4} \frac{4x^3}{\sqrt{x}}\, dx$

Lesson Summary

In this lesson, you learned how to evaluate the definite integral of a function geometrically by finding the area under a curve, and algebraically by using the Fundamental Theorem of Calculus (Part 2) and antiderivatives. You also learned that the integral of the portion of a function that is below the x-axis will have a negative value. In the next lesson, you will use what you now know about definite integrals to calculate the total area under curves that are sometimes above and sometimes below the x-axis.

Assignment 4.3

Definite Integral

Total: 18 marks

1. Evaluate the following definite integrals.

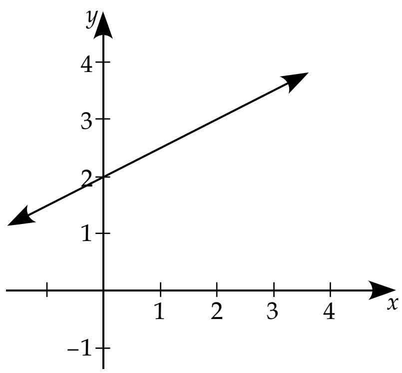
   a) $\int_{-1}^{4} 3x^2 \, dx$ *(3 marks)*

   b) $\int_{-3}^{3} \left(2x - x^3\right) dx$ *(4 marks)*

continued

Assignment 4.3: Definite Integral (continued)

2. Use the graph of $f(x) = \frac{1}{2}x + 2$ (shown) to evaluate the definite integral, $\int_0^3 f(x)\,dx$, geometrically. Show your work. *(2 marks)*



continued

Assignment 4.3: Definite Integral (continued)

3. Simplify the integrand before evaluating the following definite integrals.

   a) $\int_{0}^{2} 4(x-1)^2\,dx$ *(5 marks)*

   b) $\int_{1}^{4} \frac{3}{\sqrt{x^3}}\,dx$ *(4 marks)*

Notes

Lesson 4: Area under a Curve

Lesson Focus

In this lesson, you will

- ❏ evaluate the definite integral of functions geometrically where parts of the function may be below the x-axis.
- ❏ determine the total area bounded by a function curve above and below the x-axis using the definite integral
- ❏ relate the total area bounded by a function curve, $f(x)$, on interval $[a, b]$ to the definite integral of the absolute value of the function, $\int_a^b |f(x)|\,dx$

Lesson Introduction

You have been learning about calculating perimeter and area since elementary school. In high school, you extended your knowledge of area to include surface area of three-dimensional objects, such as cylinders, cones, and spheres. As you have learned, calculus can be used to calculate the area of unusual shapes. If you take another calculus course at university, you will learn techniques for using calculus to find the volume of unusual shapes. Calculating areas and volumes of unusual shapes is used to survey irregularly shaped property or to calculate the volume of material needed to build an irregularly shaped object.

The focus of the next two lessons is on using the definite integral to find the areas. The focus of this lesson is finding the area in a specified interval that is bounded by the x-axis and a function curve. As you will see, when using a definite integral, the area calculation needs to be adjusted if some of the function curve is below the x-axis.

Integration to Find Areas

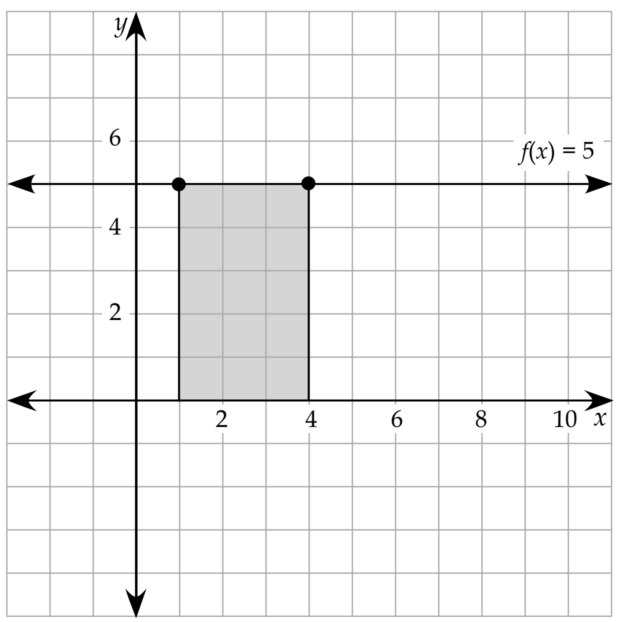
The area bounded by a **positive function**, $f(x)$, and the x-axis on a closed interval, $[a, b]$, can be determined by integrating the definite integral from a to b of $f(x)$ with respect to x.

If $f(x) \geq 0$ on $[a, b]$, then

area above the x-axis from a to $b = \int_a^b f(x)\,dx$.

Verifying Definite Integrals with Geometric Areas

Let's verify the relationship between the geometric area and the definite integral of the function $f(x) = 5$.



Geometrically	Algebraically
$A = l \cdot w$ Let l = height w = width $A = 5 \times (4 - 1) = 5 \times 3 = 15 \text{ units}^2$	$\int_1^4 5\,dx = [5x]_1^4$ $= 5(4) - 5(1) = 20 - 5 = 15 \text{ units}^2$

The area of the rectangle, found geometrically, bounded by the function, $f(x) = 5$, the x-axis, and the vertical lines $x = 1$ and $x = 4$, is equivalent to the value of the definite integral using antiderivative values.

Provided the function is positive (entirely above the x-axis) on the closed interval, the area under the curve can be evaluated by integrating the definite integral on the closed interval.

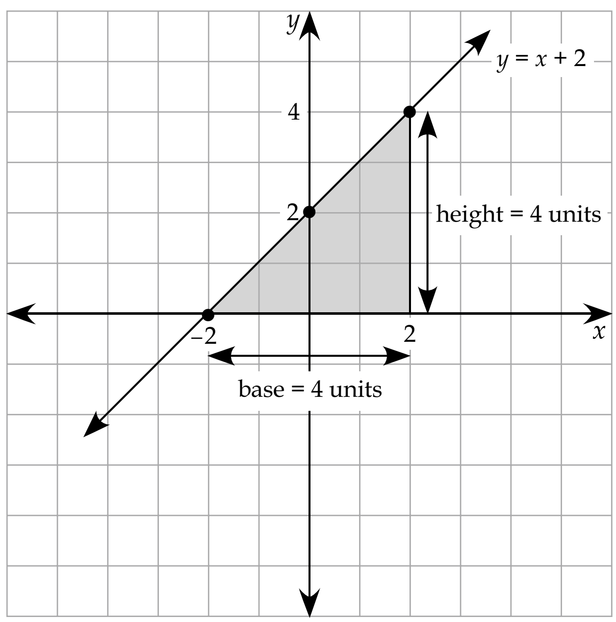
Example 1

Determine the area bounded by the curve $y = x + 2$ and the x-axis on the interval $[-2, 2]$,

a) geometrically

b) algebraically

Solution



a) The area under the curve is the area of a triangle. This area can be found by solving the equation $A = \frac{1}{2}bh$, with $b = 4$ units and $h = 4$ units.

$$A = \frac{1}{2}(4)(4) = 8 \text{ units}^2$$

b) The area under the curve can also be determined using the Fundamental Theorem of Calculus by evaluating the definite integral in the bounded interval.

$$\int_{-2}^{2} (x+2)\,dx = \left[\frac{1}{2}x^2 + 2x\right]_{-2}^{2} = \left[\frac{1}{2}(2)^2 + 2(2)\right] - \left[\frac{1}{2}(-2)^2 + 2(-2)\right]$$

$$= \left[\frac{1}{2}\cdot 4 + 4\right] - \left[\frac{1}{2}\cdot 4 - 4\right]$$

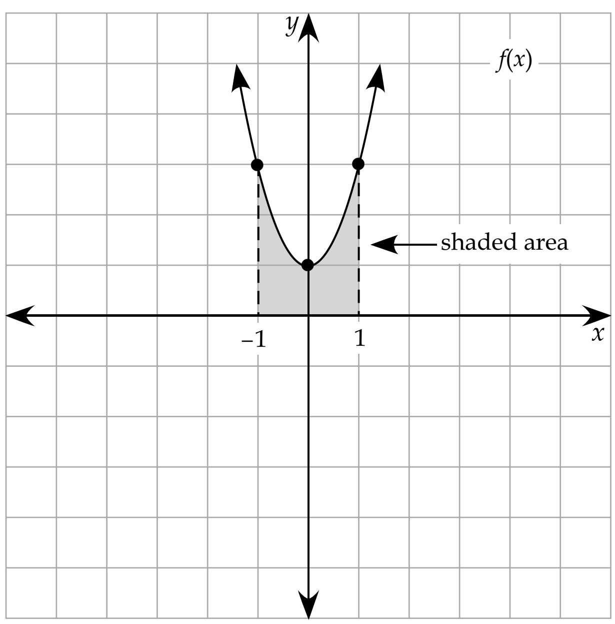
$$= 2 + 4 - 2 + 4 = 8 \text{ units}^2$$

Definite integrals can be used and are particularly useful to determine the area under a curve that is not easily found geometrically, as in the next example.

Example 2

Determine the area bounded by the function $f(x) = 2x^2 + 1$, and the x-axis on the interval $[-1, 1]$. Sketch a graph of the function and the area to be determined.

Solution



The area under a curve bounded by a closed interval can be determined using the Fundamental Theorem of Calculus and antiderivative values, as shown below.

$$\int_{-1}^{1}\left(2x^2+1\right)dx=\left[\frac{2}{3}x^3+x\right]_{-1}^{1}=\left[\frac{2}{3}(1)^3+1\right]-\left[\frac{2}{3}(-1)^3-1\right]$$

$$=\frac{2}{3}+1+\frac{2}{3}+1=\frac{4}{3}+2=\frac{10}{3}$$

The area under the curve is $\frac{10}{3}$ units^2.

Determining the area of functions that are below the x-axis on the interval need to be considered differently because area is always a positive quantity but the definite integral is not. Let's explore this challenge with the next example.

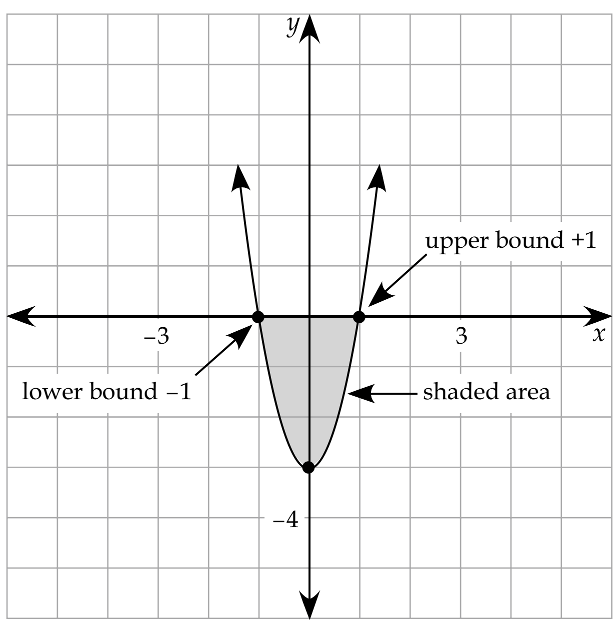
Example 3

Determine the area bound by the function $y = 3x^2 - 3$ and the x-axis.

Solution

Determine the x-intercepts and y-intercept to sketch the graph.

x-intercepts ($y = 0$)	y-intercept ($x = 0$)
$0 = 3x^2 - 3$ $3 = 3x^2$ $\quad x = \pm 1$ $1 = x^2$	$y = 3(0)^2 - 3$ $y = -3$



From the sketch of the graph, you can see that the shaded area is bounded by the interval [−1, 1], so you can use the definite integral to determine the area bounded by the curve and the x-axis.

$$\int_{-1}^{1} \left(3x^2 - 3\right) dx = \left[x^3 - 3x\right]_{-1}^{1} = \left[1^3 - 3(1)\right] - \left[(-1)^3 - 3(-1)\right]$$

$$= 1 - 3 + 1 - 3 = -4$$

The definite integral yields a negative number but *area is always a positive quantity*. As you saw in the previous lesson, the reason this happens is that the process of integration is really adding up the areas of many vertical rectangles between the curve and the x-axis. The base of the rectangles, Δx, is a positive number but the heights of the rectangles (in this case) are negative numbers because the heights are the function values that are below the x-axis. If you want to describe the area, you need to use the magnitude of the definite integral and ignore the sign.

Therefore, the area bounded by a **negative function** (below the x-axis) and the x-axis on a closed interval, $[a, b]$, is equal to the opposite sign of your definite integral from a to b of $f(x)$ with respect to x, as shown below.

If $f(x) \leq 0$ on $[a, b]$, then

area below the x-axis from a to $b = -\int_{a}^{b} f(x)\,dx$.

What if the function is both positive and negative on the closed interval to be integrated? It is important that we study the graph of a function to determine where it is positive and negative so that you can adjust the definite integral accordingly. You will study this in the next example.

Example 4

Given the function $f(x) = x^3 - 4x$.

a) Determine the value of the definite integral, $\int_{-2}^{2} f(x)\,dx$.

b) Sketch the graph of $f(x)$, including relative maxima and minima.

c) Determine the area bounded by the curve of $f(x)$ and the x-axis on the closed interval $[-2, 2]$.

Solution

a) Find an antiderivative, $F(x)$ and use the Fundamental Theorem of Calculus,

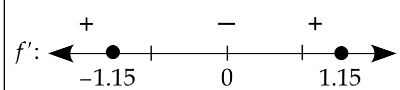
$$\int_{-2}^{2} f(x)\,dx = F(2) - F(-2).$$

An antiderivative is $F(x) = \frac{x^4}{4} - 2x^2$.

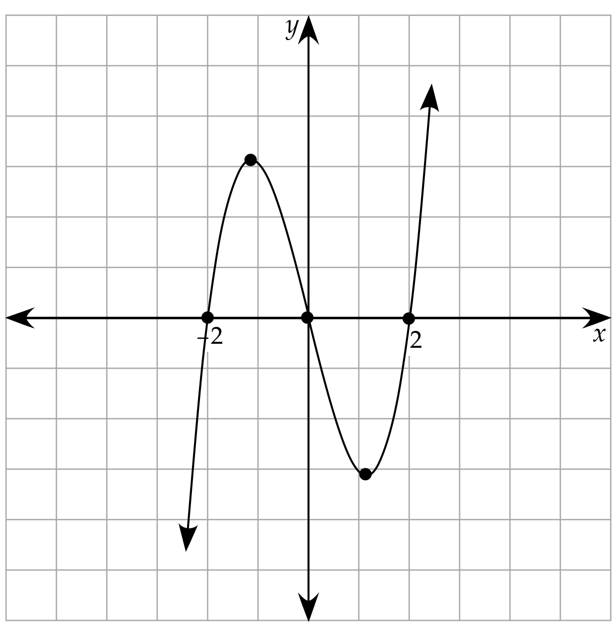
$$\begin{aligned} F(2) - F(-2) &= \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{2} \\ &= \left[\frac{(2)^4}{4} - 2(2)^2\right] - \left[\frac{(-2)^4}{4} - 2(-2)^2\right] \\ &= [4 - 8] - [4 - 8] \\ &= 0 \end{aligned}$$

$$\int_{-2}^{2} f(x)\,dx = 0$$

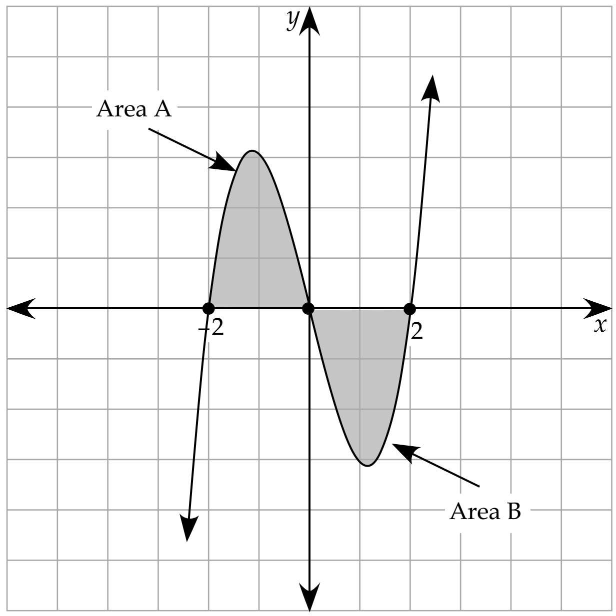
b) To sketch the graph, you need to determine its zeros, end behaviour, and relative extreme values.

Zeros or x-intercepts ($y = 0$)	**End Behaviour**	**Extreme Values**
$0 = x^3 - 4x$ $0 = x\left(x^2 - 4\right)$ $0 = x(x-2)(x+2)$ $x = 0, \pm 2$	Odd degree and positive leading coefficient, so the function is decreasing to the left and increasing to the right.	$f'(x) = 3x^2 - 4$ Extreme values are where $f'(x) = 0$, so solve $3x^2 - 4 = 0$. $3x^2 = 4$ $x = \pm\sqrt{1.33\dot{3}}$ $x \approx +1.15$ or $x \approx -1.15$ Use a sign diagram of $f'(x)$ to determine whether maximum or minimum. Test the signs of $f'(-2)$, $f'(0)$, and $f'(2)$.  Maximum at -1.15 since $f'(x)$ goes from + to −. Minimum at 1.15 since $f'(x)$ goes from − to +. Evaluate $f(-1.15)$. Maximum at $(-1.15, 3.08)$ Evaluate $f(1.15)$. Minimum at $(1.15, -3.08)$

Sketch the x-intercepts and relative extreme values with appropriate end behaviour.



c) As you can see, some of the function curve is above the x-axis (Area A) and some of the function curve is below the x-axis (Area B).



Area above the x-axis, area A, is bounded by the interval [–2, 0].

Area A can be determined by the definite integral:

$$\int_{-2}^{0}\left(x^3-4x\right)dx=\left[\frac{1}{4}x^4-2x^2\right]_{-2}^{0}=\left[\frac{1}{4}(0)^4-2(0)^2\right]-\left[\frac{1}{4}(-2)^4-2(-2)^2\right]$$

$$=0-4+8=4$$

Area below the x-axis, area B, is bounded by the interval [0, 2].

Area B can be determined by the definite integral:

$$\int_{0}^{2}\left(x^3-4x\right)dx=\left[\frac{1}{4}x^4-2x^2\right]_{0}^{2}=\left[\frac{1}{4}(2)^4-2(2)^2\right]-\left[\frac{1}{4}(0)^4-2(0)^2\right]$$

$$=4-8=-4$$

Notice the definite integral of the area below the x-axis comes out as a negative value. As a result, the definite integral from −2 to 2 is deceiving. Its value is:

$$\int_{-2}^{2}\left(x^3-4x\right)dx=0$$

So, when finding the total area from −2 to 2, you need to separate the part of the function that is below the x-axis from the part that is above the x-axis. You can use the absolute values of the definite integrals so that you are always considering the areas as positive numbers.

The **total area** bounded by the curve and x-axis on the closed interval [−2, 2] is the sum of area A and B.

$4 + | -4 | = 8 \text{ units}^2$

From the previous example, you noticed that you were able to determine when the function was positive and negative in the integration interval by finding the x-intercepts and sketching the function. You could also use a sign diagram for $f(x)$ with the x-intercepts as the critical values to determine the sign of the function.

For $f(x) \geq 0$ on interval $[a, b]$ and $f(x) < 0$ on interval $[c, d]$, then the area bounded by $f(x)$ and the x-axis is defined by:

$$\text{total area} = \int_a^b f(x)\,dx + \left| \int_c^d f(x)\,dx \right|$$

Alternatively, you could consider the graph of $y = | f(x) |$, which will be entirely above the x-axis.

Regardless of the sign of $f(x)$, the area bounded by $f(x)$ and the x-axis on $[a, b]$ is defined by:

$$\text{Total Area} = \int_a^b |f(x)|\,dx$$

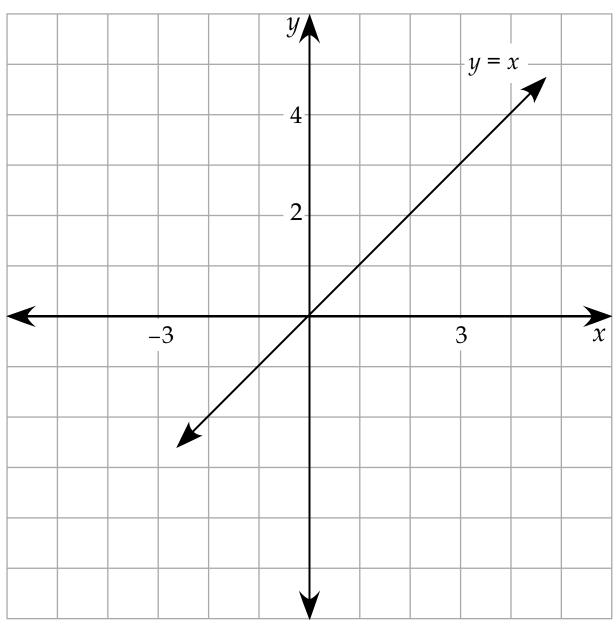
Learning Activity 4.4

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Determine the area of a rectangle with a width of 5 cm and a length of 10 cm.
2. Determine the area of a triangle with a base of 3 mm and a height of 6 mm.
3. Determine the area of a square with a side length of 5 cm.
4. Evaluate the definite integral, $\int_0^3 (x)\,dx$, geometrically.



5. Determine an antiderivative of $\sqrt[3]{x^2}$.
6. Determine the x-intercepts of $y = 3x^2 - 12$.
7. Determine the x-intercepts of $y = x(x - 1)(x + 2)$.
8. Factor: $x^2 - 3x + 2$

continued

Learning Activity 4.4 (continued)

Part B: Area under a Curve

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Determine the value of the definite integral $\int_0^3 (2x)\,dx$ in two ways,

   a) geometrically by determining the area bounded by the function and the x-axis.

   b) algebraically using the antiderivative.

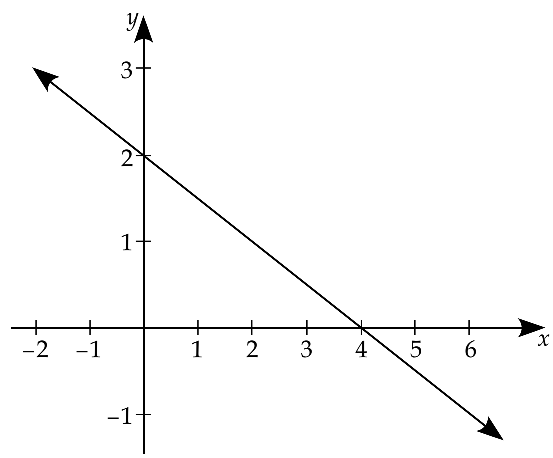
2. Determine the area bounded by the curve $y = x^2 + 2$ and the x-axis on $[-1, 2]$.

3. Find the values of each definite integral geometrically using the sketch of $f(x)$ as shown.

   a) $\int_0^4 f(x)\,dx$

   b) $\int_{-2}^4 f(x)\,dx$

   c) $\int_0^6 f(x)\,dx$



4. Determine the area bounded by the function $y = x^3 - 2x^2 - 5x + 6$ and the x-axis on $[-1, 2]$.

Lesson Summary

In this lesson, you verified that the geometric area bounded by a positive function curve and the x-axis on a closed interval is the same as the value of the definite integral of the curve's function on the same closed interval. In addition, you learned how to determine the area bound by the function and the x-axis for functions that are both positive and negative on a closed interval, using the absolute value of the definite integral for parts of the function below the x-axis. In the next lesson, you will learn how to determine the area bounded by two curves on a closed interval.

Notes

Assignment 4.4

Area under a Curve

Total: 23 marks

1. Determine the value of the definite integral, $\int_0^6 3x \, dx$, in two ways:

   a) geometrically by sketching the function, $y = 3x$, and determining the area of the appropriate region. *(3 marks)*

   b) algebraically using the antiderivative *(3 marks)*

continued

Assignment 4.4: Area under a Curve (continued)

2. Sketch the curve and determine the area bounded by $y = x^2 + 3$ and the x-axis on $[-2, 1]$. *(4 marks)*

continued

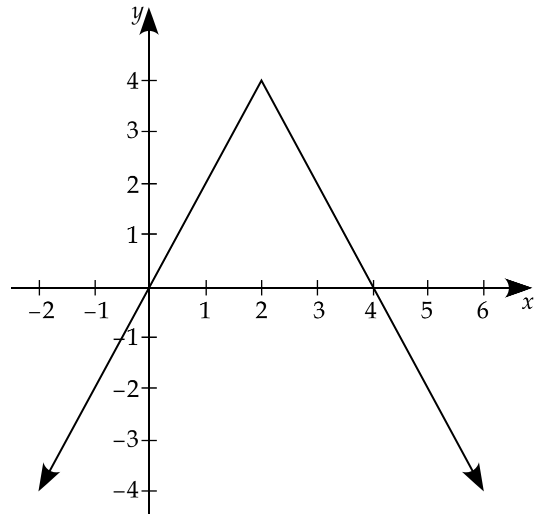
Assignment 4.4: Area under a Curve (continued)

3. Find the values of each definite integral, geometrically, using the sketch of $f(x)$ as shown. *(6 marks)*

a) $\int_{0}^{2} f(x)\, dx$

b) $\int_{-2}^{2} f(x)\, dx$

c) $\int_{0}^{6} f(x)\, dx$



continued

Assignment 4.4: Area under a Curve (continued)

4. Sketch the function and determine the area bounded by the function $y = 3x^2 - 8x - 3$ and the x-axis on [2, 4]. *(7 marks)*

Lesson 5: Area between Two Functions

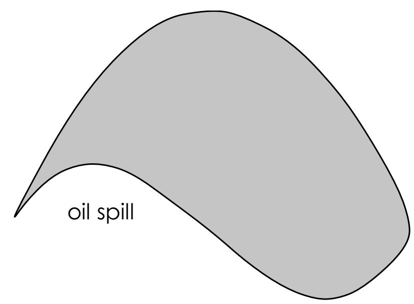
Lesson Focus

In this lesson, you will

- ❑ determine the area between two lines on an interval
- ❑ determine the area between any two function curves on an interval
- ❑ determine the area between two intersecting functions with no interval given

Lesson Introduction

You have learned to calculate the area bounded by a function curve and the x-axis. Often, applications of integral calculus require that you find the area of shapes that are irregular on all sides. In this lesson, you will learn to find the area of shapes that can be modelled by a wide variety of function curves. For example, you could find the area of an oil spill shaped as shown. The top part of the shape could be approximated with a quadratic function and the bottom of the shape could be approximated with a cubic or sinusoidal function. In this lesson, with the use of definite integrals, you will learn how the geometric relationship of compounded areas can be used to determine the area between two function curves. You will learn how to evaluate the area bounded by two curves on a given interval. Additionally, you will learn how the intersection points of two function curves can be used to determine the interval boundaries of a definite integral.



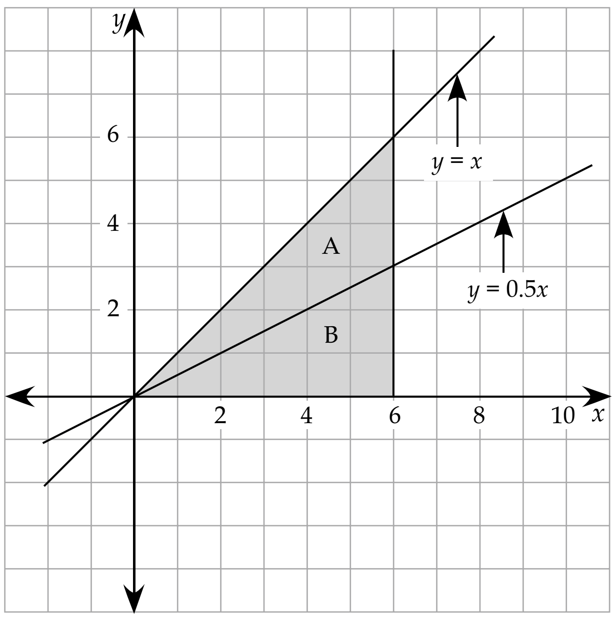
Compounded Areas

Let's review how to determine the area between two shapes using a difference calculation. The following example uses triangles to showcase this difference.

Example 1

Determine the area between the two lines $y = x$ and $y = 0.5x$ on the interval [0, 6].

Solution



The area could be calculated using the formula for the area of a triangle. However, this illustration shows you how to use antiderivatives.

$$\text{Area (A + B)} = \int_0^6 x\,dx = \left[\frac{1}{2}x^2\right]_0^6 = \frac{1}{2}(6)^2 - 0 = 18 \text{ units}^2$$

$$\text{Area B} = \int_0^6 0.5x\,dx = \left[\frac{1}{4}x^2\right]_0^6 = \frac{1}{4}(6)^2 - 0 = 9 \text{ units}^2$$

$$\text{Area A} = \int_0^6 x\,dx - \int_0^6 0.5x\,dx = 18 - 9 = 9 \text{ units}^2$$

The above graph shows the relationship between the three areas. Area B is the area bounded by the line $y = 0.5x$ and the x-axis on the interval [0, 6]. Area (A + B) is the area bounded by the line $y = x$ and the x-axis on the interval [0, 6]. Now, Area A is the difference between Area (A + B) and Area B, and more specifically the area bounded by the two lines $y = x$ and $y = 0.5x$ on the interval [0, 6].

As shown above, the area between two function curves $f(x)$ and $g(x)$ on an interval $[a, b]$ where $f(x) \geq g(x)$ can be found by subtracting the values of the two definite integrals.

$$\text{Area} = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

As long as $f(x) \geq g(x)$, it doesn't matter if $f(x)$ or $g(x)$ is sometimes negative on the interval since it is the difference between the function values that you need in order to define the height of each of the many narrow rectangles. Sometimes it is convenient to combine the two integrals according to the Sum Rule for Integration that you learned in Lesson 1 of this module. The following example illustrates these concepts.

Example 2

Given $f(x) = x^2 + 1$ and $g(x) = x^2 - 1$, find the area bounded by these two curves on the interval [0, 2].

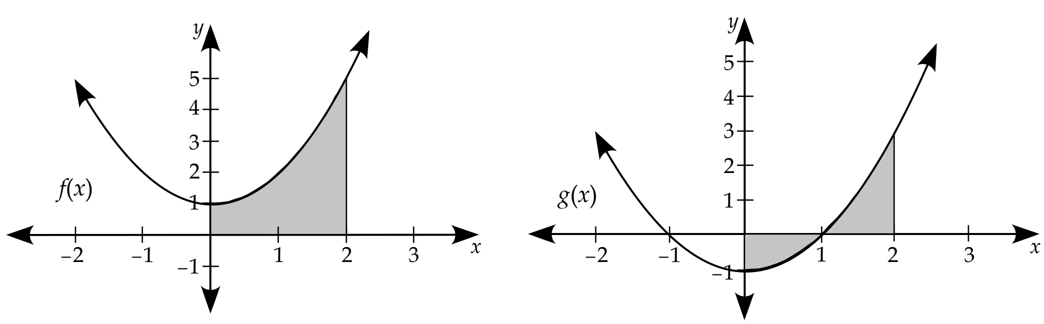
Solution

In this case, it is not necessary to draw a sketch of the functions to confirm that $f(x) > g(x)$ everywhere on the interval since $x^2 + 1$ will always be 2 units greater than $x^2 - 1$. However, the sketches are drawn to illustrate what is happening.

You could find the integrals of each area separately and then subtract those values.

$$\text{Area} = \int_0^2 (x^2 + 1)\,dx - \int_0^2 (x^2 - 1)\,dx$$

You can evaluate the integrals separately as an exercise if you so choose. The shaded regions for $f(x)$ and $g(x)$ are shown.



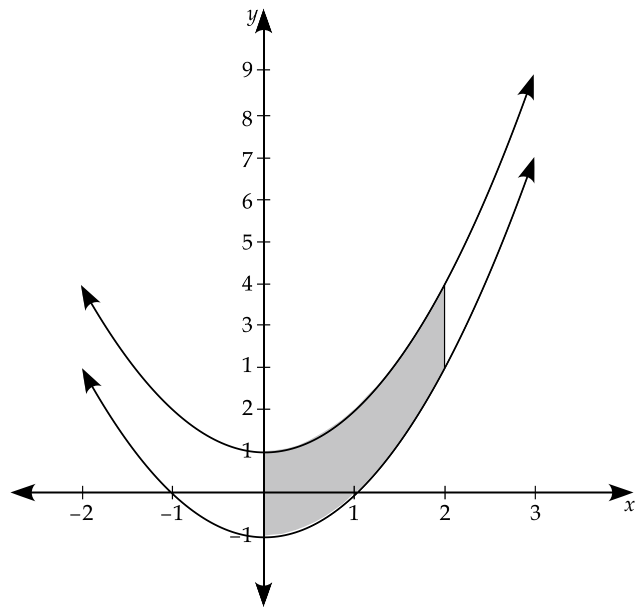
Instead, the definite integrals can be combined to create a simplified expression.

$$\text{Area} = \int_0^2 \left(x^2 + 1\right) - \left(x^2 - 1\right)\, dx$$

$$\text{Area} = \int_0^2 2\, dx$$

$$\text{Area} = \left[\frac{2x^1}{1}\right]_0^2 = [2x]_0^2$$

$$\text{Area} = 4 - 0 = 4$$



The integrals $\int_0^2 \left(x^2 + 1\right)\, dx$ and $\int_0^2 \left(x^2 - 1\right)\, dx$ could be calculated separately and then subtracted, but, as you can see, simplifying the integral first can make the work a little easier. Furthermore, you can see that there was no need to worry about the fact that the definite integral of some of $g(x)$ will be negative since it is below the x-axis. What is required to find the height of each vertical rectangle when finding the integral is the difference between the function values $f(x) - g(x)$. For intervals where $g(x)$ is negative, such as at $x = 0$, $f(0) = 1$ and $g(0) = -1$, so the difference is $f(0) - g(0) = 1 - (-1) = 2$. For intervals where $g(x)$ is positive, such as at $x = 2$, $f(2) = 5$ and $g(2) = 3$, so the difference is $f(0) - g(0) = 5 - 3 = 2$. The difference is consistent whether the area is above the x-axis, below the x-axis, or a little of both.

Area between Two Curves

If $a < b$ and $f(x) > g(x)$ on the interval $[a, b]$,
then the area, A, of the region bounded by $y = f(x)$ and $y = g(x)$
and the lines $x = a$ and $x = b$:

$$A = \int_a^b \left[f(x) - g(x)\right]dx$$

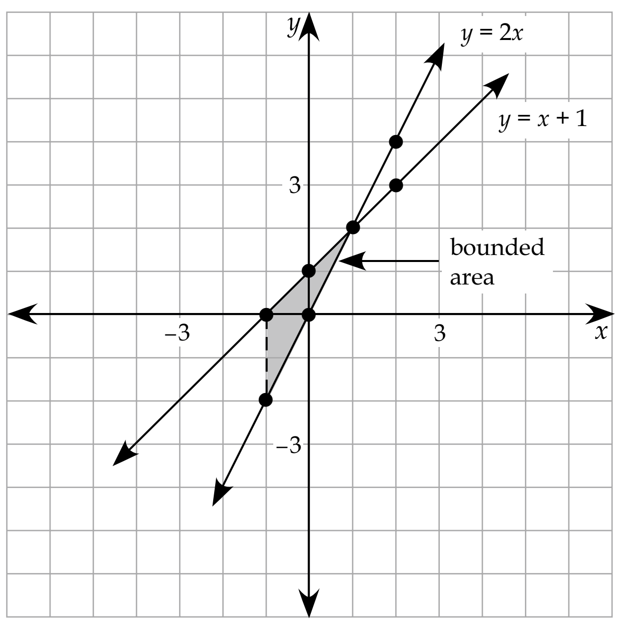
Essentially, the area between the two curves is determined by evaluating the definite integral of the difference of the two functions. It is important to notice that $f(x)$ must lie above $g(x)$ on $[a, b]$. Furthermore, the functions can lie above or below the x-axis with no additional algebraic manipulation required for the area calculation.

Example 3

Determine the area bounded by the lines $y = x + 1$, $y = 2x$, $x = -1$, and $x = 1$.

Solution

Sketch the two lines to determine which function is greater, and then set up the definite integral.



The first function is always greater on the interval, according to the sketch.

$$x + 1 > 2x \text{ on } [0, 1]$$

The area bounded between the two lines.

$$\int_{-1}^{1} [(x + 1) - 2x]\,dx = \int_{-1}^{1} (-x + 1)\,dx$$

Evaluate the definite integral.

$$= \left[-\frac{1}{2}x^2 + x\right]_{-1}^{1} = \left[-\frac{1}{2}(1)^2 + 1\right] - \left[-\frac{1}{2}(-1)^2 + (-1)\right]$$

$$= -\frac{1}{2} + 1 + \frac{1}{2} + 1 = 2 \text{ units}^2$$

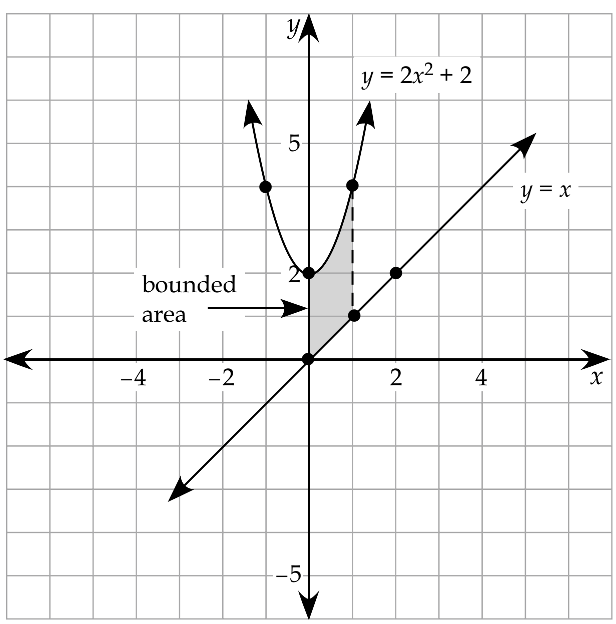
Example 4

Determine the area bounded by the graphs of $y = 2x^2 + 2$ and $y = x$ on the interval [0, 1].

Solution

Sketch the functions to determine which function is greater, and determine their point(s) of intersection (if any). Determine the area by evaluating the corresponding definite integral.

There is **no** point of intersection.



The first function is always greater on the interval, according to the sketch.

$$2x^2 + 2 > x \text{ on } [0, 1]$$

Area bounded between the two lines.

$$\int_0^1 \left[\left(2x^2 + 2\right) - x\right] dx = \int_0^1 \left(2x^2 - x + 2\right) dx$$

Evaluate the definite integral.

$$= \left[\frac{2}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_0^1$$

$$= \left[\frac{2}{3}(1)^3 - \frac{1}{2}(1)^2 + 2(1)\right] - \left[\frac{2}{3}(0)^3 - \frac{1}{2}(0)^2 + 2(0)\right]$$

$$= \frac{2}{3} - \frac{1}{2} + 2 - 0 = \frac{4}{6} - \frac{3}{6} + \frac{12}{6} = \frac{13}{6} \text{ units}^2$$

Functions Cross on the Interval

What happens if the functions cross on the interval? You will explore this in your next example by determining the intersection point.

Example 5

Determine the area bounded by the graphs of $f(x) = x + 2$ and $g(x) = -x + 3$ on the interval [0, 3].

Solution

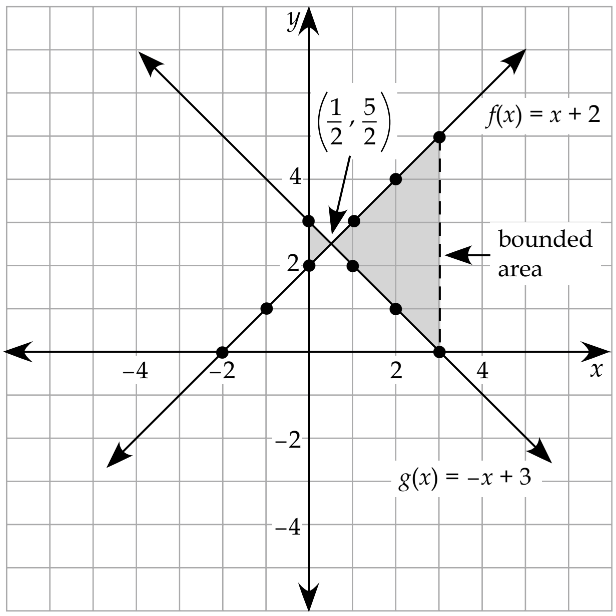
Sketch the function graphs on the same coordinate plane and determine the intersection points of the two functions.

$$x + 2 = -x + 3$$

$$2x = 1 \quad \rightarrow \quad y = -\frac{1}{2} + 3 = \frac{5}{2} \quad \rightarrow \quad \left(\frac{1}{2}, \frac{5}{2}\right)$$

$$x = \frac{1}{2}$$

The functions intersect at $\left(\frac{1}{2}, \frac{5}{2}\right)$.



Since $f(x) > g(x)$ on $\left[\frac{1}{2}, 3\right]$ and $g(x) > f(x)$ on $\left[0, \frac{1}{2}\right]$, split the interval where the functions cross.

Determine the total area by evaluating the sum of the definite integrals.

$$\text{Total area} = \int_0^{0.5} \left(g(x) - f(x)\right) dx + \int_{0.5}^{3} \left(f(x) - g(x)\right) dx$$

$$\text{Total area} = \int_0^{0.5} \left[(-x + 3) - (x + 2)\right] dx + \int_{0.5}^{3} \left[(x + 2) - (-x + 3)\right] dx$$

Simplify each definite integral.

$$\text{Total area} = \int_0^{0.5} (-2x + 1)\, dx + \int_{0.5}^{3} (2x - 1)\, dx = \left[-x^2 + x\right]_0^{0.5} + \left[x^2 - x\right]_{0.5}^{3}$$

$$= \left[-(0.5)^2 + 0.5\right] - 0 + \left[3^2 - 3\right] - \left[0.5^2 - 0.5\right]$$

$$= -0.25 + 0.5 + 9 - 3 - 0.25 + 0.5 = 6.5 \text{ units}^2$$

Interval Is Not Explicitly Stated

Sometimes the interval is not defined when you are asked to determine the area bounded by two curves. You need to determine their intersection points to use as interval boundaries, as shown in the next example.

Example 6

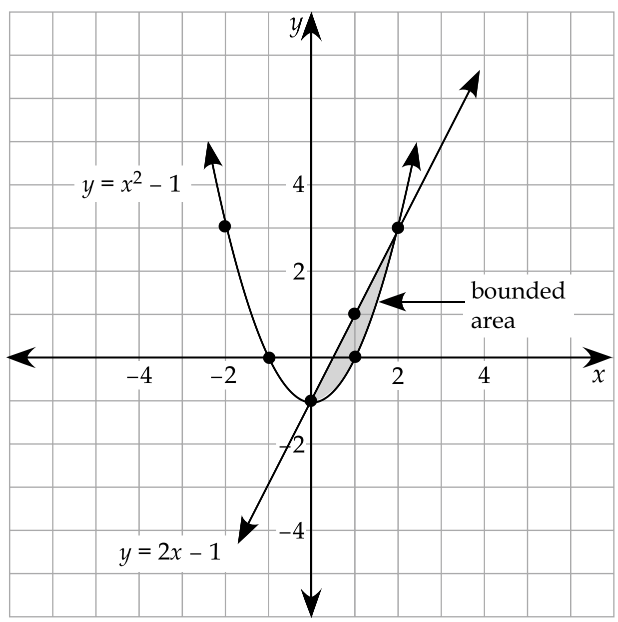
Determine the area bounded by the graphs of $y = x^2 - 1$ and $y = 2x - 1$.

Solution

Determine the intersection points to be used as the upper and lower bounds.

$$x^2 - 1 = 2x - 1$$
$$x^2 - 1 - 2x + 1 = 0 \rightarrow x(x-2) = 0$$
$$x^2 - 2x = 0 \qquad x = 0, 2$$

Sketch the graph of the two functions on the same coordinate plane to help you write the definite integral expression.



On the interval [0, 2], $2x - 1 > x^2 - 1$, so Area $= \int_0^2 \left[(2x-1) - (x^2-1)\right]dx$.

Evaluate the definite integral in order to determine the bounded area.

$$\text{Area} = \int_0^2 \left(-x^2 + 2x\right)dx = \left[-\frac{1}{3}x^3 + x^2\right]_0^2$$

$$= \left[-\frac{1}{3}(2)^3 + 2^2\right] - 0 = -\frac{8}{3} + 4 = -\frac{8}{3} + \frac{12}{3} = \frac{4}{3} \text{ units}^2$$

Example 7

Determine the area bounded by the graphs of $y = 2x^2$ and $y = 16\sqrt{x}$.

Solution

Determine the intersection points to be used as the upper and lower bounds.

$$2x^2 = 16\sqrt{x}$$

$$4x^4 = 256x$$

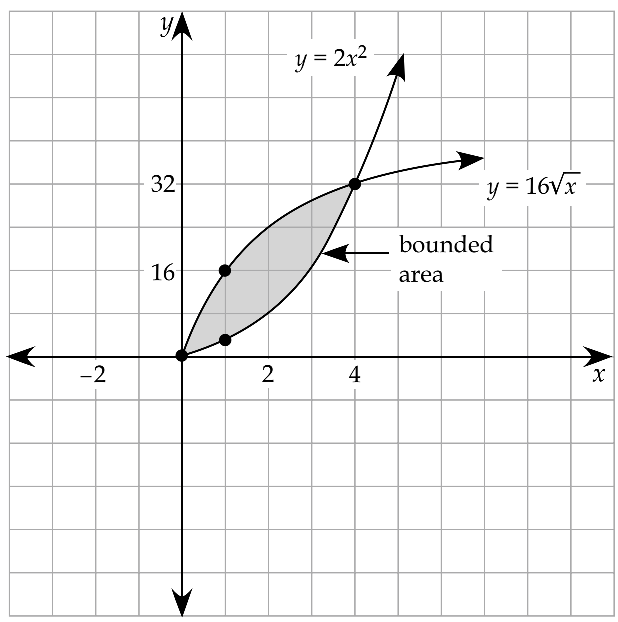
$$4x^4 - 256x = 0$$

$$4x\left(x^3 - 64\right) = 0$$

$$x = 0, 4$$

The bounded area is found on the interval [0, 4].

Sketch the functions on the same coordinate plane to help you write the definite integral expression.



On the interval [0, 4], $16\sqrt{x} > 2x^2$, so Area $= \int_0^4 \left(16\sqrt{x} - 2x^2\right) dx$.

Evaluate the definite integral in order to determine the bounded area.

$$\text{Area} = \int_0^4 \left(16x^{\frac{1}{2}} - 2x^2\right)dx = \left[\frac{32}{3}x^{\frac{3}{2}} - \frac{2}{3}x^3\right]_0^4$$

$$= \left[\frac{32}{3}(4)^{\frac{3}{2}} - \frac{2}{3}(4)^3\right] - 0 = \frac{32}{3}(8) - \frac{2}{3}(64)$$

$$= \frac{256}{3} - \frac{128}{3} = \frac{128}{3} \text{ units}^2$$

Learning Activity 4.5

Complete the following, and check your answers in the Learning Activity Answer Keys found at the end of this module.

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

Use $f(x) = x^2 + 4$ and $g(x) = 2x + 3$ to answer Questions 1 to 5.

1. Simplify: $f(x) - g(x)$
2. Simplify: $g(x) - f(x)$
3. Create a definite integral that will determine the area under the curve $f(x)$ on [1, 4].
4. Create a definite integral that will determine the area under the line $g(x)$ on [1, 4].
5. Determine x-value(s) of the intersection(s) of $f(x)$ and $g(x)$.
6. Determine the roots of $2x^2 - 3x + 1 = 0$.
7. What is the indefinite integral of $4\sqrt{x}$?
8. What is the indefinite integral of $\frac{3}{x^2}$?

continued

Learning Activity 4.5 (continued)

Part B: Area between Two Functions

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Determine the area bounded by the two lines $y = 2x - 1$ and $y = x + 2$ on $[-2, 3]$.
2. Determine the area bounded by the two lines $y = -x + 3$ and $y = x + 1$ on $[-1, 2]$.
3. Determine the area bounded by the graphs of $y = 3x - x^2$ and $y = -x$.
4. Determine the area bounded by the graphs of $y = 4 - x^2$ and $y = 2x^2 - 8$.

Lesson Summary

In this lesson, you learned how to determine the area bounded by the graph of two functions by integrating the definite integral of the difference of their respective functions. The difference of the larger function and the smaller function eliminated the problem of the definite integral of a negative function not yielding a positive area. In addition, you learned how to define the interval boundaries using the intersection points of the functions. In future courses, you will apply integration to solve a variety of problem types, including finding the volumes of irregularly shaped three-dimensional objects.

Assignment 4.5

Area between Two Functions

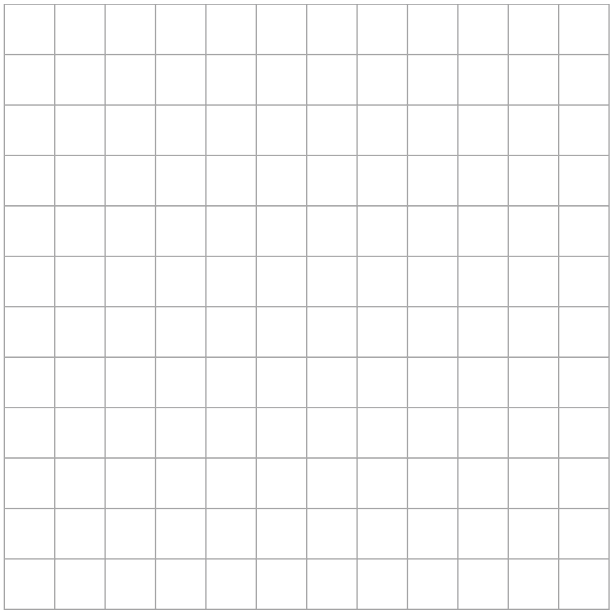
Total: 20 marks

1. Determine the area bounded by the graphs of $y = x^2 + x - 6$ and $y = -x + 2$ on $[-2, 2]$. *(7 marks)*

continued

Assignment 4.5: Area between Two Functions (continued)

2. Determine the area bounded by the graphs of $y = 3x^2 - 4x$ and $y = 2x$. *(7 marks)*



continued

Assignment 4.5: Area between Two Functions (continued)

3. Determine the area bounded by the graphs of $y = 4\sqrt{x}$ and $y = 2x$. *(6 marks)*

Notes

MODULE 4 SUMMARY

Congratulations, you have finished the fourth and final module in the course. In this module, you learned the inverse operation of differentiation, integration. You determined indefinite integrals or a family of antiderivatives of functions, keeping in mind the constant of integration. In addition, you evaluated the definite integral and explored its relationship to the area under the curve. Lastly, you learned how to use the definite integral to determine the area bounded by two functions.

You have now finished the course. In preparation for your examinaation, you should review all your assignments and summarize the important concepts of the course.

Submitting Your Assignments

It is now time for you to submit Assignments 4.1 to 4.5 to the Distance Learning Unit so that you can receive some feedback on how you are doing in this course. Remember that you must submit all the assignments in this course before you can receive your credit.

Make sure you have completed all parts of your Module 4 assignments and organize your material in the following order:

- ❑ Module 4 Cover Sheet (found at the end of the course Introduction)
- ❑ Assignment 4.1: Antidifferentiation and Integration
- ❑ Assignment 4.2: Differential Equations
- ❑ Assignment 4.3: Definite Integral
- ❑ Assignment 4.4: Area under a Curve
- ❑ Assignment 4.5: Area between Two Functions

For instructions on submitting your assignments, refer to How to Submit Assignments in the course Introduction.

Final Examination

Congratulations, you have finished Module 4 in the course. The final examination is out of 100 marks and worth 45% of your final mark. In order to do well on this examination, you should review all of your learning activities and assignments from Modules 1 to 4.

You will complete this examination while being supervised by a proctor. You should already have made arrangements to have the examination sent to the proctor from the Distance Learning Unit. If you have not yet made arrangements to write it, then do so now. The instructions for doing so are provided in the Introduction to this module.

You will need to bring the following items to the examination: pens/pencils (2 or 3 of each), blank paper, and a scientific calculator. A maximum of 3 hours is available to complete your final examination. When you have completed it, the proctor will then forward it for assessment. Good luck!

Final Practice Examination and Answer Key

To help you succeed in your examination, a practice examination can be found in the learning management system (LMS). The final practice examination is very similar to the actual examination that you will be writing. The answer key is also included so that, when you have finished writing the practice examination, you can check your answers. This will give you the confidence that you need to do well on your examination. If you do not have access to the Internet, contact the Distance Learning Unit at 1-800-465-9915 to get a copy of the practice examination and the answer key.

To get the most out of your final practice examination, follow these steps:

1. Study for the final practice examination as if it were an actual examination.
2. Review those learning activities and assignments from Modules 1 to 4 that you found the most challenging. Reread those lessons carefully and learn the concepts.
3. Ask your learning partner and your tutor/marker if you need help.
4. Review your lessons from Modules 1 to 4, including all of your notes, learning activities, and assignments.
5. Bring the following things to the final practice examination: some pens, pencils, paper, and a scientific calculator.
6. Write your final practice examination as if it were an actual examination. In other words, write the entire examination in one sitting and do not check your answers until you have completed the entire examination.

7. Once you have completed the entire practice examination, check your answers against the answer key. Review the questions that you got wrong. For each of those questions, you will need to go back into the course and learn the concepts that you have missed.

Notes

Grade 12 Introduction to Calculus (45S)

Module 4
Integration

Learning Activity Answer Keys

Module 4: Integration

Learning Activity 4.1

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Predict which of the following functions have the same derivative and then find the derivatives.

   a) $f(x) = 7x^4 - 1$

   b) $f(x) = x^{-27} + 8$

   c) $f(x) = 7x^4 + 3$

   d) $f(x) = x^4 - 5$

2. Simplify: $8 \cdot \dfrac{x^{-5+1}}{-5+1}$

3. Simplify: $-2 \cdot \dfrac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$

4. Evaluate: $-3(4-2)^3$

5. Evaluate: $\dfrac{5+3}{4-2} + 2$

Answers:

1. Prediction: $7x^4 - 1$ and $7x^4 + 3$ will have the same derivative.

   a) $f'(x) = 28x^3$

   b) $f'(x) = -27x^{-28}$

   c) $f'(x) = 28x^3$

   d) $f'(x) = 4x^3$

2. $-2x^{-4}$ $\left(8 \cdot \dfrac{x^{-5+1}}{-5+1} = \dfrac{8}{-4}x^{-4}\right)$

3. $-4x^{\frac{1}{2}}$ $\left(-2 \cdot \dfrac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \dfrac{-2}{\frac{1}{2}}x^{\frac{1}{2}}\right)$

4. -24 $\left(-3(4-2)^3 = -3(2)^3 = -3 \cdot 8\right)$

5. 6 $\left(\dfrac{5+3}{3-1} + 2 = \dfrac{8}{2} + 2 = 4 + 2\right)$

Part B: Antidifferentiation and Integration

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Match each function with a possible antiderivative function.

	Functions
A	$2x + 1$
B	$4x^3 + 2$
C	$-2x^{-3} + 1$
D	$-x^{-2} + x$

Antiderivatives
$x^{-2} + x + 100$
$x^{-1} + \frac{1}{2}x^2 + 1$
$x^2 + x - 1$
$x^4 + 2x - 3$

Answer:

Antiderivatives	Functions (Derivative of the Antiderivative)	
$x^{-2} + x + 100$	$\frac{d}{dx}\left(x^{-2} + x + 100\right) = -2x^{-3} + 1$	C
$x^{-1} + \frac{1}{2}x^2 + 1$	$\frac{d}{dx}\left(x^{-1} + \frac{1}{2}x^2 + 1\right) = -x^{-2} + x$	D
$x^2 + x - 1$	$\frac{d}{dx}\left(x^2 + x - 1\right) = 2x + 1$	A
$x^4 + 2x - 3$	$\frac{d}{dx}\left(x^4 + 2x - 3\right) = 4x^3 + 2$	B

2. Determine the indefinite integral (the general antiderivative function) for the following functions:

   a) $f(x) = x^6 - x^{-2}$

   Answer:

   $$\int \left(x^6 - x^{-1}\right) dx = \frac{x^{6+1}}{6+1} - \frac{x^{-2+1}}{-2+1} + C = \frac{1}{7}x^7 + x^{-1} + C$$

   b) $f(x) = 2x^3 + 9x^2 - 3$

   Answer:

   $$\begin{aligned}\int \left(2x^3 + 9x^2 - 3\right) dx &= 2 \cdot \frac{x^{3+1}}{3+1} + 9 \cdot \frac{x^{2+1}}{2+1} - 3 \cdot \frac{x^{0+1}}{0+1} + C \\ &= \frac{2}{4}x^4 + \frac{9}{3}x^3 - 3x + C \\ &= \frac{1}{2}x^4 + 3x^3 - 3x + C\end{aligned}$$

   c) $f(x) = x^{-1}$

   Answer:

   Using the power rule:

   $$\int \left(x^{-1}\right) dx = \frac{x^{-1+1}}{-1+1} = \frac{x^0}{0}$$

   This is an undefined value; the integral of x^{-1} cannot be found using the power rule. Look back in the lesson and you will see that the power rule is not defined for x^{-1}.

   The integral of x^{-1} is beyond the scope of this course, but you may be interested (maybe even surprised) to know $\int \left(x^{-1}\right) dx = \ln|x| + C$.

3. Write an expression to represent each integral.

a) $\int x\, dx$

Answer:

Write the antiderivative of x. You could use the power rule or think about what family of functions have a derivative equal to x.

$$\int x\, dx = \frac{x^2}{2} + C$$

b) $\int 1\, dx$

Answer:

Write the antiderivative of 1, with respect to x. You could use the power rule or think about what family of functions have a derivative equal to 1.

$$\int 1\, dx = \frac{x^1}{1} + C = x + C$$

c) $\int dy$

Answer:

Write the antiderivative of 1, with respect to y (notice the 1 is implied here).

$$\int dy = \int 1\, dy = y + C$$

Learning Activity 4.2

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

Questions 1 to 3

Evaluate if $f(x) = 3x^2 - 2x - 1$.

1. $f(-1)$
2. $f(0)$
3. $f(2)$

Questions 4 to 6

If $g(x) = -2x^2 + C$, determine the value of C for each condition that follows.

4. $g(0) = -1$
5. $g(0) = 4$
6. $g(-2) = 3$
7. If $h'(x) = 1$, determine two possible antiderivatives that are 5 units apart shifted vertically.
8. If $h'(x) = 2x$, determine two possible antiderivatives that are 5 units apart shifted vertically.

Answers:

1. 4 ($f(-1) = 3(-1)^2 - 2(-1) = 3 + 2 - 1 = 4$)
2. −1 ($f(0) = 3(0)^2 - 2(0) - 1 = -1$)
3. 7 ($f(2) = 3(2)^2 - 2(2) - 1 = 12 - 4 - 1 = 7$)
4. −1 ($g(0) = -2(0) + C = -1 \rightarrow C = -1$)
5. 4 ($g(0) = -2(0) + C = 4 \rightarrow C = 4$)
6. 11 ($g(-2) = (-2)(-2)^2 + C = 3 \rightarrow -8 + C = 3 \rightarrow C = 11$)
7. The family of functions is represented by $h(x) = x + C$, so $h(x) = x$ and $h(x) = x + 5$ are an example of two functions that are vertically 5 units apart.
8. The family of functions is represented by $h(x) = x^2 + C$, so $h(x) = x^2 - 1$ and $h(x) = x^2 + 4$ are an example of two functions that are vertically 5 units apart.

Part B: Differential Equations

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Find the equation of the curve $f(x)$, which passes through the point $(-1, 1)$, for each of the differential equations below.

   a) $f'(x) = \frac{2}{3}$

   Answer:

   $f'(x) = \frac{2}{3}$ — Integrate the differential equation to find $f(x)$.

   $\int f'(x)\,dx = \int \frac{2}{3}\,dx$

   $f(x) = \frac{2}{3}x + C$ — Indefinite integral.

   $1 = \frac{2}{3}(-1) + C$

   $1 + \frac{2}{3} = C$ — Substitute given conditions to solve for the constant of integration.

   $C = \frac{5}{3}$

   The specific equation at the given condition is:

   $$f(x) = \frac{2}{3}x + \frac{5}{3}$$

   Notice this is a linear equation with a constant slope of $\frac{2}{3}$.

b) $f'(x) = x^3 + 5x - 1$

Answer:

$$f'(x) = x^3 + 5x - 1$$

$$\int f'(x)\,dx = \int \left(x^3 + 5x - 1\right)dx$$

Integrate the differential equation to find *f*(*x*).

$$f(x) = \frac{x^{3+1}}{4} + \frac{5x^{1+1}}{2} - x + C$$

$$f(x) = \frac{1}{4}x^4 + \frac{5}{2}x^2 - x + C$$

Determine indefinite integral using the constant times power rule of integration.

$$1 = \frac{1}{4}(-1)^4 + \frac{5}{2}(-1)^2 - (-1) + C$$

$$1 = \frac{1}{4} + \frac{5}{2} + 1 + C$$

$$1 = \frac{1}{4} + \frac{10}{4} + \frac{4}{4} + C$$

$$C = 1 - \frac{15}{4} = \frac{4}{4} - \frac{15}{4} = \frac{-11}{4}$$

Substitute given conditions to solve for the constant of integration.

The specific equation at the given condition is:

$$f(x) = \frac{1}{4}x^4 + \frac{5}{2}x^2 - x - \frac{11}{4}$$

c) $f'(x) = \dfrac{1}{8x^2}$

Answer:

$$f'(x) = \frac{1}{8x^2} = \frac{1}{8}x^{-2}$$

$$f(x) = \int \left(\frac{1}{8}x^{-2}\right) dx$$

Integrate the differential equation to find $f(x)$.

$$f(x) = \frac{1}{8} \cdot \frac{x^{-2+1}}{-1} + C$$

$$f(x) = -\frac{1}{8}x^{-1} + C$$

Determine indefinite integral using the constant times power rule of integration.

$$1 = f(-1) = -\frac{1}{8}(-1)^{-1} + C$$

$$1 = \frac{1}{8} + C$$

$$C = \frac{7}{8}$$

Substitute given conditions to solve for the constant of integration.

The specific equation at the given condition is:

$$f(x) = -\frac{1}{8}x^{-1} + \frac{7}{8}$$

2. Find the formula for distance s at any time t, using the given velocity function and specific information regarding s and t:

   $v = 1 + 2t$ and $s = 3$ m, when $t = 0$ s

   Answer:

   Since velocity is the derivative of displacement, displacement is the indefinite integral of velocity.

$s' = v = 1 + 2t$	Velocity is written as a differential equation.
$s = \int v\,dt = \int (1 + 2t)\,dt$	Integrate velocity to find displacement.
$s = t + \frac{2t^2}{2} + C$ $s = t^2 + t + C$	Indefinite integral.
$3 = (0)^2 + (0) + C$ $3 = C$	Substitute the initial conditions $s = 3$ m at $t = 0$ s to solve for the constant.

   Specific equation for displacement at any time is:

   $$s = t^2 + t + 3$$

3. Find the formula for velocity at any time t and displacement at any time t, using the given acceleration function and specific information regarding v, s, and t:

   $a = 2 + 36t$ and $v_0 = 3$ m/s, $s = 0$ m, when $t = 0$ s

   Answer:

   Since acceleration is the derivative of velocity, the velocity is the indefinite integral of acceleration. Similarly, since velocity is the derivative of displacement, then displacement is the indefinite integral of velocity.

$v' = a = 2 + 36t$	Acceleration written as a differential equation.
$v = \int a\,dt = \int (2 + 36t)\,dt$	Integrate acceleration to find velocity.
$v = 2t + \frac{36t^2}{2} + C = 2t + 18t^2 + C$	Indefinite integral.
$3 = 2(0) + 18(0)^2 + C$ $C = 3$	Substitute the initial conditions to confirm $v_0 = 3$ m/s.

   Specific equation for velocity at any time is:

   $v = 18t^2 + 2t + 3$

   Now find the formula for displacement.

$s' = v = 18t^2 + 2t + 3$	Velocity written as a differential equation.
$s = \int v\,dt = \int \left(18t^2 + 2t + 3\right)dt$	Integrate velocity to find displacement.
$s = \frac{18t^3}{3} + \frac{2t^2}{2} + 3t + C$	Indefinite integral.
$s = 6t^3 + t^2 + 3t + C$ $0 = 6(0)^3 + (0)^2 + 3(0) + C$ $0 = C$	Substitute the initial conditions $s = 0$ m at $t = 0$ s to solve for the constant.

   Specific equation for displacement at any time is:

   $s = 6t^3 + t^2 + 3t$

4. An object is shot upward from the ground with an initial velocity of 30 m/s. The acceleration of gravity is −9.8 m/s^2.

   a) Calculate the elapsed time from the initial shot until the object returned to the ground.

      Answer:

      Since acceleration is the derivative of velocity, the velocity is the indefinite integral of acceleration. Similarly, since velocity is the derivative of displacement (height), then height is the indefinite integral of velocity.

$v' = a = -9.8$	Acceleration written as a differential equation.
$v = \int a\, dt = \int (-9.8)\, dt$	Integrate acceleration to find velocity.
$v = -9.8t + C$	Indefinite integral.
$30 = (-9.8)(0) + C$ $C = 30$	Substitute the initial conditions to confirm v_0 = 30 m/s.

      Specific equation for velocity at any time is:

      $v = -9.8t + 30$

      Now find the equation for displacement.

$s' = v = -9.8t + 30$	Velocity written as a differential equation.
$s = \int v\, dt = \int (-9.8t + 30)\, dt$	Integrate velocity to find displacement.
$s = \dfrac{-9.8t^2}{2} + 30t + C$ $s = -4.9t^2 + 30t + C$	Indefinite integral.
$0 = -4.9(0)^2 + 3(0) + C$ $0 = C$	Substitute the initial conditions s = 0 m at t = 0 s to solve for the constant.
$s = -4.9t^2 + 30t + 0$	

      Specific equation for displacement at any time is:

      $s = -4.9t^2 + 30t$

When the object returns to its starting point, $s = 0$. You can, therefore, solve the distance function above for t by setting $s = 0$. You can solve this by factoring or by using the quadratic formula.

$$s = -4.9t^2 + 30t = 0$$

$$t(-4.9t + 30) = 0$$

$$-4.9t + 30 = 0$$

$$-4.9t = -30$$

$$t = 0 \qquad t = 6.12s$$

Since a time of 0 seconds represents the start, the object returns to the ground 6.12 seconds later.

b) Calculate the maximum height to which the object would rise.

Answer:

The object reaches its maximum height when the velocity is zero.

$$0 = v = -9.8t + 30$$

$$-9.8t = -30$$

$t = 3.06$ s to reach the maximum height

$$s = -4.9(3.06)^2 + 30(3.06)$$

$$s = 45.9 \text{ m}$$

Substitute the time into the height function.

The object reaches a maximum height at about 45.9 metres.

Learning Activity 4.3

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Determine an antiderivative of $5x$.
2. Given $F(x) = 3x - 1$, evaluate $F(3) - F(-1)$.
3. Given $F(x) = 3x^3 + 4x$, evaluate $F(1) - F(-1)$.
4. Given $F(x) = \sqrt{x}$, evaluate $F(4) - F(0)$.
5. Simplify $\dfrac{x^3 + x^2}{x^2}$.
6. Simplify $5(x + 1)^2$ by expanding.
7. Simplify $\left(1 + \sqrt{x}\right)^2$ by expanding.
8. Simplify $\dfrac{1}{\sqrt[3]{x^4}}$ with negative rational exponents.

Answers:

1. $\dfrac{5}{2}x^2$
2. 12 $\left(\begin{aligned} F(3) - F(-1) &= [3(3) - 1] - [3(-1) - 1] \\ &= 8 - (-4) = 8 + 4 = 12 \end{aligned}\right)$
3. 14 $\left(\begin{aligned} &F(1) - F(-1) \\ &= \left[3(1)^3 + 4(1)\right] - \left[3(-1)^3 + 4(-1)\right] \\ &= (3 + 4) - (-3 - 4) = 7 - (-7) \\ &= 7 + 7 = 14 \end{aligned}\right)$
4. 2 $\left(F(4) - F(0) = \sqrt{4} - \sqrt{0} = 2 - 0 = 2\right)$
5. $x + 1,\ x \neq 0$ $\left(\dfrac{x^3 + x^2}{x^2} = \dfrac{x^3}{x^2} + \dfrac{x^2}{x^2} = x + 1, \text{ npv of } x = 0\right)$

6. $5x^2 + 10x + 5 \left(5(x+1)^2 = 5\left(x^2 + 2x + 1\right) = 5x^2 + 10x + 5\right)$

7. $1 + 2\sqrt{x} + x$

8. $x^{-\frac{4}{3}} \left(\frac{1}{\sqrt[3]{x^4}} = \frac{1}{x^{\frac{4}{3}}} = x^{-\frac{4}{3}} \right)$

Part B: Definite Integral

Remember, these questions are similar to the ones that will be on your
Remember, these questions are similar to the ones that will be on your
assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Evaluate the definite integral $\int_1^4 \left(4x^3\right) dx$.

Answer:

$$\int_1^4 4x^3\,dx = \left[\frac{4x^{3+1}}{3+1}\right]_1^4 = \left[x^4\right]_1^4$$

The antiderivative, $F(x)$, with the lower bound of 1 and an upper bound of 4.

$$= F(4) - F(1)$$

$$= \left[4^4\right] - \left[1^4\right] = 256 - 1 = 255$$

The integral evaluated using the FTC.

2. a) Evaluate the definite integral $\int_{-2}^{2} \left(x + x^3\right) dx$.

Answer:

$$\int_{-2}^{2} \left(x + x^3\right) dx = \left[\frac{1}{2}x^2 + \frac{1}{4}x^4\right]_{-2}^{2}$$

The antiderivative, $F(x)$, with the lower bound of −2 and an upper bound of 2.

$$= F(2) - F(-2)$$

$$= \left[\frac{1}{2}(2)^2 + \frac{1}{4}(2)^4\right] - \left[\frac{1}{2}(-2)^2 + \frac{1}{4}(-2)^4\right]$$

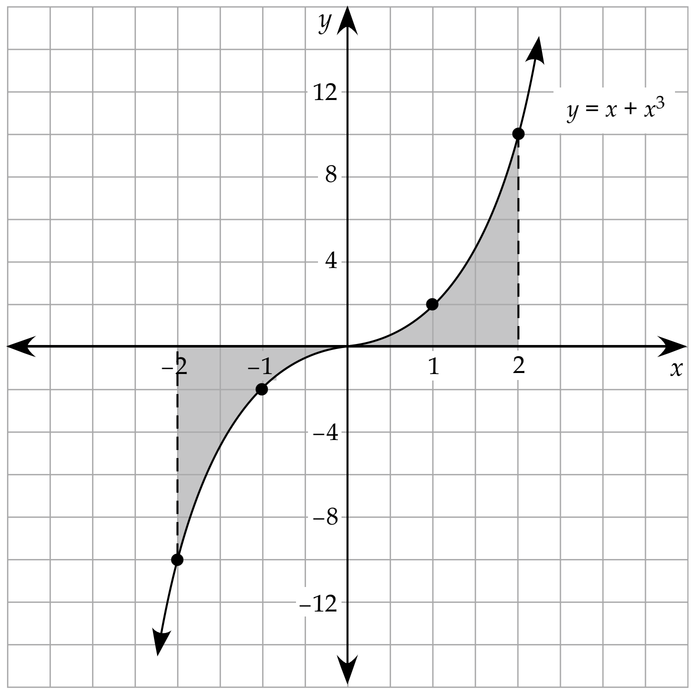
$$= (2 + 4) - (2 + 4) = 0$$

The integral evaluated using the FTC.

b) Sketch $y = x + x^3$ and explain why the definite integral on the interval from −2 to 2 can be equal to zero.

Answer:

The integral is zero because the area above the x-axis is equal to the area below the x-axis. The integral for the interval above is positive and the integral for the interval below is negative.



3. Simplify the integrand before evaluating the following definite integral:

a) $\int_0^1 4x(x+1)^2\,dx$

Answer:

$$\int_0^1 4x(x+1)^2\,dx = \int_0^1 4x\left(x^2+2x+1\right)dx$$

$$= \int_0^1 \left(4x^3+8x^2+4x\right)dx$$

Simplify the integrand.

$$= \left[x^4 + \frac{8}{3}x^3 + 2x^2\right]_0^1$$

$$= \left[(1)^4 + \frac{8}{3}(1)^3 + 2(1)^2\right] - \left[(0)^4 + \frac{8}{3}(0)^3 + 2(0)^2\right]$$

$\left[x^4 + \frac{8}{3}x^3 + 2x^2\right]_0^1$ is the antiderivative with the lower bound of 0 and an upper bound of 1.

$$= \left(1 + \frac{8}{3} + 2\right) - 0 = 3 + \frac{8}{3} = \frac{17}{3}$$

The integral evaluated using the FTC.

b) $\int_1^4 \frac{4x^3}{\sqrt{x}}\,dx$

Answer:

$$\int_1^4 \frac{4x^3}{\sqrt{x}}\,dx = \int_1^4 \frac{4x^3}{x^{\frac{1}{2}}}\,dx = \int_1^4 4x^{\frac{5}{2}}\,dx$$

Simplify the integrand.

$$= \left[4 \cdot \frac{2}{7}x^{\frac{7}{2}}\right]_1^4 = \left[\frac{8}{7}x^{\frac{7}{2}}\right]_1^4$$

$\left[\frac{8}{7}x^{\frac{7}{2}}\right]_1^4$ is the antiderivative with the lower bound of 1 and an upper bound of 4.

$$= \left[\frac{8}{7}(4)^{\frac{7}{2}}\right] - \left[\frac{8}{7}(1)^{\frac{7}{2}}\right]$$

$$= \frac{8}{7} \cdot 128 - \frac{8}{7} = \frac{1024-8}{7} = \frac{1016}{7}$$

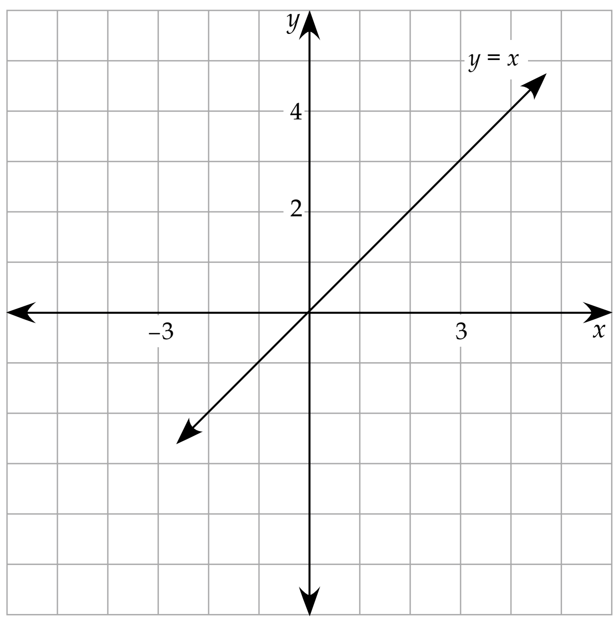
The integral evaluated using the FTC.

Learning Activity 4.4

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

1. Determine the area of a rectangle with a width of 5 cm and a length of 10 cm.
2. Determine the area of a triangle with a base of 3 mm and a height of 6 mm.
3. Determine the area of a square with a side length of 5 cm.
4. Evaluate the definite integral, $\int_0^3 (x)\,dx$, geometrically.



5. Determine an antiderivative of $\sqrt[3]{x^2}$.
6. Determine the x-intercepts of $y = 3x^2 - 12$.
7. Determine the x-intercepts of $y = x(x - 1)(x + 2)$.
8. Factor: $x^2 - 3x + 2$

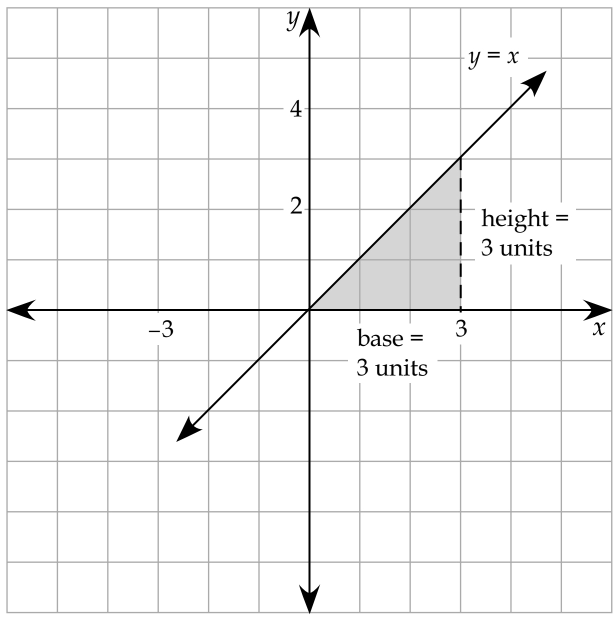
Answers:

1. 50 ($A = l \cdot w = 10 \cdot 5 = 50 \text{ cm}^2$)

2. 9 $\left(A = \dfrac{b \cdot h}{2} = \dfrac{3 \cdot 6}{2} = \dfrac{18}{2} = 9 \text{ mm}^2\right)$

3. 25 ($A = s^2 = 5^2 = 25 \text{ cm}^2$)

4. 4.5



(The bounded area is a triangle with both its base and height being 3 units each. $A = \dfrac{b \cdot h}{2} = \dfrac{3 \cdot 3}{2} = \dfrac{9}{2} = 4.5 \text{ units}^2$. Therefore, $\displaystyle\int_0^3 (x)\, dx = 4.5.$)

5. $\dfrac{3}{5}x^{\frac{5}{3}}$ $\left(y' = \sqrt[3]{x^2} = x^{\frac{2}{3}} \rightarrow y = \dfrac{5}{3}x^{\frac{2}{3}+1} = \dfrac{3}{5}x^{\frac{5}{3}}\right)$

6. $x = \pm 2$ $\left(\begin{aligned} 0 &= 3x^2 - 12 \\ 12 &= 3x^2 \\ 4 &= x^2 \end{aligned}\right)$

7. 0, 1, –2 $\left(0 = x(x-1)(x+2)\right)$

8. $(x-1)(x-2)$

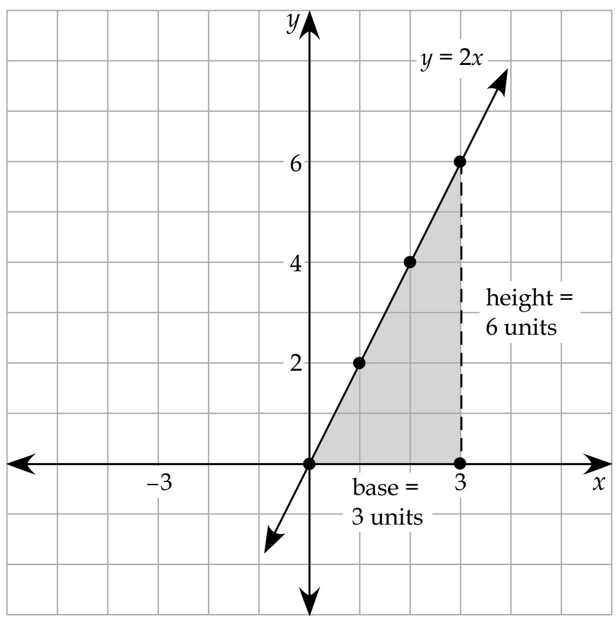
Part B: Area under a Curve

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Determine the value of the definite integral $\int_0^3 (2x)\,dx$ in two ways,

   a) geometrically by determining the area bounded by the function and the x-axis.

      Answer:

      The bounded area is a triangle with a base of 3 units and a height of 6 units.

      $$A = \frac{b \cdot h}{2} = \frac{3 \cdot 6}{2} = \frac{18}{2} = 9 \text{ units}^2$$

   b) algebraically using the antiderivative.

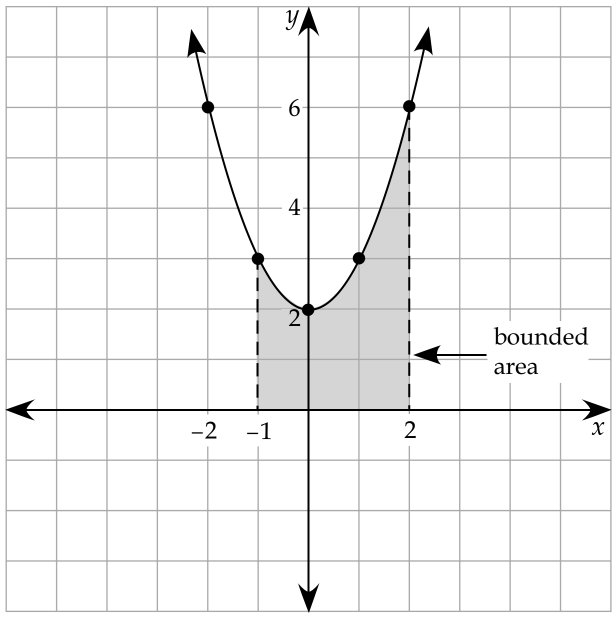
      Answer:

      $$\text{Area above } x\text{-axis} = \int_0^3 2x\,dx = \left[x^2\right]_0^3 = 3^2 - 0^2 = 9 - 0 = 9 \text{ units}^2$$

2. Determine the area bounded by the curve $y = x^2 + 2$ and the x-axis on $[-1, 2]$.

Answer:

Sketch the parabola using its vertex (0, 2) and direction of opening up.



Notice that the bounded area is above the x-axis.

$$\text{Area above the } x\text{-axis} = \int_{-1}^{2} \left(x^2 + 2\right) dx = \left[\frac{1}{3}x^3 + 2x\right]_{-1}^{2}$$

$$= \left[\frac{1}{3}(2)^3 + 2(2)\right] - \left[\frac{1}{3}(-1)^3 + 2(-1)\right]$$

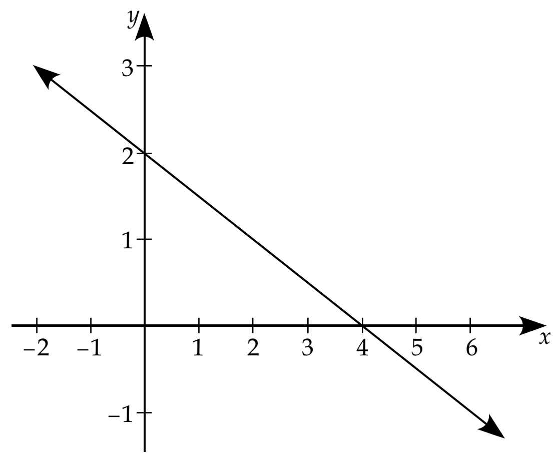
$$= \frac{8}{3} + 4 + \frac{1}{3} + 2 = \frac{9}{3} + 6 = 3 + 6 = 9 \text{ units}^2$$

3. Find the values of each definite integral geometrically using the sketch of *f*(*x*) as shown.

a) $\int_{0}^{4} f(x)\,dx$

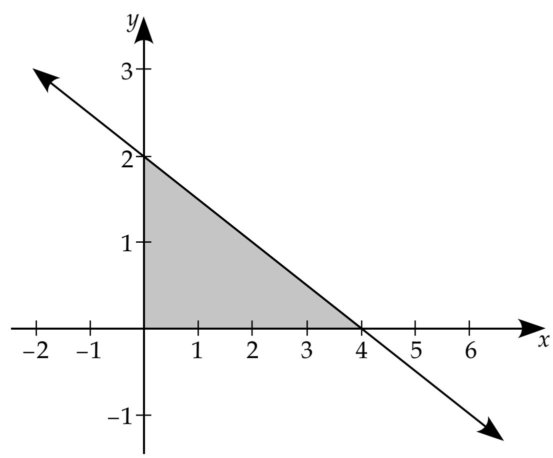
b) $\int_{-2}^{4} f(x)\,dx$

c) $\int_{0}^{6} f(x)\,dx$



Answer:

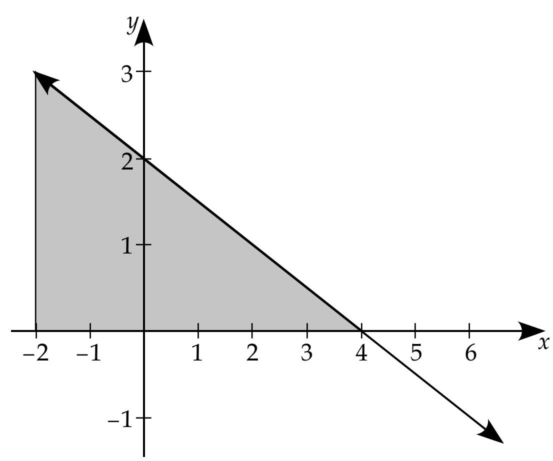
a)



Area = 4 × 2 ÷ 2 = 4

Therefore, $\int_{0}^{4} f(x)\,dx = 4$.

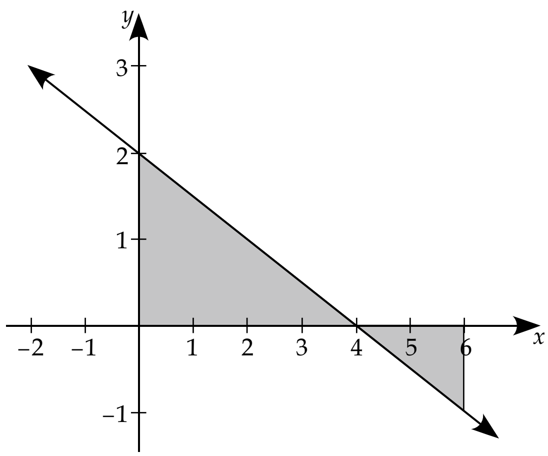
b)



Area = 6 × 3 ÷ 2 = 9

Therefore, $\int_{-2}^{4} f(x)\,dx = 9$.

c)



Upper Area = $4 \times 2 \div 2 = 4$

Lower Area = $2 \times 1 \div 2 = 1$

Therefore, $\int_0^6 f(x)\,dx = 4 + (-1) = 3.$

The definite integral evaluates areas bound by the function under the x-axis as negative numbers. If the total area was asked for, it would be $4 + |-1| = 5$.

4. Determine the area bounded by the function $y = x^3 - 2x^2 - 5x + 6$ and the x-axis on $[-1, 2]$.

Answer:

Determine the zeros of the function using the factor theorem.

Check which values make $0 = x^3 - 2x^2 - 5x + 6$ by testing the factors of the constant.

$y = 1^3 - 2(1)^2 - 5(1) + 6 = 1 - 2 - 5 + 6 = 0$ $\quad$ $x = 1$ is a zero.

$y = (-1)^3 - 2(-1)^2 - 5(-1) + 6 = -1 - 2 + 5 + 6 = 8 \neq 0$

$y = 2^3 - 2(2)^2 - 5(2) + 6 = 8 - 8 - 10 + 6 = -4 \neq 0$

$y = (-2)^3 - 2(-2)^2 - 5(-2) + 6 = -8 - 8 + 10 + 6 = 0$ $\quad$ $x = -2$ is a zero.

$y = 3^3 - 2(3)^2 - 5(3) + 6 = 27 - 18 - 15 + 6 = 0$ $\quad$ $x = 3$ is a zero.

Since the polynomial function has an odd degree with a positive leading coefficient, then the end behaviour is decreasing to the left and increasing to the right.

Determine the relative extreme values using the derivative. The critical values are where the derivative equals zero (or doesn't exist).

$$y' = 3x^2 - 4x - 5$$

Solve $0 = 3x^2 - 4x - 5$ using the quadratic formula. The critical values are $x = 2.12$ or $x = -0.79$.

You could use a sign diagram for y' or you could use the second derivative test to determine whether there is a maximum or a minimum at the critical values. Here's how to use the second derivative test.

$$y'' = 6x - 4$$

Evaluate the second derivative at the critical points to determine the signs (+ or −) there.

$$y''(2.12) = 6(2.12) - 4 > 0$$

$$y''(-0.79) = 6(-0.79) - 4 < 0$$

Since y'' is positive at $x = 2.12$, the function is concave up and there is a minimum there.

Since y'' is negative at $x = -0.79$, the function is concave down and there is a maximum there.

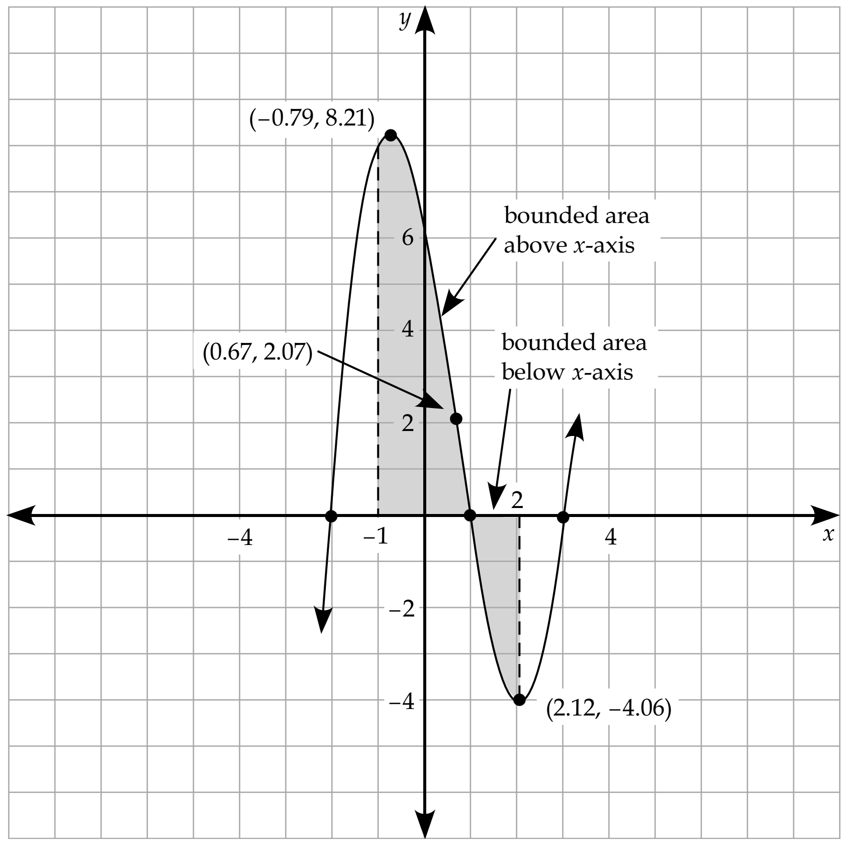
Find the function values by substitution to go along with the critical x-values.

$y(2.12) = (2.12)^3 - 2(2.12)^2 - 5(2.12) + 6 = -4.06$ minimum at (2.12, −4.06)

$y(-0.79) = (-0.79)^3 - 2(-0.79)^2 - 5(-0.79) + 6 = 8.21$

minimum at (−0.79, 8.21)

Furthermore, for completeness, you could find inflection points by solving $y'' = 0$. Solve $0 = 6x - 4$ so that a possible inflection point occurs at $x = \frac{4}{6} \approx 0.67$. Find the y-value by substitution; there is an inflection point at (0.67, 2.07). Remember, this inflection point is where the function changes from concave down to concave up. Now use each of these points to draw an accurate sketch of the function.



From the above sketch, notice that the function is above the x-axis on $[-1, 1]$ but below the x-axis on $[1, 2]$. Since the definite integral of areas below the x-axis are negative values, determine the area bounded by the function and the x-axis on the interval $[-1, 2]$ by separating the areas in two definite integrals.

Total area = Area above the x-axis + | Area below the x-axis |

$$= \int_{-1}^{1} \left(x^3 - 2x^2 - 5x + 6\right) dx + \left| \int_{1}^{2} \left(x^3 - 2x^2 - 5x + 6\right) dx \right|$$

$$= \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{5}{2}x^2 + 6x\right]_{-1}^{1} + \left|\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{5}{2}x^2 + 6x\right|_{1}^{2}$$

$$= \left(\left[\frac{1}{4}(1)^4 - \frac{2}{3}(1)^3 - \frac{5}{2}(1)^2 + 6(1)\right] - \left[\frac{1}{4}(-1)^4 - \frac{2}{3}(-1)^3 - \frac{5}{2}(-1)^2 + 6(-1)\right]\right) +$$

$$\left|\left[\frac{1}{4}(2)^4 - \frac{2}{3}(2)^3 - \frac{5}{2}(2)^2 + 6(2)\right] - \left[\frac{1}{4}(1)^4 - \frac{2}{3}(1)^3 - \frac{5}{2}(1)^2 + 6(1)\right]\right|$$

$$= \left(\frac{1}{4} - \frac{2}{3} - \frac{5}{2} + 6 - \frac{1}{4} - \frac{2}{3} + \frac{5}{2} + 6\right) + \left|4 - \frac{16}{3} - 10 + 12 - \frac{1}{4} + \frac{2}{3} + \frac{5}{2} - 6\right|$$

$$= \left(12 - \frac{4}{3}\right) + \left|0 - \frac{29}{12}\right|$$

$$= \frac{144}{12} - \frac{16}{12} + \frac{29}{12}$$

$$= \frac{157}{12} \text{ units}^2$$

Learning Activity 4.5

Part A: BrainPower

The BrainPower questions are provided as a warm-up activity for your brain before trying the questions in Part B. Try to complete each question quickly, without the use of a calculator and without writing many steps on paper.

Use $f(x) = x^2 + 4$ and $g(x) = 2x + 3$ to answer Questions 1 to 5.

1. Simplify: $f(x) - g(x)$
2. Simplify: $g(x) - f(x)$
3. Create a definite integral that will determine the area under the curve $f(x)$ on [1, 4].
4. Create a definite integral that will determine the area under the line $g(x)$ on [1, 4].
5. Determine x-value(s) of the intersection(s) of $f(x)$ and $g(x)$.
6. Determine the roots of $2x^2 - 3x + 1 = 0$.
7. What is the indefinite integral of $4\sqrt{x}$?
8. What is the indefinite integral of $\frac{3}{x^2}$?

Answers:

1. $x^2 - 2x + 1 \left(\left(x^2 + 4\right) - (2x + 3) = x^2 + 2x + 1\right)$
2. $-x^2 + 2x - 1 \left((2x + 3) - \left(x^2 + 4\right) = -x^2 + 2x - 1\right)$
3. $\int_1^4 \left(x^2 + 4\right) dx$ since $f(x)$ is always positive
4. $\int_1^4 (2x + 3)\, dx$ since $g(x)$ is always positive on the interval
5. $1 \begin{pmatrix} x^2 + 4 = 2x + 3 \\ x^2 - 2x + 1 = 0 \\ (x - 1)^2 = 0 \\ x = 1 \end{pmatrix}$

6. $\frac{1}{2}$ and 1

$$\left(\begin{array}{l} 2x^2 - 3x + 1 = 0 \\ (2x - 1)(x - 1) \\ x = \frac{1}{2} \qquad x = 1 \end{array} \right)$$

7. $\int 4\sqrt{x}\,dx = \int 4x^{\frac{1}{2}}\,dx = \frac{8}{3}x^{\frac{3}{2}} + C$

8. $\int \frac{3}{x^2}\,dx = \int 3x^{-2}\,dx = -3x^{-1} + C$

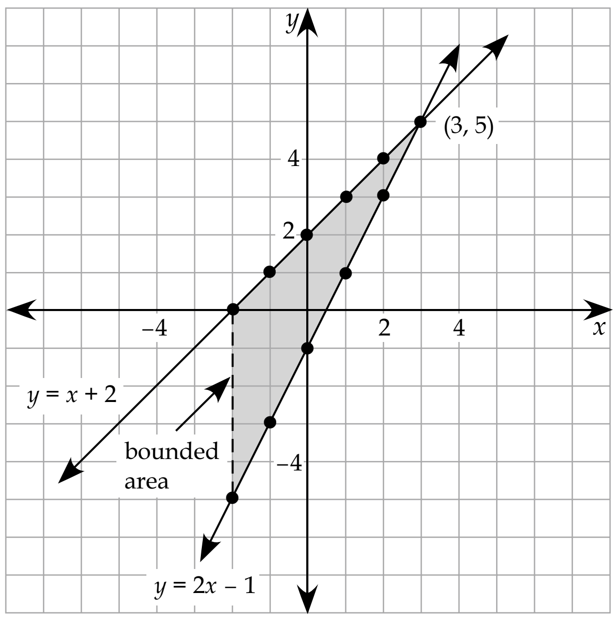
Part B: Area between Two Functions

Remember, these questions are similar to the ones that will be on your assignments and examination. So, if you were able to answer them correctly, you are likely to do well on your assignments and examination. If you did not answer them correctly, you need to go back to the lesson and learn the necessary concepts.

1. Determine the area bounded by the two lines $y = 2x - 1$ and $y = x + 2$ on $[-2, 3]$.

   Answer:

   Sketch the functions on the same coordinate plane to determine which function is greater.



$x + 2 > 2x - 1$ on $[-2, 3]$ — According to the sketch.

$$\int_{-2}^{3} [(x+2) - (2x-1)]\,dx = \int_{-2}^{3} (-x+3)\,dx$$

Area bounded by two functions.

$$= \left[-\frac{1}{2}x^2 + 3x\right]_{-2}^{3}$$

$$= \left[-\frac{1}{2}(3)^2 + 3(3)\right] - \left[-\frac{1}{2}(-2)^2 + 3(-2)\right]$$

Evaluate the definite integral.

$$= -\frac{9}{2} + 9 + 2 + 6 = -4.5 + 17 = 12.5 \text{ units}^2$$

2. Determine the area bounded by the two lines $y = -x + 3$ and $y = x + 1$ on $[-1, 2]$.

Answer:

Sketch the functions on the same coordinate plane to determine which function is greater on which interval. Determine the intersection of these two lines.

$$x + 1 = -x + 3$$

$$2x = 2$$

$$x = 1$$

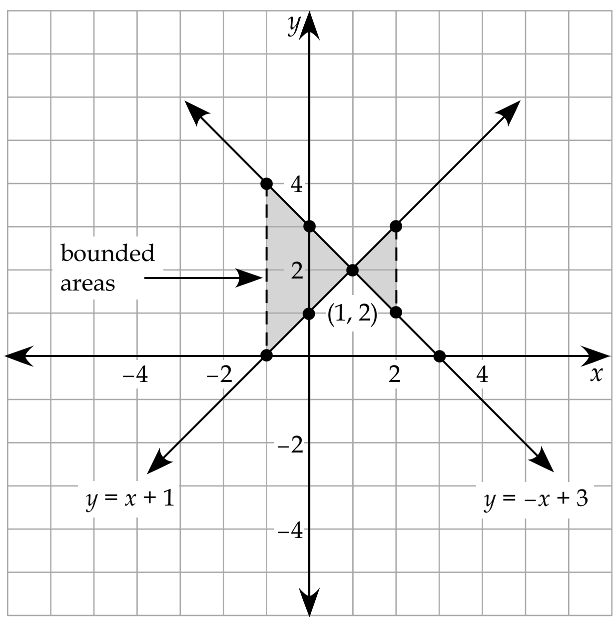
Find the y-value.

$$y = x + 1$$

$$y = 1 + 1$$

$$y = 2$$

The coordinates are (1, 2).



Split the interval where the functions cross.

$-x + 3 > x + 1$ on $[-1, 1]$ $\qquad$ $x + 1 > -x + 3$ on $[1, 2]$

Determine the total area by evaluating the sum of the following integrals.

$$\text{Total area} = \int_{-1}^{1} [(-x + 3) - (x + 1)]dx + \int_{1}^{2} [(x + 1) - (-x + 3)]dx$$

Simplify each definite integral.

$$\text{Total area} = \int_{-1}^{1} (-2x + 2)\,dx + \int_{1}^{2} (2x - 2)\,dx$$

$$= \left[-x^2 + 2x\right]_{-1}^{1} + \left[x^2 - 2x\right]_{1}^{2}$$

$$= \left[-1^2 + 2(1)\right] - \left[-(-1)^2 + 2(-1)\right] + \left[2^2 - 2(2)\right] - \left[(1)^2 - 2(1)\right]$$

$$= -1 + 2 + 1 + 2 + 4 - 4 - 1 + 2 = 5 \text{ units}^2$$

3. Determine the area bounded by the graphs of $y = 3x - x^2$ and $y = -x$.

Answer:

Find the intersection points to determine the upper and lower bounds.

$3x - x^2 = -x$

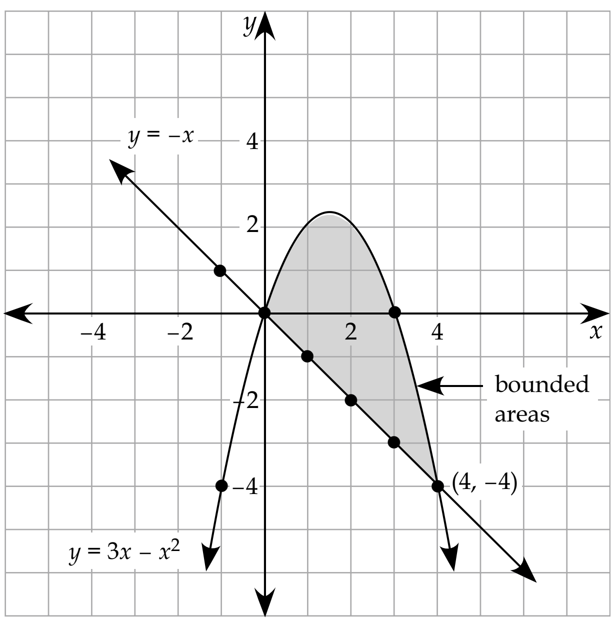
$x^2 - 4x = 0$

$x(x - 4) = 0$

$x = 0, 4$

Then, using $y = -x$, the coordinates are at (0, 0) and (4, −4).

Sketch the function curves to confirm which function is greater than the other.



On the interval [0, 4], $3x - x^2 > x$.

Evaluate the definite integral in order to determine the bounded area.

$$\int_0^4 \left[\left(3x - x^2\right) - (-x)\right] dx = \int_0^4 \left(-x^2 + 4x\right) dx = \left[-\frac{1}{3}x^3 + 2x^2\right]_0^4$$

$$= \left[-\frac{1}{3}(4)^3 + 2(4)^2\right] - 0 = -\frac{1}{3} \cdot 64 + 32 = -\frac{64}{3} + \frac{96}{3} = \frac{32}{3} \text{ units}^2$$

4. Determine the area bounded by the graphs of $y = 4 - x^2$ and $y = 2x^2 - 8$.

Answer:

Find the intersection points to determine the upper and lower bounds.

$2x^2 - 8 = 4 - x^2$

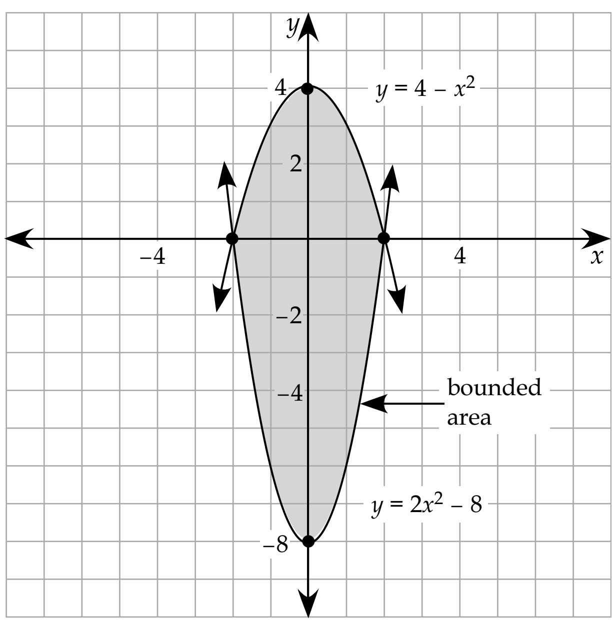
$3x^2 = 12$

$x^2 = 4$

$x = \pm 2$

The coordinates are $(-2, 0)$ and $(2, 0)$.

Sketch the function curves to determine which function is greater than the other.



On the interval $[-2, 2]$, $4 - x^2 > 2x^2 - 8$.

Evaluate the definite integral in order to determine the bounded area.

$$\int_{-2}^{2}\left[\left(4-x^2\right)-\left(2x^2-8\right)\right]dx=\int_{-2}^{2}\left(-3x^2+12\right)dx=\left[-x^3+12x\right]_{-2}^{2}$$

$$=\left[-(2)^3+12(2)\right]-\left[-(-2)^3+12(-2)\right]$$

$$=-8+24-8+24=32\text{ units}^2$$

Grade 12 Introduction to Calculus (45S)

Final Practice Exam

GRADE 12 INTRODUCTION TO CALCULUS

Final Practice Exam

Name: ____________________________________

Student Number: ____________________________

Attending ❑ Non-Attending ❑

Phone Number: ______________________________

Address: __________________________________

____

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For Marker's Use Only

Date: ___________________________________

Final Mark: ________ /100 = ________ %

Comments:

Instructions

The final examination will be weighted as follows:

Module 1: Limits	21 marks
Module 2: Derivatives	26 marks
Module 3: Applications of Derivatives	32 marks
Module 4: Integration	21 marks
	100 marks

Time allowed: 3.0 hours

Note: You are allowed to bring the following to the exam: pencils (2 or 3 of each), blank paper, a ruler, and a scientific calculator.

Show all calculations and formulas used. Use all decimal places in your calculations and round the final answers to the correct number of decimal places. Include units where appropriate. Clearly state your final answer.

Name: ____

Module 1: Limits *(21 marks)*

1. Determine $\lim_{x \to 4} \frac{3 - \sqrt{x}}{x - 9}$. *(1 mark)*

2. Given: $\lim_{x \to 9} \frac{3 - \sqrt{x}}{x - 9}$ *(4 marks)*

   a) Evaluate the limit algebraically.

b) Explain why evaluating the limit $\lim_{x \to 9} \frac{3 - \sqrt{x}}{x - 9}$ requires an algebraic manipulation, while $\lim_{x \to 4} \frac{3 - \sqrt{x}}{x - 9}$ does not.

3. Evaluate the following limits. *(6 marks)*

a) $\lim_{x \to 2} \left(\frac{x^2 - 3x + 2}{x - 2} \right)$

Name: ____

b) $\lim_{x \to \infty} \left(\frac{x^2 - 7x + 1}{x^3 + 2} \right)$

c) $\lim_{x \to 2^+} \frac{x + 1}{(x - 2)^2}$

4. Complete the tasks below to determine if $g(x) = \begin{cases} -x+1, & x \le 1 \\ x^2 - 1, & x > 1 \end{cases}$ is continuous at $x = 1$.

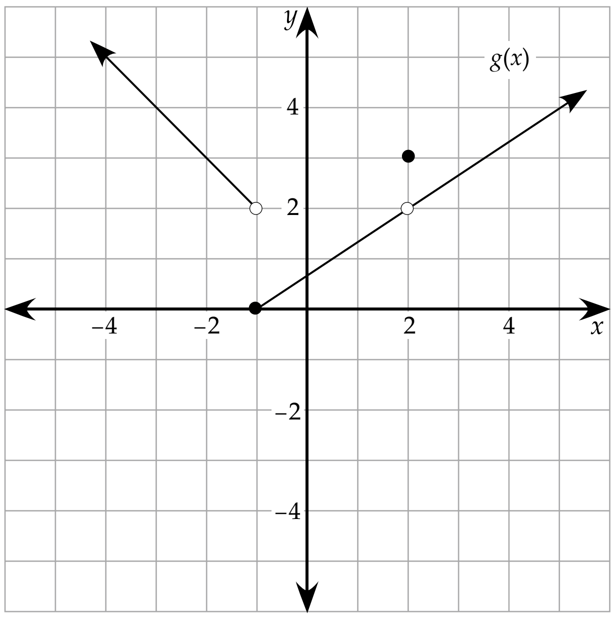
(4 marks)

a) Verify that $\lim\limits_{x \to 1} g(x)$ exists.

b) Is $g(x)$ continuous? Explain.

Name: ___________________________________

5. Use the graph of $g(x)$ below to evaluate each expression. *(6 marks)*



a) $\lim_{x \to 2^-} g(x)$

b) $\lim_{x \to 2^+} g(x)$

c) $g(2)$

d) $\lim_{x \to -1^-} g(x)$

e) $\lim_{x \to -1^+} g(x)$

f) $\lim_{x \to -1} g(x)$

Module 2: Derivatives *(26 marks)*

1. Given: $g(x) = -2x^2 + 3$ *(10 marks)*

   a) Determine the slope of the secant lines PR, PS, and PT to the curve, given the coordinates P(1, 1), R(4, −29), S(3, −15), T(1.1, 0.58).

   b) Using the values from part (a) above, describe what is happening to the value of the slope of the secant line from a point (x, y) as the point approaches P.

   c) Estimate the slope of the tangent line at point P.

Name: ____

d) Determine the derivative of $g(x) = -2x^2 + 3$ at $x = 1$ using the limit definition of the derivative and the difference quotient, $g'(1) = \lim_{h \to 0} \frac{g(1+h) - g(1)}{h}$.

e) Determine the equation of the tangent line to $g(x) = -2x^2 + 3$ at $x = 1$.

Name: ____

2. Use derivative rules to differentiate the following and **do not** simplify your derivative. *(10 marks)*

a) $f(x) = 2x^{-4} - 5x^{\frac{2}{3}} + 7$

b) $g(x) = 6\left(2x^5 - 1\right)^3$

c) $h(x) = \dfrac{6x - 3}{2x^3 + 1}$

d) $k(x) = \left(\sqrt{x}\right)\left(5x^2 + 1\right)$

3. Determine $\frac{dy}{dx}$ in terms of x and y for the equation $x + xy^2 - y = 3$. *(4 marks)*

4. Determine y'' for $y = 3x^5 - 5x^3 - 2x^2 - 1$. *(2 marks)*

Name: ____

Module 3: Applications of Derivatives *(32 marks)*

1. A ball is thrown upward so that its height above the ground after time t is $h = 20t - 5t^2$, where h is measured in metres and t is measured in seconds. *(6 marks)*

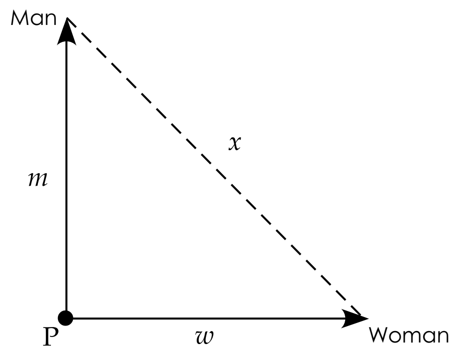
   a) Determine the equation that represents the velocity of the ball.

   b) Determine when the ball reaches its maximum height.

c) Determine the velocity of the ball when it is 15 metres high on its way down.

Name: ____

2. A man starts walking north at a speed of 1.5 m/s and a woman starts at the same point P at the same time walking east at a speed of 2 m/s. *(6 marks)*



a) How far is the man, m, from his starting point after one minute?

b) How far is the woman, w, from her starting point after one minute?

c) How far apart are the man and the woman, x, from each other after one minute?

d) At what rate is the distance between the man and the woman increasing at the instant they have been walking for one minute?

Name: ____________________________________

3. Given: $g(x) = x^3 + 6x^2 + 9x + 4$ *(13 marks)*

   a) Find the intervals where the function is increasing and decreasing.

   b) Find the coordinates where the relative extreme values occur and identify each of them as a relative maximum or minimum.

c) Find the intervals of concavity and the coordinates of any points of inflection.

d) Sketch the graph of the function and label its extreme values and point(s) of inflection.

Name: ____

4. The sum of two positive numbers is 12. If the product of one number cubed and the other number is a maximum, find the two numbers. *(7 marks)*

Module 4: Integration *(21 marks)*

1. If $f'(x) = 4x^5 - 2x^3 + x - 2$, and $f(0) = 3$, determine the function equation for $f(x)$. *(4 marks)*

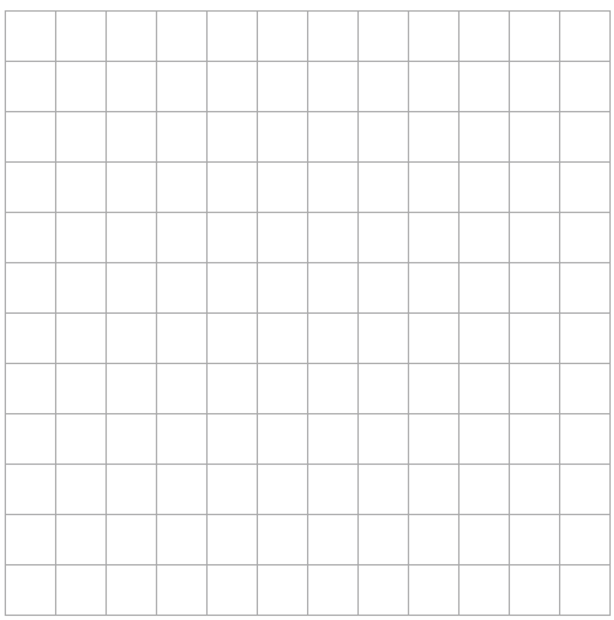
2. Evaluate algebraically $\int_{-2}^{0} \left(5x^4 - 2x^2 - 1\right) dx$. *(3 marks)*

Name: ______________________________

3. Write the general function equation represented by the indefinite integral $\int \left(3x^6 - 3x^{-2}\right) dx$. *(2 marks)*

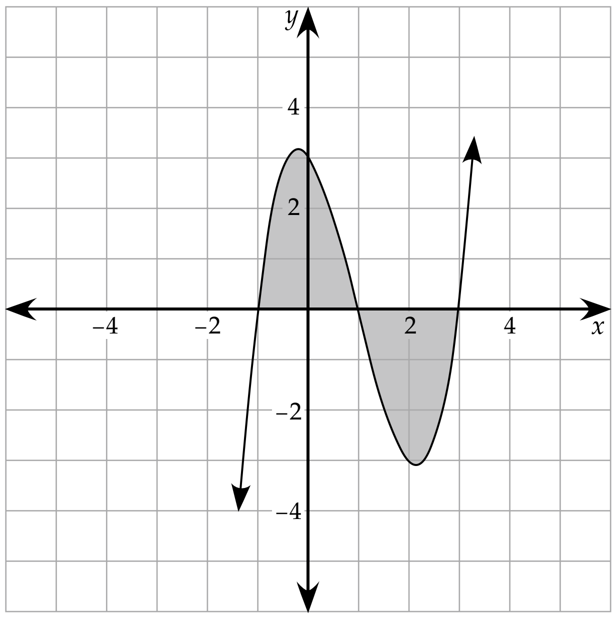
4. Sketch and determine the area bounded by the line $y = -x + 1$ and the x-axis on the closed interval [0, 1]: *(3 marks)*

   a) geometrically, using a graph of the function



   b) algebraically, using the antiderivative

5. Determine the area bounded by the curve $y = x^3 - 3x^2 - x + 3$ and the x-axis. *(5 marks)*



Name: ____

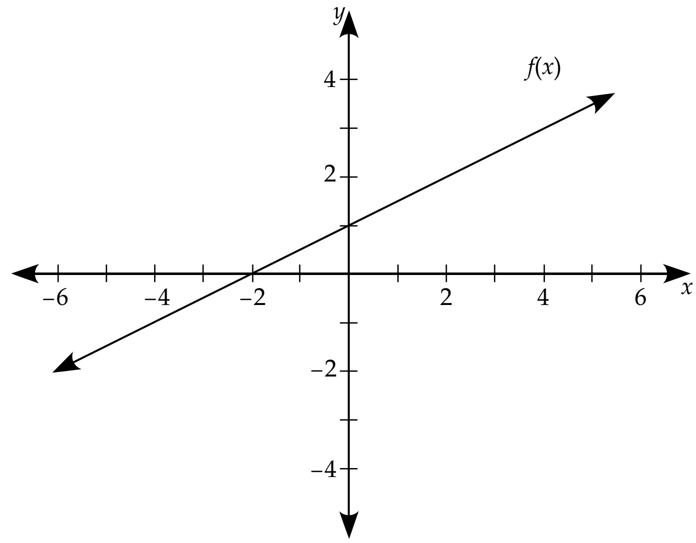
6. Find the values of each definite integral geometrically using the sketch of $f(x)$ as shown. *(4 marks)*.

a) $\int_{-2}^{4} f(x)\,dx$

b) $\int_{-4}^{0} f(x)\,dx$

c) $\int_{2}^{4} f(x)\,dx$

d) $\int_{-4}^{0} \left| f(x) \right| dx$



Grade 12 Introduction to Calculus (45S)

Final Practice Exam
Answer Key

GRADE 12 INTRODUCTION TO CALCULUS

Final Practice Exam
Answer Key

Name: ___________________________________

Student Number: ___________________________

Attending ❑ Non-Attending ❑

Phone Number: ____________________________

Address: _________________________________

____

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For Marker's Use Only

Date: ______________________________

Final Mark ________ /100 = ________ %

Comments:

Instructions

The final examination will be weighted as follows:

Module 1: Limits	21 marks
Module 2: Derivatives	26 marks
Module 3: Applications of Derivatives	32 marks
Module 4: Integration	21 marks
	100 marks

Time allowed: 3.0 hours

Note: You are allowed to bring the following to the exam: pencils (2 or 3 of each), blank paper, a ruler, and a scientific calculator.

Show all calculations and formulas used. Use all decimal places in your calculations and round the final answers to the correct number of decimal places. Include units where appropriate. Clearly state your final answer.

Name: ____

Module 1: Limits *(21 marks)*

1. Determine $\lim_{x \to 4} \frac{3 - \sqrt{x}}{x - 9}$. *(1 mark)*

   Answer: (Lesson 3)

   (1 mark for evaluation of limit)

   $$\lim_{x \to 4} \frac{3 - \sqrt{x}}{x - 9} = \frac{3 - \sqrt{4}}{4 - 9} = \frac{3 - 2}{-5} = -\frac{1}{5}$$

   The limit is $-\frac{1}{5}$.

2. Given: $\lim_{x \to 9} \frac{3 - \sqrt{x}}{x - 9}$ *(4 marks)*

   a) Evaluate the limit algebraically.

   Answer: (Lesson 4)

   (2 marks for the algebraic manipulation of the limit)

   (1 mark for evaluating the limit)

   $$\lim_{x \to 9} \frac{3 - \sqrt{x}}{x - 9} = \frac{0}{0} \qquad \text{I.F.}$$

   $$\lim_{x \to 9} \frac{3 - \sqrt{x}}{x - 9} = \lim_{x \to 9} \left[\frac{\left(3 - \sqrt{x}\right)}{(x - 9)} \cdot \frac{\left(3 + \sqrt{x}\right)}{\left(3 + \sqrt{x}\right)} \right] = \lim_{x \to 9} \frac{9 - x}{(x - 9)\left(3 + \sqrt{x}\right)} = \lim_{x \to 9} \frac{-1}{3 + \sqrt{x}}$$

   $$= \frac{-1}{3 + \sqrt{9}} = \frac{-1}{3 + 3} = \frac{-1}{6} \text{ or} -0.1\overline{6}$$

   The limit is $-\frac{1}{6}$.

b) Explain why evaluating the limit $\lim_{x \to 9} \frac{3 - \sqrt{x}}{x - 9}$ requires an algebraic manipulation, while $\lim_{x \to 4} \frac{3 - \sqrt{x}}{x - 9}$ does not.

Answer: (Lesson 4)

(1 mark for explanation)

Possible explanation could reference that the $x = 9$ is a non-permissible value that makes the numerator and denominator zero or that the resultant substitution is in the indeterminate form.

3. Evaluate the following limits. *(6 marks)*

a) $\lim_{x \to 2} \left(\frac{x^2 - 3x + 2}{x - 2} \right)$

Answer: (Lesson 3)

(1 mark for simplification of limit)

(1 mark for evaluating the limit)

$$\lim_{x \to 2} \left(\frac{x^2 - 3x + 2}{x - 2} \right) = \lim_{x \to 2} \left(\frac{(x-2)(x-1)}{x - 2} \right) = \lim_{x \to 2} (x - 1)$$

$$= (2 - 1) = 1$$

Name: ___________________________________

b) $\lim_{x\to\infty}\left(\frac{x^2 - 7x + 1}{x^3 + 2}\right)$

Answer: (Lesson 6)

(1 mark for dividing top and bottom by the highest power of x in the denominator)

(1 mark for simplifying both the numerator and denominator)

(1 mark for evaluating the limit)

$$\lim_{x\to\infty}\left(\frac{x^2 - 7x + 1}{x^3 + 2}\right) = \frac{\infty}{\infty} \quad \text{I.F.}$$

$$\lim_{x\to\infty}\left(\frac{x^2 - 7x + 1}{x^3 + 2}\right) = \lim_{x\to\infty}\left(\frac{x^2 - 7x + 1}{x^3 + 2}\right)\cdot\left(\frac{\frac{1}{x^3}}{\frac{1}{x^3}}\right) = \lim_{x\to\infty}\left(\frac{\frac{1}{x} - \frac{7}{x^2} + \frac{1}{x^3}}{1 + \frac{2}{x^3}}\right)$$

$$= \frac{0 + 0 + 0}{1 + 0} = \frac{0}{1} = 0$$

c) $\lim_{x\to 2^+}\frac{x + 1}{(x - 2)^2}$

Answer: (Lesson 6)

(1 mark for answer)

Illustrate using $x = 2.01$.

$$\lim_{x\to 2^+}\frac{x + 1}{(x - 2)^2} \approx \frac{2.01 + 1}{(2.01 - 2)^2} = \frac{3.01}{(0.01)^2} = \frac{3.01}{0.0001} = 30100$$

$$\lim_{x\to 2^+}\frac{x + 1}{(x - 2)^2} = \frac{3}{\text{very small positive number}} \text{ and approaches } \infty$$

As x approaches 2 from the right, the right limit approaches ∞.

$$\lim_{x\to 2^+}\left(\frac{x + 1}{(x - 2)^2}\right) = \infty$$

4. Complete the tasks below to determine if $g(x) = \begin{cases} -x + 1, & x \le 1 \\ x^2 - 1, & x > 1 \end{cases}$ is continuous at $x = 1$.

(4 marks)

a) Verify that $\lim_{x \to 1} g(x)$ exists.

Answer: (Lesson 5)

(1 mark for setting up the one-side limits)

(1 mark for evaluating the limit)

$$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} (-x + 1) = -1 + 1 = 0$$

$$\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} \left(x^2 - 1\right) = 1^2 - 1 = 1 - 1 = 0$$

$$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^+} g(x)$$

So, $\lim_{x \to 1} g(x) = 0$.

b) Is $g(x)$ continuous? Explain.

Answer: (Lesson 7)

(1 mark for evaluating the function value)

(1 mark for using the definition of continuity to explain why the function is continuous)

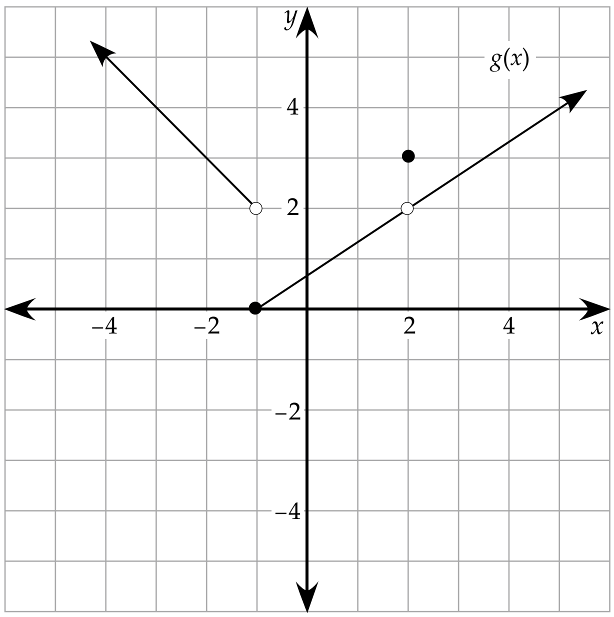
For the function to be continuous at $x = 1$, both the function and the limit values have to exist and be equal to each other at $x = 1$.

$$g(1) = -1 + 1 = 0$$

Since $g(1) = 0$ and $\lim_{x \to 1} g(x) = 0$, then the function is defined at $x = 1$ and equal to the limit value, so the function is continuous at $x = 1$.

Name: ____

5. Use the graph of $g(x)$ below to evaluate each expression. *(6 marks)*



(6 × 1 mark per correct limit) (Lesson 5)

a) $\lim_{x \to 2^-} g(x)$

Answer:

$\lim_{x \to 2^-} g(x) = 2$

b) $\lim_{x \to 2^+} g(x)$

Answer:

$\lim_{x \to 2^+} g(x) = 2$

c) $g(2)$

Answer:

$g(2) = 3$

d) $\lim_{x \to -1^-} g(x)$

Answer:

$\lim_{x \to -1^-} g(x) = 2$

e) $\lim_{x \to -1^+} g(x)$

Answer:

$\lim_{x \to -1^+} g(x) = 0$

f) $\lim_{x \to -1} g(x)$

Answer:
Does not exist since left-hand and right-hand limits are not equal.

Module 2: Derivatives *(26 marks)*

1. Given: $g(x) = -2x^2 + 3$ *(10 marks)*

   a) Determine the slope of the secant lines PR, PS, and PT to the curve, given the coordinates P(1, 1), R(4, −29), S(3, −15), T(1.1, 0.58).

   Answer: (Lesson 1)

   (2 marks for evaluating slope properly)

   $$m_{PR} = \frac{-29-1}{4-1} = \frac{-30}{3} = -10$$

   $$m_{PS} = \frac{-15-1}{3-1} = \frac{-16}{2} = -8$$

   $$m_{PT} = \frac{0.58-1}{1.1-1} = \frac{-0.42}{0.1} = -4.2$$

   b) Using the values from part (a) above, describe what is happening to the value of the slope of the secant line from a point (x, y) as the point approaches P.

   Answer: (Lesson 1)

   (1 mark for a reasonable description)

   The slopes of the secant lines are increasing since they are smaller negative numbers.

   c) Estimate the slope of the tangent line at point P.

   Answer: (Lesson 1)

   (1 mark for a reasonable estimate)

   The slope of the tangent line is approaching −4.2 (or is near −4.2).

Name: ____________________________________

d) Determine the derivative of $g(x) = -2x^2 + 3$ at $x = 1$ using the limit definition of the derivative and the difference quotient, $g'(1) = \lim_{h \to 0} \frac{g(1+h) - g(1)}{h}$.

Answer: (Lesson 2)

(1 mark for correct substitution into the limit definition of the derivative)

(1 mark for simplifying the limit)

(1 mark for evaluating the limit and finding the derivative at x = 1)

$$g'(1) = \lim_{h \to 0} \frac{g(1+h) - g(1)}{h} = \lim_{h \to 0} \frac{\left(-2(1+h)^2 + 3\right) - \left(-2(1)^2 + 3\right)}{h}$$

$$= \lim_{h \to 0} \frac{-2 - 4h - 2h^2 + 3 + 2 - 3}{h}$$

$$g'(1) = \lim_{h \to 0} \frac{-4h - 2h^2}{h} = \lim_{h \to 0} \frac{h(-4 - 2h)}{h} = \lim_{h \to 0} (-4 - 2h)$$

$$= -4 - 2(0) = -4 - 0 = -4$$

The derivative of the function at $x = 1$ is -4.

e) Determine the equation of the tangent line to $g(x) = -2x^2 + 3$ at $x = 1$.

Answer: (Lesson 2)

(1 mark for determining the coordinates of the point of tangency)

(1 mark for appropriate algebraic work)

(1 mark for determining the equation of the tangent line)

Use the slope and a point on the tangent line to determine the equation of the tangent line.

$$y = g(1) = -2(1)1 + 3 = 1$$

$$m_T = -4 \text{ and } (1, 1)$$

Using slope–intercept form:

$$y = mx + b$$

$$1 = (-4)(1) + b$$

$$b = 1 + 4 = 5$$

Or, using point–slope form:

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -4(x - 1)$$

The equation of the tangent line is $y = -4x + 5$.

Name: ____

2. Use derivative rules to differentiate the following and **do not** simplify your derivative. *(10 marks)*

a) $f(x) = 2x^{-4} - 5x^{\frac{2}{3}} + 7$

Answer: (Lesson 3)

(3 marks for proper use of the power rule on each of the three terms)

$$f'(x) = -8x^{-5} - \frac{10}{3}x^{-\frac{1}{3}}$$

b) $g(x) = 6\left(2x^5 - 1\right)^3$

Answer: (Lesson 5)

(1 mark for derivative of outside function in chain rule)

(1 mark for derivative of inside function in chain rule)

$$g'(x) = 18(\text{inside})^2 \cdot \text{inside}'$$

$$g'(x) = 18\left(2x^5 - 1\right)^2\left(10x^4\right)$$

c) $h(x) = \dfrac{6x - 3}{2x^3 + 1}$

Answer: (Lesson 4)

(2 marks for correctly writing the numerator, 1 mark for each term)

(1 mark for correctly writing the denominator using the quotient rule)

$$h'(x) = \frac{\left(2x^3 + 1\right)(6) - (6x - 3)\left(6x^2\right)}{\left(2x^3 + 1\right)^2}$$

d) $k(x) = (\sqrt{x})(5x^2 + 1)$

Answer: (Lesson 4)

(2 marks for using the product rule, one mark for each term)

$$k'(x) = \left(\frac{1}{2}x^{-\frac{1}{2}}\right)(5x^2 + 1) + (\sqrt{x})(10x)$$

3. Determine $\frac{dy}{dx}$ in terms of x and y for the equation $x + xy^2 - y = 3$. *(4 marks)*

(Lesson 6)

Answer:

(1 mark for power rule and constant rule)

(1 mark for product rule)

(1 mark for implicit differentiation)

(1 mark for isolating y')

$$\frac{d}{dx}(x + xy^2 - y) = \frac{d}{dx}(3)$$

$$1 + (1)y^2 + x(2yy') - y' = 0$$

$$2xyy' - y' = -1 - y^2$$

$$y'(2xy - 1) = -1 - y^2$$

$$\frac{dy}{dx} = y' = \frac{-1 - y^2}{2xy - 1}$$

4. Determine y'' for $y = 3x^5 - 5x^3 - 2x^2 - 1$. *(2 marks)*

Answer: (Lesson 5)

(1 mark for the first derivative)

(1 mark for the second derivative)

$$y = 3x^5 - 5x^3 - 2x^2 - 1$$

$$y' = 15x^4 - 15x^2 - 4x$$

$$y'' = 60x^3 - 30x - 4$$

Name: ____________________________________

Module 3: Applications of Derivatives *(32 marks)*

1. A ball is thrown upward so that its height above the ground after time t is $h = 20t - 5t^2$, where h is measured in metres and t is measured in seconds. *(6 marks)*

   a) Determine the equation that represents the velocity of the ball.

   Answer: (Lesson 2)

   (1 mark for determining velocity)

   $h = 20t - 5t^2$

   $v = h' = 20 - 10t$

   b) Determine when the ball reaches its maximum height.

   Answer: (Lesson 2)

   (1 mark for setting velocity equal to zero)

   (1 mark for determining when the ball reaches the maximum height)

   $v = 20 - 10t$

   $0 = 20 - 10t$

   $10t = 20$

   $t = 2$

   The ball reaches its maximum height after 2 seconds.

c) Determine the velocity of the ball when it is 15 metres high on its way down.

Answer: (Lesson 2)

(1 mark for solving the quadratic equation)

(1 mark for determining the time on the way down)

(1 mark for determining the velocity)

$15 = 20t - 5t^2$

$0 = -5t^2 + 20t - 15$

$0 = t^2 - 4t + 3$

$0 = (t - 3)(t - 1)$

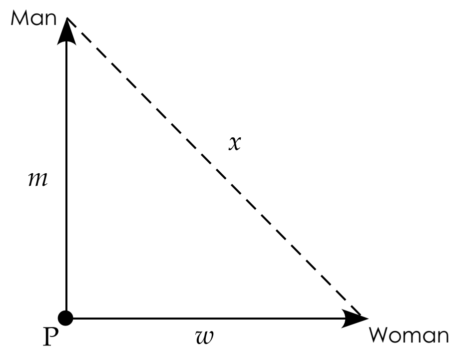
The ball is 15 metres high at $t = 1$ s or $t = 3$ s. It is on its way down when $t = 3$ s.

$v = 20 - 10(3) = 10 - 30 = -10$ m/s

Since the ball is travelling downward, its velocity is negative, and the velocity of the ball on its way down at a height of 15 metres is -10 m/s.

Name: ____

2. A man starts walking north at a speed of 1.5 m/s and a woman starts at the same point P at the same time walking east at a speed of 2 m/s. *(6 marks)*



a) How far is the man, m, from his starting point after one minute?

Answer: (Lesson 6)

(1 mark for evaluating the man's displacement after one minute)

$$m = (1.5 \text{ m/s}) \times (60 \text{ s}) = 90 \text{ m}$$

The man is 90 metres from point P after one minute.

b) How far is the woman, w, from her starting point after one minute?

Answer: (Lesson 6)

(1 mark for evaluating the woman's displacement after one minute)

$$w = (2 \text{ m/s}) \times (60 \text{ s}) = 120 \text{ m}$$

The woman is 120 metres from point P after one minute.

c) How far apart are the man and the woman, x, from each other after one minute?

Answer: (Lesson 6)

(1 mark for evaluating the displacement between the man and woman after one minute)

$$w^2 + m^2 = x^2$$

$$120^2 + 90^2 = x^2$$

$$14400 + 8100 = x^2$$

$$22500 = x^2$$

Use the Pythagorean theorem:

$$x = \sqrt{22500} = 150$$

The man and woman are 150 metres apart from one another after one minute.

d) At what rate is the distance between the man and the woman increasing at the instant they have been walking for one minute?

Answer: (Lesson 6)

(1 mark for implicit differentiation)

(1 mark for correct substitution)

(1 mark for evaluating rate at which the distance between the man and woman changes)

Differentiate the equation that relates the variables that are all dependent on time—that is, $w^2 + m^2 = x^2$.

$$\frac{d}{dt}\left(w^2 + m^2\right) = \frac{d}{dt}\left(x^2\right)$$

$$2w \cdot \frac{dw}{dt} + 2m \cdot \frac{dm}{dt} = 2x \cdot \frac{dx}{dt}$$

Given:

$$\frac{dw}{dt} = 2 \text{ m/s}$$

$$\frac{dm}{dt} = 1.5 \text{ m/s}$$

When $w = 120$ m, $m = 90$ m, and $x = 150$ m.

$$2(120) \cdot (2) + 2(90) \cdot (1.5) = 2(150) \cdot \frac{dx}{dt}$$

$$300\frac{dx}{dt} = 480 + 270 = 750$$

$$\frac{dx}{dt} = \frac{750}{300} = 2.5 \text{ m/s}$$

At the instant they have been walking for one minute, the distance between the man and woman is increasing at 2.5 m/s.

Name: ____________________________________

3. Given: $g(x) = x^3 + 6x^2 + 9x + 4$ *(13 marks)*

   a) Find the intervals where the function is increasing and decreasing.

   Answer: (Lesson 3)

   (1 mark for determining the first derivative)

   (1 mark for determining the critical values)

   $g'(x) = 3x^2 + 12x + 9$

   Solve:

   $g'(x) = 0$

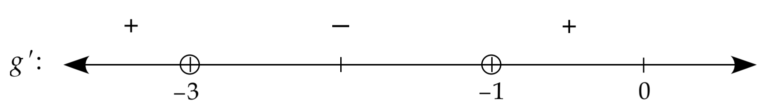
   $3x^2 + 12x + 9 = 0$

   $x^2 + 4x + 3 = 0$

   $(x + 1)(x + 3) = 0$

   The critical values when $g'(x) = 0$ are -1, -3.

   (1 mark for creating sign diagram and identifying boundary values)



   (1 mark for interpreting the sign diagram to determine the positive and negative intervals)

   The function is increasing on $(-\infty, -3) \cup (-1, \infty)$ because the first derivative is positive; and the function is decreasing on $(-3, -1)$ because the first derivative is negative.

b) Find the coordinates where the relative extreme values occur and identify each of them as a relative maximum or minimum.

Answer: (Lesson 3)

Method 1

(1 mark for using the first derivative test to determine whether maximum or minimum)

(1 mark for determining the coordinates of the extreme values)

Since the function changes from increasing to decreasing at $x = -3$, then there is a local maximum there. Also, since $g(-3) = (-3)^3 + 6(-3)^2 + 9(-3) + 4$ $= -27 + 54 - 27 + 4 = 4$, then the coordinates of the local maximum are $(-3, 4)$.

Since the function changes from decreasing to increasing at $x = -1$, then there is a local minimum there. Also, since $g(-1) = (-1)^3 + 6(-1)^2 + 9(-1) + 4$ $= -1 + 6 - 9 + 4 = 0$, then the coordinates of the local minimum are $(-1, 0)$.

Or

Method 2

(1 mark for using the second derivative test to determine whether maximum or minimum)

(1 mark for determining the coordinates of the extreme values)

$g'' = 6x + 12$

Use the second derivative test at $x = -3$ and $x = -1$.

$g''(-3) = 6(-3) + 12 = -6$

Since the second derivative is negative at $x = -3$, then the function is concave down at that x-value and there is a local maximum there.

As in Method 1 above, the local maximum coordinates are $(-3, 4)$.

$g''(-1) = 6(-1) + 12 = +6$

Since the second derivative is positive at $x = -1$, then the function is concave up at that x-value and there is a local minimum there.

As in Method 1 above, the local minimum coordinates are $(-1, 0)$.

Name: ______________________________________

c) Find the intervals of concavity and the coordinates of any points of inflection.

Answer: (Lesson 5)

(1 mark determining the second derivative)

(1 mark for determining the critical values of $g''(x)$)

$g''(x) = 6x + 12$

$0 = g''(x) = 6x + 12$

$6x = -12$

$x = -2$

(2 × 1 mark for describing each concavity interval)

(1 mark for determining the point of inflection)

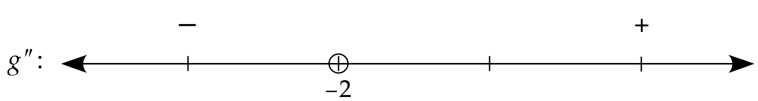
The test intervals are $(-\infty, -2)$ and $(-2, \infty)$.

Select $x = -3$ as a test point for $(-\infty, -2)$ and substitute it into the second derivative:

$g''(-3) = 6(-3) + 12 = -18 + 12 = -6 < 0$

Select $x = 0$ as a test point for $(-2, \infty)$ and substitute it into the second derivative:

$g''(0) = 6(0) + 12 = 12 > 0$



Since the second derivative is negative, then the function is concave down on $(-\infty, -2)$.

Since the second derivative is positive, then the function is concave up on $(-2, \infty)$.

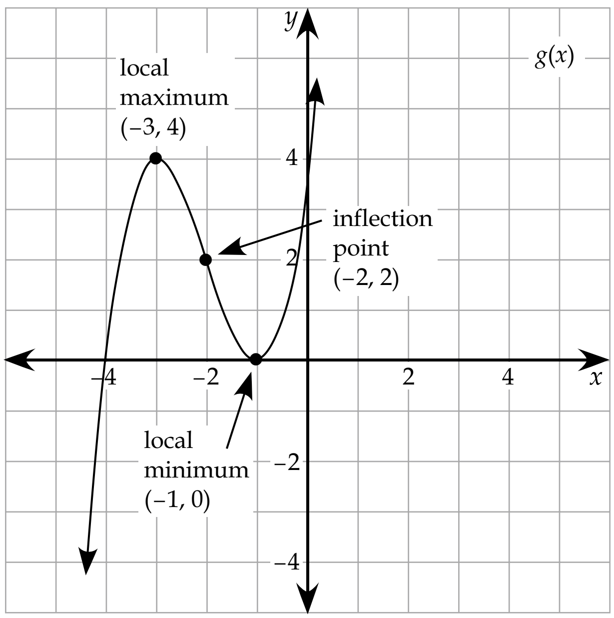
Since the function changes concavity at $x = -2$ and $g(-2) = (-2)^3 + 6(-2)^2 + 9(-2) + 4 = -8 + 24 - 18 + 4 = 2$, then there is an inflection point at $(-2, 2)$.

d) Sketch the graph of the function and label its extreme values and point(s) of inflection.

Answer: (Lesson 5)

(1 mark for plotting the extreme values and inflection point)

(1 mark for sketching the correct curve behaviour)



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4. The sum of two positive numbers is 12. If the product of one number cubed and the other number is a maximum, find the two numbers. *(7 marks)*

   Answer: (Lesson 4)

   (1 mark for defining the sum of two natural numbers)

   (1 mark for creating a product equation as a function of one variable with the sum equation)

   $x + y = 12$

   $x = 12 - y$

   You need an expression for the product to find the maximum. Substitute for x to write P in terms of y.

   $P = x \cdot y^3 \qquad P = (12 - y) \cdot y^3 = 12y^3 - y^4$

   (1 mark for determining the derivative of the product equation)

   $P' = 36y^2 - 4y^3$

   (1 mark for determining the critical value)

   $0 = P' = 36y^2 - 4y^3$

   $0 = 4y^2(9 - y)$

   $y = 0, 9$

   Only $y = 9$ is a possible critical value because the number must be positive.

   (1 mark for determining the second derivative or creating a sign diagram for P')

   $P'' = 72y - 12y^2$

   $P''(9) = 72(9) - 12(9)^2 = -324$

   (1 mark for using the second derivative or interpreting the sign diagram for P' to determine if there is a maximum)

   Since the second derivative is negative when $x = 9$, then the function is concave down and there is a maximum at that critical value.

   Alternatively, you can earn the 2 marks above by creating and interpreting a sign diagram for P'. You will notice that P' is positive to the left when $y = 8$ and negative to the right when $y = 10$. There is a maximum at the critical value, $y = 9$, since P' goes from + to − or, said another way, P goes from increasing to decreasing.

   (1 mark for determining the two natural numbers)

   $x = 12 - 9 = 3$

   The two positive numbers are 9 and 3 and the product $3(9)^3 = 2187$ is the maximum.

Module 4: Integration *(21 marks)*

1. If $f'(x) = 4x^5 - 2x^3 + x - 2$, and $f(0) = 3$, determine the function equation for $f(x)$. *(4 marks)*

   Answer: (Lesson 2)

   (1 mark for determining the general antiderivative of f ′(x))

   (1 mark for substituting initial conditions into the general antiderivative)

   (1 mark for solving for constant of variation)

   (1 mark for determining the specific antiderivative)

   $$f(x) = \frac{4}{6}x^6 - \frac{2}{4}x^4 + \frac{1}{2}x^2 - 2x + C$$

   $$f(0) = \frac{4}{6}(0)^6 - \frac{2}{4}(0)^4 + \frac{1}{2}(0)^2 - 2(0) + C = 3$$

   $$C = 3$$

   Therefore, $f(x) = \frac{2}{3}x^6 - \frac{1}{2}x^4 + \frac{1}{2}x^2 - 2x + 3.$

2. Evaluate algebraically $\int_{-2}^{0}\left(5x^4 - 2x^2 - 1\right) dx$. *(3 marks)*

   Answer: (Lesson 3)

   (1 mark for determining the antiderivative)

   (1 mark for substituting the upper and lower bounds)

   (1 mark for evaluating the definite integral)

   $$\int_{-2}^{0}\left(5x^4 - 2x^2 - 1\right) dx = \left[x^5 - \frac{2}{3}x^3 - x\right]_{-2}^{0}$$

   $$= \left[(0)^5 - \frac{2}{3}(0)^3 - (0)\right] - \left[(-2)^5 - \frac{2}{3}(-2)^3 - (-2)\right]$$

   $$= [0] - \left[(-32) - \frac{2}{3}(-8) + 2\right] = 0 + 32 - \frac{16}{3} - 2 = \frac{90}{3} - \frac{16}{3} = \frac{74}{3} \text{ or } 24\frac{2}{3}$$

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3. Write the general function equation represented by the indefinite integral $\int \left(3x^6 - 3x^{-2}\right) dx$. *(2 marks)*

Answer: (Lesson 1)

(1 mark for determining the antiderivative)

(1 mark for stating the family of indefinite integrals with constant)

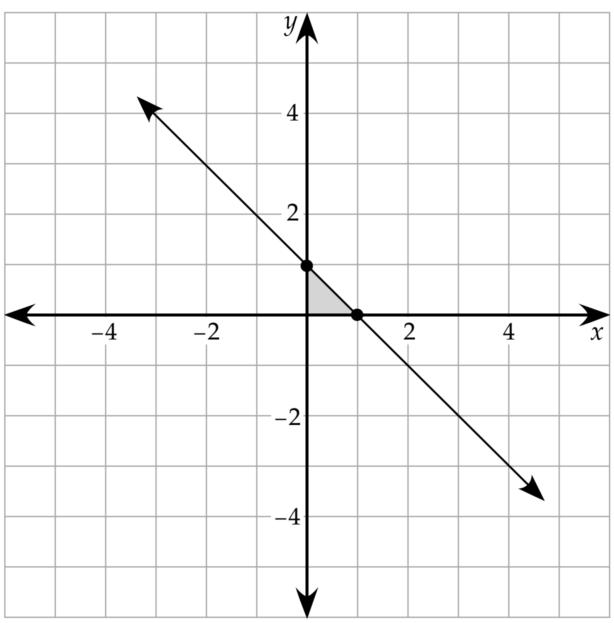
$$\int \left(3x^6 - 3x^{-2}\right) dx = \frac{3}{7}x^7 + 3x^{-1} + C = \frac{3}{7}x^7 + \frac{3}{x} + C$$

4. Sketch and determine the area bounded by the line $y = -x + 1$ and the x-axis on the closed interval [0, 1]: *(3 marks)*

a) geometrically, using a graph of the function

Answer: (Lesson 4)

(1 mark for sketching the area and calculating the area)



Area under the curve = area of triangle

$$= \frac{1}{2}(1)(1) = 0.5 \text{ units}^2$$

b) algebraically, using the antiderivative

Answer: (Lesson 4)

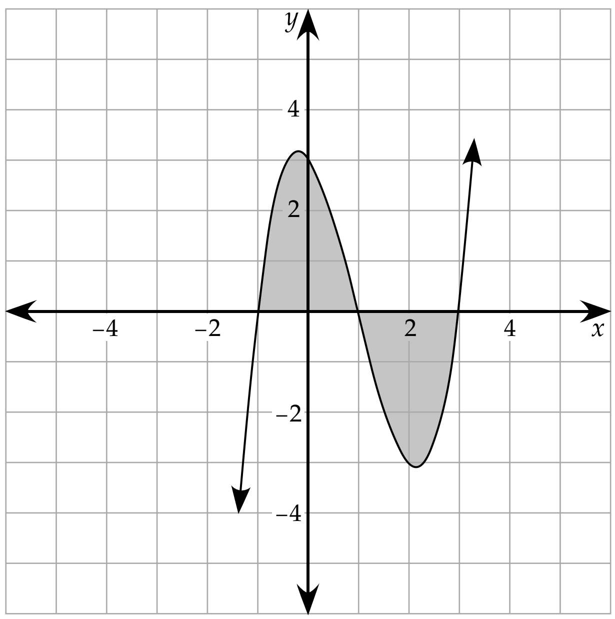
(1 mark for setting up the definite integral)

(1 mark for evaluating the antiderivative difference at x = 1 and x = 0)

$$\text{Area} = \int_0^1 (-x+1)\,dx = \left[-\frac{1}{2}x^2 + x\right]_0^1 = \left(-\frac{1}{2}(1)^2 + 1\right) - \left((0)^2 + 0\right)$$

$$= -\frac{1}{2} + 1 - 0 = 0.5 \text{ units}^2$$

5. Determine the area bounded by the curve $y = x^3 - 3x^2 - x + 3$ and the x-axis. *(5 marks)*



Answer: (Lesson 5)

(1 mark for separating the area above the axis from the area below the axis)

(1 mark for distinguishing the bounds for each area)

(1 mark for setting up the definite integral that will determine the area of the bounded region)

Total Area = Integral of Positive Interval + |Integral of Negative Interval|

$$= \int_{-1}^{1}\left(x^3 - 3x^2 - x + 3\right)dx + \left|\int_1^3\left(x^3 - 3x^2 - x + 3\right)dx\right|$$

(1 mark for determining the antiderivative)

(1 mark for evaluating the definite integral to determine the bounded area)

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$$\text{Total Area} = \int_{-1}^{1}\left(x^3 - 3x^2 - x + 3\right)dx - \int_{1}^{3}\left(x^3 - 3x^2 - x + 3\right)dx$$

$$= \left[\frac{1}{4}x^4 - x^3 - \frac{1}{2}x^2 + 3x\right]_{-1}^{1} - \left[\frac{1}{4}x^4 - x^3 - \frac{1}{2}x^2 + 3x\right]_{1}^{3}$$

$$= \left(\frac{1}{4}(1)^4 - (1)^3 - \frac{1}{2}(1)^2 + 3(1)\right) - \left(\frac{1}{4}(-1)^4 - (-1)^3 - \frac{1}{2}(-1)^2 + 3(-1)\right)$$

$$-\left(\frac{1}{4}(3)^4 - (3)^3 - \frac{1}{2}(3)^2 + 3(3)\right) + \left(\frac{1}{4}(1)^4 - (1)^3 - \frac{1}{2}(1)^2 + 3(1)\right)$$

$$= \frac{1}{4} - 1 - \frac{1}{2} + 3 - \frac{1}{4} - 1 + \frac{1}{2} + 3 - \frac{81}{4} + 27 + \frac{9}{2} - 9 + \frac{1}{4} - 1 - \frac{1}{2} + 3$$

$$= 8 \text{ units}^2$$

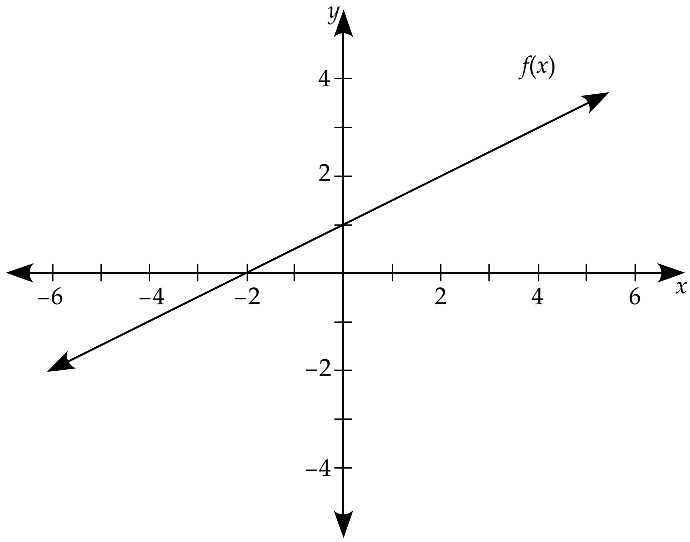
6. Find the values of each definite integral geometrically using the sketch of $f(x)$ as shown. *(4 marks)*.

a) $\int_{-2}^{4} f(x)\,dx$

b) $\int_{-4}^{0} f(x)\,dx$

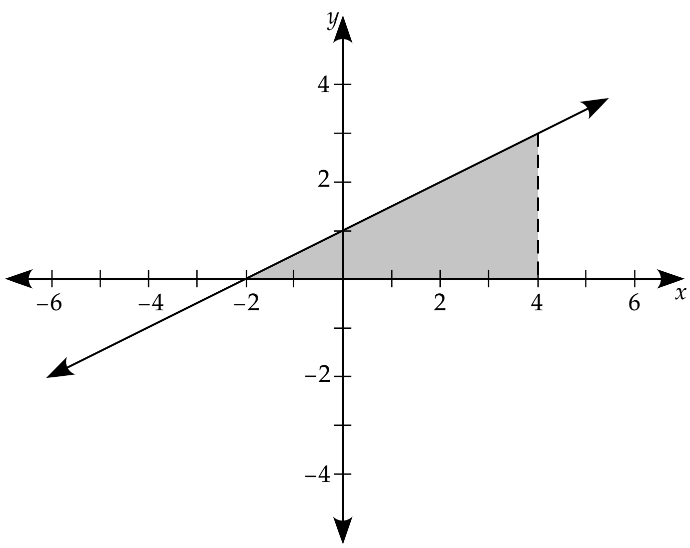
c) $\int_{2}^{4} f(x)\,dx$

d) $\int_{-4}^{0} |f(x)|\,dx$



Answers: (Lesson 4)

a)



Area = $6 \times 3 \div 2 = 9$

Therefore, $\int_{-2}^{4} f(x)\,dx = 9$.

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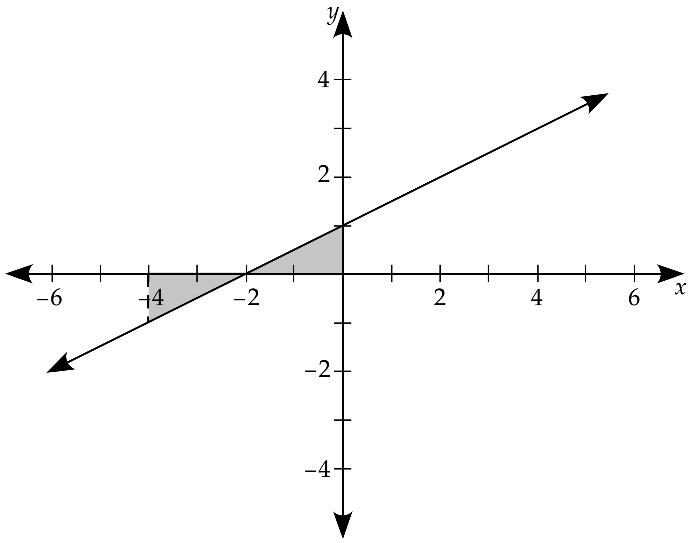
b)



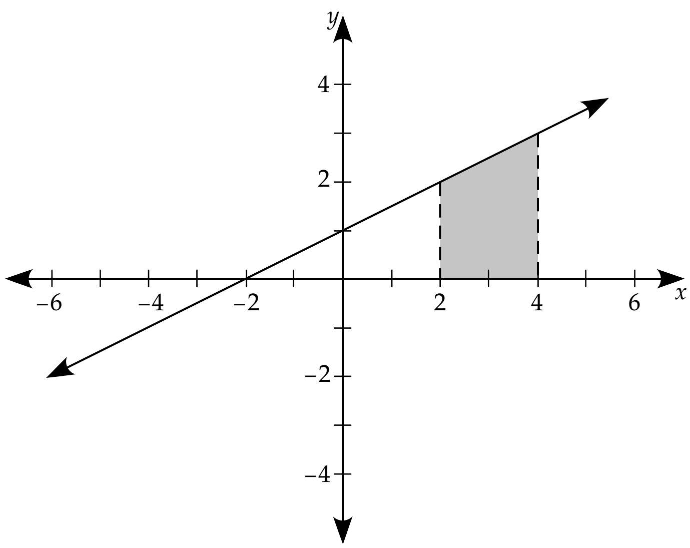
Area below x-axis
$= 2 \times 1 \div 2 = 1$

Area above x-axis
$= 2 \times 1 \div 2 = 1$

Area below x-axis has a negative definite integral, so

$$\int_{-4}^{0} f(x)\,dx = 1 + (-1) = 0.$$

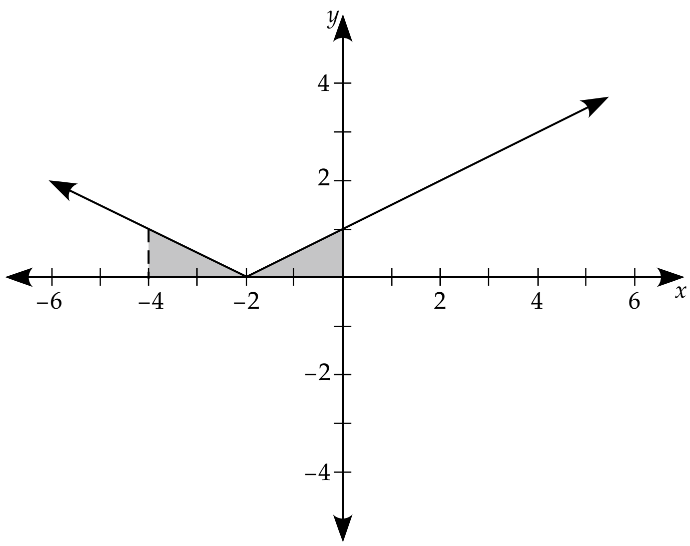
c)



Area = (2 × 2) + (2 × 1 ÷ 2) = 5

Therefore, $\int_{2}^{4} f(x)\,dx = 5.$

d)



The absolute value of the function is entirely above the x-axis.

Area = (2 × 1 ÷ 2) +
(2 × 1 ÷ 2) = 2

Therefore,

$$\int_{-4}^{0} \left| f(x) \right| dx = 1 + 1 = 2.$$